Chapter 1

Preliminaries

In this chapter, we first introduce some notations and recall related theory of *Sobolev Spaces*. We then collect some fundamental theories for partial differential equations, and recall the definitions of sub/super solutions and the theory on principal eigenvalue of the periodic-parabolic problems. Finally, we recall the well-known Krein-Rutman theorem.

1.1 Basic notations

In this section, we collect some notations. We shall write $\mathbb{R}$ for the set of real numbers, $\mathbb{R}^+$ for the set of positive real numbers, and N for the number of spatial dimension. Thus, $\mathbb{R}^N$ denotes the N-dimensional Euclidean space; $x = (x_1, ..., x_N)$ denotes an arbitrary point in the space, with norm $\|x\| = (\sum_{i=1}^{N}|x_i|^2)^{1/2}$.

Throughout this thesis, unless otherwise specified, we always use the symbol Ω to denote a bounded domain in $\mathbb{R}^N$, that means an arbitrary open connected set contained in some ball of sufficiently large radius; $\overline{\Omega}$ denotes the closure of Ω, $\partial\Omega$ denotes the boundary of Ω, so that $\overline{\Omega} = \Omega \cup \partial\Omega$. We use $|\Omega|$ and $\operatorname{diam}\Omega$ to represent the Lebesgue measure and the diameter of Ω, respectively. We also use ν to denote an outwardly directed unit vector normal to $\partial\Omega$ at any point on $\partial\Omega$, and ∂_ν denote the derivative in the direction ν.

Unless otherwise stated, X, Y and E will denote Banach spaces with norm denoted $\|\cdot\|_X$, etc., or simply $\|\cdot\|$ when the underlying space is clear. Moreover, $B_r(z) := \{x \in E : \|x - z\| < r\}$ represents the open ball in E, with center z and radius r.

If D is a subset of a Banach space E and $x \in E$ an arbitrary point, $d(x, D) := \inf\{\|x - y\| : y \in D\}$ denotes the distance of x from the set D. We also denote by $\text{int}(D)$ the set of all interior points of D.

It is understood that all the functions and quantities considered in this thesis are real-valued.

1.2 Sobolev spaces

1.2.1 Sobolev spaces only involving spatial variables

In this section, we recall the related theory of *Sobolev Spaces*. For more details concerning Sobolev spaces, we refer to [2, 51].

First of all, let us collect some classical function spaces that we shall encounter.

The vector space $C(\overline{\Omega})$ which consists of all bounded and uniformly continuous functions on Ω is a Banach space with norm given by

$$\|u\|_{C(\overline{\Omega})} := \sup_{x\in\Omega}|u(x)|.$$

The Banach space $C^m(\overline{\Omega})$ consists of all functions that are m times continuously differentiable over $\overline{\Omega}$ with norm given by

$$\|u\|_{C^m(\overline{\Omega})} = \sum_{|\beta|\leq m}|D^\beta u|_{C(\overline{\Omega})}.$$

Here, $\beta = (\beta_1, \cdots, \beta_N)$ is a multi-index with β_i nonnegative integers and $|\beta| = \Sigma_{i=1}^N \beta_i$, and

$$D^\beta u := \frac{\partial^{|\beta|}u}{\partial x_1^{\beta_1} \cdots \partial x_N^{\beta_N}}.$$

To avoid possible confusion later, let us remark that Du will be used to stand for the gradient of u:

$$Du(x) = (u_{x_1}, \cdots, u_{x_N})$$

and

$$|Du| = \left(\sum_{i=1}^{N} (u_{x_i})^2\right)^{1/2}.$$

We denote by $C_0^m(\Omega)$ the set of all functions with continuous partial derivatives of order up to m and with compact supports in Ω. If Ω is bounded, we also denote $C_0^m(\overline{\Omega})$ to be the set of all functions with continuous partial derivatives of order up to m and $u = 0$ on $\partial\Omega$. Obviously, in this case, we have $C_0^m(\Omega) \subset C_0^m(\overline{\Omega})$.

A function u is said to be Hölder continuous with exponent $\beta \in (0, 1]$ provided that, for all $x, y \in \Omega$,

$$|u(x) - u(y)| \leq C|x - y|^{\beta}$$

for some constant C. The β-th Hölder seminorm of such a function u is defined as

$$[u]_{C^{0,\beta}(\overline{\Omega})} := \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^{\beta}},$$

and the β-th Hölder norm is

$$\|u\|_{C^{0,\beta}(\overline{\Omega})} := \|u\|_{C(\overline{\Omega})} + [u]_{C^{0,\beta}(\overline{\Omega})}.$$

The Hölder space $C^{m,\beta}(\overline{\Omega})$ is a Banach space which consists of those functions u that are m-times continuously differentiable and whose m-th partial derivatives are Hölder continuous with exponent $\beta \in (0, 1]$ with norm given by

$$\|u\|_{C^{m,\beta}(\overline{\Omega})} := \sum_{|\alpha| \leq m} \|D^{\alpha}u\|_{C(\overline{\Omega})} + \sum_{|\alpha| = m} [D^{\alpha}u]_{C^{0,\beta}(\overline{\Omega})}.$$

As usual, $L^p(\Omega)$ denotes the Banach space of all functions in Ω that are measurable and p-summable with respect to Ω with $p \geq 1$. The norm in this space is defined by

$$\|u\|_{L^p(\Omega)} = \left(\int_{\Omega} |u|^p\right)^{1/p}.$$

For convenience, we sometimes just write $\|\cdot\|_p$ to represent the norm $\|\cdot\|_{L^p(\Omega)}$ when the underlying domain Ω is obvious.

Measurability and summability are always understood in the sense of Lebesgue. The elements in $L^p(\Omega)$ are the classes of equivalent functions in Ω. We denote by $L^{\infty}(\Omega)$ the space consisting of all functions u that are essentially bounded on Ω, functions being once again identified if they are equal a.e. on Ω. It is easily verified that the functional $\|\cdot\|_\infty$ defined by

$$\|u\|_{L^\infty(\Omega)} = \text{ess sup}_{x\in\Omega}|u|$$

is a norm of $L^\infty(\Omega)$.

Let Ω be a bounded domain in $\mathbb{R}^N$, u be a locally integrable function in Ω and β any multi-index. Then, a locally integrable function v is called the $\beta-$th weak derivative of u if it satisfies

$$\int_\Omega \phi v = (-1)^{|\beta|}\int_\Omega u D^\beta \phi$$

for all $\phi \in C_0^{|\beta|}(\Omega)$. We write $v = D^\beta u$. It should be noted that $D^\beta u$ is uniquely determined up to sets of measure zero.

For $p \geq 1$ and m a nonnegative integer, we let $W^{m,p}(\Omega)$ denote the Banach space of all elements of $L^p(\Omega)$ which have weak derivatives of the first m orders that are p-summable over Ω. The norm in $W^{m,p}(\Omega)$ is defined by

$$\|u\|_{W^{m,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha|\leq m}\int_\Omega |D^\alpha u|^p dx\right)^{1/p} & \text{if } 1 \leq p < \infty. \\ \sum_{|\alpha|\leq m}\text{ess sup}_{x\in\Omega}|D^\alpha u| & \text{if } p = \infty. \end{cases} \tag{1.2.1}$$

It is also well known that the subspace $C^\infty(\Omega)\cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$. Furthermore, it is clear that $C_0^\infty(\Omega) \subset W^{m,p}(\Omega)$ for any nonnegative integer m and constant $p \geq 1$. We denote by $W_0^{m,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

Concerning the domains in $\mathbb{R}^N$, we write $D' \subset\subset D$ for $D', D \subset \mathbb{R}^N$ if the closure $\overline{D'}$ of D' is a compact subset of D.

Before introducing the embedding theorems in Sobolev spaces, we need to give the following definitions concerning the domain Ω.

Definition 1.2.1 *We say that Ω satisfies the interior ball condition at some point $x_0 \in \partial\Omega$ if there exists a ball B_{r_0} of radius $r_0 > 0$ such that $B_{r_0} \subset \overline{\Omega}$ and $\overline{B_{r_0}} \cap \partial\Omega = \{x_0\}$. Furthermore, Ω is said to satisfy the uniform interior ball condition if there exists an $R > 0$ such that for any $x \in \partial\Omega$, there is a ball $B_{R,x}$ of radius R such that $B_{R,x} \subset \overline{\Omega}$ and $\overline{B_{R,x}} \cap \partial\Omega = \{x\}$.*

Definition 1.2.2 *Let $k \in N$. We say $\partial\Omega$ is C^k if for each point $x \in \partial\Omega$ there exist $r > 0$ and a C^k function $f : \mathbb{R}^{N-1} \to \mathbb{R}$ such that, upon relabeling and reorienting the coordinate axes if necessary, we have*

$$\Omega \cap B_r(x) := \{x \in B_r(x) \mid x_N > f(x_1, ..., x_{N-1})\}$$

for $k \in \mathbb{N}$. Likewise, $\partial\Omega$ is C^∞ if $\partial\Omega$ is C^k for all $k \in \mathbb{N}$.

The most important properties of Sobolev spaces are the embedding theorems and Sobolev inequalities; they provide the vital links between weak or strong solutions and classical solutions through the regularity analysis of weak or strong solutions. A Banach space X_1 is said to be continuously imbedded into a Banach space X_2, written by $X_1 \hookrightarrow X_2$, if $X_1 \subset X_2$ and the injection mapping from X_1 to X_2 is continuous. The embedding is said to be compact, written by $X_1 \hookrightarrow\hookrightarrow X_2$, if the injection mapping is moreover compact.

Theorem 1.2.1 *(see Theorem 7.10 and Corollary 7.11 of [51])*

$$W_0^{1,p}(\Omega) \hookrightarrow L^{Np/(N-p)}(\Omega), \;\; \textit{if } 1 \le p < N; \tag{1.2.2}$$

$$W_0^{1,p}(\Omega) \hookrightarrow C^\lambda(\overline{\Omega}), \; \lambda = 1 - \frac{N}{p}, \;\; \textit{if } p > N.$$

Moreover, there exits a constant $C = C(N, p)$ such that, for any $u \in W_0^{1,p}(\Omega)$,

$$\|u\|_{Np/(N-p)} \le C\|Du\|_p \;\; \textit{if } 1 \le p < N \tag{1.2.3}$$

and

$$\|u\|_\infty \le C|\Omega|^{1/N-1/p}\|Du\|_p, \quad [u]_\lambda \le C[1 + (\operatorname{diam}\Omega)^\lambda]\|Du\|_p, \;\; \textit{if } p > N.$$

Remark 1.2.1 *The embedding* (1.2.2) *and the inequality* (1.2.3) *are true also when Ω is unbounded. If $p = N$, then $W_0^{1,p}(\Omega) \hookrightarrow L^\phi(\Omega)$, where $\phi = \exp(|s|^{N/(N-1)}) - 1$ and $L^\phi(\Omega)$ is the Orlicz space. See Theorem 7.15 of [51] for more details.*

By iterating the result of Theorem 1.2.1, we can arrive at an extension to the space $W_0^{m,p}(\Omega)$.

Theorem 1.2.2

$$W_0^{m,p}(\Omega) \hookrightarrow L^{Np/(N-mp)}(\Omega), \quad \textit{if } mp < N;$$

$$W_0^{m,p}(\Omega) \hookrightarrow C^{k,\lambda}(\overline{\Omega}), \ \textit{if } 0 \le k < m - \frac{N}{p} < k+1, \ \lambda = m - \frac{N}{p} - k.$$

Remark 1.2.2 *In general, in Theorem 1.2.2, $W_0^{m,p}(\Omega)$ can not be replaced by $W^{m,p}(\Omega)$. However, this replacement can be made for a large class of domains Ω, which includes domains with C^1 boundaries.*

Theorem 1.2.3 *(see Theorem 7.22 of [51])*

$$W_0^{m,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega), \ \textit{if } mp < N, \ q < \frac{Np}{N-mp};$$

$$W_0^{m,p}(\Omega) \hookrightarrow C^{k,\mu}(\overline{\Omega}), \ \textit{if } 0 \le k < m - \frac{N}{p} < k+1, \ \mu < m - \frac{N}{p} - k.$$

Remark 1.2.3 *Theorem 1.2.3 is still valid when $W_0^{m,p}(\Omega)$ is replaced by $W^{m,p}(\Omega)$ if Ω has C^1 boundary. See Theorem 7.26 of [51].*

1.2.2 Sobolev spaces involving spatial and temporal variables

Let Q be a domain in space and time. We denote by $C^{2,1}(Q)$ the space of functions which are twice continuously differential in the spatial variable x and once in the time variable t. If $u \in L^p(Q)$, then $u_t, \ D_x u$ and $D_x^2 u$ denote the time derivative, and first and second spatial derivatives of u in the sense of distribution. Sometimes, we also use the notation ∇u and $D^2 u$ for $D_x u$ and $D_x^2 u$.

We denote by $W^{2,1;p}(Q)$ the space of functions $u \in L^p(Q)$ satisfying $u_t, \ D_x u, \ D_x^2 u \in L^p(Q)$, endowed with the norm

$$\|u\|_{2,1;p} = \|u\|_{2,1;p;Q} := \|u\|_{p;Q} + \|D_x u\|_{p;Q} + \|D_x^2 u\|_{p;Q} + \|u_t\|_{p;Q}.$$

We now consider a special case that Q is a cylinder, that is, $Q = Q_T = \Omega \times (0,T)$ where Ω is an arbitrary domain in $\mathbb{R}^N$ and $0 < T < \infty$. Given $\alpha \in (0,1]$, we set

$$[u]_{\alpha;Q_T} = \sup\left\{ \frac{|u(x,t) - u(y,s)|}{|x-y|^\alpha + |t-s|^{\alpha/2}} \ : \ x, y \in \Omega, \ t, s \in (0,T), \ (x,t) \neq (y,s) \right\}.$$

Let k be a nonnegative integer, $\alpha \in (0,1)$ and $a = k + \alpha$. Then, we put

$$\|u\|_{a;\,a/2;\,Q_T} = \sum_{|\beta|+2j \le k} \max_{\overline{Q}} |D_x^\beta D_t^j u| + \sum_{|\beta|+2j=k} [D_x^\beta D_t^j u]_{\alpha;\,Q_T},$$

and $C^{a,a/2}(\overline{Q_T}) := \{u : \ \|u\|_{a;\,a/2;\,Q_T} < \infty\}$. It is easy to see that $C^{a,a/2}(Q_T)$ is a Banach space with the norm $\|\cdot\|_{a;\,a/2;\,Q_T}$.

Note that, if $p > N+2, \ a < 2-(N+2)/p$ and Ω is a smooth bounded domain (for example, if Ω satisfies the so-called uniform interior cone condition), then we have

$$W^{2,1;p}(Q_T) \hookrightarrow C^{a,a/2}(\overline{Q_T}).$$

For the above statement and more general embedding theorems for anisotropic spaces, see Lemmas II.3.3, II.3.4 of [65] and Theorem 6.9 of [74].

1.3 Basic theorems for partial differential equations

In this section, we collect some fundamental theories for partial differential equations, including regularity theories for strong and classical solutions, maximum principles, comparison principles for linear parabolic partial differential equations and semigroup theory.

We assume that Ω is an arbitrary domain and $T > 0$. Set $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$ and $P_T = S_T \cup (\overline{\Omega} \times \{0\})$. We consider the second-order parabolic differential operators of the form

$$Au = -\sum_{i,j=1}^{N} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u,$$

where measurable coefficients $a_{ij}, \ b_i$ and c satisfy the uniform parabolic condition

$$\sum_{i,j} a_{ij}(x,t)\xi_i\xi_j \ge \lambda|\xi|^2 \quad \text{for all } (x,t) \in Q_T, \ \xi \in \mathbb{R}^N, \tag{1.3.1}$$

with $\lambda > 0$ and a uniform bound

$$|a_{ij}|, \ |b_i|, \ |c| \le \Lambda. \tag{1.3.2}$$

Now, we consider the linear problem:

$$u_t + Au = f(x,t), \tag{1.3.3}$$

where f is a given function in Q_T.

By a classical solution $u(x,t)$, we mean that $u(x,t)$ satisfies (1.3.3) pointwise in Q_T. A strong solution of (1.3.3) is a function $u(x,t) \in W^{2,1;1}(Q_T)$ satisfying (1.3.3) a.e. in Q_T.

The following result (see, for example, [65, 66, 83]) contains the basic interior and global parabolic L^p-estimates, and the assertion of existence and uniqueness for solutions.

Theorem 1.3.1 *(see Theorems 7.13, 7.15, 7.17 and Corollary 7.16 of [66]) Let Ω be an arbitrary bounded domain in $\mathbb{R}^N$. Assume that* (1.3.1) *and* (1.3.2) *hold. Let $u \in W^{2,1;p}_{loc} \cap L^p(Q_T)$, $1 < p < \infty$, be a strong solution of* (1.3.3)*, where $a_{ij} \in C(\overline{Q_T})$ and $f \in L^p(Q_T)$.*

(i) *If $Q' \subset Q_T$ and $dist(Q', P_T) > 0$, then*

$$\|u\|_{2,1;p;Q'} \leq C(\|u\|_{p;Q_T} + \|f\|_{p;Q_T}), \tag{1.3.4}$$

where C depends only on $n, p, Q_T, \lambda, \Lambda$ and the moduli of continuity of the coefficients $a_{i,j}$.

(ii) *If Ω is of class C^2 and either Σ is an open subset of S_T or $\Sigma = P_T$. Assume that $u \in W^{2,1;p}(Q_T)$ and $u = 0$ on Σ. Let $Q' \subset Q_T$, $\operatorname{dist}(Q', P_T \setminus \Sigma) > 0$ if $\Sigma \neq P_T$. Then* (1.3.4) *is true, where C depends also on Σ.*

(iii) *If Ω is of class C^2, $\varphi \in W^{2,1;p}(Q_T)$, $f \in L^p(Q_T)$. Then, there exists a unique strong solution u of* (1.3.3) *satisfying $u = \varphi$ on P_T. Moreover, u satisfies the estimates*

$$\|u\|_{2,1;p;Q_T} \leq C(\|f\|_{p;Q_T} + \|\varphi\|_{p;Q_T}).$$

The following result contains the basic interior-boundary parabolic Schauder estimates and the existence and uniqueness statements. We restrict ourselves to the global estimates; local estimates can be easily derived by applying this theorem to the function $u\psi$ where ψ is a smooth cut-off function.

Theorem 1.3.2 *(see Theorems 4.28 and 5.14 of [66]) Let* $\alpha \in (0,1)$ *and* Ω *be an arbitrary bounded domain of class* $C^{2+\alpha}$. *Assume that* (1.3.1) *holds and* $a_{ij},\ b_i,\ c,\ f \in C^{\alpha,\alpha/2}(Q_T)$, $\varphi \in C^{2+\alpha,1+\alpha/2}(Q_T)$.

(i) *If* $u \in C^{2+\alpha,1+\alpha/2}(Q_T)$ *is a solution of* (1.3.3) *satisfying* $u = \varphi$ *on* P_T, *then*

$$\|u\|_{2+\alpha,1+\alpha/2;\,Q_T} \le C(\|u\|_{\infty;\,Q_T} + \|f\|_{\alpha;Q_T} + \|\varphi\|_{2+\alpha,1+\alpha/2;\,Q_T}),$$

where C *depends only on* n, α, Q_T, λ *and the norms of* $a_{ij},\ b_i,\ c$ *in* $C^{\alpha,\alpha/2}(Q_T)$.

(ii) *There exists a unique solution* $u \in C(\overline{Q_T}) \cap C^{2,1}(Q_T)$ *of* (1.3.3) *satisfying* $u = \varphi$ *on* P_T. *If* $\varphi_t + A\varphi = f$ *on* $\partial\Omega \times \{0\}$, *then* $u \in C^{2+\alpha,1+\alpha/2}(Q_T)$ *and*

$$\|u\|_{2+\alpha,1+\alpha/2;\,Q_T} \le C(\|f\|_{\alpha;\,Q_T} + \|\varphi\|_{2+\alpha,1+\alpha/2;\,Q_T}).$$

Remark 1.3.1 *When it comes to Neumann boundary conditions, under the assumptions that* Ω *is bounded,* (1.3.1) *and* (1.3.2), $a_{ij} \in C(\overline{Q_T}),\ 1 < p < \infty$ *and* $f \in L^p(Q_T)$, *if* $u \in W^{2,1;p}_{loc} \cap L^p(Q_T)$ *is a strong solution of* (1.3.3) *and satisfies* $\partial_\nu u = 0$ *on* S_T *and* $u = 0$ *on* $\Omega \times \{0\}$, *then we have the estimates*

$$\|u\|_{2,1;\,p;\,Q_T} \le C\|f\|_{p;\,Q_T}.$$

Similarly, Theorem 1.3.2 remains valid if the condition $u = \varphi$ *on* P_T *is replaced by* $\partial_\nu u = \partial_\nu \varphi$ *on* S_T *and* $u = \varphi$ *on* $\Omega \times \{0\}$. *Theses facts follow from Theorem 7.20 of [67] (or Theorem 8.2 of [29]) and Theorem 4.31 of [67], respectively. For existence and uniqueness results analogous to Theorem 1.3.2 (ii), one may refer to see Theorem 4.31 of [67].*

We next present the classical maximum principle and comparison principle for parabolic equations.

Theorem 1.3.3 *(see Theorem 2 of [81]) Assume that* (1.3.1) *holds,* a_{ij}, b_i *are bounded in* Q_T, *and* $c(x,t) \ge 0$ *in* Q_T *and is bounded from above. Let* $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ *and satisfy*

$$u_t + Au \le 0 \ \ (\ge 0) \quad \text{in } Q_T.$$

If $u \leq M$ $(u \geq m)$ *on* $\overline{Q_T}$ *and there exists* $(x_1, t_1) \in Q_T$ *such that* $u(x_1, t_1) = M$ $(u(x_1, t_1) = m)$. *Assume further* $M \geq 0$ $(m \leq 0)$ *if* $c(x,t) \not\equiv 0$. *Then,*

$$u(x,t) = M \ \ (u(x,t) = m) \quad \text{in } Q_{t_1}.$$

The following form of maximum principle is usually referred to as the Hopf boundary Lemma.

Theorem 1.3.4 *(see Theorem 3 of [81]) Assume that* (1.3.1) *holds,* a_{ij}, b_i *are bounded in* Q_T*, and* $c(x,t) \geq 0$ *in* Q_T *and is bounded from above. Let* $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ *and satisfy*

$$u_t + Au \leq 0 \ \ (\geq 0) \quad \text{in } Q_T.$$

If $u \leq M$ $(u \geq m)$ *on* $\overline{Q_T}$ *and there exists* $(x_2, t_2) \in S_T$ *such that* $u(x_2, t_2) = \max_{\overline{Q_T}} u(x,t) = M$ $(u(x_2, t_2) = \min_{\overline{Q_T}} u(x,t) = m)$. *Assume further* $M \geq 0$ $(m \leq 0)$ *if* $c(x,t) \not\equiv 0$. *If* $\partial_\nu u(x_2, t_2)$ *exists and* $\partial\Omega$ *satisfies the interior ball condition at* x_2*, then*

$$\partial_\nu u(x_2, t_2) > 0 \ \ (\partial_\nu u(x_2, t_2) < 0).$$

Finally, we recall the useful comparison principle for parabolic equations as follows.

Theorem 1.3.5 *(see Theorem 8 of [81]) Assume that* (1.3.1) *and* (1.3.2) *hold. Let* $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ *and satisfy*

$$u_t + Au \geq 0 \ \text{ in } Q_T, \ \ a\partial_\nu u + b(x,t)u \geq 0 \ \text{ on } S_T \ \text{ and } \ u(x,0) \geq 0 \ \text{ in } \Omega.$$

where, either $a = 0,\ b = 1$ *or* $a = 1$ *and* $b(x,t) \geq 0$. *If* $a = 1$*, we further assume that* $\partial\Omega$ *satisfies the interior ball condition. Then,*

$$u(x,t) \geq 0 \quad \text{in } Q_T.$$

If further $u(x,0) \not\equiv 0$ *in* Ω*, then*

$$u(x,t) > 0 \quad \text{in } Q_T.$$

In what follows, we recall some semigroup theories on the initial boundary value problem of linear parabolic equations.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $C^{2+\theta}$ boundary $\partial\Omega$, and let $A = A(x,t,D)$ be the second-order parabolic differential operator given as before. Assume that $A(x,t,D)$ is uniformly elliptic for each $t \in [0,T]$, where $T > 0$ is a given positive number. We also assume that

$$a_{ij}, b_i, c \in C^{\theta,\theta/2}(\overline{Q}_T), \; Q_T = \Omega \times [0,T].$$

Let $B = B(x,D)$ be given by

$$B(x,D)v = v \quad \text{or} \quad B(x,D)v = \partial_\nu v + b_0(x)v,$$

where $\nu : \partial\Omega \to \mathbb{R}^N$ is an outward pointing, nowhere tangential vector field of class $C^{1+\theta}$, and $b_0 : \partial\Omega \to \mathbb{R}$ is of class $C^{1+\theta}$. We notice that $B = B(x,D)$ is independent of t.

Consider the initial-boundary value problem

$$\begin{cases} \partial_t u + A(x,t,D)u = f(x,t) & \text{in } \Omega \times (0,T], \\ Bu = 0 & \text{on } \partial\Omega \times (0,T], \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.3.5}$$

where

$$f \in C^{\theta,\theta/2}(\overline{Q}_T) \;\text{ and }\; u_0 \in X_0 := L^p(\Omega) \;\text{ for some }\; p > 1.$$

Let

$$X_1 := W_B^{2,p}(\Omega) := \{v \in W^{2,p}(\Omega) : Bv = 0\}.$$

Then there exist a family of Banach spaces X_α, $0 \le \alpha \le 1$, defined by the fractional power A^α of the differential operator A, with the properties:

(i) $0 \le \alpha < \beta \le 1$ implies that X_β embeds into X_α compactly;

(ii) if $0 < \alpha < 1$, then for any given $\epsilon > 0$, there exists $C = C(\epsilon) > 0$ such that $\|v\|_{X_\alpha} \le \epsilon\|v\|_{X_1} + C\|v\|_0 \; \forall v \in X_1$;

(iii) X_α compactly embeds into $C_B^{1+\lambda}(\overline{\Omega}) := \{v \in C^{1+\lambda}(\overline{\Omega}) : Bv = 0\}$ if $\frac{1}{2} + \frac{N}{2p} < \alpha \leq 1$ and $0 \leq \lambda < 2\alpha - 1 - \frac{N}{p}$.

By Theorem 1.2.1 on page 43 of Amann [4], problem (1.3.5) has a unique solution $u \in C^\theta((0,T], X_1) \cap C^{1+\theta}((0,T], X_0)$. Moreover, if $u_0 \in X_1$, then $u \in C^1([0,T], X_0)$.

Therefore, for $t > 0$, $u(\cdot,t) \in X_1 = W_B^{2,p}(\Omega) \hookrightarrow C^{1+\lambda}(\overline{\Omega})$ if $p > N$. One can actually use the Hölder theory to see that $u \in C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega} \times (0,T])$.

The unique solution of (1.3.5) can be expressed by a constant of variation formula:

$$u(\cdot,t) = U(t,0)u_0 + \int_0^t U(t,\tau)f(\cdot,\tau)d\tau \ (0 \leq t \leq T), \tag{1.3.6}$$

where $U(t,0)u_0$ is the unique solution to (1.3.5) with $f \equiv 0$, and $U(t,\tau)$ satisfies:

(i) for each $v \in X_0$

$$U(\cdot)v : \Delta := \{(t,\tau) : 0 \leq \tau \leq t \leq T\} \to X_0 \text{ is continuous,}$$

(ii) $U(t,t) = I, \ \ U(s,t)U(t,\tau) = U(s,\tau) \ (0 \leq \tau \leq t \leq s \leq T)$,

(iii) $U(t,\tau) \in L(X_0, X_1)$ for $0 \leq \tau < t \leq T$,

(iv) for $0 \leq \tau < t \leq T$,

$$\begin{aligned} &\|U(t,\tau)\|_{L(X_\alpha,X_\beta)} \leq C(\alpha,\beta) && \text{for } 0 \leq \beta < \alpha \leq 1, \\ &\|U(t,\tau)\|_{L(X_\alpha,X_\beta)} \leq C(\alpha,\beta,\gamma)(t-\tau)^{-\gamma} && \text{for } 0 \leq \alpha \leq \beta, \beta - \alpha < \gamma < 1, \end{aligned}$$

(v) for $0 \leq \alpha < \beta \leq 1, 0 \leq \gamma < \beta - \alpha$ and $(t,\tau),(s,\tau) \in \Delta$,

$$\|U(t,\tau) - U(s,\tau)\|_{L(X_\beta,X_\alpha)} \leq C(\alpha,\beta,\gamma)|t-s|^\gamma,$$

(vi) for $0 \leq \alpha < 1$, $g \in C([0,T], X_0)$ and $0 \leq \gamma < 1 - \alpha$,

$$\left\| \int_0^t U(t,\tau)g(\tau)d\tau - \int_0^s U(s,\tau)g(\tau)d\tau \right\|_{X_\alpha} \leq C(\alpha,\gamma)|t-s|^\gamma \max_{0\leq\tau\leq T} \|g(\tau)\|_{X_0}.$$

1.4 Sub-super solutions and eigenvalues of periodic-parabolic problems

Let $A(x,t,D)$ and B be defined as in the preceding section. In this section, we recall the definitions of sub/super solutions and principal eigenvalue of the periodic-parabolic problems, which will be used frequently later.

The first one concerns the super- and sub-solutions to

$$\begin{cases} \partial_t u + A(x,t,D)u = f(x,t,u) & \text{in } \Omega \times [0,T], \\ Bu = 0 & \text{on } \partial\Omega \times [0,T], \\ u(x,0) = u(x,T) & \text{in } \Omega, \end{cases} \tag{1.4.1}$$

where f is continuous and $f(\cdot,\cdot,u)$ is of class $C^{\theta,\theta/2}(\overline{\Omega} \times [0,T])$ uniformly for u in bounded subsets of $\mathbb{R}$, $\partial_u f$ is continuous on $\overline{\Omega} \times [0,T] \times \mathbb{R}$, and there exists a continuous function $c : (0,\infty) \to (0,\infty)$ such that

$$|f(x,t,u)| \leq c(\rho) \ \forall \rho > 0, \ \forall (x,t,u) \in \overline{\Omega} \times [0,T] \times [-\rho,\rho].$$

Following Hess [53], a function

$$\underline{u} \in C^{1+\theta,\frac{1+\theta}{2}}(\overline{\Omega} \times [0,T]) \cap C^{2,1}(\overline{\Omega} \times (0,T])$$

is called a subsolution for the $T-$periodic problem (1.4.1) if

$$\begin{cases} \partial_t \underline{u} - \Delta \underline{u} \leq f(x,t,\underline{u}) & \text{in } \Omega \times (0,T], \\ B\underline{u} \leq 0 & \text{on } \partial\Omega \times (0,T], \\ \underline{u}(x,0) \leq \underline{u}(x,T) & \text{in } \Omega. \end{cases}$$

A supersolution $\overline{u}$ is defined by reversing the inequality signs.

By Theorem 22.3 of [53], we know that if $\underline{u} \leq \overline{u}$ is a pair of sub- and super-solutions to (1.4.1), then (1.4.1) has a solution u satisfying $\underline{u} \leq u \leq \overline{u}$.

The above definition and existence result can be easily extended to the case that the boundary condition $Bu = 0$ is replaced by $Bu = B\psi$, where $\psi \in C^{2+\theta,1+\theta/2}(\overline{\Omega} \times [0,T])$. In such a case we simply let $v = u - \psi$ and the problem reduces to the standard case. A situation that arises frequently later in the present thesis is that $\partial\Omega$ has two components Γ_1 and Γ_2, and the boundary condition is given by $u|_{\Gamma_1} = \xi$, $\partial_\nu u|_{\Gamma_2} = 0$, where $\xi \in C^{2+\theta,1+\theta/2}(\overline{\Omega} \times [0,T])$. In such a case, we may choose a smooth function $\sigma(x)$ such that $\sigma = 1$ near Γ_1 and $\sigma = 0$ near Γ_2, and let $\psi = \sigma\xi$. Then it is easily seen that the given boundary condition is equivalent to $B_0 u = B_0\psi$ on $\partial\Omega$, where $B_0 u = u$ on Γ_1 and $B_0 u = \partial_\nu u$ on Γ_2.

Let us also recall the theory of the principal eigenvalue for a linear periodic-parabolic eigenvalue problem. For any given $T-$periodic function $g \in C^{\theta,\theta/2}(\overline{\Omega} \times \mathbb{R})$, we consider the eigenvalue problem:

$$\begin{cases} \partial_t\varphi - \Delta\varphi + g(x,t)\varphi = \lambda\varphi & \text{in } \Omega \times \mathbb{R}, \\ \partial_\nu\varphi = 0 & \text{on } \partial\Omega \times \mathbb{R}, \\ \varphi(x,t) = \varphi(x,t+T) & \text{in } \Omega \times \mathbb{R}. \end{cases} \tag{1.4.2}$$

By Proposition 14.4 of [53], we know that (1.4.2) has a principal eigenvalue $\lambda = \lambda_1(g)$, which corresponds to a positive eigenfunction $\varphi \in C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega} \times \mathbb{R})$. Such a function φ is usually called a principal eigenfunction.

1.5 The Krein-Rutman Theorem

In this section, we collect the well-known Krein-Rutman theorem and some of its consequences in an ordered Banach space.

Let E be an ordered Banach space, i.e., a real Banach space with a partial ordering induced by an order cone P (a closed convex set such that $P \cap (-P) = \{0\}$). For $x,\ y \in E$, we write

$$x \geq y \quad \text{if } x - y \in P,$$

and

$$x > y \quad \text{if } x - y \in P \setminus \{0\}.$$

If P has a nonempty interior int(P), we also write

$$x \gg y \quad \text{if } x - y \in \text{int}(P).$$

Let (E, P) be an ordered Banach space. In the dual space E^*, set

$$P^* := \{x^* \in E^* \,:\, (x^*, x) \geq 0,\ \forall x \in P\}.$$

Then, P^* is a closed convex set with vertex at 0, but it is in general not an order cone, i.e., $P^* \cap (-P^*) \neq \{0\}$. We write

$$x^* \geq 0 \quad \text{if } x^* \in P^*,$$

$$x^* > 0 \quad \text{if } x^* \in P^* \ \text{ and } \exists x \in P \text{ such that } (x^*, x) > 0.$$

and

$$x^* \gg 0 \quad \text{if } (x^*, x) > 0 \ \text{ for all } x \in P \setminus \{0\}.$$

The order cone $P \subset E$ is said to be total if $E = \overline{P - P}$ and be solid if $E = P - P$. In the case that P is total, P^* becomes an order cone.

Let $\mathcal{L}(E)$ be the set of all bounded linear operators from E to itself. Assume that $K \in \mathcal{L}(E)$. Then, K is called positive if $K(P) \subset P$. By the spectrum $\sigma(K)$, we understand the spectrum in the sense of complexification (see, for example, [22]). Denote by spr(K) the spectral radius of K.

Now, we recall the well-known Krein-Rutman theorem.

Theorem 1.5.1 *(see Theorem 7.1 of [53]) Let (E, P) be an ordered Banach space where the order cone is total, and let $K \in \mathcal{L}(E)$ be compact and positive. Assume $r :=$ spr$(K) > 0$. Then, r is an eigenvalue of K and of the adjoint operator K^*, with eigenfunctions $x > 0,\ x^* > 0$.*

If the order cone P has nonempty interior, hence P is solid, the further properties can be deduced. In this case, $K \in \mathcal{L}(E)$ is called strongly positive if $K(P \setminus \{0\}) \subset$ int(P). In what follows, by an algebraically simple eigenvalue, we mean that the algebraic multiplicity of an eigenvalue is one.

Theorem 1.5.2 *(see Theorem 7.2 of [53]) Let (E, P) be an ordered Banach space with $int(P) \neq \emptyset$, and let $K \in \mathcal{L}(E)$ be compact and strongly positive. Then, $r := spr(K) > 0$ and r is the unique eigenvalue of K having a positive eigenfunction x. Moreover, $x \gg 0$ and r is an algebraically simple eigenvalue. Further, r is also an algebraically simple eigenvalue of the adjoint operator K^* with eigenfunctions $x^* \gg 0$. Finally, $|\lambda| < r$ for all $\lambda \in \sigma(K)$, $\lambda \neq r$.*

We point out that, in general, $r = \text{spr}(K)$ is called the principal eigenvalue of K.

We consider now the inhomogeneous equation

$$\lambda u - Ku = h \quad \text{in } E, \quad h > 0. \tag{1.5.1}$$

Theorem 1.5.3 *(see Theorem 7.3 of [53]) Under the assumptions of Theorem 1.5.2, the following assertion holds.*

(i) (1.5.1) *has a unique solution u if $\lambda > r$, and $u \gg 0$;* (1.5.1) *has no positive solution if $\lambda \leq r$.*

(ii) (1.5.1) *admits no solution if $\lambda = r$.*

For the proofs of the results, we refer to [27, 63, 64].

Chapter 2

Background and outline of research

2.1 General introduction

A fundamental goal of theoretical ecology is to understand how the interactions of individual organizations with each other under a certain inhabiting environment determine the spatiotemporal structure of distribution of populations. Empirical evidence suggests that the spatial dispersal and the structure of environment can influence population interaction and distribution. In the past decades, many biologists and applied mathematicians have focused their attention on the role of spatial effects in maintaining biodiversity (especially the formation of so-called Turing pattern). See, for example, [13, 73]. Human activities frequently alter environments by fragmenting habitats and creating edges. Reaction-diffusion models provide a way to present a global description concerning the persistence or extinction of populations and the coexistence of interacting species, and have been intensively investigated in the past three decades for this purpose, see [13, 14, 72, 73, 84, 88] and the references therein.

In this thesis, our focus will be on the most basic population model: the reaction-diffusion logistic model, which is used to describe the spatial and temporal growth of a single species occupying an isolated habitat fragment. This model plays a key role in understanding the dynamics of various kinds of multispecies models such as Lotka-Volterra competition and predator-prey models. We will be concerned with the situation where the spatial variation and temporal periodicity both occur and discuss various qualitative properties of solutions to the periodic logistic equation under some degenerate assumptions on the parameters.

Our investigation shows that the temporal degeneracy causes a fundamental change of the dynamical behavior of the equation only when spatial degeneracy also exists. Moreover, by comparing all three basic cases (namely, only temporal degeneracy, only spatial degeneracy and both spatial and temporal degeneracies), we find that whether or not temporal degeneracy appears in the equation, the spatial degeneracy always induces fundamental changes of the behavior of the equation, though such changes differ significantly according to whether there is temporal degeneracy or not. Therefore, our study reveals the qualitative influence of degenerate spatial and temporal environments on the dynamical behaviors of the equation.

In the degenerate situation, it turns out that the mathematical analysis is much more involved than the classical case where no spatiotemporal degeneracy appears. In particular, we have to overcome several new and technical difficulties caused by the combination of spatial and temporal degeneracies, which require us to establish some new theories on an eigenvalue problem for periodic-parabolic operators over a varying cylinder and on some parabolic boundary blow-up problems.

2.2 The logistic model

The first serious attempt to model population dynamics is due to the work of Malthus (1798, [69]). Malthus assumed that human population can be expected to increase geometrically with time but the amount of land available to support them can only be expected to increase at most arithmetically. Based on this idea, Malthus characterized the population growth of a single species by using a density-independent or linear growth model of ordinary differential equation, now called the Malthus model.

Let $u(t)$ be the density of the single species at time t. Then, in the simplest form, the Malthus model is governed by

$$\begin{cases} \dfrac{du}{dt} = au, \quad t > 0; \\ u(0) = u_0. \end{cases} \tag{2.2.1}$$

Here, the constant a represents the intrinsic growth rate of the species, and the positive number u_0 accounts for the initial data of the density of the species u.

Obviously, for equation (2.2.1), the solution is given by

$$u(t) = u_0 e^{at}, \quad t \geq 0.$$

Hence, the model predicts the exponential growth if $a > 0$ or decay if $a < 0$ for the population.

The second significant step to population modeling is the introduction of population self-regulation. In 1838, Verhulst [86] found that the density of a population should affect its growth rate. Specifically, the logistic equation arises from the assumption that as the population density increases, the effects of crowding and resource depletion cause the birth rate to decrease and the death rate to increase. Taking into account these factors, still in the simplified version, the logistic equation was mathematically formulated as

$$\begin{cases} \dfrac{du}{dt} = a\Big(1 - \dfrac{u}{K}\Big)u, \quad t > 0; \\ u(0) = u_0 > 0. \end{cases} \tag{2.2.2}$$

Equation (2.2.2) is the standard form used in the biology literature to describe the density growth of a single species. Here, a is the intrinsic growth rate of the species and the positive constant K accounts for the carrying capacity of the habitat for the species.

Simple analysis shows that, equation (2.2.2) has a unique solution $u(t)$, which exists and is positive in $[0, \infty)$. Moreover, $u(t) \to K$ as $t \to \infty$. This implies that, the population will eventually stabilize at the unique positive equilibrium K.

In the above two models (2.2.1) and (2.2.2), we always assume that all individuals experience the same homogeneous environment. However, in reality, individual organizations are distributed in space and typically interact with the physical environment and other organizations. Many physical aspects of environment such as climate, chemical composition or physical structure can vary from place to place. As was pointed out in [13], there is considerable evidence from laboratory experiments showing that space can affect the dynamics of populations.

There are many ways that spatial effect can be included in mathematical models. One important and widely used class of spatial models is given by reaction-diffusion equations, where one treats space as a continuum and describes the distribution of populations in terms of densities that vary deterministically in time. In the present thesis, we will follow this line of approach.

Let $u(x,t)$ be the density of a single species at location x and time t. Thus, if the spatial and temporal variation of the environment is taken into account, the logistic equation (2.2.2) should be replaced by the following one

$$\begin{cases} \partial_t u - \operatorname{div}(d(x,t)\nabla u) = a(x,t)u - b(x,t)u^2, & x \in \Omega,\ t > 0, \\ \partial_\nu u = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

This equation describes the population density $u(x,t)$ of a single species with initial density $u_0(x)$, the diffusion rate d and intrinsic growth rate a in a habitat Ω that has carrying capacity a/b. The Neumann boundary condition means that the species is enclosed in Ω with no population flux across its boundary $\partial\Omega$.

In order to stress the main point of our approach and to avoid unnecessary complication with the notations, we take $d(x,t) = 1$ and $a(x,t) = a$ to be a positive constant. Furthermore, in the realistic world, since the natural living environment of the species is typically periodic in time (for example, daily, seasonal or yearly), we here consider the situation that $b(x,t)$ is a T-periodic, nonnegative continuous function. We also replace u^2 by u^p for some $p > 1$ since the mathematical treatment is the same. With these simplifications and generalizations, the above reaction-diffusion equation can be written as

$$\begin{cases} \partial_t u - \Delta u = au - b(x,t)u^p, & x \in \Omega,\ t > 0, \\ \partial_\nu u = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq, \not\equiv 0, & x \in \Omega. \end{cases} \tag{2.2.3}$$

When the inhabiting environment of the single species only involves spatial variation (that is, $b(x,t) \equiv b(x)$ is independent of t), there has been a great amount of research work on the nonnegative steady state solutions and their dynamical behaviors of (2.2.3). If $b(x) > 0$ on $\overline{\Omega}$, it is easy to prove that the nonnegative steady state solutions and their dynamical behaviors are very simple and similar to those of (2.2.2). If $b(x)$ vanishes in a proper subset of Ω, a good deal of work has been devoted to the studies of this degenerate logistic equation; see, e.g., [36, 46, 49], where it was revealed that such degeneracy can cause fundamental and qualitative changes in the properties of solutions to the logistic equation.

It should be mentioned that the region where b vanishes represents the extreme environmental situation that the species experiences no self-limitation for its growth there. A good understanding of such an extreme case is important in order to understand the scope of the possible behavior of the logistic model as the environment varies heterogeneously.

Moreover, in reality, the carrying capacity a/b of the habitat for the species should be heterogeneous in space. Therefore, it is more reasonable to assume that the coefficient $b(x,t)$ is a function of two variables x and t with period T in t for some given constant $T > 0$.

With the above assumption, in the case that $b(x,t) > 0$, a rather complete understanding on the existence of T-periodic positive solution and its dynamical behavior was obtained in [53]. In such a case it turns out that the qualitative properties of solutions to (2.2.3) is essentially the same as that of (2.2.2), and so the environmental periodicity will not cause any fundamental change of dynamical behaviors.

However, to our knowledge, there is no work to consider the situation where $b(x,t)$ vanishes in a subdomain of $\Omega \times (0,T)$ and to investigate the impact of this degeneracy on the dynamics of the periodic-parabolic logistic equation (2.2.3). The focus of this thesis will be on such a case. In particular, we will examine the effect of a combination of spatial and temporal degeneracies on the behavior of (2.2.3), and reveal some new phenomena caused by the inclusion of temporal degeneracy in the model.

The remainder of this chapter is arranged as follows. In section 2.3, we will first recall some basic results of (2.2.3) in the case of $b(x,t) \equiv b(x)$ and $b(x) = 0$ in a proper subdomain of Ω. Then, in section 2.4, we collect the results of (2.2.3) in the case of $b(x,t) > 0$ on $\overline{\Omega} \times [0,T]$.

Finally, in section 2.5, we shall outline the main investigation of this thesis.

Before ending this section, we would like to mention some further existing research work related to (2.2.3). When $b(x,t) = b(x)$ only depends on the spatial variable x and $b(x)$ is sign-changing, the behavior of (2.2.3) is completely different; see, for example, [1, 5, 8, 9, 10, 12, 85]. When the domain Ω is replaced by the entire space $\mathbf{R}^N$, the existing work can be found in, for example, [15, 35]. On the other hand, when $b(x,t)$ is T-periodic and $b(x,t) < 0$, [7, 25, 47, 48, 54, 75, 82] dealt with the spatially bounded domains while [21, 23, 24] for spatially unbounded domains. There is an extensive research on the multispecies population models; see, for example [16, 17, 19, 26, 30, 31, 32, 33, 39, 40, 41, 42, 43, 44, 45, 52, 55, 56, 57, 58, 59, 68, 78, 79, 80] and the references therein.

2.3 The logistic model with spatial degeneracies

In this section, we recall the existing results on the logistic equation in the case that the parameter function $b(x,t)$ depends only on the spatial variable x, that is $b(x,t) = b(x)$. Then, (2.2.3) becomes

$$\begin{cases} \partial_t u - \Delta u = au - b(x)u^p, & x \in \Omega,\ t > 0, \\ \partial_\nu u = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq, \not\equiv 0, & x \in \Omega. \end{cases} \tag{2.3.1}$$

The corresponding steady state problem is given by

$$\begin{cases} -\Delta u = au - b(x)u^p, & x \in \Omega, \\ \partial_\nu u = 0, & x \in \partial\Omega. \end{cases} \tag{2.3.2}$$

Here, a is a real parameter, $b(x) \in C^\theta(\overline{\Omega})$ is a nonnegative function, the habitat Ω is a smooth and bounded domain. We would like to point out that the semilinear elliptic equation (2.3.2) is also related to some prescribed curvature problems in Riemannian geometry (see, e.g., [61] and [76, 77]). According to the physical meaning of our problems, only nonnegative solutions of

(2.3.1) and (2.3.2) make sense. Moreover, it is clear that, for any given initial data $u_0(x) \in C(\overline{\Omega})$ satisfying $u_0(x) \geq, \not\equiv 0$, (2.3.1) has a unique solution, which exists for all $t > 0$.

If $b(x) > 0$ on $\overline{\Omega}$, then the equations are known as the classical logistic equations with diffusion, and the following result is well known.

Theorem 2.3.1 *Assume that* $b(x) > 0$ *on* $\overline{\Omega}$*. Then, problem* (2.3.2) *has a positive solution if and only if* $a > 0$*. If it exists, the positive solution is also unique, denoted by* $u_a(x)$*. Furthermore, for any given admissible initial value* $u_0 \in C(\overline{\Omega})$*, the unique solution* $u(x,t)$ *of* (2.3.1) *exists for all* $t > 0$*, and as* $t \to \infty$*,* $u(x,t) \to 0$ *uniformly on* $\overline{\Omega}$ *if* $a \leq 0$ *while* $u(x,t) \to u_a(x)$ *uniformly on* $\overline{\Omega}$ *if* $a > 0$*.*

If $b(x) \equiv 0$ on $\overline{\Omega}$, then the equations reduce to the linear Malthusian models with diffusion. It follows from some basic theory of linear parabolic equations (see, for example, [50, 65]) that, as $t \to \infty$, the unique solution $u(x,t)$ of (2.3.1) uniformly converges to 0 on $\overline{\Omega}$ if $a \leq 0$ while $u(x,t)$ grows to infinity at an exponential rate as $t \to \infty$ on $\overline{\Omega}$ if $a > 0$.

If $b(x) \geq, \not\equiv 0$, but $b(x)$ vanishes in some subset of Ω, we say that $b(x)$ or the equation (2.3.1) or (2.3.2) is degenerate or has a degeneracy. In this situation, there are many research works devoted to the problems (2.3.1) and (2.3.2). We now recall some of the existing results in this direction.

Let us define

$$\overline{\Omega}_0 = \{x \in \Omega \,:\, b(x) = 0\}$$

and assume that Ω_0 is a nonempty and connected subset of Ω with $\overline{\Omega}_0 \subset\subset \Omega$ and the same smoothness as Ω. Biologically, the model becomes a mixture of logistic and Malthusian type. The subhabitat Ω_0 where $b(x)$ vanishes may be regarded as an ideal environment so that the species living there has no limitation for its population growth.

Let $\lambda_1^D(\Omega_0)$ denote the first eigenvalue of the Dirichlet problem:

$$-\Delta\varphi = \lambda\,\varphi \ \text{ in } \Omega_0, \qquad u = 0 \ \text{ on } \partial\Omega_0.$$

Under the above assumptions, the problems (2.3.1) and (2.3.2) were studied in [3, 6, 18, 28, 49, 76, 77]. Their results can be summarized as follows.

Theorem 2.3.2 *In the degenerate case described above, the following assertions hold.*

(i) *Problem* (2.3.2) *has a positive solution if and only if* $a \in (0, \lambda_1^D(\Omega_0))$. *If the positive solution of* (2.3.2) *exists, it is also unique, denoted by* $u_a(x)$. *Moreover,* u_a *is a continuous and strictly increasing mapping in* a *from* $(0, \lambda_1^D(\Omega_0))$ *to* $C^{2+\mu}(\overline{\Omega})$, *and* $\|u_a\|_{L^\infty(\Omega)} \to \infty$ *as* $a \to \lambda_1^D(\Omega_0)$.

(ii) *If* $0 < a < \lambda_1^D(\Omega_0)$, *the unique positive solution* u_a *attracts all the solutions of* (2.3.1) *with admissible nonnegative initial data; if* $a \leq 0$, *then* 0 *attracts all the solutions of* (2.3.1) *with admissible nonnegative initial data; and if* $a \geq \lambda_1^D(\Omega_0)$, *then any solution* $u(x,t)$ *of* (2.3.1) *blows up:* $\lim_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$.

The subsequent work of Du and Huang [36] and Du and Yamada [46] gave a better understanding to these two problems. In these works, the authors considered all three types of boundary conditions; but we only deal with the Neumann boundary condition here to better describe the results in [36, 46].

We need to consider the following boundary blow-up problem, which plays a key role in determining the asymptotic behavior of solutions to (2.3.2):

$$\begin{cases} -\Delta u = au - b(x)u^p & \text{in } \Omega \setminus \overline{\Omega}_0, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \\ u = \infty & \text{on } \partial\Omega_0. \end{cases} \tag{2.3.3}$$

Here, $u = \infty$ on $\partial\Omega_0$ means that

$$u(x) \to \infty \ \text{ as } x \in \Omega \setminus \overline{\Omega}_0 \text{ and } d(x, \partial\Omega_0) \to 0.$$

As for problem (2.3.3), the following conclusions were derived in [36, 46].

Theorem 2.3.3 *Under the assumptions of Theorem 2.3.2, the following assertions hold true.*

(i) *For any given* $a \in (-\infty, \infty)$, *problem* (2.3.3) *has a minimal positive solution* $\underline{U}_a$ *and a maximal positive solution* $\overline{U}_a$.

(ii) *If there exist constants* $\alpha \geq 0$ *and* $\beta_2 > \beta_1 > 0$ *such that for all* $x \in \Omega \setminus \overline{\Omega}_0$ *near* Ω_0,

$$\beta_1[d(x, \partial\Omega_0)]^\alpha \leq b(x) \leq \beta_2[d(x, \partial\Omega_0)]^\alpha,$$

then problem (2.3.3) *has a unique positive solution.*

Theorem 2.3.4 *Under the assumptions of Theorem 2.3.2, the following assertions hold true.*

(i) *For any fixed* $x \in \overline{\Omega}_0$, $u_a(x) \to \infty$ *as* $a \to \lambda_1^D(\Omega_0)$.

(ii) *For any fixed* $x \in \overline{\Omega} \setminus \overline{\Omega}_0$, $u_a(x) \to \underline{U}_a(x)$ *as* $a \to \lambda_1^D(\Omega_0)$.

Theorem 2.3.5 *Let* $a \geq \lambda_1^D(\Omega_0)$. *Then for any given admissible initial data* $u_0(x)$, *the unique solution* $u(x,t)$ *of* (2.3.1) *satisfies:*

(i) *For any fixed* $x \in \overline{\Omega}_0$, $u(x,t) \to \infty$ *as* $t \to \infty$.

(ii) *For any fixed* $x \in \overline{\Omega} \setminus \overline{\Omega}_0$, $u(x,t) \to \underline{U}_a(x)$ *as* $t \to \infty$. *Moreover, the limit is locally uniform in* x *for* $x \in \overline{\Omega} \setminus \overline{\Omega}_0$.

As in [36], we would like to make some comments on the above results.

Remark 2.3.1

(i) *The above theorems give a complete bifurcation picture for the problems* (2.3.1) *and* (2.3.2). *In fact, let us define* $\tilde{U}_a(x) = \underline{U}_a(x)$ *for* $x \in \overline{\Omega} \setminus \overline{\Omega}_0$ *and* $\tilde{U}_a(x) = \infty$ *for* $x \in \overline{\Omega}_0$, *and regard* $\tilde{U}_a(x)$ *as a solution of* (2.3.2) *at "infinity" . Then, the unique positive solution branch*

$$\Sigma = \{(a, u_a) \ : \ 0 < a < \lambda_1^D(\Omega_0)\}$$

bifurcates from the branch of trivial solutions

$$\Sigma_0 = \{(a, 0) \ : \ -\infty < a < \infty\}$$

at $a = 0$ *and it joins the branch of positive solutions at "infinity"*

$$\Sigma_\infty = \{(a, \tilde{U}_a) \ : \ -\infty < a < \infty\}$$

at $a = \lambda_1^D(\Omega_0)$. *Furthermore, when* $a \le 0$, *the trivial solution on* Σ_0 *is globally attractive for* (2.3.1)*, when* $0 < a < \lambda_1^D(\Omega_0)$, *the positive solution on* Σ *is globally attractive, and when* $a \ge \lambda_1^D(\Omega_0)$, *the solution at " infinity" on* Σ_∞ *is globally attractive.*

(ii) *If* $b(x)$ *vanishes on the closure of a finite number of disjoint subdomains* $\Omega_1, \cdots, \Omega_\ell$, *all with smooth boundaries, the obviously according assertions of the above ones remain true.*

(iii) *All of the results in the above theorems can be extended to the more general case where the Laplacian operator* $-\triangle$ *can be replaced by a general self-adjoint second-order uniformly elliptic operator and the nonlinear term* u^p *can be replaced by a more general nonlinear function of the similar type.*

(iv) *In the above theorems, we always assume that* $\Omega_0 \subset\subset \Omega$*; for the case of* $\partial\Omega_0 \cap \partial\Omega \ne \emptyset$, *further technical difficulties arise and the recent work by Du and Guo [37] discussed this situation.*

To understand how the solution of (2.3.2) behaves as the environment changes from the classical case ($b(x) > 0$ on $\overline{\Omega}$) to the degenerate case as in Theorem 2.3.2, the work of Du and Li in [38] investigated the following perturbed problem of (2.3.2):

$$\begin{cases} -\Delta u = au - [b(x) + \epsilon]u^p & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3.4}$$

It is clear from Theorem 2.3.1 that, for any given $\epsilon > 0$, (2.3.4) admits a unique positive solution if and only if $a > 0$.

Before presenting the main results of Du and Li, we recall some notations and assumptions. First, for each $j = 1, 2, \cdots, \ell$, let us assume that Ω_j is an open subdomain of Ω and has the smooth boundary $\partial\Omega_j$ satisfying $\overline{\Omega}_j \subset \Omega$ and $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ for $i \ne j$. For the function $b(x)$, we also assume that

$$b(x) = 0 \ \text{ on } \Omega_* := \cup_{j=1}^{\ell} \Omega_j \ \text{ and } \ b(x) > 0 \ \text{ on } \overline{\Omega} \backslash \overline{\Omega}_*. \tag{2.3.5}$$

By rearranging the subscripts if necessary, we may assume that

$$\lambda_1^D(\Omega_1) \le \lambda_1^D(\Omega_2) \le \cdots \le \lambda_1^D(\Omega_\ell).$$

It is well known that $0 < \lambda_1^D(\Omega) < \lambda_1^D(\Omega_1)$.

With the above hypotheses, Du and Li [38] proved the following conclusions.

Theorem 2.3.6 *Let $a > 0$ and $u_\epsilon^a(x)$ be the unique positive solution to* (2.3.4).

(i) *If $\lambda_1^B(\Omega) < a < \lambda_1^D(\Omega_1)$, then as $\epsilon \to 0$, u_ϵ^a converges uniformly on $\overline{\Omega}$ to the unique positive solution u_0 of* (2.3.4) *with $\epsilon = 0$.*

(ii) *If $\lambda_1^D(\Omega_k) \le a < \lambda_1^D(\Omega_{k+1})$ for some $1 \le k \le \ell - 1$, then*

(ii-a) $\lim_{\epsilon\to 0} u_\epsilon^a(x) = \infty$ *uniformly on $\cup_{j=1}^k \overline{\Omega}_j$ and*

(ii-b) $\lim_{\epsilon\to 0} u_\epsilon^a(x) = U(x) < \infty$ *uniformly on any compact subset of $(\overline{\Omega}\backslash \cup_{j=1}^k \overline{\Omega}_j)$, where $U(x)$ is the minimal positive solution to the boundary blow-up problem*

$$\begin{cases} -\Delta u = au - b(x)u^p & \text{in } (\Omega\backslash \cup_{j=1}^k \overline{\Omega}_j), \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \\ u = \infty & \text{on } \partial\Omega_j,\ j = 1, \cdots, k. \end{cases} \tag{2.3.6}$$

(iii) *If $a \ge \lambda_1^D(\Omega_\ell)$, then*

(iii-a) $\lim_{\epsilon\to 0} u_\epsilon^a(x) = \infty$ *uniformly on $\overline{\Omega}_*$;*

(iii-b) $\lim_{\epsilon\to 0} u_\epsilon^a(x) = U(x) < \infty$ *uniformly on any compact subset of $(\overline{\Omega}\backslash\overline{\Omega}_*)$, where $U(x)$ is the minimal positive solution to the boundary blow-up problem*

$$\begin{cases} -\Delta u = au - b(x)u^p & \text{in } (\Omega\backslash\overline{\Omega}_*), \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \\ u = \infty & \text{on } \partial\Omega_*. \end{cases} \tag{2.3.7}$$

Letting $v_\epsilon^a(x) = \epsilon^{\frac{1}{p-1}} u_\epsilon^a(x)$, they showed the profile of v_ϵ as follows.

Theorem 2.3.7 *Let $a > 0$ and v_ϵ^a be defined as above.*

(i) *If $0 < a \le \lambda_1^D(\Omega_1)$, then as $\epsilon \to 0$, v_ϵ^a converges to zero uniformly on $\overline{\Omega}$.*

(ii) *If* $\lambda_1^D(\Omega_k) < a \le \lambda_1^D(\Omega_{k+1})$ *for some* $1 \le k < \ell - 1$, *then*

(ii-a) $\lim_{\epsilon\to 0} v_\epsilon^a(x) = 0$ *uniformly on* $\overline{\Omega}\backslash\cup_{j=1}^k\Omega_j$ *and*

(ii-b) $\lim_{\epsilon\to 0} v_\epsilon^a(x) = \theta_a^i(x) < \infty$ *uniformly on each* $\overline{\Omega}_j$, $j = 1, 2, \cdots, k$, *where* $\theta_a^j(x)$ *is the unique positive solution to the elliptic problem*

$$-\Delta v = av - v^p \quad \text{in } \Omega_j; \quad v = 0 \quad \text{on } \partial\Omega_j. \tag{2.3.8}$$

(iii) *If* $a > \lambda_1^D(\Omega_\ell)$, *then*

(iii-a) $\lim_{\epsilon\to 0} v_\epsilon^a(x) = 0$ *uniformly on* $\overline{\Omega}\backslash\Omega_*$ *and*

(iii-b) $\lim_{\epsilon\to 0} v_\epsilon^a(x) = \theta_a^j(x) < \infty$ *uniformly on each* $\overline{\Omega}_j$, $j = 1, 2, \cdots, \ell$, *where* $\theta_a^j(x)$ *is the unique positive solution to the elliptic problem* (2.3.8).

We first need to point out that, the results in [38] dealt with the general boundary condition. Then, as commented in [38], we notice that the above results show, for any given $a > 0$, u_ϵ^a is globally stable as the stationary solution of the corresponding parabolic problem, and also for small ϵ, it has peaks concentrating exactly on Ω_*. Moreover, by properly choosing $\Omega_1, \cdots, \Omega_\ell$, the set of peaks of the solution u_ϵ^a can develop a rather arbitrary spatial pattern given on the set Ω. Since Ω_* has a positive measure, we therefore see that the measure of the set of peaks of u_ϵ^a tends to a positive number as $\epsilon \to 0$. As a consequence, the peak solutions generated by the logistic equation (2.3.4) are significantly different from those obtained in many other superlinear problems such as in [20, 73, 87], in which the peaks of the solutions concentrate at isolated points in the underlying domain and so the measure of the set of peaks goes to zero as $\epsilon \to 0$. For the latter case, the peak solutions are usually unstable and exhibit the so-called spike-layers. Moreover, according to Theorem 2.3.7, the rescaled solution v_ϵ^a has a limit $\tilde{v}$ as $\epsilon \to 0$ satisfying $\tilde{v} > 0$ in the interior of Ω_* but $\tilde{v} \equiv 0$ on $\overline{\Omega} \setminus \Omega_*$. Thus, v_ϵ^a exhibits no sharp layers at all.

2.4 The periodic-parabolic logistic model

In population ecology, the data usually depend periodically on time (seasonal or daily variation). So it is natural to assume that the function $b(x, t)$ in (2.2.3) is a nonnegative and $T-$periodic function with T being a fixed positive number.

We will consider the following problems:

$$\begin{cases} \partial_t u - \Delta u = au - b(x,t)u^p & \text{in } \Omega \times (0,\infty), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geq, \not\equiv 0 & \text{in } \Omega \end{cases} \tag{2.4.1}$$

and

$$\begin{cases} \partial_t u - \Delta u = au - b(x,t)u^p & \text{in } \Omega \times (0,\infty), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u(x,T) & \text{in } \Omega. \end{cases} \tag{2.4.2}$$

In the classical case, that is, when $b(x,t)$ is a $T-$periodic function and satisfies $b(x,t) > 0$ on $\overline{\Omega} \times [0,T]$, a complete understanding to the long-time dynamical behavior for (2.4.1) is well known; see, e.g., Hess [53]. To be precise, the following result holds.

Theorem 2.4.1 *Suppose that* $b(x,t) > 0$ *on* $\overline{\Omega} \times [0,T]$.

(i) *Problem* (2.4.2) *has a positive* $T-$*periodic solution if and only if* $a > 0$, *which is also unique, denoted by* $u_a(x,t)$.

(ii) *For any given admissible and continuous initial data* $u_0(x)$, $u_a(x,t)$ *is globally attractive in the sense that* $\lim_{n\to\infty} u(x,t+nT) \to u_a(x,t)$ *uniformly on* $\overline{\Omega} \times [0,T]$ *if* $a > 0$, *while* 0 *is globally attractive if* $a \leq 0$.

We observe that in (ii) of Theorem 2.4.1, it is clear that $\lim_{n\to\infty} u(x,t+nT) \to u_a(x,t)$ uniformly on $\overline{\Omega} \times [0,T]$ means $\lim_{t\to\infty}[u(x,t) - u_a(x,t)] = 0$ uniformly on $\overline{\Omega}$.

However, when the function $b(x,t)$ is T-periodic, nonnegative, and vanishes (i.e., has a degeneracy) in some subdomain of $\Omega \times \mathbb{R}$, to our best knowledge, there is no result for (2.4.1) and (2.4.2) in the existing literature. The main part of the thesis is devoted to the understanding of this case. Such an investigation will play a significant role in the study of the multispecies population models with periodic degenerate environment parallel to [19, 30, 31, 32, 39, 43, 45].

2.5 Outline of our investigation

We are mainly concerned with the periodic-parabolic logistic equations (2.4.1) and (2.4.2) for the case that the function $b(x,t)$ is T-periodic in t, nonnegative, and vanishes in some subdomain of $\Omega \times \mathbb{R}$. We want to examine the effects of various natural spatial and temporal degeneracies of $b(x,t)$ on the long-time dynamical behavior of (2.4.1), which requires a good understanding of the T-periodic positive solution of (2.4.2). In the degenerate cases, as mentioned before, the region where $b(x,t)$ vanishes represents the extreme environmental situation that the species experiences no self-limitation for its growth there. A good understanding of such an extreme case is important in order to understand the scope of the possible behavior of the model as the environment varies heterogeneously.

In chapter 3, we first consider problem (2.4.2) and derive some new sufficient and necessary conditions for the existence and uniqueness of T-periodic positive solutions. We also study how the unique T-periodic positive solution varies with the parameter a. Then, we determine the long-time dynamical behavior of the solution to (2.4.1).

In order to obtain a good understanding of these equations when both spatial and temporal degeneracies appear in the model, we have to overcome several technical difficulties. First of all, we require a precise characterization for the existence condition of the unique T-periodic positive solution in the case of spatiotemporal degenerate environment. In doing so, we need to establish new theories on an eigenvalue problem for periodic-parabolic operators over a varying cylinder, which has never been studied before. Our results give the accurate lower and upper bounds of the principal eigenvalue and the regularity of the corresponding positive eigenfunction. Secondly, to determine the long-time dynamics of the solutions to (2.4.1), we need to analyze the asymptotic behavior of the unique T-periodic positive solution with respect to the parameter a and develop new theories on certain parabolic initial and boundary blow-up problems. These are achieved by making use of various parabolic estimates and comparison principles.

Our investigation shows that the temporal degeneracy causes a fundamental change of the dynamical behavior of the equation only when spatial degeneracy also exists; but in sharp contrast, whether or not temporal degeneracy appears in the equation, the spatial degeneracy always induces fundamental changes of the behavior of the equation, though such changes differ significantly according to whether there is temporal degeneracy or not.

In chapter 4, we consider the perturbed problem of (2.4.2) with $b(x,t)$ replaced by $b(x,t)+\epsilon$:

$$\begin{cases} \partial_t u - \Delta u = au - [b(x,t)+\epsilon]u^p & \text{in } \Omega \times (0,\infty), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u(x,T) & \text{in } \Omega. \end{cases} \tag{2.5.1}$$

We will study the asymptotic behavior of the unique T-periodic positive solution of problem (2.5.1) as $\epsilon \to 0$. Our results show that, in sharp contrast to the case where only spatial degeneracies exist (that is, problem (2.3.4)), the perturbed periodic-parabolic logistic equation (2.5.1) can generate some very different spatiotemporal pattern (a precise description of pattern will be given in chapter 4) only when both the spatial and temporal degeneracies occur.

In discussing the asymptotic behavior of the unique T-periodic positive solution $u_\epsilon(x,t)$ of (2.5.1), we are led to investigate an eigenvalue problem, and some parabolic initial and boundary blow-up problems over a multi-connected cylinder. To determine the rescaled solution $v_\epsilon(x,t) = \epsilon^{\frac{1}{p-1}} u_\epsilon(x,t)$ in the case of spatiotemporal degeneracies, our analysis relies heavily on the understanding of a new periodic-parabolic equation over a varying and multi-connected cylinder. For such a periodic-parabolic equation, we derive the existence, uniqueness and regularity of T-periodic positive solutions.

Chapter 3

The degenerate periodic-parabolic logistic equation

3.1 Introduction

In this chapter, we will consider in detail the degenerate periodic-parabolic logistic equation:

$$\begin{cases} \partial_t u - \Delta u = au - b(x,t)u^p & \text{in } \Omega \times (0,\infty), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geq, \not\equiv 0 & \text{in } \Omega. \end{cases} \tag{3.1.1}$$

We assume that $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with $C^{2+\theta}$ boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$, and $b(x,t)$ is a function in $C^{\theta,\theta/2}(\overline{\Omega} \times \mathbb{R})$ $(0 < \theta < 1)$, which is T-periodic in t and satisfies $b(x,t) \geq, \not\equiv 0$ in $\Omega \times \mathbb{R}$.

We remark that the techniques developed here work as well if a is smooth positive function that is T-periodic in t, but we choose to sacrifice such generality in order to keep the notations and presentation concise and transparent.

If $b(x,t) > 0$ in $\overline{\Omega} \times \mathbb{R}$, from Theorem 2.3.1, we know that

$$\lim_{n\to\infty} u(x,t+nT) = \begin{cases} 0 & \text{uniformly for } x \in \Omega \text{ and } t \in [0,T] \text{ if } a \leq 0, \\ u_a(x,t) & \text{uniformly for } x \in \Omega \text{ and } t \in [0,T] \text{ if } a > 0, \end{cases} \tag{3.1.2}$$

where u_a is the unique positive T-periodic solution of

$$\begin{cases} \partial_t u - \Delta u = au - b(x,t)u^p & \text{in } \Omega \times \mathbb{R}, \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times \mathbb{R}, \end{cases} \tag{3.1.3}$$

which exists if and only if $a > 0$.

Our main interest here is to examine the case that $b(x,t)$ vanishes in a proper subset of $\Omega \times \mathbb{R}$. We will call such a case a degeneracy in the logistic equation. More precisely, we will examine the effect of a combination of spatial and temporal degeneracies on the behavior of (3.1.1), and reveal some new phenomena caused by the inclusion of temporal degeneracy in the model. Our results are best described in the special case that

$$b(x,t) = p(x)q(t),$$

where $p(x)$ and $q(t)$ are Hölder continuous nonnegative functions, and q is T-periodic.

We distinguish three different cases:

$$\begin{array}{lll} \text{(i)} & \text{No spatial degeneracy :} & p(x) > 0 \text{ in } \overline{\Omega} \text{ and } q(t) \geq, \not\equiv 0; \\ \text{(ii)} & \text{No temporal degenecary :} & q(t) > 0 \text{ in } \mathbb{R}, \{p(x) = 0\} = \overline{\Omega}_0 \subset \Omega; \\ \text{(iii)} & \text{Full degeneracy :} & \{p(x) = 0\} \text{ is as in (ii)}, \{q(t) = 0\} \cap [0,T] = [0,T^*]. \end{array} \tag{3.1.4}$$

Here Ω_0 is a connected open set with $C^{2+\theta}$ boundary and $T^* \in (0,T)$.

We will show that in case (i), the long-time behavior of (3.1.1) is similar to (3.1.2). In case (ii), it is analogous to that in subsection 2.3 of chapter 2 with $b(x,t) \equiv b(x)$. But in case (iii), we will prove that new behavior arises.

We now briefly describe the new behavior in case (iii).

Firstly we show that there exists $a_* \in (0, \lambda_1^D(\Omega_0))$ such that (3.1.3) with $b(x,t) = p(x)q(t)$ has a unique positive periodic solution u_a if $a \in (0, a_*)$ and it has no positive periodic solution otherwise. Moreover we show that a_* is the principal eigenvalue of the following eigenvalue problem over a varying cylinder:

$$\begin{cases} \partial_t\varphi - \Delta\varphi = \lambda\varphi & \text{in } (\Omega\times(0,T^*])\cup(\Omega_0\times(T^*,T]), \\ \partial_\nu\varphi = 0 & \text{on } \partial\Omega\times(0,T^*], \\ \varphi(x,t) = 0 & \text{on } (\partial\Omega_0\times(T^*,T])\cup((\Omega\backslash\Omega_0)\times\{0\}), \\ \varphi(x,0)=\varphi(x,T) & \text{in } \Omega_0. \end{cases} \tag{3.1.5}$$

Secondly we show that the unique solution $u(x,t)$ of (3.1.1) with $b(x,t)=p(x)q(t)$ satisfies

(a) $\lim_{t\to\infty} u(x,t) = 0$ when $a\leq 0$,

(b) $\lim_{n\to\infty} u(x,t+nT) = u_a(x,t)$ when $a\in(0,a_*)$,

(c) when $a\geq a_*$, $\lim_{n\to\infty} u(x,t+nT)=\infty$ locally uniformly on $(\overline{\Omega}\times(0,T^*])\cup(\overline{\Omega}_0\times[T^*,T])$, $\lim_{n\to\infty} u(x,t+nT) = U_a(x,t)$ uniformly on any compact subset of $(\overline{\Omega}\backslash\overline{\Omega}_0)\times(T^*,T)$,

where U_a is the minimal positive solution of the following parabolic boundary blow-up problem

$$\begin{cases} \partial_t u - \Delta u = au - p(x)q(t)u^p & \text{in } (\Omega\setminus\overline{\Omega}_0)\times(T^*,T), \\ \partial_\nu u = 0 & \text{on } \partial\Omega\times(T^*,T), \\ u=\infty & \text{on } \partial\Omega_0\times(T^*,T), \\ u=\infty & \text{on } (\overline{\Omega}\setminus\Omega_0)\times\{T^*\}. \end{cases} \tag{3.1.6}$$

Here by $u=\infty$ on $\partial\Omega_0\times(T^*,T)$, we mean that

$$u(x,t)\to\infty \quad \text{as } d(x,\Omega_0)\to 0 \quad \text{for each } t\in(T^*,T).$$

By $u=\infty$ on $(\overline{\Omega}\setminus\Omega_0)\times\{T^*\}$, we mean

$$u(x,t)\to\infty \quad \text{as } t \text{ decreases to } T^* \text{ for each } x\in\overline{\Omega}\setminus\Omega_0.$$

Comparing case (i) with case (ii) in (3.1.4), we notice that the temporal degeneracy causes a fundamental change of the dynamical behavior of the equation only when spatial degeneracy also exists. In sharp contrast, by comparing all three cases in (3.1.4), one finds that whether or not temporal degeneracy appears in the equation, the spatial degeneracy always induces fundamental changes of the behavior of the equation, though such changes differ significantly according to whether there is temporal degeneracy or not.

The rest of this chapter is organized as follows. In section 3.2, we prove the existence and uniqueness results of positive periodic solutions to (3.1.3) in the general setting. In section 3.3, we study the properties of the eigenvalue $\lambda_1(\infty)$ (to be defined below) in three cases. In particular, in the case that both spatial and temporal degeneracies occur, we shall show how the eigenvalue problem (3.1.5) arises from the existence problem of positive periodic solutions of (3.1.3). In section 3.4, we examine the long-time behavior of (3.1.1) by making use of some parabolic boundary blow-up problems such as (3.1.6).

Finally, we also want to mention that, the techniques and ideas developed in this chapter can be modified to treat a much more general version of (3.1.1). For example, the differential operator $\partial_t - \Delta$ can be replaced by one of the form $\partial_t + A(x,t,D)$ as given in section 1.3 of chapter 1 but with A in divergence form, the nonlinear function $au - b(x,t)u^p$ can be replaced by a general function of the form $f(x,t,u)$ with the same key features, and the Neumann boundary operator can be replaced by a general boundary operator of the form Bu given in section 1.3 of chapter 1.

3.2 Existence and uniqueness of positive periodic solutions

In this section, we will prove the basic existence and uniqueness result for the positive periodic solution to (3.1.3) and its global stability property as an element of the omega limit set of (3.1.1).

First of all, we consider the eigenvalue problem:

$$\begin{cases} \partial_t\varphi - \Delta\varphi + \mu b(x,t)\varphi = \lambda\varphi & \text{in } \Omega\times\mathbb{R}, \\ \partial_\nu\varphi = 0 & \text{on } \partial\Omega\times\mathbb{R}, \\ \varphi(x,t) = \varphi(x,t+T) & \text{in } \Omega\times\mathbb{R}. \end{cases} \tag{3.2.1}$$

With $b(x,t)$ as before, namely it belongs to $C^{\theta,\theta/2}(\overline{\Omega}\times\mathbb{R})$, is T-periodic in t and $b \geq, \not\equiv 0$, for each $\mu\in\mathbb{R}$, by Proposition 14.4 of [53], we know that (3.2.1) has a principal eigenvalue

$\lambda = \lambda_1(\mu b)$, which corresponds to a positive eigenfunction $\varphi \in C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega} \times \mathbb{R})$. Moreover, by Lemmas 15.5 and 15.7 of [53], $\mu \mapsto \lambda_1(\mu b)$ is a strictly increasing continuous function with $\lambda_1(\mu b) > \lambda_1(0) = 0$ when $\mu > 0$. Therefore, we can define

$$\lambda_1(\infty) := \lim_{\mu\to\infty} \lambda_1(\mu b) \in (0, \infty]. \tag{3.2.2}$$

Our first result is the following theorem.

Theorem 3.2.1 *Problem* (3.1.3) *admits a unique positive* $T-$*periodic solution* $u_a(x,t)$ *if*

$$0 < a < \lambda_1(\infty). \tag{3.2.3}$$

It has no positive periodic solution otherwise. Moreover, if (3.2.3) *holds, then the unique solution of* (3.1.1) *satisfies*

$$\lim_{n\to\infty} u(x, t + nT) = u_a(x,t) \textit{ uniformly in } x \in \overline{\Omega} \textit{ and } t \in [0,T].$$

If $a \leq 0$,

$$\lim_{t\to\infty} u(x,t) = 0 \textit{ uniformly in } \overline{\Omega}.$$

Proof. Assume that (3.1.3) has a positive $T-$periodic solution $u^*(x,t)$. Set

$$m = \max_{\overline{\Omega}\times[0,T]} u^*(x,t).$$

Clearly $m > 0$ and by the uniqueness and monotonicity properties of the principle eigenvalues we see that

$$a = \lambda_1(bu^{p-1}),$$

and

$$0 = \lambda_1(0) < \lambda_1(bu^{p-1}) \leq \lambda_1(bm^{p-1}) < \lambda_1(\infty).$$

Hence, we have $0 < a < \lambda_1(\infty)$.

On the other hand, if (3.2.3) holds, we set

$$\overline{u} = M\varphi_\mu,$$

where $\varphi_\mu(x,t)$ is a positive principal eigenfunction corresponding to $\lambda_1(\mu b)$. We may fix $\mu > 0$ sufficiently large such that $a < \lambda_1(\mu b)$. We then take M so large that $(M\varphi_\mu)^{p-1} \geq \mu$ on $\overline{\Omega} \times [0,T]$. With such μ and M, it is easy to check that $\overline{u} := M\varphi_\mu$ is a positive supersolution to (3.1.3). One also easily checks that any small positive constant $\underline{u}$ is a subsolution of (3.1.3). Thus (3.1.3) has a positive T-periodic solution.

Using the concavity of the nonlinearity in (3.1.3), one can follow a standard argument (see Theorem 27.1 in [53]) to show that the positive periodic solution of (3.1.3) is unique and attracts all the positive solutions of (3.1.1).

Finally suppose that $a \leq 0$. Then one can follow the argument in the proof of Theorem 28.1 in [53] to conclude that $\lim_{n\to\infty} u(x,t+nT) = 0$ uniformly in $x \in \overline{\Omega}$ and $t \in [0,T]$. It follows that $\lim_{t\to\infty} u(x,t) = 0$ uniformly in x.

The proof is complete. □

In view of the above theorem, to obtain a full understanding of the long-time dynamical behavior of (3.1.1), we need to find a better description of $\lambda_1(\infty)$, and more importantly, we need to know the long-time behavior of the solution of (3.1.1) when $a \geq \lambda_1(\infty)$. The rest of this chapter is devoted to answering these questions under suitable further conditions on $b(x,t)$.

3.3 Characterization of $\lambda_1(\infty)$

In this section we characterize $\lambda_1(\infty)$ under suitable assumptions on $b(x,t)$. We will consider three basic cases: no spatial degeneracy, spatial degeneracy but no temporal degeneracy, and both spatial and temporal degeneracy. In the last case, we will show how this leads to a periodic-parabolic eigenvalue problem over a varying cylinder. Again let us recall that $b(x,t) \in C^{\theta,\theta/2}(\overline{\Omega} \times \mathbb{R})$ is T-periodic in t and $b \geq, \not\equiv 0$.

3.3.1 Case 1: temporal degeneracy

The easiest case to handle is when $b(x,t)$ has no spatial degeneracy at some point in time, that is,

$$b(x,t_0) > 0 \quad \text{for all } x \in \overline{\Omega} \text{ and some } t_0 \in \mathbb{R}. \tag{3.3.1}$$

Under (3.3.1), a temporal degeneracy is possible.

Without loss of generality, we may assume that $t_0 \in (0,T]$. Clearly case (i) in (3.1.4) belongs to this situation. We show that in this case $\lambda_1(\infty) = \infty$, and hence Theorem 3.2.1 gives a full description of the long-time dynamical behavior of (3.1.1), which is the same as in the classical case (i.e., the case that no degeneracy occurs in the logistic equation).

Indeed, when (3.3.1) holds, $\min_{\overline{\Omega}} b(x,t_0) > 0$ and hence,

$$\int_0^T \min_{\overline{\Omega}} b(x,t)dt > 0.$$

It follows that, for any given $M > 0$, there exists a large μ_0 such that

$$\int_0^T \max_{\overline{\Omega}}(M - \mu b(x,t))\mathrm{d}t = \int_0^T [M - \mu \min_{\overline{\Omega}} b(x,t)]\mathrm{d}t < 0,$$

for all $\mu \geq \mu_0$. Hence, by Lemma 15.6 in [53],

$$\lambda_1(\mu b) - M = \lambda_1(\mu b - M) > 0,$$

for $\mu \geq \mu_0$, which implies

$$\lambda_1(\infty) = \infty.$$

We thus have

Theorem 3.3.1 *Assume that* (3.3.1) *holds; then* $\lambda_1(\infty) = \infty$.

3.3.2 Case 2: spatial degeneracy

Next, we consider a case that includes but generalizes case (ii) in (3.1.4), namely

$$c_1 p(x) \leq b(x,t) \leq c_2 p(x), \tag{3.3.2}$$

where $c_1,\ c_2$ are positive constants and $p(x)$ is as in case (ii) of (3.1.4).

We will show that in this case $\lambda_1(\infty) = \lambda_1^D(\Omega_0)$. Our argument is based on the properties of the first eigenvalues for elliptic operators.

To the end, we need to introduce some more notations. Let O be a bounded domain, and $f(x)$ be an $L^\infty(O)$ function. We denote by $\lambda_1^D(f, O)$ and $\lambda_1^N(f, O)$ the first eigenvalue of the operator $-\Delta + f$ over O, with Dirichlet and Neumann boundary conditions, respectively. We also use the convention that

$$\lambda_1^D(0, O) = \lambda_1^D(O),\ \ \lambda_1^N(0, O) = \lambda_1^N(O).$$

For convenience of later use, we list some well known properties:

(1) $\lambda_1^D(f, O) > \lambda_1^N(f, O)$;

(2) $\lambda_1^B(f_1, O) > \lambda_1^B(f_2, O)$ if $f_1 \geq f_2$ and $f_1 \not\equiv f_2$, for $B = D$ or $B = N$;

(3) $\lambda_1^D(f, O_1) \geq \lambda_1^D(f, O_2)$ if $O_1 \subset O_2$.

Let us also note that by the uniqueness property of the principal eigenvalue of the periodic-parabolic operator

$$Lu = \partial_t u - \Delta u - g(x,t)u,$$

when $g(x,t) = g(x)$ is a function independent of the time variable t, then

$$\lambda_1(g) = \lambda_1^N(g, \Omega).$$

Let

$$\bar{b}(x) = \max_{[0,T]} b(x,t) \quad \text{and} \quad \underline{b}(x) = \min_{[0,T]} b(x,t).$$

Then,

$$\{\bar{b}(x) = 0\} = \{\underline{b}(x) = 0\} = \overline{\Omega}_0.$$

By the monotonicity of the principal eigenvalues, we have

$$\lambda_1^N(\mu\underline{b}, \Omega) = \lambda_1(\mu\underline{b}) \leq \lambda_1(\mu b) \leq \lambda_1(\mu\bar{b}) = \lambda_1^N(\mu\bar{b}, \Omega).$$

By Theorem 2.4 of [49],

$$\lim_{\mu\to\infty} \lambda_1^N(\mu\underline{b},\Omega) = \lim_{\mu\to\infty} \lambda_1^N(\mu\bar{b},\Omega) = \lambda_1^D(\Omega_0).$$

Thus we have proved the following result:

Theorem 3.3.2 *Under the assumption* (3.3.2), *we have* $\lambda_1(\infty) = \lambda_1^D(\Omega_0)$.

3.3.3 Case 3: spatial and temporal degeneracies

We now consider the third case where both spatial and temporal degeneracies exist. This case includes but generalizes case (iii) in (3.1.4), namely

$$c_1 p(x)q(t) \le b(x,t) \le c_2 p(x)q(t) \text{ on } \overline{\Omega}\times\mathbb{R}, \tag{3.3.3}$$

where p and q are as in case (iii) of (3.1.4), and c_1, c_2 are positive constants. It turns out that this case is much more difficult to handle.

Our first main result on $\lambda_1(\infty)$ is the following:

Theorem 3.3.3 *When* (3.3.3) *holds, we have* $\lambda_1(\infty) < \lambda_1^D(\Omega_0)$. *Moreover, there exists a function* $\varphi(x,t)$ *which is continuous in* $\big(\overline{\Omega}\times[0,T]\big)\setminus\big[(\overline{\Omega}\setminus\Omega_0)\times\{T^*\}\big]$, *and satisfies*

$$\varphi > 0 \text{ in } (\overline{\Omega}\times(0,T^*])\cup(\Omega_0\times(T^*,T]),\ \varphi = 0 \text{ in } \big(\overline{\Omega}\setminus\Omega_0\big)\times(T^*,T], \tag{3.3.4}$$

$$\varphi \in C^{2+\theta,1+\frac{\theta}{2}}\Big(\big[(\overline{\Omega}\times(0,T^*])\cup(\overline{\Omega}_0\times[T^*,T])\big]\setminus\big[\partial\Omega_0\times\{T^*\}\big]\Big), \tag{3.3.5}$$

and

$$\begin{cases} \partial_t\varphi - \Delta\varphi = \lambda_1(\infty)\varphi & \text{in } (\Omega\times(0,T^*])\cup(\Omega_0\times(T^*,T]),\\ \partial_\nu\varphi = 0 & \text{on } \partial\Omega\times(0,T^*],\\ \varphi(x,t) = 0 & \text{on } (\partial\Omega_0\times(T^*,T])\cup(\Omega\backslash\Omega_0\times\{0\}),\\ \varphi(x,0) = \varphi(x,T) & \text{in } \Omega_0. \end{cases} \tag{3.3.6}$$

Proof. Let $\varphi = \varphi_\mu$ be a positive principal eigenfunction corresponding to $\lambda_1(\mu b)$. Then

$$\begin{cases} \partial_t\varphi - \Delta\varphi + \mu b(x,t)\varphi = \lambda_1(\mu b)\varphi & \text{in } \Omega \times (0,T), \\ \partial_\nu \varphi = 0 & \text{on } \partial\Omega \times (0,T), \\ \varphi(x,0) = \varphi(x,T) & \text{in } \Omega. \end{cases} \tag{3.3.7}$$

By (3.3.3) and Theorem 2.4 of [49], we have

$$\lambda_1(\mu b) \le \lambda_1(\mu c_2 Q p) \to \lambda_1^D(\Omega_0), \;\; \text{as } \mu \to \infty, \tag{3.3.8}$$

where $Q = \max_{[0,T]} q(t) > 0$. It follows that

$$\lambda_1(\infty) \le \lambda_1^D(\Omega_0).$$

Without loss of generality, we may assume that

$$\max_{\overline{\Omega}\times[0,T]} \varphi_\mu = 1.$$

Since

$$0 \le \varphi_\mu(x,t) \le 1 \;\; \text{on} \;\; \overline{\Omega} \times [0,T],$$

we can find a sequence $\mu_n \to \infty$ as $n \to \infty$ such that

$$\varphi_{\mu_n}(x,t) \to \varphi^*(x,t) \;\; \text{weakly in} \;\; L^2(\Omega \times (0,T)) \;\; \text{as } n \to \infty,$$

and

$$0 \le \varphi^*(x,t) \le 1 \;\; \text{a.e. in} \;\; \Omega \times (0,T).$$

For the sake of convenience, we will write $\varphi_n(x,t)$ instead of $\varphi_{\mu_n}(x,t)$.

In the following, we will investigate the properties of φ^* through improved understanding of the convergence of φ_n. For clarity, the long discussions below are divided into several steps.

Step 1: $\varphi^*(x,t) \not\equiv 0$ in $\Omega \times (0,T)$.

We proceed by a contradiction argument. Suppose that

$$\varphi_n(x,t) \to 0 \quad \text{weakly in } L^2(\Omega \times (0,T)). \tag{3.3.9}$$

We multiply (3.3.7) (with $(\mu, \varphi) = (\mu_n, \varphi_n)$) by $\varphi_n(x,t)$ and then integrate it over Ω to obtain that

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \varphi_n^2 + \int_\Omega |\nabla \varphi_n|^2 + \mu_n \int_\Omega b(x,t)\varphi_n^2 = \lambda_1(\mu_n b)\int_\Omega \varphi_n^2. \tag{3.3.10}$$

As $\int_\Omega \varphi_n^2$ is a T–periodic function of t, for each $n \geq 1$ there exists $\tau_n \in [0,T)$ such that

$$\frac{\partial}{\partial t}\int_\Omega \varphi_n^2 \Big|_{t=\tau_n} = 0.$$

In view of (3.3.8) and (3.3.10), this implies

$$\int_\Omega |\nabla \varphi_n(x,\tau_n)|^2 + \int_\Omega \varphi_n^2(x,\tau_n) \leq C_0, \tag{3.3.11}$$

where C_0 is a positive constant independent of n. It follows that by passing to a subsequence if necessary, we may assume that

$$\varphi_n(\cdot,\tau_n) \to \varphi_0^* \text{ in } L^2(\Omega). \tag{3.3.12}$$

For any fixed $n \geq 1$, we next consider the auxiliary problem:

$$\begin{cases} \partial_t \psi - \Delta \psi = \lambda_1(\infty)\varphi_n(x,t+\tau_n) & \text{in } \Omega \times (0,\infty), \\ \partial_\nu \psi = 0 & \text{on } \partial\Omega \times (0,\infty), \\ \psi(x,0) = \varphi_n(x,\tau_n) & \text{in } \Omega. \end{cases} \tag{3.3.13}$$

For any fixed n, (3.3.13) admits a unique solution $\psi_n(x,t) \in C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega} \times (0,\infty))$. Furthermore, the comparison principle for parabolic equations immediately infers

$$\varphi_n(x,t+\tau_n) \leq \psi_n(x,t) \quad \text{in } \Omega \times (0,\infty) \quad \text{for each } n \geq 1. \tag{3.3.14}$$

Since $0 \leq \varphi_n(x,t) \leq 1$, by comparing ψ_n with the unique solution of (3.3.13) but with initial data $\varphi_n(x,\tau_n)$ replaced by 1 and right-hand side $\lambda_1(\infty)\varphi_n(x,t+\tau_n)$ replaced by $\lambda_1(\infty)$, we find that for any given $\hat{T} > 0$, there exists $C_{\hat{T}} > 0$ such that $0 \leq \psi_n \leq C_{\hat{T}}$ in $\overline{\Omega} \times [0,\hat{T}]$. For any given constant $0 < \tau < \hat{T}$, let $\xi(t)$ be a smooth function on $[0,\hat{T}]$ such that $\xi \geq 0$, $\xi = 1$ over

$[\tau, \hat{T}]$ and $\xi = 0$ in $[0, \tau/2]$. Then $\psi = \xi\psi_n$ satisfies

$$\begin{cases} \partial_t \psi - \Delta\psi = \xi'(t)\psi_n + \xi(t)\lambda_1(\infty)\varphi_n(x, t + \tau_n) & \text{in } \Omega \times (0, \infty), \\ \partial_\nu \psi = 0 & \text{on } \partial\Omega \times (0, \infty), \\ \psi(x, 0) = 0 & \text{in } \Omega. \end{cases}$$

Since the right-hand side of the first equation above has a bound in $L^\infty(\Omega \times [0, \hat{T}])$ that is independent of n, by standard parabolic L^p estimates, we have, for any $p > 1$,

$$\|\psi_n\|_{W_p^{2,1}(\Omega\times[\tau,\hat{T}])} \le C_0$$

for some constant C_0 independent of n. Taking p large enough and applying the Sobolev embedding result (see section 1.2.2 or [65] Lemma II3.3), we obtain

$$\|\psi_n\|_{C^{1+\theta, \frac{1+\theta}{2}}(\overline{\Omega}\times[\tau,\hat{T}])} \le C = C_{\tau, \hat{T}}.$$

Therefore by passing to a subsequence we can assume that

$$\psi_n \to \psi^* \quad \text{in } C^{1,\frac{1}{2}}(\overline{\Omega} \times [\tau, \hat{T}]).$$

By this conclusion and a standard diagonal argument, we can pass to a further subsequence so that

$$\psi_n \to \psi^* \ \text{ in } \ C^{1,\frac{1}{2}}(\overline{\Omega} \times [\tau, \hat{T}]) \ \text{ for any } \ \tau \in (0, \hat{T}).$$

In the following, we show that (3.3.9) implies

$$\psi^*(x, t) \equiv 0 \quad \text{on } \overline{\Omega} \times [0, \infty).$$

To achieve this goal, for an arbitrary test function $\eta(x) \in C^2(\overline{\Omega})$ satisfying $\eta(x) \ge 0$ on $\overline{\Omega}$, we set

$$h_n(t) = \int_\Omega \varphi_n(x, t)\eta(x)dx.$$

We have

$$\begin{aligned} h_n'(t) &= \int_\Omega \partial_t\varphi_n(x,t)\eta(x)dx \\ &= \int_\Omega [\Delta\varphi_n(x,t) - \mu_n b(x,t)\varphi_n(x,t) + \lambda_1(\mu)\varphi_n(x,t)]\eta(x)dx \\ &\le \int_\Omega [\varphi_n(x,t)\Delta\eta(x) + \lambda_1(\mu)\varphi_n(x,t)\eta(x)]dx + \Big|\int_{\partial\Omega} \varphi_n(x,t)\partial_\nu\eta(x)dS_x\Big| \\ &\le M_0, \end{aligned} \tag{3.3.15}$$

for some M_0 independent of n.

In view of (3.3.9) and (3.3.15), we find that $h_n(t)$ has the following properties:

(i) $h_n(t)$ is a T-periodic function and $h_n(t) \ge 0$;

(ii) $h_n'(t) \le M_0$ for $t \in \mathbb{R}$;

(iii) $\int_0^T h_n(t) \to 0$ as $n \to \infty$.

Based on these properties, by an elementary argument we conclude that

$$h_n(t) \to 0 \quad \text{uniformly on } \mathbb{R} \ \text{ as } n \to \infty. \tag{3.3.16}$$

Indeed, due to properties (i) and (iii), it is well known that, taking a subsequence of $\{h_n(t)\}$, denoted by itself again, $h_n(t) \to 0$ a.e. in $(0,T)$ as $n \to \infty$. Then, it follows from (ii) and the T-periodicity of $\{h_n(t)\}$ that for any fixed $t \in [0,T]$, $h_n(t) \to 0$ as $n \to \infty$. Now, letting $\epsilon > 0$, we can find a large $N_0 > 0$ such that $n \ge N_0$, $h_n(0) \le \epsilon/2$. On the other hand, for any $t \in [0, \epsilon/(2M_0)]$, we have

$$h_n(t) - h_n(0) \le M_0 t \le \epsilon/2 \quad \text{for any } n \ge N_0.$$

Thus, $h_n(t) \le h_n(0) + \epsilon/2 \le \epsilon$ for $n \ge N_0$. This, together with the finite covering argument and the T-periodicity of $\{h_n(t)\}$, implies (3.3.16).

Clearly (3.3.16) and (3.3.12) imply that $\varphi_0^* = 0$. We now use the weak formulation of (3.3.13) to show that $\psi^*(x,t)$ satisfies weakly (and then classically)

$$\begin{cases} \partial_t\psi - \Delta\psi = 0 & \text{in } \Omega \times (0,\infty), \\ \partial_\nu\psi = 0 & \text{on } \partial\Omega \times (0,\infty), \\ \psi(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{3.3.17}$$

which implies $\psi^*(x,t) = 0$ on $\overline{\Omega} \times [0,\infty)$.

For any given $\hat{T} > 0$, let V be the space of all functions $u(x,t)$ in $L^2(\Omega \times [0,\hat{T}])$ such that $|\nabla_x u| \in L^2(\Omega \times [0,\hat{T}])$, $u(\cdot,t) \in L^2(\Omega)$ for all $t \in [0,\hat{T}]$, and the norm defined by

$$\|u\|_V^2 = \int_{\Omega\times[0,\hat{T}]} |\nabla_x u|^2 dxdt + \sup_{t\in[0,\hat{T}]} \int_\Omega u^2 dx$$

is finite. Following Lieberman [66] (page 136), $u \in V$ is called a weak solution of the initial boundary value problem

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \Omega \times (0,\hat{T}], \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0,\hat{T}], \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3.3.18}$$

with $f \in L^2(\Omega \times [0,\hat{T}])$ and $u_0 \in L^2(\Omega)$, if for all $v \in C^1(\overline{\Omega} \times [0,\hat{T}])$ satisfying $v(x,\hat{T}) = 0$, we have

$$\int_{\Omega\times[0,\hat{T}]} [-u\partial_t v + \nabla_x u \cdot \nabla_x v - fv] dxdt = \int_\Omega u_0(x) v(x,0) dx.$$

Moreover, if u is a weak solution of (3.3.18), by Theorem 6.38 in [66],

$$\|u\|_V \leq Ce^{C\hat{T}}(\|f\|_{L^2(\Omega\times[0,\hat{T}])} + \|u_0\|_{L^2(\Omega)}) \tag{3.3.19}$$

for some constant C independent of f, u_0 and u.

We now apply (3.3.19) to (3.3.13) and obtain

$$\|\psi_n\|_V \leq M \tag{3.3.20}$$

for some $M = M_{\hat{T}} > 0$ independent of n. Next we use (3.3.20), the weak formulation of (3.3.13) and $\varphi_n(\cdot,\tau_n) \to 0$ in $L^2(\Omega)$ to show that ψ^* is a weak solution of (3.3.17) and hence $\psi^* \equiv 0$ due to (3.3.19). Firstly we show that $\psi^* \in V$. Indeed, for any $\tau \in (0,\hat{T})$, by (3.3.20),

$$\int_{\Omega\times[\tau,\hat{T}]} |\nabla_x \psi^*|^2 dxdt + \sup_{t\in[\tau,\hat{T}]} \int_\Omega (\psi^*)^2 dx = \lim_{n\to\infty} \Big(\int_{\Omega\times[\tau,\hat{T}]} |\nabla_x \psi_n|^2 dxdt + \sup_{t\in[\tau,\hat{T}]} \int_\Omega \psi_n^2 dx \Big) \leq M.$$

It follows that

$$\int_{\Omega\times[0,\hat{T}]} |\nabla_x \psi^*|^2 dxdt + \sup_{t\in[0,\hat{T}]} \int_\Omega (\psi^*)^2 dx \leq M.$$

Hence $\psi^* \in V$. It remains to show that for all $v \in C^1(\overline{\Omega} \times [0,\hat{T}])$ satisfying $v(x,\hat{T}) = 0$,

$$\int_{\Omega\times[0,\hat{T}]} [-\psi^* \partial_t v + \nabla_x \psi^* \cdot \nabla_x v] dxdt = 0.$$

Using the weak formulation of (3.3.13) we have, for every $v \in C^1(\overline{\Omega} \times [0,\hat{T}])$ satisfying $v(x,\hat{T}) = 0$,

$$\int_{\Omega\times[0,\hat{T}]} [-\psi_n \partial_t v + \nabla_x \psi_n \cdot \nabla_x v - \lambda_1(\infty)\varphi_n(x,t+\tau_n)v] dxdt = \int_\Omega \varphi_n(x,\tau_n)v(x,0)dx. \quad (3.3.21)$$

Since $\varphi_n \to 0$ weakly in $L^2(\Omega \times [0,T])$ and φ_n is T-periodic in t, and since $\varphi_n(\cdot,\tau_n) \to 0$ in $L^2(\Omega)$, we immediately see that, as $n \to \infty$,

$$\int_{\Omega\times[0,\hat{T}]} [-\lambda_1(\infty)\varphi_n(x,t+\tau_n)v] dxdt \to 0$$

and

$$\int_\Omega \varphi_n(x,\tau_n)v(x,0)dx \to 0.$$

For any $\tau \in (0,\hat{T})$, using $\psi_n \to \psi^*$ in $C^{1,1/2}(\overline{\Omega} \times [\tau,\hat{T}])$, we deduce

$$\int_{\Omega\times[\tau,\hat{T}]} [-\psi_n \partial_t v + \nabla_x \psi_n \cdot \nabla_x v] dxdt \to \int_{\Omega\times[\tau,\hat{T}]} [-\psi^* \partial_t v + \nabla_x \psi^* \cdot \nabla_x v] dxdt$$

as $n \to \infty$. On the other hand,

$$|\int_{\Omega\times[0,\tau]} [-\psi_n \partial_t v + \nabla_x \psi_n \cdot \nabla_x v] dxdt| \leq \|\psi_n\|_V \Big(\int_{\Omega\times[0,\tau]} [(\partial_t v)^2 + |\nabla_x v|^2] dxdt\Big)^{1/2} \to 0$$

as $\tau \to 0$ due to (3.3.20). It is also clear that

$$\int_{\Omega\times[0,\tau]} [-\psi^* \partial_t v + \nabla_x \psi^* \cdot \nabla_x v] dxdt \to 0$$

as $\tau \to 0$. Hence by letting $n \to \infty$ in (3.3.21), we can deduce

$$\int_{\Omega\times[0,\hat{T}]} [-\psi^* \partial_t v + \nabla_x \psi^* \cdot \nabla_x v] dxdt = 0,$$

as we wanted. Thus $\psi^* \equiv 0$, and due to (3.3.14) and $0 \leq \tau_n < T$, we find that $\varphi_n \to 0$ uniformly for $(x,t) \in \overline{\Omega} \times [T+1,\hat{T}]$. Since φ_n is T–periodic in t, the above conclusion implies $\varphi_n \to 0$ uniformly for $(x,t) \in \overline{\Omega} \times [0,T]$ provided that we have chosen $\hat{T} > 2T+1$. But this contradicts our assumption that $\max_{\overline{\Omega}\times[0,T]}\varphi_n = 1$. This contradiction proves that $\varphi^* \not\equiv 0$, and the proof of Step 1 is complete.

Next, we determine the differential equation satisfied by $\varphi^*(x,t)$. To this end, it is convenient to consider φ^* over the regions $\overline{\Omega}\times(0,T^*]$ and $\overline{\Omega}\times(T^*,T]$ separately.

Step 2: φ^* in the range $(x,t)\in\overline{\Omega}\times(0,T^*]$.

In this range, $\psi = \varphi_n$ is the unique solution of

$$\begin{cases} \partial_t\psi - \Delta\psi = \lambda_1(\mu_n b)\varphi_n(x,t) & \text{in } \Omega\times(0,T^*], \\ \partial_\nu\psi = 0 & \text{on } \partial\Omega\times(0,T^*], \\ \psi(x,0) = \varphi_n(x,0) & \text{in } \Omega. \end{cases} \tag{3.3.22}$$

As before we may use a cut-off function $\xi(t)$ and the equation satisfied by $\xi\varphi_n$ (which is a variation of (3.3.22) above) to deduce that for any $\tau\in(0,T^*)$, there exists $C = C_\tau$ such that

$$\|\varphi_n\|_{C^{1+\theta,\frac{1+\theta}{2}}(\overline{\Omega}\times[\tau,T^*])} \le C.$$

Therefore by passing to a subsequence and also using a diagonal argument, we can assume that $\varphi_n \to \varphi_*$ in $C^{1,\frac{1}{2}}(\overline{\Omega}\times[\tau,T^*])$ for any $\tau\in(0,T^*)$. We necessarily have $\varphi_* = \varphi^*$. Hence φ^* satisfies weakly

$$\begin{cases} \partial_t\varphi^* - \Delta\varphi^* = \lambda_1(\infty)\varphi^*(x,t) & \text{in } \Omega\times(0,T^*], \\ \partial_\nu\varphi^* = 0 & \text{on } \partial\Omega\times(0,T^*]. \end{cases} \tag{3.3.23}$$

By standard parabolic regularity we know that $\varphi^*\in C^{2+\theta,1+\theta}(\overline{\Omega}\times(0,T^*])$ and satisfies the above equation in the classical sense.

Step 3: φ^* in the the range $(x,t)\in\overline{\Omega}\times(T^*,T]$.

This case turns out to be difficult to handle. We first prove that

$$\varphi^* = 0 \text{ a.e. in } (\overline{\Omega}\setminus\Omega_0)\times(T^*,T].$$

Take $v(x,t)$ to be a smooth T-periodic function on $\overline{\Omega} \times \mathbb{R}$ with $v = 0$ near $\partial\Omega \times \mathbb{R}$. Multiplying (3.3.7) by v and then integrating over $\Omega \times (0,T)$, we derive

$$\int_0^T \int_\Omega \{-\varphi_n v_t - \varphi_n \Delta v + \mu_n b(x,t)\varphi_n v\} = \lambda_1(\mu_n b) \int_0^T \int_\Omega \varphi_n v.$$

Dividing the above identity by μ_n and then letting $n \to \infty$, we obtain

$$\int_0^T \int_\Omega b(x,t)\varphi^*(x,t)v(x,t) = 0.$$

Due to the arbitrariness of v, we necessarily have

$$b(x,t)\varphi^*(x,t) = 0 \quad \text{a.e in } \Omega \times (0,T). \tag{3.3.24}$$

Since $b(x,t) > 0$ in $\overline{\Omega}\backslash\overline{\Omega}_0 \times (T^*,T)$, it follows that

$$\varphi^*(x,t) = 0 \quad \text{a.e. in } \Omega\backslash\Omega_0 \times (T^*,T). \tag{3.3.25}$$

Secondly we prove that restricted to $\Omega_0 \times \mathbb{R}$, $\varphi_n \to \varphi^*$ in $C^{2,1}_{\text{loc}}(\Omega_0 \times \mathbb{R})$. Indeed, in this range, $\varphi_n(x,t)$ satisfies

$$\begin{cases} \partial_t \varphi_n - \Delta\varphi_n = \lambda_1(\mu_n b)\varphi_n & \text{in } \Omega_0 \times \mathbb{R}, \\ \varphi_n(x,0) = \varphi_n(x,T) & \text{in } \Omega_0. \end{cases} \tag{3.3.26}$$

Since

$$0 < \lambda_1(\mu_n b) \le \lambda_1(\infty) \text{ and } 0 \le \varphi_n \le 1,$$

by standard interior estimates (see, e.g., [65] or [66]), for any compact subset $K \subset \Omega_0 \times \mathbb{R}$, there exists a positive constant $C = C_K$ independent of n such that

$$\|\varphi_n(x,t)\|_{C^{2+\theta,1+\theta/2}(K)} \le C.$$

Therefore, by passing to a subsequence of $\{\varphi_n(x,t)\}$ and a diagonal argument, we may assume that

$$\varphi_n \to \varphi_* \quad \text{in } C^{2,1}_{\text{loc}}(\Omega_0 \times \mathbb{R}).$$

As before we necessarily have $\varphi_* = \varphi^*$. Clearly φ^* satisfies

$$\begin{cases} \partial_t \varphi^* - \Delta\varphi^* = \lambda_1(\infty)\varphi^* & \text{in } \Omega_0 \times \mathbb{R}, \\ \varphi^*(x,0) = \varphi^*(x,T) & \text{in } \Omega_0. \end{cases} \tag{3.3.27}$$

Thirdly we determine the boundary condition satisfied by $\varphi^*|_{\Omega_0 \times (T^*,T]}$ over $\partial\Omega_0 \times (T^*,T]$. Integrating (3.3.10) for t over $[0,T]$ we obtain

$$\int_0^T \int_\Omega |\nabla\varphi_n|^2 dxdt \le \lambda_1(\mu_n b) \int_0^T \int_\Omega \varphi_n^2 dxdt \le \lambda_1(\infty) T|\Omega|.$$

It follows that

$$\int_0^T \int_\Omega |\nabla\varphi_n|^2 dxdt + \int_0^T \int_\Omega \varphi_n^2 dxdt \le M_0 := [\lambda_1(\infty)+1]T|\Omega|. \tag{3.3.28}$$

That is, $\{\varphi_n\}$ is a bounded set in the Hilbert space $W_2^{1,0}(\Omega \times [0,T])$ with inner product

$$(u,v) = \int_0^T \int_\Omega \nabla u \cdot \nabla v dxdt + \int_0^T \int_\Omega uv dxdt.$$

Hence by passing to a subsequence $\varphi_n \to \varphi_*$ weakly in $W_2^{1,0}(\Omega \times [0,T])$. Necessarily $\varphi_* = \varphi^*$. Thus $\varphi^* \in W_2^{1,0}(\Omega \times [0,T])$, and hence for a.e. $t \in [0,T]$, $\varphi^*(\cdot,t) \in H^1(\Omega)$. By (3.3.25), for a.e. $t \in (T^*,T]$, $\varphi^*(\cdot,t) = 0$ over $\Omega \setminus \Omega_0$. Since $\partial\Omega_0$ is smooth (actually Lipschitz is enough here), it follows that

$$\varphi^*(\cdot,t)|_{\Omega_0} \in H_0^1(\Omega_0) \quad \text{for a.e. } t \in (T^*,T].$$

From (3.3.28) we deduce

$$\int_0^T \int_\Omega |\nabla\varphi^*|^2 dxdt + \int_0^T \int_\Omega (\varphi^*)^2 dxdt \le M_0.$$

As a consequence,

$$\int_{T^*}^T \int_{\Omega_0} |\nabla\varphi^*|^2 dxdt \le M_0.$$

Using this and $0 \le \varphi^* \le 1$, we obtain

$$\int_{T^*}^T \int_{\Omega_0} |\nabla\varphi^*|^2 dxdt + \sup_{t\in[T^*,T]} \int_{\Omega_0} (\varphi^*)^2 dxdt \le M_0 + |\Omega_0|.$$

By the above facts for φ^* and the fact that $\varphi_n \to \varphi^*$ in $C^{2,1}_{\text{loc}}(\Omega_0 \times \mathbb{R})$, we easily see that $\psi = \varphi^*$ is the unique weak solution of

$$\begin{cases} \partial_t \psi - \Delta \psi = \lambda_1(\infty)\varphi^* & \text{in } \Omega_0 \times (T^*, T], \\ \psi = 0 & \text{on } \partial\Omega_0 \times (T^*, T], \\ \psi(x, T^*) = \varphi^*(x, T^*) & \text{in } \Omega_0. \end{cases} \tag{3.3.29}$$

By standard regularity theory for weak solutions (see [66]) the weak solution of (3.3.29) belongs to $C^{\theta,\theta/2}(\overline{\Omega}_0 \times [\tau, T])$ for any $\tau \in (T^*, T)$. Hence $\varphi^* \in C^{\theta,\theta/2}(\overline{\Omega}_0 \times [\tau, T])$ and we can use the Hölder theory to conclude that $\varphi^* \in C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega}_0 \times (T^*, T])$.

To better understand the behavior of φ^* near $\overline{\Omega} \setminus \overline{\Omega}_0 \times \{T\}$, we need

Step 4: φ_n converges to 0 uniformly on any compact subset of $\overline{\Omega} \setminus \Omega_0 \times (T^*, T]$.

Since $\varphi_n \to \varphi^*$ weakly in $L^2(\Omega \times [0, T])$ and $\varphi^* = 0$ over $\Omega \setminus \Omega_0 \times (T^*, T]$, if we define

$$\xi_n(t) = \int_{\Omega \setminus \Omega_0} \varphi_n(x, t) dx,$$

then $\xi_n \to 0$ in $L^1([T^*, T])$. From the well-known Lusin's theorem, it follows that for any given $\epsilon > 0$, there exists a measurable set $S_\epsilon \subset [T^*, T]$ such that $|S_\epsilon| < \epsilon$ and $\xi_n(t) \to 0$ uniformly in $[T^*, T] \setminus S_\epsilon$. Thus we can find a sequence t_k decreasing to T^* such that $\xi_n(t_k) \to 0$ as $n \to \infty$ for each $k \geq 1$. Hence

$$0 \leq \int_{\Omega \setminus \Omega_0} \varphi_n(x, t_k)^2 dx \leq \int_{\Omega \setminus \Omega_0} \varphi_n(x, t_k) dx \to 0$$

as $n \to \infty$ for each $k \geq 1$.

Due to the conclusions proved in Step 3, for any given small $\delta > 0$ and $k \geq 1$, we can find $\sigma > 0$ small such that $0 < \varphi^*(x, t) < \delta/2$ for (x, t) satisfying $x \in \Omega_0 \setminus \Omega^\sigma$, $t \in [t_k, T]$, where $\Omega^\sigma = \{x \in \Omega_0 : d(x, \partial\Omega_0) > \sigma\}$. Since $\varphi_n \to \varphi^*$ in $C^{2,1}_{\text{loc}}(\Omega_0 \times \mathbb{R})$, for all large n, $\varphi_n < \delta$ on $\partial\Omega^\sigma \times [t_k, T]$. We now consider the auxiliary problem

$$\begin{cases} \partial_t v - \Delta v = \lambda_1(\infty)\varphi_n & \text{in } \Omega \setminus \overline{\Omega^\sigma} \times (t_k, T], \\ \partial_\nu v = 0 & \text{on } \partial\Omega \times (t_k, T], \\ v = \delta & \text{on } \partial\Omega^\sigma \times (t_k, T], \\ v(x, t_k) = \varphi_n(x, t_k) & \text{in } \Omega \setminus \overline{\Omega^\sigma}. \end{cases} \tag{3.3.30}$$

Let v_n denote the unique solution of (3.3.30); a simple comparison consideration shows that for all large n,

$$\varphi_n \le v_n \quad \text{in } \Omega\setminus\overline{\Omega^\sigma}\times(t_k,T].$$

Much as before, we can show that, by passing to a subsequence,

$$v_n \to v^* \quad \text{in } C^{1+\theta,\frac{1+\theta}{2}}(\overline{\Omega}\setminus\Omega^\sigma\times(\tau,T])\ (\forall \tau\in(t_k,T))$$

and

$$v = v^*$$

is a weak solution of

$$\begin{cases} \partial_t v - \Delta v = \lambda_1(\infty)\varphi^* & \text{in } \Omega\setminus\overline{\Omega^\sigma}\times(t_k,T],\\ \partial_\nu v = 0 & \text{on } \partial\Omega\times(t_k,T],\\ v=\delta & \text{on } \partial\Omega^\sigma\times(t_k,T],\\ v(x,t_k)=v_0(x) & \text{in } \Omega\setminus\overline{\Omega^\sigma}, \end{cases} \tag{3.3.31}$$

where $v_0(x)=0$ in $\Omega\setminus\overline{\Omega}_0$, and $v_0(x)=\varphi^*(x,t_k)$ for $x\in\Omega_0\setminus\Omega^\sigma$. Since

$$0\le\varphi^*\le\delta \ \text{ in } \Omega\setminus\overline{\Omega^\sigma}\times(t_k,T],$$

and $v_0\le\delta$ in $\Omega\setminus\Omega^\sigma$, a direct calculation shows that

$$\tilde v(x,t) := [\lambda_1(\infty)t+1]\delta$$

is a supersolution of (3.3.31). Hence

$$v^*\le\tilde v\le[\lambda_1(\infty)T+1]\delta \ \text{ in } \overline{\Omega}\setminus\Omega^\sigma\times(t_k,T].$$

It follows that, for all large n,

$$\varphi_n\le v_n\le v^*+\delta\le\tilde v+\delta\le[\lambda_1(\infty)T+2]\delta$$

in $\overline{\Omega}\setminus\Omega^\sigma\times[t_{k+1},T]$. This implies that

$$\varphi_n\to 0 \quad \text{uniformly in } \overline{\Omega}\setminus\Omega_0\times[t_{k+1},T] \ \text{ as } n\to\infty,$$

for each $k \geq 1$. Since $t_k \to T^*$, this proves Step 4.

Step 5: Summary and positivity of φ^*.

To summarize, we have shown that, by passing to a subsequence,

- over $\overline{\Omega} \times (0, T^*]$, $\varphi_n \to \varphi^*$ locally in the $C^{2,1}$ norm,
- over $\Omega_0 \times \mathbb{R}$, $\varphi_n \to \varphi^*$ locally in the $C^{2,1}$ norm,
- over $\overline{\Omega} \setminus \Omega_0 \times (T^*, T]$, $\varphi_n \to 0 = \varphi^*$ locally uniformly,
- $\varphi^* \in C^{2,1}(\overline{\Omega}_0 \times (T^*, T])$ and $\varphi^* = 0$ on $\partial\Omega_0 \times (T^*, T]$.

These properties imply in particular that $\varphi_n(x, 0) = \varphi_n(x, T) \to \varphi^*(x, 0)$ in the $L^2(\Omega)$ norm (actually the convergence is in $C(\overline{\Omega})$), and we see from (3.3.22) that $v = \varphi^*$ is the unique weak solution of the problem

$$\begin{cases} \partial_t v - \Delta v = \lambda_1(\infty) v & \text{in } \Omega \times (0, T^*], \\ \partial_\nu v = 0 & \text{on } \partial\Omega \times (0, T^*], \\ v(x, 0) = \varphi^*(x, 0) & \text{in } \Omega. \end{cases} \tag{3.3.32}$$

As $\varphi^*(x, 0)$ is continuous over $\overline{\Omega}$ and equals 0 near $\partial\Omega$, and $\partial\Omega$ is smooth, by standard theory for parabolic equations (see Theorem 9 on page 69 of [50]) we know that $\varphi^* \in C^{2+\theta, 1+\frac{\theta}{2}}(\overline{\Omega} \times (0, T^*]) \cap C(\overline{\Omega} \times [0, T^*])$. Hence φ^* is a continuous function over $\overline{\Omega} \times [0, T]$ except a possible discontinuity along $\overline{\Omega} \setminus \Omega_0 \times \{T^*\}$.

We now use the strong maximum principle to show that $\varphi^* > 0$ in $\{\overline{\Omega} \times (0, T^*]\} \cup \{\Omega_0 \times (T^*, T]\}$. Indeed we must have $\varphi^*(\cdot, 0) \not\equiv 0$ in Ω_0. Otherwise $\varphi^*(\cdot, 0) \equiv 0$ and hence $v = 0$ is the unique solution of (3.3.32). It follows that $\varphi^* = 0$ over $\overline{\Omega} \times (0, T^*]$. Due to (3.3.29), $v = \varphi^*$ is the unique solution of

$$\begin{cases} \partial_t v - \Delta v = \lambda_1(\infty) v & \text{in } \Omega_0 \times (T^*, T], \\ v = 0 & \text{on } \partial\Omega_0 \times (T^*, T], \\ v(x, T^*) = \varphi^*(x, T^*) & \text{in } \Omega_0. \end{cases} \tag{3.3.33}$$

Since now $\varphi^*(\cdot,T^*)\equiv 0$, clearly $v=0$ solves (3.3.33), and we deduce $\varphi^*=0$ over $\Omega_0\times(T^*,T]$. As we already know that $\varphi^*=0$ over $\Omega\setminus\Omega_0\times(T^*,T]$, we see that $\varphi^*\equiv 0$ over $\Omega\times(T^*,T]$. Hence $\varphi^*\equiv 0$ over $\Omega\times[0,T]$, contradicting our earlier conclusion that $\varphi^*\not\equiv 0$. This proves that $\varphi^*(\cdot,0)\geq,\not\equiv 0$ in Ω_0. Thus we can apply the strong maximum principle to (3.3.32) to conclude that $\varphi^*(x,t)>0$ for $(x,t)\in\overline{\Omega}\times(0,T^*]$. We may then apply the strong maximum principle to (3.3.33) to see that $\varphi^*>0$ in $\Omega_0\times(T^*,T]$. Hence,

$$\varphi^*(x,0)=\varphi^*(x,T)>0 \quad \text{in } \Omega_0.$$

Let us note that the above conclusions show that $\varphi^*(x,t)$ does have a jumping discontinuity across $\overline{\Omega}\setminus\Omega_0\times\{T^*\}$.

Thus we find that

$$\begin{aligned}\varphi^*\in\ & C^{2+\theta,1+\frac{\theta}{2}}\Big(\big(\overline{\Omega}\times(0,T^*]\big)\cup\big(\overline{\Omega}_0\times[T^*,T]\big)\setminus\partial\Omega_0\times\{T^*\}\Big)\\ &\cap C^0\Big(\big(\overline{\Omega}\times[0,T]\big)\setminus\big[(\overline{\Omega}\setminus\Omega_0)\times\{T^*\}\big]\Big),\\ \varphi^*>0 \text{ in } & (\overline{\Omega}\times(0,T^*])\cup(\Omega_0\times(T^*,T]),\ \varphi=0 \text{ in } (\overline{\Omega}\setminus\Omega_0)\times(T^*,T],\end{aligned}$$

and

$$\begin{cases}\partial_t\varphi^*-\Delta\varphi^*=\lambda_1(\infty)\varphi^* & \text{in } (\Omega\times(0,T^*])\cup(\Omega_0\times(T^*,T]),\\ \partial_\nu\varphi^*=0 & \text{on } \partial\Omega\times(0,T^*],\\ \varphi^*(x,t)=0 & \text{on } (\overline{\Omega}\backslash\Omega_0\times\{0\})\cup(\partial\Omega_0\times(T^*,T]),\\ \varphi^*(x,0)=\varphi^*(x,T) & \text{in } \Omega_0.\end{cases} \tag{3.3.34}$$

Step 6: $\lambda_1(\infty)<\lambda_1^D(\Omega_0)$.

Let $\varphi_*(x)$ be the corresponding eigenfunction of $\lambda_1^D(\Omega_0)$ with $\varphi_*(x)>0$, that is, $\varphi_*(x)$ satisfies:

$$-\Delta\varphi_*=\lambda_1^D(\Omega_0)\varphi_*,\ \varphi_*>0 \ \text{ in } \Omega,\quad \varphi_*=0 \ \text{ on } \partial\Omega_0.$$

Then, we multiply the equation in (3.3.34) by $\varphi_*(x)$ and integrate the resulting identity over $\Omega_0\times(0,T)$ to derive

$$\int_0^T\int_{\Omega_0}\partial_t\varphi^*\varphi_*-\int_0^T\int_{\Omega_0}\Delta\varphi^*\varphi_*=\lambda_1(\infty)\int_0^T\int_{\Omega_0}\varphi^*\varphi_*. \tag{3.3.35}$$

By the $T-$periodic property of $\varphi^*(x,t)$, it is clear that the first term in the left-hand side is zero. For the second term in the left-hand side, integrating by parts we have

$$\begin{aligned}
-\int_0^T\int_{\Omega_0}\Delta\varphi^*\varphi_* &= -\int_0^{T^*}\int_{\Omega_0}\Delta\varphi^*\varphi_* - \int_{T^*}^T\int_{\Omega_0}\Delta\varphi^*\varphi_* \\
&= -\int_0^{T^*}\int_{\Omega_0}\varphi^*\Delta\varphi_* + \int_0^{T^*}\int_{\partial\Omega_0}\varphi^*\partial_{\nu_0}\varphi_* - \int_{T^*}^T\int_{\Omega_0}\varphi^*\Delta\varphi_* \\
&= \lambda_1^D(\Omega_0)\int_0^T\int_{\Omega_0}\varphi^*\varphi_* + \int_0^{T^*}\int_{\partial\Omega_0}\varphi^*\partial_{\nu_0}\varphi_* \\
&< \lambda_1^D(\Omega_0)\int_0^T\int_{\Omega_0}\varphi^*\varphi_*,
\end{aligned}$$

where ν_0 denotes the unit normal of $\partial\Omega_0$ pointing inward of Ω_0. Hence, it follows from (3.3.35) that $\lambda_1(\infty) < \lambda_1^D(\Omega_0)$, which completes the proof of Step 6.

The theorem is now completely proved. □

Consider the eigenvalue problem

$$\begin{cases}
\partial_t\varphi - \Delta\varphi = \lambda\varphi & \text{in } (\Omega\times[0,T^*])\cup(\Omega_0\times(T^*,T]), \\
\partial_\nu\varphi = 0 & \text{on } \partial\Omega\times(0,T^*], \\
\varphi(x,t) = 0 & \text{on } (\overline{\Omega}\backslash\Omega_0\times\{0,T\})\cup(\partial\Omega_0\times(T^*,T]), \\
\varphi(x,0) = \varphi(x,T) & \text{in } \Omega_0.
\end{cases} \tag{3.3.36}$$

Theorem 3.3.4 *The eigenvalue problem* (3.3.36) *admits a principal eigenvalue* $\lambda = \lambda_1 > 0$ *which corresponds to a positive eigenfunction* $\varphi_1(x,t)$ *satisfying* (3.3.4) *and* (3.3.5)*. Conversely, if* (3.3.36) *has a solution* φ *satisfying* (3.3.4) *and* (3.3.5)*, then necessarily* $\lambda = \lambda_1$*, the principal eigenvalue of* (3.3.36)*, and* $\varphi = c\varphi_1$ *for some constant c.*

Proof. For any given $u \in C_0^1(\overline{\Omega}_0)$, we extend it by 0 to $\overline{\Omega}$, and denote the resulting function by $\tilde{u}$. Clearly $\tilde{u} \in C(\overline{\Omega})$. Let $v(x,t)$ be the unique solution of the problem

$$\begin{cases}
\partial_t v - \Delta v = 0 & \text{in } \Omega\times(0,T^*], \\
\partial_\nu v = 0 & \text{on } \partial\Omega\times(0,T^*), \\
v(x,0) = \tilde{u}(x) & \text{in } \Omega.
\end{cases} \tag{3.3.37}$$

By the standard regularity theory and the imbedding theorems for parabolic equations (see section 1.3 of chapter 1), we know that $v \in C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega} \times (0,T^*]) \cap C(\overline{\Omega} \times [0,T^*])$.

We then consider the problem

$$\begin{cases} \partial_t w - \Delta w = 0 & \text{in } \Omega_0 \times (T^*,T], \\ w = 0 & \text{on } \partial\Omega_0 \times (T^*,T], \\ w(x,T^*) = v(x,T^*) & \text{in } \Omega_0. \end{cases} \tag{3.3.38}$$

By the existence result recalled in section 3.2 we know that this problem has a unique solution $w \in C^{\theta}((T^*,T],X_1) \cap C^{1+\theta}((T^*,T],X_0)$, where $X_0 = L^p(\Omega_0)$ and $X_1 = W_0^{2,p}(\Omega_0)$, $p > 1$. We may choose p large enough such that $W_0^{2,p}(\Omega_0)$ embeds compactly into $E := C_0^1(\overline{\Omega}_0)$.

With u and w as above, we define the operator $K_0 : E \to E$ by

$$K_0 u = w(\cdot,T).$$

It is easily seen that K_0 is a linear operator. We show next that K_0 is compact. Suppose that $\{u_n\}$ is a bounded sequence in E. Then there exists $C > 0$ such that $-C \le \tilde{u}_n \le C$ in Ω. If we denote by v_n the unique solution of (3.3.37) with $\tilde{u}$ replaced by $\tilde{u}_n$, then a simple comparison consideration gives $-C \le v_n \le C$ in $\Omega \times (0,T^*]$. In particular, $-C \le v_n(x,T^*) \le C$ in Ω_0. We may then apply the comparison principle to deduce that $-C \le w_n \le C$ in $\Omega_0 \times (T^*,T]$, where w_n is the unique solution of (3.3.38) with $v(x,T^*)$ replaced by $v_n(x,T^*)$. We may now use a suitable cut-off function $\xi(t)$ and apply the standard L^p estimates to the equation satisfied by ξw_n, much as in Step 1 of the proof of Theorem 3.3.3, to conclude that, for any $p > 1$ and $\tau \in (T^*,T)$, there exists $C_0 > 0$ such that

$$\|w_n\|_{W_p^{2,1}(\Omega_0 \times [\tau,T])} \le C_0 \text{ for all } n \ge 1.$$

By the Sobolev embedding result (see, for example, section 1.2.2 or Lemma II 3.3 in [65]) we deduce

$$\|w_n\|_{C^{1+\theta,\frac{1+\theta}{2}}(\overline{\Omega}_0 \times [\tau,T])} \le C$$

for some constant C and all $n \ge 1$. In particular, $\{w_n(\cdot,T)\}$ is bounded in $C^{1+\theta}(\overline{\Omega}_0)$. Hence it has a convergent subsequence in E. This proves the compactness of K_0.

Let P denote the cone of nonnegative functions in E, and P^o the interior of P. It is easily seen that P is solid, namely, $E = P - P$. We show that K_0 is strongly positive, that is, $K_0(P\backslash\{0\}) \subset P^o$. Indeed, if $u \geq 0$ and $u \not\equiv 0$ in E, then by the strong maximum principle we know that the unique solution v of (3.3.37) satisfies $v > 0$ in $\overline{\Omega} \times (0, T^*]$. It follows that the unique solution w of (3.3.38) satisfies $w > 0$ in $\Omega_0 \times (T^*, T]$. By the Hopf boundary lemma we deduce $\partial_{\nu_0} w < 0$ on $\partial\Omega_0 \times (T^*, T]$, where ν_0 denotes the unit outward normal of $\partial\Omega_0$. In particular we have

$$w(x,T) > 0 \ \text{ in } \Omega_0 \quad \text{and} \quad \partial_{\nu_0} w(x,T) < 0 \ \text{ on } \partial\Omega_0.$$

This implies that $w(\cdot, T) \in P^o$. Hence K_0 is strongly positive.

With the above properties for K_0, the Krein-Rutman theorem (namely, Theorem 1.5.2) applies and hence the spectral radius $r(K_0)$ of K_0 is positive, it corresponds to an eigenvector $u_0 \in P^o$. Moreover, if $K_0 u_1 = r u_1$ for some $u_1 \in P^o$, then necessarily $r = r(K_0)$ and $u_1 = c u_0$ for some constant c.

Let us now see how K_0 and $r(K_0)$ are related to the eigenvalue problem (3.3.36). Let $u_0 \in P^o$ be an eigenvector of K_0 corresponding to $r(K_0)$: $K_0 u_0 = r(K_0) u_0$. Let $U_0(x,t)$ be defined by

$$U_0(x,t) = v_0(x,t) \text{ in } \overline{\Omega} \times [0, T^*], \ U_0(x,t) = w_0(x,t) \text{ in } \overline{\Omega}_0 \times (T^*, T],$$

where v_0 denotes the unique solution of (3.3.37) with $\tilde{u}_0$ in place of $\tilde{u}$, and w_0 is the unique solution of (3.3.38) with $v(x, T^*)$ replaced by $v_0(x, T^*)$.

By definition,

$$U_0(\cdot, T) = K_0 u_0 = r(K_0) u_0 \quad \text{in } \overline{\Omega}_0.$$

We now define

$$\varphi_0(x,t) = e^{\lambda t} U_0(x,t) \text{ with } \lambda = -\frac{1}{T} \ln r(K_0).$$

Then clearly φ_0 satisfies (3.3.4) and (3.3.5). Moreover, a direct calculation shows that φ_0 satisfies (3.3.36).

Conversely, if (3.3.36) has a solution φ satisfying (3.3.4) and (3.3.5)., then let

$$r_0 = e^{-\lambda T} \quad \text{and} \quad \psi(x,t) = e^{-\lambda t} \varphi(x,t).$$

We easily see that ψ satisfies (3.3.37) with $\tilde{u}$ replaced by $\varphi(x,0)$ in $\overline{\Omega}\times[0,T^*]$, and it satisfies (3.3.38) with $v(x,T^*)$ replaced by $\psi(x,T^*)$. Moreover,

$$K_0\psi(\cdot,0)=\psi(\cdot,T)=e^{-\lambda T}\varphi(\cdot,T)=r_0\varphi(\cdot,0)=r_0\psi(\cdot,0).$$

Hence $u:=\psi(\cdot,0)|_{\overline{\Omega}_0}=\varphi(\cdot,T)\in P^o$ satisfies $K_0u=r_0u$. By the Krein-Rutman theorem, we necessarily have $r_0=r(K_0)$ and $u=cu_0$ for some constant c. It follows that

$$\varphi=c\varphi_0.$$

Our proof is complete. □

Remark 3.3.1 *By Theorem 3.3.4 we know that the limiting function φ^* in Theorem 3.3.3 is uniquely determined by* (3.3.6)*. It follows that the limit $\lim_{\mu\to\infty}\varphi_\mu$ exists and equals φ^*.*

If we denote by $\lambda_1=\lambda_1(\Omega,\Omega_0,T,T^*)$ the principal eigenvalue of (3.3.36), then it follows from Theorem 3.3.3 that $\lambda_1<\lambda_1^D(\Omega_0)$. We now give a lower bound for λ_1, which will be used in the next section.

Theorem 3.3.5 $\lambda_1(\Omega,\Omega_0,T,T^*)\geq\left(1-\frac{T^*}{T}\right)\lambda_1^D(\Omega_0)$.

Proof. Firstly we observe that the linear operator K_0 defined in the proof of Theorem 3.3.4 can be extended to a compact linear operator $\tilde{K}_0$ over $X_0=L^2(\Omega_0)$. Indeed, for any $u\in L^2(\Omega_0)$ we define $\tilde{u}$ as the extension of u by 0 to Ω, and let v be the unique solution of (3.3.37); then we have

$$v(\cdot,T^*)=U_1(T^*,0)\tilde{u},$$

where U_1 is the operator in (1.3.6) associated with (3.3.37). Similarly the unique solution w of (3.3.38) is given by

$$w(\cdot,t)=U_2(t-T^*,0)v(\cdot,T^*)|_{\Omega_0},$$

where U_2 is the operator in (1.3.6) associated with (3.3.38). By the properties of U_1 and U_2, we know that $U_1(0,T^*)$ and $U_2(T-T^*,0)$ are compact operators on $L^2(\Omega)$ and $L^2(\Omega_0)$, respectively. It follows easily that

$$\tilde{K}_0=U_2(T-T^*,0)\circ I\circ U_1(T^*,0)\circ J$$

is compact from $L^2(\Omega_0)$ to itself, where $Ju = \tilde{u}$ is the extension operator, and $Iv = v|_{\Omega_0}$ is the restriction operator. By the maximum principle we know that $\tilde{K}_0$ is also a positive operator: $\tilde{K}_0 u \geq 0$ if u is a nonnegative function in $L^2(\Omega_0)$. Since the positive cone in $L^2(\Omega_0)$ is reproducing, we can apply the Krein-Rutman theorem to conclude that $r(\tilde{K}_0) \geq r(K_0)$ is an eigenvalue that corresponds to a positive eigenfunction: $\tilde{K}_0\phi = r(\tilde{K}_0)\phi$. Using the regularity of $\tilde{K}_0$ and the Sobolev embedding theorem we can easily deduce from an iteration argument that $\phi \in C_0^1(\overline{\Omega}_0)$ and hence $\tilde{K}_0\phi = K_0\phi$. It follows that

$$K_0\phi = r(\tilde{K}_0)\phi.$$

However, since K_0 is a strongly positive operator, the above equality implies that

$$r(\tilde{K}_0) = r(K_0).$$

Clearly,

$$r(\tilde{K}_0) \leq \|\tilde{K}_0\|.$$

We now estimate $\|\tilde{K}_0\|$. Let $\lambda_k^N(\Omega)$ be the eigenvalues of $-\Delta$ over Ω with Neumann boundary conditions, with corresponding eigenfunctions ϕ_k, $k \geq 1$; and let $\lambda_k^D(\Omega_0)$ denote the eigenvalues of $-\Delta$ over Ω_0 with Dirichlet boundary conditions, with corresponding eigenfunctions ψ_k, $k \geq 1$. We may assume that the eigenfunctions are orthonormal:

$$\int_\Omega \phi_k\phi_j = \delta_{kj}, \quad \int_{\Omega_0} \psi_k\psi_j = \delta_{kj}.$$

Then for any $u \in L^2(\Omega_0)$, we have

$$Ju = \tilde{u} = \Sigma_{k=1}^\infty a_k\phi_k,$$

and

$$\|u\|_{L^2(\Omega_0)} = \|Ju\|_{L^2(\Omega)} = \left(\Sigma_{k=1}^\infty a_k^2\right)^{1/2}.$$

It is easily seen that with $\tilde{u}$ expressed this way,

$$v(\cdot,t) = U_1(t,0)\tilde{u} = \Sigma_{k=1}^\infty a_k e^{-\lambda_k^N(\Omega)t}\phi_k,$$

and hence

$$\begin{aligned} \|v(\cdot,T^*)\|_{L^2(\Omega)} &= \left(\Sigma_{k=1}^{\infty} a_k^2 e^{-2\lambda_k^N(\Omega)T^*}\right)^{1/2} \\ &\leq \left(\Sigma_{k=1}^{\infty} a_k^2\right)^{1/2} \\ &= \|u\|_{L^2(\Omega_0)}. \end{aligned}$$

Similarly, we can write

$$Iv(\cdot,T^*) = v(\cdot,T^*)|_{\Omega_0} = \Sigma_{k=1}^{\infty} b_k \psi_k,$$

and hence

$$\|Iv(\cdot,T^*)\|_{L^2(\Omega_0)} = \left(\Sigma_{k=1}^{\infty} b_k^2\right)^{1/2},$$

$$\begin{aligned} w(\cdot,t) &= U_2(t-T^*,0)v(\cdot,T^*)|_{\Omega_0} \\ &= \Sigma_{k=1}^{\infty} b_k e^{-\lambda_k^D(\Omega_0)(t-T^*)}\psi_k(x). \end{aligned}$$

It follows that

$$\begin{aligned} \|w(\cdot,T)\|_{L^2(\Omega_0)} &= \left(\Sigma_{k=1}^{\infty} b_k^2 e^{-2\lambda_k^D(\Omega_0)(T-T^*)}\right)^{1/2} \\ &\leq e^{-\lambda_1^D(\Omega_0)(T-T^*)}\|Iv(\cdot,T^*)\|_{L^2(\Omega_0)}. \end{aligned}$$

We thus obtain

$$\begin{aligned} \|w(\cdot,T)\|_{L^2(\Omega_0)} &\leq e^{-\lambda_1^D(\Omega_0)(T-T^*)}\|Iv(\cdot,T^*)\|_{L^2(\Omega_0)} \\ &\leq e^{-\lambda_1^D(\Omega_0)(T-T^*)}\|v(\cdot,T^*)\|_{L^2(\Omega)} \\ &\leq e^{-\lambda_1^D(\Omega_0)(T-T^*)}\|u\|_{L^2(\Omega_0)}. \end{aligned}$$

This implies that $\|\tilde{K}_0\| \leq e^{-\lambda_1^D(\Omega_0)(T-T^*)}$ and hence

$$r(K_0) = r(\tilde{K}_0) \leq e^{-\lambda_1^D(\Omega_0)(T-T^*)}.$$

From the proof of Theorem 3.3.4, we have

$$\begin{aligned}\lambda_1(\Omega, \Omega_0, T, T^*) &= -\tfrac{1}{T}\ln r(K_0)\\ &\geq -\tfrac{1}{T}\ln e^{-\lambda_1^D(\Omega_0)(T-T^*)}\\ &= \left(1-\tfrac{T^*}{T}\right)\lambda_1^D(\Omega_0).\end{aligned}$$

The proof is complete. □

3.4 Long-time dynamical behavior of solutions

Suppose that $\lambda_1(\infty) < \infty$, we now study the long-time behavior of the positive solution of (3.1.1). Recall that for $a < \lambda_1(\infty)$, the behavior of the solution is already given in Theorem 3.2.1.

We first consider the case that (3.3.3) holds, and then discuss the case (3.3.2). As we will see below, the limit $\lim_{a\to\lambda_1(\infty)} u_a$, where u_a is the unique positive T-periodic solution of (3.1.3), which exists if and only if $a \in (0, \lambda_1(\infty))$ (see Theorem 3.2.1), will play a key role in our analysis. This limit turns out to be determined by certain boundary blow-up solutions, and the boundary blow-up problems are fundamentally different between the case (3.3.3) and the case (3.3.2).

3.4.1 The case that (3.3.3) holds

Throughout this subsection we assume that (3.3.3) holds. We first discuss the asymptotic behavior of $u_a(x,t)$ as $a \uparrow \lambda_1(\infty)$. For simplicity, we denote $a_\infty = \lambda_1(\infty)$. By a simple comparison and sub- and super-solution argument it is easily seen that $u_a(x,t)$ is strictly increasing in a for $a \in (0, a_\infty)$. Hence, it suffices to consider a sequence a_n with $a_n \to a_\infty$. In the discussions below, we also denote

$$u_n(x,t) = u_{a_n}(x,t) \quad \text{and} \quad \Omega^* := \Omega \setminus \overline{\Omega}_0$$

for simplicity.

Theorem 3.4.1 *$u_a(x,t) \to \infty$ uniformly on every compact subset of $(\overline{\Omega} \times (0,T^*]) \cup (\overline{\Omega}_0 \times [0,T])$ as $a \to a_\infty$.*

The proof of this theorem requires the following result.

Lemma 3.4.1 *Let $m(x,t)$ be a given positive T-periodic function on $\overline{\Omega^*} \times [0,T]$ that belongs to the space $C^{2+\theta,1+\theta/2}(\overline{\Omega^*} \times [0,T])$. Then, for any $a \in (-\infty,\infty)$, the following periodic problem*

$$\begin{cases} \partial_t u - \Delta u = au - b(x,t)u^p & \text{in } \Omega^* \times [0,T], \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times [0,T], \\ u = m(x,t) & \text{on } \partial\Omega_0 \times [0,T], \\ u(x,0) = u(x,T) & \text{in } \Omega^* \end{cases} \tag{3.4.1}$$

has a unique T-periodic solution $u_a^m \in C^{2,1}(\overline{\Omega^*} \times [0,T])$. Moreover $u_a^m(x,t) > 0$ on $\overline{\Omega^*} \times [0,T]$, and $u_a^m(x,t)$ is a strict increasing function with respect to $m(x,t)$ and a in the sense that $u_a^{m_1} > u_a^{m_2}$ in $\Omega^* \times [0,T]$ if $m_1(x,t) \geq, \not\equiv m_2(x,t)$ on $\partial\Omega_0 \times [0,T]$, and $u_{a_1}^m > u_{a_2}^m$ if $a_1 > a_2$.

Proof. For small $\delta > 0$ we define

$$\Omega_0^\delta := \{x \in \Omega_0 : d(x, \partial\Omega_0) < \delta\}.$$

We then choose a C^θ function $p_\delta(x)$ which is positive in $\Omega_0 \setminus \overline{\Omega_0^\delta}$ and vanishes on $\partial\Omega_0^\delta \cap \Omega_0$. Then define

$$b_\delta(x,t) = p_\delta(x)q(t) \ \text{ for } (x,t) \in \left(\Omega_0 \setminus \Omega_0^\delta\right) \times \mathbb{R}^N, \ b_\delta(x,t) = b(x,t) \ \text{ elsewhere.}$$

It is clear that $b_\delta(x,t)$ satisfies a condition similar to (3.3.3) but with Ω_0 replaced by Ω_0^δ. By Theorem 3.2.1, problem (3.1.3) with b replaced by b_δ has a unique positive T-periodic solution u_a^δ if and only if

$$0 < a < \lambda_1^\delta(\infty) := \lim_{\mu \to \infty} \lambda_1(\mu b_\delta).$$

Moreover, by Theorems 3.3.3, 3.3.4 and 3.3.5,

$$\left(1 - \frac{T^*}{T}\right) \lambda_1^D(\Omega_0^\delta) \leq \lambda_1^\delta(\infty) < \lambda_1^D(\Omega_0^\delta).$$

Since $\lambda_1^D(\Omega_0^\delta) \to \infty$ as $\delta \to 0$, for any given $a \in (0, \infty)$, we can find a $\delta > 0$ such that

$$(1 - T^* T^{-1})\lambda_1^D(\Omega_0^\delta) > a$$

and hence u_a^δ exists. It is easily checked that for sufficiently large $M > 1$,

$$\overline{u} := M u_a^\delta|_{\Omega^* \times [0,T]}$$

is a super-solution to (3.4.1). On the other hand, clearly $\underline{u} := 0$ is a sub-solution. Hence, from the results stated in section 1.4 of chapter 1, (3.4.1) has a nonnegative T-periodic solution. The strong maximum principle then implies that the solution is positive.

If $a \leq 0$, then 0 is a sub-solution and any positive constant $M > \max m(x,t)$ is a super-solution. Hence (3.4.1) has a nonnegative T-periodic solution in the order interval $[0, M]$. Since $m > 0$, by the strong maximum principle the solution is positive.

We now prove the uniqueness and monotonicity properties of the positive T-periodic solution. Suppose that (3.4.1) has two positive T-periodic solutions $u_1(x,t)$ and $u_2(x,t)$. We may choose $M_0 > 1$ such that $M_0^{-1}u_1(x,t) < u_i(x,t) < M_0 u_1(x,t)$ for $i = 1, 2$. It is easily seen that $M_0 u_1$ is a supersolution of (3.4.1) and $M_0^{-1}u_1$ is a sub-solution. Hence there exist a minimal and a maximal solution in the order interval $[M_0^{-1}u_1, M_0 u_1]$, which we denote by $u_*(x,t)$ and $u^*(x,t)$, respectively. Thus $u_*(x,t) \leq u_i(x,t) \leq u^*(x,t)$ for $i = 1, 2$. Hence it suffices to show that $u_*(x,t) = u^*(x,t)$.

Define

$$\sigma_* := \inf\{\sigma \in \mathbb{R} : u^* \leq \sigma u_*\}.$$

Clearly $\sigma_* \geq 1$ and $u^* \leq \sigma_* u_*$. To prove $u^* = u_*$, it is enough to show $\sigma_* = 1$. Suppose for contradiction that $\sigma_* > 1$. Then for $w(x,t) := \sigma_* u_*(x,t) - u^*(x,t)$ we have $w \geq 0$, $w(x,0) = w(x,T)$,

$$\begin{aligned} \partial_t w - \Delta w &= aw - b(x,t)[\sigma_*(u_*)^p - (u^*)^p] \\ &\geq aw - b(x,t)(u^*)^{p-1} w \end{aligned}$$

for $(x,t) \in \Omega^* \times [0,T]$, and $\partial_\nu w = 0$ on $\partial\Omega \times [0,T]$, $w = (\sigma_* - 1)m > 0$ on $\partial\Omega_0 \times [0,T]$. Hence we can use the strong maximum principle to deduce that $w(x,t) > 0$ on $\overline{\Omega^*} \times [0,T]$. It follows that $w(x,t) \geq \epsilon u^*(x,t)$ for some $\epsilon > 0$ small, and hence

$$u^* \leq (1+\epsilon)^{-1}\sigma_* u_*,$$

which contradicts the definition of σ_*. This contradiction shows that we must have $\sigma_* = 1$, and the uniqueness conclusion is proved.

We next show the monotonicity of $u^m = u_a^m$ with respect to m. Assume that $m_1(x,t) \geq, \not\equiv m_2(x,t)$ on $\partial\Omega_0 \times [0,T]$. Then, u^{m_1} is a strict super-solution to the equation that u^{m_2} satisfies, and so the super-sub solution argument and the above proved uniqueness result indicate $u^{m_1} \geq u^{m_2}$ in $\Omega^* \times [0,T]$. Consequently, combined with the T-periodicity, the well-known maximum principle for parabolic equations and the Hopf boundary lemma we deduce $u^{m_1} > u^{m_2}$ in $\Omega^* \times [0,T]$. The monotonicity of u_a^m with respect to a is proved similarly. The proof is now complete. ☐

Proof of Theorem 3.4.1. For fixed $\mu > 0$, as in the proof of Theorem 3.3.3, let $\varphi_\mu(x,t)$ be the eigenfunction corresponding to $\lambda_1(\mu b)$ with the properties $\varphi_\mu(x,t) \geq 0$ on $\overline{\Omega} \times [0,T]$ and $\max_{\overline{\Omega}\times[0,T]} \varphi_\mu = 1$.

By the monotonicity of u_a with respect to a, we only need to prove the desired conclusion along a sequence $a_n \to a_\infty$. Since $\lambda_1(\mu b) \to a_\infty$, we take $a_n = \lambda_1(\mu_n b)$. For simplicity, we also denote u_{a_n} by u_n and φ_{μ_n} by φ_n, where μ_n increases to ∞ as $n \to \infty$.

A simple computation shows that

$$\underline{u}(x,t) = \mu_n^{\frac{1}{p-1}} \varphi_n(x,t) \;\text{ and }\; \overline{u}(x,t) = M\varphi_n(x,t)$$

form a pair of sub and super solutions of (3.1.3), where M satisfies

$$M^{p-1}[\varphi_n(x,t)]^{p-1} \geq \mu_n.$$

Then by the uniqueness of u_n it immediately follows that

$$\mu_n^{\frac{1}{p-1}} \varphi_n(x,t) \leq u_n(x,t) \leq M\varphi_n(x,t) \quad \text{on } \overline{\Omega} \times [0,T].$$

On the other hand, by Remark 3.3.1 and Step 5 in the proof of Theorem 3.3.3, we see that for any compact subset $K \subset (\overline{\Omega} \times (0,T^*]) \cup (\Omega_0 \times \mathbb{R})$,

$$\varphi_n(x,t) \to \varphi^*(x,t) \;\text{ in } C^{2,1}(K) \quad \text{as } n \to \infty,$$

where $\varphi^*(x,t) > 0$ in K. Hence

$$u_n(x,t) \geq \mu_n^{\frac{1}{p-1}} \varphi_n(x,t) \to \infty \text{ uniformly in } K.$$

It remains to show that

$$u_n(x,t) \to \infty \text{ uniformly on } \overline{\Omega}_0 \times [0,T] \text{ as } n \to \infty.$$

We follow an argument in the spirit of the proof of Lemma 3.3 and Lemma 3.4 of [36]. Note that $u_n(x,t)$ satisfies

$$\partial_t u_n - \Delta u_n = a_n u_n > 0 \text{ in } \Omega_0 \times [0,T],$$

$u_n(x,t) > 0$ on $\overline{\Omega}_0 \times [0,T]$, $u_n(x,t) \to \infty$ uniformly on any compact subset of $\Omega_0 \times \mathbb{R}$, and $u_n(x,T^*) \to \infty$ uniformly on $\overline{\Omega}_0$. By the maximum principle, it is sufficient to prove

$$u_n(x_n,t_n) = \min_{\partial\Omega_0 \times \mathbb{R}} u_n(x,t) \to \infty \quad \text{as } n \to \infty, \tag{3.4.2}$$

where we may choose $(x_n,t_n) \in \partial\Omega_0 \times [T^*, T+T^*]$.

To verify (3.4.2), we shall use a contradiction argument. We suppose on the contrary that (3.4.2) is false. Then, we may assume that $u_n(x_n,t_n) \leq C$ for all $n \geq 1$ and some positive constant C. We may now use the maximum principle and the fact that $u_n(x,T^*) \to \infty$ uniformly on $\overline{\Omega}_0$ to conclude that for all large n,

$$u_n(x_n,t_n) = \min_{\overline{\Omega}_0 \times [T^*, T+T^*]} u_n(x,t).$$

Without loss of generality we assume that this holds for all $n \geq 1$.

Since $\partial\Omega_0$ is smooth, it enjoys the uniform interior ball property, that is, we can find a small $R > 0$ such that for any $x \in \partial\Omega_0$, there exists a ball $B_{x,R}$ of radius R such that $B_{x,R} \subset \overline{\Omega}_0$ and $B_{x,R} \cap \partial\Omega_0 = \{x\}$.

To produce a contradiction, we first claim that: there is a constant $\delta > 0$ and a sequence of constants c_n satisfying $c_n \to \infty$, such that

$$u_n(x_n,t_n) + c_n\omega(x) \leq u_n(x,t) \text{ if } \frac{R}{2} \leq |x - y_n| \leq R,\ T^* \leq t \leq T+T^*, \tag{3.4.3}$$

where

$$\omega(x) = e^{-\delta|x-y_n|^2} - e^{-\delta R^2},$$

and y_n is the center of the ball $B_{x_n,R}$.

A simple computation gives

$$\Delta\omega + a_n\omega = (4\delta^2|x-y_n|^2 - 2N\delta + a_n)e^{-\delta|x-y_n|^2} - a_n e^{-\delta R^2}.$$

Thus, we can take a large $\delta > 0$ such that

$$\Delta\omega + a_n\omega \geq 0 \quad \forall\, x \in B_{x_n,R} \backslash B_{R/2}(y_n),$$

where

$$B_{R/2}(y_n) = \{x \in R^N \,:\, |x-y_n| < R/2\}.$$

We now choose a compact set $K \subset\subset \Omega_0$ such that $K \supset \cup_{n=1}^{\infty} B_{R/2}(y_n)$. By what has already been proved, $u_n(x,t) \to \infty$ uniformly in $K \times \mathbb{R}$, and hence there is a sequence c_n with $c_n \to \infty$ such that

$$u_n(x_n,t_n) + c_n(e^{-\delta R^2/4} - e^{-\delta R^2}) \leq u_n(x,t), \;\; \forall\, x \in B_{R/2}(y_n) \subset K, \; t \in [T^*, T+T^*].$$

We may further require that

$$u_n(x_n,t_n) + c_n(e^{-\delta R^2/4} - e^{-\delta R^2}) \leq u_n(x,T^*), \;\; \forall\, x \in \overline{\Omega}_0.$$

Then, as $u_n(x,t) \geq u_n(x_n,t_n)$ on $\overline{\Omega}_0 \times [T^*, T+T^*]$, we find that $u_n(x,t)$ is a super-solution of the problem

$$\begin{cases} \partial_t u - \Delta u = a_n u & \text{in } B_{x_n,R} \backslash B_{R/2}(y_n) \times [T^*, T+T^*], \\ u = u_n(x_n,t_n) & \text{on } \partial B_{x_n,R} \times [T^*, T+T^*], \\ u = u_n(x_n,t_n) + c_n(e^{-\delta R^2/4} - e^{-\delta R^2}) & \text{on } \partial B_{R/2}(y_n) \times [T^*, T+T^*], \\ u(x,T^*) = u_n(x,T^*) & \text{in } \{R/2 < |x-y_n| < R\}. \end{cases} \tag{3.4.4}$$

One also sees that $u_n(x_n,t_n) + c_n\omega(x)$ is a subsolution to (3.4.4). The comparison principle for parabolic equations then yields (3.4.3). Consequently, as $n \to \infty$, we find

$$\partial_{\nu_n} u_n|_{(x_n,t_n)} \geq c_n \partial_{\nu_n}\omega|_{x_n} = 2c_n\delta R e^{-\delta R^2} \to \infty, \tag{3.4.5}$$

where $\nu_n = (y_n - x_n)/|y_n - x_n|$.

On the other hand, for any $n \geq 1$, the following $T-$periodic problem

$$\begin{cases} \partial_t u - \Delta u = a_n u - b(x,t)u^p & \text{in } \Omega\backslash\overline{\Omega}_0 \times [0,T], \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times [0,T], \\ u = u_n(x_n, t_n) & \text{on } \partial\Omega_0 \times [0,T], \\ u(x,0) = u(x,T) & \text{in } \Omega\backslash\overline{\Omega}_0. \end{cases} \tag{3.4.6}$$

admits a unique positive solution $v_n(x,t)$ (see, Lemma 3.4.1). Furthermore, $u_n(x,t)$ is a supersolution of (3.4.6). Due to Lemma 3.4.1, we have $v_n(x,t) \leq u_n(x,t)$ on $\overline{\Omega}\backslash\Omega_0 \times [0,T]$. If we replace a_n by a_∞ and replace $u_n(x_n,t_n)$ by its upper bound C in (3.4.6), we obtain a unique positive solution of (3.4.6), denoted by $U_0(x,t)$. By Lemma 3.4.1 again,

$$v_n(x,t) \leq U_0(x,t) \text{ on } \overline{\Omega}\backslash\Omega_0 \times [0,T].$$

In particular, $\|v_n\|_{L^\infty(\overline{\Omega}\backslash\Omega_0\times[0,T])}$ is bounded. Thus, the L^p- estimates and Sobolev embedding theorem imply that $\{v_n\}$ is bounded in $C^{1+\theta,\theta/2}(\overline{\Omega}\backslash\Omega_0 \times [0,T])$, and so $\|\nabla v_n(x_n,t_n)\| \leq C_0$ for some $C_0 > 0$. Since

$$v_n(x,t) \leq u_n(x,t)\ \forall (x,t) \in \overline{\Omega}\backslash\Omega_0 \times [T^*, T+T^*] \text{ and } u_n(x_n,t_n) = v_n(x_n,t_n),$$

we conclude

$$\partial_{\nu_n} u_n|_{(x_n,t_n)} \leq \partial_{\nu_n} v_n|_{(x_n,t_n)} \leq C_0. \tag{3.4.7}$$

Clearly (3.4.5) and (3.4.7) contradict each other, which indicates that (3.4.2) is true. The proof of Theorem 3.4.1 is now complete. ☐

Theorem 3.4.2 *Let $a_\infty = \lambda_1(\infty)$. Then, as a increases to a_∞, $u_a(x,t) \to U_\infty(x,t)$ uniformly on any compact subset of $\overline{\Omega}\backslash\overline{\Omega}_0 \times (T^*, T)$, where $U_\infty(x,t)$ is the minimal positive solution of*

$$\begin{cases} \partial_t u - \Delta u = a_\infty u - b(x,t)u^p & \text{in } \Omega\backslash\overline{\Omega}_0 \times (T^*,T), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (T^*,T), \\ u = \infty & \text{on } \partial\Omega_0 \times (T^*,T), \\ u(x,T^*) = \infty & \text{in } \overline{\Omega}\backslash\Omega_0. \end{cases} \tag{3.4.8}$$

Proof. As before, since u_a is increasing in a, we only need to consider the limit of $u_n := u_{a_n}$ along an increasing sequence a_n which converges to a_∞ as $n \to \infty$.

Assume that $\epsilon > 0$ is small. Let $\Omega_\epsilon = \{x \in \Omega : d(x, \Omega_0) < \epsilon\}$. Since $b(x,t) > 0$ in $\overline{\Omega}\backslash\overline{\Omega}_0 \times (T^*, T)$, we may assume that $b(x,t) \geq M_\epsilon$ on $\overline{\Omega}\backslash\Omega_\epsilon \times [T^* + \epsilon, T - \epsilon]$ for some positive constant M_ϵ.

Consider the problem:

$$\begin{cases} \partial_t u - \Delta u = a_\infty u - M_\epsilon u^p & \text{in } \Omega\backslash\overline{\Omega}_\epsilon \times (T^* + \epsilon, T - \epsilon], \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (T^* + \epsilon, T - \epsilon), \\ u = u_n(x,t) & \text{on } \partial\Omega_\epsilon \times (T^* + \epsilon, T - \epsilon), \\ u(x, T^* + \epsilon) = u_n(x, T^* + \epsilon) & \text{in } \Omega\backslash\overline{\Omega}_\epsilon. \end{cases} \tag{3.4.9}$$

It is clear that $u_n(x,t)$ is a subsolution of (3.4.9).

In what follows, we find a supersolution of (3.4.9). For this purpose, we consider the following two auxiliary problems:

$$w_t = a_\infty w - M_\epsilon w^p, \quad t > T^* + \epsilon; \quad w(T^* + \epsilon) = \infty, \tag{3.4.10}$$

and

$$\begin{cases} -\Delta z = a_\infty z - M_\epsilon z^p \quad \text{in } \Omega\backslash\overline{\Omega}_\epsilon, \\ \partial_\nu z = 0 \;\; \text{on } \partial\Omega, \quad z = \infty \;\; \text{on } \partial\Omega_\epsilon. \end{cases} \tag{3.4.11}$$

The unique solution $w(t)$ of (3.4.10) can be explicitly written as

$$w(t) = \left(\frac{a_\infty}{M_\epsilon}\right)^{\frac{1}{p-1}} e^{a_\infty t} \left[e^{a_\infty (p-1) t} - e^{a_\infty (p-1)(T^* + \epsilon)}\right]^{\frac{1}{1-p}}, \;\; t > T^* + \epsilon.$$

And by the result of [35, 36], we know that problem (3.4.11) also admits a unique positive solution, which we denote by $z(x)$.

For any fixed n, we have

$$w(t) + z(x) > u_n(x,t) \;\; \text{in } \partial\Omega_\epsilon \times (T^* + \epsilon, T - \epsilon)$$

and

$$w(T^*+\epsilon) > u_n(x, T^*+\epsilon) \text{ on } \overline{\Omega}\backslash\Omega_\epsilon.$$

We can also easily check that $w(t)+z(x)$ satisfies the required differential inequality for a supersolution of (3.4.9). Hence, for all $n \geq 1$, by the comparison principle for parabolic equations, we have $u_n(x,t) \leq w(t)+z(x)$ on $\overline{\Omega}\backslash\Omega_\epsilon \times [T^*+\epsilon, T-\epsilon]$. Observe that, for fixed small $\epsilon > 0$, $w(t)+z(x)$ is bounded on $\overline{\Omega}\backslash\Omega_{2\epsilon} \times [T^*+2\epsilon, T-\epsilon]$. As a result, by the standard regularity argument, it is clear that $u_n(x,t) \to U_\infty(x,t)$ uniformly on any compact subset of $\overline{\Omega}\backslash\overline{\Omega}_0 \times (T^*, T)$ as $n \to \infty$, where $U_\infty(x,t)$ satisfies the first equation of (3.4.8), and $\partial_\nu U_\infty = 0$ on $\partial\Omega \times (T^*, T)$.

Next we show that

$$\lim_{t \downarrow T^*} U_\infty(x,t) = \infty \text{ uniformly for } x \in \overline{\Omega} \setminus \Omega_0, \tag{3.4.12}$$

$$\lim_{d(x,\Omega_0)\to 0} U_\infty(x,t) = \infty \text{ uniformly for } t \in [T^*, T). \tag{3.4.13}$$

Since u_n increases to U_∞ as $n \to \infty$, we have $U_\infty > u_k$ for all $k \geq 1$. Suppose for contradiction that there exist sequences $x_n \in \overline{\Omega} \setminus \Omega_0$ and t_n decreasing to T^* such that $U_\infty(x_n, t_n) \leq M$ for all $n \geq 1$ and some constant $M > 0$, then

$$u_k(x_n, t_n) \leq M \ \forall n \geq 1, \ \forall k \geq 1. \tag{3.4.14}$$

On the other hand, by Theorem 3.4.1 we know that $u_k(x_n, T^*) \to \infty$ as $k \to \infty$ uniformly in $n \geq 1$. Thus there exists k_0 large such that $u_{k_0}(x_n, T^*) \geq 3M$ for all $n \geq 1$. Since the function $u_{k_0}(x,t)$ is uniformly continuous in its variables, and $t_n \to T^*$, we deduce

$$|u_{k_0}(x_n, t_n) - u_{k_0}(x_n, T^*)| \to 0 \text{ as } n \to \infty.$$

Thus for all large n,

$$u_{k_0}(x_n, t_n) \geq u_{k_0}(x_n, T^*) - M \geq 2M,$$

which is in contradiction to (3.4.14). This proves (3.4.12). The proof of (3.4.13) is similar, where we use $u_n \to \infty$ on $\partial\Omega_0 \times [0, T]$ (by Theorem 3.4.1), and $u_k < U_\infty$.

Thus U_∞ is a solution to (3.4.8). It remains to show that U_∞ is the minimal positive solution of (3.4.8). Let U be any positive solution of (3.4.8). Then applying the parabolic comparison principle we easily see that $u_n < U$ in $\Omega \setminus \overline{\Omega}_0 \times (T^*, T)$. Letting $n \to \infty$ we deduce $U_\infty \leq U$. Hence U_∞ is the minimal positive solution. □

For later use, we also need to consider the following more general version of (3.4.8), where a_∞ is replaced by an arbitrary $a \in (-\infty, \infty)$:

$$\begin{cases} \partial_t u - \Delta u = au - b(x,t)u^p & \text{in } \Omega\backslash\overline{\Omega}_0 \times (T^*, T), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (T^*, T), \\ u = \infty & \text{on } \partial\Omega_0 \times (T^*, T), \\ u(x, T^*) = \infty & \text{in } \overline{\Omega}\backslash\Omega_0. \end{cases} \tag{3.4.15}$$

Theorem 3.4.3 *For any* $a \in (-\infty, \infty)$, (3.4.15) *has a minimal positive solution* $\underline{U}_a$ *and a maximal positive solution* $\overline{U}_a$, *in the sense that if* U *is any positive solution of* (3.4.15), *then* $\underline{U}_a \leq U \leq \overline{U}_a$ *in* $\Omega \setminus \overline{\Omega}_0 \times (T^*, T)$.

Proof. For $\epsilon \geq 0$ small, we define Ω_ϵ as in the proof of Theorem 3.4.2 and then for each integer $n \geq 1$ consider the following initial boundary value problem:

$$\begin{cases} \partial_t u - \Delta u = au - b(x,t)u^p & \text{in } \Omega\backslash\overline{\Omega}_\epsilon \times (T^* + \epsilon, T), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (T^* + \epsilon, T), \\ u = n & \text{on } \partial\Omega_\epsilon \times (T^* + \epsilon, T), \\ u(x, T^* + \epsilon) = n & \text{in } \overline{\Omega}\backslash\Omega_\epsilon. \end{cases} \tag{3.4.16}$$

Let u_n denote the unique positive solution of (3.4.16). By the same argument used in the proof of Theorem 3.4.2, we find that u_n increases to U_ϵ as $n \to \infty$, and U_ϵ is the minimal positive solution of

$$\begin{cases} \partial_t u - \Delta u = au - b(x,t)u^p & \text{in } \Omega\backslash\overline{\Omega}_\epsilon \times (T^* + \epsilon, T), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (T^* + \epsilon, T), \\ u = \infty & \text{on } \partial\Omega_\epsilon \times (T^* + \epsilon, T), \\ u(x, T^* + \epsilon) = \infty & \text{in } \overline{\Omega}\backslash\Omega_\epsilon. \end{cases} \tag{3.4.17}$$

Taking $\epsilon = 0$ we obtain the minimal positive solution of (3.4.15).

Using the parabolic comparison principle we easily deduce that

$$U_{\epsilon_1} \geq U_{\epsilon_2} \geq U_0 \ \text{ when } \epsilon_1 > \epsilon_2 > 0.$$

Hence there is a decreasing sequence ϵ_n converging to 0 such that $U_{\epsilon_n} \to \overline{U}$ as $\epsilon_n \to 0$ and $\overline{U}$ is a positive solution of (3.4.15). We show that $\overline{U}$ is the maximal positive solution of (3.4.15). Indeed, if U is any positive solution of (3.4.15), then we can apply the parabolic comparison principle to deduce $U_{\epsilon_n} > U$ for each n. Letting $n \to \infty$ we obtain $\overline{U} \geq U$. Hence $\overline{U}$ is the maximal positive solution of (3.4.15). The proof is complete. □

We are now ready to state and prove the long-time asymptotic behavior of the unique positive solution of (3.1.1) for $a \geq a_\infty$.

Theorem 3.4.4 *Assume that* $a \geq a_\infty$, $u_0 \in C(\overline{\Omega})$ *and* $u_0 \geq, \not\equiv 0$. *Then, the unique solution* $u(x,t)$ *of* (3.1.1) *satisfies*

$$\lim_{n\to\infty} u(x, t + nT) = \begin{cases} \infty & \text{locally uniformly on } \ (\overline{\Omega} \times (0, T^*]) \cup (\overline{\Omega}_0 \times [0, T]), \\ \underline{U}_a(x,t) & \text{locally uniformly on } \ \overline{\Omega}\backslash\overline{\Omega}_0 \times (T^*, T). \end{cases}$$

Proof. For any given $\epsilon > 0$, let $u^\epsilon(x,t)$ denote the unique solution of the problem

$$\begin{cases} \partial_t u - \Delta u = (a_\infty - \epsilon)u - b(x,t)u^p & \text{in } \Omega \times (0, \infty), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \Omega \times (0, \infty). \end{cases} \tag{3.4.18}$$

Since $a > a_\infty - \epsilon$, it is obvious that $u(x,t)$ is a supersolution to (3.4.18) and thus

$$u^\epsilon(x,t) \le u(x,t) \quad \text{on } \overline{\Omega} \times [0,\infty). \tag{3.4.19}$$

Let $u_{a_\infty-\epsilon}(x,t)$ be the unique $T-$periodic positive solution to

$$\begin{cases} \partial_t u - \Delta u = (a_\infty - \epsilon)u - b(x,t)u^p & \text{in } \Omega \times (0,T), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u(x,T) & \text{in } \Omega. \end{cases}$$

By Theorem 3.2.1, we have

$$u^\epsilon(x,t+nT) \to u_{a_\infty-\epsilon}(x,t) \ \text{ uniformly on } \overline{\Omega} \times [0,T] \quad \text{as } n \to \infty. \tag{3.4.20}$$

Using (3.4.19) and (3.4.20) we obtain that

$$\liminf_{n\to\infty} u(x,t+nT) \ge u_{a_\infty-\epsilon}(x,t)$$

for all small $\epsilon > 0$, uniformly on $\overline{\Omega} \times [0,T]$. Letting $\epsilon \to 0$ in the above inequality and using Theorems 3.4.1 and 3.4.2, we deduce that

$$\lim_{n\to\infty} u(x,t+nT) = \infty \text{ locally uniformly on } (\overline{\Omega} \times (0,T^*]) \cup (\overline{\Omega}_0 \times [0,T]),$$

and

$$\liminf_{n\to\infty} u(x,t+nT) \ge \underline{U}_{a_\infty}(x,t) \text{ locally uniformly on } \overline{\Omega}\backslash\overline{\Omega}_0 \times (T^*,T). \tag{3.4.21}$$

On the other hand, by the parabolic comparison principle, we easily see that for every $n \ge 1$, $u(x,t+nT) < \underline{U}_a(x,t)$ in $\Omega \setminus \overline{\Omega}_0 \times (T^*,T)$, and hence

$$\limsup_{n\to\infty} u(x,t+nT) \le \underline{U}_a(x,t) \text{ uniformly on } \overline{\Omega}\backslash\overline{\Omega}_0 \times (T^*,T). \tag{3.4.22}$$

Using this upper bound for $\tilde{u}_n(x,t) := u(x,t+nT)$ and standard parabolic estimates and (3.4.21), we easily see that, by passing to a subsequence, $\tilde{u}_n(x,t) \to \tilde{U}_a(x,t)$ which is a positive solution of (3.4.15). Hence $\tilde{U}_a \geq \underline{U}_a$. Together with (3.4.22), this implies that $\tilde{U}_a = \underline{U}_a$. Hence the entire original sequence converges and

$$\lim_{n\to\infty} u(x, t+nT) = \underline{U}_a(x,t).$$

By standard parabolic estimates, the above convergence is locally uniform in $\overline{\Omega}\backslash\overline{\Omega}_0 \times (T^*, T)$. The proof is thus complete. $\square$

Remark 3.4.1 *The following questions arise naturally:*

(Q1) Does (3.4.15) *have at most one positive solution?*

(Q2) If U is a positive solution to (3.4.15)*, what is the asymptotic behavior of $U(x,t)$ as t increases to T?*

We will address these and related questions in future work.

3.4.2 The case that (3.3.2) holds.

In this subsection, we suppose that (3.3.2) holds, and show that the long-time dynamical behavior of the positive solution to (3.1.1) is analogous to Theorems 2.3.2 and 2.3.4 in chapter 2. Let us recall that by Theorem 3.3.2, $\lambda_1(\infty) = \lambda_1^D(\Omega_0)$.

Our approach in this subsection follows the lines of the previous subsection. We start with the following result.

Lemma 3.4.2 *The conclusions in Lemma 3.4.1 remain valid under condition* (3.3.2).

Proof. We only give the proof for existence; the other conclusions are proved in the same way as in Lemma 3.4.1.

We shall again use a super-sub solution argument. It is obvious that $\underline{u}(x,t) = 0$ is a sub-solution to (3.4.1). Next, we construct a super-solution. It is well-known that the following elliptic problem

$$\begin{cases} -\Delta u = au - \underline{b}(x)u^p & \text{in } \Omega^*, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \\ u = \max\limits_{\partial\Omega_0 \times [0,T]} m(x,t) & \text{on } \partial\Omega_0 \end{cases}$$

has a unique positive solution $\overline{u}(x) \in C^2(\overline{\Omega^*})$ (see, e.g., Lemma 2.3 in [36]), and we easily see that $\overline{u}(x)$ is a supersolution to (3.4.1). Thus by using the standard super-sub solution iteration argument, (3.4.1) admits a positive T-periodic solution. □

Remark 3.4.2 *By exactly the same proof, we see that when* $b(x,t) > 0$ *on* $\overline{\Omega^*} \times [0,T]$*, Lemma 3.4.2 remans valid.*

Theorem 3.4.5 *For any* $a \in (-\infty, \infty)$*, the following boundary blow-up problem*

$$\begin{cases} \partial_t v - \Delta v = av - b(x,t)v^p & \text{in } \Omega^* \times [0,T], \\ \partial_\nu v = 0 & \text{on } \partial\Omega \times [0,T], \\ v = \infty & \text{on } \partial\Omega_0 \times [0,T], \\ v(x,0) = v(x,T) & \text{in } \Omega^* \end{cases} \tag{3.4.23}$$

has a minimal positive solution $\underline{V}_a(x,t)$ *and a maximal positive solution* $\overline{V}_a(x,t)$ *in the sense that any positive solution* $V(x,t)$ *of* (3.4.23) *satisfies* $\underline{V}_a(x,t) \le V(x,t) \le \overline{V}_a(x,t)$ *in* $\Omega^* \times [0,T]$*. Moreover, both the minimal and maximal solutions are nondecreasing in* a*.*

Proof. For small $\epsilon \ge 0$, we define $\Omega_\epsilon = \{x \in \Omega : x \in \Omega_0 \text{ or } \mathrm{d}(x,\Omega_0) < \epsilon\}$. By Lemma 1.2.5 of [62], for small ϵ, $\partial\Omega_\epsilon$ has the same smoothness as $\partial\Omega_0$. In (3.4.1), we take $m(x,t) = m$ and replace Ω_0 by Ω_ϵ. By Lemma 3.4.2 and Remark 3.4.2, we know that the modified (3.4.1) has a unique positive T-periodic solution $u_a^m(x,t) = u_a^{m,\epsilon}(x,t)$. We claim that $\underline{V}_a^\epsilon(x,t) = \lim_{m\to\infty} u_a^m(x,t)$ is a minimal positive solution of (3.4.23) with Ω_0 replaced by Ω_ϵ.

To prove this, we first show that for any fixed small $\delta > 0$, $u_a^m(x,t)$ is uniformly bounded on

$\overline{\Omega}\backslash\Omega_{\epsilon+\delta} \times [0,T]$. To this end, we consider the elliptic problem

$$\begin{cases} -\Delta u = au - \underline{b}(x)u^p & \text{in } \Omega\backslash\overline{\Omega}_\epsilon, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \\ u = m & \text{on } \partial\Omega_\epsilon. \end{cases} \tag{3.4.24}$$

By Lemma 2.3 in [36], problem (3.4.24) has a unique positive solution, which we denote by $\underline{u}_a^m(x)$. Moreover $\underline{u}_a^m(x)$ is strictly increasing in m, and

$$U_a(x) := \lim_{m\to\infty} \underline{u}_a^m(x)$$

exists and is the minimal positive solution of (3.4.24) with m replaced by ∞. This implies in particular that $\underline{u}_a^m < U_a$ for all $m \geq 1$.

As in the proof of Lemma 3.4.1, we can use a super-sub solution argument, together with the uniqueness of $u_a^m(x,t)$, to show that

$$u_a^m(x,t) \leq \underline{u}_a^m(x) \text{ on } \overline{\Omega}\backslash\Omega_\epsilon \times [0,T] \text{ for each } m \geq 1.$$

Therefore, we have

$$u_a^m(x,t) \leq U_a(x) \text{ on } \overline{\Omega}\backslash\Omega_\epsilon \times [0,T] \text{ for all } m \geq 1.$$

This proves the required uniform boundedness of $u_a^m(x,t)$. Hence

$$\underline{V}_a^\epsilon(x,t) := \lim_{m\to\infty} u_a^m(x,t)$$

exists. Moreover, using standard regularity theorem for parabolic equations and the embedding theorem, we can easily conclude that $\underline{V}_a^\epsilon = \lim_{m\to\infty} u_a^m$ holds in $C^{2,1}(K \times [0,T])$ for any compact subset K of $\overline{\Omega^*}\backslash\overline{\Omega}_\epsilon$, and $V_a^\epsilon(x,t)$ satisfies (3.4.23) with Ω_0 replaced by Ω_ϵ. Since each u_a^m is increasing in a, $\underline{V}_a^\epsilon$ is nondecreasing in a.

We next show that $\underline{V}_a^\epsilon(x,t)$ obtained above is the minimal positive solution. Assume that $V(x,t)$ is an arbitrary positive solution of (3.4.23) with Ω_0 replaced by Ω_ϵ. Since $\lim_{d(x,\Omega_\epsilon)\to 0} V(x,t) = \infty$ and $u_a^m = m$ on $\partial\Omega_\epsilon \times [0,T]$, there exists $\delta_m > 0$ sufficiently small such that $V > u_a^m$ on

$\partial\Omega_{\epsilon+\delta} \times [0,T]$ for all $\delta \in (0,\delta_m]$. Hence for all such δ, V is a supersolution to the following problem:

$$\begin{cases} \partial_t u - \Delta u = au - b(x,t)u^p & \text{in } \Omega\backslash\overline{\Omega}_{\epsilon+\delta} \times [0,T], \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times [0,T], \\ u = u_a^m(x,t) & \text{on } \partial\Omega_{\epsilon+\delta} \times [0,T], \\ u(x,0) = u(x,T) & \text{in } \Omega\backslash\overline{\Omega}_{\epsilon+\delta}. \end{cases} \tag{3.4.25}$$

Since clearly 0 is a sub-solution to this problem, we conclude that (3.4.25) has a positive solution satisfying $u \le V$. Since clearly $u_a^m(x,t)$ solves (3.4.25), and by Remark 3.4.2 it is the unique positive solution, we deduce $u_a^m(x,t) \le V(x,t)$ for all $m \ge 1$ and $(x,t) \in \Omega \setminus \Omega_{\epsilon+\delta} \times [0,T]$. Since $\delta > 0$ can be arbitrarily small we deduce

$$u_a^m(x,t) \le V(x,t) \quad \text{for all } m \ge 1 \text{ and } (x,t) \in \Omega \setminus \overline{\Omega}_\epsilon \times [0,T].$$

Letting $m \to \infty$ we obtain $\underline{V}_a^\epsilon \le V$. This proves that $\underline{V}_a^\epsilon$ is the minimal positive solution. Taking $\epsilon = 0$ we know that (3.4.23) has a minimal positive solution, and it is nondecreasing in a.

To show the existence of a maximal positive solution, we notice that for any small ϵ_1, ϵ_2 with $0 < \epsilon_1 < \epsilon_2$, we can use a comparison argument as in the last paragraph to deduce that, for any positive solution V of (3.4.23),

$$\underline{V}_a^{\epsilon_2}(x,t) \ge \underline{V}_a^{\epsilon_1}(x,t) \ge V(x,t) \;\; \text{in } \Omega\backslash\overline{\Omega}_{\epsilon_2} \times [0,T].$$

It follows that

$$\overline{V}_a(x,t) := \lim_{\epsilon\to 0} \underline{V}_a^\epsilon(x,t) \ge V(x,t),$$

exists, and moreover, $\overline{V}_a(x,t)$ is a positive solution of (3.4.23). Since $V(x,t) \le \overline{V}_a(x,t)$, we conclude that $\overline{V}_a(x,t)$ is the desired maximal positive solution. Since each $\underline{V}_a^\epsilon$ is nondecreasing in a, so is $\overline{V}_a$. □

Remark 3.4.3 *For the boundary blow-up problem* (3.4.23)*, if we assume that*

$$\sigma_1 d(x,\Omega_0)^\alpha \le b(x,t) \le \sigma_2 d(x,\Omega_0)^\alpha$$

for some constants $\sigma_1 > 0$, $\sigma_2 > 0$, $\alpha > -2$, *and for all* x *close to* $\partial\Omega_0$ *and* $t \in [0, T]$, *then one can make use of Corollary 6.17 in [35], and a convex function trick due to Marcus and Véron [70, 71] as in the proof of Theorem 6.18 of [35] to show that* (3.4.23) *has a unique positive solution. Some details of this idea are given in the proof of Theorem 3.4.7 below.*

Theorem 3.4.6 *Let* $a_\infty = \lambda_1(\infty)$. *Then, as* a *increases to* a_∞, *the unique positive* T*-periodic solution of* (3.1.3) *satisfies*

(i) $u_a(x, t) \to \infty$ *uniformly on* $\overline{\Omega}_0$;

(ii) $u_a(x, t) \to \underline{V}_{a_\infty}(x, t)$ *in* $C^{2,1}(K \times [0, T])$ *for any compact set* $K \subset \overline{\Omega} \backslash \overline{\Omega}_0$.

Proof. As before, by a simple super-sub solution argument, we find that $u_a(x, t)$ is strictly increasing in a for $a \in (0, a_\infty)$.

From [36], we know that the following problem

$$\begin{cases} -\Delta u = au - \bar{b}(x)u^p & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique positive solution if and only if $a \in (0, a_\infty)$; we denote it by $\underline{u}_a(x)$. Moreover, Theorem 1.2 in [36] tells us that $\underline{u}_a(x) \to \infty$ uniformly on $\overline{\Omega}_0$ as $a \to a_\infty$. On the other hand, by a simple sub-super solution argument we deduce that $\underline{u}_a(x) \le u_a(x, t)$. As a result, we can use Theorem 3.6 of [36] to obtain

$$u_a(x, t) \to \infty \text{ uniformly on } \overline{\Omega}_0, \text{ as } a \to a_\infty.$$

Furthermore, by the comparison argument we used in the proof of Theorem 3.4.5 to deduce $u_a^m \le V$ through (3.4.25), we can easily show that

$$u_a(x, t) \le \underline{V}_a(x, t) \le \underline{V}_{a_\infty}(x, t) \text{ in } \Omega^* \times [0, T].$$

Since $u_a(x, t)$ is increasing in a for $a \in (0, a_\infty)$, $u_*(x, t) := \lim_{a \to a_\infty} u_a(x, t)$ exists. Moreover, $u_*(x, t) \le \underline{V}_{a_\infty}(x, t)$ and it satisfies (3.4.23) with $a = a_\infty$. Hence, by Theorem 3.4.5, it is necessary that $u_*(x, t) = \underline{V}_{a_\infty}(x, t)$. Using the Sobolev embedding theorems and the interior estimates (see, section 1.2.2. or [65, 66]), we easily see that, as $a \to a_\infty$,

$$u_a(x, t) \to \underline{V}_{a_\infty}(x, t) \text{ in } C^{2,1}(K \times [0, T]) \text{ for any compact set } K \subset \overline{\Omega} \backslash \overline{\Omega}_0.$$

This completes the proof. ☐

Theorem 3.4.7 *Assume that* $a \geq a_\infty$, $u_0 \in C(\overline{\Omega})$ *and* $u_0 \geq, \not\equiv 0$. *Then, the unique solution* $u(x,t)$ *of* (3.1.1) *satisfies that*

(i) $u(x,t) \to \infty$ *uniformly on* $\overline{\Omega}_0$ *as* $t \to \infty$;

(ii) $u(x, t+nT) \to \underline{V}_a(x,t)$ *in* $C^{2,1}(K \times [0,T])$ *as* $n \to \infty$ *for any compact set* $K \subset \overline{\Omega}\backslash\overline{\Omega}_0$.

Proof. By the same argument as in the proof of Theorem 3.4.4, we can use Theorem 3.4.6 to obtain

$$\lim_{n\to\infty} u(x, t+nT) = \infty \text{ uniformly on } (\overline{\Omega}_0 \times [0,T]),$$

and

$$\liminf_{n\to\infty} u(x, t+nT) \geq \underline{V}_{a_\infty}(x,t) \text{ in } \overline{\Omega} \setminus \overline{\Omega}_0 \times [0,T].$$

Conclusion (i) in the theorem is thus proved.

We next prove (ii). Let us denote by $u_a^m(x,t)$ the unique positive solution of (3.4.1) with $m(x,t) = m > 0$ being a constant. We first verify that for any $v_0 \in C(\overline{\Omega}^*)$, $v_0 \geq 0$, the solution $v^m(x,t)$ of

$$\begin{cases} \partial_t v - \Delta v = av - b(x,t)v^p & \text{in } \Omega^* \times (0,\infty), \\ \partial_\nu v = 0 & \text{on } \partial\Omega \times (0,\infty), \\ v = m & \text{on } \partial\Omega_0 \times (0,\infty), \\ v(x,0) = v_0(x) & \text{in } \Omega^* \end{cases} \tag{3.4.26}$$

satisfies

$$v^m(x, t+nT) \to u_a^m(x,t) \text{ uniformly for } (x,t) \in \overline{\Omega^*} \times [0,T], \text{ as } n \to \infty. \tag{3.4.27}$$

In fact, for any constant $M > 1$, $Mu_a^m(x,t)$ is a supersolution of (3.4.1), while 0 is a subsolution. We may choose $M > 1$ large enough so that $Mu_a^m(x,0) > v_0(x)$ on $\overline{\Omega^*}$. Let $\underline{v}^m(x,t)$ denote the unique solution of (3.4.26) with $v(x,0) \equiv 0$, and let $\overline{v}^m(x,t)$ be the unique solution of (3.4.26) with $v(x,0) = Mu_a^m(x,0)$. Then, the well-known comparison principle for parabolic equations infers that

$$\underline{v}^m(x,t) \le v^m(x,t) \le \overline{v}^m(x,t).$$

Moreover, by standard iteration procedure from sub- and super-solutions for periodic-parabolic problems (as in [53]), we see that, as $n \to \infty$, $\underline{v}^m(x,t+nT)$ increases to a positive T-periodic solution of (3.4.1) with $m(x,t) \equiv m$, and $\overline{v}^m(x,t+nT)$ decreases to such a solution. Since $u_a^m(x,t)$ is the unique positive T-periodic solution of (3.4.1) with $m(x,t) \equiv m$ by Lemma 3.4.1, we must have

$$\underline{v}^m(x,t+nT),\ \overline{v}^m(x,t+nT) \to u_a^m(x,t) \ \text{ uniformly for } (x,t) \in \overline{\Omega^*} \times [0,T], \ \text{ as } n \to \infty.$$

Clearly (3.4.27) is a consequence of this fact.

As before we know that as $m \to \infty$, $u_a^m(x,t)$ converges to $\underline{V}_a(x,t)$. Moreover, this convergence is uniform on $K \times [0,T]$ for any compact subset $K \subset \overline{\Omega} \backslash \overline{\Omega}_0$. Hence, for any given $\epsilon > 0$, we can find $m_\epsilon > 0$ large such that

$$u_a^{m_\epsilon}(x,t) \ge \underline{V}_a(x,t) - \epsilon/2 \ \ \forall (x,t) \in K \times [0,T]. \tag{3.4.28}$$

In view of conclusion (i) proved above, one can find a large $N_\epsilon > 0$ such that $u(x,t) > m_\epsilon$ for $t \ge N_\epsilon T$ and $x \in \partial\Omega_0$. Consequently, $u(x,t+N_\epsilon T)$ is a supersolution of (3.4.26) with $m = m_\epsilon$ and $v_0(x) = 0$. It follows that

$$u(x,t+nT) \ge \underline{v}^{m_\epsilon}(x,t+(n-N_\epsilon)T) \ge u^{m_\epsilon}(x,t) - \epsilon/2$$

uniformly on $K \times [0,T]$ for all large $n \ge N_\epsilon$. Applying (3.4.28), it follows that for all large $n \ge N_\epsilon$,

$$u(x,t+nT) \ge \underline{V}_a(x,t) - \epsilon \ \text{ uniformly on } K \times [0,T],$$

from which we derive

$$\liminf_{n\to\infty} u(x,t+nT) \ge \underline{V}_a(x,t) \ \text{ uniformly on } K \times [0,T]. \tag{3.4.29}$$

We show next that

$$\lim_{n\to\infty} u(x,t+nT) = \underline{V}_a(x,t) \text{ uniformly on } K\times[0,T]. \tag{3.4.30}$$

To this end we consider the auxiliary problem

$$\begin{cases} \partial_t w - \Delta w = aw - \underline{b}(x)v^p & \text{in } \Omega\times(0,\infty), \\ \partial_\nu w = 0 & \text{on } \partial\Omega\times(0,\infty), \\ w(x,0)=u_0(x) & \text{in } \Omega. \end{cases} \tag{3.4.31}$$

Since $\underline{b}\le b$, by the parabolic comparison principle we deduce

$$u(x,t)\le w(x,t) \text{ in } \Omega\times(0,\infty).$$

By the main result in [46], for $a\ge a_\infty$ and $x\in\overline{\Omega}\setminus\overline{\Omega}_0$,

$$\lim_{t\to\infty} w(x,t) = \tilde{W}_a(x),$$

where the limits are locally uniform in $\overline{\Omega}\setminus\overline{\Omega}_0$, and $\tilde{W}_a$ is the minimal positive solution of

$$-\Delta W = aW - \underline{b}(x)W^p \text{ in } \Omega\setminus\overline{\Omega}_0,\ \partial_\nu W|_{\partial\Omega}=0,\ W|_{\partial\Omega_0}=\infty.$$

Since $u(x,t)\le w(x,t)$, we necessarily have

$$\underline{V}_a(x,t)\le \liminf_{n\to\infty} u(x,t+nT)\le \limsup_{n\to\infty} u(x,t+nT)\le \tilde{W}_a(x) \tag{3.4.32}$$

locally uniformly for $x\in\overline{\Omega}\setminus\overline{\Omega}_0$.

Using the above bounds for $\tilde{u}_n(x,t):=u(x,t+nT)$ and standard parabolic estimates, we easily see that by passing to a subsequence $\tilde{u}_n(x,t)\to\hat{V}_a(x,t)$, and $\hat{V}_a$ satisfies

$$\begin{cases} \partial_t v - \Delta v = av - b(x,t)v^p & \text{in } \Omega^*\times\mathbb{R}, \\ \partial_\nu v = 0 & \text{on } \partial\Omega\times\mathbb{R}, \\ v=\infty & \text{on } \partial\Omega_0\times\mathbb{R}. \end{cases} \tag{3.4.33}$$

By (3.4.32), we deduce

$$\underline{V}_a\le\hat{V}_a.$$

On the other hand, if we choose $k > 1$ large enough such that $k\underline{V}_a(x, 0) \geq u_0(x)$ in $\overline{\Omega} \setminus \overline{\Omega}_0$, then we can apply the parabolic comparison principle to deduce that $u(x, t) \leq k\underline{V}_a(x, t)$ in $(\overline{\Omega} \setminus \overline{\Omega}_0) \times (0, \infty)$. It follows that

$$\hat{V}_a \leq k\underline{V}_a.$$

To prove (3.4.30) it suffices to verify that $\underline{V}_a = \hat{V}_a$. Arguing by contradiction, we assume that $\underline{V}_a(x, t) \leq, \not\equiv \hat{V}_a(x, t)$ in $\Omega^* \times \mathbb{R}$. Then, by the well-known strong maximum principle for parabolic equations, it is easily seen that $\underline{V}_a(x, t) < \hat{V}_a(x, t)$ in $\Omega^* \times \mathbb{R}$.

We now define

$$U(x, t) = \underline{V}_a(x, t) - (2k)^{-1}(\hat{V}_a(x, t) - \underline{V}_a(x, t)),$$

and use a convex function trick introduced by Marcus and Véron [70, 71] as in Theorem 6.18 of [35]. Simple direct computations show that

$$\underline{V}_a > U \geq \frac{k+1}{2k}\underline{V}_a \quad \text{in } \Omega^* \times \mathbb{R}, \tag{3.4.34}$$

and

$$\frac{2k}{2k+1}U + \frac{1}{2k+1}\hat{V}_a(x, t) = \underline{V}_a(x, t). \tag{3.4.35}$$

It is clear that

$$f(x, t, v) = -av + b(x, t)v^p$$

is convex with respect to v in $(0, \infty)$. Hence, by virtue of (3.4.35), we obtain

$$f(x, t, \underline{V}_a(x, t)) \leq \frac{2k}{2k+1}f(x, t, U) + \frac{1}{2k+1}f(x, t, \hat{V}_a(x, t)).$$

It follows that

$$\partial_t U - \Delta U = -\frac{2k+1}{2k}f(x, t, \underline{V}_a(x, t)) + \frac{1}{2k}f(x, t, \hat{V}_a(x, t)) \geq -f(x, t, U),$$

from which and (3.4.34), we deduce

$$\begin{cases} \partial_t U - \Delta U \geq aU - b(x, t)U^p & \text{in } \Omega^* \times \mathbb{R}, \\ \partial_\nu U = 0 & \text{on } \partial\Omega \times \mathbb{R}, \\ U = \infty & \text{on } \partial\Omega_0 \times \mathbb{R}. \end{cases}$$

Note that due to the periodicity of $b(x,t)$ in t, for each $n \geq 1$, $U(x,t-nT)$ also satisfies the above system. Hence we may use the parabolic comparison principle to deduce that $U(x,t-nT) \geq u_m(x,t)$ in $\Omega^* \times (0,\infty)$ for all $m,\ n \geq 1$, where u_m is the unique solution to

$$\begin{cases} \partial_t u - \Delta u = au - b(x,t)u^p & \text{in } \Omega^* \times (0,\infty), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u = m & \text{on } \partial\Omega_0 \times (0,\infty), \\ u(x,0) = m_* & \text{in } \Omega^*, \end{cases} \tag{3.4.36}$$

with $m_* = \inf U \geq \frac{k+1}{2k} \min \underline{V}_a > 0$. As before we know that $\lim_{n\to\infty} u_m(x,t+nT) = u_a^m(x,t)$, which is the unique positive T-periodic solution of (3.4.1) with $m(x,t) \equiv m$. Thus

$$u_a^m(x,t) = \lim_{n\to\infty} u_m(x,t+nT) \leq U(x,t).$$

Letting $m \to \infty$ we deduce

$$\underline{V}_a(x,t) \leq U(x,t).$$

But this is a contradiction with (3.4.34). This proves (3.4.30).

Using standard parabolic regularity theory and embedding theorems, we easily see that the convergence in (3.4.30) holds in $C^{2,1}(K \times [0,T])$ for any compact subset K of $\overline{\Omega} \setminus \overline{\Omega}_0$. This finishes the proof of the theorem. □

Chapter 4

The perturbed periodic-parabolic logistic equation

4.1 Introduction

In this chapter, we are interested in the formation of spatiotemporal patterns of the perturbed periodic-parabolic logistic equation. To be more precise, we shall be concerned with the following problem:

$$\begin{cases} \partial_t u - \Delta u = au - [b(x,t) + \epsilon]u^p & \text{in } \Omega \times (0,T), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u(x,T) & \text{in } \Omega. \end{cases} \tag{4.1.1}$$

As in the preceding chapters, in (4.1.1), a and $p > 1$ are constants, and the function $b(x,t) \in C^{\theta,\theta/2}(\overline{Q_T})$ $(0 < \theta < 1)$ is T-periodic in t, nonnegative and vanishes (i.e., has a degeneracy) in some subdomain of Q_T, and ν is the outward unit normal vector on $\partial\Omega$. The constant $\epsilon > 0$ is a small perturbation parameter.

It is known from Theorems 3.2.1 and 3.3.1 that, for any given constant $\epsilon > 0$, (4.1.1) has a unique positive T- periodic solution, which is a global attractor of the corresponding initial-boundary value problem

$$\begin{cases} \partial_t u - \Delta u = au - [b(x,t) + \epsilon]u^p & \text{in } \Omega \times (0, \infty), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) \geq, \not\equiv 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1.2}$$

As shown in chapter 3, when $\epsilon = 0$, the dynamical behavior of (4.1.2) is fundamentally different. So it is interesting to examine the change of the dynamical behavior of (4.1.2) as $\epsilon \to 0$, which then leads us to consider the asymptotic behavior of the unique positive T- periodic solution u_ϵ of (4.1.1) as $\epsilon \to 0$.

Our investigation is also motivated by the study of Du and Li [38] . In their work, the following perturbed elliptic logistic equation

$$\begin{cases} -\Delta u = au - [b(x) + \epsilon]u^p & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1.3}$$

was used to see how the spatial patterns arise. They assumed that $b(x)$ vanishes in some subregions of Ω. The analytical results in [38] show that the peak solutions generated by the perturbed elliptic logistic equation (4.1.3) is significantly different from those obtained in many superlinear problems such as in [20, 73, 87]. In the latter case, the peaks of the solutions concentrate at isolated points on the underlying domain and so the measure of the set of peaks goes to zero as $\epsilon \to 0$; moreover, the peak solutions are usually unstable and exhibit a concentration of mass. However, Theorems 2.2 and 2.7 of [38] tell us that, not only the set of peaks of the unique and globally stable positive steady-state solution u_ϵ^a to (4.1.3) can develop a rather arbitrary spatial pattern, but also the rescaled solution $v_\epsilon^a(x) = \epsilon^{\frac{1}{p-1}} u_\epsilon^a(x)$ exhibits no layers at all.

To capture the possible spatiotemporal pattern of (4.1.1) (a precise definition of spatiotemporal pattern will be given in the coming section), we will have to determine the asymptotical behavior of the unique positive T- periodic solution $u_\epsilon(x,t)$ of (4.1.1) as $\epsilon \to 0$. Our findings here will reveal some new and interesting phenomena on the formation of spatiotemporal patterns. As in chapter 3, we will deal with three basic cases: temporal degeneracy, spatial degeneracy and both spatial and temporal degeneracies.

Roughly speaking, we will show that, spatiotemporal pattern occurs only if at least spatial degeneracy occurs in $b(x,t)$. In other words, temporal degeneracy alone can not generate such patterns. On the other hand, once spatial degeneracy exists, the spatiotemporal patterns generated by (4.1.1) vary significantly, depending on whether temporal degeneracy exists or not. If only spatial degeneracy is imposed, our conclusions show that the pattern is similar in nature to that generated by (4.1.3). In fact, in this case, as $\epsilon \to 0$, the peaks of the T-periodic solution $u_\epsilon(x,t)$ of (4.1.1) only concentrate on certain degenerate regions, which can vary and is completely determined by the value of a. In sharp contrast, when $b(x,t)$ is both spatially and temporally degenerate, the patterns are fundamentally different, and as $\epsilon \to 0$, the peaks of $u_\epsilon(x,t)$ will concentrate on the whole degenerate region, independent of the value of a. Furthermore, for the rescaled solution $v_\epsilon(x,t) = \epsilon^{\frac{1}{p-1}} u_\epsilon(x,t)$, it also exhibits new properties. See the main results in section 4.3 for more details.

The remainder of this chapter is arranged as follows. In section 4.2, some necessary notations and definitions are introduced and section 4.3 states the main results. Section 4.4 gives the preliminary results while section 4.5 is devoted to the proofs of the main results.

4.2 Definitions and preliminaries

Throughout this chapter, we assume that T is a given positive constant, m is a given positive integer and $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$.

For convenience, we always set $Q_T = \Omega \times [0,T]$. Let $\mathcal{D} = \{D_1, D_2, \cdots, D_m\}$ be a finite set of disjoint subdomains of Q_T, such that each D_j is connected, $\overline{D}_j \subset Q_T$, $\overline{D}_i \cap \overline{D}_j = \emptyset$ when $i \neq j$.

We first define what we mean by a spatiotemporal pattern as follows.

Definition 4.2.1 *Let $\mathcal{D}$ be as above. A one parameter family of functions $\phi_\epsilon(x,t) \in C(\overline{Q_T})$, $\epsilon \in (0,\epsilon_0)$ is said to have spatiotemporal pattern $\mathcal{D}$ as $\epsilon \to 0$, if $\phi_\epsilon > 0$ in Q_T and for any compact* *subset* D_0 *of* $\{\overline{\Omega}\times(0,T)\}\backslash\overline{H}$ *with* $H=D_1\cup D_2\cdots\cup D_m$ *and any compact subset* H_0 *of* H*, there holds*

$$\lim_{\epsilon\to 0}\frac{\min_{(x,t)\in H_0}\phi_\epsilon(x,t)}{\max_{(x,t)\in D_0}\phi_\epsilon(x,t)}=\infty.$$

For our later purpose, as in chapter 3, we need some more notations. Let O be a bounded domain, and $f(x)$ be an $L^\infty(O)$ function. We denote by $\lambda_1^D(f,O)$ and $\lambda_1^N(f,O)$ the first eigenvalue of the operator $-\Delta+f$ over O, with Dirichlet and Neumann boundary conditions, respectively. We also use the convention that $\lambda_1^A(0,O)=\lambda_1^A(O)$ for $A=D,\ N$.

Let us also recall the theory of the principal eigenvalue for a linear periodic-parabolic eigenvalue problem. For any given T-periodic function $g\in C^{\theta,\theta/2}(\overline{Q_T})$, we consider the eigenvalue problem:

$$\begin{cases}\partial_t\varphi-\Delta\varphi+g(x,t)\varphi=\lambda\varphi & \text{in } \Omega\times(0,T),\\ \partial_\nu\varphi=0 & \text{on } \partial\Omega\times(0,T),\\ \varphi(x,0)=\varphi(x,T) & \text{in } \Omega.\end{cases}\tag{4.2.1}$$

By Proposition 14.4 of [53], we know that (4.2.1) has a principal eigenvalue $\lambda=\lambda_1(g)$, which corresponds to a positive eigenfunction $\varphi\in C^{2+\theta,1+\frac{\theta}{2}}(\overline{Q_T})$. Such a function φ is usually called a principal eigenfunction.

Assume that $b(x,t)\geq,\not\equiv 0$. Then, according to chapter 3, $\mu\mapsto\lambda_1(\mu b)$ is a strictly increasing continuous function with $\lambda_1(\mu b)>0$ when $\mu>0$, which then allows us to define

$$\lambda_1(\infty):=\lim_{\mu\to\infty}\lambda_1(\mu b)\in(0,\infty].\tag{4.2.2}$$

When $\epsilon=0$, according to Theorem 3.2.1, we have

Theorem 4.2.1 *Problem* (4.1.1) *with* $\epsilon=0$ *admits a positive solution if and only if* $0<a<\lambda_1(\infty)$*. Moreover, if it exists, the positive solution is also unique, denoted by* $u_0(x,t)$*.*

When $\epsilon>0$, from Theorem 3.3.1, the following conclusion holds.

Theorem 4.2.2 *For each $\epsilon > 0$,* (4.1.1) *has a unique positive solution $u_\epsilon(x,t)$ if and only if $a > 0$. Moreover, $u_\epsilon(x,t)$ is a global attractor of the corresponding initial-boundary value problem* (4.1.2).

We also mention that, from simple comparison arguments, combined with the uniqueness of positive T-periodic solution, one can easily prove that $u_\epsilon(x,t)$ increases as ϵ decreases in the sense that $u_{\epsilon_1}(x,t) < u_{\epsilon_2}(x,t)$ on $\overline{Q_T}$ if $0 \le \epsilon_2 < \epsilon_1$.

In the forthcoming sections, our attention will be paid to the study of the asymptotic behavior of $u_\epsilon(x,t)$ when $a > 0$ as $\epsilon \to 0$, which thereby induces the possible formation of spatiotemporal patterns of (4.1.1).

4.3 Statement of main results

Let us recall that $b(x,t) \in C^{\theta,\theta/2}(\overline{Q_T})$ is T-periodic in t and $b(x,t) \ge, \not\equiv 0$. As in the preceding chapter, in order to better understand the behavior of the solution to (4.1.1), we shall consider three basic cases of $b(x,t)$.

The first case is the simplest to handle, where $b(x,t)$ has no spatial degeneracy at some point in time, that is,

$$b(x,t_0) > 0 \quad \text{for all } x \in \overline{\Omega} \text{ and some } t_0 \in [0,T). \tag{4.3.1}$$

In this case, due to Theorem 3.3.1, we see that (4.1.1) with $\epsilon = 0$ has a unique positive solution $u_0(x,t)$ if and only if $a > 0$. Moreover, it is clear from the comparison principle of parabolic equations that $u_\epsilon(x,t) < u_0(x,t)$ on $\overline{Q_T}$ for all $\epsilon > 0$. As a result, by the standard regularity theory for parabolic equations and the embedding theorems, one can easily deduce the following result.

Theorem 4.3.1 *Assume that* (4.3.1) *holds. Then, $u_\epsilon(x,t) \to u_0(x,t)$ uniformly on $\overline{Q_T}$ as $\epsilon \to 0$.*

By virtue of the definition of spatiotemporal pattern given in the preceding section, we have

Remark 4.3.1 *In the case of* (4.3.1)*, no spatiotemporal pattern can be observed in* (4.1.1).

Next we discuss the second case where only spatial degeneracy exists. We assume that Ω has m subdomains Ω_j $(i = 1, \cdots, m)$ with each Ω_j having smooth boundary $\partial\Omega_j$, and $\overline{\Omega}_j \subset \Omega$ and $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ for $i \neq j$. Moreover, we assume there exist two positive constants $c_1,\ c_2$, such that

$$c_1 p(x) \le b(x,t) \le c_2 p(x), \quad \forall x \in \Omega,\ t \in [0,T], \tag{4.3.2}$$

with $p(x) \in C^\theta(\overline{\Omega})$ satisfying

$$p(x) = 0 \ \text{ on } \Omega_* := \cup_{j=1}^m \Omega_j \quad \text{and} \quad p(x) > 0 \ \text{ on } \overline{\Omega}\backslash\overline{\Omega_*}. \tag{4.3.3}$$

By rearranging the subscripts if necessary, we may assume that

$$\lambda_1^D(\Omega_1) \le \lambda_1^D(\Omega_2) \le \cdots \le \lambda_1^D(\Omega_m).$$

It is well known that $0 < \lambda_1^D(\Omega) < \lambda_1^D(\Omega_1)$.

Under the hypotheses of (4.3.2) and (4.3.3), we are able to prove the following conclusions.

Theorem 4.3.2 *Let $a > 0$ and $u_\epsilon(x,t)$ be the unique positive solution to* (4.1.1).

(i) *If $\lambda_1^B(\Omega) < a < \lambda_1^D(\Omega_1)$, then as $\epsilon \to 0$, $u_\epsilon(x,t)$ converges uniformly on $\overline{Q_T}$ to the unique positive solution $u_0(x,t)$ of* (4.1.1) *with $\epsilon = 0$.*

(ii) *If $\lambda_1^D(\Omega_k) \le a < \lambda_1^D(\Omega_{k+1})$ for some $1 \le k \le m-1$, then*

(ii-a) $\lim_{\epsilon\to 0} u_\epsilon(x,t) = \infty$ *uniformly on* $\cup_{j=1}^k \overline{\Omega}_j \times [0,T]$ *and*

(ii-b) $\lim_{\epsilon\to 0} u_\epsilon(x,t) = U(x,t) < \infty$ *uniformly on any compact subset of* $(\overline{\Omega}\backslash\cup_{j=1}^k \overline{\Omega}_j) \times [0,T]$*, where $U(x,t)$ is the minimal positive solution to the boundary blow-up problem*

$$\begin{cases} \partial_t u - \Delta u = au - b(x,t)u^p & \text{in } (\Omega\backslash\cup_{j=1}^k \overline{\Omega}_j) \times (0,T), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0,T), \\ u = \infty & \text{on } \partial\Omega_j \times (0,T),\ j = 1,\cdots,k, \\ u(x,0) = u(x,T) & \text{in } \Omega\backslash\cup_{j=1}^k \overline{\Omega}_j. \end{cases} \tag{4.3.4}$$

(iii) *If* $a \geq \lambda_1^D(\Omega_m)$*, then*

(iii-a) $\lim_{\epsilon\to 0} u_\epsilon(x,t) = \infty$ *uniformly on* $\overline{\Omega_*} \times [0,T]$*;*

(iii-b) $\lim_{\epsilon\to 0} u_\epsilon(x,t) = U(x,t) < \infty$ *uniformly on any compact subset of* $(\overline{\Omega}\backslash\overline{\Omega_*}) \times [0,T]$*, where* $U(x,t)$ *is the minimal positive solution to the boundary blow-up problem*

$$\begin{cases} \partial_t u - \Delta u = au - b(x,t)u^p & \text{in } (\Omega\backslash\overline{\Omega_*}) \times (0,T), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (0,T), \\ u = \infty & \text{on } \partial\Omega_* \times (0,T), \\ u(x,0) = u(x,T) & \text{in } \Omega\backslash\overline{\Omega_*}. \end{cases} \tag{4.3.5}$$

Remark 4.3.2 *We would like to make some remarks as follows.*

(i) *Unless each* Ω_j *is simply connected,* $(\Omega\backslash\cup_{j=1}^k\Omega_j) \times [0,T]$ *may have more than one component. By a positive solution of* (4.3.4) *or* (4.3.5)*, we mean a solution which is positive on each component of the underlying region. A similar remark also holds whenever it applies below.*

(ii) *If* $\lambda_1^D(\Omega_k) \leq a < \lambda_1^D(\Omega_{k+1})$ *for some* $1 \leq k \leq m-1$*, then the assertion* (i) *of Theorem 4.3.2 shows that the unique positive solution* $u_\epsilon(x,t)$ *of* (4.1.1) *exhibits the spatiotemporal pattern* $\mathcal{D} = \{\Omega_1 \times [0,T], \cdots, \Omega_k \times [0,T]\}$ *as* $\epsilon \to 0$*; and if* $a \geq \lambda_1^D(\Omega_m)$*, then the assertion* (ii) *of Theorem 4.3.2 shows that the unique solution* $u_\epsilon(x,t)$ *exhibits the spatiotemporal pattern* $\mathcal{D} = \{\Omega_1 \times [0,T], \cdots, \Omega_m \times [0,T]\}$*, as* $\epsilon \to 0$*.*

Letting $v_\epsilon(x,t) = \epsilon^{\frac{1}{p-1}} u_\epsilon(x,t)$, we can establish the profile of v_ϵ as follows.

Theorem 4.3.3 *Let* $a > 0$ *and* $v_\epsilon(x,t)$ *be defined as above. The following assertions hold true.*

(i) *If* $0 < a \leq \lambda_1^D(\Omega_1)$*, then as* $\epsilon \to 0$*,* v_ϵ *converges to zero uniformly on* $\overline{Q_T}$*.*

(ii) *If* $\lambda_1^D(\Omega_k) < a \leq \lambda_1^D(\Omega_{k+1})$ *for some* $1 \leq k < m-1$*, then*

(ii-a) $\lim_{\epsilon\to 0} v_\epsilon(x,t) = 0$ *uniformly on* $\overline{\Omega}\backslash\cup_{j=1}^k\Omega_j \times [0,T]$ *and* (ii-b) $\lim_{\epsilon\to 0} v_\epsilon(x,t) = \theta_a^i(x) < \infty$ *uniformly on each* $\overline{\Omega}_j \times [0,T],\ j = 1,2,\cdots,k$*, where* $\theta_a^j(x)$ *is the unique positive solution to the elliptic problem*

$$-\Delta v = av - v^p \quad \text{in } \Omega_j; \quad v = 0 \quad \text{on } \partial\Omega_j. \tag{4.3.6}$$

(iii) *If* $a > \lambda_1^D(\Omega_m)$*, then*

(iii-a) $\lim_{\epsilon\to 0} v_\epsilon(x,t) = 0$ *uniformly on* $\overline{\Omega}\backslash\Omega_* \times [0,T]$ *and*

(iii-b) $\lim_{\epsilon\to 0} v_\epsilon(x,t) = \theta_a^j(x) < \infty$ *uniformly on each* $\overline{\Omega}_j \times [0,T],\ j = 1,2,\cdots,m$*, where* $\theta_a^j(x)$ *is the unique positive solution to the elliptic problem* (4.3.6).

Finally, we consider the third case where both spatial and temporal degeneracies exist. That is, we assume that

$$c_1 p(x)q(t) \le b(x,t) \le c_2 p(x) q(t) \quad \text{on } \overline{Q_T}, \tag{4.3.7}$$

where c_1, c_2 are positive constants, $p(x)$ is as in (4.3.2) and $q(t) \in C^{\theta/2}([0,T])$ is a T-periodic nonnegative function satisfying

$$q(t) = 0 \ \text{ on } [0,T^*] \ \text{ and } \ q(t) > 0 \ \text{ in } (T^*,T), \quad \text{where } 0 < T^* < T.$$

In this case, as in chapter 3, we can prove that $\lambda_1(\infty) < \lambda_1^D(\Omega_1)$. In the next section, we will give a useful characterization of $\lambda_1(\infty)$ and some of its properties.

Our result below shows that in the third case, the asymptotic behavior of $u_\epsilon(x,t)$ is significantly different from the previous two cases as $\epsilon \to 0$.

Theorem 4.3.4 *Let* $a > 0$ *and* $u_\epsilon(x,t)$ *be the unique positive solution to* (4.1.1). *The following assertions hold true.*

(i) *If* $0 < a < \lambda_1(\infty)$*, then as* $\epsilon \to 0$*,* $u_\epsilon(x,t)$ *converges uniformly on* $\overline{Q_T}$ *to the unique positive solution* $u_0(x,t)$ *of* (4.1.1) *with* $\epsilon = 0$.

(ii) *If* $a \ge \lambda_1(\infty)$*, then*

(ii-a) $\lim_{\epsilon\to 0} u_\epsilon(x,t) = \infty$ *uniformly on any compact subset of* $(\overline{\Omega} \times (0,T^*]) \cup (\overline{\Omega_*} \times (T^*,T])$ *and* (ii-b) $\lim_{\epsilon\to 0} u_\epsilon(x,t) = U(x,t) < \infty$ *uniformly on any compact subset of* $(\overline{\Omega}\backslash\overline{\Omega_*}) \times (T^*, T)$, *where* $U(x,t)$ *is the minimal positive solution to the initial boundary blow-up problem*

$$\begin{cases} \partial_t u - \Delta u = au - b(x,t)u^p & \text{in } (\Omega\backslash\overline{\Omega_*}) \times (T^*, T), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (T^*, T), \\ u = \infty & \text{on } \partial\Omega_* \times (T^*, T), \\ u(x, T^*) = \infty & \text{in } \overline{\Omega}\backslash\overline{\Omega_*}. \end{cases} \tag{4.3.8}$$

Remark 4.3.3 *Under the assumptions of* (4.3.7), *we notice that, if* $a \geq \lambda_1(\infty)$, *the assertion* (ii) *of Theorem 4.3.4 shows that the unique positive solution* $u_\epsilon(x,t)$ *of* (4.1.1) *exhibits the spatiotemporal pattern* $\mathcal{D} = (\overline{\Omega} \times (0, T^*]) \cup (\overline{\Omega_*} \times (T^*, T])$ *as* $\epsilon \to 0$.

Letting $v_\epsilon(x,t) = \epsilon^{\frac{1}{p-1}} u_\epsilon(x,t)$, we have

Theorem 4.3.5 *Let* $a > 0$ *and* v_ϵ *be defined as above. The following assertions hold true.*

(i) *If* $0 < a < \lambda_1(\infty)$, *then as* $\epsilon \to 0$, $v_\epsilon(x,t)$ *converges to zero uniformly on* $\overline{Q_T}$.

(ii) *If* $a > \lambda_1(\infty)$, *then*

(ii-a) $\lim_{\epsilon\to 0} v_\epsilon(x,t) = 0$ *uniformly on any compact subset of* $(\overline{\Omega}\backslash\overline{\Omega_*}) \times (T^*, T)$ *and*

(ii-b) $\lim_{\epsilon\to 0} v_\epsilon(x,t) = v^*(x,t) < \infty$ *uniformly on any compact subset of* $(\overline{\Omega} \times (0, T^*]) \cup (\Omega_* \times (T^*, T))$, *where* $v^*(x,t)$ *is the unique positive solution to the problem*

$$\begin{cases} \partial_t v - \Delta u = av - v^p & \text{in } (\Omega \times [0, T^*]) \cup (\Omega_* \times (T^*, T)), \\ \partial_\nu v = 0 & \text{on } \partial\Omega \times (0, T^*], \\ v = 0 & \text{on } (\overline{\Omega}\backslash\overline{\Omega_*} \times \{0, T\}) \cup (\partial\Omega_* \times (T^*, T)), \\ v(x, 0) = v(x, T) & \text{in } \Omega_*. \end{cases} \tag{4.3.9}$$

We need to point out that in (ii-b) of Theorem 4.3.5, the unique positive T-periodic solution $v^*(x,t)$ to (4.3.9) satisfies

$$v^* \in C^{2+\theta,1+\frac{\theta}{2}}\Big(\big[(\overline{\Omega}\times(0,T^*])\cup(\Omega_*\times(T^*,T])\big]\backslash[\partial\Omega_*\times\{T^*\}]\Big)\cap\Big(L^\infty\Big([\Omega\times(0,T^*)]\cup[\Omega_*\times(T^*,T)]\Big)\Big)$$

and

$$v^* > 0 \ \text{ in } \ (\overline{\Omega}\times(0,T^*])\cup(\Omega_*\times(T^*,T]).$$

Such solution exists if and only if $a > \lambda_1(\infty)$. Notice that, $v^*(x,t)$ is discontinuous only on $\partial\Omega_* \times \{T^*\}$.

We would also like to mention that, if $a = \lambda_1(\infty)$, by some more delicate analysis, it can be shown that $v_\epsilon(x,t) \to 0$ uniformly on $\overline{Q_T}$ as $\epsilon \to 0$.

4.4 Intermediate results

In this section, we state and prove some intermediate results, which will be used in the proof of the main results in section 4.3. Most of the results are also of independent interests.

To prove Theorem 4.3.2, we need to recall a result obtained in [38] (see Lemma 2.6 there).

Lemma 4.4.1 *Assume that $p(x)$ satisfies* (4.3.3) *and $c > 0$ is a given constant. Then, the following assertions hold true.*

(i) *Let $1 \le k \le m-1$. For any fixed $a \in (-\infty, \lambda_1^D(\Omega_{k+1}))$, the boundary blow-up problem*

$$\begin{cases} -\Delta u = au - cp(x)u^p & \text{in } \Omega\backslash\cup_{j=1}^k\overline{\Omega}_j, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \\ u = \infty & \text{on } \partial\Omega_j,\ j = 1,\cdots,k \end{cases} \tag{4.4.1}$$

has a minimal positive solution $\underline{U}(x)$ and a maximal positive solution $\overline{U}(x)$ in the sense that any positive solution $u(x)$ of (4.4.1) *satisfies $\underline{U}(x) \le u(x) \le \overline{U}(x)$ in $\overline{\Omega}\backslash\cup_{j=1}^k\overline{\Omega}_j$; while for $a \ge \lambda_1^D(\Omega_{k+1})$,* (4.4.1) *has no positive solution.*

(ii) *For any fixed $a \in (-\infty, \infty)$, the boundary blow-up problem*

$$\begin{cases} -\Delta u = au - cp(x)u^p & \text{in } \Omega\backslash\overline{\Omega_*}, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \\ u = \infty & \text{on } \partial\Omega_* \end{cases} \tag{4.4.2}$$

has a minimal positive solution $\underline{U}(x)$ and a maximal positive solution $\overline{U}(x)$.

In the periodic-parabolic case, we can prove an analogue to Lemma 4.4.1.

Lemma 4.4.2 *Assume that $b(x,t)$ satisfies* (4.3.2) *and* (4.3.3)*. Then, the following assertions hold true.*

(i) *Let $1 \leq k \leq m-1$. For any fixed $a \in (-\infty, \lambda_1^D(\Omega_{k+1}))$, the boundary blow-up problem* (4.3.4) *has a minimal positive solution $\underline{U}(x,t)$ and a maximal positive solution $\overline{U}(x,t)$ in the sense that any positive solution $u(x,t)$ of* (4.3.4) *satisfies $\underline{U}(x,t) \leq u(x,t) \leq \overline{U}(x,t)$ in $(\overline{\Omega}\backslash\cup_{j=1}^k\overline{\Omega}_j) \times [0,T]$; while for $a \geq \lambda_1^D(\Omega_{k+1})$,* (4.3.4) *has no positive solution.*

(ii) *For any fixed $a \in (-\infty, \infty)$, the boundary blow-up problem* (4.3.5) *has a minimal positive solution $\underline{U}(x,t)$ and a maximal positive solution $\overline{U}(x,t)$.*

Proof. We first prove the assertions in (i). If $a \geq \lambda_1^D(\Omega_{k+1})$, we verify that (4.3.4) has no positive solution. In fact, assume that $u(x,t)$ is a positive solution of (4.3.4). Then, as $b(x,t) = 0$ in $\Omega_{k+1} \times (0,T)$, it is clear that $u(x,t)$ satisfies

$$\begin{cases} \partial_t u - \Delta u = au & \text{in } \Omega_{k+1} \times (0,T), \\ u > 0 & \text{on } \partial\Omega_{k+1} \times (0,T), \\ u(x,0) = u(x,T) & \text{in } \Omega_{k+1}. \end{cases}$$

Choose $\varphi(x)$ to be a positive eigenfunction corresponding to $\lambda_1^D(\Omega_{k+1})$. Then,

$$\varphi(x)|_{\partial\Omega_{k+1}} = 0 \text{ and } \partial_\nu\varphi(x)|_{\partial\Omega_{k+1}} < 0.$$

Multiplying the above equation by $\varphi(x)$ and integrating over $\Omega_{k+1}\times(0,T)$ by parts, we find

$$\lambda_1^D(\Omega_{k+1})\int_0^T\int_{\Omega_{k+1}}u\varphi > a\int_0^T\int_{\Omega_{k+1}}u\varphi,$$

which shows $a<\lambda_1^D(\Omega_{k+1})$. This contradiction confirms our claim.

We next prove that (4.3.4) has a minimal positive solution when $a\in(-\infty,\lambda_1^D(\Omega_{k+1}))$. Note that $\partial\Omega$ and all $\partial\Omega_j$ are smooth. Thus, $\Omega\backslash\cup_{j=1}^k\Omega_j$ has finitely many components. Let us denote by $\omega_1,\cdots,\omega_\ell$ these components with $\partial\Omega\subset\overline{\omega}_1$.

For each $n\geq 1$, we consider the following ℓ problems:

$$\begin{cases}\partial_t u-\Delta u=au-b(x,t)u^p & \text{in } \omega_1\times(0,T),\\ \partial_\nu u=0 & \text{on } \partial\Omega\times(0,T),\\ u=n & \text{on } (\partial\omega_1\backslash\partial\Omega)\times(0,T),\\ u(x,0)=u(x,T) & \text{in } \omega_1\end{cases}\tag{4.4.3}$$

and

$$\begin{cases}\partial_t u-\Delta u=au-b(x,t)u^p & \text{in } \omega_j\times(0,T),\\ u=n & \text{on } \partial\omega_j\times(0,T),\\ u(x,0)=u(x,T) & \text{in } \omega_j,\ \ j=2,\cdots,\ell.\end{cases}\tag{4.4.4}$$

In what follows, we show that, for each $n\geq 1$, problems (4.4.3) and (4.4.4) each has a unique positive solution, which increases in n. To do this, for $i=1,2$, we consider the following auxiliary problems:

$$\begin{cases}-\Delta u=au-c_ip(x)u^p & \text{in } \omega_1,\\ \partial_\nu u=0 & \text{on } \partial\Omega,\\ u=n & \text{on } \partial\omega_1\backslash\partial\Omega,\end{cases}\tag{4.4.5}$$

and

$$\begin{cases}-\Delta u=au-c_ip(x)u^p & \text{in } \omega_j,\\ u=n & \text{on } \partial\omega_j,\ \ j=2,\cdots,\ell.\end{cases}\tag{4.4.6}$$

By Lemma 2.3 in [36], we know that (4.4.5) and (4.4.6) have a unique positive solution, denoted by $u_n^{(i,j)}$, $i = 1, 2; j = 1, \cdots, \ell$. On the other hand, thanks to Lemma 2.3 of [38], $u_n^{(2,j)} \leq u_n^{(1,j)}$ for each $j = 1, \cdots, \ell$; and for each fixed $i = 1, 2$ and $j = 1, \cdots, \ell$, $u_n^{(i,j)}$ is increasing with respect to n. Moreover, from the proof of Lemma 2.6 in [38], one sees that, $\lim_{n\to\infty} u_n^{(i,j)}$ is the minimal solution to (4.4.1) on the component ω_j with c replaced by c_i.

On the other hand, we observe that, for each fixed n, $(u_n^{(1,1)}, u_n^{(2,1)})$ is a pair of super-sub solutions of (4.4.3) and $(u_n^{(1,j)}, u_n^{(2,j)})$ is a pair of super-sub solutions of (4.4.4), $j = 2, \cdots, \ell$. Thus, from the well-known super-sub solution iteration argument, it follows that both (4.4.3) and (4.4.4) have at least one positive solution. The uniqueness and monotonicity in n of the positive solution can be verified similarly as in the proof of Lemma 3.4.1. We denote by $w_n^j(x,t)$ $(j = 1, \cdots, \ell)$, the unique positive solution of (4.4.3) and (4.4.4).

Since $w_n^j(x,t) \leq u_n^{(1,j)}(x,t), j = 1, \cdots, \ell$, by the regularity theory and Sobolev embedding theorems, it is easily seen that $\underline{U}^j(x,t) = \lim_{n\to\infty} w_n^j(x,t)$ exists and is a positive solution of (4.3.4) over the component $\omega_j \times (0,T)$. Furthermore, by its construction and the comparison principle for parabolic equations, we easily see that $\underline{U}^j(x,t)$ is the minimal positive solution to (4.3.4). The existence of the maximal positive solution $\overline{U}^j(x,t)$ on ω_j can be obtained in the same way as in the proof of Theorem 3.4.3, which is the limit of the minimal positive solutions on a sequence of increasing domains approaching ω_j.

Finally, we define $\underline{U}(x,t) = \underline{U}^j(x,t)$ and $\overline{U}(x,t) = \overline{U}^j(x,t)$ for $x \in \omega_j$, $j = 1, \cdots, \ell$. Then, $\underline{U}(x,t)$ and $\overline{U}(x,t)$ are the desired minimal and maximal positive solutions to (4.3.4).

By making use of (ii) of Lemma 4.4.1, the assertion (ii) can be proved in a similar manner; we omit the details. $\square$

In order to prove Theorem 4.3.4, we first investigate an eigenvalue problem over a varying multi-connected cylinder which is analogous to (3.3.36) where the degenerate subdomain is connected, and prove the monotonicity and continuity of the principal eigenvalue with respect to the domain; secondly, we study an initial-boundary value problem with both initial and boundary values infinity.

The eigenvalue problem is given by

$$
\begin{cases}
\partial_t\varphi - \Delta\varphi = \lambda\varphi & \text{in } (\Omega \times (0,T^*]) \cup (\Omega_* \times (T^*,T]),\\
\partial_\nu\varphi = 0 & \text{on } \partial\Omega \times (0,T^*],\\
\varphi(x,t) = 0 & \text{on } (\overline{\Omega}\backslash\overline{\Omega_*} \times \{0,T\}) \cup (\partial\Omega_* \times (T^*,T]),\\
\varphi(x,0) = \varphi(x,T) & \text{in } \Omega_*.
\end{cases}
\tag{4.4.7}
$$

Let $\lambda_1(\mu b)$ be defined as before, and let $\varphi_\mu(x,t)$ be the corresponding positive eigenfunction with $\max_{\overline{Q_T}}\varphi_\mu = 1$. Similarly to Theorems 3.3.3 and 3.3.4, we have the following result.

Proposition 4.4.1 *The eigenvalue problem* (4.4.7) *admits a principal eigenvalue* $\lambda = \lambda_1^*$ *with* $0 < \lambda_1^* < \lambda_1^D(\Omega_1)$, *which corresponds to a positive eigenfunction* $\varphi_1^*(x,t)$, *satisfying*

$$\varphi_1^* > 0 \text{ in } (\overline{\Omega} \times (0,T^*]) \cup (\Omega_* \times (T^*,T]),\ \ \varphi_1^* = 0 \text{ in } \left(\overline{\Omega} \setminus \Omega_*\right) \times (T^*,T], \tag{4.4.8}$$

$$\varphi_1^* \in C\Big((\overline{\Omega} \times [0,T])\backslash(\overline{\Omega}\backslash\Omega_* \times \{T^*\})\Big), \tag{4.4.9}$$

$$\varphi_1^* \in C^{2+\theta,1+\frac{\theta}{2}}\Big(\left[(\overline{\Omega} \times (0,T^*]) \cup (\Omega_* \times [T^*,T])\right] \setminus \left[\partial\Omega_* \times \{T^*\}\right]\Big), \tag{4.4.10}$$

and

$$\sup_{(\overline{\Omega}\times[0,T^*])\cup(\Omega_*\times(T^*,T])} \varphi_1^* = 1. \tag{4.4.11}$$

Conversely, if (4.4.7) *has a solution* φ *satisfying* (4.4.8)-(4.4.11), *then necessarily* $\lambda = \lambda_1^*$, *the principal eigenvalue of* (4.4.7), *and* $\varphi = c\varphi_1^*$ *for some constant c.*

Proposition 4.4.2 *Assume that* $b(x,t)$ *satisfies* (4.3.7). *Then, the following assertions hold:*

(i) $\lim_{\mu\to\infty}\lambda_1(\mu b) =: \lambda_1(\infty) = \lambda_1^*$;

(ii) As $\mu \to \infty$, *the normalized principal eigenfunction* $\varphi_\mu(x,t)$ *corresponding to* $\lambda_1(\mu b)$ *satisfies that* $\varphi_\mu \to \varphi_1^*$ *locally in the* $C^{2,1}$ *norm over* $\overline{\Omega} \times (0,T^*]$, $\varphi_\mu \to \varphi_1^*$ *locally in* $C^{2,1}$ *norm over* $\Omega_* \times [T^*,T]$, $\varphi_\mu \to 0 = \varphi_1^*$ *locally uniformly over* $\overline{\Omega} \setminus \overline{\Omega_*} \times (T^*,T]$.

Proof. The proofs of the above two propositions are the same as that of Theorems 3.3.3 and 3.3.4. The only difference is that the connected open set Ω_0 is replaced by Ω_*, which is the union of m disjoint subdomains $\Omega_1, \cdots, \Omega_m$ of Ω. However, this change does not affect the proof there.

As in the proof of Theorem 3.3.3, first of all, we use Theorem 2.4 of [49] to deduce

$$\lambda_1(\mu b) \le \lambda_1(\mu c_2 Qp) \to \min\{\lambda_1^D(\Omega_1), \cdots, \lambda_1^D(\Omega_m)\} = \lambda_1^D(\Omega_1), \text{ as } \mu \to \infty,$$

where $Q = \max_{[0,T]} q(t) > 0$. This thereby implies $\lambda_1(\infty) \le \lambda_1^D(\Omega_1)$. Then, except for step 6 of the proof of Theorem 3.3.3 (which is devoted to the verification of $\lambda_1(\infty) < \lambda_1^D(\Omega_1)$), we just repeat the proofs of Theorems 3.3.3 and 3.3.4 with Ω_0 there replaced by each and every component of Ω_*. To obtain $\lambda_1(\infty) < \lambda_1^D(\Omega_1)$, in step 6 of the proof of Theorem 3.3.3 there, we only need to replace Ω_0 there by Ω_1 here. □

Next, we establish the monotonicity and continuity of $\lambda_1(\infty)$ with respect to the variation of the domain; though we will not directly use these results to prove the main theorems in this chapter, they are of independent interests. To emphasize the dependence of $\lambda_1(\infty)$ on Ω_* and T^* below, we denote it by $\lambda_1(\infty; \Omega_*, T^*)$. Let $\epsilon_0 > 0$ be a given small number such that for any $0 \le \epsilon \le \epsilon_0$,

$$\Omega_{*,\epsilon} := \cup_{i=1}^m \Omega_{i,\epsilon} \subset\subset \Omega$$

with $\Omega_{i,\epsilon} = \{x \in \Omega : d(x, \Omega_i) < \epsilon\}$ satisfying $\overline{\Omega}_{i,\epsilon} \cap \overline{\Omega}_{j,\epsilon} = \emptyset$ for $i \ne j$. Clearly, $\Omega_{*,0} = \Omega_*$. Since Ω_* is assumed to be smooth, we can require that for all $0 < \epsilon \le \epsilon_0$, $\Omega_{*,\epsilon}$ has the same smoothness as Ω_*.

We are able to obtain the following

Proposition 4.4.3 *Under the above assumptions, $\lambda_1(\infty; \Omega_{*,\epsilon}, T^*)$ is a continuous function of $\epsilon \in [0, \epsilon_0]$ and $T^* \in (0, T)$. Moreover, if $0 \le \epsilon_1 \le \epsilon_2 \le \epsilon_0$ and $0 < T_1^* \le T_2^* < T$, then*

$$\lambda_1(\infty; \Omega_{*,\epsilon_2}, T_2^*) \le \lambda_1(\infty; \Omega_{*,\epsilon_1}, T_1^*), \tag{4.4.12}$$

and equality holds if and only if $\epsilon_1 = \epsilon_2$ and $T_1^* = T_2^*$.

Proof. Since the proof is quite long, for the sake of clarity, we divide it into several steps.

Step 1. We first prove (4.4.12). As in the introduction, let $\lambda_1(g)$ be the principal eigenvalue of (4.2.1). We choose $p_i(x) \in C^\theta(\overline{\Omega})$ such that $p_i(x) = 0$ on Ω_{*,ϵ_i}, $p_i(x) > 0$ on $\overline{\Omega}\backslash\overline{\Omega_{*,\epsilon_i}}$ $(i = 1, 2)$, and $p_1(x) \geq p_2(x)$. We also take $q_i(t) \in C^{\theta/2}([0, T])$, $(i = 1, 2)$ to be T-periodic functions which satisfy $q_i(t) = 0$ on $[0, T_i^*]$, $q_i(t) > 0$ in (T_i^*, T) and $q_1(t) \geq q_2(t)$ on $[0, T]$.

For any $\mu > 0$, using the monotonicity of $\lambda_1(g)$ in the weight function $g(x, t)$, we obtain

$$\lambda_1(\mu p_2(x)q_2(t)) \leq \lambda_1(\mu p_1(x)q_1(t)).$$

Thus, from Proposition 4.4.2 it follows

$$\lambda_1(\infty; \Omega_{*,\epsilon_2}, T_2^*) = \lim_{\mu\to\infty} \lambda_1(\mu p_2(x)q_2(t)) \leq \lim_{\mu\to\infty} \lambda_1(\mu p_1(x)q_1(t)) = \lambda_1(\infty; \Omega_{*,\epsilon_1}, T_1^*),$$

as we wanted.

Step 2. We then verify

$$\lambda_1(\infty; \Omega_{*,\epsilon_2}, T_2^*) < \lambda_1(\infty; \Omega_{*,\epsilon_1}, T_1^*) \ \text{ if } (\epsilon_2, T_2^*) \neq (\epsilon_1, T_1^*). \tag{4.4.13}$$

Obviously, to prove (4.4.13), it suffices to prove the following two inequalities:

$$\lambda_1(\infty; \Omega_{*,\epsilon_2}, T_2^*) < \lambda_1(\infty; \Omega_{*,\epsilon_1}, T_2^*) \quad \text{if } \ \epsilon_1 < \epsilon_2 \tag{4.4.14}$$

and

$$\lambda_1(\infty; \Omega_{*,\epsilon_2}, T_2^*) < \lambda_1(\infty; \Omega_{*,\epsilon_2}, T_1^*) \quad \text{if } T_1^* < T_2^*. \tag{4.4.15}$$

Before going further, we need some preparations. Fix $i, j = 1, 2$. For any given $u \in E_i =: \oplus_{k=1}^m C_0^1(\Omega_{k,\epsilon_i})$, we extend it by 0 to $\overline{\Omega}$, and denote the resulting function by $\tilde{u}$. It is obvious that $\tilde{u} \in C(\overline{\Omega})$. Let $v_{i,j}(x, t)$ be the unique solution of the problem

$$\begin{cases} \partial_t v - \Delta v = 0 & \text{in } \Omega \times (0, T_j^*], \\ \partial_\nu v = 0 & \text{on } \partial\Omega \times (0, T_j^*), \\ v(x, 0) = \tilde{u}(x) & \text{in } \Omega. \end{cases} \tag{4.4.16}$$

By [66] we know that $v_j \in C^{2+\theta,1+\frac{\theta}{2}}(\overline{\Omega} \times (0, T_j^*]) \cap C(\overline{\Omega} \times [0, T_j^*])$.

We next consider the problem

$$\begin{cases} \partial_t w - \Delta w = 0 & \text{in } \Omega_{*,\epsilon_i} \times (T_j^*, T], \\ w = 0 & \text{on } \partial\Omega_{*,\epsilon_i} \times (T_j^*, T], \\ w(x, T_j^*) = v_j(x, T_j^*) & \text{in } \Omega_{*,\epsilon_i}. \end{cases} \tag{4.4.17}$$

This is actually a system of m problems, with each one in $\Omega_{k,\epsilon_i} \times (T_j^*, T], k = 1, \cdots, m$. As in chapter 3, we know that the above problem has a unique solution $w_{i,j} \in C^{\theta}((T_j^*, T], X_{1,i}) \cap C^{1+\theta}((T_j^*, T], X_{0,i})$, where $X_{0,i} = \oplus_{k=1}^m L^p(\Omega_{k,\epsilon_i})$ and $X_{1,i} = \oplus_{k=1}^m W_0^{2,p}(\Omega_{k,\epsilon_i})$, $p > 1$. We may choose p large enough such that $X_{1,i}$ embeds compactly into E_i.

Now, we define the operator $K_{i,j} : E_i \to E_i$ as follows

$$K_{i,j} u = w_{i,j}(\cdot, T).$$

Applying the same analysis as in the proof of Theorem 3.3.4, for fixed $i, j = 1, 2$, $K_{i,j}$ is a compact linear operator. Furthermore, if P_i denotes the cone of nonnegative functions in E_i, and P_i^o the interior of P_i, then $K_{i,j}$ is strongly positive, that is, $K_{i,j}(P_i \setminus \{0\}) \subset P_i^o$. These properties for $K_{i,j}$ enable us to use the well-known Krein-Rutman theorem to conclude that the spectral radius $r(K_{i,j})$ of $K_{i,j}$ is positive, and it corresponds to an eigenvector $u_{i,j}^0 \in P_i^o$. Moreover, if $K_{i,j} h = rh$ for some $h \in P_i^o$, then necessarily $r = r(K_{i,j})$ and $h = c u_{i,j}^0$ for some constant $c > 0$.

For simplicity, we denote $\lambda_{i,j} = \lambda_1(\infty; \Omega_{*,\epsilon_i}, T_j^*)$. Let $\varphi_{i,j}(x,t)$ $(i, j = 1, 2)$ be the positive eigenfunction corresponding to $\lambda_{i,j}$. Then, $r_{i,j} := r(K_{i,j}) = e^{-\lambda_{i,j} T}$ and $\psi_{i,j}(x,t) = e^{-\lambda_{i,j} t} \varphi_{i,j}(x,t)$ satisfy

$$K_{i,j} \psi_{i,j}(\cdot, 0) = \psi_{i,j}(\cdot, T) = e^{-\lambda_{i,j} T} \varphi_{i,j}(\cdot, T) = r_{i,j} \varphi_{i,j}(\cdot, 0) = r_{i,j} \psi_{i,j}(\cdot, 0).$$

We are now ready to prove (4.4.14) and (4.4.15). We first verify (4.4.14) by an indirect argument. Suppose that $\lambda_{2,2} = \lambda_{1,2} =: \lambda_0$ and so $r_{2,2} = r_{1,2} =: r_0$.

To derive a contradiction, we consider $K_{2,2} \psi_{1,2}(\cdot, 0) - r_0 \psi_{1,2}(\cdot, 0)$. We observe $\psi_{1,2}(x, 0) \in E_2$. Thus $K_{2,2} \psi_{1,2}(\cdot, 0)$ is well-defined. Let $v_{2,2}^*(x,t)$ be the unique solution of problem (4.4.16) with $j = 2$ and $\tilde{u}(x) = \psi_{1,2}(x, 0)$, and let $w^*_{2,2}(x, t)$ be the unique solution of problem (4.4.17) with $i = j = 2$ and $w^*_{2,2}(x, T^*_2) = v^*_{2,2}(x, T^*_2)$. Clearly, in this case, $\psi_{1,2}(x, t)$ also satisfies (4.4.16) with $j = 2$ and $\tilde{u}(x) = \psi_{1,2}(x, 0)$. Hence, the uniqueness gives $v^*_{2,2}(x, t) = \psi_{1,2}(x, t)$ and so $v^*_{2,2}(x, T^*_2) = \psi_{1,2}(x, T^*_2)$ in Ω_{*,ϵ_2}.

We note that $\psi_{1,2}(x, t)$ satisfies the equation

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega_{*,\epsilon_1} \times (T^*_2, T), \\ u = 0 & \text{on } \partial\Omega_{*,\epsilon_1} \times (T^*_2, T), \\ u(x, T^*_2) = \psi_{1,2}(x, T^*_2) & \text{in } \Omega_{*,\epsilon_1}. \end{cases} \tag{4.4.18}$$

On the other hand, we have $w^*_{2,2}(x, T^*_2) = v^*_{2,2}(x, T^*_2) = \psi_{1,2}(x, T^*_2)$ on $\overline{\Omega_{*,\epsilon_1}}$, $w^*_{2,2}(x, t) > 0$ on $\partial\Omega_{*,\epsilon_1} \times (T^*_2, T)$ and $w^*_{2,2}(x, t)$ satisfies the first equation in (4.4.18). Hence, from the comparison principle for parabolic equations, it follows

$$w^*_{2,2}(x, t) > \psi_{1,2}(x, t) \quad \text{in } \Omega_{*,\epsilon_1} \times (T^*_2, T].$$

This shows that in Ω_{*,ϵ_1}, the following holds:

$$K_{2,2}\psi_{1,2}(\cdot, 0) = w^*_{2,2}(\cdot, T) > \psi_{1,2}(\cdot, T) = e^{-\lambda_0 T}\varphi_{1,2}(\cdot, T) = e^{-\lambda_0 T}\varphi_{1,2}(\cdot, 0) = r_0\psi_{1,2}(\cdot, 0).$$

Furthermore, by the definition, $\psi_{1,2}(\cdot, 0) = 0$ and $w^*_{2,2}(\cdot, T) = K_{2,2}\psi_{1,2}(\cdot, 0) \geq 0$ on $\overline{\Omega}\backslash\Omega_{*,\epsilon_1}$. As a consequence, we obtain

$$r_0(-\psi_{1,2}(\cdot, 0)) - K_{2,2}(-\psi_{1,2}(\cdot, 0)) > 0 \quad \text{on } \Omega_{*,\epsilon_2}. \tag{4.4.19}$$

Recall that $K_{2,2} : E_2 \to E_2$ is a strongly positive and compact operator and r_0 is its principal eigenvalue. Thus, Theorem 1.4.3 concludes that the inhomogeneous equation

$$r_0 u - K_{2,2}u = h \quad \text{in } E_2$$

has no solution for any $h \in P_2\backslash\{0\}$. This contradicts with (4.4.19) and thus (4.4.14) holds.

We can prove (4.4.15) in a similar way. We use a contradiction analysis again. Suppose that $\lambda_{2,2} = \lambda_{2,1} =: \lambda_0$ and so $r_{2,2} = r_{2,1} =: r_0$. We consider $K_{2,1}\psi_{2,2}(\cdot, 0) - r_0\psi_{2,2}(\cdot, 0)$.

Obviously, $K_{2,1}\psi_{2,2}(\cdot,0)$ is well-defined. As before, let $v^*_{2,1}(x,t)$ be the solution of problem (4.4.16) with $j = 1$ and $\tilde{u}(x) = \psi_{2,2}(x,0)$, and $w^*_{2,1}(x,t)$ be the solution of problem (4.4.17) with $(i,j) = (2,1)$ and $w^*_{2,1}(x,T^*_1) = v^*_{2,1}(x,T^*_1)$. It is easily seen that $v^*_{2,1}(x,t) = \psi_{2,2}(x,t)$ and so $w^*_{2,1}(x,T^*_1) = v^*_{2,1}(x,T^*_1) = \psi_{2,2}(x,T^*_1)$ in Ω_{*,ϵ_2}.

Observe that $w_{2,1}(x,t)$ satisfies the equation

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega_{*,\epsilon_2} \times (T^*_1, T^*_2), \\ u = 0 & \text{on } \partial\Omega_{*,\epsilon_2} \times (T^*_1, T^*_2), \\ u(x,T^*_1) = \psi_{2,2}(x,T^*_1) & \text{in } \Omega_{*,\epsilon_2}. \end{cases} \tag{4.4.20}$$

On the other hand, we also note that $\psi_{2,2}(x,t) > 0$ on $\partial\Omega_{*,\epsilon_2} \times (T^*_1, T^*_2)$ and $\psi_{2,2}(x,t)$ satisfies the first equation in (4.4.20). Hence, the comparison principle for parabolic equations guarantees

$$\psi_{2,2}(x,t) > w^*_{2,1}(x,t) \quad \text{in } \Omega_{*,\epsilon_2} \times (T^*_1, T^*_2].$$

In particularly, we have

$$\psi_{2,2}(x,T^*_2) > w^*_{2,1}(x,T^*_2) \quad \text{in } \Omega_{*,\epsilon_2}. \tag{4.4.21}$$

Finally, let us consider the following auxiliary problem

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega_{*,\epsilon_2} \times (T^*_2, T), \\ u = 0 & \text{on } \partial\Omega_{*,\epsilon_2} \times (T^*_2, T), \\ u(x,T^*_2) = w^*_{2,1}(x,T^*_2) & \text{in } \Omega_{*,\epsilon_2}. \end{cases} \tag{4.4.22}$$

Problem (4.4.22) has the unique solution $w^*_{2,1}(x,t)$. As above, together with (4.4.21), simple comparison argument shows $\psi_{2,2}(x,t) > w^*_{2,1}(x,t)$ in $\Omega_{*,\epsilon_2} \times (T^*_2, T]$, which thereby indicates $\psi_{2,2}(x,T) > w^*_{2,1}(x,T)$ in Ω_{*,ϵ_2}. Therefore, we get

$$r_0\varphi_{2,2}(\cdot,0) = r_0\psi_{2,2}(\cdot,0) = \psi_{2,2}(\cdot,T) > w^*_{2,1}(\cdot,T) = K_{2,1}\psi_{2,2}(\cdot,0) \quad \text{in } \Omega_{*,\epsilon_2}.$$

Arguing as in step 2, as $K_{2,1} : E_2 \to E_2$ is a strongly positive and compact operator and r_0 is its principal eigenvalue, we get a contradiction against Theorem 1.4.3. Thus, (4.4.15) holds true.

Step 3. We now verify the continuity of $\lambda_1(\infty;(\Omega_{*,\epsilon},T^*))$ with respect to $\epsilon\in[0,\epsilon_0]$ and $T^*\in(0,T)$. Let $K(\epsilon,T^*)$ be the operator $K_{i,j}$ defined in the proof of step 2 with Ω_{*,ϵ_i} and T_j^* replaced by $\Omega_{*,\epsilon}$ and T^* respectively. Then, by a standard technique, for any given $(\hat\epsilon,\hat T)$ with $\hat\epsilon>0$ small and $\hat T\in(0,T)$, for $(\epsilon,\tilde T)$ close to $(\hat\epsilon,\hat T)$, one can use a suitable change of variables to transform problem (4.4.7) with $\Omega=\Omega_\epsilon$ and $T^*=\tilde T$ into one with $\Omega=\Omega_{\hat\epsilon}$ and $T^*=\hat T$ but with Δ replaced by a general elliptic operator $\mathcal{L}(\epsilon,\tilde T)$ whose coefficients are smooth for such $(\epsilon,\tilde T)$ such that $\mathcal{L}(\epsilon,\tilde T)$ reduces to the Laplacian operator Δ as $\epsilon\to\hat\epsilon$ and $\tilde T\to\hat T$. One can then easily show that $K(\epsilon,T^*)$ varies continuously with respect to $\epsilon\in[0,\epsilon_0]$ and $T^*\in(0,T)$. Consequently, the classical regular perturbation theory of compact operators in [60] concludes the spectral radius $r(K(\epsilon,T^*))$ of $K(\epsilon,T^*)$ is also a continuous function of ϵ and T^*, so is the principal eigenvalue $\lambda_1(\infty;(\Omega_{*,\epsilon},T^*))=-\frac{1}{T}\ln r(K(\epsilon,T^*))$.

Our proof of Proposition 4.4.3 is thus complete. □

According to the characterization of $\lambda_1^*=\lambda_1(\infty;\Omega^*,T^*)$ given in the proof of Proposition 4.4.3, using the analysis similar to that of Theorem 3.3.5, we can also obtain a lower bound of λ_1^* in the case that $b(x,t)$ satisfies (4.3.7).

Proposition 4.4.4 *Assume that* $b(x,t)$ *satisfies* (4.3.7). *Then,*

$$\lambda_1^*\geq\Big(1-\frac{T^*}{T}\Big)\lambda_1^D(\Omega_1).$$

Proof. We follow the idea in the proof of Proposition 4.4.3. Since some necessary modifications are required, we include the details here.

We first denote $X_0=\oplus_{i=1}^m L^2(\Omega_i)$ with the norm $\|f\|_{X_0}=(\sum_{i=1}^m\|f_i\|^2_{L^2(\Omega_i)})^{1/2}$ for any $f=\oplus_{i=1}^m f_i$ and $f_i\in L^2(\Omega_i)$. Here, by $f=\oplus_{i=1}^m f_i$, we mean $f(x)=f_i(x)$ for $x\in\Omega_i$. We denote K to be the operator defined as in Step 2 of the proof in Proposition 4.4.3, where Ω_{*,ϵ_i} and T_j^* are replaced by Ω_* and T^*. So, for fixed Ω_* and T^*, the operator $K:\ E:=\oplus_{k=1}^m C_0^1(\Omega_k)\to E$ is linear, compact and strongly positive.

Arguing as in the proof of Proposition 4.4.3, one can easily see that the linear operator K can be extended to a compact linear operator $\tilde K=\oplus_{i=1}^m\Big(U_{2,i}(T-T^*,0)\circ I_i\Big)\circ U_1(T^*,0)\circ J$ over X_0. Here, U_1 is the same as in the proof of Proposition 4.4.3, $U_{2,i}$ is the operator in (1.3.6) associated with (3.3.38) with Ω_0 there replaced by each Ω_i $(i = 1, \cdots, m)$, and $I_i v = v|_{\Omega_i}$ is the restriction operator and $Ju = \tilde{u}$ is the extension operator such that, for any given $u \in X_0$, $\tilde{u} = u$ in each Ω_i and $\tilde{u} = 0$ in $\Omega \backslash \Omega_*$. Moreover, the argument as there shows $r(\tilde{K}) = r(K)$. Note that $r(\tilde{K}) \leq \|\tilde{K}\|$.

We next estimate $\|\tilde{K}\|$. Let $\lambda_k^N(\Omega)$ be the eigenvalues of $-\Delta$ over Ω with Neumann boundary conditions, with corresponding eigenfunctions ϕ_k, $k \geq 1$; and let $\lambda_{k,i}^D(\Omega_i)$ denote the eigenvalues of $-\Delta$ over Ω_i with Dirichlet boundary conditions, with corresponding eigenfunctions $\psi_{k,i}$, $k \geq 1,\ i = 1, \cdots, m$. According to our previous notation, $\lambda_{1,i}^D(\Omega_i) = \lambda_1^D(\Omega_i)$. We may assume that the eigenfunctions are orthonormal:

$$\int_\Omega \phi_k \phi_j = \delta_{kj}, \ \int_{\Omega_i} \psi_{k,i} \psi_{j,i} = \delta_{kj}.$$

Hence, for any $u \in X_0$, we have

$$Ju = \tilde{u} = \Sigma_{k=1}^\infty a_k \phi_k,$$

and

$$\|u\|_{X_0} = \|Ju\|_{L^2(\Omega)} = \Big(\Sigma_{k=1}^\infty a_k^2\Big)^{1/2}.$$

Furthermore, we have

$$v(\cdot, t) = U_1(t, 0)\tilde{u} = \Sigma_{k=1}^\infty a_k e^{-\lambda_k^N(\Omega)t} \phi_k,$$

which immediately yields

$$\|v(\cdot, T^*)\|_{L^2(\Omega)} = \Big(\Sigma_{k=1}^\infty a_k^2 e^{-2\lambda_k^N(\Omega)T^*}\Big)^{1/2} \leq \Big(\Sigma_{k=1}^\infty a_k^2\Big)^{1/2} = \|u\|_{X_0}.$$

On the other hand, we also write

$$I_i v(\cdot, T^*) = v(\cdot, T^*)|_{\Omega_i} = \Sigma_{k=1}^\infty b_{k,i} \psi_{k,i},$$

and hence

$$\| \oplus_{i=1}^m I_i v(\cdot, T^*)\|_{X_0} = \Big(\Sigma_{i=1}^m \Sigma_{k=1}^\infty b_{k,i}^2\Big)^{1/2},$$

$$w(\cdot, t) = \oplus_{i=1}^m \Big(U_{2,i}(t - T^*, 0) \circ I_i v(\cdot, T^*)\Big) = \oplus_{i=1}^m (\Sigma_{k=1}^\infty b_{k,i} e^{-\lambda_{k,i}^D(\Omega_i)(t-T^*)} \psi_{k,i}(x)).$$

As a result, one gets that

$$\|w(\cdot,T)\|_{X_0} = \left(\Sigma_{i=1}^{m}\Sigma_{k=1}^{\infty} b_{k,i}^2 e^{-2\lambda_{k,i}^D(\Omega_i)(T-T^*)}\right)^{1/2} \leq e^{-\lambda_1^D(\Omega_1)(T-T^*)}\| \oplus_{i=1}^m I_i v(\cdot,T^*)\|_{X_0}.$$

Therefore, we obtain

$$\begin{aligned}\|w(\cdot,T)\|_{X_0} &\leq e^{-\lambda_1^D(\Omega_1)(T-T^*)}\| \oplus_{i=1}^m I_i v(\cdot,T^*)\|_{X_0} \\ &\leq e^{-\lambda_1^D(\Omega_1)(T-T^*)}\|v(\cdot,T^*)\|_{L^2(\Omega)} \leq e^{-\lambda_1^D(\Omega_1)(T-T^*)}\|u\|_{X_0},\end{aligned}$$

which shows that $\|\tilde{K}\| \leq e^{-\lambda_1^D(\Omega_1)(T-T^*)}$ and henceforth

$$r(K) = r(\tilde{K}) \leq e^{-\lambda_1^D(\Omega_1)(T-T^*)}.$$

In view of $\lambda_1^* = -\frac{1}{T}\ln r(K)$, we have

$$\lambda_1^* \geq -\frac{1}{T}\ln e^{-\lambda_1^D(\Omega_1)(T-T^*)} = \left(1-\frac{T^*}{T}\right)\lambda_1^D(\Omega_1).$$

The proof is complete. □

In the sequel, we consider the following initial-boundary value problem in which both initial values and partial boundary values blow up. We can prove that

Proposition 4.4.5 *Assume that* $b(x,t)$ *satisfies* (4.3.7)*. Then, for any fixed* $a \in (-\infty,\infty)$*, the initial-boundary blow-up problem*

$$\begin{cases} u_t - \Delta u = au - b(x,t)u^p & \text{in } \Omega\backslash\Omega_* \times (T^*,T), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (T^*,T), \\ u = \infty & \text{on } \partial\Omega_* \times (T^*,T), \\ u(x,T^*) = \infty & \text{in } \overline{\Omega}\backslash\overline{\Omega_*} \end{cases} \tag{4.4.23}$$

has a minimal positive solution $\underline{U}(x,t)$ *and a maximal positive solution* $\overline{U}(x,t)$*.*

The proof of Proposition 4.4.5 is similar to that of Theorem 3.4.3; we omit the details. Here, we only want to mention that the minimal positive solution $\underline{U}(x,t)$ and the maximal positive solution $\overline{U}(x,t)$ can be constructed in the following manner.

Let $\Omega_{*,\epsilon}$ be defined as before. For $\epsilon \geq 0$, we consider the problem:

$$\begin{cases} u_t - \Delta u = au - b(x,t)u^p & \text{in } \Omega\backslash\overline{\Omega_{*,\epsilon}} \times (T^* + \epsilon, T), \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times (T^* + \epsilon, T), \\ u = n & \text{on } \partial\Omega_{*,\epsilon} \times (T^* + \epsilon, T), \\ u(x, T^* + \epsilon) = n & \text{in } \overline{\Omega}\backslash\overline{\Omega_{*,\epsilon}}. \end{cases} \tag{4.4.24}$$

It is clear that problem (4.4.24) admits a unique positive solution, which we denote by $u_{n,\epsilon}(x,t)$. Then, as in the argument of Theorem 3.4.3, as $n \to \infty$, we have that $u_{n,\epsilon}(x,t)$ increases to the minimal positive solution $\underline{U}_\epsilon(x,t)$ of (4.4.23) with T^* and Ω_* replaced by $T^* + \epsilon$ and $\Omega_{*,\epsilon}$ respectively, and $\underline{U}_0(x,t) = \underline{U}(x,t)$. On the other hand, by the parabolic comparison principle, we also easily see that $U_{\epsilon_1}(x,t) \geq U_{\epsilon_2}(x,t) \geq \underline{U}(x,t)$ when $\epsilon_1 > \epsilon_2 > 0$. Hence there is a decreasing sequence ϵ_n converging to 0 such that $U_{\epsilon_n}(x,t) \to \overline{U}(x,t)$ as $\epsilon_n \to 0$ and $\overline{U}(x,t)$ is the maximal positive solution of (4.4.23).

In order to prove Theorem 4.3.5, we need to consider the following problem

$$\begin{cases} \partial_t v - \Delta v = av - v^p & \text{in } (\Omega \times (0, T^*]) \cup (\Omega_* \times (T^*, T]), \\ \partial_\nu v = 0 & \text{on } \partial\Omega \times (0, T^*], \\ v = 0 & \text{on } (\overline{\Omega}\backslash\overline{\Omega_*} \times \{0, T\}) \cup (\partial\Omega_* \times (T^*, T]), \\ v(x,0) = v(x,T) & \text{in } \Omega_*. \end{cases} \tag{4.4.25}$$

For this problem, we have the following result.

Proposition 4.4.6 *Problem* (4.4.25) *admits a positive solution* $v(x,t)$ *satisfying*

$$v \in C^{2+\theta,1+\frac{\theta}{2}}\Big(\big[(\overline{\Omega}\times(0,T^*])\cup(\Omega_*\times(T^*,T])\big]\backslash[\partial\Omega_*\times\{T^*\}]\Big)\cap\Big(L^\infty\Big([\Omega\times(0,T^*)]\cup[\Omega_*\times(T^*,T)]\Big)\Big)$$

and

$$v > 0 \quad \textit{in} \;\; (\overline{\Omega} \times (0, T^*]) \cup (\Omega_* \times (T^*, T])$$

if and only if $a > \lambda_1^*$, *where* λ_1^* *was given in Proposition 4.4.1. If such positive solution exists, it is unique, denoted by* $v^*(x,t)$.

Proof. We break our proof into three steps.

Step 1. Necessity of existence of positive solution. Assume that (4.4.25) has a positive solution $v(x,t)$. Set

$$z(x,t) = e^{-at}v(x,t).$$

Then, $z(x,t)$ satisfies

$$\begin{cases} \partial_t z - \Delta z = -e^{a(p-1)t}z^p < 0 & \text{in } (\Omega \times (0,T^*]) \cup (\Omega_* \times (T^*,T]), \\ \partial_\nu z = 0 & \text{on } \partial\Omega \times (0,T^*], \\ z = 0 & \text{on } (\overline{\Omega}\backslash\overline{\Omega_*} \times \{0,T\}) \cup (\partial\Omega_* \times (T^*,T]), \\ z(x,0) = v(x,0) & \text{in } \Omega_*. \end{cases} \tag{4.4.26}$$

Let $K(\Omega_*,T^*)$ be the operator defined as in Step 2 of the proof in Proposition 4.4.3, where Ω_{*,ϵ_i} and T_j^* are replaced by Ω_* and T^*. For such fixed Ω_* and T^*, the operator $K(\Omega_*,T^*) : E := \oplus_{k=1}^m C_0^1(\Omega_k) \to E$ is linear, compact and strongly positive. Let r^* be the spectral radius of $K(\Omega_*,T^*)$. Then, $r^* = e^{-\lambda_1^* T}$. Furthermore, arguing similarly as there, one can easily prove

$$K(\Omega_*,T^*)z(\cdot,0) > z(\cdot,T) = e^{-aT}v(\cdot,T) = e^{-aT}v(\cdot,0) = e^{-aT}z(\cdot,0) \quad \text{in } \Omega_*.$$

That is,

$$e^{-aT}(-z(\cdot,0)) - K(\Omega_*,T^*)(-z(\cdot,0)) > 0 \quad \text{in } \Omega_*.$$

Note that $-z(\cdot,0) < 0$ in Ω_*. Hence, due to the Krein-Rutman theorem (namely, Theorem 7.3 in [53]), it is necessary that $e^{-aT} < r^* = e^{-\lambda_1^* T}$, which in turn infers $a > \lambda_1^*$.

Step 2. Sufficiency of existence of positive solution. To prove the existence of positive solution, we use the so-called Poincaré mapping as follows.

Let $v_0 \in E$. We extend v_0 to be 0 on $\overline{\Omega}\backslash\Omega_*$, and denote the extended function by $\tilde{v}_0$. Then, we consider the initial boundary value problem:

$$\begin{cases} \partial_t v - \Delta v = av - v^p & \text{in } (\Omega \times (0,T^*]) \cup (\Omega_* \times (T^*,T]), \\ \partial_\nu v = 0 & \text{on } \partial\Omega \times (0,T^*], \\ v = 0 & \text{on } \partial\Omega_* \times (T^*,T], \\ v(x,0) = \tilde{v}_0(x) & \text{in } \Omega, \end{cases} \tag{4.4.27}$$

and denote by $v(x,t)$ the unique solution of (4.4.27).

We now define the Poincaré mapping:

$$Sv_0 = v(x,T), \quad v_0 \in E.$$

Much as in the argument of the proof of Theorem 3.3.4, one can easily show that $S: E \to E$ is a compact and strongly order-preserving operator. By strongly order-preserving, we mean that $v_1,\ v_2 \in E$ and $v_2 - v_1 \in P\backslash\{0\}$ imply $Sv_2 - Sv_1 \in P^o$, where P is the cone of nonnegative functions in E and P^o is the interior of P. Moreover, we can employ the analysis similar to that in Section 21 of Chapter III of [53] to conclude that S admits a fixed point in the natural order interval $\{g \in P:\ \underline{v}(x,0) \le g \le \overline{v}(x,0)\}$, if $\underline{v}(x,t)$ and $\overline{v}(x,t)$ with $\underline{v} \le \overline{v}$ is a pair of sub-super solutions of (4.4.25). Equivalently, this implies that (4.4.25) has a T-periodic solution. Here, by saying $\underline{v}(x,t)$ is a subsolution to (4.4.25), we mean that

$$\underline{v}(x,t) \in C^{2+\theta,1+\frac{\theta}{2}}\Big(\big[(\overline{\Omega}\times(0,T^*])\cup(\Omega_*\times(T^*,T])\big]\backslash[\partial\Omega_*\times\{T^*\}]\Big)\cap\Big(L^\infty\Big([\Omega\times(0,T^*)]\cup[\Omega_*\times(T^*,T)]\Big)\Big),$$

$\underline{v}(x,t) \ge 0$ a.e. in $(\Omega \times (0,T^*)) \cup (\Omega_* \times (T^*,T))$ and satisfies

$$\begin{cases} \partial_t\underline{v} - \Delta\underline{v} \le a\underline{v} - (\underline{v})^p & \text{in } (\Omega \times [0,T^*]) \cup (\Omega_* \times (T^*,T]), \\ \partial_\nu\underline{v} \le 0 & \text{on } \partial\Omega \times (0,T^*], \\ \underline{v} \le 0 & \text{on } (\overline{\Omega}\backslash\Omega_* \times \{0\}) \cup (\partial\Omega_* \times (T^*,T]), \\ \underline{v}(x,0) \le \underline{v}(x,T) & \text{in } \Omega_*. \end{cases}$$

A supersolution $\overline{v}(x,t)$ is defined similarly by reversing the above inequalities. In proving this fact along the direction in [53], we just want to stress that, instead of using the Schauder theory and the classical strong maximum principle for parabolic equations like in Section 21 of Chapter III of [53], in our present situation, we have to turn to the well-known L^p regularity theory and the maximum principle of the so-called strong solution (namely, $W_p^{2,1}$-solution) for parabolic equations.

Now, we choose $\underline{v} = \epsilon\varphi_1^*$ and $\overline{v} = a^{\frac{1}{p-1}}$, where φ_1^* was given in Proposition 4.4.1. For sufficiently small $\epsilon > 0$, a simple calculation shows that $\underline{v}$ and $\overline{v}$ are respectively the sub- and supersolutions to (4.4.25). So (4.4.25) has at least one positive solution.

Step 3. Uniqueness of positive solution. Suppose that (4.4.25) has two positive T-periodic solutions $v_1(x,t)$ and $v_2(x,t)$. To prove $v_1(x,t) = v_2(x,t)$, we first claim that there exists a large constant $M > 1$ such that

$$M^{-1}v_1(x,t) < v_2(x,t) < Mv_1(x,t) \quad \text{in } (\Omega \times (0,T^*]) \cup (\Omega_* \times (T^*,T]). \tag{4.4.28}$$

Since $v_i(x,t)$ $(i = 1,2)$ satisfies (4.4.25) on $\Omega_* \times (T^*,T]$, the Hopf boundary lemma infers $v_i(x,T)|_{\overline{\Omega^*}} \in P^o$, $(i = 1,2)$. Therefore, we can find $M_1 > 1$ such that

$$v_2(x,0) = v_2(x,T) < M_1v_1(x,T) = M_1v_1(x,0) \quad \text{in } \Omega_*.$$

Thanks to $v_i(x,0) = 0$ on $\overline{\Omega}\backslash\Omega_*$, we then have

$$v_2(x,0) \leq, \not\equiv M_1v_1(x,0) \quad \text{on } \overline{\Omega}.$$

We set

$$w(x,t) := M_1v_1(x,t) - v_2(x,t).$$

Then, for $(x,t) \in (\overline{\Omega} \times (0,T^*]) \cup (\overline{\Omega_*} \times (T^*,T])$, $w(x,t)$ satisfies

$$\begin{aligned} \partial_t w - \Delta w &= aw - [M_1v_1^p - v_2^p] \\ &> aw - [(M_1v_1)^p - v_2^p] \\ &= [a - \xi(x,t)]w \end{aligned}$$

where $\xi(x,t)$ is a bounded function on $(\overline{\Omega} \times [0,T^*]) \cup (\overline{\Omega_*} \times (T^*,T])$. Moreover, $\partial_\nu w = 0$ on $\partial\Omega \times (0,T^*]$, $w = 0$ on $\partial\Omega_* \times (T^*,T]$, and $w(x,0) >, \not\equiv 0$ on $\overline{\Omega}$. Hence we can use the maximum principle for parabolic equations to deduce that

$$w(x,t) > 0 \quad \text{in } (\overline{\Omega} \times (0,T^*]) \cup (\Omega_* \times (T^*,T]),$$

which confirms $v_2(x,t) < M_1v_1(x,t)$. Similarly, we can find a large $M_2 > 1$ such that $v_1(x,t) < M_2v_2(x,t)$ in $(\overline{\Omega} \times (0,T^*]) \cup (\Omega_* \times (T^*,T])$. Thus, we have found the desired $M = \max\{M_1, M_2\}$ such that (4.4.28) holds.

Note that, for the above chosen M, $M^{-1}v_1(x,t)$ is a subsolution and $Mv_1(x,t)$ is a supersolution of (4.4.25). Therefore, adopting the standard iteration argument, we know that (4.4.25) has a minimal and maximal solution in $[M^{-1}v_1, Mv_1]$, denoted by $u_*(x,t)$ and $u^*(x,t)$, respectively, such that

$$u_*(x,t) \le v(x,t) \le u^*(x,t) \text{ in } (\overline{\Omega} \times (0,T^*]) \cup (\Omega_* \times (T^*,T])$$

for any solution $v(x,t)$ satisfying $M^{-1}v_1(x,t) \le v(x,t) \le Mv_1(x,t)$. Thus

$$u_*(x,t) \le v_i(x,t) \le u^*(x,t) \text{ for } i = 1,2.$$

Hence, to prove the uniqueness, it suffices to show that $u_*(x,t) = u^*(x,t)$.

Let us define

$$\sigma_* := \inf\{\sigma \in \mathbb{R} : u^* \le \sigma u_* \text{ in } (\overline{\Omega} \times (0,T^*]) \cup (\Omega_* \times (T^*,T])\}.$$

Clearly, $\sigma_* \ge 1$ and $u^* \le \sigma_* u_*$.

To prove $u^* = u_*$, it is enough to show $\sigma_* = 1$. Suppose by contradiction that $\sigma_* > 1$. Set

$$w(x,t) := \sigma_* u_*(x,t) - u^*(x,t).$$

Then, as before, by applying the maximum principle for parabolic equations, we derive that

$$w(x,t) > 0 \text{ in } (\overline{\Omega} \times (0,T^*]) \cup (\Omega_* \times (T^*,T]).$$

In fact, the strong maximum principle and the Hopf boundary lemma guarantees

$$w(x,t) \ge \sigma u^*(x,t) \text{ on } (\overline{\Omega} \times [T^*/2,T^*]) \cup (\overline{\Omega_*} \times [(T^*+T)/2,T])$$

for some small $\sigma > 0$. In particular,

$$w(x,0) = w(x,T) \ge \sigma u^*(x,T) = \sigma u^*(x,0) \text{ on } \overline{\Omega}.$$

This allows us to use the argument as in proving the existence of the previously defined M again to assert

$$w(x,t) \ge \sigma u^*(x,t) \text{ in } (\overline{\Omega} \times [0,T^*]) \cup (\overline{\Omega_*} \times (T^*,T])$$

for the same σ. This implies

$$u^* \le (1+\sigma)^{-1}\sigma_* u_*,$$

which contradicts the definition of σ_*. Therefore, it is necessary that $\sigma_* = 1$, and the uniqueness conclusion is thus proved. $\square$

4.5 Proofs of main results

This section is devoted to the detailed proof of the main results presented in section 4.3.

Proof of Theorem 4.3.2. We first prove assertion (i). With the assumption (4.3.2), by the monotonicity of the principal eigenvalues, we note that

$$\lambda_1^N(\mu c_1 p, \Omega) = \lambda_1(\mu c_1 p, \Omega) \leq \lambda_1(\mu b) \leq \lambda_1(\mu c_2 p, \Omega) = \lambda_1^N(\mu c_2 p, \Omega).$$

Under our assumption on $p(x)$, Theorem 2.4 of [49] shows

$$\lim_{\mu\to\infty} \lambda_1^N(\mu c_1 p, \Omega) = \lim_{\mu\to\infty} \lambda_1^N(\mu c_2 p, \Omega) = \min\{\lambda_1^D(\Omega_1), \cdots, \lambda_1^D(\Omega_m)\} = \lambda_1^D(\Omega_1).$$

Therefore,

$$\lambda_1(\infty) = \lim_{\mu\to\infty} \lambda_1(\mu b) = \lambda_1^D(\Omega_1).$$

As a consequence, from Theorem 4.2.1, it follows that problem (4.1.1) with $\epsilon = 0$ has a unique positive T-periodic solution $u_0(x,t)$ if and only if

$$0 < a < \lambda_1^D(\Omega_1).$$

On the other hand, we know that $u_\epsilon(x,t)$ increases as ϵ decreases and $u_\epsilon(x,t) < u_0(x,t)$. Thus, $\lim_{\epsilon\to 0} u_\epsilon(x,t) = u_*(x,t)$ is a positive function. Furthermore, the standard regularity theory for parabolic equations and Sobolev embedding theorems imply that $u_\epsilon(x,t) \to u_*(x,t)$ in $C^{2,1}(\overline{Q_T})$, and hence u_* is a positive solution to (4.1.1) with $\epsilon = 0$. The uniqueness of $u_0(x,t)$ implies $\lim_{\epsilon\to 0} u_\epsilon(x,t) = u_0(x,t)$.

Next, we verify assertion (ii). Let $u_\epsilon^1(x)$ and $u_\epsilon^2(x)$ be the respective unique positive solution to

$$-\Delta u = au - [c_1 p(x) + \epsilon]u^p \quad \text{in } \Omega, \quad \partial_\nu u = 0 \quad \text{on } \partial\Omega \tag{4.5.1}$$

and

$$-\Delta u = au - [c_2 p(x) + \epsilon]u^p \quad \text{in } \Omega, \quad \partial_\nu u = 0 \quad \text{on } \partial\Omega. \tag{4.5.2}$$

Clearly, $u_\epsilon^2 \le u_\epsilon^1$ and $(u_\epsilon^1, u_\epsilon^2)$ is a pair of sup-sub solutions to (4.1.1). Hence,

$$u_\epsilon^2(x) \le u_\epsilon(x,t) \le u_\epsilon^1(x) \text{ on } \overline{Q_T}.$$

Moreover, by Theorem 2.2 of [38], as $\epsilon \to 0$, we have $u_\epsilon^2(x) \to \infty$ uniformly on $\cup_{j=1}^k \overline{\Omega}_j$, and $u_\epsilon^1(x) \to \underline{U}(x)$ uniformly on any compact subset of $\overline{\Omega} \backslash \cup_{j=1}^k \overline{\Omega}_j$, where $\underline{U}(x)$ is the minimal positive solution of (4.4.1) with $c = c_1$. Therefore, it follows that

$$u_\epsilon(x,t) \to \infty \text{ uniformly on } \cup_{j=1}^k \overline{\Omega}_j \times [0,T]$$

as $\epsilon \to 0$ and $u_\epsilon(x,t)$ is uniformly bounded from above on any compact subset of $(\overline{\Omega} \backslash \cup_{j=1}^k \overline{\Omega}_j) \times [0,T]$. So the standard interior regularity theory for parabolic equations shows that

$$u_\epsilon(x,t) \to \tilde{U}(x,t) \text{ locally uniformly on } (\overline{\Omega} \backslash \cup_{j=1}^k \overline{\Omega}_j) \times [0,T],$$

where $\tilde{U}(x,t)$ is a positive solution to (4.3.4). By Lemma 4.4.2, (4.3.4) has a minimal positive solution $\underline{U}(x,t)$. Hence, $\underline{U}(x,t) \le \tilde{U}(x,t)$.

It remains to prove $\tilde{U}(x,t) = \underline{U}(x,t)$. As in the proof of Lemma 4.4.2, let us denote by $\omega_1, \cdots, \omega_\ell$ the components of $\Omega \backslash \cup_{j=1}^k \overline{\Omega}_j$ with $\partial\Omega \subset \omega_1$. For any fixed ϵ, we can choose $n = n(\epsilon)$ to be so large that

$$u_\epsilon(x,t) \le n \text{ on } (\cup_{j=1}^\ell \partial\omega_j) \times [0,T].$$

For this fixed n, let $w_n^j(x,t)$ $(j = 1, \cdots, \ell)$ be the unique positive solution of (4.4.3) and (4.4.4), and $u_n^{i,j}(x)$ $(i = 1,2; j = 1, \cdots, \ell)$ be the unique positive solution of (4.4.5) and (4.4.6). Furthermore, we have

$$u_\epsilon(x,t) \le u_n^{1,j}(x) \text{ on } (\cup_{j=1}^\ell \omega_j) \times [0,T].$$

Thus, $(u_\epsilon(x,t), u_n^{1,j}(x))$ is a pair of sub-sup solutions to (4.4.3) and (4.4.4) on each component ω_j. As a result, combined with the uniqueness of positive solutions, the sub-super solution argument indicates

$$u_\epsilon(x,t) \le w_n^j(x,t) \quad \text{on } \omega_j \times [0,T].$$

Letting $n \to \infty$, according to the construction of $\underline{U}(x,t)$ in the proof of Lemma 4.4.2, we obtain $u_\epsilon(x,t) \le \underline{U}(x,t)$. Thus, $\lim_{\epsilon\to 0} u_\epsilon(x,t) \le \underline{U}(x,t)$, that is, $\tilde{U}(x,t) \le \underline{U}(x,t)$. Hence, $\tilde{U}(x,t) = \underline{U}(x,t)$, as wanted. This finishes the proof of the assertion in (ii).

The proof of (iii) proceeds similarly by using (ii) of Lemma 4.4.2 here and assertion (iii) of Theorem 2.2 in [38]. □

Proof of Theorem 4.3.3. The assertion (i) of Theorem 4.3.3 is obvious by making use of (i) in Theorem 4.3.2. It remains to verify (ii) and (iii). We only prove (ii) since (iii) can be derived in a similar way.

We first observe that $v_\epsilon(x,t) = \epsilon^{\frac{1}{p-1}} u_\epsilon(x,t)$ satisfies

$$\begin{cases} \partial_t v_\epsilon - \Delta v_\epsilon = a v_\epsilon - [\epsilon^{-1} b(x,t) + 1] v_\epsilon^p & \text{in } \Omega \times (0,T), \\ \partial_\nu v_\epsilon = 0 & \text{on } \partial\Omega \times (0,T), \\ v_\epsilon(x,0) = v_\epsilon(x,T) & \text{in } \Omega. \end{cases}$$

Let $u_\epsilon^1(x)$ and $u_\epsilon^2(x)$ be the respective unique positive solution to (4.5.1) and (4.5.2). We also denote $v_\epsilon^i(x) = \epsilon^{\frac{1}{p-1}} u_\epsilon^i(x), (i = 1,2)$. Then, from the proof of Theorem 4.3.2, we know that $u_\epsilon^2(x) \le u_\epsilon(x,t) \le u_\epsilon^1(x)$ on $\overline{Q_T}$, which in turn shows $v_\epsilon^2(x) \le v_\epsilon(x,t) \le v_\epsilon^1(x)$ on $\overline{Q_T}$. Furthermore, according to Theorem 2.7 and Remark 2.8 of [38], for each $i = 1,2$, as $\epsilon \to 0$, we see that $v_\epsilon^i(x) \to 0$ uniformly on $\overline{\Omega} \backslash \cup_{j=1}^k \Omega_j$ and $v_\epsilon^i(x) \to \theta_a^j(x)$ uniformly on each $\overline{\Omega}_j,\ j = 1,2,\cdots,k$. Hence,

$$v_\epsilon(x,t) \to 0 \ \text{ uniformly on } \overline{\Omega} \backslash \cup_{j=1}^k \Omega_j \times [0,T]$$

and

$$v_\epsilon(x,t) \to \theta_a^j(x) \ \text{ uniformly on each } \overline{\Omega}_j \times [0,T],\ j = 1,2,\cdots,k.$$

Therefore, the assertion (ii) of Theorem 4.3.2 holds. □

We now prove Theorem 4.3.4.

Proof of Theorem 4.3.4. Since (4.1.1) has a unique positive solution if $0 < a < \lambda_1(\infty; \Omega_*, T^*)$, the assertion (i) of Theorem 4.3.4 can be proved in the same way as in Theorem 4.3.2.

Next, we prove assertion (ii) of Theorem 4.3.4. For clarity, we break it into several steps.

Step 1. We first show $\lim_{\epsilon \to 0} u_\epsilon(x,t) = \infty$ uniformly on any compact subset of $(\overline{\Omega} \times (0,T^*]) \cup (\Omega_* \times (T^*,T])$. Recall that $\lambda_1(\infty) = \lambda_1^* := \lambda_1(\infty; \Omega_*, T^*)$.

Suppose that $a \geq \lambda_1^*$. Let $\lambda_1(\mu b(x,t))$ be defined as before and $\varphi_\mu(x,t)$ be the corresponding positive eigenfunction with $\max_{\overline{Q_T}} \varphi_\mu = 1$. We set

$$\underline{u}(x,t) = \mu^{\frac{1}{p-1}} \varphi_\mu(x,t) \quad \text{and} \quad \overline{u} = \max\left\{\mu^{\frac{1}{p-1}},\ \left(\frac{a}{\epsilon}\right)^{\frac{1}{p-1}}\right\}.$$

Note that, for any $\mu > 0$, we have $\lambda_1(\mu b(x,t)) < a$. Then, it is easily verified that $(\underline{u}, \overline{u})$ is a pair of sub-super solutions of (4.1.1) if $\epsilon < \mu^{-1}(a - \lambda_1(\mu b(x,t)))$. Due to the uniqueness of positive solutions of (4.1.1), we have

$$\underline{u}(x,t) = \mu^{\frac{1}{p-1}} \varphi_\mu(x,t) < u_\epsilon(x,t) \quad \text{on } \overline{Q_T}.$$

Hence,

$$\mu^{\frac{1}{p-1}} \varphi_\mu(x,t) \leq \lim_{\epsilon \to 0} u_\epsilon(x,t) \quad \text{on } \overline{Q_T}.$$

Thus, by letting $\mu \to \infty$, we find from Propositions 4.4.1 and 4.4.2 that $\underline{u}(x,t) \to \infty$ uniformly on any compact subset of $(\overline{\Omega} \times (0, T^*]) \cup (\Omega_* \times (T^*, T])$. This completes the proof of Step 1.

Step 2. We next show that $u_\epsilon(x,t) \to \infty$ uniformly on $\overline{\Omega_*} \times [0,T]$ as $\epsilon \to 0$. We only consider the case that Ω_* is connected; the case that Ω_* has more than one component can be similarly handled. What is more, since $u_\epsilon(x,t)$ increases as ϵ decreases to zero, it suffices to prove the above conclusion along a sequence $\epsilon_n \to 0$. To do this, we use a simple modification of the argument in the proof of Theorem 3.4.1. For completeness, we give the details below.

Let $u_n(x_n,t_n) = \min_{\overline{\Omega_*} \times [0,T]} u_n(x,t)$, where we may choose $(x_n,t_n) \in \overline{\Omega_*} \times [T^*, T+T^*]$. We shall use a contradiction argument. Suppose on the contrary that our claim does not hold true. Then, we may assume that $u_n(x_n,t_n) \leq C$ for all $n \geq 1$ and some positive constant C. By the result of step 1 and the boundedness of $u_n(x_n,t_n)$, it is necessary that $d(x_n, \partial\Omega_*) \to 0$ as $n \to \infty$. Let

$$\Omega_*^n = \{x \in \Omega_* \,:\, d(x, \partial\Omega_*) \geq d(x_n, \partial\Omega_*)\}.$$

Clearly, for $x_n \in \partial\Omega_*$, $\Omega_*^n = \Omega_*$, and for $x_n \notin \partial\Omega_*$, $\overline{\Omega_*^n} \subset \Omega_*$ but Ω_*^n approaches Ω_* as $n \to \infty$. Thus, in any case, for all large n, Ω_*^n and Ω_* have the same smoothness.

Since Ω_*^n is smooth for all large n, it enjoys the uniform interior ball property, that is, we can find a small $R > 0$ such that, for all large n and for any $x \in \partial\Omega_*^n$, there exists a ball $B_{x,R}$ of radius R such that $B_{x,R} \subset \Omega_*^n$ and $\overline{B_{x,R}} \cap \partial\Omega_*^n = \{x\}$.

To derive a contradiction, we first claim that: there is a constant $\delta > 0$ and a sequence of constants c_n satisfying $c_n \to \infty$, such that

$$u_n(x_n, t_n) + c_n h(x) \le u_n(x, t) \text{ if } \frac{R}{2} \le |x - y_n| \le R,\ T^* \le t \le T + T^*, \tag{4.5.3}$$

where $h(x) = e^{-\delta|x-y_n|^2} - e^{-\delta R^2}$, and y_n is the center of the ball $B_{x_n,R}$. A simple computation gives

$$\Delta h = (4\delta^2|x - y_n|^2 - 2N\delta)e^{-\delta|x-y_n|^2}.$$

We choose $\tilde{c}_n \to \infty$ satisfying $\epsilon_n(C + \tilde{c}_n)^p \le M_0 < \infty$ for all $n \ge 1$ and some constant M_0. For all $n \ge 1$, we define

$$\tilde{w}_n = u_n(x_n, t_n) + \tilde{c}_n h(x).$$

Then, for $x \in B_{x_n,R} \backslash B_{R/2}(y_n)$ and $t \in [T^*, T + T^*]$, where $B_{R/2}(y_n) = \{x \in R^N : |x - y_n| < R/2\}$, by direct computations, we have

$$\begin{aligned}
&\partial_t \tilde{w}_n - \Delta \tilde{w}_n - a\tilde{w}_n + \epsilon_n(\tilde{w}_n)^p \\
&= \tilde{c}_n[\partial_t h - \Delta h] - a\tilde{w}_n + \epsilon_n(\tilde{w}_n)^p \\
&\le -\tilde{c}_n[4\delta^2|x - y_n|^2 - 2N\delta]e^{-\delta|x-y_n|^2} + M_0 \\
&\le -(\eta_1\delta^2 - \eta_2\delta)\tilde{c}_n e^{-\delta R^2/4} + \eta_3.
\end{aligned}$$

Here, η_i $(i = 1, 2, 3)$ are positive constants independent of n. So we can find a fixed $\delta > 0$ such that $\eta_1\delta^2 - \eta_2\delta > 0$. Then, noting that $\tilde{c}_n \to \infty$, we obtain

$$\partial_t \tilde{w}_n - \Delta \tilde{w}_n - a\tilde{w}_n + \epsilon_n(\tilde{w}_n)^p < 0 \text{ in } B_{x_n,R} \backslash B_{R/2}(y_n) \times [T^*, T^* + T] \tag{4.5.4}$$

for all large n.

We now choose a compact set $K \subset\subset \Omega_*$ such that $K \supset \cup_{n=1}^{\infty} B_{R/2}(y_n)$. By step 1, $u_n(x, t) \to \infty$ uniformly in $K \times [T^*, T + T^*]$, hence we can find c_n with $c_n \le \tilde{c}_n$ and $c_n \to \infty$ as $n \to \infty$, such that

$$u_n(x_n, t_n) + c_n(e^{-\delta R^2/4} - e^{-\delta R^2}) \le u_n(x, t),\ \ \forall\, x \in \partial B_{R/2}(y_n) \subset K,\ t \in [T^*, T + T^*].$$

Since $u_n(x,T^*) \to \infty$ uniformly on $\overline{\Omega_*}$, we may further require that

$$u_n(x_n,t_n) + c_n(e^{-\delta R^2/4} - e^{-\delta R^2}) \le u_n(x,T^*), \ \ \forall\, x \in \{R/2 < |x-y_n| < R\}.$$

Then, as $u_n(x,t) \ge u_n(x_n,t_n)$ on $\partial B_{x_n,R} \times [T^*, T+T^*]$, we find that $u_n(x,t)$ is a supersolution of the problem

$$\begin{cases} \partial_t u - \Delta u = au - \epsilon_n u^p & \text{in } B_{x_n,R} \backslash B_{R/2}(y_n) \times [T^*, T+T^*], \\ u = u_n(x_n,t_n) & \text{on } \partial B_{x_n,R} \times [T^*, T+T^*], \\ u = u_n(x_n,t_n) + c_n(e^{-\delta R^2/4} - e^{-\delta R^2}) & \text{on } \partial B_{R/2}(y_n) \times [T^*, T+T^*], \\ u(x,T^*) = u_n(x,T^*) & \text{in } \{R/2 < |x-y_n| < R\}. \end{cases} \tag{4.5.5}$$

On the other hand, by the above choice of δ and c_n, it is not hard to see that $w_n(x,t) = u_n(x_n,t_n) + c_n h(x)$ also satisfies the inequality (4.5.4) and so for all large n, w_n is a subsolution to (4.5.5). Thus, (4.5.3) follows from the comparison principle for parabolic equations. As $h(x_n) = 0$ and so $u(x_n,t_n) = u(x_n,t_n) + h(x_n)$, we obtain that

$$\partial_{\nu_n} u_n|_{(x_n,t_n)} \ge c_n \partial_{\nu_n} h|_{x_n} = 2c_n \delta R e^{-\delta R^2} \to \infty, \tag{4.5.6}$$

as $n \to \infty$, where $\nu_n = (y_n - x_n)/|y_n - x_n|$.

Next, with Ω_*^n defined as before, we consider the following T-periodic problem

$$\begin{cases} \partial_t u - \Delta u = au - [b(x,t) + \epsilon_n] u^p & \text{in } \Omega \backslash \overline{\Omega_*^n} \times [0,T], \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times [0,T], \\ u = u_n(x_n,t_n) & \text{on } \partial\Omega_*^n \times [0,T], \\ u(x,0) = u(x,T) & \text{in } \Omega \backslash \overline{\Omega_*^n}. \end{cases} \tag{4.5.7}$$

For any large n, in view of Lemma 3.4.1, (4.5.7) admits a unique positive solution, denoted by $v_n(x,t)$. Furthermore, since $u_n(x,t)$ is a supersolution of (4.5.7), we have $v_n(x,t) \le u_n(x,t)$ on $\overline{\Omega} \backslash \Omega_*^n \times [0,T]$ due to Lemma 3.4.1 again.

On the other hand, if we replace $u_n(x_n,t_n)$ by its upper bound C and take $\epsilon_n = 0$ in (4.5.7), much as in the proof of Lemma 3.4.1, one can deduce that this problem has a unique positive solution, denoted by $U_0^n(x,t)$. Obviously, $v_n(x,t) \leq U_0^n(x,t)$ on $\overline{\Omega}\backslash\Omega_*^n \times [0,T]$. Furthermore, we can claim that $\|U_0^n\|_{L^\infty(\overline{\Omega}\backslash\Omega_*^n\times[0,T])}$ has a bound independent of n, so does $\|v_n\|_{L^\infty(\overline{\Omega}\backslash\Omega_*^n\times[0,T])}$. In fact, to derive the uniform boundedness of $\|U_0^n\|_{L^\infty(\overline{\Omega}\backslash\Omega_*^n\times[0,T])}$, we can adapt the argument of Lemma 3.4.1 in proving the existence of positive T-periodic solutions. By using Proposition 4.4.4, for given small $\delta > 0$ and large $M > 1$ (both depend only on a and C here), we can find the function $\overline{u}(x,t) = Mu_a^\delta(x,t)$ which is defined in $\Omega\backslash\overline{\Omega_*}\times[0,T]$ (see the proof of Lemma 3.4.1 for the construction of such similar $u_a^\delta(x,t)$), such that $\overline{u}(x,t)$ is a supersolution of problem (4.5.7) with $u_n(x_n,t_n)$ replaced by its upper bound C and $\epsilon_n = 0$. Since $\underline{u} = 0$ is the subsolution of such problem, it follows $U_0^n(x,t) \leq \overline{u}(x,t) = Mu_a^\delta(x,t)$ on $\overline{\Omega}\backslash\Omega_*^n \times [0,T]$ for all large n, which thus implies the uniform boundedness of $\|U_0^n\|_{L^\infty(\overline{\Omega}\backslash\Omega_*^n\times[0,T])}$. Hence, from the L^p-estimates for parabolic equations up to the boundary and the embedding theorems, we can conclude that $\{v_n\}$ also has a bound independent of n in $C^{1+\theta,\theta/2}(\overline{\Omega}\backslash\Omega_*^n \times [0,T])$, and so $\|\nabla v_n(x_n,t_n)\| \leq C_0$ for some $C_0 > 0$. Since

$$v_n(x,t) \leq u_n(x,t)\ \forall(x,t) \in \overline{\Omega}\backslash\Omega_*^n \times [T^*, T+T^*] \ \text{ and } \ u_n(x_n,t_n) = v_n(x_n,t_n),$$

we obtain

$$\partial_{\nu_n} u_n|_{(x_n,t_n)} \leq \partial_{\nu_n} v_n|_{(x_n,t_n)} \leq C_0. \tag{4.5.8}$$

Clearly, (4.5.6) and (4.5.8) contradict each other. This finishes the proof of step 2.

Step 3. It remains to verify that $u_\epsilon(x,t) \to \underline{U}(x,t)$ locally uniformly on $\overline{\Omega}\backslash\overline{\Omega_*} \times (T^*,T]$ as $\epsilon \to 0$. By virtue of the construction of $\underline{U}(x,t)$ (see (4.4.24)), one immediately deduces $u_\epsilon(x,t) \leq \underline{U}(x,t)$. Therefore, since u_ϵ increases as ϵ decreases to zero, by the standard regularity theory for parabolic equations, it easily follows that $u_\epsilon(x,t) \to u^*(x,t)$ locally uniformly on $\overline{\Omega}\backslash\overline{\Omega_*} \times (T^*,T]$ and $u^*(x,t)$ satisfies

$$\partial_t u^* - \Delta u^* = au - b(x,t)(u^*)^p.$$

Moreover, $u^*(x,t) \leq \underline{U}(x,t)$. By the conclusions obtained in steps 1 and 2, $u^*(x,T^*) = \infty$ in $\Omega\backslash\overline{\Omega_*}$ and $u^*(x,t) = \infty$ in $\partial\Omega_* \times (T^*,T)$. Henceforth, $u^*(x,t) = \underline{U}(x,t)$, which completes the whole proof of Theorem 4.3.4. □

Finally, we prove Theorem 4.3.5.

Proof of Theorem 4.3.5. The assertion (i) is obvious if $0 < a < \lambda_1(\infty)$. We then verify assertion (ii) for $a > \lambda_1(\infty)$.

First of all, we note that $v_\epsilon = \epsilon^{\frac{1}{p-1}} u_\epsilon(x,t) < \epsilon^{\frac{1}{p-1}} U(x,t)$, where $U(x,t)$ was given in Theorem 4.3.4. So, as $\epsilon \to 0$, $v_\epsilon \to 0$ uniformly on any compact subset of $\overline{\Omega}\backslash\overline{\Omega_*} \times (T^*, T]$.

We also note that v_ϵ satisfies

$$\begin{cases} \partial_t v_\epsilon - \Delta v_\epsilon = a v_\epsilon - [\epsilon^{-1} b(x,t) + 1] v_\epsilon^p & \text{in } \Omega \times (0,T), \\ \partial_\nu v_\epsilon = 0 & \text{on } \partial\Omega \times (0,T), \\ v_\epsilon(x,0) = v_\epsilon(x,T) & \text{in } \Omega. \end{cases}$$

As $b(x,t) = 0$ in $(\overline{\Omega} \times [0,T^*]) \cup (\overline{\Omega_*} \times [T^*,T])$ and $v_\epsilon > 0$ on $\overline{Q_T}$, v_ϵ is a supersolution to (4.4.25). Furthermore, for any fixed $\epsilon > 0$, we can find small $\delta = \delta(\epsilon) > 0$ such that $\delta\varphi_1^*(x,t)$ is a subsolution of (4.4.25) with $\delta\varphi_1^*(x,t) \le v_\epsilon$ on $(\overline{\Omega} \times (0,T^*]) \cup (\overline{\Omega_*} \times (T^*,T])$, where $\varphi_1^*(x,t)$ was defined in Proposition 4.4.1. Thus, from the argument of sub-super solutions, combined with the uniqueness of positive solutions to (4.4.25), it follows that $v^*(x,t) \le v_\epsilon$ in $(\overline{\Omega} \times [0,T^*]) \cup (\overline{\Omega_*} \times [T^*,T])$. Additionally, it is easy to see that $v_\epsilon \le a^{\frac{1}{p-1}}$ on $\overline{Q_T}$.

With the properties stated above, we can use the standard regularity theory for parabolic equations and the embedding theorems to show that, by passing to a sequence,

$$v_\epsilon(x,t) \to \tilde{v}(x,t) \ \text{ weakly in } \ L^2(Q_T) \ \text{ as } \epsilon \to 0,$$

and $v_\epsilon \to \tilde{v}(x,t)$ locally in the $C^{2,1}$ norm over $(\overline{\Omega} \times (0,T^*]) \cup (\Omega_* \times [0,T])$. Moreover,

$$\tilde{v}(x,t) > 0 \ \text{ in } (\overline{\Omega} \times (0,T^*]) \cup (\Omega_* \times (T^*,T]),$$

and $\tilde{v}(x,t)$ solves the first equation of (4.4.25). Recall that it has been proved that $v_\epsilon \to 0$ uniformly on any compact subset of $\overline{\Omega}\backslash\overline{\Omega_*} \times (T^*,T]$. Using the argument similar to step 3 in the proof of Theorem 3.3.3, we can easily assert that

$$\tilde{v}(\cdot,t)|_{\Omega_*} \in H_0^1(\Omega_*) \ \text{ a.e. } t \in (T^*,T].$$

Therefore, $\tilde{v}(x,t)$ satisfies (4.4.25) in the week sense. By standard regularity theory, we have $\tilde{v}(x,t)$ satisfies (4.4.25) in the classical sense. Since v^* is the unique positive solution of (4.4.25), we necessarily have $\tilde{v} = v^*$. This implies that $v_\epsilon \to v^*$ as $\epsilon \to 0$. The proof of Theorem 4.3.5 is complete. □

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