Chapter 9

Results and Discussion: Qualitative Analyses

Introduction

The data presented here are text data, based on student verbal responses, and descriptions of their physical actions as they work with MAB to solve subtraction problems, presented so as to complement the previously reported quantitative data. The data were collected from videotaping teachers and students at work during the teaching sequence, and from interviews with students conducted immediately after the post test was administered and again immediately after the retention test. There are approximately 32 videotapes of lessons, sections of each which have been transcribed, and 48 interviews resulting from 24 students each interviewed twice, all of which have been transcribed.

The earlier quantitative data provided a broad picture concerning the similarities and differences of a high isomorphism index teaching approach with that of a low isomorphism index approach, and provided a comparison of outcomes related to high and low levels of both simultaneous and successive processing. As Molloy and Das (1980, and citing others) mention, one of the problems associated with quantitative measures is that they mask the more subtle details associated with individuals and with experimental treatments, that is, the qualitative differences. This chapter will concentrate on providing a more detailed picture, one that reflects a microcosm rather than the whole system, the individual's mathematical ability and language rather than the quantitative analysis of large groups. In particular, this chapter will examine the kinds of mathematical activities used in each of the teaching approaches, especially as it is reflected in the language used by teachers. It will also look at the procedures described by students as they solve mathematical questions and the explanations they give for these procedures.

First, the kinds of learning that mathematics educators might want to take place in typical classrooms will be considered. There will then be an analysis of the data gathered from high and low isomorphism index classrooms, involving words and actions of both teachers and students. Next there will be an analysis of the interviews of high and low simultaneous and successive processors as they solve algorithms. Finally, interactions of teaching approach and cognitive processing will be considered.

Mathematical learning

What kind of learning do we want in mathematics lessons in typical classrooms, or more particularly, what kinds of meanings do we want students to make as we teach them about subtraction algorithms? For example, are we primarily concerned with students obtaining the correct answer or are we also concerned that they adopt an efficient and logical procedure leading to that answer? Is there something we expect them to learn beyond the procedure, do we expect them to have insights into the logical relationships within an algorithm, do we expect them to understand *why* and not just *how*? What are the roles of concrete materials that are used as learning aids? Are these materials simply non-electronic calculators used to help children gain correct answers, or are students' actions on these materials supposed to represent important mathematical operations? Are students expected to understand the relationship between actions on these materials and the structure of written algorithms? What kinds of classroom approaches lead to these various outcomes?

In the transcript below the student is completing the algorithm 653-472.

R: Do you reckon you are right?
S: Yes.
R: Good, well I think you are too. Now I want you to tell me how come that 1 is there.
S: Because 3 take away 2 and you have 1 left.
R: Now why did you cross out that 6?
S: Because you can't take 7 away from 5 because 5 is less than 7, so you cross off the 6 and put a 5, and put the 1 next to the five. Now it's higher and you can take away.

In this transcript, it is clear that the student is able to describe the correct procedure efficiently. She does not need prompting to move onto the next step, and provides a full and accurate account of decomposing 6 hundreds into 5 hundreds and 10 tens. The student knows the steps to obtain a correct answer, and she is able to recognise the need to trade and to complete the trade correctly. However, this transcript does not tell us about her broader understandings. In the context of completing the algorithm, she appears to understand why as well as how, but is she able to represent the mathematics involved here using concrete materials? Does she understand the relationship between the materials and the algorithm? Now consider the next transcript where another student is completing the algorithm 83-37.

R: What is that asking you to do?
S: It's a minus and you've got to take 3 from 7 and that one is 5.
R: Good, write the answer there.
S: And then you've got to take 8 from 3 and you can't do it. You can't do it.

Here the student uses 'minus' instead of 'subtract' or 'take away'. This use of a direction may be problematic since it may demonstrate that the student has a limited understanding of subtraction. He immediately subtracts 3 from 8 in the tens column, and says "and that one is five", but this recognition is short-lived. He immediately wants to take the 'top' digit in the vertical algorithm from the 'bottom' digit in each case. He calculates 7-3 correctly, not recognising it as the wrong calculation, and is stopped by the inability to complete 3-8, even though he has already answered this correctly. It seems he has little understanding of the procedure, but his realisation that you cannot take 8 from 3 has some significance. The third case is a student using MAB materials to answer the question 651-293.

(Student makes up 651 with MABs correctly, then begins to make up 293.)

R: OK, what have you got there?

S: I haven't got it all out yet but I put the top bit which is 651 and then I've got 2 hundred .... (pause) and nine. I've got to get 9 of these (tens). And I've got 9 of them, and I have to get 3 (units) there. And now I've got to take away 3 from 1 which I can't do so I trade (trades one of the five tens for ten units). I have got ten (units) to make up for 1 of these (tens) and then I count them all and it equals 11. And 3 of these and I have to take away 3 from 11. Put them all together, 11, 12, 13, 14 (adds 3 units in subtrahend to 11 units in minuend). I have to take away 3 from 14 which equals 11 (counts units).

R: Right so what do you do with the 11 now?
(Pause while student looks at algorithm, confused. Then writes 1 in answer space and 1 next to the 9.)

S: Well you can't put down 11 so I put the 1 there and the other 1 there, so that when I count up there is going to be one more.

In this case, the student has represented both the minuend and the subtrahend with MAB. She recognises the need to trade, which she does correctly, but then combines all the remaining units before subtracting. She does not recognise that her actions neutralise the subtraction. The student appears to have some understandings of subtraction, and is clear about what has to be answered, but her use of concrete materials is problematic. Her confusion as to what to do with the 11 suggests she has some understanding, and her solution contains a good deal of correct mathematical ideas.

Mathematics educators are likely to want learning outcomes that do not contain those kinds of explanations provided by the second and third transcripts above. Do these students really know so little mathematics, or at least so little about subtraction algorithms? Do we accept that the student in the first transcript has sufficient mathematical understanding concerning subtraction algorithms? A more detailed analysis of transcripts follows.

Learning outcomes from high and low isomorphism index teaching approaches

The analysis here, of the way students talk as they complete subtraction questions, is particularly concerned with any patterns of talk that may exist associated with the teaching approach in which students participated. This consideration of patterns of talk and teaching approach leads to the following research questions:

> will participants in a high isomorphism index treatment talk differently, and make different meanings, about written subtraction procedures, than participants in a low isomorphism index treatment;
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> will participants in a high isomorphism index treatment talk differently, and make different meanings about MAB procedures involving subtraction, than participants in a low isomorphism index treatment; and
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> will participants in a high isomorphism index treatment be more able than participants in a low isomorphism index treatment, to relate the use of MABs to the development and structure of written algorithms?

The following transcript is from the posttest interview with Nancy, where she is initially solving 547-169 as a written algorithm. This student has a good understanding of the procedure required to complete the algorithm, uses correct terminology, identifies facts and logical relationships in the algorithm, corrects herself when she makes an error, and gives full and accurate descriptions of sections of the algorithm without requiring further prompting.

R:	Can you tell me why that one is there?	The 1 above the 7
S:	Because that one ten was traded and so you could put it here.	Explains why, using a *logical relationship*
R:	So you had one ten and it was traded and ten came there and that number became what?	
S:	Seventeen units.	*Identification* of fact and relationship
R:	Seventeen units, good girl. Now you crossed that four out and you wrote something up here. Now why did you cross the four out and write something at the top?	
S:	Because I needed to trade so I traded one of the tens for ten units and then I only had three tens left.	Establishes *logical relationship,* and *technical* terms. "I" implies identification with action and as agent.
R:	Good girl, that's very good, you had four tens and you needed to trade one so you wrote the three up there and one went over here. Now what about here, why did you have to cross that five out?	
S:	Because I couldn't take six from three so I had to trade one of the five tens for a four *(pause)* I mean five hundreds for a four and then I put one of the ten units in that one so you could take six from thirteen.	*Recognises* the reason to trade, uses *"I"*. *Corrects* herself. Full description unprompted. Uses *technical* terms - trade, tens.

The transcript of this interview continues below, where the student now uses MAB materials to solve the question 653-472.

R:	I want you to use this equipment to do that problem and as you are doing it I want you to write down what the materials are showing. Do you know what I mean? You have a go at that.	Makes up 653. Relates MAB to numbers
R:	Good, now what do we do next? We are going to do the subtraction aren't we? So what do we do now?	
S:	We take two from three and we have one left, put that down there and then we have seven from five so we can't do it and we put that back. Then you take seven.	Completes units place value. Recognises and completes trade. Establishing logical relationship,
R:	How do you write that down?	*then...so*.
S:	You have eight tens so you put that down in the tens column. Then you have five hundreds left so you put that in the hundreds column.	
R:	Now let's go back a step all right. Let's go back to where you had that one. You have done the subtraction. We'll write the one, we have done that subtraction, we have taken two from three and we have got one. Now we want to take seven from five; now you tell me what you think we do now. You were right but I just want you to tell me again.	'Now' is temporal, child responds, but gives reasoning too
S:	Since you can't take seven from five you trade one of these.	
R:	Now as soon as you do that, you've got to trade, now how do you write that down? Right, what we'll do is we'll just write a little five. So now you put that in to trade and then I interrupted you, you've got to finish the trade haven't you?	Establishes logical relationship, *since*. Then trades 1 hundred for 10 tens. Writes 5. Algorithm corresponds to materials.
R:	So how many tens have you got there now?	Works with materials
S:	Fifteen.	
R:	How do you know it is fifteen?	Identifies fact
S:	Because we had five tens before and when we add another ten it is fifteen.	Identifies *relationship*
R:	Good. So now we are going to take something away aren't we? What do we take away?	
S:	Seven.	
R:	We have to do one more subtraction yet haven't we? Good girl, you did that very well.	Completes subtraction

The student uses materials in an effective manner, leading to a correct solution. However, there is much more than this happening. The student recognises and applies the relationship between the actions on the materials and the written algorithm. Her descriptions and explanations contain many logical relationships, she uses *then*, *so* and *since*, to describe her logic. And she does this with the minimum of prompting. Indeed her answers go beyond what one might expect. For example, when asked to say what happens now, she not only describes the next step, but also explains why. That is, she does not limit her explanations to the temporal, or the sequence, but elaborates so as to explain why. The interview continued with a word problem, giving the algorithm 85-49.

R:	So see if you can read that question. Do you want me to help you?	
S:	No I can read it.	
R:	What is it asking you to do?	
S:	Eighty-five minus forty-nine.	Correctly converts word problem to an arithmetic operation.
R:	And why does it say to take away?	
S:	Because forty-nine boxes were sold from eighty-five.	Understands subtraction *relationship*.
R:	Now if I say to you I want you to write the algorithm that corresponds to that what would you write down?	Writes 85-49 in vertical algorithm. Establishes *relationship*. Completes written algorithm.

R: Do you reckon you are right?	
S: I don't know yet.	
R: You don't know yet, how would you know if you were right?	
S: Check it.	
R: How could you check it.	
S: You could go over it again.	*Identifies checking alternatives.*
R: Is there another way of checking it?	
S: Use a calculator and the materials to check it.	

The student continues to show that she understands the concept of subtraction, is able to write and to complete algorithms correctly. She establishes a range of relationships within the algorithm, and between the algorithm and the word problem. She is also clear as to ways of checking if her answer is correct.

Nancy's transcripts illustrate some essential elements in not only successfully completing algorithms, but also in understanding a range of mathematical relationships within and beyond the algorithm. For example, the student identifies meanings and establishes relationships within the algorithm, and between the algorithm and the teaching aid, in this case MAB materials. In this last set of transcripts, there are many extensions of what was the minimum required, that is, instances where the student says something that broadens or deepens some construct or something said previously. Such extensions typically use complex language structures, make reference to words such as *then..so*, and *since*, and are an important way of identifying that the student is consciously aware of the process of making meaning. That is, she explains why a particular action has to be taken, by providing a logical relationship rather than by referring to a rule. Indeed it seems her understanding goes beyond the procedural, to one involving an understanding of relationships. The following example, taken from the above text, is especially informative.

> Because I couldn't take six from three so I had to trade one of the five tens for a four *(pause)* I mean five hundreds for a four and then I put one of the ten units in that one so you could take six from thirteen.

Here the student establishes a relationship *I couldn't take six from three so I had to*, and then recognises an error and corrects herself *I mean....* She provides a complex logical relationship, *(b)ecause...so...so that.* Finally, the whole text is brief but accurate, full of meaning for her and for the reader, and goes beyond simple mechanical description to elaborate on the whole process.

In the following transcript another student uses a complex language structure to explain the difficulty of subtracting 8 from 4. He provides an exemplar *Just say you had four leaves*, and then proceeds to explain the logic of the situation.

R: Four take away what?
S: Eight
R: Now why can't you do that?

S: Because if you've got four. Just say you had four leaves or something and then if you have four, you can't take eigh away because you haven't got enough.
R: Right. So what do you do Kieran?
S: You cross out the seven and put a six on top. And put a one next to the four and that's six fourteen and then you take eight away from the fourteen which leaves you with six.

Again, this is an instance of a logical relationship, where his response to the researcher's *why* begins with *(b)ecause*. It is also an instance of understanding, beyond the procedural, where he calls upon a concrete referent to establish the mathematical relationship. He does not revert to a statement or paraphrase of a simple rule, such as "you can't take 8 from 4", and he attempts to provide a genuinely meaningful response that is both logical and mathematically correct. Here is the same student explaining the need to trade, calling on logical necessity, and referring to the correct mathematical construct.

R: You had eighty five and you crossed out the eight and you put a seven there and a one there, now why did you do that?
S: Because if I had five I could not take nine away, I haven't got enough.
T: So what do you have to do?
S: Trade.

The two students whose transcripts are shown above[1] exhibit those qualities that mathematics educators in general, and certainly this mathematics educator, would hope students exhibit in their learning outcomes. In particular, these students have highly effective procedural knowledge, both with written algorithms and with the use of MAB materials. They have insights into the logical relationships implicit within subtraction algorithms, and insights concerning the mathematical properties of MAB materials and their actions upon such materials, as well as insights into the relationships between the use of these materials and the corresponding written algorithms. Clearly not all students have such insights: let us now consider another set of transcripts.

Here is Damien using MABs to calculate 83-37. His use of the materials is inconsistent, and reflects poor proceduralisation of the materials and a lack of understanding of the relationship between the different sized pieces.

R: So Damien what have you got there?
S: Eighty-three.
R: So what do we do now? Tell me what you just did. You've got your pen so write your answer down, all right?
R: Now when you had this (83 in MAB), you had that (8 tens) and the first thing you did was that (remove 3 tens). Now can you tell me why did you do that?
S: Because I do it.
R: No, I'm not saying you are wrong. In fact what you did (pause, rephrases question) you got the correct answer that way, so why did you do that? Why did you take those three away? What are they?
S: Tens.
R: So whereabouts on that (the written algorithm), what you have got written there, are these things?
S: (inaudible)
R: Then you took an extra one (1 ten) and you put those down (10 units).

[1] The transcript in column format and the one immediately following, pages 127 to 130.

S: Because I didn't have enough units to take seven from.
R: So what did you do then?
S: Took one of them away and got seven, eight, nine, ten, then got three units and put them there.
R: Good, now that is the right answer.

The above transcript shows Damien subtracting three tens from the eight tens as his first action after forming 83. This may be arithmetically correct, but it does not help establish the procedure for completing a written algorithm. His lack of a deeper understanding of the materials, his actions on them and the various interrelationships that exist is reflected in his explanation *Because I do it*. He is unable to answer a series of *why* questions, but can then answer a *what* question, when he labels the MAB tens. He refers to the trade action as *took one of them away*, but doesn't trade correctly, counts *seven*, then brings a unit from the bank for each of *eight, nine, ten*. That is, he calculates 10-7, then adds the existing 3 units to give the answer 6. Again these methods reflect an idiosyncratic procedure, which may arrive at the correct answer, but do not reflect either the way the student writes the algorithm, or the way a written algorithm is efficiently completed. This question (83-37) is repeated by the same student, this time as a written algorithm, without any use of materials.

R: I'll take the materials away and you can write the answer down there (Pause while child works).
S: I could put the seven there.
R: You write the seven where you think it goes. You try and do that now. (Pause) What is the problem, why is that hard for everyone to do? Why can't they just work it out like the first one?
S: Because three is bigger than the seven.
R: Which is bigger?
S: Seven is bigger than the three.
R: So what does that mean?
S: You can't take three from seven (brief pause) I mean seven from three.
R: You can't take seven from three, so what do you do about that?
S: You trade.
R: Good, that is the right word. Now what does it mean?
S: To regroup something.
R: So you crossed the eight off and you wrote seven. Now that's right. Now what is the other bit that you do?
S: Put a one there (writes 1 next to the 3 in the subtrahend) and then it's thirteen take away seven.
R: Do you want to try that? So you don't mean that you want me to have that one there, is that right?
S: Yes.
R: You don't want it there?
S: No.
R: So you have got a seven there and a one there, is that okay?
S: Yes.
R: Now, why do you cross the eight out and why do you write seven there?
S: (no response)
R: If I asked you to finish that now to do the subtraction, you try and finish it now but not with the material (Pause while child works).
R: What is the subtraction you are trying to work out Damien?
S: Thirteen take away 7.
R: Try it again. Tell me which one you are trying to subtract.
S: That would be six down there.
R: Write it down. Why is it six down there?
S: Thirteen take away seven is six.
R: Thirteen take away seven is six. You tell me how you got six Show me.
S: On my fingers?
R: Yes if that's how you did it.

S: Well, I had ten fingers here like that and then I got them three fingers and made out that that was thirteen and took three, four, five six, seven and put them down and then I got seven take away three leaves four.

Damien recognises that the 8 has to be reduced to 7 but is initially uncertain where to write it, he is unwilling to do so, so the questioner asks about the general problem here. The student responds *three is bigger than seven*, and then immediately corrects himself, but quickly repeats the error *you can't take three from seven*, and corrects himself again. He is able to recall the word *trade*, then crosses off the 8 and writes 7 next to it correctly, but then writes a 1 beside the 3 in the subtrahend immediately below the 8. So even though he has said trade, he has not proceduralised the actions correctly. The researcher then intervenes by placing the traded 1 in the correct position, the student appears to agree, but it could be under duress. In any event the student is unable to state any relationship between the 7 and the 1. That is, as in the previous transcript he is unable to answer 'why' questions. His use of fingers is also problematic in that it may help him obtain more correct answers, but it is likely to prevent automatisation of the procedure: because the procedure is constantly interrupted by the need for him to use finger counting. The principal point here is that, even though the student is able to achieve a correct answer, his knowledge of mathematical concepts and ideas, and his knowledge of how these constructs interact, is limited. He is sometimes procedurally correct, for example he later completes 74-46 correctly, and explains his work in this way.

S: First I noticed that four is smaller than six and it's at the top so you can't take it away, so I went to the seven and crossed it out, put a six there and a one there and that becomes fourteen from forty-six, then I put fourteen take away six is eight, eight there and I put six take away four is two there.
R: Good. That's right. Now can you tell me why do you cross that out? Why do you cross the seven out?
S: Because four is smaller than six and it's on the top so you've got to cross that out and make it one lower and then put it there so you can work the answer out.

This text suggests that his explanation is totally procedural. He focuses on *the top* and *one lower*, to identify where and how something has to be altered. It seems he acts as he does, because that's the way you do it. He is unable to explain his actions in terms of mathematical constructs, he makes no reference to technical terms, to the possibility of using an analogy, or to any other possible explanations.

Casey is completing the written algorithm 547-169, which she finishes without particular difficulty, but her explanations are procedural rather than involving relationships.

R: So what is the first thing you try and do?
S: You can't take nine from seven so you have got to cross out the four and put a three. You put a one next to the seven.
R: Now why do you do that?
S: Because if you try and take nine from seven you can't do it but if you do do it then you come up with the wrong answer.
R: And why did you cross the four out and put a three?
S: Because you go to the next column to get more out.
R: What does the three there mean?
S: Because you have four tens but you crossed it out and you put a three because you had to take one out and put it back in the ones column.

R: And what does that one there mean?
S: One little thing that I just put in
R: So now what do we do?
S: Take nine from seventeen.
R: What is that?
S: Eight.
R: That's good so now what is the next thing you do?
S: You've got to take six from three but you can't do it because there's not enough up there so you have to cross out the five and put the four down and put one, then you take six from thirteen which is eight and then you put eight down and you can take one from four and it comes to three.
R: Now how did that four come to be there?
S: Because you had to cross out the five and put a four and put another ten in the tens column.
R: Right.

Casey recognises the need to decompose, describes it correctly, but when giving her reason gives a rule-based response rather than referring to relationships. This acceptance of rules is reflected in her contradiction *you can't do it but if you do do it then you come up with the wrong answer*. That is, her actions are rule and procedure based, not based on a deeper understanding of logical or mathematical relationships. Her description of the trading process emphasises the procedure of taking out from somewhere, *to get more out* and *you had to take one out*, rather than describing mathematical relationships, eventually giving as a reason for placing the carried 10 units where she does as *One little thing that I just put in.* Apart from a number bond error, where she says 13-6=8, she has obtained the correct answer using her procedure. When the same question is repeated with MAB materials, she misuses the materials through an incorrect trading process and obtains an incorrect answer. That is, in written form, Casey has a procedure that leads to a generally correct response: but she is able to give only a limited explanation of the procedure. She does not have a correct procedure for using MABs, and so is unable to relate their use to a written algorithm. Casey is inconsistent, she shows confusion over a question not involving trading, but completes a complex subtraction correctly. In her class workbook, and in the various tests, she showed little evidence of successful trading, sometimes taking the smaller digit from the larger regardless of position, and at least at times she seemed to add on to gain an answer. In completing 651-293 here is what she says

R: Now what does this say?
S: You take one (that is, 3-1).
R: How do you do that?
S: You go one take away (pause) take one away from three.
R: Do you want to write that down? (Writes 2) The next one?
S: You've got to take five from nine and that leaves four and then it says take six from two but you can't do it.
R: You can't do it.

She appears first to subtract smaller-from-larger, but then recognises 2 - 6 cannot be completed. And a little later, in 86-35, her confusion continues.

S: Eighty-six take away thirty-five.
R: So what is the first thing you are trying to subtract?
S: Take six from five.
R: How do you take six from five?
S: You can't, so you cross that out and put seven.

R: Now why did you cross that eigh. out?
S: Because if you try and take six from five you'll get the wrong answer, you can't do it, so I took one ten out of there and put it into th: ones column.
R: So you try and work out the answ·er to that one (pause while child works).
S: I can't do it.
R: Tell me what you are trying to do.
S: Trying to take eight from three.
R: Trying to take eight from three. Now what is the one here?
S: Because I came up with eleven ai d put one here.
R: So what is just one and that seve 1, what do you do with those?
S: You add the one to the seven.
R: Let's go back a step. What did this become?
S: Sixteen.
R: Okay and what did you do then?
S: I took five from sixteen.
R: And what did you write down?
S: Eleven.
R: You wrote down eleven. So where is the eleven written?
S: There.
R: So what happens now?
S: You take eight from three and you can't do it.

Here, Casey states the question correctly, but tries to take 6 from 5. She trades so as to do this, crossing off the 8 and writing 7, and says 16-5=11. The 1 unit is written in the correct position, and the ten is carried to the 7. She is actually correct in arithmetic terms, though her procedure is inefficient. She does not recognise that she almost has the correct answer, simply by taking 5 from 8 (the 7+1), and she continues to gives reasons for her actions in terms of rules or fear of the wrong answer. She explains that, in taking 6 from 5, "you'll get the wrong answer", as she continues to subtract digits in the wrong order. Towards the end of the transcript, she recognises that the decomposed tens have made 6 into sixteen, this is correct, but she is confused, does not recognise her errors and does not know how to proceed.

Both students whose transcripts have been used over the last few pages have learning outcomes that are undesirable. In particular, they are unable to regularly and efficiently obtain correct answers to subtraction algorithms, or to provide explanations for their actions that make reference to logical relationships or to mathematical constructs. Also worrisome is their poor use of MAB materials, and their lack of understandings about the relationships within the materials and their actions upon them or in the way the materials relate to written algorithms.

Up to this point, two sets of transcripts have been presented: one set where learning outcomes would generally be regarded as pleasing; and, a second set where learning outcomes are problematic in terms of either incorrect answers or limited mathematical insights. The important point to be taken from these sets of transcripts is that the first set are taken from, and are typical of, students who participated in the high isomorphism index teaching approach, while the second set are taken from, and are again typical of, students who participated in the low isomorphism index teaching approach. That is, there appears to be a link between teaching approach and the way students complete algorithms, in the way they use and make sense of concrete materials, and the manner in which they speak and think about subtraction. It appears from these and other texts, that in comparison with their peers involved in the teaching approach using a low isomorphism index, high isomorphism index students are likely to score more highly, make more effective use of MAB materials to establish corresponding algorithms, have better procedural knowledge, have knowledge of mathematical relationships beyond the procedure, and are able to explain *why* in addition to *how*.

Take the case of Jessica who had a poor score on her pretest (69), but who showed considerable improvement by the posttest (108), and who improved further on the retention test (129). Jessica does not use many extensions in her language, but she is good at identifying the important parts of algorithms and MABs, uses MAB materials in an efficient manner, and is able to establish relationships between action on MABs and the written algorithm.

R:	Let's try that one.	Completes written algorithm, 746-382.
R:	Can you tell me why you crossed that seven out and wrote that number there?	Writes 6, and 1 next to 8
S:	Because you can't take eight away from four.	Identification.
R:	So what do you do?	
S:	Cross out the seven and change it to a six and put a one next to the four so it becomes fourteen.	Relationship.
R:	So what is the six that is there now? Is that six units, six tens or whatever?	
S:	Hundreds.	Identification of fact
R:	What about the one that is there?	
S:	Ten.	Identification of fact
R:	So how many tens have you got there all together?	
S:	Fourteen.	Identification of fact
R:	Fourteen and you take away eight tens, is that the idea?	
S:	Yes.	
R:	Good. Let's try that one, this time I want you to do that one except I want you to do it with these materials. What have you got in front of you now?	651-293 with MAB Makes 651 with MAB
S:	Six hundred and fifty-one.	Identifies MAB/number match
R:	Six hundred and fifty-one. So what are you being asked to take away?	
S:	Two hundred and ninety-three.	Identifies subtrahend
R:	So what do you do?	
S:	You can't take three away from one, so you put one of them here and you get ten of these.	Logical relationship and correct action
R:	What is that called, do you remember?	
S:	Trading. Then I take three away.	Identifies technical term, continues without prompting
R:	Before you take three away, if I asked you to write down what you just did, what would you do?	
S:	Cross out the five then write a four and put a one next to it.	Identifies logical and mathematical relationship between MAB and algorithm
R:	Good, keep going. As you go through this I want you to write it out.	Works with MAB
R:	Good, you did that very well. Can you remember when you crossed out this (5) and wrote 4 there and 1 there (next to 1 in units), why you did that?	
S:	Because you can't take three away from one.	Recognises relationship
R:	Right keep going. You can't take three away from one, so what did you actually have when you used the equipment? When you started how many tens did you have here?	
S:	Five.	Identification
R:	Five and you had a 1 there. You wanted to take three away. You couldn't take three from one so what happened?	

S: I crossed out the five and changed it to a four and put a one next to it.	Correct actions
R: But when you used the equipment you didn't cross out the four, you used the equipment first, so what did you do with the equipment?	
S: Put one of these up, and got ten of these, then took away three.	Repeats relationship between MAB and materials
R: That's exactly right. Okay Jessica, that's terrific, you did that really well.	

In this text, Jessica showed a good understanding of the procedures required in both algorithms and MABs, and a high awareness of the relationship within and between each of the procedures. Her use of *because* shows an awareness of the logic and relationships involved. That is, Jessica is another case of a student from the high isomorphism index teaching approach who shows such characteristics.

There are other students whose transcripts showed these skills and insights. For example, Laura, Natasha, Emily, Kieran, Ann and Scott. Of course, this is not to claim that all students in the high isomorphism teaching approach can do all these things, and are superior to all students in the low isomorphism teaching group. For example, the following students from the high index teaching approach all exhibit problematic learning outcomes. David can clearly describe the procedure for solving 547-169. In describing the procedure in the tens column he says *three take away six you can't do so I crossed off the five, put a four and put a one next to it and take away six equals seven* which is both correct and a clear advance on his pretest procedure of taking the smaller digit from the larger. At the same time, David experiences some difficulties both in using the MAB materials (subtracts from largest place value first) and in relating the materials to the algorithm. Marie, who is very efficient in completing procedures, and scores well on the tests, makes little sense of the MAB materials and misuses them. For example, she will successfully complete a written procedure, but when asked to use materials to solve a similar question introduces an extra ten whenever she needs to, instead of making a trade. Further, Chris is not at all successful in either making links between MAB and algorithm structure, or completing written algorithms.

In spite of these exceptions, the overall picture from an analysis of students' transcripts indicates that students who participated in the high isomorphism index teaching approach were more able than students in the low index approach, to correctly complete algorithms, to use MABs in a meaningful manner, and to complete both these procedures efficiently and accurately. They were better able to recognise, remember and recall mathematical and logical relationships within each kind of procedure, and between the MAB and written procedure. That is, students from high isomorphism index teaching approaches have qualitatively superior learning outcomes to those from the low isomorphism index teaching approach.

Lesson transcripts

It has been shown previously that there are some quantitative differences in learning outcomes between the different teaching approaches, and it has been argued above that there are also qualitative differences. In what way do these different outcomes reflect the teaching approaches used? That is, will examining lesson transcripts indicate a connection between what the teacher says and does, and what students say and do? Here is a text with a teacher interacting with students in a

T: Oh, 42 is it? One, two, three, 42, ten, 20, 30, 40. Two. I'm working upside down. What do I do now?	Teacher counts out 4 longs, starts counting 1,2,3, corrects herself, 10, 20, 30, 40. The takes 2 units.
S4: Take the ten away and you get ten ones.	
T: Why?	
S4: You can't trade.	
T: What do I need to trade?	
S: Because you can't take away seven.	
T: Oh. Excellent boy. One, two, three, four, stop fiddling,..ten. I've now traded, two four, six, eight, ten. What do I do now?	Teacher holds a ten in her left hand, takes 10 units from the bank with her right hand, places them next to the tens, counts as she does it, then puts the ten in the bank.
S5: Take away seven from the ten	
T: Exactly. One, two, three, four, five, six, seven. What's my remainder?	Teacher takes seven units away from the line of ten
S: Thirty-five.	
T: 10, 20, 30, one, two, three, four, five	Teacher counts tens and units, verifies answer.

low isomorphism index teaching approach, completing the calculation 42 - 7, then 42 - 17, and using MABs.

The point here is that the teacher subtracted from the 10 units that were traded. That is, she did not emphasise joining the units together and renaming them. This is an acceptable part of a low isomorphism teaching approach, and one common in classrooms. What then do students do when they are asked to complete subtractions using MABs? The extent to which they copy the teacher's actions are shown in the samples below.

Student 1	91-38	= 9 tens - 38
		= 80+10-38 trades one 10 for ten 1s
		= 80+10+1-38 realises needs 1 in units place
		= 90+10+1-38 (adds one ten), long pause
		= 90+ (11-8 -30 (subtracts 8 from 1 unit and traded 10)
		= 80+3+10-30 (trades 1 ten for 10 units)
		= 90+3-30 (undoes trade, subtracts three 10s)
		= 60+3
		= 63 (incorrect)

Here the student starts with 9 tens representing 90, and trades for 10 units, then realises the need for another unit, to represent the units in 91, but keeps it separate from the traded units. The student adds another ten, and after a time combines the units and subtracts 8 from 11. The extent of the error is related to the additional ten taken from the bank. The problems here are related to poor representation of numbers using MAB, and an inadequate proceduralisation of when to trade and the trading process. That is, the student's actions here are inefficient, since they do not lead to a correct answer, and because they cannot be used to establish an effective written algorithm. In such cases, the actions on MABs are unhelpful in establishing an analogy between concrete materials and the meanings of symbols used in algorithms. In the second example below, the student correctly represents 75 with MABs, then trades 1 ten for 10 units.

Student 2 75-46
= 60+10+5-46 (three piles, six 10s, ten 1s, five 1s)
= 60+(10-6) + 5 -40 (subtracts 6 from 10)
= 60+4+5-40 (subtracts four 10s, does not combine units)
= 20+4+5 (long pause, counts ten 1s from bank, holds then for some seconds, returns them to bank)
= 20+9 (combines 1s)
= 29 (correct) T: know how to do it now?
S: Yes. (Returns to seat)

After trading the student did not join the units together, and subtracted 6 from 10. The units were still not combined when the student subtracted 4 tens from 6 tens. The units are then combined and the student achieved the correct answer. The point here is that the answer was correct, but the manipulation of MABs did not help to establish an efficient vertical written algorithm. That is, the level of correspondence between actions on blocks and the written algorithm was low. Further, this subtraction from the traded 10 units was very common, with the teacher's procedure copied as it was presented even though she made no special emphasis of it. Here are procedures used by other students. The arithmetic represents their actions as they manipulated MABs.

83-59	72-39	134-27
= 70+10+3-59	= 60+10+2-39	= 100+30+4-27
= 70+(10-9)+3-50	= 60+(10-9)+2-30	= 100+20+(4-4)+(10-3)-20
= 70+1+3-50	= 60+1+2-30	= 100+20+7-20
= 70+4-50	= 60+3-30	= 100+7 (correct)
= 20+4	= 30+3	= 1 7 (Writes 1_7, with space between digits)
= 24 (Writes 24, correct)	= 33 (Writes 33, correct)	

In each case, these students obtain the correct answer, but equally in each case, their methods involve actions for which there can be no correspondence in an efficient vertical written algorithm. The written algorithms reflect their actions; to the extent that this also reflects their cognitive processing, they have an inefficient cognitive network. In every case, it seems the blocks are being used as a calculator, not as a vehicle for furthering mathematical understandings.

The argument here, then, is that the approaches used by students typically reflect the words and actions of the teacher. Indeed, students' actions will reflect both what the teacher intends and what may not be intended. The teacher in this case did not emphasise joining and renaming, so some students joined traded units with existing units, others didn't. Some students subtracted from the joined units, some subtracted from the traded tens, some found other possibilities. This has implications for the development of effective teaching approaches, especially where the context requires algorithmic procedures.

Here's another text from a low isomorphism index teaching approach, where the teacher is taking students through the subtraction 54 - 19, using MABs, and relating the use of MABs to the written algorithm.

T: All right take nine. What's wrong with that, why can't I take nine? Yes please.
S: Because four is smaller than nine.
T: Because four is smaller than nine. so it says four subtract nine; I can't take nine from four, what do I do please?
S: You trade.
T: I trade okay. I trade so I swap. What do I trade that for?
S: Ones.
T: How many ones?
S: Ten.
T: Ten ones, okay. So you count and check that I've got it okay. One, two, three, four, five, six, seven, eight, nine, ten. Okay, there they are, all there. What do I subtract now? Please look at the question on the board. Up the back look at the question on the board.
S: Nine.
T: I subtract nine, that's right. So, one, two, three, four, five, six, seven, eight, nine. Okay, hands up please, how many units are left there please. Hands up, units left please?
S: Five.
T: And so now I can write five there Now what else can you tell me I should write down, because how many tens have I got here now?
S: Four.
T: Four so what do I do to show I've got four tens? What step have I left out? When I write it down I've left a step out when I write it down, what do I do?
S: You cross out five and you write the four and...
T: I cross out the five and I write the four.
S: ...and you put the one next to the four.
T: And put the one next to the four, so nine from fourteen was five, right, what do I have to take away now please? Those down the back who aren't watching, what do I have to take away now please? Come on I've taken the nine away, what else do I have to take away now please? Yes?
S: One.
T: One what?
S: Four take away one.
T: Four take away one, okay, so here's four, I take away one, so I take away one of these, how many are left there?
S: Three.
T: Three and who can tell me the answer please?
S: Thirty-five.

Here the teacher has gone through the correct procedural steps. He joined the traded units to the existing units but did not rename them, and he did not emphasise the crossing out of 5 and writing 4 as operating on the tens place value. The children seem to be following the work, since there are correct answers from them. In the following text, a teacher is leading students through the calculation 52 - 33, based on a word problem.

T: So how many longs do you have?
S: Five
T: And how many units do you have?
S: Two.
T: Do you agree with the sum we've got to do? What have you got to do? Fifty-two, we've got to take away what?
S: Take away thirty-three.

T: All right, we've got a problem, we have fifty-two and we have to take away thirty-three. How do we do that?
S: Take one long away and (pause)
T: Put one long into the box, what do we take out.
S: Units
T: Everybody do that for me. Be certain you put them in the right column. Wait, there are just a couple of people not quite ready. So, I have traded one ten, one long and now I have four tens. What do I have in the unit column?
S: Twelve.
T: Twelve units. Everyone double check if you have twelve units. If you don't have twelve, fix it up. Now if it was as before, fifty and two, equals fifty-two, then what is it now?
S: It's still fifty-two.
T: It is fifty-two, very true, but what do you have in front of you in the text column?
S: You have four longs.
T: Can you give me another name for four longs?
S: Forty twelve.

The text above comes from a highly structured, high isomorphism index lesson, where the teacher emphasised the trade concept and correct procedures in trading, place value and the written algorithm. At each stage she checked students are using materials and representing numbers in a particular manner. She asked many questions to check this arrangement and to check that students understood why. For example, when 52 was decomposed into 40 + 12 she checked that students recognised that it was still 52, and that it could be called 'forty twelve'. Here is the same teacher in a later lesson.

T: Even if you have not finished, stop. Fold your arms please. (Teacher draws up place value chart, asks students to show me the number she writes, 2H 3T 1U.)
(Children making up number with MAB).
T: Read the number to me please John.
S: 231
T: So there are 2 hundreds, 3 tens and 1 unit. Now I'm going to put something on the board that is different from what we have done so far. Someone very clever may be able to tell me what's different about it. (Teacher writes 231-40 on board in place value chart.) What's different about that sum? Something we haven't done before.
S: You are not borrowing
T: Not renaming where?
S: In the units.
T: Good. We are not renaming in the units. Are we renaming somewhere else?
S: Yes, you are borrowing from the hundreds.
T: And putting it into what column?
S: The tens.
T: So we rename, we're trading from the hundreds to the tens. Would you write that (231-40) for me in your books, on a new page.
S: Do we set it out the same?
T: Yes, how we usually set it out, you do that. Except one person knows he's on a diet because they take up too much room. (Slight laughter from children.)
I just want you to copy it down, I don't want you to do it yet (pause).

In the section of the lesson immediately above, the teacher has modelled the correct terminology, and illustrates how the tens/units principle can be transferred to the hundreds/tens columns. The lesson continues.

T: Put your answer box so I can see where your answer's going to be (pause). Okay, whether you're ready or not just put your pencils down. I can see this table's almost completely ready. Just a couple of people at this table aren't quite ready and we're still waiting. I want to know that you're concentrating and looking at the board, and ready for me to continue. (Teacher standing at board, next to algorithm that children have copied into book.)

T: So we have been told by some very clever people that we are not renaming in the units column. So will you tell me what does it say here (points to units column, 1-0)?
S: One take away nought.
T: Is?
S: One. (teacher writes 1 in answer space.)
T: Now what do you say in this column (points to tens column, 3-4)? You tell me?
S: Three take away four you can't do it.
T: So we have 3 tens, take away 4 tens, we can't do that. What do I do?
S: You trade the um...
T: You're right, you trade. What are you trading?
S: Two
T: Can someone help him?
S: You cross off the three (corrects himself) I mean you cross off the two and put a... just one (teacher does this).
T: That's one what?
S: One hundred.
T: (disciplinary remark)
S: And you put the one next to the three (Teacher does this).
T: What's this now?
S: Thirteen.
T: Thirteen what?
S: Take away...
T: No. Thirteen what? Thirteen units?
S: Thirteen tens.
T: Thirteen tens. Now, the only thing that's different is that we've moved over into the tens column, the same pattern (Teacher at board using both hands to show how pattern for 10s/1s is moved over to the 100s/10s). We do the same thing we did before. Now, I want you to work with a partner for this. Now wait. The reason I want you to work with a partner is that you have to get 13 tens and that's a lot of tens (teacher at board pointing to 13 tens in algorithm), so I want you to trade and have for me one hundred, thirteen tens and one unit. Do that for me(pause, children make up this number).

The teacher again illustrates how the analogy between the tens/units principle and the hundreds/tens columns. In the last sentence she has stated the relationships specifically.

T: Right some people are ready. (Teacher moving about class checking children are working in pairs and have enough blocks.) Put the blocks on your card (children using a place value chart). Don't put your answer in, show me that you know (pause).
T: Right, so you should have this number. One hundred, 13 tens and 1 unit (teacher saying number, emphasising each part with a pause between place values). You don't have that and you don't, put it in front of you now. One hundred, 13 tens and 1 unit. (Video shows two more pairs of boys successfully completing this exactly as teacher asked). Put yours in the middle please, so I can see that you're working with .. (teacher insisting children work in pairs, that blocks are equally accessible to each child in pair and that non-relevant block are in box/bank). That's a good girl. No, all of it, so I can see who you are working with. (Teacher has been moving among pupils and now moves to board.)
T: Now I had this number to start (teacher writes 231 on board a short distance away from the 231 in the algorithm). I had two hundred and thirty-one. Now I have another number. I have a hundred plus what? (teacher writes 100+ below the 2 she has just written.) Who can tell me what 13 tens are? Thirteen tens.
S: A hundred and thirty.
T: I have a hundred and thirty plus one (teacher writes 130+1 next to the 100+). Is that the same number or a different number? Is it more or less or the same?
S: The same.
T: I haven't changed the value. The value is the same. I've just traded into this column (points to tens column, and pauses). What do I do next please? (teacher moves slightly away from board). What do I do next? We've said one take away zero is one. Now we have 13 tens (teacher moves back to board, and indicates 13 tens with circular hand movements). How many are we taking?
S: Four
T: You're taking four tens. Everyone do that? Show me in blocks that your answer is there. Don't tell me (pause). How many do you have now please?
S: Nine.

T: Nine tens. Who agrees, they have nine tens? Who doesn't? (teacher writes 9 in answer space) All right, what do I say next please? (lesson continues).

In this lesson, the teacher insisted on particular actions with the MABs, and with the way the algorithm was written down. She was meticulous in her movement about the class, so as to check that students were doing as she asked. Her questions directed the actions and writing patterns she wanted, but also emphasised the relationships within the blocks and between the blocks and the written algorithm. This was also the case in the text below, where another teacher is insisting on specific actions, both with blocks and in writing the algorithm.

T: Trading between not the units and the tens but between the tens and the hundreds. We're going to try that again this morning. You had a go so I'm going to let you do this one with me. Put out your first tray, you'll want five hundred, three tens, wake up, five hundred, three tens and how many units?
T: Seven. Okay, looking at the first column, the units column. Seven take away five, can you do that? Does five go from seven? Okay, take five away and you are left with?
S: Two.
T: You can write this down today, and you can do it out there please. Okay let's look at the tens column. We have three tens and we are asked to take away how many?
S: Eight.
T: Eight. Can you take eight from three?
S: No.
T: Okay, we can't, so what are we going to do? We are going to?
S: Trade.
T: We are going to trade a hundred for how many tens?
S: Ten.
T: Ten tens. We have ten tens and three tens, we join them together and we have how many?
S: Thirteen.
T: Thirteen. Don't forget to write it in. Thirteen tens, write it in and show how we traded out one of our hundreds so that five becomes a seven. All right, we now have thirteen tens and you need to take eight away. We have, let's see how many. Take eight away.
S: I'm taking away.
T: You have already taken one of those away. Did you put the other ten down? Did you have thirteen there? Okay and you took away the eight? How many did you have left?
S: Five.
T: Okay, who had five as their answer? Good and the last column we did have five, now we have four and we take away?
S: One.
T: One. We should have three, write down the answer. So the answer is three hundred and fifty-two.

Here the teacher emphasised each step in the procedure. She insisted that children watch and follow her instructions, continually asked questions to gauge if children understood, and emphasised place value, decomposition and renaming. She went to great lengths to establish the procedure – both written and with MABs, she gave reasons for these procedures, and she clearly described the relationship between the written algorithm and actions on the MABs.

So these texts show, that in the high isomorphism treatment teachers emphasise every aspect of the use of MABs and the written algorithm. Through explanation, demonstration and questioning teachers' show a correct algorithm, an effective use of MABs, and emphasise relationships within the MABs, within the algorithm and between actions on the MABs and the written algorithm. The low isomorphism index teaching approach helps students use MABs in a correct manner, but does not insist on a specific MAB procedure that could lead logically to a written algorithm. This approach encourages the completion of written algorithms correctly, and shows some relationships within and between these activities. Examples were provided of the kinds of language and actions teachers use, particularly in the low isomorphism index approach, and students actions with MABs. A fuller account of students' actions and explanations has been provided in this chapter, prior to this section on teacher transcripts, and more will be provided immediately below.

Further interview transcripts

In the preceding section, it was argued that there was a link between teacher pedagogy, especially as it relates to language and to the use of MAB, and the way students speak about subtraction algorithms, act on MABs and make mathematical meanings. Here are some more examples of students who participated in the low isomorphism index teaching approach. Nadia is calculating 653-472 using a written algorithm.

R:	Which one of those would you like to have a go at. All right, see how you go with that.	Completes written algorithm 653-472
R:	Can you tell me what your answer is?	
S:	Two hundred and seventy-one.	Incorrect
R:	Tell me how you did it?	(answer should be 181)
S:	I put three take away two is one and five take away seven you can't do so I traded and I crossed out the six, made that five and I called that fifteen, fifteen take away seven.	*Identification* Correct procedure
R:	So this has become fifteen. Why did you cross the six out?	
S:	Because you can't do it.	Procedural, not relational
R:	Why did you write five there, why didn't you write four or three or something like that?	*No logical relationship*
S:	Because the one before six is five.	
R:	Now, when you said fifteen take away seven, what did you get for the answer.	*Identification*
S:	Seven.	
R:	Do you think that is right?	
S:	Not quite.	
R:	What is it then?	*No elaboration*
S:	Eight.	
R:	You are not sure? You tell me how you are doing it? Are you using your fingers? Put your fingers up so I can see how you are doing it? Say it out loud.	
S:	Fifteen, fourteen, thirteen, twelve, ten, nine, eight, seven (pause)	
R:	So what have you got left?	Fingers to model subtraction, but
S:	Seven.	incorrectly completed.
R:	I'm confusing you. Let's skip that part. Now when you cross the six out and you've got five there, now what subtraction have you done on the last column?	6 in 100s crossed out, replaced by 5
S:	(inaudible)	
R:	Take your hand away and tell me again.	
S:	I had three take away (pause)	Focusing on wrong position.
R:	No, no you said six take away four and you got two instead of five take away four. So what should that two on the end be?	Redirecting question
S:	Three. One.	Calculates 5-2, then realises
R:	Right. You write one there	should be 5-4

Nadia is able to state the correct procedure, and she carries out most of the written procedure correctly, but makes a mistake in completing the subtraction 15-7. She corrects the mistake once it's pointed out, but appears uncertain as to whether this new answer is now correct. In the hundreds column, she calculated 6-4 even though there had been a trade and the 6 was now 5, and she had written it in the algorithm. The text below indicates she can state what has to be done, so she has declarative knowledge, but there are two errors in her written algorithm.

> I put three take away two is one and five take away seven you can't do so I traded and I crossed out the six, made that five and I called that fifteen, fifteen take away seven.

One of her errors is an incorrect number combination (she says 15-7=7), that is, an arithmetic fact: the second error is more procedural when she says 6-4 in the hundreds column, instead of 5-4. That is, though she is correct declaratively, her procedural knowledge is incomplete and she is not able to operationalise what she says. Further, when asked why she crossed out the 6, her response of *Because you can't do it* refers more to a rule than to any underlying mathematical understanding. This is an instance, then, of a student from a low isomorphism index teaching approach where the low index is likely to have led to an incomplete proceduralisation, and to a paucity of mathematical insights. Reference to the larger text above shows that there are few language extensions, that she has to be prompted to attempt explanations, and that her explanations tend to be statements of procedure rather than statements of relationships. That is, the teaching approach has not assisted this student to move to a correct procedure, and certainly not to a fuller, relational understanding of the steps in the procedure. The text below shows the same student solving a multi-digit subtraction using MAB materials.

R:	You have got five hundred and forty-seven and you want to take away one hundred and sixty-nine. Here is the material, you use the material; you don't have to work it out in your head.	547-169 Works with materials
R:	What are you up to? What are you trying to do? So what have you got in your hand there?	
S:	Seven.	Holding 7 units
R:	Seven. What are you trying to do?	
S:	I took a hundred for that and got ten of them.	Traded 1 hundred for 10 tens
R:	Are these still all in it?	
S:	No.	Unwanted materials increase likelihood of error
R:	So we've got a box of tens here, a seven in your hand there. So now what are we trying to do now?	
S:	Take seven from nine...nine from seven.	*Identification* Corrects herself
R:	How am I going to do that?	
S:	I'm not sure.	
R:	How many have you got there?	Uncertainty
S:	Seven.	
R:	You can't take nine from that can you, so what do you do?	Hint
S:	Trade.	*Identification*
R:	What does that mean?	
S:	Trade nine of them.	Misinterpretation

R:	How many do I trade?	
S:	I don't know.	
R:	What are they?	
S:	Tens.	*Identification*
R:	So how many in one of those?	
S:	Ten. I'll trade one of them for another ten.	*Relationship*
R:	Now what did you just put there?	Works with MABs
S:	Ten. I traded for one of these.	
R:	Where should that one go?	
S:	There.	
R:	Are you sure? Did it come from there?	
S:	Yes.	
R:	Should it stay there? Did you trade already?	
S:	Yes.	
R:	So where does that one belong, in there or over there? I think it belongs there. I think you have already traded.	
S:	That can't be right.	
R:	No, it can't be right.	Uncertainty

In this text, Nadia continues to show her incomplete knowledge of the concept of trading, and her incomplete knowledge of using MABs to complete subtractions. She has not automatised the sequence used with MAB materials, and shows little in the way of understanding relationships. She shows much uncertainty, ***I'm not sure*** and ***I don't know***; she requires a hint to arrive at the need to trade, and then trades incorrectly. That is, not only is she confused, but in both her use of MABs and in the earlier written algorithm, Nadia's methodology reflects the low isomorphism teaching approach where links between MAB and the written algorithm, and mathematical relationships within both actions on the MABs and the algorithm, were mentioned but not investigated in detail.

The low isomorphism index teaching approach used MAB materials where the relationship between actions on the materials and the written algorithm was described in words, pointed out in gestures and illustrated in materials, but was not given the step-by-step detail insisted on in the high isomorphism teaching approach One outcome of such low isomorphism index approaches appears to be a lack of proceduralisation of the use of the MAB. In the above case, the student understands some of the relationships, such as trading 1 hundred for 10 tens, but the use of MABs falls well short of operationalisation and automatisation. Yet this student's scores on the mathematics tests were above average for the low isomorphism group. Clearly, though, the identification of important constructs are limited, there are few relationships recognised and her explanations are brief, rely on rules rather than mathematical relationships, and require some prompting by the researcher.

Her lack of mathematical insight, of relational understanding, was apparent at the retention interview too, even where she correctly completed an algorithm. In the text below, Nadia calculated 746 - 382 as a written algorithm, and obtained the correct answer. She stated that you cannot take 8 from 4, providing a circular explanation rather than giving an example or more mathematical explanation *Because you just can't....Because you can't.* She then reduced the 7 to 6 and placed a 1 beside the 4 to make it fourteen, so identifying the need to cross out a number, reduce it by one, and put a one somewhere else. However, apart from this procedure she sees no relationship between these actions: Q: *is there a connection between those?* A: *no.* She appears to have no understanding of trading, and is unaware of any meanings beyond the procedural.

R: You couldn't do four take away eight, why not?
S: Because you just can't take four from eight, eight from four.
R: Right, you just can't take eight from four, so what did you do then?
S: I crossed out the seven and made that a six and I turned the four into fourteen.
R: Now, why did you cross out the seven and make it six?
S: Because you can't do four take away eight.
R: Right, now what is the connection between this one and that six there, is there a connection between those?
S: No.

These lack of understandings of the procedure appear again when she tries to use MAB materials to answer 651 - 293. She realises she has to exchange materials in some way but seems uncertain as to what to do and what it means. That is, even when she uses the materials in effective ways for part of the algorithm, she appears to have little insight into the mathematical meanings of her actions, and the relationship between these actions and the written algorithm.

What of other students in this teaching approach? Are there similar incomplete understandings? Megan was correct procedurally when she answered questions during the posttest interview, though she made mistakes when asked to use the MAB materials. Her procedures with these materials were incomplete. Later, in the retention interview Megan explains how she completes the written algorithm 746 - 382.

R: Tell me how you did that 746-382?
S: I took two away from six and then I couldn't take eight away from four so I put a one there and I crossed out the seven and put a six and that makes fourteen and then I took away eight and fourteen equals six and then I had six take away three.
R: Okay now this one here, where did that one there come from?
S: From the seven.
R: How can you do that and why can you do that?
S: Because you can't really take away eight from four which would equal nothing instead you get another one from the next number.

Her procedure above is correct, but there seems to be little understanding beyond the processes required by the procedure, particularly as illustrated by her final statement in the text above. She says that you cannot take 8 from 4, that would be nought, *instead you get another one from the next number*. Here she appeals only to the rule, *you cannot...*, and explains the need *to get another one* without indicating why. She does not appear to have a more relational understanding. Not unexpectedly as soon as a zero is introduced, Megan's procedure fails. In 302 - 145 she trades incorrectly, first changing 1 hundred into 10 units, leaving 2 in the hundreds column, then another one of these is traded for 10 tens in the tens column. To use the word 'trade' is a misnomer here. She appears to think she is trading, but she is doing little more than rearranging digits in a systematic, but incorrect manner. She simply writes a 1 next to the 2, crosses off the 3 and writes a 2 above it, and then writes a 1 beside the zero, and crosses off another hundred. So 302 - 145, has become 1 - 1 = 0, 10 - 4 = 6, and 12 - 5 = 4 in the place value columns, so she gives the answer 64. The text of this interview, part of which is shown below, suggested her actions and explanations were mechanical procedures, not calling upon relationships or other mathematical insights.

> S: I couldn't take away five from two so I got one from here and put it there.
> R: Where did you get it from? Just point to where you got it from?
> S: From the three. So I crossed that out and put a two and then I put a one there and that equals up to twelve so then I took away five and that equals four and then I couldn't take away four from nought so I took away one from two and crossed out the two and put a one at the top and a one here and that equalled ten and then I took away four which equals six and then I had to take away one which equals nought. So that equals sixty-four.

Megan was not unusual in this difficulty with zero. There had been very few questions involving zero in the teaching sequences; it was being used as an informal test to see if students could generalise from a typical procedure to a more specialised case. It appeared that students from the low isomorphism index teaching approach had particular difficulties in correctly generalising, from their learned subtraction procedures, to solve algorithms involving a subtraction from zero. Since they had poor insights into the relationships between MABs and the written algorithm, they had little basis upon which to hypothesise how to subtract from zero.

Megan's use of materials was arbitrary and problematic, as the text below indicates. In the retention interview, as she used MABs to answer 651-293, she continually combined a form of trading with adding all the materials she had placed in front of her, then completing a subtraction. In the classroom, materials were only ever placed out for the original number, so subtraction was interpreted as 'taking from this number'. Megan scored only a little less than average for her group on the various mathematics tests, but she clearly has problems with using MABs, and although she seems to know that there has to be some relationship between the MAB and the algorithm, she is inconsistent in proceduralising this. Further, there was no recognition that she was not actually subtracting with the materials, her actions were a - b = (a + b) - b = a, and no consideration that 651-293 was unlikely to be 641.

R: Okay what have you got there?	
S: I haven't got it all yet but I put the top bit which is six hundred and fifty-one and then I've got two hundred (pause) And nine and I've got to get nine of these. (pause) And I've got nine of them and I have to get three there (does this). And now I have to take away three from one	Represents 651 with MABs, then represents 293. Places them all in appropriate place value position. *Identification*

Transcript	Observations
which I can't do so I trade. I have got ten to make up for one of these and then I count them all and it equals eleven and three of these and I have to take away three from eleven. Put them all together, eleven, twelve, thirteen, fourteen. I have to take away three from fourteen which equals eleven.	Trades one ten correctly. Correct declarative. Wrong procedure, she adds all the units, then subtracts three.
R: Right so what do you do with the eleven now?	Pause as she looks at algorithm, seems confused.
R: What did you do just then? S: Well you can't put down one eleven so I put the one there and the other one there so when I count that up there is going to be one more.	Places 1 in answer position, carries second 1 to the 9 in the tens column. Describes tens column.
R: Okay tell me what you did then? S: I didn't have enough. I have got four left so I put them in together and then I have to take away nine. That equals three plus one equals four and then I've got to do the hundreds.	Trades 1 hundred for 10 tens Undoes trade, and adds all tens (9+4), subtracts 9. But misplaces 1 ten, then adds the carried ten by taking 1 ten from bank. Writes 4. Looks at hundreds, finds the misplaced ten, counts out tens from the bank, appears confused.
R: What are you doing now? S: Well there was one over here. R: Why don't we stop that part for a minute. Now you just tell me about the hundreds, we won't worry about that one, just put those down. Tell me about the hundreds.	Puts tens down
S: Well there are six here and two here and so I put them all together which equals eight then I had to take away two equals six.	Adds all the hundreds (6+2), counts 8, then subtracts 2, writes 6. Answer 641

The above children were not special, there were many other misrepresentations, misinterpretations and incorrect procedures associated with the low isomorphism index approach. Presumably children make these mistakes in their everyday classrooms, and teachers correct them wherever possible. These corrections may be at the procedural level, but even if they use MABs or similar materials, the research literature does not give one great hope that typical teaching approaches will alleviate such errors. While the statistical analysis of the data indicated the significance of the differences in mathematics achievement test scores between the high and low isomorphism index teaching approaches, it is these qualitative data that show the form of errors, and the range of misinterpretations.

Glenn (low isomorphism index) misinterprets the subtraction algorithm, then uses materials but is confused, and eventually comes to a correct decision, but not one that will help him understand the relationship between the materials and the written algorithm.

R: What is that asking you to do? (83-37)
S: It's a minus and you've got to take three from seven and that one is five.
R: Good, write the answer there.
S: And then you've got to take eight from three, you can't do it. You can't do it.
R: Now you have to think about this a bit more. Let's start it again. So, it says eighty three take away thirty seven. So what is the first thing you do?
S: Take three from seven and that leaves five and you've got to take eight from three which you can't do.
R: What if you try these, use these and show me eighty three. Can you do that. (Student uses MAB materials)
R: Eighty three, so how many of these have you got.
S: Three.

R: And what are they called?
S: Ones or units.
R: And what are these called?
S: Tens or one long.
R: And how many of these have you got?
S: Eight.
R: Eight of those. Now if you take away thirty seven what do you do?
S: You take three of them.
R: Which of these numbers are you trying to take away?
S: Thirty seven.
R: How do you go about doing that?
S: Three longs (pause...then removes three tens from 83).
R: Which part is that that you have just taken away?
S: Thirty.

That is, Glenn is misreading the algorithm. He is partly correct, you cannot take a larger number from a smaller one, but he has the minuend and subtrahend confused. The materials are of some help, but he still has poor procedural knowledge, in terms of both solving algorithms and manipulating materials in a meaningful manner. Glenn has some declarative knowledge, that is, he can say what he has to do, but cannot apply it. For example he says correctly *4 is more than one so you can't take 4 from one*, but he also says *you trade eight for seven then make the three to 13*, and this shows confusion as to the meaning of trade. That is, Glenn has a limited conceptual understanding of the process and meaning of subtraction. He reads 653-472 as *you've got to take away six hundred and fifty three*, and then proceeds to take the smaller digit from the larger in each place value position, but the confusion continues *I can't take five from seven* meaning *I'm not allowed to*. Immediately after correctly answering a question using blocks, he falls back onto the incorrect procedure of smaller digit from larger. These difficulties and confusions occurred at both the posttest and the retention tests interviews, where he was unable to correctly answer any subtraction algorithms.

Once again, it is clear, as it was in the high isomorphism teaching approach, that there are exceptions, and not all students in the low isomorphism teaching approach were like this. The descriptions above include students with a range of mathematical abilities, and they are typical of the majority of students in the low isomorphism teaching approach. Eric achieved high scores in the mathematics tests in this research, during lessons he frequently expressed his dislike of MABs and all other aids. He simply wanted to get on with the job of completing written algorithms, at which he was very good. In the text below Eric is using MABs to answer 547-169.

R: Good. I'm going to get you to do another question. I'm going to get you to do that one, you'll love this Eric, I know what your answer is going to be to this one. See that. Why did I say that Eric?
S: I have to use the blocks
R: What do you think about that? What did you tell me the other day?
S: I don't like using blocks.
R: You don't like using blocks. Ah, That s right!
(Makes 547 correctly with MABs.)
R: What have you got there?
S: Five hundred and forty-seven.

R: What are we going to do now? What are we trying to do? Tell me first of all what we are being asked to do?
S: I've got five hundred and forty-seven take away one hundred and sixty-nine.
R: Right. So what are you doing with those? (Takes 10 tens from bank, counts out nine and puts them back in bank - his represented the subtraction. There was no trade, should have used units, and there was no attempt to combine blocks from the bank with those already there.)
S: You've got to put them back because you had to take nine away.
R: I see, right.
S: If you put that there you have to cross that one out (the 4).
R: Well you do it, how you think the materials are working (crosses out 4, writes 3, places 1 next to 7.) Where is your seventeen?
S: There (correctly shows position in algorithm).
R: Where is your seventeen down there (in MABs)? Have you got it there?
S: No.
R: You got ten of these out didn t you and you counted ten. What was that for, why did you do that?
S: Because you had to move the n there, because you can't take nine from ten.
R: You keep doing that and I will be quiet. You keep working (takes out 10 tens from bank, combines with 4, subtracts 9. Looks at algorithm, probably calculates 14-6).
S: That should leave eight, but it only leaves five.
R: Right, what is happening?
S: I'm not doing it right.

Eric was clearly confused with the use of materials. He did not recall the correct procedure for trading. Instead of trading he simply introduced another ten, as 10 tens, and subtracted from them. He did not use the correct place value block, did not combine MABs to show decomposition correctly. Further, instead of using the materials to help in the calculation, he checks the calculation to see if the blocks are correct. That is, Eric sees little use for the blocks, does not understand the procedure or the purpose for them, and they appear to be a hindrance to his success. He recognises that he is wrong, but has too little skill and insight with the MABs to help himself.

Zachary is also an atypical member of this group. He tended to ignore what the teacher said, appeared not to see any purpose for MABs, and proceeded with his own idiosyncratic method. He did not link the written algorithm to the actions on the materials. Indeed in the problem solving questions although he wrote an algorithm corresponding to what was asked in the word problem, he did not attempt to solve the algorithm through any written procedure, he simply used tally marks. At the same time Zachary scored well above average in all tests. In a lengthy section of the posttest interview, he was able to use materials to correctly answer 653-472, but had to be prompted constantly as to what the materials showed, and how the algorithm would be written as the materials were manipulated. In a later question, he was able to correctly complete 85-49. He traded correctly, but used yet another idiosyncrasy, by crossing out both the 8 and the 5, then wrote 7 and 15. This continued in the retention test interview. For 73-28, the 7 was crossed out and 6 was written, and the 3 was crossed out and 13 written. 651-293 became 5 14 11 through a series of crossings out and rewriting. He actually describes this process in words in the posttest. For 83-48 he wrote *cross out t(h)e 8 and 3. put 7 w(h)ere the*

8 is. put 13 w(h)ere the 3 is. From a teaching and learning point of view the problem is one of unnecessary symbolic manipulations that may well lead to errors. The problem could be exaggerated, but for students having difficulties in subtracting why support notions that may interfere with effective learning? His description of the problem of the zero in 302-145 is long and complex even if there are many misconceptions.

> I couldn't take two away from five so I had to cross out the zero there. And because there's no numbers with zero I had to cross out the three as well and put a two above the three and put a ten above the zero and cross it out again and put a zero above it. Then you had to cross out the two and put two there then twelve there and then put the answer down there.

Zachary shows some appropriate strategies for subtracting from zero. For example, trading 1 hundred for 10 tens is correct, though he referred to it as *cross out.* He then writes *10* above the zero, which is correct arithmetically, though may lead to confusion with too many digits in the algorithm. Then he crosses out the *10* and puts a *0* instead of nine. That is, his idiosyncratic methods allow him to complete algorithms correctly, but when the procedure needs to vary, or when MABs are used he is unable to provide those necessary incremental changes to his algorithm procedure to lead to the correct answer.

The text below focuses on a different aspect of low isomorphism approaches, it describes the outcome of irrelevancies introduced by teachers so as to 'help' students overcome specific learning or procedural difficulties. Work on Bobis (1992), who showed that aids to learning could in fact make learning more difficult, has been mentioned earlier. In the case of the text below, the teacher taught – or at least the student believed the teacher taught – the trick of underlining the ten that was carried and the digit next to it. The case below is where 1 of the six hundreds is traded for 10 tens, the tens are represented by the digit 1 placed beside the 5, for the existing 5 tens. However, it is written so that the 1 becomes an 'L' with the base of the L shape continuing under the 5.

> R: I just want to ask you another question about this. When you wrote seventeen there, that's the one; now what's that thing there?
> S: We used to do that in first grade.
> R: Did you? Why did you do that?
> S: I don't know. The teacher told us to put a line underneath it.
> R: She didn't explain it. She just asked you to put a line underneath the number.
> S: That's what she used to put on the board.
> R: I see, is that why you did it there again?
> S: Yes.
> R: So you know that's really a what?
> S: Fifteen.

The importance of the above example is that not only do high isomorphism index teaching approaches lead to better learning outcomes than lower index approaches, but that there are more ways to decrease the value of the isomorphism index than there are of increasing it. That is, in the context of subtraction with MAB and algorithms, there are few highly effective teaching approaches, but there is no shortage of ineffective approaches.

In this chapter so far, interviews with students have been analysed, not by describing or summarising what they said or what I think they meant, but through a detailed analysis of the interview transcripts. Some transcripts from lessons have also been analysed, to show the connection between teachers' language and actions and the learning outcomes of students. In particular, it has been argued that what teachers say, together with their use of MAB materials, influences the language of students, the procedures they use and the mathematical insights they develop beyond the procedural. Most importantly, it has been shown that students who participate in different teaching approaches complete subtraction procedures differently, with different degrees of success. These different teaching approaches lead to different learning outcomes, not only in terms of scores on achievement tests, but especially as they relate to the manner in which students talk, and presumably think, about algorithms, MABs, the connections between MABs and algorithms, and the range of mathematical relationships existing within and between algorithms and actions with MABs.

Addressing the research questions posed earlier in this chapter, it appears that participants in a high isomorphism index treatment talk differently, and make different meanings, about written subtraction procedures, than participants in a low isomorphism index treatment. Further, these participants from the high isomorphism index treatment talk differently, and make different meanings about MAB procedures involving subtraction, than participants from low isomorphism index treatment, and seem more able than participants from the low isomorphism index treatment, to relate the use of MABs to the development and structure of written algorithms.

Cognitive processing

The previous section of this chapter concentrated on the way in which teaching approaches influenced students' achievement scores in subtraction and the ways they talked about subtraction. Another variable in this research relates to cognitive processing, so the question arises as to the kinds of discourses used by students, and how these discourses differ among students who process data differently. That is, aspects of simultaneous and successive processing will now be investigated through text analysis. In particular, this analysis will focus on the ways in which students with various combinations of high and low simultaneous and successive processing complete subtraction algorithms, and talk about their understandings of the algorithms, the use of MABs and the various mathematical relationships that exist in such contexts. This analysis will address the following research questions:

> will high simultaneous-high successive, high simultaneous-low successive, low simultaneous-high successive, and low simultaneous-low successive processors differ from one another in the way they talk about, and the kinds of meanings they make of, written subtraction procedures;
>
> will high simultaneous-high successive, high simultaneous-low successive, low simultaneous-high successive, and low simultaneous-low successive processors differ from one another in the way they talk about, and the kinds of meanings they make of, MAB procedures involving subtraction; and
>
> will high simultaneous-high successive, high simultaneous-low successive, low simultaneous-high successive, and low simultaneous-low successive processors differ from one another in the way they talk about, and the kinds of meanings they make of, the relationships between their actions on MABs and their written subtraction algorithms?

High simultaneous and high successive processors

The high simultaneous and high successive processors scored the highest marks on mathematics achievement tests, used in this research to measure subtraction skills and understandings. What can we learn from interviewing these students, particularly with regard to differences from their peers in their cognitive processing? The text below shows Laura completing the written algorithm 653-472 during the posttest interview.

R:	What about that one?	653-472
S:	Three take away two is one. Five take away seven you can't do so you trade, five, fifteen; fifteen take away seven is eight and five take away four is one.	Identifies need to trade Extended response Correct answer 181
R:	Good. Why did you have to trade?	
S:	Because five is less than seven.	*Relationship*
R:	Now when you traded you crossed that out. So why did you cross that out?	
S:	So I could put the five at the top and put the ten from there and make fifteen.	*Relationship*

R:	So why is that five and not four or three or two or something different?	
S:	Because the next number down from six is five.	*Relationship*

Laura is a high simultaneous and a high successive cognitive processor. The transcript indicates she has high procedural knowledge. She is able to describe the process of completing the algorithm briefly and accurately, identifying all the important aspects. That is, she recognises the need to trade, decomposes correctly and writes the algorithm procedures correctly. This appears to be a reflection of her high successive processing ability. She also gives reasons for these steps, and explains why she does a particular thing and not anything else. In particular, not only does she identify mathematical properties (***five take away seven you can't do so you trade***) , but she recognises relationships (***because five is less than seven***), and establishes a logical explanation through the use of ***because*** and ***so*** in various combinations. That is, she recognises mathematical relationships, and in her interview provides extended explanations on various components of the algorithm, without requiring continual prompting. Taken all together, this seems to mean she has a good understanding, a relational understanding, of the algorithm sequence, and the interrelatedness of all its components. It appears that these qualities are possible because of her high simultaneous processing capability. Laura's interview continues below.

R:	You did that easily. Now this time I want you to do this one except I want you to use the material. So what is this one asking you to do?	547-169 with MAB
S:	Trade.	
R:	No, I didn't mean that, I mean what are the numbers?	*Identification*
S:	Five hundred and forty-seven take away one hundred and sixty-nine.	
R:	Right, so you use the materials to do that. If ever you want to write anything down, you can write it down if you want but you use the materials as well.	*Identification*
R:	So what have we got there?	
S:	Five hundred and forty-seven.	Works with MAB
R:	Have you got enough space there to work that out? Do you want to put them any other way or is it all right like that? So we have got five hundred and forty-seven and what are we going to take away?	547 with MAB
S:	One hundred and sixty-nine.	
R:	So how are we going to do that?	
S:	Take one of these away and you have to trade... there's only forty there so you have to get one of these and get ten of these.	*Identification*
S:	Then you get three hundred and seventy-eight.	
R:	Three hundred and seventy-eight. Now I would like you to do it without that.	*Relationship* *Elaboration*
S:	Eight, three take away six you can't do so you have to trade again. Thirteen take away six is seven and four take away one is three.	Works with MAB Correct answer
R:	Is that the same number you got with the material?	Repeats question as a written algorithm.
S:	Yes.	
R:	It is, good.	Succinct, accurate

In this text, Laura has been asked to complete a similar question to the first, except this time to use MAB materials. She did this quickly and correctly, her procedures appear to have been automatised: able to be interpreted as another instance of high successive processing. Laura appeared very able to describe her working processes, and provided extended explanations. For example, she first described what to do *Take one of these away*, then gave it its technical term *trade*, then unasked, provided a reason for this *there's only forty there so...* This high language proficiency, and the giving of explanations beyond the immediate answer to the question asked, seems a feature of high simultaneous processors. Laura easily identified each step in the MAB process, she recognised the need to trade, and described the process. She identified facts or skills, she recognised relationships, and she elaborated on these: all signs of high level simultaneous and successive processing. When she repeated this second question using pen and paper only, she had no hesitation in predicting that her answer was the same as when she used MAB materials.

Later, in the retention interview, her written algorithm ability remained high. In the transcript of a section of this interview shown below, she appeared to use efficient procedures, and raced through the question. In this case she may have gone too fast because she made a subtraction error (11 - 3 = 9). This is a minor point in comparison with the correctness of her procedure, and in the light of her understanding of mathematical relationships. She also used what would seem to be a unnecessary crutch. When she traded she wrote a small '10' above the number to where the ten was taken, as some form of a reminder. Unlike other students and other hints and crutches, Laura knew precisely why the number was there and used it correctly. That is, it did not prevent her being procedurally correct, and automatising the procedure. This recognition of the crutch's purpose was a further instance of her high simultaneous processing ability.

R:	Just tell me out loud what you did?	651-293
S:	Well, one take away three you can't do so you cross the five out and put that into eleven and put the five as four and eleven take away three is nine. Four take away nine you can't do so you cross the six out, put the four as fourteen, put the six as five fourteen take away nine is five and five take away two is three.	*Identification* *Relationship* *Elaboration* Error with 11-3=9 Answer 359, should be 358
R:	So two things. Where did this one come from here to make that eleven?	
S:	From five.	*Relationship*
R:	Now, where did the one come from here that made this fourteen?	
S:	Six.	*Relationship*
R:	Okay. What are these up here?	
S:	I just put that as ten because I take ten of them away from there.	Effective invention of crutch
R:	So is it just to remind you, is that why it's there?	
S:	Yes.	

Laura's long sentence as she described the procedure, presented as if a chant, was not unique to her. It was also used by other students, and tended to reflect high successive processing. What made Laura and other high simultaneous processors different from the high successive processors was the understanding of mathematical relationships that Laura and students like her exhibited. They knew both *how* and *why*. From Laura's interview we see that she was able to identify specific instances in an algorithm where a particular step has to be taken, and this appears to be a reflection of a high successive processor. She was also able to recognise relationships, and not only identified them but explained them by reference to mathematical or logical relationships, and this seems to be a characteristic of high simultaneous processors. The speed with which she completed an algorithm, and her high level of accuracy were reflections of both high simultaneous and high successive processing.

(Jarrad, 746 - 382)	
R: Can you tell me, just do it again and tell me what you have done?	Completes 746-382
S: Six take away two equals four and four take away eight you can't do so you cross off the seven and put a six. Then you put a one next to the four, and then fourteen take away eight leaves six. And six take away three is three.	Correct procedure
R: Why did you cross out the seven?	
S: Because you can't take eight from four so you cross off the seven. You give another one to the four too so you can take eight.	*Logical relationship* Extends answer
R: What is this six here?	
S: Six hundred.	*Identifies fact*
R: What is the one here?	
S: That's a ten.	*Identifies fact*
R: So this says six hundreds; how many tens are there here?	
S: Fourteen.	*Identifies fact*
R: Good.	

The qualities identified from Laura's transcripts were not uniquely hers. Other high simultaneous and high successive processors illustrated similar qualities. Consider the previous transcript where Jarrad is solving the written algorithm 746 - 382 during the retention interview.

Jarrad was detailed and accurate in his description of the subtraction procedure. He worked quickly, and his descriptions and answers to the researcher's questions showed no hesitations. He identified mathematical facts or steps, described relationships in the algorithm and provided extended descriptions. These were the same characteristics shown by Laura. Jarrad then attempted a harder question, a subtraction from zero. This question was typically answered very poorly by students during the interview. Some were unable to take any significant steps towards achieving a correct answer, others were able to generalise from the knowledge of the algorithm procedure and the interrelationships between the steps. Jarrad was one of those who was able to generalise correctly.

R: Tell me what you did again?	Completes 302-145
S: You can't take five from two. And you look at the nought and you can't take nothing away from the nought so you cross off the three and one less is two so you put two there and you make the nought into a nine and put a one next to the two to make a twelve and then you can take five away.	*Identification* Recalls procedure for subtracting from zero *Logical relationship* implied by use of *so*
R: When you cross off this two, there's got to be one left somewhere. Now why do you write nine there?	
S: So that I can put the nine there and then I can still carry the one there.	More than procedural, by use of *so...I can still*
R: So what is this nine here and that one there? How are they related?	

S: I don't know.	
R: Is that what the teacher told you to do, is that what you mean?	
S: Yes.	Recalls procedure not relationship

Jarrad has developed a correct procedure for completing subtraction from zero: decompose across the zero, make the zero 9 and carry the other 1 to the next column where it becomes a ten. This procedure combines the two steps of decomposing a number, taking a ten to the next column on the right, then decomposing that ten into a nine and a one with the one being taken to the next column on the right where it now becomes a ten[2]. Jarrad has automatised this response, and this is likely to reflect his high level successive processing. At the same time he is not perfect, he can not remember the relationship between the nine and the one. However, his language suggests that he is operating at a level beyond the procedural, particularly when he uses *so*, which implies a logical relationship, and when he says ***I can still***, which implies understanding of the process.

Jarrad attempted the same question with MAB materials. The text below shows his knowledge and confidence. He immediately changed the hundred for 10 tens, then 1 ten for 10 units. His descriptions are full and accurate, he provides answers beyond the minimum without prompting, the *now I'm trading* shows a confidence that he knows what he is saying and doing, it shows command of technical terms and suggests knowledge beyond the procedural.

R: Now you take those and I'm going to cover your answer up so you can't use that at all. Now I want you to do it again but this time I want you to use the materials.	302-145 with MAB
R: Tell me what you are doing Jarrad?	
S: I'm trading a hundred blocks for 10 tens	*Identifies* name and action
R: So what number have you got there now?	
S: Three hundred and two. I've got 2 hundred blocks, 10 tens and 2 ones. Now I'm trading the ten for 10 ones. Now I take away five.	Extended explanation.
R: Before you take away five would you put those five back for a minute. Now what number have you got there now?	
S: Two hundred and... three hundred. Three hundred and two.	
R: You've got three hundred and two. Now you shouldn't be surprised at that because that is what you started with. Now if you had to write down what you've got in front of you now by crossing down or however you do it, you write it down what you think you've got there now. Do you know what I mean?	
S: No.	Cannot generalise decomposition to zero
R: You know how before you crossed out; you said you had a three, you crossed it out and you wrote something down. What I want you to do is to do that, so you write down here what you think these numbers are. So you crossed out the three. What should you write there? So show me where that two is? Now what is that number?	Crosses out 3, writes 2 *Identifies* the 2 hundred MABs
S: Nine.	and the 9 tens

[2] For those who thought subtraction from zero was easy, or that it was easy to teach by telling students what to do, consider the complexities of the language here. Place value labels are given then modified, a ten becomes a nine and a one, and then one becomes a ten.

R: What are they?	*Identifies* they are tens
S: Tens.	
R: What have we got here?	*Identification*
S: Units.	
R: Good. Now doing the subtraction using the equipment, what is your answer?	Correct
S: One hundred and fifty-seven.	Writes answer
R: You write the answer down. Now if I were to lift this up is it going to be the same answer?	Confident
S: Yes.	

Of course not all high simultaneous and high successive processors showed these capabilities. For example, Casey is able to give only procedural reasons in explanations; she does not give reasons involving mathematical relationships. Here is Casey explaining her method for 547 - 169.

> S: You can't take nine from seven so you have got to cross out the four and put a three. You put a one next to the seven.
> T: Now why do you do that?
> S: Because if you try and take nine from seven you can't do it. But if you do it then you come up with the wrong answer.
> T: And why did you cross the four out and put a three?
> S: Because you go to the next column to get more out

She explained that you cannot do 7-9, *Because if you try and take nine from seven you can't do it.* Further, she continued in this vein of calling up rules *But if you do it then you come up with the wrong answer*, rather than providing a mathematically-based explanation. At the same time, Casey described the correct procedure, and apart from a simple number bond error would have obtained the correct answer to 547 - 169.

> T: That's good so what is the next thing you do?
> S: You've got to take six from three but you can't do it because there's not enough up there. So you have to cross out the five and put the four down and put one, then take six from thirteen which is eight. And then you put eight down and you can take one from four and it comes to three.

Casey was able to explain clearly, but she made errors in the interview, and many more in the actual tests. Eric is unable to use the MAB materials effectively, he prefers not to use them. His high scores reflect his skill with algorithms, probably due to successive processing, since his understanding of mathematical relationships is not good. Even though there are the kinds of exceptions described above, in general, students who are both high simultaneous and high successive processors will have highly automatised procedures. They will explain their actions by referring to mathematical relationships, will be able to explain why as well as how, and they will be able to link the use of MAB materials to written algorithms.

High simultaneous and low successive processors

If the group of students who were high simultaneous and low successive processors are now examined, what characteristics will they display in interviews concerned with solving subtraction algorithms? Take the case of Natasha. In the posttest interview, Natasha showed she was able to complete algorithms correctly, and was able to use MABs effectively. The text below begins with Natasha having just completed 653 - 472, and obtaining the correct answer 181.

R: Now Natasha, tell me how you got the eight in the middle down the bottom?
S: I crossed off the six, put a five at the top and turned that into fifteen.
R: Why did you cross the six off?
S: Because you can't take seven away from five so I crossed off the six and turned it into a five.
R: Why did you turn it into a five and not say a four or a three or something different?
S: Because that's the number that was left.
R: So where did this one come from there?
S: It came from the six.
R: It came from the six. When you do that, do you remember what that is called?
S: Renaming.
R: Renaming and there is another name too. Do you remember that one?
S: Trading.
R: Good girl.

The text shows that Natasha could complete algorithms and give appropriate reasons for her actions. In the same interview she showed a good knowledge of the use of MAB materials, and an understanding of the mathematical relationships they represent. In the text below, Natasha is part way through completing the algorithm 547 - 169, when she described the procedure correctly.

R: So tell me what you have done so far. You had five hundred and forty-seven and then you made some changes. What have you done so far?
S: I traded one of these for ten of these, and then I traded one of these for ten of these.

At the retention interview Natasha was able to use the materials effectively, but had some procedural difficulties with written algorithms, as is shown in the following text.

R: Can you tell me what you have got in front of you there now?	746-382
S: Seven hundred and forty-six.	746 in MAB
R: Good, what do you have to do now?	Identification
S: Take away three of these and eight of them and two of these.	Identifies the subtraction
R: Off you go and do that.	correctly
R: Hold on, just before you do that, what have you got there now?	Using MABs
S: I've got fourteen there.	14 tens
R: You've got fourteen. How did you get fourteen there?	
S: Well, I took away one of these flats and I got ten longs and now I'm taking away eight of them.	Identification and explanation Writes 360

R:	So you go ahead and do that, all right.	
R:	You read out your answer for me.	
S:	Three hundred...	Uncertainty
R:	Which is your answer?	
S:	I haven't done it yet.	Not identifying step she is up to, or all MABs she has
R:	No, you are right, it's just that that is your answer.	
R:	How many units have you got there?	
S:	No units.	Not identifying
R:	Are you sure?	
S:	Four units.	Identification
R:	Can you write a little four there next to that? Now do you reckon you are right?	Writes 364
S:	Yes.	
R:	Well, you are right. That is the right answer. I'm just going to ask you some questions. We had seven hundred and forty-six. Now here's what you did. You said I'm going to take away three and then you said I'll take this one and change it for ten of these. There they are. Now is there any way that you know of writing that down along here somewhere?	Not recognising relationship between MAB and algorithm
S:	I don't know.	
R:	Well, I'll show you. I want you to do this again, I'll just take that away but this time I want you to do it just in your head with the pen, no materials.	746-382
S:	Four hundred and forty.	Not identifying procedure
R:	Now you tell me how you got that?	Has taken smaller digit from larger
S:	I took six away from two which is four and...	
R:	What have you realised?	Hint
S:	I forgot to trade.	Identifies error

In this text, Natasha has generally used the MAB materials efficiently, but has some procedural weaknesses when it comes to the algorithm. In particular, even though she had the correct answer, she did not immediately recognise it, which is surprising for high level simultaneous processors, who generally are good at seeing the relationship between materials and algorithm. Further, when asked to repeat the question as a written algorithm, she fell back upon an earlier incorrect procedure of subtracting the smaller digit from the larger, in each place value position. She did not immediately recognise her error, but was given a hint, then recognised it was a subtraction requiring trading. This shows that Natasha had not automatised the subtraction procedure, yet she was quite good with materials. A plausible interpretation here is that Natasha's low level successive processing capacity has an impact on her ability to remember and reproduce the procedures necessary to solve algorithms, and has hindered her in automatising such procedures. Yet her ability to use the MABs relatively well, suggests that her high simultaneous processing capabilities come to the fore here.

Trent performed well on the mathematics tests, but dropped to his pretest score on the retention test. He had some errors that continued throughout the teaching sessions, for example in 653 - 472 he said the reason for the trade, and there had to be a trade, was *you can't take 5 away from 7*. That is, he read the subtraction the wrong way round, and later he read 9 - 4 in the tens column as *9 take away 4* but comments *you can't do it*. His procedures were sometimes confused. During the posttest interview, when asked to read a word problem, he read it out aloud, and was able to immediately write the corresponding algorithm (85 - 49), which he solved efficiently with MAB materials. This is shown in the following text, where he once again said the subtraction of single digits back to front. He appears to operate well with the MABs, and has insights into the relationships between the numbers they represent and his actions on them, but he continued to have problems with the algorithm procedure.

R: Let's see you do that with the equipment. Start with eighty-five.
R: What number have you got there?
S: Eighty-five.
R: What number are you going to take away?
S: Forty-nine.
R: So you take away forty-nine How are you going to do that?
S: You can't take five away from nine so you trade ten for ten units (pause) now you can take nine away from five.
R: Tell me that again.
S: Now you can take nine away from five.
R: How many are really here?
S: Fifteen.
R: Do you know why there are fifteen there? How many did you start with?
S: Five.
R: What did you do then?
S: Added ten.
R: So now you have to take away how many?
S: Nine.
R: So you take that away. So how many units have you got left now?
S: Six.
R: Six, good so what else do you have to take away now?
S: Seven fours.
R: Sorry, what are you taking away?
S: Four from seven.
R: So what do you think the answer is?
S: Thirty-six.
R: Look, that is what you got before so you must be right, good boy.

He recognised that he had to take 49 from 85, but then said *you can't take 5 away from 9 so you trade ten for 10 units*, and later he referred to 7-4 as *seven fours*. Yet his use of MAB was efficient. That is, he appeared to understand the use of MAB, and the mathematical relationships that come from action on them, but he had some procedural difficulties related to the meaning of the algorithm. It seems high level simultaneous processing allows him to use MABs effectively: he understood how to operate on them and from this capability generated further understandings. However, his low level successive processing made it difficult for him to complete algorithms correctly, and to automatise either the algorithm or the use of MABs. These points are also illustrated in the text below, taken from his retention interview.

R: That's fine. Now let's try this.	746-382, written
Okay good Trent, why did you cross that out?	Crosses out
S: Because four take away eight you can't do.	Identification
R: So what do you do?	
S: You cross out the seven and add a six on the top and then put a one next to the four.	Identification
R: Why did you write six there? Why don' you write five or nine or something like that?	

S:	Because you have taken one from that and given the one to the tens	Relationship
R:	Good. Now what if I asked you to do that one except I want you to use that.	302-145, MAB Works with MAB
R:	Now tell me, what is it you are trying to do? What have you got there?	
S:	Three hundred and two and one hundred and forty-five.	
R:	All right take me through it and see if you can do the subtraction.	Works with MAB
R:	Now stop for a second and tell me exactly what you have done so far.	
S:	You can't take two away from five and you can't take zero away from four so I took out a hundred and traded it for ten tens and then traded one ten for ten units.	Identification Says 5-2 and 4-0, means opposite Correct use of MAB, correct
R:	Now if you were to write down here what you've got there, what would you write down?	decomposition
S:	I would cross out the three, then add a two, then make it twelve and twelve take away five.	Problems with written algorithm
R:	Hold on, what about this? Does this say what you have got there? Is this what you have got there?	Ignoring MAB to get written procedure. Given hint.
S:	No.	
R:	What else is there then?	
S:	Nine tens.	Identification

In this transcript, Trent was able to correctly complete the algorithm 746 - 382. He traded as was required, and explains *You cross out the seven and add a six on the top and then put a one next to the four*. This is a procedural explanation, so the researcher asks *why*, and Trent replies *Because you have taken one from that and given the one to the tens*, which has some hint at mathematical relationships. Trent then appears to have no difficulty in decomposing across a zero with MABs. This probably implies confidence in the use of materials, but more importantly an understanding of the materials, the relationship between the different blocks and the manner in which they are used to represent mathematical constructs. That is, Trent's use of the MAB here was likely to reflect his high simultaneous processing capacity. At the same time he continued to say things such as *you can't take zero away from four*, which indicated he still had problems with the procedural aspects of subtraction algorithms, and which I take to mean reflects his low level successive processing.

In this interview, Trent achieved a very positive outcome for the 302 - 145 question. He uses the materials and recognises the need to trade a hundred for tens, one of which he traded immediately for units. He has decomposed sufficiently so that he can now subtract. It is most likely his high level simultaneous processing capacity that allows this invention. It appears that if he has to solve a number problem, he works quite well with MABs, but he seems not to have automatised this procedure. Moreover, he may regard working with MABs as a problem solving activity; seeing each question as an individual problem to be solved. His solutions to these problems would be assisted by his high level simultaneous processing. He has two difficulties here: the one relates to a lack of automatisation of the MAB procedure; and the other relates to his inability to deal with written algorithms efficiently. His low level successive processing may hinder his ability with MABs, and account for a lack of automatisation in both the written algorithm and in MAB procedures. For example, as Trent continued with this problem, he again illustrated his difficulty in recognising a subtraction and dealing with it.

R: This says nine subtract four, so what are you trying to take away?
S: You can't do it...
R: Yes you can.
S: I'm trying to take four from nine.
R: Exactly. So take four away.
R: Good, what are you left with?
S: Five.

The researcher states the question (9 - 4), Trent claims you cannot do it. He is apparently confused, and has to be reminded by the researcher that you can. He tries to work out how this can be done, then removes 4 MAB units. That is, he is inconsistent in recognising the points when trades are necessary, and this is part of the bigger picture of his weakness in procedures.

From these two examples (Natasha and Trent), it seems that high simultaneous processors who are also low successive processors, will generally be quite able when using materials, and will often give elaborate and correct descriptions and explanations as to what they are doing. However, their low successive processing may lead to mistakes in the procedural aspect of what they do. For example, even though they give the correct description for using MAB materials, they may not actually put that description into practice, and without materials they may perform written procedures incorrectly.

This generalisation is also correct in the cases of both Jessica and Nadia. Jessica used MABs correctly and appears to know the relationships shown in and by action on them, and she described the procedure correctly but completed written procedures incorrectly at times when she took the smaller digit from the larger in the particular place value column. Nadia appears to use her high simultaneous processing to elaborate on her answers, and she described procedures correctly but did not always complete the sequence correctly. For example, in 653 - 472 she traded the 6 hundreds correctly, to make the 5 tens 15 tens, and the 6 hundreds became 5 hundreds. However, then she said 15-7=7 in the tens column, and in the hundreds column instead of 5-4, she calculated 6-4.

Ann knew the procedure, the relationships within the procedure, and the relationships between the MAB materials and the written procedure. She consistently had near-perfect scores on the mathematics tests. Ann is the exception, though, rather than the rule. That is, in general, the combination of high simultaneous and low successive processing means that these students have some procedural weaknesses, but they have a good understanding of mathematical relationships. That is, they can explain *why*, but may not be so good on the *how*, since they make procedural errors. They may be able to see the conceptual links between actions on MABs and written algorithms, but again may produce errors here, because of a lack of automatised procedures.

Up to this point, text data concerned with those students who were high level simultaneous processors and high level successive processors has been examined, as have the transcripts of those students who were high level simultaneous processors and low level successive processors. The analysis so far suggests the possibility of a pattern where high simultaneous processors are able to build mathematical relationships, particularly through the use of MAB materials, which assist them in building knowledge about algorithms by making links between the materials, their actions on them and the structure of algorithms. It seems that these students are more likely than their peers to understand *why*, and are more likely than others to recognise and correct errors If they also have high successive processing then the MAB and written procedures are likely to be automatised, they understand *how*. If they have low level successive processing, they will not be so good at automatising procedures and will, therefore, not be so good on the *how* aspect of algorithm solution.

Low simultaneous and high successive processors

Cases of low simultaneous processors are now investigated: firstly when this processing is combined with high level successive processing, then with low successive processing. Here is the case of Marie, who is attempting the written algorithm 547-169 during the posttest interview and obtains the answer 398 She begins well by taking one of the four tens to the units column, making the 4 into 3, and the 7 units into 17. Or as she says *You go over to the four and you take a ten off there and put it with the seven and call it seventeen.* She mentions the terms *regrouping* and *renaming*, but does not identify *trading*. When asked for a reason for doing this she was silent.

R: So you think the answer is three hundred and ninety-eight. The first thing you did was you looked at this and you wrote a three (above the 4 tens) and a one there (next to the 7 units). Now can you tell me why you did that?
S: Because seven is a smaller number than nine.
R: So you can't take nine from seven so what do you do then?
S: You go over to the four and you take a ten off there and put it with the seven and call it seventeen.
R: Now what is that called when you do that?
S: Regrouping, renaming.
R: There is another word too. Can you think of another name? You said regrouping didn't you?
S: Yes.
R: And you said renaming as well. That's good. So now why did you (pause) you had five here (5 hundreds) and you wrote four above the 5), so why did you write the four there?
S: Because (pause).
R: You are right. Now I'm just asking you the reason. There was five there and you write a four here and why did you do that?
S: Because the four you can't take away from six.
R: So you had a six here and a four here. What about this three?
S: That's for the seven.
R: So what subtraction are you doing in this column here? Are you doing four subtract six or three subtract six?
S: None. I'm doing fourteen.
R: You are doing fourteen subtract six? So when you wrote four here what happened to the rest of the five? You had five and then you wrote four, why did you do that?
S: Because I had to take a ten off the five and put it with the four to make it fourteen and then I had to take fourteen.
R: Good.

The researcher reassures her, *you are right*, she says *the four you can't take away from six*, which has the digits reversed but identified the correct place value column. She knows that in someway the 3, renamed from 4 tens, is connected to the seven units, and says *that's for the seven*. The next question gives her a hint, *Are you doing four subtract six or three subtract six?* She then says *I'm doing fourteen*, and reasserts this in the next question. She then subtracts 6 from 14 and writes 9 as her answer. In this case the student has some of the procedure correct, she has the "general" idea but makes errors, but she seems to have little understanding of why her sets of actions are appropriate. That is, her only procedural error is calculating 14-6 instead of 13-6; she has the correct digit in the units and in the hundreds position. She is able to explain the steps in the procedure *Because I had to take a ten off the five and put it with the four to make it fourteen*; but it is a procedural explanation not one calling on mathematical relationships. In the next section of the interview, she repeats the question 547-169, but using MABs this time.

R: Now I want you to put out five hundred and forty-seven in the materials. (Pause while child works.)	547 represented correctly with MAB.
R: Now you've got five hundred and forty-seven. Now I want you to take away one hundred and sixty-nine. (Pause while child works.)	Takes extra 10 units from bank, without trading.
R: How many have you got there all together?	
S: Seventeen.	The ten from the bank and the 7 units.
R: Are you sure you have counted them? (Pause while child counts.)	
R: So how many are there all together now?	
S: Seventeen.	
R: Where did these ten come from?	Points to bank
Do you have to do anything else?	
S: Gotta take away 9	
R: Do you want to write that part of the answer down?S: Eight.	Researcher hinting at need to write algorithm
R: Now keep going with the subtraction.	
S: I've got a ten.	Counts out 10 units, adds 10 units to 4 tens, subtracts 6
4 take away 6	
14 take away 6	
R: So what do you think you have done?	
S: (inaudible)	Takes 10 tens from bank, does not trade. Researcher intervenes
R: You put the ones back.	
So how many tens did you get there?	
S: Ten.	
R: Now when you took the ten tens out from here where did they come from?	The bank
S: Here.	Answer 488 (incorrect)
R: Keep going. Good. So that time you got four hundred and eighty-eight, all right?	

Marie recognises that 7-9 in the units position creates a difficulty, she knows the 7 has to become 17, so she goes to the bank for an additional 10 units. However, she does not trade. This is repeated later when she recognises the difficulty with 4-6, and goes to the bank for 10 tens. There are two problems for Marie here. Firstly, she has a procedure but it is incorrect, and secondly she has no understanding of either what trading is or of its proceduralisation.

In taking into account both of the above transcripts, in looking at her class workbook, in considering her above average test scores and in looking at specific items from these tests, Marie has considerable ability with written algorithms. That is, she frequently completes written algorithms correctly. In particular, she was generally correct in questions involving three digit subtractions and trading. The 14-6 instead of 13-6 is likely to have been a chance error or nervousness at being interviewed, rather than a regular strategy. Her test papers and her class workbook suggest she has automatised the subtraction procedure. That is, in spite of the algorithm errors in the interview, Marie seems to use her high successive processing skills to enable her to correctly answer subtraction algorithms. The second transcript shows something different though. It strongly suggests that Marie has little knowledge about the use of MAB materials - she has not automatised the procedure, and is unable to use the materials effectively. This seems to be a case of low simultaneous processing being insufficient to interpret the use of MAB, particularly with regard to recognising the materials and actions on them as linked with an algorithm and its procedure leading to a correct solution.

At the retention interview Marie completes the written algorithm 651-293 correctly, except she says 14-9 is 6 instead of 5, but her actual procedure is correct. This gives some weight to my interpretation of her as an effective solver of written algorithms, through automatisation of the procedure.

R: Now can you tell me out loud how you did it?
S: One take away three you can't do, because the one is smaller than three, so you go over to the five and cross that out and you say that is four. Then you take the one ten and you take it over to the one and make it eleven. And then eleven take away three is eight and then four take away nine you can't do it. So you cross out the six, take one ten and make that fourteen; fourteen take away nine is six and five take away two is three.
R: Good. Now can you tell me where the five is crossed out. Why did you cross the five out?
S: I had to cross that out because the five (pause) I had to take one ten from the five to make the one unit.
R: That's a good reason.

Here she states the whole and correct procedure for 651-293, in one long uninterrupted and non-prompted description. At the same time, she does not use correct place value, referring to *one ten* instead of *ten tens*, and her explanation is procedural. That is, she seems to be saying you do it because you have to, because that's the way to get the correct answer. Her explanation makes no reference to mathematical relationships. Her errors with materials are repeated in the transcript below, also taken from the retention interview, as she tries to complete 746 - 382 using MABs.

Dialogue	Observations
R: Now just stop for a second and tell me what you just did then. S: Four couldn't take away eight so I took one of these and got ten and that was fourteen. R: You took one of those and got ten of them back, is that right?	Represents 6 units, subtracts 2 writes 4 in answer sheet. Represents 4 tens (long pause, looks at question, then blocks). Trades 1 ten for 10 tens from the bank. Counts out 8 tens from the 10, places 1 of the remaining 2 tens next to the original 3 tens to make 4. Writes 4 in answer position, ignores the fifth ten.
S: Yes. I had fourteen and then I took eight away. (pause)	She actually had 13.
R: What is the next step now? So how many of those did you have? S: Seven. R: And how many did you take away?	Counts out 7 hundreds, then subtracts 3.
S: Three.	Writes 4 in answer space.
There should be only three.	Alters 4 to 3 in tens answer position.

Marie has just completed a written subtraction algorithm and described the procedure correctly. However, as soon as the materials were used she made errors. For example, instead of representing 746, she represented only the 6 units. She correctly subtracted 2 and wrote the answer 4 in the units place. She next represented 4 tens, but recognised the problem of subtracting 8, so exchanged 1 ten for 10 tens. That is, she was not using correct place value, and not trading correctly. This may have been exacerbated by her decision not to represent the whole number initially. She subtracted 8 tens from the 10 tens, but then appeared confused. She joined one of these two remaining tens to the 3 she originally had, making 4, and ignored the other ten, and wrote 4 in the answer position. The data in this retention interview confirm my earlier interpretation. That is, Marie appears able to automatise and complete written algorithms successfully because of her high successive processing skills, but her use of MABs, particularly as to the meanings that her actions have on them and the manner in which they relate to the written algorithm, is ineffective because of her low level simultaneous processing.

Here's Emily calculating 653-472 as a written algorithm during the posttest interview, to obtain the correct answer 181. Emily is a better than average student in terms of her ability to complete subtraction algorithms correctly. Her test scores were much higher than average, and her class workbook showed many correctly completed algorithms.

R: Now I want you to tell me how come that one is there (1 unit in answer)?
S: Because three take away two and you have one left.
R: Now why did you cross out that six?
S: Because you can't take seven away from five because five is less than seven, so you cross off the six and put a five and put the one next to the five, now it's higher and you can take away.
R: So when you do that can you think of a word that describes that?
S: Renaming.
R: Renaming, that's right. Now when you crossed off the six here (6 hundreds), how come you didn't put three or four or two?
S: Because that wasn't the number before the six.
R: So it's got to be the number just before the six, does it? So when you cross the six off you write five there, now what happens to the rest of the six where does it go?
S: You don't have it, it's crossed off.
R: I crossed the six off and I write five and there is one left over. So what has happened here? What has the five changed to (the 5 tens)?

S: Fifteen.
R: So it's fifteen. So how did you get eight there (8 tens in answer)?
S: Because fifteen take away seven is eight.
R: Now you have got one there (1 hundred in answer), you had six here but six take away four is not one is it? I'm tricking you here. So how come it's a one?
S: Because five take away four is one. That is the number that is there now.

Emily has automatised the procedure effectively. She can explain in procedural terms each step in the sequence. She also knows information beyond the procedural, for example, she correctly uses the term renaming, and identifies the need for trading *you can't take seven away from five because five is less than seven.* At the same time, her explanations do not involve place value statements or mathematical relationships. This may be too harsh an interpretation, so I will turn to another section of her posttest interview, where Emily is using materials to calculate 653-472. Her answer of 141 is incorrect.

R: You had six hundred and fifty-three, you took away the two and it gave you one, so now you want to take away that seven don't you? So what are you going to do now?
S: Take one of them away (a hundred...seeks reassurance).
R: You tell me all right.
S: And get three of these (tens).
R: Why are you going to get three of them?
S: I forget how to do this (nervous laugh).
R: You are right. There are lots of different ways you can do it; so you have done this part of the subtraction haven't you (units), so you don't need that any more, so now you are trying to take seven away but you have only got five haven't you, so how do you overcome that?
(long pause: then trades 1 hundred for 5 tens, combines tens. There are now 10 tens, subtracts 6, leaving 4 tens.)
S: Then you have to get one, two, three, four, five hundred (subtracts 5 hundreds).
R: Now what do you think your answer is there now?
S: One hundred and forty-one.
R: Now look at what you got before.
S: One hundred and eighty-one.
R: Which one is right?
(pause)
S: That one (181).
R: Why do you think that one is right?
S: I don't know. Because I forgot how to do that (MABs) so I probably did it wrong.
R: Which one, this one or that one?
S: This one (MAB).
R: Yes you made a mistake in that one (MAB). That one is right (181).

In the above transcript, Emily has misused MABs. She traded incorrectly and was not sure what the materials represent, or how her actions on them moved toward the completion of the problem. She admitted to forgetting, she took some inexplicable steps, and had little confidence in her final answer. Her confusion was still evident at the time of the retention test. Even though she was able to complete written algorithms correctly at this time (for example, 746 - 382= 364), her work with MABs created confusion and error, as can be seen from the transcript below where she was trying to answer 302 - 145.

R: Okay tell me what you have got. Tell me what you have done so far.	Puts out 302. Hesitantly puts 1 hundred into bank, takes it back. Pauses, puts 1 hundred into bank, takes back 9 tens and 5 units. That is, takes 5 from 100, by putting 1 hundred in bank and withdrawing 9 tens and 5 units. Gathers 4 units from 7 remaining.
S: I put in the three, the one hundred and I got ninety-five out.	
R: Why did you get ninety-five out?	
S: Because I took away five.	
R: Oh, OK. (pause)	
R: What are you trying to do now?	
S: That one.	0 - 4
R: Wait a minute, now you have taken the five away; you put a hundred in, you got ninety-five out and took the five away, right. How many units does that leave you with?	Refocussing question
S: It should leave you with seven.	
R: Have you got seven there? Well you write the seven part down in the answer	Nods yes
S: There.	Suggests tens, immediately corrects herself, writes 7 in units.
R: Is that the right spot? (pause)	
S: I don't know.	Uncertainty with block place value.
R: Come on yes you do, what are these?	
S: Units.	
R: Right, is that units. You are right. Good, good. Now what is the next thing you try to do?	Nods yes
S: Take away four from the zero.	Identifies correct procedure
R: Okay so how do you go about taking away the four?	
S: I don't know because there should be zero tens and two (units) but there is some tens left.	Forgetting earlier actions on blocks. Uncertainty
R: Okay but how come there are some tens there. What did you do before to make the tens come back?	
S: I made a mistake.	She is correct but does not recognise it.
R: You didn't actually.	

Here we have a low simultaneous processor again confused with manipulation of the MABs. She did not automatise procedures with the blocks, made mistakes with trading, forgot earlier actions on the blocks, and was uncertain of what she did. Later in this retention interview she again failed to remember an earlier trade, and so was confused when the blocks had altered from their original number and position. When prompted she recalled the trade. At the end she did not think her answer was correct, but it was.

Here is Emily during the retention interview, explaining how to answer 302-145 as a written algorithm, prior to the above transcript with its use of MAB materials.

S: Well two take away five you can't do it, so I crossed off the zero and put another zero because it can't be anything else, and put a one to the two so you can take away five from that number. And then zero take away four you can't do that so I crossed off the three and put a two and then I put the one under the zero to make ten so I can take away four from ten.
R: Right. Now tell me again about when you crossed out the zero and put the zero up here?
S: Well I couldn't take away two from five so I had to cross off the zero and put the zero there.
R: Why did you do that?
S: Because you can't do anything else.
R: But where did this one come from?
S: I don't know.

She was not completely correct, but her procedure was generally accurate, except for the trading across zero. The point here is not how nearly correct she may be, it's the quality of her explanations, explanations which were rule-based *(b)ecause you can't do anything else*, or a failure of explanation *I don't know*. So Emily also falls into the category of a student who scores well on written procedures, because of her high level successive processing, but one who does not have well developed insights into mathematical relationships. In particular she does not use MABs efficiently, does not see their purpose, and is unable to relate them to written algorithms - all characteristics which have been argued previously as reflecting low level simultaneous processing.

Do other low simultaneous and high successive processors also have these kinds of learning outcomes? Consider the case of Scott who scored above average on each of the three mathematics achievement tests. He is able to complete a range of questions successfully, but his language is not elaborated. He answers questions in a brief manner, is procedurally correct, but does not describe all the relationships that exist. His answers are typically one word, one phrase, one sentence, whatever is the minimum response.

T: tell me again why that five is crossed cut. (the 5 in the 100s)
A: because I'm putting the 100s into the 10s
T: so why is that four, and not three or six or seven or something like that?
A: because you took the 100s and add them into the 10s and now that is four.
T: And this number is not just three any more is it? What number is that?
A: 130
T: and this six isn't really just six, what is it?
A: 60

The exception for Scott is when he is asked to explain how to complete a procedure. During the retention interview he described the completion of 651-293.

> One take away three you can't deduct so you cross out the five and make five a four and give a ten to the units and make it eleven. And then you say eleven take away three, and eleven take away three is nine and then you add four take away...you take away four take away nine and you can't do it, so you cross out the six and make it a five and give a hundred to the four and make it a hundred and forty and then you say...sort of like the units way, four take away nine is five.

Scott arrives at correct answers but his explanations contain errors. That is, his high successive processing allows him to complete the procedure, but his low simultaneous processing limits his ability to build a complex cognitive network where he can call on mathematical relationships, or models of materials when he is unsure of how to move along in a procedure. This view of an incomplete network is supported by Scott's misuse of MAB materials during the retention interview where he arranged the materials in an idiosyncratic manner, and where he did not trade. That is, he scores well because he is a high level successive processor, but he does not have the complex cognitive network that high level simultaneous processors are likely to have.

Finally, Damien relies on fingers to perform all subtractions, and scores below average on the mathematics tests. He completed some procedures correctly, and was partly correct in others. His use of MABs was inconsistent. He appeared unable to make use of his high successive processing skill in the completion of algorithms. It may be due to his lack of number combinations, so he has constantly to interrupt the procedure's flow while his short term memory is used up in calculating number bonds, and perhaps forgets the sequence of the procedure. Further, he had low level simultaneous processing, and so was unable to effectively use materials or relate their meanings beyond the materials themselves.

From these data, it appears that low simultaneous and high successive processors are likely to have automatised written procedures, but are unable to make effective use of MAB materials, and are unable to establish relationships between the materials, their actions on them, and written algorithms. They may do well in mathematics tests, but this will be a result of their high successive processing. If there are problems with a procedure, they are unlikely to be able to develop a novel sequence since they rely on memory of learned procedures.

Low simultaneous and low successive processors

Students in this category scored least well, of all the categories, on the various mathematics tests. What can we learn from the transcripts of such students as to why this occurred? Consider the case of David who scores a little below average on the various tests. There is a number of interesting aspects to David's work. In the pretest algorithms, he frequently subtracted the smaller digit from the larger, simply disregarding place value. In later tests, this strategy was less frequent, but within adjacent algorithms he would answer one using a correct procedure, and in the next revert to his smaller digit from the larger strategy. In the problem solving questions, after he had written the algorithm corresponding to the word problem, rather than solve the algorithm, he would literally use a tally where one stroke was used to represent each number in the minuend, and the appropriate numbers were then crossed off. For example, in the problem where a milk truck delivers to 96 houses, but has only been to 59, he calculated how many remained by writing ninety six strokes, and crossing off fifty nine of these. His class workbook showed that his algorithms were correct, and that trading has been used, at least according to marks on the page.

Here he is completing the written algorithm 547-169 and obtaining the correct answer (169), during the posttest.

R: Eventually when you crossed out the four (4 tens) you had a three here didn't you? So what was the next subtraction you had to do?

S: Three take away six you can't do so I crossed off the five, put a four and put a one next to it and take away six equals seven.

R: Now when you cross five off why do you put four? Why don't you put three or two or one or something like that?
(pause)
S: Because the four goes there and whatever you put a one next to it, whatever you crossed off.
R: What is that called, sometimes people use a special name for it? You cross the five, you write four and then you move one over there. What is that called?
S: Trading.

In the above text, it seems that David has a good understanding of the procedure. He recognises the need to trade, carries out the correct written actions and obtains the correct answer. However, when asked to explain why the 5 hundreds became 4 hundreds, and not some other number, there was a delay before he responded, and his explanation was *(b)ecause the four goes there*. This is not an explanation, but reference to a rule or statement which is used to support itself, so the argument is circular. It almost certainly indicates that even though David was able to recall and identify the correct instance of *trade*, he had little meaning of the concept beyond the written action. He was asked to repeat the question, but to use MABs. The episode is transcribed below.

R: Now I want you to use the equipment to do this and as you use the equipment I want you to write down what you are doing. Do you know what I mean?	Works with equipment. Takes tens from box, counts out 5 tens.
R: Can you tell me what you are doing?	
S: There are four tens.	
I'm getting 5 tens, 4 tens and 7 tens.	Incorrect, ignores place value.
R: Now can you tell me what that number says?	
S: Five hundred and forty-seven.	But has used tens to represent all digits.
R: So what is that five? Five what?	5 hundreds in question.
S: Five hundred.	
R: What have you got? (pause)	
S: Five hundred.	Has five tens.
R: Have you? Now what does that four say?	
S: Four tens.	
R: How many have you got there?	
S: Four.	
R: Now what is different between that four and that five?	
(pause)	Uncertainty

R: We'll do it another way. Let's just leave that for a minute.	
Now what's the seven?	7 units in question.
S: Units.	Immediately replaces 7 tens with 7 units, smiling as he recognises error.
R: What have you got there?	
S: Tens.	Pointing to units in question.
R: What have you got there now ?	Pointing to tens
S: Seven units.	
R: What have you got there?	Pointing to hundreds
S: Four tens.	Immediately replaces with 5 hundreds
R: What have you got there?	
S: Five tens.	Correct
R: Now tell me the number you have got, all up?	
S: 547	
R: How come before you just put all the tens out?	
S: I got confused.	

The first problem here for the student is that he used MAB materials as counters, 5 to represent 500, 4 to represent 40 and 7 to represent 7. That is, he had not remembered that there are different blocks for each of these place values. However, it is not as simple as this; this incident also suggests that he had little knowledge of the materials and how they related to algorithms, and that he had little conceptualisation of what the materials or his actions on them meant. This interpretation is supported by the lengths the researcher had to go to, the hints given and the time taken, to have the student recognise that his use of tens only was an incorrect method for representing 547. The text below illustrates the next stage of David's subtraction (547-169).

R:	Let's see you take one hundred and sixty-nine away now? (Pause)	Works with materials. Counts 6 backward on a ten. Puts a ten in bank, takes out 4 units. Likely to be 10-6=4. Puts all units together, now 11. Counts 9 of these units, and
R:	I don't think I can follow that, can you do that again? Let's start again.	places them in bank. This is subtracting the 9 units indicated in the question.
	There's your five hundred and forty-seven.	Placed by researcher.
	Now you tell me what you are doing next, what is the next thing you are going to do? I'm not saying you are right or wrong, I just didn't follow what you did.	Places 1 hundred in bank.
R:	So the one you took away there. Was that the one there, is that it?	The 1 hundred to be subtracted.
S:	Yes.	So working from left hand side.
R:	So what are you doing now? Are you taking away six now?	
S:	Yes.	Counts 6 from a ten, ignores place value in calculating 10-6.
R:	Six away there (from a ten) and you put four back there (in units), is that what you did?	Has 4 units, joins them to the 7
S:	Yes.	units.
R:	Right, now what are you going to do?	
S:	Seven take away nine.	Counts 9 units from pile of 11.
R:	Good; now you write your answer down for me.	Writes 432

The student has a nearly consistent procedure here, but it is full of errors. For example, in the first attempt, he began by subtracting 6 from 10: to calculate the tens place digit, then to use the leftover units in the units column. When asked to repeat the question, he subtracted the 1 hundred first, then carried out the same erroneous procedure as he had completed at the first attempt. This attempt is wrong in so many ways: it reflects little knowledge of the MABs and little knowledge of applying place value concepts. It is difficult to believe that this student had just completed several weeks of instruction using MABs for subtraction algorithms. The MAB and actions on them seem to make no sense to him whatever. He was then asked to read out a problem, write a corresponding algorithm, and complete the calculation. He wrote the correct algorithm (85-49), completed the algorithm efficiently and gained the correct answer (36). However, when asked to use MAB materials, he made the same errors as before.

R: Good, now this time I want you to make eighty-five with this material for me
(Takes 8 tens from the bank).
R: What have you got there?
S: Eighty (pause, then takes 5 units from bank) five.
R: So you have got eighty-five. What do you have to take away?
S: Forty-nine.
(Counts 9 off 1 ten, puts the ten into the bank, and brings back 6 units:
so has calculated 15-4=6. This means there are 11 MAB units now)
R: So what have you taken away so far?
S: Four.
R: What else do you have to take away now?
(Counts 9 units from units pile.)
S: Nine.
R: So what is your answer there now?
S: 72

There was a pause between making 8 tens represent 80 and then 5 units to represent the 5. That is, he remained confused about place value with MABs. He used 1 ten to subtract 4 units leaving 6 units, and this was arithmetically correct, but he thought he had subtracted the tens. He counted *one, two, three,...nine*, when he used the MAB ten block to subtract 6, so his thoughts are further confused.

Although during the interview his written subtraction procedures were generally correct, assessing his knowledge is problematic. His class workbook contains very few errors, but this is not the case in the mathematics tests; perhaps he received a good deal of help in class, or he looked at the work of others, or perhaps he does not do well in tests. There seems little doubt, though, that he lacks knowledge of MAB materials, what they represent and how they may be used. In addition to using a ten for calculating simple subtraction combinations as demonstrated above, he also regularly used his fingers, and regularly failed to recognise correct place value both in algorithms and in MAB. In the later retention interview, the student continued to exhibit the same inconsistencies and errors. Here he is calculating 651-293 as a written algorithm.

R: Tell me why you crossed this five out and wrote four?
S: Because I was trading and one take away thirty you can't do so I crossed off the five and put a four with the one next to it.
R: Okay this four here, is that four hundreds, four tens or four units?
S: Four tens.
R: Okay this one here the big one there (the 1 in the question), is that one ten, one hundred or one unit?
S: One unit.
R: Now this one here (the 1 ten that was traded from the 6 tens), that little one, is that a ten, a unit or a hundred?
S: A unit (it's 1 ten).
R: What is this number here altogether?
S: Eleven (fails to see the inconsistency in 1 unit and 1 unit making 11).

Although he completed the algorithm quickly and correctly, the text illustrates his lack of understanding. He seems to explain correctly, but made errors in place value identification, and does not recognise his own inconsistencies. He was then asked to answer 746-382 with MABs, and he calculated 436. That is, he continued to misuse the MABs.

R:	Tell me what you have in front of you there now.	Works with materials. Places 3 hundreds in bank (subtracting the 3 hundreds in question), then takes them back. Places 2 units in bank (6-2). Uses 1 ten, counts 8, places in bank, brings back 2 units, places with other 4 units to make 6 units. Then places 3 hundreds in bank, counts answer, and writes 436.
S:	746	
R:	So now you have to take away. So how are you going to do that?	
R:	You write the answer down here. Let's go back a step and we'll have 746. You check, is there 746?	Researcher makes 746 again.
S:	Yes	
R:	When you are taking 382 what do you take away first?	
S:	Two from the units.	
R:	You take two from the units so you put them somewhere.	Places 2 units off to the right, but not in the bank.
	If you take away 8 what is left?	Counts 8 from a ten, takes out 2 units, places with existing 4 units. Pointing to 4-6.
R:	So what has happened in that one?	
S:	I've taken that one away.	
R:	Right so it has gone.	
	S takes 3 hundreds to bank.	
R:	Have you got the same answer both times?	
S:	Yes.	

David continued to use a ten as a calculating aid, and did not realise that he was subtracting 8 units rather than 8 tens. His actions continued to be consistent but wrong, and he appeared to make no connection between his written algorithms and his use of MAB materials. He acted as though the materials were an exercise in themselves, for their own benefit. It appeared, by both his words and actions, that the meanings he associated with the use of the materials, and the meanings he associated with the written algorithm, were not in the least related.

How do other low simultaneous and low successive processors handle these kinds of written algorithms, and the MAB materials? Glenn appeared to have declarative knowledge, but could not apply it. He did not link the materials to the written algorithm. He was confused as to how to read an algorithm, and generally subtracted the smaller digit from the larger in any place value position, ignoring the whole number. Glenn's workbook showed digits out of their place value position in algorithms, and there were very few instances of trading marks. His language was narrow and unlinked, he made no elaborations and described no relationships.

In the post test interview, Glenn was troubled by 8 - 3 in the units column, saying *you've got to take 8 from 3, you can't do it, you can't do it.* That is, he read 8-3 as 8 from 3 which led to uncertainty. Later in the retention interview 3-8 appeared, which he read as

R What do you do first?
S Take away 8 from 3 but you can't do it (pause).
R So what do you do?

(pause)
S Trade.

He has read the question correctly, but did not realise the solution. After a considerable pause, and a prompt, he responded *trade*. This took place as Glenn answered a word problem leading to the algorithm 73-28, and as he used MAB materials to help answer the question. He obtained the correct answer, but when asked to repeat the question using the algorithm only, he was unable to proceed. Even after prompting by the researcher, where Glenn said he traded with the blocks, he was unable to continue. This confusion is prevalent throughout Glenn's work. For example, in the posttest he appeared to be adding-on in algorithms, since there were no trading marks, and even though he correctly described the trading process in the explanation section of the posttest, he did not operationalise it consistently. In some of the posttest questions he traded correctly, but in others he took the smaller digit from the larger.

It seems, then, that the reason low simultaneous and low successive students score poorly on subtraction tests relates to their inability to operationalise procedures correctly. Their low level successive processing means they are inconsistent in their procedures, adopting correct procedures at times, but incorrect procedures at other times. Further, their low level simultaneous processing seems to limit their capacity to understand the role and use of MAB equipment. In effect, such students seem to separate written algorithms from actions on MAB, as though the two were quite independent.

These kinds of errors were also produced by Chris. For example, in his workbook there were a number of cases where he had taken the smaller digit from the larger, so that 31-25 = 14, but there were also many cases where trading was completed correctly. In the posttest, he sometimes took the smaller digit from the larger in the particular place value column, so that 503-75=572, but more frequently he adopted the strategy 'small digit subtract large digit is zero', so that 563-385=200, and this mix of strategies continued at the retention test. At the posttest interview he was uncertain in the use of MAB materials, particularly when and how to trade with them. Here he is answering the vertical algorithm 83-37, where he obtained the correct answer (46). He attempted this question immediately, with no hesitation, traded immediately, and then used fingers to calculate number bonds.

R: Tell me again. You crossed this out, now why did you cross it out?	The 8 tens
S: Because you can't take away three from seven.	In the units column, but read back-to-front.
R: Now when you crossed it out you wrote seven there and one there, why did you do that?	
S: Because I had to make this number larger than the bottom number.	Recognises mathematical relationship, the need to make the 3 units larger than the 7.
R: Right, so you put that one there, so that number became what then?	A 1 next to the 3 units.
S: Thirteen.	Correct.
R: Thirteen. Now why did you write seven there? Why couldn't I write five or six or nine or something like that?	

S: Because you take away a ten from the eight to put with the units.	Recognises relationship.
R: And how much is that going to leave there then?	In the tens
S: That leaves seven.	
R: Seven, so if this had been five and I crossed it out, what would I have to write up there?	Establishing principle, by conjecturing that the 7 was a 5.
S: Four.	
R: Good boy, that's well done.	Correct

The point to note here is that the student traded quickly and completed the algorithm correctly. He used his fingers to calculate individual subtractions, had automatised the trading procedure and appeared to have insights into the structure of the algorithm. He described the mathematical relationship of trading a ten for 10 units. He was procedurally correct and a teacher would have some confidence in believing he had learned effectively. His level of understanding of place value, and of the structure of the algorithm, was explored further in the interview.

R: This seven isn't really just seven is it?
S: No.
R: What is it?
S: It's a seven tens.
R: So what could we call it?
S: Seventy.
R: Seventy.

The student was able to correctly identify the place value of the 7. The student has declarative knowledge, in that he is able to state what has to be done, and appears to have procedural knowledge. However, in explaining 653 - 472 in the following text, the paucity of his explanations suggests otherwise.

R: Good, now, here it said five take away seven. Why is five take away seven different from three take away two?	
S: Because five (pause) I'm not sure.	Not establishing important mathematical relationship.
R: I think you are right, I mean you just tell me because you were doing it correctly so you obviously know. So here you say three take away two is one and here you say five take away seven and you go "no I can't, it's wrong" so why is it different?	
S: Because five you can't take away from seven and three and two you can.	Poor expression of mathematical relationship, suggests confusion.
R: So what do you do next then?	
S: You cross out the six and change it into a five and you get the ten and put it with the five, then you get fifteen.	Correct procedure. Certainly has declarative knowledge.
R: So now there's five here; is it five or fifty or five hundred or five thousand?	Interviewer changes focus to place value.
S: It's fifty.	It's 500, wrong place value.
R: Now that one there, is that a one, a ten or a hundred, what is it?	
S: It's a ten.	It's 100.
R: So this number here is fifteen, all right, but there is another name for that number. Is it fifteen or is it fifteen hundred or is it a hundred and fifty. What else could it be?	The 5 tens together with the carried 10 tens.
S: Five units.	Incorrect, uncertainty.

It could be argued that the student is confused and that the interviewer is increasing this confusion. However, the point of the interview is to find out, in detail, what the child knows and how he processes subtraction concepts and algorithms. He has finished several weeks of instruction, and is familiar with the interviewer, who several times jokes with him, and chats about non-mathematical topics. This uncertainty about place value suggests he completed algorithms as procedures, not as meaningful mathematical activities. Indeed, he appears to have automatised the procedure, but relies on remembering the procedure rather than any deeper level of understanding. He has not automatised the use of MAB materials, and remains confused as to their meanings. Here is a description of his actions as he used them to calculate 547-169.

> Represents 547 correctly;
> (long pause);
> crosses off 4 tens in written algorithm, writes 3;
> takes 1 ten to bank, brings back 10 units;
> (pause).
> Counts 17 units
> writes 7 in units place in answer space, together with small 1;
> writes 1 next to 6 tens in written algorithm (has carried 1 ten);
> calculates ten column, 3-16=10, by addition.

Not only is the procedure not automatised, since there were pauses between steps, but his actions were incorrect, with the materials, in the written algorithm, and in the confused way he related one to the other. At the completion of this series of actions, the interviewer intervenes, and has the student repeat the question.

R: So now we've got five hundred and forty-seven and we are going to take away that nine, so what do we do?	547-169
S: We have to trade.	Recognises relationship
R: So how many units have you got there?	
S: Ten.	Trades, but has 9 units
R: So you did the trade, good. What happens now?	
S: I have to trade one ten for one hundred.	Trades 100 for 10 tens, now has 400+130+16
R: Now you had three, didn't you?	Interviewer wants student to check work.
S: Yes.	
R: You had four and then you traded so there were three there. Now how many more should there be here when you trade one of these? (Pause, uncertainty)	That is, the 4 tens became 3 tens and 10 units.
S: Ten.	
R: Are there ten there?	
S: Yes.	
R: You count them.	
S: There's nine.	
R: I don't want to trick you or anything because you are getting it right. You had four to start off with didn't you?	
S: Yes.	
R: Then you did a trade.	

The student is confused, but the discussion continues as the interviewer attempted to have the student identify the correct procedures. The student was uncertain of the procedure for using the materials, continually wanted to add instead of subtract, confused the trading process with the subtraction process, and wrote digits in their incorrect place value positions. He apparently saw no connection between his actions on the MAB and what he wrote.

R: Now look, there's five there. You can't take six from three so what do you do?	
S: Trade.	An immediate response. Declarative knowledge, well learned: that if you cannot subtract, you trade.
R: You trade. When you trade, what do you do?	
S: Take away.	Incorrect mathematical relationship.
R: What do you actually write when you trade?	
S: What you are taking away and what you come up with.	Incorrect mathematical relationship

The student has good declarative knowledge, as soon as the interviewer says *you can't take 6 from 3 so what do you do?* the student immediately responds *you trade*. This reflects the extent to which this declarative knowledge has been learned, but he has not proceduralised the knowledge. For example, there is a clear instance in the interview, where he thinks the trading process represents subtraction, rather than a preparation for subtraction: in response to the question *When you trade, what do you do?* he answers *Take away.*

That is, Chris is uncertain as to the appropriate strategy, either in the written algorithm or in the use of MAB materials. This confusion is also apparent in his posttest and retention tests. That is, low successive processing seems to limit his capabilities in completing written algorithms and in using MABs, and his low level of simultaneous processing prevents him from understanding the relationship between MABs and written procedures. This combination of cognitive processing also appears to increase the likelihood of inconsistencies in procedures; particularly the situation where a student's strategy will alter from one problem to the next, in an illogical and unconnected manner.

The exception to the above low simultaneous and low successive processors is Zachary whose class workbook is well set out, with algorithms correctly aligned according to their place value. He has an idiosyncratic way of showing the subtraction symbol, on the right hand side separating the numerals in the units place. There is some indication that he trades correctly in written procedures, but often there are no trading marks; and he appears to have adopted another strategy here, perhaps adding on. Zachary has not related the procedure in the written algorithm to the use of MAB materials. In problem solving instead of writing and solving algorithms he uses tallies. In the text below, at the retention interview, Zachary is calculating 302-145 as a written algorithm.

S: I couldn't take two away from five so I had to cross out the zero there and because there's no numbers with zero I had to cross out the three as well and put a two above the three and put a ten above the zero and cross it out again and put a zero above it. Then you had to cross out the two and put the two there then twelve there and then put the answer down there.
R: So what you said was that you crossed this three out, you wrote two then you wrote ten there. Then you crossed this ten out. Now what did you write once you crossed that ten out?
S: A zero.
R: Why did you write a zero?
S: Because if you subtract one ten from that it goes to zero and you put the ten there.
R: I see, so this is ten and you are taking that ten and putting it over there, is that right?
S: Yes.
R: Good. Now do you reckon you are right?
S: No.
R: Where do you think you might not be right?
S: Like I've got...it's meant to be a hundred there, it's not actually a one hundred, it's just meant to be...
R: So the one you cross out here, is it one unit, one ten or one hundred?
S: One hundred.
R: Now that ten there?
S: I know what I did wrong. I should have put a nine there where the zero was instead.

He completed the question, but did not accept the answer; there seemed to be something wrong. He recognised his error: he had traded incorrectly, crossing off the 3 hundreds and making it 2, then crossing off that 2 and making it 1. The 0 tens became 10 tens, then 0 tens, then 10 tens. He repeated the question, then obtained the correct answer, but the manner in which he identified the error suggests he viewed the activity as a procedural one, not a conceptual one. His statement *I should have put a nine where the zero is* can be interpreted as a recall of a procedure, not a place value or trading concept. Later in the same interview, he correctly used MABs to answer 651-293, and was able to explain the trading process.

Once again, though, while acknowledging the exception of Zachary, the more typical low simultaneous and low successive processors had declarative knowledge, but lacked procedural knowledge. That is, they made procedural errors on a regular basis. Because they generally had not been able to automatise the procedure, they had difficulty with the *how* when it came to solving algorithms. Further, their lack of simultaneous processing capability made it problematic for them to understand *why* algorithms were structured and completed the way they were. They were generally unable to see any significant mathematical relationships that linked written algorithms and actions on MABs, they were not likely to recognise and correct errors.

In considering the research questions posed earlier in this section, it appears that high simultaneous-high successive, high simultaneous-low successive, low simultaneous-high successive, and low simultaneous-low successive processors differ from one another in the way they talk about, and the kinds of meanings they make of, written subtraction procedures; in their talk about MAB procedures involving subtraction, and the kinds of meanings they make of their actions on these materials; and in the way they talk about, and the kinds of meanings they make of the relationships between their actions on MABs and the corresponding written subtraction algorithms.

In this chapter, text data has been analysed: data gathered from students through interviews conducted immediately after posttest and retention tests, and data gathered from teachers interacting with students during class lessons. The patterns in the data seem to suggest that, at least for the topic of subtraction, teaching approaches used by teachers are reflected in the manner in which students complete and talk about subtraction questions. Different teaching approaches lead to different ways in which students complete subtraction questions, act on concrete materials and talk about both written and manipulative actions. Students appear to use language similar to the language used by the teacher in the class; differences between teachers' language about subtraction are reflected in differences in student talk and explanation.

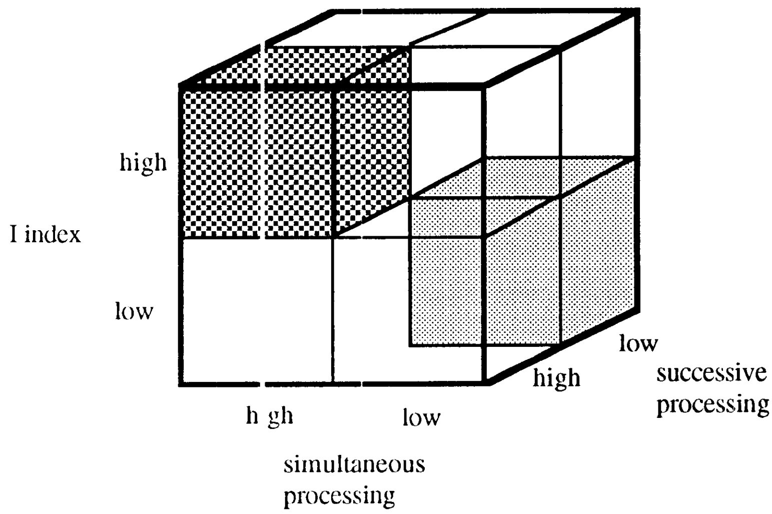

Figure 9.1: Some patterns of interactions between cognitive processing and high and low isomorphism index teaching approaches.

In terms of cognitive processing abilities, it appears that the high simultaneous-high successive processors will have automatised procedural knowledge, will be able to use MAB materials effectively, and will understand the relationship between the materials and the target written subtraction algorithm. The high simultaneous-low successive processors will have some procedural weaknesses, with both MAB and written procedures, but will have insights into the relationships between MAB representations and actions on them, and on the written algorithm. Low simultaneous-high successive processors will have automatised procedures, but will have poorly developed understandings of mathematical relationships. They are likely to be able to effectively complete algorithms but will have limited insights into the link between MABs and algorithms, and what the algorithm actually means. Finally, low simultaneous-low successive processors will have declarative knowledge, will have difficulties with procedures, and will have few insights into mathematical relationships within written or MAB procedures, and between these representation systems. Some of these patterns are shown in Figure 9.1.

Figure 9.1 shows a 2-by-2 cube, where the three dimensions are the level of isomorphism index (I index), the level of simultaneous processing, and the level of successive processing: and each dimension is measured as either high or low. This figure shows, in effect, all the possible combinations of these three variables: and is an initial attempt to represent their interactions diagrammatically. The shaded area at the front-top-left represents high-simultaneous and high-successive processors participating in the high isomorphism index teaching approach: these are the most able students in terms of scores on subtraction tests, and in terms of the mathematical meanings they make, as reflected in their language. Whereas the data in this research is clear about this set of interactions, the data is not so clear about the back-bottom-right shaded area. This area represents low-simultaneous and low-successive students participating in a low isomorphism index teaching approach. Certainly, the low-simultaneous and low-successive combination of cognitive processing led to students with the lowest scores on subtraction tests, and to those least able to make meaning of mathematical relationships: but that this occurs largely in the low isomorphism index treatment, rather than in the high isomorphism index treatment, is less certain.

The extent to which these findings will occur in other mathematics topics and with students of other ages is uncertain, and clearly provides an area worthy of further research. Also worthy of further investigation is the way dimensions of cognitive processing and levels of isomorphism index treatments interact. For example, there are patterns in both the quantitative data and the text data, suggesting that understanding the relationships within and between the MAB representation and the written algorithm representation, requires high simultaneous processing. There is the suggestion that low simultaneous processors will find it difficult to make the necessary cognitive connections that are the very essence of a high isomorphism index treatment. Further, the data hint that high-level simultaneous processors were not able to effectively apply this cognitive capability in low isomorphism index treatments.

Levels of successive processing appear to be more related to the completion of a procedure, than to the understanding of relationships. So a student with low simultaneous processing has the possibility of completing algorithms effectively through a high level of successive processing. Further, low simultaneous-low successive students are unlikely to be able to complete procedures correctly, and will have few insights into mathematical relationships. These findings can be seen only as a possibility, there needs to be further investigations into these relationships.

Synthesising quantitative and qualitative results

Results from an analysis of the quantitative data gathered during this investigation have been reported in Chapter 8, and the present chapter has reported an analysis of the qualitative data. There is now a need to synthesise these findings to see the extent to which the two sets of results complement one another.

The results reported in the chapter concerned with quantitative measures initially focussed on the possibility of applying the procedural analogy theory to mathematics instruction, with the results giving some support to the value of using the procedural analogy theory when planning instruction. The results indicated cases of a high isomorphism teaching approach leading to better learning outcomes than a low isomorphism index teaching approach. The qualitative data in the present chapter support the existence of these patterns. Interpretations based on the qualitative results seem to suggest that, at least for the topic of subtraction, teaching approaches used by teachers are reflected by students in the manner in which they complete and talk about subtraction questions. In particular, students from the high isomorphism index teaching approaches appear to act in more efficient ways when using concrete materials, and their talk about both written symbols and manipulative actions seem to involve greater understanding of mathematical relationships, than is the case for students from the low isomorphism index teaching approach. Further, all students appear to use language similar to that used by the teacher, so that the detail of instruction given by teachers in a high isomorphism index treatment leads to greater understanding and clearer explanation by students from the high isomorphism index treatment than for those in the low isomorphism index treatment.

Analysis of the quantitative data gave little support to the possibility that different patterns of competence on the two dimensions of cognitive processing lead to differences in scores on subtraction tests, the results were therefore unable to clarify the roles of simultaneous and successive processing in mathematics learning. It may be that these different cognitive processes lead to different kinds of learning outcomes, for example, high levels of successive processing may lead to correct algorithms, but high levels of simultaneous processing may lead to a more complete understanding of the process. Or it may be that both simultaneous and successive processing are needed, but at different times, for example, successive processing when learning a new procedure, but simultaneous processing when relating this procedure to other mathematical knowledge. Conclusions drawn from the quantitative results are unable to clarify these possibilities, but interpretations of the qualitative results were much more informative.

The qualitative results indicated that there were patterns related to dimensions of cognitive processing, the ability to effectively manipulate and make meaning from concrete materials, the extent of mathematical understandings related to subtraction and the quality of students talk and explanation concerning subtraction. For example, students who were both high simultaneous and high successive processors had highly automatised subtraction procedures, were able to explain why as well as how, explained their actions by referring to mathematical relationships, and were able to link the use of MAB materials to written algorithms. High simultaneous and low successive processors exhibited some procedural weaknesses, but demonstrated good understanding of mathematical relationships. They were able to explain why, but made procedural errors. They demonstrated conceptual links between actions on MABs and written algorithms, but lacked automatised procedures.

Low simultaneous and high successive processors had automatised written procedures, but were unable to make effective use of MAB materials, and unable to establish relationships between the materials, their actions on them, and written algorithms. They may do well in mathematics tests, but this is likely to be a result of their high successive processing. If there are problems with a procedure, they are unlikely to be able to develop a novel sequence since they rely on memory of learned procedures. Finally, typical low simultaneous and low successive processors had declarative knowledge, but lacked procedural knowledge, and were not able to automatise procedures. Their lack of simultaneous processing capability made it problematic for them to understand the structure of algorithms, and made it difficult for them to link written algorithms and actions on MABs.

These results support an earlier interpretation, which suggested simultaneous processing appeared necessary for understanding the structure of subtraction algorithms, and successive processing supported their efficient completion. Further, this may clarify the lack of differences in subtraction test scores between high simultaneous-low successive and low simultaneous-high successive processors. It seems that high simultaneous students use their understanding of structure to complete algorithms correctly, whereas high successive students rely on remembering the procedure. Each group have different strengths that allow them to be successful in subtraction algorithms, but they are unlikely to be as successful as high simultaneous-high successive processors who both understand the structure of the algorithm and have automatised its completion.

The quantitative results from the present investigation generally indicated that there were few differences in the various interactions between teaching approach and dimension of cognitive processing, but there were exceptions. These exceptions, together with some qualitative results suggested the possibility that high isomorphism teaching approaches favour, or require, high simultaneous processing, and that high simultaneous processors are unable to use this capability in a low isomorphism teaching approach. Consequently, the quality of learning high simultaneous-high successive processors achieve, in a low isomorphism index teaching approach, may be inferior to their high simultaneous-high successive processing counterparts in a high isomorphism index teaching approach. However, this relationship is far from certain, and points to the need further investigation rather than to any certainty.