Chapter 3

Concrete materials in mathematics education: Philosophy and research

Introduction

For quite some years now, and especially since the 1960s, mathematics educators have emphasised the need to teach and learn mathematics through the use of concrete materials. Many researchers and teachers have suspected that the abstract nature of mathematics is a major cause of learners' difficulties. The pedagogical thoughts behind this embodiment idea come from a range of views including the widely-accepted principle that human beings learn best when they are active, that acting on materials is important to learning, that structure is the essence of mathematical knowledge, and concrete materials are essential for the understanding of this structure. There's no doubt in many teachers' and teacher educators' minds that the value in this kind of approach is more-or-less self evident. Certainly the numerous articles appearing in a range of journals over the years, especially the kind describing "how to...." or "how I....", and the anecdotal accounts one hears both at organised conferences and more casually from classroom practitioners, suggest that the principle is both well accepted and widely practised.

However, if it is well accepted, where in the research literature is the body of empirical information indicating its value? What research indicates that concrete representations of mathematical concepts are actually valuable, that they do represent the concepts teachers intend, and that they do have a real and measurable impact on students' learning? Further, where are the data indicating exactly how concrete representations allow the learner to better arrange his or her cognitive structure so that learning is more effective? Where did this idea of concrete representation begin? What kind of meanings are associated with the use of concrete materials? What was the rationale behind their use? Exactly what kind of mathematical ideas were these materials supposed to represent, what kind of learning was envisaged and how were they to help learners?

This chapter addresses these questions. The chapter is particularly necessary so that the reader understands the context in which the kind of activities that are the subject of this research actually occur in typical classrooms. For example, though research may be used to inform classroom practice, classroom practice is seldom based on an unemotional interpretation of research findings. That is, teachers, and educators in general, are likely to base their practices on past experiences and on emotive issues, and on their own values and beliefs. More specifically, the use of concrete materials in the teaching and learning of school mathematics, and recommendations for their use by educational authorities and experts, have a lot to do with presenting a position about the nature of teaching and learning mathematics. It is important for the reader to have at least some inkling into the strength of these values within sections of the mathematics education community.

For example, Lesh, Post and Behr (1987) argue that teacher use of concrete materials requires them to have a firm belief in the ultimate payoff for their students, a clear understanding about the higher-level misunderstandings that may be avoided, and confidence that misunderstandings that emerge can be corrected. Berlin and White (1986) claim that mathematics educators believe in concept development through progression from concrete objects, to representations and finally to symbolic representations of these abstract concepts. Learners participate in concrete experiences with manipulative materials to gain experience, then move to semi-concrete representations where two dimensional pictures are used to show the same concepts, and finally to internalising the concept as an abstract thought. Earlier, Fennema (1972) had argued that the use of concrete materials was supported by the work of Piaget, where assimilation and accommodation were encouraged through the actions of learners on the environment. This view of the influence of Piaget as providing a theoretical base for the use of concrete materials was supported by Scott and Neufeld (1976), and by Williams and Kamii (1986) who used Piagetian theory to explain the value of manipulative materials, in terms of Piaget's (1967) physical and logicomathematical knowledge. There are dissenters, though. For example, Friedman (1978) wrote about the exaggerated claims for the use of manipulative materials, and though sympathetic to supporting their use, he argued that more rigorous support needed to be found, and referred to their status as "this latest Pied Piper".

The background to these beliefs is discussed in the present chapter, initially through examining a range of educational philosophies that emphasise child-centred education, especially the idea of 'learning by doing'. This philosophical approach covers the period up to the mid-nineteenth century. There is then an analysis of the literature on mathematics education for the century beginning about 1850, and this is followed by ideas related to the period from the beginning of 'New Mathematics' up to present-day interpretations of concrete embodiments.

Philosophies of Education

At first glance, one may be surprised to find that concrete representation is a very old idea. One then quickly realises that centuries ago people used everyday items to represent numbers. We are all familiar with the idea of tally sticks and with the idea of an abacus, though few of us may have actually used either recently. Bowen (1972) suggests that as early as 3000BC the Mesopotamians were using a one-to-one correspondence to perform counting, with small lumps of clay as the counters. In Rome, about 100AD counting was taught with the aid of a child's fingers, and a bag of pebbles (*calculi*) used on a flat wooden board, the abacus.

There are two points to be made here. Firstly, the idea of representing a number by some physical object is indeed very old. The other point, though, is that there is no evidence up to the period of the Renaissance that the idea of physical representation was carried any further, that is, as a device or methodology to make mathematical concepts more easily understood. Certainly, the abacus was used, but it was not intended to explain to the user the structure and relationships between numbers, it was simply a calculator.

The University of Paris during the thirteenth century represented the best of European education with studies in theology, law and medicine. Teaching at the time was far removed from any pragmatic considerations; the best learning was seen to have been about ideas, through discussion and book learning, or the notion of mental discipline dating back to Plato (Schoenfeld, 1992). This was applied to all learning, so to the extent that mathematics was taught – and there seems to have been little of it – it was book learning unrelated to the world about teacher and learner. This is not as disconcerting as the situation in medicine where Bowen (1975) states "the medical teachings of the time...were never expected to conform to any pragmatic tests of effectiveness. Such an attitude held little place in the learning of the day" (p.124). The point here, is that there was little value placed on learning practical skills, so little consideration was given to the value of mathematics being made easier to learn through representations.

Smith (1913) includes in his work a number of quotations from Janicke. These quotations from about 1700 give some idea of teaching mathematics at that time.

> In the matter of arithmetic, it is impossible to form classes; hence the teacher shall use a printed book and shall teach the subject from it...He shall go around among the children and give help where it is necessary.
>
> The teacher is to write the first nine numbers, then pronounce them four or five times, then let the boys, one after another, repeat them.
>
> (Smith, 1913 quoting Janicke)

This quotation confirms the point made before, and repeated on a number of occasions below, that the teaching of mathematics was, until quite recently, carried out with little regard to students' understanding of the mathematical concepts and principles involved. Bowen (1981) suggested that it was only in the seventeenth century that book learning as the basis of all education at all levels was effectively challenged throughout Europe. All was not as bleak as this may sound, though, for Smith (1913) noted that somewhere around 1780 Christian Trapp worked out a scheme to teach addition and subtraction using objects with "the effort being made to teach numbers rather than figures".

A variety of sources (Bowen, 1981; Boyd & King, 1966; Good &Teller, 1969; Mayer, 1973) indicate that the sequence of events leading away from education based on book learning to a more pragmatic approach took place in the seventeenth century through Bacon's work on empiricism, with Descartes' writings on rationalism, Newton's scientific empiricism and Locke's idea of Tabula Rasa. All these views contained aspects of learning through experience. In particular, Locke rejected the concept of original sin that had formed the underlying principle of so much bible-based, rote-learned education. He claimed learning came about through experience. These notions of experience, doing rather than listening, and the rejection of rote learning may all be interpreted as giving weight to the view of the importance of manipulative materials in learning mathematics. Schoenfeld's view, where he states "it would seem short-sighted to ignore the past 2000 years of philosophy and psychology related to mathematical thinking and problem solving" (1992, p. 345), supports the points I'm making here.

According to Bowen (1981) two important features led to changes in education in the second half of the eighteenth century: the increased participation of governments in education, and the emergence of a new conception of people and society. Cajori (1890) had noted much earlier, that while most mathematics texts intended for school learners and published between 1776 and 1815 emphasised rules that were "ill arranged and disconnected", from 1815 the influence of Pestalozzi began to take effect. He noted that Pestalozzi's idea "to begin with the concrete and known, instead of the abstract and unknown" had some impact. In particular, Cajori instances the success of Warren Colburn's *Intellectual Arithmetic upon the Inductive Method of Instruction*, also known as the *First Lesson*, which sold some 3.5 million copies at about this time.

Rousseau published *Emile* in 1762, a treatise on education and on the wretchedness of society. Rousseau's ideas can be traced back to Locke's influence and can be seen as part of the stream of Enlightenment thinking. He changed educational thought and practice: he "expounded more successfully than anyone else a theory of education that accorded with the trends of Enlightenment thought' (Bowen, 1981, p. 197). "Reaction and hostility from conservative forces was instantaneous" (Bowen, 1981, p. 197), with his views becoming "a significant theoretical component of the groundswell of demands for the rights of man in the late eighteenth century" (Bowen, 1981, p. 202). Mayer agrees with this analysis: Rousseau "was the father of modern child psychology; he laid the foundations for a new curriculum; he emphasised the importance of play activities" (1960, p. 243).

Rousseau saw childhood as a time for learning through experiences, through doing. In particular, he argued that things learned by oneself were likely to be clearer and more convincing than those acquired from someone else's teaching. So there is some support in what Rousseau says for modern approaches to teaching mathematics. In particular, Good reports, that according to Rousseau "geometry, drawing and music should also be taught to the young child through experience, projects and active doing" (1960, p. 214). Pestalozzi followed on from Rousseau's ideas, suggesting that whatever materials were available should be used to teach numbers. According to Smith, Pestalozzi was the first educator "scientifically to make perception the basis for all number work" (1913, p. 78), and "to recognise its value to the full, and to put it to practical use in teaching" (1913, p. 78). He went on to suggest "we can learn more today from Pestalozzi than from any other one teacher of the subject". More particularly, Bowen reports Pestalozzi's view that "the elements of number, or preparatory exercises in calculation, should always be taught by submitting to the eye of the child certain objects representing the units" (1981, p. 228), and that "to begin by abstract notions is absurd and detrimental, instead of being educative" (1981, p. 228). Pestalozzi gave suggestions for the teaching of a number of topics. He recommended that "the operation must always begin with objects" (Bowen, 1981, p. 228). After numbers up to ten are known, "then the teacher can repeat the operations on a blackboard using dots or strokes in place of objects" (Bowen, 1981, p. 228). "So with fractions: rectangles or squares should be subdivided by having the child fold paper or break up patterns of blocks" (Bowen, 1981, p. 228).

Kant and Froebel both believed that the best way to understand was by doing. Later Dewey and Montessori used the work of these philosophers to develop their views on education: Dewey to emphasise the school as a laboratory for experimental and inquiry learning; and, Montessori to emphasise the handling of objects as a way of understanding not only mathematics but of understanding the world (Bowen, 1981). There is, then, a strong line of thought, extending back into the seventeenth century, supporting what we would call today the use of concrete materials, and supporting what we call 'conceptual learning' as opposed to 'procedural learning'. This range of philosophies provides contemporary teachers with strong support to adopt combinations of child-centred teaching, experiential learning and the use of manipulative materials. They also provide the basis for later philosophical developments, classroom practices and research directions in mathematics education.

Mathematics Education, 1850-1950

The start of the nineteenth century saw a greater need for mathematics, in order to cope with and further the Industrial Revolution. Yet schools of the time gave little emphasis to mathematics, and certainly little in the way of appropriate, practical mathematical tools (Bowen, 1971). In the grammar schools of England in the 1870s mathematics was taught as a mental discipline, with a strong emphasis on rote learning. This view of learning dated from the time of Plato, where learning through speaking was seen to be superior to any form of learning involving written notations. Much the same was happening in the USA and elsewhere.

There were some supporters, though, for the view of learning mathematics through concrete materials and with understanding. Bidwell and Clason (1970) expressed the view that Pestalozzi's work influenced a number of American educators. They noted that the *Method of Teaching Arithmetic*, written by August Grube in 1842, and based on Pestalozzi's ideas, was translated into English and sold in the USA from 1888. In this book Grube expressed views about the use of concrete materials:

> By the use of objects the child is brought to see the relations of numbers until he is able to reproduce the relations without objects (cited in Bidwell & Clason, p. 105);

and

> Elementary teaching of number should proceed from observation, or better, it should proceed form things (cited in Bidwell & Clason, p. 107).

Davies (1851) suggested that numbers should be introduced through things. In referring to the manner in which learners best acquire number concepts, he recommended the following procedure:

> How do we attain unto the significance of such expressions? By first presenting to the mind through the eye, a single thing, and calling it ONE. Then presenting two things, and naming them TWO: then three things, and naming them THREE; and so forth for other numbers. Thus, we acquire primarily, in a concrete form, our elementary notions of number, by perception, comparison and reflection (p. 101).

In general, Davies recommended using concrete materials to introduce an elementary concept, but did not hold to this belief for rules. In such cases, he suggested the kind of method so regularly employed in today's classrooms, especially in secondary schools.

> The mind should never be forced through a long series of examples, without explanation. One or two examples should always precede the statement of an abstract principle, or the laying down of a rule, so as to make the language of the principle or rule intelligible (p. 208).

By 1910 there had been some significant changes to mathematics education in England, and there was, by then, a much wider range of topics taught, in a variety of ways, and supported for a wide range of reasons including pragmatism. However, there was little change in teaching methodology from that which had existed throughout the nineteenth century. In the USA, and throughout Europe, there was little evidence as to how to teach mathematics, or argument as to why one approach was better than another. Those educators setting out alternative possibilities for the teaching of mathematics remained unheard. The content of school mathematics was to remain largely unaltered, as were teaching methods, for the great bulk of students from this time until the 1950s and 60s, though, at times, there were challenges to content and teaching methodology by mathematical teacher organisations and various government reports (Ellerton & Clements, 1988).

The 1911 report, *Mathematics in the Elementary Schools of the United States*, encouraged "direct mathematical training through concrete experiences with Froebel's educational materials" (Bidwell & Clason, 1970, p. 284). As recorded by Bidwell and Clason, the report indicated that

> the very nature of materials familiarises the child...with the mental processes involved in the solution of problems in fractions and all elementary activities of mathematics (1970, p. 285).

While there were exceptions, manipulative materials were generally ignored, by both teachers and researchers. For example, Reeve (1936) outlined twenty one areas for research in mathematics education. He mentioned mathematics for gifted children, teacher training, the problem of transfer and mastery tests, but nowhere was there the slightest hint of the need for research into the use of concrete materials in teaching mathematics. Articles by Spear (1938), Hartung (1939) and Thompson (1941) studied ways to improve students' abilities in calculations, but none mentioned the use of materials. There was considerable concern with students applying rules and procedures in order to obtain a high proportion of correct answers as quickly as possible (Buckington, 1928; Eaton, 1938; Fuller, 1949; Greene, 1930; Johnson, 1939; Johnson, 1944; Lutes, 1926; Osborn, 1927; Pauli, 1928; Swenson, 1944; Vandevelde, 1948). Indeed, publications were still appearing on this topic well into the 1950s: for example, de Moraes (1954), Weaver (1954) and Buswell (1959). None of these papers mentioned concrete materials. Even Hartung, writing about progressive education, with its emphasis on child-centred education, stated that in typical Progressive Schools "the content of the mathematics curriculum is quite the same as it is in thousands of other (traditional) schools" and that "the pupils are learning geometry and algebra as they are developed in the well known textbooks". The contrast between the textbook and rule method, and the concrete materials method is no where better illustrated than in the *Mathematics Teacher* issue where Hartung's article appears; five pages after his conservative article, Mahacher's (1939) paper describes a range of innovative teaching materials for use in junior high school mathematics.

Further, it is not as though articles supporting the use of concrete materials were new. For example, Schultze (1912) outlined a number of teaching approaches in mathematics, including a discovery approach, and emphasised moving "from the concrete to the abstract". In 1929 Durell had described a range of advantages in the use of concrete materials, including the saving of time, better retention and transfer of learning. In particular, Durell mentioned "the use of splits tied up in bundles of ten, the use of diagrams to illustrate fractions and the use of graphs". Breslich (1933) and Taylor (1938) also published articles supportive of the use of concrete materials, Christofferson's (1937) ideas about the use of rectangles divided in particular ways in the teaching of fractions pre-empted quite a number of current day writers, and a University of the State of New York report (1937) argued that

> Psychologically we learn better through seeing and learning together than through either singly. Hence concrete materials have an important value not only in primary education but also for higher age groups (p. 24).

This report supported the use of concrete representations because of the mathematical concepts and principles the material represented, and because such materials were likely to increase student understandings of these concepts and principles. It was clearly the case that concrete materials as aids to teaching and learning mathematics had supporters throughout both Europe and North America, but their views did not hold sway and were not widely taken up.

New Mathematics

The Bourbakists, a group of eminent French mathematicians, were concerned with the structure of mathematics, and published widely under the name N. Bourbaki from the mid-1930s. Their division of mathematics into sections including set theory, algebra and topology can be seen in mathematics curriculums of the 1960s (Moon, 1986; Pitman, 1989). According to Moon (1986) "the Bourbakists provide a fascinating first link between attempts at academic redefinitions of mathematics and later reform of the school curriculum" (p. 6).

In the USA, the launch of the USSR's Sputnik initiated significant experimentation in school mathematics with "the ensuing release of federal funds for national education development creating a wave of reform" (Moon, 1986, p. 9). The School Mathematics Study Group was formed in 1958 sponsored by the American Mathematical Association, the Mathematical Association of America and the National Council of Teachers of Mathematics and so had strong support and considerable influence. The report of the Woods Hole conference (Bruner, 1960), held in 1959 and involving a range of scientists, scholars and educators, emphasised the importance of learners understanding the structure of mathematics and the interrelatedness of its underlying principles. This was seen to require changes in subject content and changes in teaching materials. The report acknowledged the influence of Piaget and maintained "what is most important for teaching basic concepts is that the child is helped to pass progressively from concrete thinking to the utilisation of more conceptually adequate modes of thought" (p. 38). Later, the report mentioned the use of teaching aids; "(they) have the function of helping the student to grasp the underlying structure of a phenomenon" (Bruner, 1963). Cuisenaire rods and Dienes blocks were seen to give visible embodiment to the mathematical idea. The Royaumont conference held in France in late 1959 focussed on new thinking in mathematics and in mathematics education, and the implementation of reform. That is, there was some psychological basis for the development of the New Mathematics. The work of Piaget, Bruner and Dienes all contributed, through a concern about children's development of logical thinking and an implication as to the value of concrete materials as teaching aids (Pitman, 1989).

This version of school mathematics reform began in the early 1960s. In the UK the Schools Mathematics Project (SMP) began in 1963 and from 1964 the Nuffield Foundation funded curriculum development projects in mathematics for 5-13 year olds. In France secondary school curriculums in mathematics were reformed from 1966; in 1968 there was a television series to explain "new maths"; and, in 1969 the first IREMs (institutes for research in mathematics education) were established. The problem of implementation of reform was a continuing difficulty. By the mid 70s the enthusiasm had altered course somewhat. Pleas for change from earlier reforms were being based on research data, and on more scholarly arguments than the quests for reform of a decade earlier (Moon, 1986). From 1980 the reforms of the late 60s and the 70s were themselves altered to give more control back to the central education authority, with an emphasis on the role of inspectors.

Quadling (in Thwaites, 1972) when referring to the early years of SMP mathematics in the UK noted that "we acknowledge amongst our objectives the development of structure, but saw abstraction as a process in which the pupils should engage actively as a result of concrete experience, not as a system of laws to be imposed upon them". The following lines, said to be an old Chinese proverb, appear in the introduction to most books in the Nuffield mathematics series. Whether they are or not, the words are intended to capture the spirit of Nuffield mathematics.

> I hear, and I forget
> I see, and I remember
> I do and I understand.

That is, involvement in practical activities and the use of concrete representation of mathematical concepts was an integral part of the Nuffield approach. An important theoretical support for this approach was the work of Piaget. In particular, the use of materials to represent mathematical ideas was seen by the Nuffield project to support children's intuitive thinking at the preoperational level and at the concrete operational level. So there was need for "experiences" to establish invariance, and "real materials" to support learning throughout the concrete operational stage (Nuffield, 1967, p. 5).

Davis (1967) claimed that "the 'new mathematics revolution' had not taken place, but – considering the pressures that are building up – it probably will, possibly within the next ten years". He had hopes that the new technologies becoming available in classrooms, together with the manner in which many children were ill-served by schools, especially in typical mathematics classrooms which he described as "one of the most culturally deprived environments inhabited by any American child", would force curriculum and pedagogy changes. Moon (1986) believed that "New Maths perhaps more than any other curriculum reform, caught the imagination of the world at large", but as Mathews and Brown remind us "unlike mathematics itself, mathematics education is political, it is riddled with value judgements" (in Moon, 1986, p.69). So ten years later the support from industry and parts of the press had dwindled as there was a move back to a more traditional view, where basic arithmetic skills were to be taught with a renewed emphasis.

The point here is that the 1960s and 70s were periods of great reform in mathematics education, and central to these reforms were different ways of teaching and learning mathematics. One important element in these changes was the use of manipulative materials in a wide range of mathematical topics This period of reform was not accompanied by sufficient research to adequately evaluate the extent, effectiveness and implications of these changes. There were few notable investigations into the effectiveness of manipulative materials in teaching and learning. In particular, there was little attempt to investigate the theoretical bases of learning with such materials. How these materials helped learning, how learning actually occurred, and the development of effective pedagogies with these materials received too little investigation. While the reforms were not always adopted in school systems or by individual educators, political expediency soon superseded such considerations. These changes to the school mathematics curriculum came under attack by the mid 1970s, specialised educational centres lost government funding, and the old standby of standards in education became an issue.

So in spite of all these attempts at reform, and in spite of a long history of some philosophers and educators valuing manipulative materials in the teaching of school mathematics, we are still left with the fundamental questions of "what concrete embodiments best serve the purposes of mathematics, of teaching mathematics and of learning mathematics?".

Concrete materials - research

The term 'concrete' has been used in a number of ways in the literature. For example, Lewis (1985) uses the term to refer to something that can been seen and manipulated, a three-dimensional object. Fennema (1972) talks of a concrete model where a mathematical idea is represented by three-dimensional objects. She provides an example of the use of counters to develop multiplication facts, and states "(m)eaning comes directly through the manipulation of the objects" (p. 635). Although she talks of "using both concrete and symbolic models in a carefully controlled manner" (p. 636), she does not make explicit what this actually means. Scott and Neufeld (1976) note that there is no agreed definition of concrete, and that both pictorial and manipulative objects may be regarded as concrete. Dienes (1971) regarded concrete as usually meaning "our immediate contact with the real world" (p. 337), that is, our everyday contact with objects and events. Concrete materials then need to be defined in terms of their meaning for this thesis.

> For the purposes of this thesis, concrete materials are defined as two or three dimensional objects used in teaching sequences, that can be physically manipulated by learners in a way intended to enhance the quality of their learning. That is, concrete materials may be acted upon by learners so as to alter their position, size or shape.
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> Also, for the purpose of this thesis the terms *concrete materials*, *manipulatives* and *embodiments* will be used synonymously. Attention will be drawn to any case where the meanings have a deeper, particular significance.

Results from the many studies on concrete representations are equivocal, often failing to produce the positive outcomes expected (Berlin & White, 1986; Bobis, 1992; Boulton-Lewis, 1992; Dufour-Janvier, Bednarz & Belanger, 1987; Fennema, 1972; Fuson & Briars, 1990; Fuson & Willis, 1989; Hall, 1981a, 1981b; Labinowicz, 1985; Lesh, Behr & Post, 1987; Lesh, Post & Behr, 1987; Resnick & Omanson, 1987; Sowell, 1989; Suydam & Higgins, 1976; Wearne & Hiebert, 1988). Bobis (1992) reports that two-dimensional pictorial representations can be useful in teaching – "under most circumstances, learning based on illustrations is more effective than learning based on equivalent words" (p.4), but this does not appear to apply for all representations. Fennema (1972) lists a number of studies from the 1950s and 60s where results achieved through the use of concrete materials in teaching are inconclusive. Of the 15 studies reported, seven showed no significant differences between manipulative and nonmanipulative treatments, four favoured the manipulative groups, three showed mixed results and one favoured the nonmanipulative group. Suydam and Higgins (1976) surveyed 40 studies on the use and effects of manipulative materials in teaching, and reported 24 as showing positive effects on student achievement. Once again, the findings of such studies are open to question because of various methodological flaws (Raphael & Wahlstrom, 1989). Freidman (1978) noted an earlier call by Kieren (1971) for research to produce a theory on the instructional use of manipulative materials, and went on to report that there were very few such studies in the six years between Kieren's paper and his own, and none of them were especially encouraging to proponents of manipulative materials. He also noted that of the 18 doctoral dissertations in the period 1970-1977 investigating the use of manipulatives in the teaching of mathematics, only four showed significant differences favouring the manipulative groups over the nonmanipulative groups.

Scott (1983) surveyed some 800 teachers in a large urban school district to gain data on the use of concrete manipulative teaching materials in their classrooms. While noting that teacher reports were likely to rate usage as higher than actual (Weibe, 1981), he still interpreted the data as showing "(t)he use of materials is particularly low" (p. 62). He found that most teachers used few materials other than text books and that concrete material usage declined as grade level increased. Labinowicz (1985) reported little gain in third grade students using multibased arithmetic blocks (MAB) to develop computational skills, whereas Fuson and Briar (1990) reported using MAB materials to gain high levels of skills in addition and subtraction algorithms, and Wearne and Hiebert (1988) reported fourth, fifth and sixth grade students showed some gain in decimal numeration, addition and subtraction from the use of MAB. Fuson (1986) investigated young children taught addition and subtraction algorithms with MAB. When these children were interviewed after making errors, Fuson reported that most "were able to use a mental representation of the blocks" to self-correct errors (p. 183). All the same, there is merit in Raphael and Wahlstrom's (1989) claim that "relatively little detailed work has examined the use of instructional aids and their effects on students' achievement" (p.173). There is merit, too, with the scepticism of Post's claim (1980) concerning the way in which mathematics educators are building a persuasive body of literature that supports the use of manipulative materials in teaching mathematics.

Scott and Neufeld (1976) mention that "(i)n theory, at least, there is increasing support for concrete instruction in the schools" (p. 68), and reported the findings of their study in which second-grade children from three schools were taught multiplication using methods that emphasised one of concrete manipulative, pictorial or abstract approaches for twenty thirty-five minute lessons. They reported no significant differences between students taught under any of the instructional regimens tested, though students in the pictorial or concrete teaching approaches reported more positive responses to the lessons than did students in the abstract approach.

Williams and Kamii (1986) use Piagetian theory to explain what and how children learn when using manipulative materials, through what Piaget (1967) called physical knowledge – or knowledge of objects that can be observed – and logicomathematical knowledge – or knowledge stemming from the relationships between objects. According to Williams and Kamii these two forms of knowledge depend on one another and develop together. Manipulation of objects is essential for the development of physical knowledge about the objects, and the mental actions that take place during and after the physical manipulation encourages mental actions on objects as relationships are explored. This view is important in that it separates out the materials, the concepts they represent, and the importance of action on the materials.

The 1960s were a time when the names Cuisenaire, Montessori, Dienes and Piaget were fashionable and used as the theoretical base to argue for an active approach to the teaching and learning of school mathematics through the use of concrete materials, but their views appeared to have had little impact on classroom practice (Lesh, Post & Behr, 1987). Dienes' constructive principle reflects the importance that he saw in learners constructing their own mathematical ideas. For him, the internalisation of mathematical structure came about through exploring relationships by operating on materials; that is, as Lesh et al. state "the role of the materials is simply to serve as a support for the student's mental activities, not to serve as the direct basis for abstraction" (p. 649). There is some disagreement, too, as to the value of multiple embodiments, another of Dienes' principles. That is, will one set of concrete materials be sufficient for learners to abstract the mathematical ideas intended? Dienes did not think so, but the research is unclear – Lesh (1979) and Lesh, Post and Behr (1987) think so, but Edge and Ashlock (1982) do not. From a teacher's perspective, Lesh et al. (1987) maintain that the purposes of the materials is not clear, teachers do not realise that the materials themselves are insufficient, and that the mathematical structure must be imposed on them. They argue too, that teachers see mathematics as "as a collection of isolated rules for manipulating symbols" rather than an interconnected whole (p.652), and that the learners' misconceptions shown up through the use of concrete materials is taken as a weakness in the materials by teachers, who revert to "superficial computational proficiency" (p.652).

Doll (1981) described a curriculum development activity involving "structural arithmetic" where he claimed the structures of arithmetic were integrated with the structures of learners, resulting in the transfer and transformation of knowledge in new contexts. Over the three years of the activity results suggested that this approach led to scores on national achievement and cognitive abilities tests superior to scores of a control group pursing the already-existing mathematics curriculum. Doll reported one experimenter as stating "(t)he knowledge the children have is their own knowledge, while those in the control group are always trying to follow the teacher" (p. 35), and reported increased confidence, innovation and intellectual sophistication in the experimental group. Here we have results that seem quite positive, yet we know little about the content and teaching methods employed in either the experimental or the control groups, we know little about assessment techniques, and we know little about the transfer and transformation claimed, nor anything about the measures of confidence, innovation and intellectual sophistication, whatever they might be. That is, we have a seemingly positive statement concerning a structured approach to arithmetic, but we do not know what "structured" means, and we have too little information to replicate the study. An enthusiastic teacher may take some positive views from the paper, but in reality it adds little to the literature, it provides neither support nor damnation to a particular view, there is no way to check the claims or to fit this paper in to a broader literature with any confidence.

Lewis (1985) discussed her investigation with MAB materials, where she worked individually with children in her role as an elementary school resource teacher. She mentioned that many children taught with manipulative materials seemed not to have made the connection between the materials and the pen-and-paper calculations they represent. She then optimistically claims that "(m)athematics teachers can rejoice, for there are ways to make the critical connection real to students" (p. 371). The process she outlined emphasised both exploring manipulatives and the language associated with them. She refers to this process as *concrete* "because it is based upon materials perceived by the senses" (p. 372). The next step described was where mathematical ideas were represented by words, symbols and pictures which she referred to as *representational*, and the later stage where symbols alone were used as *abstract*. Lewis had some sound ideas, for example, returning to the concrete as new topics are introduced rather than using concrete materials only in lower grades. However, there were some weaknesses in her work. One instance is the labelling she uses, for example, pictures could be considered as concrete objects able to be manipulated, but it is not clear that she would accept this interpretation. Though she provided some instructional advice, such as the benefits of one representation rather than many when introducing a topic, she did not explain either the method of making the bridge between concrete and abstract or its cognitive or learning significance. Her assertions were well intended, and one assumes, based on experience and successful teaching, but her methodologies were vague rules-of-thumb lacking prescription, not open to empirical investigation and not considering the learner's mental actions. The more positive aspect of Lewis' paper was where she mentioned the need to use the representational phase to bridge the concrete to the abstract, particularly by writing symbols corresponding to the physical representations, and where she discussed some points concerning the selection of suitable materials to provide concrete representations.

Resnick and Omanson (1987) sought to establish the relationship between performing arithmetic and understanding it, especially by illustrating procedural learning with "well-grounded mathematical principles" (p. 42). They developed a *mapping instruction* in which they maintained "a step-by-step correspondence between the blocks and written symbols throughout the problem" (p. 71). They had 80 fourth, fifth and sixth grade students perform tasks, both written and using MAB materials, where representations of numbers were constructed and decomposed, and where activities involved addition with carrying, and subtraction with decomposition. After a period of instruction, posttest scores showed children taught with the mapping instruction did not differ significantly from children in the comparison group, but in a delayed posttest the mapping instruction group gained higher scores. The researchers expressed disappointment at children's levels of achievement, and concluded that the mapping instruction was not effective in curing subtraction bugs. These findings suggest that at least some of our beliefs about the value of concrete materials are questionable. Mathematics educators need to be concerned with these outcomes, especially since Resnick and

Omanson's research was well designed, is frequently cited, was conducted by well known researchers, and employed what appeared to be a detailed and sensible pedagogy – yet the use of concrete materials did not appear to have led to much in the way of positive outcomes.

Fuson and Briars (1990) examined the use of MAB materials in the addition and subtraction of four-digit numbers by first and second grade pupils. This study was reported in great detail, and made explicit many aspects of teacher preparation, teaching approaches and the novel lesson content which would not normally be covered until later in elementary school when children were some years older. The researchers used a number of strategies to emphasise the link between action on the blocks and written symbols. For example, action with the blocks was immediately followed by the equivalent written symbol, and there was much verbalisation about the blocks, in everyday English and in base ten terms. Blocks were placed on large cardboard calculating sheets, where both sets of numbers were arranged for both addition and subtraction. They reported children performing these calculations at a level well above what is normally expected of their age group; they showed little of the small-digit-from-the-large error, were able to label digits in their place values, were able to change word names to numerals and vice versa, and selected the correct digits in trading which they were able to describe in terms of its place value. No specific detail is available on the actual classroom experiences of these learners since the lessons were not videotaped, and there clearly would have been differences between groups in their instructional environment. All the same, this research provides strong support for the use of concrete materials, and for the need to use them in particular ways to meet specific objectives. According to Fuson and Briars, their methodologies emphasised materials that reflected the relative size of the numbers involved, that reflected the positional nature of place value, and focussed on both understanding and procedural competence. However, it is still useful to remember Lesh, Post and Behr's claim that "'doing the procedure' frequently has little to do with 'doing mathematics', nor is it a good indicator of a student's understanding about the underlying ideas" (1987, p. 679). There is some literature to support this view - that is, that many students complete whole number algorithms correctly, they have procedural competence, but do not understand crucial aspects of the procedure such as why trading is necessary (Cobb & Wheatley, 1988; Fuson & Briars,1990; Labinowicz, 1985; Resnick & Omanson, 1987).

In another important paper, Sowell (1989) reported a meta-analysis of 60 studies designed to assess the value of manipulative materials in mathematics instruction. The studies ranged from those involving kindergarten children to those in which tertiary students participated, and employed a wide range of manipulatives and mathematics topics. Sowell found that treatment lasting a school year or longer favoured the manipulative groups but only for the use of concrete materials and not for pictorial representations. Treatments for shorter periods showed no difference between the manipulative and nonmanipulative groups on either posttest or delayed posttest scores. Another interesting aspect of Sowell's paper was her report on those findings where instruction with concrete materials resulted in greater gains in learning than the alternative instruction. In such cases, students were often taught by university academics or teachers involved in long-term training on the use of such materials for instruction. There is then the question of what these teachers did, that others did not do. We cannot say with any certainty that they were more knowledgeable about mathematics, whether they were more expert teachers, or whether their pedagogies were consistently different from colleagues in other investigations. Sowell acknowledged the view, one that was expressed previously, that individual papers often lacked detail as to who did the teaching, what training they undertook relevant to the study, and what the teaching treatments actually involved. Fennema (1972) came to a similar view much earlier when she noted that studies involving concrete materials were often inconclusive, and referred to experimental and control groups where she noted they were "often defined no better than this" (p. 636), that is, without a detailed description of the teaching strategies employed and without other distinguishing features of each group. Scott and Neufeld (1976) also noted that many studies gave little emphasis to the nature of the 'concreteness' used.

Reference to a wider range of research literature simply confirms the uncertain value of concrete materials (for example, Hart, 1989; Hiebert and Carpenter, 1992). Treatment time is an element in teaching and learning, and so is pedagogy, but one of the difficulties in analysing the literature on concrete materials is the lack of detail given about the actual teaching methods employed. Statements about an experimental teaching approach contrasted with a traditional approach give insufficient detail as to the intricacies and nuances of the learners' experiences. It seems likely that the pedagogy used in some studies could be improved simply through more attention to detail in the teaching-learning process. For example, the procedural analogy theory outlined later is likely to provide one set of guidelines for improving instruction.

Larkin and Simon (1987) noted that in their problem solving, humans used both internal representations, stored in their brains, and external representations stored on any of a range of media, as a form of external memory. The notion of calculators and computers as external memories enabling students to increase the processing capacities of their short term memories is of increasing importance as information technologies increasingly find their way into learning contexts (Cooper & Hall, 1989). Larkin and Simon investigated the difference between *sentential* representation where a sequence of statements is created so that the sequence describes in step-by-step detail the solution to a particular problem, and *diagrammatic* representation where the components of a diagram represent the problem situation. The sentential representation altered natural language presentations into a sequence of propositions and a set of production rules, and both representations required search and recognition components. Larkin and Simon agreed with Anderson (1984), that the distinction between representations lies not in their symbol systems but in the operations used on them. In the first example they analysed a pulley problem where two weights were supported by three pulleys, in order to find the ratio of the weights. The second example concerned the proof that two triangles formed by a pair of parallel lines and two intersecting transversals were congruent. The diagrammatic solution to the pulley problem takes some seven steps, more if errors are made, but the sentential solution requires 138 steps. In the geometry problem the sentential approach requires just over 100 steps compared to the 60 or so steps in the diagrammatic solution. As Larkin and Simon state, there is a huge benefit to the solution of problems through diagrammatic solutions rather than language solutions - diagrams group together all the relevant information and support the human propensity for perceptual inferences, while a section of a diagram typically focuses information about that element. All this lessens cognitive load through indexing the information so as to support efficient computational processes.

The Procedural Analogy Theory, to be discussed in Chapter 4, uses analogy and simplification so as to lessen learners' cognitive loads. At that time, it will also be argued that solutions through concrete materials are superior to language solutions as a method of instruction. Such diagrammatic, or concrete, representations are likely to have the pedagogic advantage of being easier to talk about, and easier to describe and analyse than language-based solutions which are inherently abstract, and difficult to analyse in learning contexts. Indeed, this is consistent with a point made by Larkin and Simon, who note that the only psychological process their analysis refers to is perception, where cues detected by focussing attention on a specific part of a diagram are then used to recall relevant components from memory: "these assumptions agree with everything that has been learned in the past two decades about expert performance" (p. 97). Also relevant to a future aspect of this thesis is the notion put forward by Larkin and Simon that "these representations and processes could equally well be interpreted as denoting mental images and imagery processes in the brain" (p. 98). The researcher's view tends to the notion of well-learned mathematics as providing a richly-connected cognitive network of concepts and skills, empowering learners through one part of the network allowing access to other sections, as contrasted to a mathematically-weak network where concepts and skills are isolated, and where a paucity of the possible interconnections are present, because the learner has not constructed an inter-connected network, but a network of relatively isolated nodes. At the same time, consideration needs to be given to Kaput's (1985) point about computational solutions developed through some external memory having no logically necessary inference concerning the actual structure of human knowledge.

Boulton-Lewis (1992) noted that "the value of a concrete representation is that it mirrors the structure of the concept and the child should be able to use it to construct a mental model" (p. 1), and that concrete materials are useful in teaching only if learners recognise "the correspondence between the structure of the materials and the structure of the concept" (p. 10). She noted that often concrete materials did not produce anticipated gains in student outcomes, and proposed that this was the case because these investigations had not taken into account the cognitive processing loads required in their use. She maintained that processing load would be increased if children were unfamiliar with the materials or analogies used, if the analogies were inappropriate, and if children lacked declarative or procedural knowledge. This is sensible when we think of the common occurrence of children, trying to solve an algorithm and where the procedure is interrupted, as the child counts on his or her fingers in order to answer some elementary number combination. The child loses the sequence of the procedure, presumably because the cognitive load to perform both the calculation and to remember the procedural sequence is too great. That is, simple algorithms will not be 'simple' for many learners: and the flowcharts in Figure 3.1 support this.

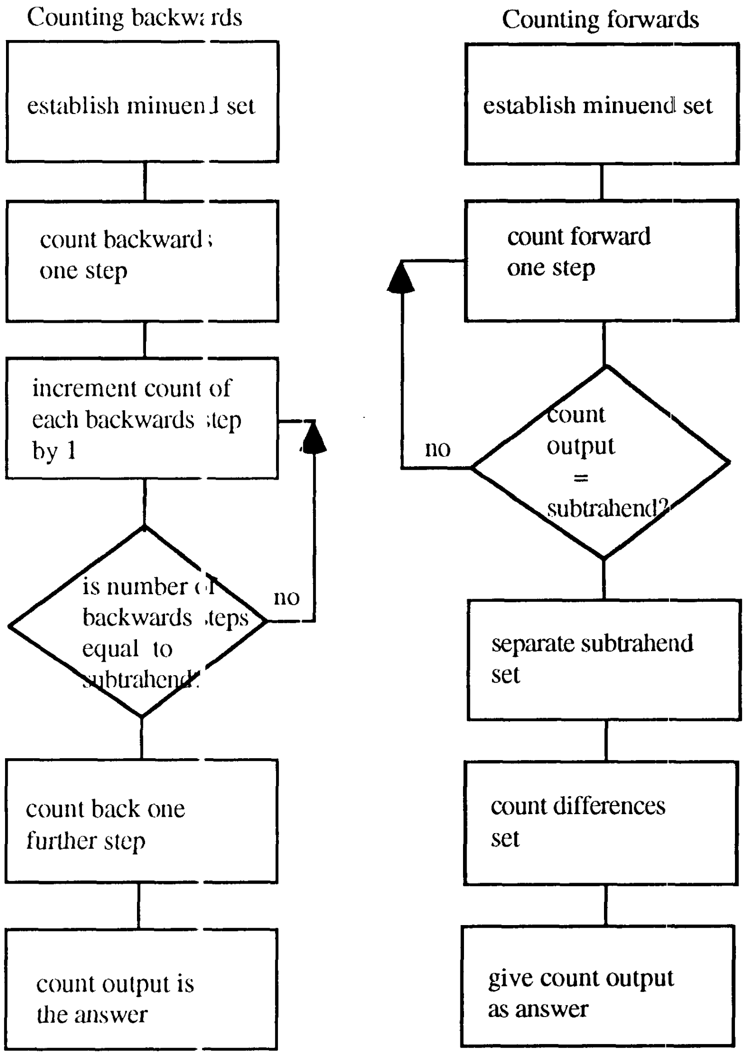

Figure 3.1: Flowcharts for processing subtraction using counting backwards and counting forwards

(Modified from Boulton-Lewis, 1992, p.12)

These kinds of limits on school learners' information processing capacities have been widely investigated (Bennett, Desforges, Ferrara & Campione, 1983; Hegland, 1991; Hiebert, Carpenter & Moser, 1982; Sweller, 1988). Boulton-Lewis (1992) reported her study of 29 infant school children, interviewed individually, and set a number of arithmetic exercises on subtraction and place value using a range of concrete materials including MABs, bundles of sticks, counters and metre rules. Included in the results were those of children who counted backwards to solve a subtraction, for example, "8 - 3 = 6 (using counters) 'I counted backwards 3 places 8, 7, 6'" (p. 7). She accounts for this by showing that in counting forward mental actions are serial, whereas in counting backwards two sets of processes must be carried out simultaneously. It is made more difficult, too, because in adding on or in subtracting, the last number provides the answer, whereas in counting backwards the answer is the next number. If a child uses more than one of these rules, confusion is understandable. Figure 3.1 shows the flowcharts for completing subtraction either by counting backwards or by counting forwards.

In the context of using concrete materials to help infants-school children learn the concept of place value, Boulton-Lewis (1992) suggested that using a range of materials to overcome learner boredom may have the positive effect of showing children that numbers can be represented in many ways but may also have the negative effect of not ensuring children have made strong mappings between a set of materials and place-value concepts. When children come to use materials to solve questions involving place value their lack of declarative and procedural knowledge with the particular materials increases their processing load as they deal with numbers unit by unit rather than recognising and using the place-value format. To minimise cognitive processing loads, learners need to recognise when a particular concrete material is appropriate, and the sequence of steps for its effective use needs to be automatic, so minimising processing load. This work, together with that of Bobis (1992), and a point made by Lesh, Post and Behr, (1987) about perceptually compelling but misleading cues, suggest that teacher-introduced representations, aimed at assisting learning, may actually make learning more problematic.

Bobis (1992) reported her intriguing study showing that there can be an over-emphasis on concrete materials, and that the "helpful hints" given by teachers are sometimes counter productive, since they impose excessive processing demands on learners. This is consistent with the views of Bennett (1989) who reported that the demands of the instructional sequence and its materials sometimes seems more demanding than the actual task. At the same time, Bennett's view must be contrasted with the view, that even though concrete materials seem more demanding than simply illustrating the procedure, it is the manipulation of the materials and the cognitive action so generated that allows for more effective learning. Concrete materials also have the benefit of being more easily talked about, and manipulated, than do learners' conceptions and misconceptions stored in the brain, and so unseeable. All the same, Bobis' work gives a sobering reminder that concrete materials have to be used with care in instruction, and that a line has to be drawn somewhere between concrete aids as a help, and such aids as a hindrance, to teaching and learning, or such materials will provide yet another misunderstood barrier between learners and their construction of mathematical knowledge.

Chapter 4

The Procedural Analogy Theory

Introduction

Earlier chapters in this thesis have been concerned with aspects of cognitive science and issues related to the use of manipulative materials in mathematics education. In particular, the previous chapter concluded by suggesting that materials may in fact be misused in mathematics classrooms, and so form a barrier to learning. The present chapter draws together ideas from cognitive science, and aspects of mathematics education. More specifically, this chapter describes the Procedural Analogy Theory[1], developed by Ohlsson and Hall (1990), and based on both cognitive science and mathematics education.

The procedural analogy theory is concerned with making explicit a theory of instruction to guide the development of effective teaching strategies, where manipulative materials, and analogy, are employed as part of the process of teaching mathematical procedures. An instructional theory needs to take into account students' learning processes, and needs to provide both a theoretical description and practical guidelines as to how the learning mechanisms involved bring about changes to students' cognitive structures. That is, the function of a procedural analogy theory, at least in part, must be to explain for a particular teaching instance which concrete materials are useful and which are not. Further, a procedural analogy theory must explain how to select the most effective approach from the options available, and should provide generalisable principles to be useful in instruction.

Teaching sequences involving the use of materials, whether highly teacher directed or more discovery orientated, generally begin with activities based on exploration of the properties of those materials, the relationships between component materials, and the mathematical concepts represented by the materials and by actions on them. Typically, there will then be a move to some form of symbolic representation where the written symbols closely reflect both

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[1] This chapter is based on the Ohlsson & Hall (1990) publication, and includes further interpretation and analysis.

the structure of the materials and the learners' actions on them. There may be some intermediary symbolic notation as the learner constructs meaning, prior to movement to a final, more conventional, target procedure. This target procedure will reflect both an efficient and expert way of completing the algorithm or problem solving sequence.

The theory described here provides specifications from which to design instruction in whole number algorithms; it explains why particular teaching sequences are likely to work and others not to work; and, it explains the cognitive function of the concrete materials for a particular teaching setting. The theory may apply in the teaching of fractions and decimal number and in other topics too, but this has not yet been explored. There are many forms of concrete materials and many other forms of representations and they may be useful in a range of topics involving a range of pedagogies. There may be other theories that lead both to effective teaching and to an effective explanation of the processes involved. The application of this theory, then, will have limitations, and may not be unique.

Declarative encoding and proceduralisation

As explained at some length previously[2], learning takes place initially through declarative encoding of data. This could occur through the discovery of a new relationship while participating in a discovery learning lesson, or during some problem solving activity putting together two ideas which had until then been independent; but most commonly in school learning, declarative encoding takes place on hearing the teacher say something. Further activity, whether it be through discovery, problem solving or teacher direction, allows consolidation of this new data by making it operational. That is, a new relationship between ideas will need to be thought about more than once to make its meaning clearer. Some steps then have to be taken to operationalise this new knowledge. Knowing what the teacher said, or what you discovered is not the same as being able to perform what the teacher described or to perform that problem solving technique again. This consolidation requires that the declarative encoding be followed by proceduralisation - to move from words to action.

Frequently, classroom instruction is based largely on the teacher talking to students. Even where teachers emphasise demonstrations, examples, gestures, and so on, their effective use requires that they be accompanied by explanations. So any theory of learning cognitive skills from instruction must explain the way in which teachers' talk impacts on students' cognitive functioning. As mentioned earlier, Anderson (1983) argued that verbal instructions were processed in two steps, declarative and procedural. Teacher instructions will first be processed as a language understanding, where the student listens to what the teacher said and makes a declarative encoding of this. Such a coding is not yet executable, and will become so only when the declarative encoding is transformed into executable, procedural code.

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[2] Chapter 2, p. 8ff.

The instructional sequence described below encompasses aspects of the use of concrete materials for mathematics learning that would be widely agreed to by members of the mathematics education community. There may be aspects that individuals would disagree with, but taken as a whole, the sequence will have many supports (Williams, 1971). In particular, this sequence has been an important guide for my work in teaching mathematics, and in teaching mathematics education students. How can this sequence be interpreted in the light of the declarative and procedural knowledge described above?

Free play
Using materials in any manner to achieve correct answer
Using materials in a prescribed manner to achieve correct answer
Using materials, writing corresponding expanded algorithm
Using materials, writing corresponding contracted algorithm
(Algorithm only, check with materials)
(Algorithm only, place value language)
Algorithm only, face value language

The first step in using materials new to a learner, regardless of the age of the learner, is to allow for a period of free play in which the learner experiences the relationships existing within the materials, through teacher direction and intervention, and through student discovery. During this period the learner gains declarative knowledge of the materials, that is, what they mean, what they represent, and the relationships between component parts. The continued manipulation of the materials allow the learner to proceduralise their use. That is, after some period using the materials, the learner, either through teacher direction or from self-discovery, follows consistent patterns in using the materials. For example, in representing a number, place value may be represented by placing subsets of the materials together, and the trading-decomposition action will become consistent. At this stage, the learner has proceduralised the actions. Further practice will eventually lead to the procedure, or sections of it, becoming automatised so that the learner does not have to spend much short term memory load on using the materials. During both the declarative and procedural learning stages, the materials are typically used to solve specific questions. In these circumstances, learning takes place through teacher explanation and learner actions on the materials. The materials may be a form of calculator, but more importantly they allow for the exploration of mathematical principles and relationships, and support the view that there may be more than one way to obtain a particular correct answer.

Once the learner is confident in using the materials, through questioning, discovery, or teacher intervention, the learner may adopt an approach which is more efficient in terms of mathematical concepts, and in terms of the relationship between action on the materials and the written algorithm corresponding to these materials. Alterations to the learner's existing set of skills and knowledge will again require periods involving declarative and procedural learning. After sufficient practice and exploration, the materials and actions on them are transferred to an algorithm; either to an expanded algorithm where each numeral written has a one-to-one correspondence with a concrete referent[3], or to a contracted algorithm. This, and the next steps, are the crucial stages in encouraging correct mathematical principles, and their efficient arrangement in learners' cognitive structures.

The expanded algorithm, again with the use of concrete materials, is made more compact: that is, contracted. Of course there are many variations to these ideas, how long one spends on each aspect, the actual questions used and directions given to students, and these aspects vary between and within mathematical topics. The next step, which is not always essential, encourages the learner to use contracted algorithms and to then check the answer using materials. Of course, using a calculator at this stage gives the correct answer, but does not show the structure and sequence of the algorithm.

If the algorithm involves place value, then the teacher may still be using 'place value language', even though the algorithm is contracted. 'Face value language' means simply giving the value of the digit, so in the number 79 the '7' is called 'seven', whereas in 'place value' language it is called 'seven tens' so emphasising the position of the numeral in the number. It is appropriate to use place value language with an expanded algorithm, but face value language may be more suitable with the contracted form, though perhaps not immediately. It is important to minimise change as the learner progresses through each step in the sequence. So we now arrive at the final step, that of the algorithm where a contracted format and face value language are used. An additional benefit of this sequence is that as (say) larger numbers are used, or as some later difficulty is experienced, the learner returns to an earlier stage in the sequence, one where the teacher can be confident that understanding exists. Every new step in learning, described in the various stages above, requires declarative knowledge, followed by practice so as to develop procedural knowledge.

The transformation from the declarative to the procedural encoding can be quite complicated (Anderson, 1982). The teacher must encourage the student to link the declarative content to the relevant goals, when many other transformations are possible. This is illustrated in the sequence described above. This complexity explains why learning by being told is insufficient; there has to be opportunity for proceduralising what is initially declarative. A number of researchers have shown that individuals cannot execute a procedure from a description, that practice of the procedure is necessary (Anderson, 1982; Mostow, 1983; Ohlsson, 1991).

It is in the movement from declarative to procedural that concrete materials, and other embodiments, are especially useful. The teacher explains the procedure using concrete materials, rather than explaining abstract ideas; the students construct a declarative encoding of

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[3] For example, if the numbers 79 and 48 are represented with MAB materials, then an expanded algorithm is (70 +9) + (40 + 8), and a contracted algorithm is 79 + 48.

that procedure which they can represent through action on the concrete materials, which provides them with a crutch to more readily manipulate their thoughts, and which makes it easier for the teacher to 'see' what the student is thinking. The argument here is not that there is only one way to learn, or that the instructional strategy involves telling students what to do at every stage. Rather, the argument is concerned with students' constructing meaning through operation on materials, and through cognitive conflict as existing cognitive structures are challenged: and the claim is that the teacher is likely to be better able to judge when to intervene, and to have a more meaningful intervention, with students using concrete materials than when teaching, learning and discussion rely solely on words. The declarative encoding is then converted to an executable procedure through practice under teacher supervision, with materials that are transparent in the sense that they show both teacher and student what the student is thinking. From the instructional sequence described previously, it is clear that there will be many declarative and procedural encodings as the learner moves through the various stages in constructing an efficient target procedure. Throughout these stages, actions on concrete materials together with discussion about them, provide the teacher with a visible analogy of the student's working memory and cognitive structure.

The initial declarative encoding is important, since we do not want students to have to unlearn incorrect information early in a teaching sequence, yet it is difficult to know when the declarative encoding has taken place. Certainly the teacher can ask students to repeat what was said, to write it down, to read it out, to paraphrase it. Yet all this is just so many words and we have little control over how students actually encode and store this new data. As we have said elsewhere "(I)f the declarative encoding does not contain the right information, the proceduralisation process cannot generate the correct procedure" (Ohlsson & Hall, 1990, p. 15), errors will quickly become apparent, with further work likely to compound misconceptions. Here the benefit of concrete materials and other forms of representations become apparent: they can be seen, manipulated and talked about; they make teaching easier for the teacher, and learning easier for the student "because embodiment procedures are easier to describe than symbolic procedures" (Ohlsson & Hall, 1990, p. 15).

> More precisely, we hypothesise that it is easier for the teacher to construct utterances which will generate declarative encodings which, in turn, will generate the correct procedural rules, if those utterances refer to actions on concrete objects than if they refer to conceptual operations on abstract objects such as numbers. Although we cannot verify this assumption within this article, the reasons for believing it include children's more extensive experience with conversations about concrete objects, and the fact that it is easy to refer unambiguously to objects in the concrete sphere (e.g., "the number of green strips", "the number of parts") without introducing complicated expressions ("the digit that stands for the number of units of ten in this number", "the number which when multiplied with the value of this fraction gives the numerator") or unfamiliar terminology ("place value", "numerator") (Ohlsson & Hall, 1990, p. 15).

At a time when it seems quicker and easier to tell students a procedure, demonstrate an example or two and then set them to work as is common in classrooms, it may not be obvious as to why a teacher would want to bother with the declarative proceduralisation steps together with the use of concrete materials. It seems that the start of a topic is difficult enough without adding what might be seen as extraneous ideas and activities. However, these views are simplistic, in particular they ignore important elements of what research tells us about learning, and they ignore students' construction of meaning and their development of reflective thinking and problem solving strategies. So the purpose of the use of concrete materials, then, is as Ohlsson and Hall state "to get the process of procedure acquisition started by providing an easy-to-learn procedure" (1990, p. 15).

Isomorphism and procedural analogy

The issue of analogy was discussed previously[4] and led to the conclusion that learning by analogy is common across a range of fields, and an important component of human learning. People learn easily and efficiently through analogies, the process is simple for humans and possible in machine learning. Especially in school learning, teachers make regular reference to analogies to assist student learning. Analogies are an essential element of the procedural analogy theory, where learning comes about through the analogy between declarative and procedural structures, and through the analogy between one set of procedures and a second set of procedures. As Ohlsson and Hall stated "(t)o construct a procedural analogy is to say, in effect, that the solution to *this* task is like the solution to *that* task, where the phrase 'is like' is to be interpreted with a particular mapping in mind" so that "a new procedure is *created* by using a previously learned procedure as a template" (1990, p. 16). For example, subtracting a three digit number from a four digit number has many similarities to subtracting a two digit number from a three digit number, as does the addition of algebraic fractions compared to addition of fractions without pronumerals, and calculating the volume of a cone when a student already knows how to calculate the volume of a pyramid. A procedure created from an already known procedure is said to be *isomorphic* with the original procedure.

A major function of procedural analogy is to move a physical procedure using concrete materials or other embodiments to the symbolic domain. That is, a symbolic procedure can be learned by analogy from a previously learned concrete materials procedure, so avoiding having to develop a symbolic procedure from the very beginning. That is, analogies serve as the cognitive equivalent of embodiment procedures, and allow bridge building between such procedures and the targeted symbolic procedure, through symbolic procedures isomorphic to the embodiment procedure. The analogical construction will require a set of substitutions, for example, the teacher might say "this is just like what we did before" and point out the specific details of the analogy. The symbolic procedure can then be practised and automated until the

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4 Chapter 2, p. 13ff.

learner has no further need of the embodiment. It needs to be noted, though, that this movement from concrete to symbolic will depend on the extent of the analogy between the two systems. That is, learning by analogy where such analogies are either not obvious or strained and artificial, are likely to be problematic. In other cases it may be necessary to invent an intermediate procedure to link the concrete system to the final symbolic system, and so provide a more obvious analogy. For example, an MAB representation involving 3 hundreds, 6 tens and 5 units can be thought of as representing the number 365. At the same time the representation could be written in an expanded form such as 300 + 60 + 5, so emphasising the place value of the number. This intermediate procedure, the expanded written numeral in this case, may assist learning through increasing the analogy between the initial concrete representation and the final symbolic representation.

One of the purposes of concrete materials is to provide an embodiment which is easy for the teacher to describe and for the learner to observe, operate on and analyse. Physical procedures can be seen and discussed more readily than symbolic equivalents, and so the teacher is more able to observe and intervene, and the learner more able to demonstrate, discuss and explain through embodiments than through symbols. By using analogy, learners develop symbolic procedures isomorphic to the embodiment procedure. While the structure of this symbolic procedure is independent of the embodiment, its structure can be explained by analogy from learners' actions on concrete materials, thus providing a visible raison d'être for the use of concrete materials in initial learning, and allowing a vehicle for remediation if the procedure is later forgotten in part or whole. Procedural learning does not stop when a cognitive procedure has been established. In particular, the original symbolic procedure may be simplified through practice and student identified short-cuts. For example, the speed of a procedure becomes faster with practice, as learners apply both chunking and automatisation. Further, through practice, discussion and analysis, learners discover short-cuts.

What has been said so far about procedural analogy? The teacher, perhaps after allowing a period of exploration, describes an embodiment procedure. Learners adopt this procedure through a modification of their own experiences and through discussion. The teacher's instructions are *declaratively encoded* by learners, and through practice the embodiment procedure is *proceduralised*, and so executable as both a cognitive and a practical procedure. The embodiment procedure is again practised and then moved to a *symbolic* representation by *procedural analogy*, where the symbolic procedure is isomorphic to the embodiment procedure. The closer the isomorphism between the embodiment procedure and the symbolic procedure, the easier the analogical learning step. If this symbolic procedure is not the target procedure, further transformations are needed, but this procedure already contains the final elements of the target procedure, so requires only *simplification*.

In this teaching approach, the learner is not required to perform complicated, abstract and seemingly inexplicable symbolic manipulations. Learning is guided by instruction, exploration and discussion. Further, because of the actions on embodiments, there is a store of procedures for resolving cognitive conflict as the movement to symbolic procedures takes place. Learning has become more of a translation and transformation of physical and cognitive structures, rather than the invention or copying of complicated symbolic patterns in a vacuum.

The procedural analogy theory predicts that successful learning is a function of four variables: the extent to which verbal descriptions of an embodiment procedure are easy to understand; the extent to which the declarative encoding of the embodiment procedure can easily be proceduralised, and generate the correct procedural rules; the extent to which the embodiment procedure is isomorphic with the targeted symbolic procedure; and the extent to which any intermediate representation contains the target procedure as a substructure, so that the final transformation can take the form of a simplification.

A Measure of Isomorphism Between Procedures

The procedural analogy theory seeks to distinguish between successful and unsuccessful uses of embodiments in arithmetic instruction, and so requires some measure of the degree of isomorphism between an embodiment procedure and a symbolic procedure. It is essential for the theory to be able to quantify the degree of isomorphism, so as to be able to predict the likely success of the embodiment instruction. If the embodiment procedure is not sufficiently isomorphic to the symbolic procedure, then the analogical transfer will fail, and the learner will not benefit from using the embodiment.

The task of explaining why an embodiment is likely to work well becomes the task of showing that action on the embodiment is isomorphic with the relevant symbolic procedure. Similarly, explaining why an embodiment does not work well, implies showing that the embodiment procedure is not isomorphic to the symbolic procedure. Analogies are unlikely to be totally isomorphic or totally non-isomorphic, so there has to be a measure of isomorphism between two procedures. Ways of representing knowledge have previously been discussed (including traces, planning nets, and procedural nets; in chapter 2) and possibilities for quantifying isomorphisms (for example, VanLehn and Brown's *closeness metric* in Chapter 2, and to a lesser extent, Resnick and Omanson's *mapping instruction* in Chapter 3). In general, these attempts require that planning nets and the like exist in the head of the learner, and that the learner both has the structure and understands the relationships between components of the structure. It is made all the more problematic since it is difficult to calculate an index to compare different representations, and different instructional sequences for any one representation. These aspects make their value somewhat doubtful, and not especially helpful in designing instruction.

The procedural analogy theory (Ohlsson & Hall, 1990) developed a measure of isomorphism between two procedures based on a comparison between expanded traces of the procedures. The expanded trace of a procedure contained the sequence of goals and actions generated when the procedure was carried out. The isomorphism between two traces was calculated by mapping the expanded traces of the two procedures onto each other, beginning with the top goal. If two goals fulfilled similar functions in the two procedures, then those goals are counted as corresponding. When a goal could not be mapped onto the goal in the other procedure, then that goal and all goals below it in the expanded trace are counted as not corresponding. The degree of isomorphism between the two procedures was expressed by the index:

$$I_{(1,2)} = \frac{(N_1 + N_2 - 2) - (D_1 + D_2)}{N_1 + N_2 - 2}$$

where N_1 and N_2 were the total number of goals and actions in the two traces, respectively, and D_1 and D_2 were the number of entries in each trace that did not have a match in the other trace. Two (2) is subtracted from the total number of entries, because of the assumption that the two top goals correspond to each other; if they did not, then the issue of the isomorphism between the two procedures would not arise. The index varies between 0 and 1, where zero represents no correspondence with each other, and one indicates complete correspondence. By comparing the number of corresponding entries in proportion to the entire set of entries, the index is independent of the size of the trace, and allows comparisons between instructional possibilities. Such comparisons are provided in Tables 4.1 and 4.2, together with various isomorphism indices calculations based on the data in these tables and modifications to them.

Table 4.1
Comparing traces for using MABs and the final target procedure, in calculating 85 - 32.

MAB procedure	**Target procedure**
0.0 85-32	0.0 85-32
0.1 Subtract 32 from 8T, 5U	
1.0 Process units	1.0 Process units
1.1 Take 2U from 5U	1.1 Take 2 from 5
1.2 Recall 5U - 2U = 3U	1.2 Recall 5 - 2 = 3
1.3 Record 3U in answer space	1.3 Record 3 in answer space
2.0 Process tens	2.0 Process tens
2.1 Take 3T from 8T	2.1 Take 3 from 8
2.2 Recall 8T - 3T = 5T	2.2 Recall 8 - 3 = 5
2.4 Record 5T in answer space	2.4 Record 5 in answer space
4.0 Read answer (5T 3U)	3.0 Read answer (53)

Table 4.1 indicates the teaching approach with MABs, the final target procedure, and the detail of how the MAB teaching approach links to the final procedure. Substituting data available in Table 4.1 into the isomorphism index formula

$$I_{(1,2)} = \frac{(N_1 + N_2 - 2) - (D_1 + D_2)}{N_1 + N_2 - 2}$$

and using values $N_1=11$, $N_2=10$, $D_1=1$ and $D_2=0$, $I_{(1,2)}$ becomes 18/19 or 0.95, a high isomorphism index. If the recall steps are left out of the MAB process – that is, if the teacher does not have students consider the arithmetic operation they are performing – then steps 1.2 and 2.2 are no longer used, and if the teacher does not have students deliberately record the answer then step 3.0 is no longer used. In these circumstances the new values become $N_1=8$, $N_2=10$, $D_1=1$ and $D_2=3$, and so $I_{(1,2)}$ becomes 12/16 or 0.75, considerably lower than the 0.95 in the first instance. If any students perform the calculations 5 - 3 and 8 - 3 on their fingers, or by any form other than an immediate mental recall, these students are using two additional steps in the target procedure, which would become 1.2.1 and 2.2.1. Here $N_1=8$, $N_2=12$, $D_1=1$ and $D_2=5$, and so $I_{(1,2)}$ becomes 12/18 or 0.67. The differences in the indices reflect plausible different teaching approaches.

Table 4.2 repeats the process in Table 4.1 for a more complex algorithm where there is trading.

Table 4.2
Comparing traces for using MABs and the final target procedure, in calculating 75 - 39.

MAB procedure		Target procedure	
0.0	75 - 39	0.0	75 - 39
0.1	Subtract 39 from 7T, 5U		
1.0	Process units	1.0	Process units
1.1	Take 9U from 5U (cannot)	1.1	Take 9 from 5 (cannot)
1.1.1	Trade for more units	1.1.1	Trade for more units
1.1.2	Move 1L from 7L to bank, bring back 10U	1.1.2	Recall 7 = 6 + 1
		1.1.3	Cross out 7, write 6
1.1.3	Join 10U and 5U	1.1.4	Write 1 next to 5
1.1.4	Recall 10U + 5U = 15U	1.1.5	Recall this is 15
1.2	Take 9U from 15U	1.2	Take 9 from 15
1.3	Recall 15U - 9U = 6U	1.3	Recall 15 - 9 = 6
1.4	Record 6U in answer space	1.4	Record 6 in answer space
2.0	Process tens	2.0	Process tens
2.1	Take 3T from 6T	2.1	Take 3 from 6
2.3	Recall 6T - 3T = 3T	2.2	Recall 6 - 3 = 3
2.4	Record 3T in answer space	2.4	Record 3 in answer space
3.0	Read answer (3T 6U)	3.0	Read answer (36)

In this case, data in Table 4.2 indicate $N_1=15$, $N_2=15$, $D_1=2$ and $D_2=2$, and so $I_{(1,2)}$ becomes 24/28 or 0.86, a high isomorphism index. If the recall steps are left out of the MAB process (steps 1.1.4, 1.3 and 2.3) , and if the joining step 1.1.3 is left out of the MAB sequence the new values become $N_1=1$, $N_2=15$, $D_1=2$ and $D_2=6$, and so $I_{(1,2)}$ becomes 16/24 or 0.67, considerably lower than the 0.86 in the first instance. If any students perform the calculations 15 - 9 and 6 - 3 on their fingers, then $N_1=11$, $N_2=17$, $D_1=2$ and $D_2=8$, and so $I_{(1,2)}$ becomes 16/26 or 0.62.

It is generally the case, that in everyday classrooms, teachers have clear ideas of what knowledge and skills students are expected to acquire. So the development of alternative instruction strategies, through which traces can be compared, is likely to be strongly influenced by these pedagogical necessities[5]. Teachers cannot know which procedures students will in fact acquire during their instruction, but through exploration, discussion, analysis, direction and examples, teachers can assist students in making meaning and in acquiring efficient cognitive networks. Indeed, teachers know what they intend, and it is these intentions that provide the target traces. While procedures can be implemented in different ways, resulting in different traces, teachers will have specific goals in mind, and will encourage students not only to construct their own meanings, but to have efficient procedures. At the same time, even with consistent goals in mind, teaching strategies can alter, so traces may be quite different. For example, subtracting without decomposition will give a different trace to one where decomposition is necessary, and these will in turn be different from a trace where there is a subtraction across a zero. Since there are many alternative strategies not only between topics but also within the one topic, and because there is a wide range of procedures differing in slight but important ways within a topic, the isomorphism index is calculated from a sample of common algorithms within the given topic.

In calculating the isomorphism index, there should be no goals with single subgoals, the trace should be consistent (the same goal generates the same subgoals), and entries with different characters (such as arithmetic operations and physical manipulations) need to be distinguished from each other. The extent to which two entries correspond will depend on the context. For example, using MABs to trade a ten for ten units, and placing these units with the existing units, is analogous to crossing off a ten in the tens column and placing a ten in the units place value column of an algorithm, but subtracting from the carried ten MAB units, then adding on the units that were originally there, is not analogous to adding ten to the units place value and subtracting from that new number. Further, to consider the trading of ten MAB units for one MAB ten as ten single-unit trading operations is inconsistent with good pedagogy and everyday classroom practice. In particular, teachers would be unlikely to proceed to written algorithms unless these kinds of exchanges had been operationalised, chunked and automatised by students. These decisions are not entirely arbitrary; they are professional judgements, and are open to empirical research, though that path has not been followed here. Further, any such arbitrariness does not necessarily negate the theory, and in any event all instructional design involves judgements by the designer and the teacher.

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[5] The practicality of the Procedural Analogy Theory, largely through its ease of implementation is discussed in Ohlsson and Hall (1990, p. 60).

Summarising what has been said so far, the procedural analogy theory argues that the pedagogical effectiveness of an embodiment is a function of four variables: the ease with which the learner can understand the description of the embodiment procedure; the ease with which that description can be proceduralised; the degree of isomorphism between the embodiment procedure and the symbolic procedure; and, the degree to which transformations of either the embodiment procedure, or any intermediate procedure, into the target procedure take place through simplifications (as opposed to transformations that require the building of new mental code). The procedural analogy theory has as one of its strengths that the initial construction of the embodiment procedure draws upon the child's extensive experience of operating in the physical domain. From that point, all learning consists of substitutions and revisions of the embodiment procedure, rather than newly introduced, seemingly arbitrary and complicated symbolic constructions. This is in considerable contrast to both learning through solved examples, where the learner has to focus on specific variables impacting on the procedure, and learning by discovery, where the learner has to derive the important principles.

The procedural analogy theory predicts that the degree of isomorphism between the embodiment procedure and the symbolic procedure is a major determinant of pedagogical effectiveness. To measure the isomorphism index for a particular algorithm involves deciding on the target symbolic procedure, writing an expanded trace for this symbolic procedure, identifying and trialling suitable embodiments and sequences, writing an expanded trace for the embodiment procedure, mapping the traces onto each other, and calculating the isomorphism index[6]. The theory allows a range of pedagogical alternatives to be considered prior to instruction commencing, and indicates that it is not useful to use embodiments in inconsistent and unguided patterns at the stage of moving from concrete representation to symbolic representation.

[6] The isomorphism index formula, its calculation, and a range of traces and applications, are discussed in Ohlsson and Hall (1990).

Chapter 5

Simultaneous and Successive Processing

Introduction

Investigating the procedural analogy theory is the central theme of this research. This theory is based on application of an isomorphism index so as to compare different teaching approaches: and this research investigates the learning outcomes of these teaching approaches. The theory, and the measurement of mathematics achievement, involve individual students. However, the resulting data will provide only limited insight into the way in which individuals differ from one another with regard to their cognitive processing. While individual test results can be analysed, and interviews with participants can be conducted and analysed, the procedural analogy theory is primarily concerned with pedagogy, with instruction of groups of students. It is not a theory of individual differences.

Consequently, the addition of a study of individual differences is likely to be beneficial in this research, so that the research, taken as a whole, considers both group instruction and individual learning. The procedural analogy theory is particularly concerned with cognitive processing: it would be helpful to be able to identify a complementary theory emphasising individual differences. Such a possibility is the theory of simultaneous and successive processing, based on Luria's clinical work. The choice of simultaneous and successive processing is especially appropriate since the procedural analogy theory is concerned with the use of concrete materials, and with both physical and symbolic representation systems, while simultaneous processing is concerned with two and three-dimensional spatial abilities. Das has argued that the cognitive processing of both logical relationships in mathematical thinking, and spatial relationships, "should share the same underlying coding process, which is simultaneous processing" (p. 121, 1988): and Solan claimed the need for simultaneous processing "in the understanding of the relationship between two or more objects" (1987, p. 241). Taking these views together indicates that a theory of individual differences involving simultaneous processing is particularly appropriate for investigation within the context of the procedural analogy theory. In particular, because the procedural analogy theory requires learners to cognitively map one set of relationships onto another set of relationships through analogy, the theory, therefore, requires a holistic view of both physical actions representing mathematical constructs and written mathematical symbols; and this requires simultaneous cognitive processing. There is, then, likely to be some commonality between concrete representations in the procedural analogy theory and levels of simultaneous processing.

Brain Functioning

Simultaneous and successive processing refer to two methods used within the brain to process data. As their names suggest, simultaneous processing involves data analysed together, at the same time, whereas successive processing involves sequential data, one item processed, then the next, and so on. These concepts derive from the work of Luria, clinical work on brain-damaged clients which has since been operationalised in the context of the broader population by a range of researchers.

The Lurian model of cognitive processing is a model of brain-action, an information processing model, one concerned with the processes of analysing data in the brain rather than with making inferences about intelligence. Luria (1966, 1973) was concerned with the impact of damage to the brain on clients' information processing abilities. By studying sufficient clinical cases Luria was able to develop a model of cognitive processing. For example, damage to the parietal-occipital[1] region indicated a loss of spatial ability, exhibited by an inability to read maps or to read the time from hands on a clock and in the performance of various arithmetic operations. This loss of simultaneous processing capacity was evident in clients Luria worked with who had temporal[2] or fronto-temporal[3] brain lesions. In such cases, clients were able to read multi-digit numbers, but were unable to recall a series of numbers. They were unable to recall the sequence either of events in a story read to them immediately before, or of events in their day-to-day lives (1966). While Luria's theory was based on clinical analysis of small numbers of brain-damaged individuals, later works were more concerned with applications of this theory, and were based on factor analysis of large numbers of individuals so as to determine the existence of particular forms of cognitive processing (Das, 1988).

As has been explained in detail previously in this thesis, information processing approaches to cognition emphasise those processes that contribute to thinking and problem solving. Das (1988) introduces simultaneous and successive processing as part of the continuous processing of data analysis taking place in the brain, which he saw as "an active and

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[1] The parietal lobe of the brain is found in the upper rear region, the occipital lobe in the middle rear region. Injury to the back of the head may thus cause damage to the parietal-occipital region.

[2] The temporal lobe is situated in the lower centre section of the brain, slightly towards the front when viewed from side on.

[3] The frontal lobe refers to the front section of the brain. Fronto-temporal lesions refer to significant damage to the front sections of the brain.

dynamic processor engaged in seeking and selecting information and integrating it in a routine or novel manner" (p. 102). Luria (1966) regarded simultaneous processing as involving a synthesis of separate elements into a coherent whole, where all aspects of the situation are taken into account. He saw successive processing as temporal, sequential and dependent. In such cases there was no overall synthesis, the place of one element was clear only in its relation to a previous or following element.

According to Das (1988) data is registered first by any of the human senses, which send the data onto a sensory register, in either a simultaneous or a successive pattern. At this point a "complex interaction between states of arousal and attention, coding of information, and planning - Luria's three functional units - determines what information is transmitted to the central processing unit" (p. 106). The buffer of the sensory register next transmits selected data to the central processing section of the brain in a serial pattern. Later Das, Naglieri and Kirby (1994) describe this second function in more detail than "coding information" when they describe it as "responsible for receiving, processing, and retaining information" (p. 14). Luria (1966) noted that "there is strong evidence for distinguishing two basic forms of integrative activity of the central cortex" (p. 74). That is, his model of cognitive processing proposes that this central processing unit of the brain has two functions: simultaneous and successive processing. Researchers have agreed to the existence and labelling of these two factors, and to the arousal and attention unit. However, other researchers have proposed that the third functional unit is not a control construct, but is actually more concerned with planning (Das, 1988; Hunt, de Lacey & Randhawa, 1987), or memory (Try, 1989), or speed (Das & Molloy, 1975; Kirby & Das, 1978), or attention (Luria, 1973; Ransley, 1981; Tulloch, 1986; Walton, 1983), or strategy formation and executive programming (Molloy & Das, 1980). Das maintains that the actual processing method selected depends on interactions between individual preference and task demands. The final brain function is to provide an output, a reaction to the sensory stimulus. Such an output may be simultaneous or successive.

The Lurian model is clear and concise, is grounded in neuropsychology and psychology, and investigates the dynamic relationship between social and cognitive acts in learning (Crawford, 1987; Elliott, 1990; Vocate, 1987). Das (1988) goes to some length to justify the measures used to investigate applications of simultaneous and successive coding. He notes that we need both to measure these processes, and to use these measures in understanding classroom learning. Of course such processes are inferred, since they clearly cannot be seen: measures of their influence rely on correlational analysis[4]. Das notes "the statistical method of correlations is not popular in neuropsychology, and syndrome analysis instead of factor analysis is frequently used to determine brain-behaviour relations (Das, Kirby & Jarman, 1979; Luria, 1966b). The evidence is correlational nonetheless" (p. 104). Research by Naglieri and

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[4] Refer to Results Summary chapter of this thesis for the establishment of simultaneous and successive constructs in this work.

Das (1988), by Naglieri, Prewett and Bardos (1989) and by Naglieri, Braden and Warwick (1991) continued to investigate the ex stence of the constructs of simultaneous and successive processing.

Lurian Theory

Luria (1970, 1973) identified th ee principal functional blocks in the brain. The first block is the brain stem, and lower sections of the brain, and is responsible for wakefulness and discrimination among incoming data. The second block is the back half of the brain, including parietal and occipital lobes, but not al of them, and is where data is sorted, organised, integrated and stored. The third block is the front section of the brain and portions of the parietal and occipital lobes, and regulates cognitive activity. Unlike the other blocks, this third block has no sensory or motor functic ns; rather its purpose is to activate the brain, to direct attention and concentration, to plan, regulate and verify action.

Luria (1966, 1973) argued that cognitive processing "takes place through the combined workings of all three brain units" (1973, p. 99). That is, through the arousal system where data is first registered in the brain, in the processing system where data is recalled, transformed and stored, and in the control or planning system where the processing of data is monitored and directed (Kirby & Robinson, 1987). Luria (1973) argued, too, that of the three principal functional systems in the brain, the first component, the arousal and attention mechanisms of the brain, was responsible for maintaining a wakeful state and cortical control, and influenced and was influenced by the regulatory nature of the cortex. It did not contain modal specificity, but responded to changes in the state of the organism. The second component of the brain received, processed and stored data, and had high modal specificity, so that, for example, there were specific components for responding to visual or auditory information. Luria argued that this component was essential for the conversion of perception into abstract thinking, and so allowed simultaneous and successive processing. This component of the brain operated so that there was diminishing specificity of the hierarchically arranged cortical zones composing it, and progressive lateralisation of functions through the hierarchy of zones. The third component of the brain-controlled cognitive actions through monitoring cognitive activities and selecting processing strategies, it organised intentions, and regulated and verified the most complex human behaviours (Luria, 1973).

That is, the first and third brain units were concerned with awareness and observation of the individual's world, and with initiating and directing actions to that world. The second brain unit processed and structured data from that world and from within the brain, evolving and applying what Luria referred to as "dynamic functional structures" (1973, p. 68). Cognitive activity in this processing system is not isolated as either a simultaneous or a successive process; rather, the two components work together, with the emphasis of the one in comparison to the other varying from individual to individual, and from context to context. Das (1988) pointed out that it did not matter how data were presented to the brain's sensory register, the manner and mode was irrelevant. Once registered, data can be processed by the brain as synthesised and related, and so dealt with simultaneously, or processed in a temporally organised fashion, and so handled successively.

Luria (1973) reported that subjects with brain lesions in the posterior regions situated between the occipital, temporal and postcentral[5] regions of the hemisphere were unable to convert continuous data into a simultaneous perception, so that they were unable to navigate their way through familiar buildings. That is, simultaneous processing took place in the occipital-parietal region of the brain. Brain lesions in this area, together with lesions effecting speech and symbolic representation led to inferior intellectual processes, for although subjects with such lesions remained intelligent they had difficulty coping with situations requiring simultaneous analysis and so were considerably handicapped in problem solving. Disturbances to successive processing were associated with damage to the fronto-temporal area.

Simultaneous and successive processing

Luria developed the constructs of simultaneous and successive processing from his clinical work on the behaviours of subjects with cortical lesions, and argued that these two types of information processing occurred in different parts of the brain and were stored differently (Luria, 1966, 1970, 1973) Simultaneous and successive processing components have long had construct validity, and have been identified in a range of cultural settings (Das, Naglieri & Kirby, 1994).

Simultaneous processing involves the brain taking data fed to it sequentially, and analysing it as a whole and at the one time. For example, perception requires simultaneous processing, as does reading, problem solving and our place value number system. All these activities require the individual to take into account a range of inputs simultaneously. Identifying meaningful relationships between a number of concepts and applying these concepts together in a problem solving situation also requires simultaneous processing. Yet another example of simultaneous processing is determining the shape of an object: a range of attributes would have to be considered together at the one point of time. That is, simultaneous processing occurs in contexts involving copying a design, recalling a design from memory, or when reasoning about a design (Das, Naglieri & Kirby, 1994). Simultaneous processing occurs, then, in situations where multi-attributed data are involved either from external stimuli or from memory retrieval where earlier patterns of relationships are recalled. Simultaneous processing will occur during perception, during memory storage and retrieval and during complex analyses. This form of processing has been investigated through figure copying, memory-design tasks and verbal tasks (Das, Leong & Williams, 1978; Green, 1977; Kirby & Das,

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[5] The postcentral section of the brain is the central region of the brain, viewed from a side-on perspective, but slightly towards the rear.

1977, 1978; Mohanty, 1994; Solan, 1987; Try, 1989; Tulloch, 1986; Varnhagen, Varnhagen & Das, 1992; Walton, 1983). Das, Naglieri and Kirby (1994) outline the key features for simultaneous processing:

> the pieces of information have to be related in some way to one another;
> the relationship, or potential to find it, must exist in long term memory;
> the resulting code takes up only one unit in working memory;
> there is no intrinsic order to the code;
> some previously coded information may be lost (p. 58).

Successive processing involves the analysis of data in sequence, and is of necessity temporally organised. That is, both ordered structure and the passing of time are elements of successive processing. Data in the processing sequence are considered in succession, they are constructed from a sequence of linked data, and cannot be considered together in the one instant of time. This processing requires each stage of the sequence to be linked with a response, which then fires the next stage of the processing sequence. The recognition of a musical tune, with notes identified one after the other, is an instance of successive processing: the analysis is sequential and temporal. Automatised routines and other cognitive activities not requiring introspection, for example the arrangement of words in a sentence, or the solving of an arithmetic algorithm, are based on successive processing (Kirby & Robinson, 1987). Successive processing will occur during sensorimotor activities such as writing and finger tapping, in retrieving and operationalising a highly automatised habit such as driving a car, and during necessarily sequential cognitive activities such as speaking and writing. These views are consistent with Luria's argument that successive processing is essential for the automatisation of skilled movements (1966b). Successive processing has been investigated using number, letter and word spans and through shape sequence tests (Angus, 1985; Green, 1977; Mohanty, 1994; Ransley, 1981; Tulloch, 1986; Varnhagen, Varnhagen & Das, 1992; Walton, 1983). The key features of successive processing are:

> only the sequential relationship is seen;
> the successive code takes up one working memory space for each unit in the sequence initially;
> with practice the sequence becomes automated, and so takes up less space in working memory;
> the order of the items is critical (Das, Naglieri & Kirby, 1994, p. 59).

Das (1988) points out that these processes are not an either/or situation, and that in processing data an individual is likely to use both simultaneous and successive processes: "(n)ot the exclusive use but, relatively speaking, a predominant utilization of either of the two coding processes" (p. 103-4). The two processes are not hierarchically related, the mode selected depends on the individual's preference for information processing, the specific task at hand (Das, 1973), and the interaction between the preferred mode and the task. That is, cognitive analysis requires both time (successive processing) and space (simultaneous processing): time provides the medium through which successive processing exists, similarly space provides the medium for the representation of simultaneous processing. In particular, Das et al. (1979), Kirby and Das (1977), and Randhawa and Hunt (1979) all agree that neither simultaneous nor successive processing by itself is sufficient for high achievement in school.

Organising the application of cognitive processing is what Luria calls cognitive control (1973). It is this control mechanism that determines what aspects of a situation are addressed by cognitive action, an executive function that organises and maintains cognitive actions as tasks are performed (Bialystok & Bouchard Ryan, 1985; Das, 1988). The significance of internalising the process of controlling cognitive attention is obvious in learning, and is based on both language development and on more general developmental factors (Luria, 1973). Thus, problem solving ability, for example, is determined by both the individual's information processing capabilities, and the degree of cognitive control of that individual. As has been mentioned before, the control construct is less well defined than the simultaneous and successive constructs. For example, the control construct has also been referred to as planning (Das, 1988; Hunt, de Lacey & Randhawa, 1987), which is consistent in meaning with control, but also as speed (Das & Molloy, 1975; Kirby & Das, 1978), memory (Try, 1989), attention (Luria, 1973; Ransley, 1981; Tulloch, 1986; Walton, 1983), and strategy formation and executive programming (Molloy & Das, 1980), none of which are synonymous with control. Das, Naglieri and Kirby (1994) have developed a model of cognitive functioning called *Planning, Attention, Simultaneous, and Successive (PASS) processing* in which they hope to expand the notion of intelligence, particularly concerning exceptional children and the development of intervention programs. One of the essential elements of this model is the planning (or control) component, that the authors claim allows simultaneous and successive processing to be applied to the focus of attention. This construct has not been considered in the present research, nor was it in a range of other studies, including Kirby and Robinson (1987) and Solan (1987).

Measuring cognitive processing abilities

Reliable measures of simultaneous and successive processing have been widely described and applied (Crawford, 1987; Das, et al., 1975, 1979; Elliott, 1990; Kirby & Robinson, 1987; Tulloch, 1986). In these works, the statistical process of principal components analysis was applied to a range of data gathered from each subject, so as to determine common variances on test scores, which were then applied to analysing individual differences in behaviours.

The value of the Lurian model, together with the value of this form of statistical analysis, in attempting to understand information processing and classroom learning is well documented (Crawford, 1987; Das, 1988; Das, Kirby & Jarman, 1975; Elliott, 1990; Green, 1977; Hunt, de Lacey & Randhawa, 1987; Kirby & Das, 1977; Solan, 1987; Try, 1989; Tulloch, 1986). For example, Das (1988) reported on work with a range of colleagues in which they found successive processing to be essential for reading achievement, but that in order to move to more advanced levels of reading, simultaneous processing was required. He reported too, that simultaneous processing was a good predictor of mathematics achievement, that successive processing was unrelated to mathematical achievement, and that there was some indication that achievement in arithmetic computation is related to cognitive control scores. However, as already described, this concept of 'control' is problematic. Further, these findings may need to be examined again since the researchers appear not to have made a clear distinction between arithmetic, problem solving in general, and other topics in mathematics. Indeed Das' notion of mathematics may be simplistic, particularly when he refers to "the two basic skills in mathematics" as if arithmetic and problem solving constitute all mathematics (p.125).

The tests used in this research to identify simultaneous and successive processing have been refined over the years by a range of researchers. For example, Ransley (1981) measured simultaneous ability through having participants copy figures and memorise shapes based on an array of dots, and these have been modified and verified by later researchers (Crawford, 1987; Elliott, 1990; Try, 1989; Tulloch, 1986). These measures appear consistent with Luria's ideas. In particular, the Shapes test, the Paper Folding test and the Matrix A test used in this research, and described in more detail in the Method chapter, required children to have a global picture involving a range of attributes. That is, in order to correctly answer items in these tests children had to identify a range of attributes together, in the one instant, and this required simultaneous processing. Measures of successive processing typically involved reproducing a list of words, letters or numerals. These tests were based on the work of others (Green, 1977; Kirby & Das, 1978; Ransley, 1981), modified later (Tulloch, 1986; Crawford, 1987), and used in this research as Number Span, Letter Span and Word Span tests.

Reported findings

Investigations into the operationalisation of the simultaneous and successive constructs have been carried out by many researchers (Angus, 1985; Crawford, 1987; Das, Cummins, Kirby & Jarman, 1979; Das, Kirby & Jarman, 1975; Das & Molloy, 1975; Elliott, 1990; Fitzgerald, 1973, 1978; Green, 1977; Mohanty, 1994; Molloy & Das, 1973, 1980; Varnhagen, Varnhagen & Das, 1992; Try, 1989). These researchers have been more concerned with typical classrooms than clinical investigations of atypical individuals. Whereas Luria was concerned with the study of brain-damaged individuals, other researchers have been more concerned with the implications of simultaneous and successive thinking for learning and for classroom practices. After Luria identified simultaneous and successive factors, other researchers used factor analytic studies to confirm the existence of these factors. Recent research has been concerned particularly with the roles of these factors in language and mathematics learning (Das, 1988; Das, Naglieri & Kirby, 1994).

This model of simultaneous and successive processing has been used in a number of investigations of cognitive processing, strategy development and teaching applications (Crawford, 1987; Cummins & Das, 1978; Das, Cummins, Kirby & Jarman, 1979; Elliott, 1990; Fitzgerald, 1973; Molloy & Das, 1979; Snart, 1979; Kirby & Robinson, 1987). Classroom investigations have shown children as young as four have simultaneous and successive processing capabilities (Angus, 1985; Crawford, 1987; Elliott, 1990; Tulloch, 1986). The impact of simultaneous and successive processing on aspects of language development, especially reading, has been a common area for research (Kirby, 1992; Leong & Sheh, 1982; Shinn-Strieker, 1989). Other studies have shown the model to have relevance in the teaching of school-level literacy and mathematics, across grade and age (Crawford, 1987; Elliott, 1990; Hunt & Fitzgerald, 1979; Try, 1989). Yet others have given more emphasis to the role of simultaneous and successive processing in mathematical learning (Crawford, 1987; Merritt & McCallum, 1984; Molloy & Das, 1980). Finally, there are isolated studies examining atypical areas in which these cognitive processing abilities may impact on student learning. For example, Winn (1986) investigated high school students' learning of electronic circuit diagrams, Sutherland and Winn (1987) examined the map reading skills of graduate education students, and Merritt and McCallum (1983) looked at grade point average and American College Testing data.

Molloy and Das (1980), Merritt and McCallum (1984) and Crawford (1987) have reported a significant relationship between scores in mathematics achievement tests and in strengths in simultaneous processing. This is consistent with Luria's claims (1966, 1973) that disturbances in simultaneous processing can result in difficulties in number concepts and mathematical operations. Tulloch (1986) found that both simultaneous and successive processing were related to achievement in number tasks, and Crawford (1987) suggested there were sex differences in simultaneous processing, but no differences in successive processing.

There were initially many studies concerned with verifying simultaneous and successive constructs. These have been followed by studies investigating the impact of simultaneous and successive dimensions of cognitive processing on young children and students of all ages, and in a range of knowledge areas. The present work will investigate relationships between simultaneous and successive processing and achievement in mathematics, and relationships between simultaneous and successive processing and a range of teaching approaches.

Chapter 6

Research questions

Following the previous chapter's consideration of a broad range of literature, this chapter is concerned with identifying a series of research questions that follow logically from the literature review. These questions are typically presented in groups, and are numbered so as to make clear which questions belong together. For example, the first research questions (1 and 1a) are concerned with using the procedural analogy theory to predict differences in learning outcomes, and questions 2a, 2b and 2c are concerned with the way students talk about subtraction procedures and the meanings they appear to have made. There are also research questions about dimensions of cognitive processing and scores on subtraction tests (question 3), about scores on subtraction tests in the context of dimensions of cognitive processing and the procedural analogy theory (question 4), and research questions concerned with the way patterns of student talk and mathematical meanings relate to dimensions of cognitive processing (question 5).

The literature reviewed to this point indicates that this study involves two major fields of research, one area is mathematics education, the other cognition. More specifically, this research seeks to investigate the procedural analogy theory, devised by Ohlsson and Hall (1990), as a basis for designing efficient instruction in mathematics, and for explaining the value of concrete materials in teaching arithmetic skills. The theory argues that learners convert teacher talk to a declarative encoding, but this is not a cognitive procedure and so not executable. Through proceduralisation this new encoding is transformed into an executable function, where it can be represented symbolically. The theory grew out of the unsuccessful development of an intelligent tutoring system, and involves concepts from cognitive science, such as the role of analogy in learning, and the roles of practice and chunking. The main research question here is:

1 can the procedural analogy theory be applied effectively in the teaching of subtraction to elementary grade students?

The use of concrete materials in the teaching of school mathematics has long been supported by mathematics educators and by teacher educators. The use of such materials in schools is common, but generally confined to use by only some teachers, typically involving younger children and then in idiosyncratic manners. The research literature is equivocal about the effectiveness of concrete materials, with contradictory studies having been reported over many years. This lack of certainty is frequently further confused by the uncertainty of the details of teaching approaches used in these investigations. Often the pedagogies used in the experimental treatment and in the comparison group are described in too little detail to know exactly what these treatments mean for day-to-day teaching practice. The procedural analogy theory provides guidelines for the planning and comparison of pedagogies before their implementation in classrooms. This guidance occurs through the use of a formula to calculate an isomorphism index, which predicts the effectiveness of a given pedagogy. The subsidiary research question here becomes:

1a will participants in a high isomorphism index treatment gain greater scores on subtraction achievement tests than participants in a low isomorphism index treatment?

This research is concerned not only with differences in scores between students who participate in different teaching approaches, it is also concerned with the different kinds of learning that take place, especially the kinds of meanings students appear to make. This leads to the following research questions concerning student meaning and the procedural analogy theory:

2a will participants in a high isomorphism index treatment talk differently, and make different meanings, about written subtraction procedures, than participants in a low isomorphism index treatment;

2b will participants in a high isomorphism index treatment talk differently, and make different meanings about MAB procedures involving subtraction, than participants in a low isomorphism index treatment; and

2c will participants in a high isomorphism index treatment be more able than participants in a low isomorphism index treatment, to relate the use of MABs to the development and structure of written algorithms?

The procedural analogy theory provides guidelines for class group instruction, and is concerned with the detail of teaching class groups of students. Further, the procedural analogy theory may have its basis in cognitive psychology, and it is concerned with cognitive processing, but it is not especially concerned with individual differences. Luria's research into brain functioning, resulting in his identification of simultaneous and successive patterns of brain functioning, appears to be an appropriate way in which to examine individual differences in the context of the application of the procedural analogy theory in classrooms. Extensions of Luria's work have been concerned with dimensions of cognitive processing, and have been applied to student learning in language and, to a lesser extent, in mathematics. In this research, a principal components analysis will be used to categorise participants into one of four cognitive processing abilities: high simultaneous-high successive, high simultaneous-low successive, low simultaneous-high successive, and low simultaneous-low successive. The major research question here is:

3 will there be differences in scores on subtraction tests between students with different patterns of competence on the two dimensions of cognitive processing?

A range of subsidiary questions follow from this question about subtraction test scores and dimensions of cognitive processing. For example:

3a will there be differences in subtraction test scores between high simultaneous-high successive processing students and low simultaneous-low successive processing students; and

3b will there be differences in subtraction test scores between high simultaneous-low successive processing students and low simultaneous-high successive processing students?

The procedural analogy theory attempts to guide instruction, the Lurian theory of simultaneous and successive processing seeks to explain brain functioning, and both the procedural analogy theory and the Lurian theory have implications for the learning taking place in those students participating in this investigation. There may be a relationship between the procedural analogy theory, especially a high isomorphism index teaching approach with its emphasis on emphasising relationships within and between concrete and symbolic representations, and simultaneous processing, concerned with individuals' skills in dealing with two and three dimensional spatial representations. The research question here is:

4 will students with different patterns of competence on the two dimensions of cognitive processing react differently to high or low isomorphism index teaching approaches, so as to produce differences in subtraction test scores according to dimensions of cognitive processing and teaching approach?

Once again, there are subsidiary questions following from this research question about interactions between teaching approaches and cognitive processing abilities. For example:

4a will subtraction tests scores by high simultaneous-high successive processing students differ from those by low simultaneous-low successive processing students, according to whether they participate in a high isomorphism index or a low isomorphism index teaching approach; and

4b will subtraction tests scores by high simultaneous-low successive processing students differ from those by low simultaneous-high successive processing students, according to whether they participate in a high isomorphism index or a low isomorphism index teaching approach?

The research questions so far have been concerned with student learning, and as will become clearer in the next chapter on method, the answers to these questions depend mainly on pen-and-paper tests. This study is also concerned with the meanings students appear to make of the mathematics they were taught. These research questions are considered:

5a will high simultaneous-high successive, high simultaneous-low successive, low simultaneous-high successive, and low simultaneous-low successive processors differ from one another in the way they talk about, and the kinds of meanings they make of, written subtraction procedures;

5b will high simultaneous-high successive, high simultaneous-low successive, low simultaneous-high successive, and low simultaneous-low successive processors differ from one another in the way they talk about, and the kinds of meanings they make of, MAB procedures involving subtraction; and

5c will high simultaneous-high successive, high simultaneous-low successive, low simultaneous-high successive, and low simultaneous-low successive processors differ from one another in the way they talk about, and the kinds of meanings they make of, the relationships between their actions on MABs and their written subtraction algorithms?

The next chapter, the Method chapter, describes the manner in which these research questions were investigated.