Chapter 7

Method

This chapter is concerned with the procedures employed to answer the research questions raised in the previous chapter. In investigating these research questions, measures need to be taken concerning the isomorphism index for a number of teaching approaches, so as to select treatment and comparison pedagogies, and measurements have to be taken of students' cognitive processing abilities. In addition, changes to students' scores on mathematics tests, before and after a period of instruction need to be measured to assess the outcomes of the instruction.

Sample

The research reported here involved 112 students, aged nine or ten, from two local primary schools[1]. These schools were situated in lower-middle class areas, where unemployment levels were high, particularly for young school leavers. Almost all the children in this study were born in Australia, though there were many whose parents or grandparents would have migrated to Australia. English would not have been the first language of some of these children's parents, but none of the children had difficulties speaking, writing or understanding English. These characteristics are typical of many schools in the Illawarra region, and typical of many urban Australian schools.

Each of the two schools used in the research had two Year 4 classes, which were arranged into three groups for this research, through systematically assigning names from the alphabetical class roll into three groups. The class rolls listed boys and girls separately: by taking the list of boys from one class, and placing names 1, 4, 7, 10, 13, 16 into one group,

[1] Two public elementary schools, situated in the Illawarra region, south of Sydney. The schools were selected because the researcher had previously worked professionally with the school principals, and both the principals and their staff agreed to participate in the research.

names 2, 5, 8, 11, 14 and 17 into a second group, and names 3, 6, 9, 12, 15 and 18 into a third group, each boy was assigned to one group. The process was repeated for the girls in the class, then for the boys and the girls in the other classes.

For each school, the two regular classroom teachers and the researcher met to select groups and teaching approaches. Each teacher selected one of the three groups of students, together with one of the three teaching approaches, the researcher took the third group[2]. All teachers participating in the research had at least seven years of elementary school teaching experience.

Experimental Design

Table 7.1 summarises the sequence of the research. It indicates that all students received a mathematics pretest, and completed tests designed to gauge cognitive processing abilities. Each of the treatment groups participated in one teaching approach, during which time lessons were videotaped by a research assistant, some field notes gathered, and students kept records of their work in a class workbook.

Table 7.1
Experimental design

Before	During	After	Later
Mathematics pretest and cognitive processing test	Teaching approach 1, 2 or 3	Mathematics posttest	Mathematics retention test
Systematic assignment to teaching approach 1, 2 or 3	Videotaping Field notes Workbooks	Interviews	Interviews

At the end of instruction all students completed a mathematics posttest, parallel to the pretest, and approximately 20% of the students were interviewed. The students to be interviewed were selected using a systematic method, based on an alphabetical list of names for treatment groups: a random number from 1 to 5 led to the selection of one student, adding 5 to the random number led to the next selection, and the process continued for each treatment group. Later, students completed a retention test, and those interviewed before were interviewed again. Data from these subtraction tests were analysed using a multivariate analysis of covariance through the SPSS^X package.

Students participated in eleven forty-minute lessons taught over a four week period, generally three lessons per week, where they were taught subtraction algorithms through the use of MAB materials. Each treatment group involved a teaching approach developed through the researcher discussing alternative teaching possibilities with these teachers, and by calculations based on the isomorphism index (I) described in the Procedural Analogy Theory

2 The researcher taught a high isomorphism index approach in one school, and a low isomorphism approach in the second school, so as to minimise any researcher-as-teacher biases.

chapter, Chapter 4. Two treatment groups followed a teaching approach consistent with a high isomorphism index (I > 0.8). One of these approaches used expanded numerals so that, for example, one hundred and sixty-five, shown in MAB materials, was initially written as 100 + 60 + 5. The second high isomorphism index teaching approach moved directly from the MAB materials to a standardised, contracted written form; where one hundred and sixty-five was written immediately as 165. The third teaching approach involved a lower isomorphism index (I < 0.7). These details are summarised in Table 7.2.

Table 7.2
Treatment groups

School	Treatment/Teaching Approach	Teacher
A	Treatment 1 – High I, expanded	1
	Treatment 2 – High I	2*
	Treatment 3 – Low I	3
B	Treatment 1 – High I, expanded	4
	Treatment 2 – High I	5
	Treatment 3 – Low I	2*

* the researcher

The teaching approaches employed with the two experimental groups differed from the teaching approach used with the comparison group in the detail of the correspondence between the actions on concrete materials and the written algorithm, and in the detail of the guidance and description given by the teacher. It should be noted that all teachers agreed that all these teaching approaches were acceptable, were typical teaching approaches, not dramatically different from existing classroom practices, and were supported by curriculum statements and textbooks. More detail of these teaching approaches is given later in this chapter.

Students' cognitive processing abilities were measured using a range of audio and pen-and-paper tests; and their mathematics achievement in subtraction algorithms was measured with three parallel tests, the first prior to instruction, at a posttest immediately after the completion of instruction, and a retention test four weeks later. After the posttest, and again after the retention test, a number of students were interviewed, mainly individually, but occasionally in groups.

As with much research conducted over some months, there were a number of missing data cells. Indeed more than 40 of the participants had at least one data cell entry missing. In order to have as many complete data entries as possible, yet without inordinately influencing

Table 7.3
Participant numbers

Treatment group	n	Levels of sim/succ processing* HH	HL	LH	LL
1	32	6	10	7	9
2	33	10	8	6	9
3	32	7	6	9	10
Totals	97	23	24	22	28

* Levels high (H) or low (L);
simultaneous first mentioned, then successive

the data gathered, the following principle was applied to the mathematics achievement test scores: for any given student, empty data cells would be replaced by the average score of the relevant treatment group for that component of the test, provided there were no more than two missing entries. No substitutions were made where students were absent from the cognitive processing tests. Table 7.3 indicates the number of students with complete data sets in the treatment groups, and in the various simultaneous-successive categories.

Measuring mathematical achievement

Students' levels of mathematical achievement in subtraction algorithms were measured using a test constructed according to a set of objectives[3] based on both the school system's syllabus for subtraction (New South Wales Department of School Education, 1989), the AM

Table 7.4
Components of subtraction tests

<u>Section 1</u> Subtraction algorithms 33 questions Total 129
Students have to complete written vertical and horizontal subtraction algorithms, scored as
1 per correct subtraction and
1 per correct decomposition
Questions valued from 1 (eg. 13-7) to 9 (eg. 93750-5867).

<u>Section 2</u> Explaining algorithms 6 questions Total 13
Students have to explain, or select an explanation, concerning the correct procedure to follow in order to solve a given subtraction algorithm, scored as
1 per correct subtraction and
1 per correct decomposition
Questions valued from 1 (eg. in 52-13, we say 3 from 2 cannot be done, give the reason for this) to 3 (eg. in 84-58, explain to a new pupil how to get the correct answer).

<u>Section 3</u> Missing digits 4 questions Total 6
Students have to complete a subtraction algorithm in which there is a missing digit, scored as
1 per correct subtraction and
1 per correct decomposition
Questions valued from 1 (eg. 16 - Δ = 9), to 2 (eg. 37Δ -154 = 218)

<u>Section 4</u> Place value 8 questions Total 8
Students had to answer a series of questions concerning the place value of digits, selecting one of four possible answers, scored as 1 each (eg. what is the value of 6 in the number 1365?: select from 6 units, 6 tens, 6 hundreds, 6 thousands.)

<u>Section 5</u> Problem solving 7 questions Total 44
Word problems involving the development and completion of subtraction algorithms, scored as
1 for algorithm development, 2 when more complex
1 per correct subtraction and
1 per correct decomposition
Questions valued from 3 (eg. one step word problem) to 15 (eg. multi-step, complex, word problem).

(Total 200)

[3] For example, two-digit subtract two-digit, no carrying, three-digit subtract three-digit with zero in minuend.

tests (ACER, 1971), the CATIM test (ACER, 1976), standardised tests used throughout Australia, and the YardSticks series of criterion referenced tests (1975). The test was constructed for the purposes of this research since there was no commercially available test that assessed the components that the research was to measure. Given the nature of these tests, particularly that they were meant to test subtraction skills, and that they were based on syllabus statements and previous standardised tests, they are likely to be both valid and reliable. There were five sections in the test, as described in Table 7.4[4].

Three parallel versions of the tests[5] were administered, the first immediately prior to instruction commencing, the next immediately after the completion of instruction, and the final version four weeks after the completion of instruction. The four-week period was based on time equivalent to the period of instruction. All these tests were administered by the students' usual class teacher, during normal lesson time, in their usual classroom, with sections 1 to 3 administered first, then sections 4 and 5 the next day. No particular time limit was set, so as to allow sufficient time for students to make their best attempts.

Measuring simultaneous and successive processing

The tests used in this research had previously all been applied widely in measuring simultaneous and successive processing: that is, they are valid and reliable. Details of each test, and previous use of the tests, are presented in Table 7.5[6]. In every case, there were standard formal instructions, and before children began each test, they were given a sample question which was then answered. These tests were administered in the students' regular classrooms by the researcher, in one session, where copying was minimised by judicious arrangement of children and furniture, and close observation by the test administrator. The tests were given in the sequence Number Span, Letter Span, Shapes, Paper Folding, Word Span and Matrix A so as to provide variety, and overcome the potential for boredom.

Number Span test

The Number Span test required children to listen to a series of from three to ten numbers, then write them in order on an answer sheet where eleven spaces were provided for writing numbers for each question. A score of one was given for each numeral placed in its correct position. In those cases where a numeral was placed incorrectly, but was the beginning of a correct sequence, no mark was awarded for the first number, but one was scored for each other

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[4] Preliminary analysis of the data suggested that only two of the five components were relevant to the analysis, that is, produced significant effects. Therefore, the analysis reported later in Chapter 8, considered only data on the algorithm and problem solving components of these tests.

[5] Copies of these tests are in Appendix A.

[6] A copy of these tests is available in Appendix B.

correct number in the sequence. If the final digit was correct it scored one regardless of its position. There were 16 questions of this type, with a maximum possible score of 82.

Table 7.5
Tests of simultaneous and successive processing

Test	Factor	Previous Use
Number span	Successive	Angus, 1984; Crawford, 1987; Das & Molloy, 1975; Gibson, 1993; Green, 1977; Klich, 1983; Walton, 1983.
Letter span	Successive	Angus, 1984; Kirby & Das, 1978; Klich, 1983; Walton, 1983.
Word span	Successive	Angus, 1984; Crawford, 1987; Gibson, 1993; Green, 1977.
Shapes	Simultaneous	Crawford, 1987; Gibson, 1993; Green, 1977; Try, 1984; Walton, 1983.
Paper folding	Simultaneous	Gibson, 1993; Luria, 1966b; Try, 1984; Walton, 1983.
Matrix A	Simultaneous	Angus, 1984; Crawford, 1987; Elliott, 1990; Fitzgerald, 1971; Green, 1977; Kirby & Robinson, 1987; Tulloch, 1986; Walton, 1983.

Letter Span test

The Letter Span test required children to listen to a series of from two to nine letters, then write them in order on an answer sheet where eleven spaces were provided for writing letters for each question. A score of one was given for each letter placed in its correct position. In those cases where a letter was placed incorrectly, but was the beginning of a correct sequence, no mark was awarded for the first letter, but one was scored for each other correct letter in the sequence. If the final letter was correct it scored one regardless of its position. There were 16 questions of this type, with a maximum possible score of 79.

Word Span test

The Word Span test required children to listen to a series of from two to eight words, then write them in order on an answer sheet where one line ran across the page for each question. A score of one was given for each word placed in its correct position. In those cases where a word was placed incorrectly but was the beginning of a correct sequence, no mark was awarded for the first word, but one was scored for each other correct word in the sequence. If the final word was correct it scored one regardless of its position. There were 12 questions of this type, with a maximum possible score of 51.

For sequences where there were at least four words, similar sounding words were scored as correct. For example, in the five word sequence in which *moss* was the second word, *toss* was a common alternative and accepted as correct. However, *air* was not accepted for *hair* in a two word sequence. Incorrect spellings were accepted. The logic behind this relates to attempting to maximise students' scores, to eliminate 'silly' errors, but not exaggerating this attempt.

Shapes test

The Shapes test required children to select any number of five components to make up a given shape. Children completed three practice examples which were then corrected before beginning the test, rather than the more usual one. Rotation of the given components was permitted, but there was to be no flipping or turning over. The test contained 12 items – four questions each relating to three shapes (a rectangle, a square and a cross) – and had to be completed in seven minutes. Answers were given by circling the appropriate components on the question sheet. Questions were scored as one for each correct combination, with a possible range of scores from zero to twelve.

Paper Folding test

In the Paper Folding test children were given a series of figures on the left hand side of a page, representing the folding of a piece of paper which then has a hole punched through it. All folds were horizontal, vertical or along a diagonal, with children completing a practice example before beginning the test. Children were asked to select which of five possibilities best represented what the sheet of paper, now with its array of holes, would look like if it were unfolded. Children indicated their selection by circling the appropriate figure on the question sheet. Questions were scored as one for each correct combination, with a possible range of scores from zero to seven.

Matrix A test

Items in the Matrix A test consisted of shapes, constructed by connecting dots in a 3 by 3 array with straight line intervals. Children were shown one such figure for five seconds, then had ten seconds to draw it on an answer sheet, where 3 by 3 arrays of dots were provided. Children completed two practice examples before beginning the test. There were eighteen questions, each scoring one if correct, with a possible range of scores from zero to eighteen.

Deriving simultaneous and successive processing factors through principal components analysis

The initial analysis required here involved a principal components analysis to derive simultaneous and successive processing factors. Establishing the principal components of a set of variables allows the researcher to view the structure of the data in terms of the minimum number of variables (SAS Institute, 1989). That is, variables were grouped together in such a way that within groups they were highly correlated, but the different groups were unrelated (Stevens, 1986; Tabachnick & Fidell, 1989).

The interpretation of these principal components is frequently problematic, not least because meanings have to be suggested by the researcher, and there is no measure possible to test the accuracy of assumed meanings. Which component related to which variable, and what it means, has to be interpreted by the researcher. Tabachnick and Fidell (1989) note that both principal components and factor analysis are interpreted in a pragmatic light, to make sensible meanings, rather than based on theoretical criteria. In the present study, the variables were selected on the basis of a structure demonstrated in a series of studies concerned with the Luria model (Angus, 1984; Crawford, 1986; Das & Molloy, 1975; Elliott, 1990; Fitzgerald, 1971; Green, 1977; Hunt, de Lacey & Randhawa, 1987; Kirby & Das, 1978; Kirby & Robinson, 1987; Molloy & Das, 1980; Try, 1989; Ransley, 1981; Tulloch, 1986; Vocate, 1987; Walton, 1983), and therefore interpretation is not likely to be problematic.

Table 7.6
Principal components analysis:
rotated components (2 factors)

Variable	Factor 1*	Factor 2	Communalities
Number span	-0.113	0.873	0.774
Letter span	-0.117	0.850	0.735
Word span	0.288	0.616	0.462
Shapes	0.770	-0.055	0.595
Paper folding	0.468	-0.063	0.223
Matrix A	0.805	0.187	0.683

* Factor 1 is simultaneous processing,
Factor 2 is successive processing.

Eigenvalues express the relative proportion to which different components account for variation in the variables (SAS Institute, 1989; Stevens, 1990; Tabachnick & Fidell, 1989), and values greater than one are usually assumed to be significant (Stevens, 1986); so the criterion for factor selection was set at eigenvalue greater than 1.0. Earlier research, and the present eigenvalues (1.915, 1.560, 0.939, 0.711, 0.543, 0.333) suggested there were two component factors here, so a form of factor analysis was carried out forcing the six components towards one of two extremes. The result of this analysis is shown in Table 7.6, where factor 1 can be interpreted as a measure of simultaneous processing, and factor 2 as a measure of successive processing.

Other data

Teachers participating in this research were asked to remain faithful to the teaching approaches agreed to throughout the lessons, to refrain from introducing other materials, to limit instruction to the lesson times set, to allow video equipment and a research assistant into their room as requested, and to allow testing and interviewing of children.

A research assistant was used in each school to videotape one of the classes each day. That is, one lesson per school, per day was videotaped. In each case, the research assistant was a parent[7] who spent time helping out at the school in a voluntary capacity, and was known to the students. The research assistants also videotaped each of the student interviews. Both lesson and interview videos were later transcribed, and the resulting text added to, so as to include descriptions of teacher and student actions with multibased arithmetic blocks (MABs) in addition to verbal exchanges. This allowed the researcher to see teachers in action in the classroom, to confirm the teaching approach, and to see students' reactions to this teaching approach, in terms of both what was said, and actions with the MAB materials.

In the days following the posttest and again in the days following the retention test, individual interviews of 5 to 10 minutes were conducted, by the researcher, with 20% of the students. Equal numbers of boys and girls were systematically selected for interview, through the use of random numbers and the selection of every fifth student on an alphabetical list from the various teaching approaches. The interviews were videotaped, then transcribed, and notes describing student actions with MABs added. During these interviews, students completed a selection of algorithm and word problems involving subtraction. The purpose here was to relate the various teaching approaches to levels of mathematical achievement, and the various cognitive processing abilities to mathematical achievement.

These transcripts, from class lessons and from individual interviews, formed the basis of a qualitative analysis of the events occurring in the lessons and interviews, including both teachers' and students' language and actions. This qualitative analysis was intended to provide finer-grained data, both to support the quantitative analysis, and to provide for the possibility of gaining insights into student learning and cognitive processing not available through the quantitative analysis.

Teaching approaches

The teaching approaches used in this research allowed comparison of approaches based on a high isomorphism index with approaches based on a low isomorphism index. At the same time, all approaches had to be able to be taught by experienced school teachers. This was not seen as especially problematic, since both the literature review and my own experiences in schools, suggested typical classroom approaches were using a low index: but this low isomorphism index could be made higher through some subtle changes in teaching methods. That is, the basis already existed in teacher classroom behaviour for high isomorphism index treatments to be implemented. In particular, it was not as though a major professional development program was needed to allow teachers to learn the necessary skills, or that high isomorphism index treatments so were not so different from existing classroom practices. Rather, it was a matter of giving emphasis to particular teacher actions and language, especially

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[7] with a child at the school, but not participating in the research.

those actions and words that would draw out the mathematical significance of teaching materials. Also emphasised were the subtleties in moving from one system of representation to another, and the way actions and words can be used to establish written algorithms.

This approach is consistent with recommendations by Lesh, Post and Behr (1987), who described a range of features necessary for concrete materials to be suitable as learning models. Lesh et al., argued that the materials must allow simplification of the original situation through focussing on the more relevant features, that there had to be a mapping between the original situation and the model, and that the model must allow prediction and mapping backwards to the original situation. They argued too that "not only is it important for youngsters to work with concrete models that illustrate mathematical concepts, it is important to focus on translations from one representational system to another" (p. 660). The MAB materials satisfy these conditions, and the teaching approaches used in this research were very much about translation from one representation to another.

These teaching approaches have some elements in common with earlier research, particularly those investigations by Resnick and Omanson (1987), and the work by Fuson and Willis (1989) and Fuson and Briars (1990)[8]. Each of these reports gave a detailed account of specific teaching approaches used in their investigations. In the Fuson and Willis study the researchers noted the different levels of enthusiasm shown by the teachers for the experimental approaches, their varying levels of commitment to the research, and the ways they implemented the agreed approaches in their classrooms. The Fuson and Briars (1990) study, where first and second grade pupils using MAB materials to solve addition and subtraction questions with four-digit numbers, emphasised a teaching approach to encourage a strong link between actions on blocks, and the pen-and-paper symbols where "each step with the blocks is immediately recorded with written marks" (p.182), and with these links strengthened by constant reference to them. The present research provides more detail of the pedagogies to be used that either of these investigations, and used videotapes of lessons to examine the detail of classroom interaction. The earlier research by Resnick and Omanson, and by Fuson and her colleagues, did not use classroom-based videotape sessions. In the present research, lessons were captured on videotape, then transcribed to take into account both words and actions, and so provide fine-grained detail of classroom situations.

In the research reported here, teachers in the high isomorphism treatment were asked to continually remind students of the relationship between written numerals and the way MAB materials were used to represent numbers. They were asked to emphasise place value, to ensure that the materials were arranged in place value position; and to continually use the terms *trade* and *rename*, to describe the action of decomposing numbers, both with materials and in written algorithms. They were to use the words *move*, *join*, *recall/remember*, *record/write* frequently,

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[8] I have noted previously, in chapter 3, that it is often the case that teaching methods are described in insufficient detail for later analysis or experimental replication.

so that every action on the materials, and every written action, were highlighted and made obvious. They continually reminded students of the relationship between actions on MABs, and changes to the written algorithm. Finally, teachers were asked to remember that the movement from the MABs to the algorithm was to emphasise not only the relationship between action on the materials and the written algorithm, but to also use language to support this: so that, for example, the place value of written numerals was mentioned initially, then gradually tapered off. Each aspect of the use of MABs to represent numbers, all actions on the materials, the movement between representation systems, and symbolic representation were all highlighted. For the teachers involved in this research, training took place in two 30 to 60 minute sessions over two days, about a week apart. The text in Table 7.7 was taken from a handout, given to the participating teachers, showing the detail of these teaching approaches; in this case the relationship between MAB and a written contracted algorithm.

Table 7.7
Treatment 2, MAB to target procedure, high isomorphism index

MAB procedure	Target procedure
0.0 542 - 263	0.0 542 - 263
0.1 Subtract 263 from 5H, 4T, 2U	
1.0 Process units	1.0 Process units
1.1 Take 3U from 2U (cannot)	1.1 Take 3 from 2 (cannot)
1.1.1 Trade for more units	1.1.1 Trade for more units
1.1.2 Move 1L from 4L to bank,	1.1.2 Recall 4 = 3 + 1
bring back 10U	1.1.3 Cross out 4, write 3
1.1.3 Join 10U and 2U	1.1.4 Write 1 next to 2
1.1.4 Recall 10U + 2U = 12U	1.1.5 Recall this is 12
1.2 Take 3U from 12U	1.2 Take 3 from 12
1.3 Recall 12U - 3U = 9U	1.3 Recall 12 - 3 = 9
1.4 Record answer, 9U in answer space	1.4 Record 9 in answer space
2.0 Process tens	2.0 Process tens
2.1 Take 6T from 4T (cannot)	2.1 Take 6 from 4 (cannot)
2.1.1 Trade for more longs	2.1.1 Trade for more tens
2.1.2 Move 1H from 5H to bank,	2.1.2 Recall 5 = 4 + 1
bring back 10T	2.1.3 Cross out 5, write 4
2.1.3 Join 10T and 4T	2.1.4 Write 1 next to 4
2.1.4 Recall 10T + 4T = 14T	2.1.5 Recall this is 14
2.2 Take 6T from 14T	2.2 Take 6 from 14
2.3 Recall 14T - 6T = 8T	2.3 Recall 14 - 6 = 8
2.4 Record answer, 8T in answer space	2.4 Record 8 in answer space
3.0 Process hundreds	3.0 Process hundreds
3.1 Take 2H from 4H	3.1 Take 2 from 4
3.2 Recall 4H - 2H = 2H	3.2 Recall 4 - 2 = 2
3.3 Record answer, 2H in answer space	3.3 Record 2 in answer space
4.0 Read answer	4.0 Read answer

Table 7.7 indicates the teaching approach with MABs, the final target procedure, and the detail of how the MAB teaching approach links to the final procedure in a high isomorphism index treatment. Substituting data available in Table 7.7 into the isomorphism index formula

$$I_{(1,2)} = \frac{(N_1 + N_2 - 2) - (D_1 + D_2)}{N_1 + N_2 - 2}$$

and using values $N_1=25$, $N_2=26$, $D_1=3$ (steps 0.1, 1.1.2, and 2.1.2) and $D_2=4$ (steps 1.1.2, 1.1.3, 2.1.2, and 2.1.3), results in $I_{(1,2)}$ becoming 42/49 or 0.86, a high isomorphism index.

In introducing this series of lessons, teachers encouraged students to spend time familiarising themselves with the MAB materials, and with relationships between the various components of these materials. There was time set aside for familiarisation, for exploration, and for various forms of games encouraging cognitive chunking and automatisation of the place values inherent in the materials. A period of 'free play', in which the learner experienced the relationships existing within the materials, was followed by teacher guidance and intervention as materials were used to solve specific questions. This familiarisation was an aspect of all teaching approaches, regardless of the isomorphism index or teaching approach emphasised.

Each day of the research, one lesson was videotaped in each school, not only to provide a vehicle for recording lessons for later analysis, but also as a method for checking that teachers were using the agreed teaching approach. The texts below are taken from lessons based on a high isomorphism index teaching approach. The first text is taken from the first lesson by teacher #2, where children are using MABs to calculate 12 - 5. They have used MABs before, but the teacher is emphasising the point of trading 1 ten for 10 units then renaming.

T: Here's ten. Why can't I take away five from here at the moment? Why can't I take away five?
S: Because there's not enough.
T: Because there's not enough, that's exactly right. Okay, so what I have to do is trade this ten for what?
S: Ten units.
T: Ten units. So everybody trade their ten for ten units. Now, how many units do you have when you've done that? How many units do you have when you've done that?
S: Twelve.
T: Twelve, what do you reckon?
S: Twelve
T: Right, twelve units, okay. Now take away five. Twelve units take away five units: and when you've done it put your hand up so I know. Twelve take away five, how many is that?
S: Seven
T: Twelve take away five is seven, okay, you should have seven. You had twelve and you took five away and that left you with seven, so let's write that down. Twelve take away five equals seven.
T: Right, next one (14-8). Now you have a ten and four units. If I want to take away eight, why can't I take away eight - let's say I have fourteen take away eight - why can't I take away eight?
S: You haven't got ten numbers.
T: You haven't got enough units so what would we do?
S: Trade
T: We'll trade. What do we trade? Quick, what do we trade?
S: Ten
T: We trade ten and when we trade one ten what do we get back?
S: Ten units.

T: Ten units okay, so what do we trade? We trade one ten for ten units. Put your hand up when you've done that and if you can tell me how many units you have?...How many units do you have?...

S: Fourteen

T: Fourteen units. Okay you've got fourteen units and now I want you to take away eight of those units so take away eight of those units. (pause)

T: When you take away eight how many do you have left? Fourteen units take away eight units how many do you have left?

S: Six.

T: Six. Who else said six? Okay good that's the right answer. So now we put down six. Right let's do another one. Let's do eleven take away that number (9) - so make up eleven with one ten and one unit. I want you to get the sets right. You do a ten and a unit; then you change the ten into ten units and then you subtract the nine....
Now who can tell me the answer please? Who can tell me the answer please - hands up, the answer is?

S: Two.

T: Two, okay. Is that right, does everybody agree? Okay, so now we write things down.

In this text the teacher is emphasising the structure of the algorithm, the need to trade, *You haven't got enough units so what would we do?* child answers *Trade*, showing how to trade, *So everybody trade their ten for ten units*, renaming *you've got fourteen units and now I want you to take away eight of those units*, using the materials, *I want you to get the sets right. You do a ten and a unit; then you change the ten into ten units*, and writing the answer, *we write that the answer is two.*

The next text[9] is teacher #1 emphasising identifying trading, the need to trade, the process of trading and renaming, and how the algorithm is written. Note that as children use incorrect language, she responds with the more appropriate terminology.

T: All right, today on your stencils you have some take away. Some you have to trade and some you don't have to trade. I want you to look for a moment - if this was your question here and you thought you had traded that question, you'd put a circle around it and if you don't have to trade you'd leave it and go on. Do that for me now, just that first row (pause). So how many longs do you have?

S: Five

T: And how many units do you have?

S: Two.

T: Do you agree with the sum we've got to do? What have you got to do? Fifty-two, we've got to take away what?

S: Take away thirty-three.

T: All right, we've got a problem, we have fifty-two and we have to take away thirty-three. How do we do that?

S: Take one long away and.......

T: Put one long into the box, what do we take out.

S: Units.

T: Everybody do that for me. Be certain you put them in the right column. Wait, there are just a couple of people not quite ready, so, I have traded one ten, one long and now I have four tens. What do I have in the unit column, Melissa?

S: Twelve.

T: Twelve units. Everyone double check if you have twelve units. If you don't have twelve, look it up. Now if it was as before, fifty and two, equals fifty-two, then what is it now Alisha?

S: It's still fifty-two.

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[9] In every case the teacher is addressing students by name. For reasons of confidentiality, the names have been omitted.

T: It is fifty-two, very true, but what do you have in front of you in the text column?
S: You have four longs.
T: Can you give me another name for four longs?
S: Forty and twelve.

In the following text, teacher #4 is at the point of helping students to move from an expanded algorithm to a contracted algorithm. He uses the same question in each case (42-36). Note how he emphasises identifying trading, and the need to rename.

T: Who can tell me what I can rename that top number two? It's forty-plus four. Rename it.
S: Thirty plus fourteen.
T: Right, okay. What did we do to forty to make thirty?
S: Traded.
T: We traded. What did we trade?
S: We took one ten.
T: We took one ten off there. Where am I going to put it? Does that say forty?
S: Yes. They are both forty.
T: It goes from there to there, so now we haven't changed anything; all we have done is we have moved ten. We have traded. We've traded one ten for ten little ones. How many little ones do we have altogether now?
S: Fourteen.
T: Can you all see that? You're all positive about that one?
S: Yes.
T: Okay, now what do we do? We're still taking away. What will we get, what answer?
S: Eight.
T: Eight is the answer. What do we do over here? Once you've got eight here, what is the next step? Once you've got eight here, what do we do now?
S: You take away thirty from thirty.
T: We go to the tens and we take away thirty from thirty and we get?
S: Nought.
T: Nought. Now who understood that?
S: Easy (transcript continues after some other classroom discussion)
T: There is a shorter way to do it. We look first of all at what fellows?
S: Units
T: Good boy, so we're going to look at the units column. What can we see in the units column?
S: We can't take away four.
T: But what does it say in the units column? What does it say?
S: Four take away six.
T: And you said we can't do it, so what are we going to do? That magic word?
S: Trade.
T: We're going to trade. What are we trading?
S: One ten.
T: If we trade one ten, what does that new ten become?
S: Three.
T: It becomes thirty or three tens, both of them we know mean the same thing. And what are we going to do with the extra ten? You would put it over here?
S: Yeah.
T: All right. We've got ten little ones. How many ones have we got altogether now?
S: Fourteen.
T: Who saw it? Who can understand it? Okay, now we can say fourteen take away six is eight. In the tens you take away three is nothing. That's a shorter form. Hands up if you understood it? All right, now it's going to be your turn.
T: Sixty-five. Who has worked out how to rename sixty-five?
S: You get the ten and you rename it. You take a ten from the sixty, put it onto the units and that becomes fifteen.

These texts are typical of these teachers and teaching approaches using high isomorphism indices. The teachers may not always use every emphasis that the researcher asked them to highlight, but they regularly give these emphases. In particular, it is clear from these transcripts that teachers attempt to help students identify the need to trade, help them trade using both materials and numbers, regularly mention renaming, show the relationship between actions on the materials and the written algorithm, and emphasise the need to follow a particular procedure. That is, the teachers provide a full and detailed account of subtraction, they try to help students come to understandings about the procedures, both with the materials and the written algorithm, and about relationships between these actions and what is written. Contrast these emphases with the low isomorphism teaching approaches that will be discussed below.

Table 7.8 shows part of a handout for teachers involved in the low isomorphism index teaching approaches, distributed during a training session for these teachers. In this approach, the minute detail of the MAB procedure, the detail of the written procedure, and the relationship within and between these procedures are not emphasised to the extent that they are in the high isomorphism teaching approach. For example, when a trade takes place with materials, the new 10 units are joined with the existing units, but this is not stressed particularly, and no mention is made of renaming these units, though they are renamed in the written algorithm. When subtraction takes place, this method emphasises the taking away of materials, but does not recall the particular subtraction facts involved. For example, steps 1.1.4 and 1.3 in the MAB

Table 7.8

Treatment 3, MAB to target procedure, low isomorphism index

MAB procedure		**Target procedure**	
0.0	542 - 263	0.0	542 - 263
0.1	Subtract 263 from 5H, 4T 2U		
1.0	Process units	1.0	Process units
1.1	Take 3U from 2U (cannot)	1.1	Take 3 from 2 (cannot)
1.1.1	Trade for more units	1.1.1	Trade for more units
1.1.2	Move 1L from 4L to bank, bring back 10U	1.1.2	Recall 4 = 3 + 1
		1.1.3	Cross out 4, write 3
1.1.3	Join 10U and 2U	1.1.4	Write 1 next to 2
		1.1.5	Recall this is 12
1.2	Take away 3U	1.2	Take 3 from 12
		1.3	Recall 12 - 3 = 9
1.3	Record answer, 9U in answer space	1.4	Record 9 in answer space
2.0	Process tens	2.0	Process tens
2.1	Take 6T from 4T (cannot)	2.1	Take 6 from 4 (cannot)
2.1.1	Trade for more longs	2.1.1	Trade for more tens
2.1.2	Move 1H from 5H to bank, bring back 10T	2.1.2	Recall 5 = 4 + 1
		2.1.3	Cross out 5, write 4
2.1.3	Join 10T and 4T	2.1.4	Write 1 next to 4
		2.1.5	Recall this is 14
2.2	Take away 6T	2.2	Take 6 from 14
		2.3	Recall 14 - 6 = 8
2.3	Record answer, 8T in answer space	2.4	Record 8 in answer space
3.0	Process hundreds	3.0	Process hundreds
3.1	Take 2H from 4H	3.1	Take 2 from 4
		3.2	Recall 4 - 2 = 2
3.2	Record answer, 2H in answer space	3.3	Record 2 in answer space
4.0	Read answer	4.0	Read answer

procedure in Table 7.7, the high index approach, are not used in the low index approach, Table 7.8. The teaching approaches exemplified by Table 7.8, are sensible strategies, likely to result in student learning: and reflect the kinds of teaching approaches typical teachers use in typical classrooms.

Table 7.8 indicates the teaching approach with MABs, the final target procedure, and the detail of how the MAB teaching approach links to the final procedure in a low isomorphism index treatment. Substituting data available in Table 7.8 into the isomorphism index formula

$$I_{(1,2)} = \frac{(N_1 + N_2 - 2) - (D_1 + D_2)}{N_1 + N_2 - 2}$$

and using values $N_1=20$, $N_2=26$, $D_1=3$ (steps 0.1, 1.1.2, 2.1.2) and $D_2=9$ (steps 1.1.2, 1.1.3, 1.1.5, 1.3, 2.1.2, 2.1.3, 2.1.5, 2.3, and 3.2), leads to $I_{(1,2)}$ becoming 32/44 or 0.72, a low isomorphism index. If steps 1.1.3 and 1.1.4 are removed, that is, if the teacher does not emphasise that the traded ten units and the existing units ought to be joined together before the relevant subtraction, then $N_1=18$, $N_2=26$, $D_1=3$ and $D_2=11$, and so $I_{(1,2)}$ becomes 28/42 or 0.67, a lower index. These indices are low in comparison to the higher index reported previously in Table 7.7. It is possible to make the index lower, but the teacher has to either leave out important steps or add unnecessary steps. It is an important element in this research that the low isomorphism treatment be seen to teachers as an acceptable teaching approach, one that is genuinely used in classrooms.

Videotapes again allowed the researcher to check that teachers were using agreed teaching approaches, and provided data for later analysis. The text below is taken from teacher #3.

T:	Now if you know how to do this watch what I'm going to do now. Instead of 42-7, I'm now going to make this 17. What do I have to do now? We're going to do the same process again except this time I'm going to take away 17. What am I going to do first?	
S7:	Make the number 42.	
T:	Oh yeh. Did I say 47?	
Ss:	No you said 42.	
T:	Righto. 10, 20, 30, 42. There's my 42. What am I going to do now?	Teacher counts out four tens (says 42) then puts out two units. Puts them in place value position.
S8:	You take one of the big long tens away and then you get ten little ones.	
T:	One, two, three, four, five, six, seven, eight, nine, ten. Good girl. What do I do next?	Teacher places a ten in the bank, counts out tens units, places them next to the tens, leaves original two units where they were.
S9:	Take away seventeen.	
T:	Seven. No. Take away seven first.	
S9:	Take away seven	

T:	One, two, three, four,, seven. One, two, three, four, five, six, seven. What am I going to do next?	Teacher counts seven units. Original two units then units in the traded ten. Shows them to S9, counts them a second time, then puts them in the bank.
S10:	Take the other number away.	
T:	Take away ten All right. Re-phrase	Teacher puts one ten into bank
T:	Look at the tens column. There is one ten that you have to take away. Remember now that it is 42-17. You have four tens, and the one ten is your going to have to take away, as well, and don't forget it. Did I take away?	Teacher replaces the ten, to make 35.
S:	Take away one ten	Teacher puts one ten into bank.
T:	There's my one that I can take away.	

In the above transcript the teacher is accurate in what she says, but there is some lack of correspondence between her words and her actions. For example, she counts out forty, as four tens, but says 42, then places the additional 2 units. She also counts out ten units as part of a trade, then leaves these separate from the already existing 2 units. There is no intention here, of being critical of this teacher: she has effectively adopted the low isomorphism index treatment agreed to for this research. At the same time, this laissez-faire approach appears typical of classrooms wherever manipulative materials are used. The research review chapter supports this notion, as does my own experience with classrooms throughout the world. The lesson continued, with the text shown below.

T:	Read me out the first algorithm, number A.	Several children put up hands.
S:	Forty-two take away seven.	
T:	Can you give me a number sentence with that in it? Forty two take away seven. Can you give me a number story about that?	
S1:	I have forty-two apples and I took away seven.	
T:	Yes, good boy. Okay, tell me what have to do with the third please.	
S2:	You put four of your (pause)	(child is confused, suggesting 4 longs?)
T:	Make the number what?	
S2:	Um...um, set this up...	
T:	No, no, you are getting mixed up. What do I have to do first?	
S3:	Make 42 up.	
T:	Oh, 42 is it? One, two, three, 42, ten, 20, 30, 40. Two. I'm working upside down. What do I do now?	Teacher counts out 4 longs, starts counting 1,2,3, corrects herself, 10,20,30,40. Then gets 2 units.
S4:	Take the ten away and you get ten ones.	
T:	Why?	
S4:	You can't trade.	
T:	What do I need to trade?	
S:	Because you can't take away seven.	
T:	Oh. Excellent boy. One, two, three. four, stop fiddling,..ten. I've now traded, two, four, six, eight, ten. What do I do now?	Teacher holds a ten in her left hand, takes 10 units from the bank with her right hand, places them next to the tens, counts as she does it, then puts the ten in the bank.
S5:	Take away seven from the ten	
T:	Exactly. One, two, three, four, five six, seven. What's my remainder?	Teacher takes seven units away from a line of ten
S:	Thirty-five.	
T:	10, 20, 30, one, two, three, four, five.	Teacher counts tens and units, verifies answer.

Here the teacher counts 1, 2, 3 as she places out the tens, then corrects herself and counts 10, 20, 30. This seems a small matter, we know she meant 10, 20, 30 when she said 1, 2, 3 and we assume children do too. Do they? Is it worth allowing this misunderstanding to occur? It's all well and good to say that most children will either understand or correct themselves very quickly; my view is that this attitude is one of the reasons mathematics is difficult to learn. That is, teachers do not actually see how difficult learning is, because they already know it and are lacking insight into the demands made on learners. For example, the teacher above is trading, but she has both 1 ten and 10 units. So she has not really traded. Then when she subtracts 7, she counts seven from the traded ten. This is a very common occurrence. That is, the MAB units are either not joined or are joined but no mention is made of this, and the subtraction is then from the ten that was traded. This does not correspond to the written algorithm, where the traded ten is joined to the existing units and is renamed. Again, these may appear as differences too subtle to matter, but that is the focus of this research.

Here is a text from a lesson given by the second teacher involved in the low isomorphism teaching approach (Teacher #2). Note that the low isomorphism approach continues to provide what seems to be quite an effective teaching approach, there are no obvious errors or omissions in the lesson. The trading was shown accurately, but while the teacher joined the new tens with the existing tens, he did not emphasise this joining, and he didn't rename them. A student renamed them, but he or she was not asked to do this, and it is questionable whether other students either heard or applied what this student said. Subtraction was then taking away 6 tens, but the teacher simply did this by counting to six, he did not emphasise that it was 6 tens, and he did not mention fourteen.

T: So the number we have in front is three hundred and forty-seven (in MABs). Okay now I've got a question on the board. Three hundred and forty-seven subtract one hundred and sixty-five, well that's easy. I start with the units because when I subtract I start subtracting from the units, so I'll take five away. One, two, three, four, five, I take five and hands up how many left please, hands up.
S: Two.
T: Two. Seven subtract five is two, that's easy. But the next one says four subtract six and why is that difficult, what's the problem there?
S: Six is bigger than four.
T: That's it. Six is bigger than four so what do I do please? What do I do?
S: You trade.
T: I trade. Now what am I going to trade then please? Quickly, what am I going to trade?
S: A hundred.
T: A hundred, okay. I'm going to trade one hundred. What do I trade it for please?
S: Ten.
T: Ten, ten and it just so happens - but remember how I tricked you the other day. I put the ten tens on top like this.
S: Yeah.
T: Just watching. This shows that ten tens and one hundred are exactly the same. So what I'm doing is I'm trading my hundred that I put away and my ten tens. What I've got now is this, now just watch carefully. What I've got now is, I no longer have three hundreds I only have two and instead of four tens I have another ten.
S: Fourteen.

T: So I put that there. Now the important part, looking this way as people are getting restless; the important part is the trading that took place. Put your hand down please. Now I can subtract six so I simply count out, two, four, six and I count how many are left. One, two, three, four, five, six, seven, eight. And now I've done a subtraction with units. I've done a subtraction of the tens. Now I have to subtract what else? What else do I have to subtract to finish it?
S: The hundreds.
T: One hundred to subtract. I take that away and that leaves me with one hundred.
S: One hundred and eighty-two.

As was the case before with the high isomorphism index approach, these texts are typical of lessons taught in the low isomorphism teaching approach. As was stated previously, this low isomorphism index approach differed only subtly from the high index approach. These differences were achieved through omission of terminology or less emphasis of constructs or relationships, or through manipulation of materials in a manner that was arithmetically correct, but which did not lead to an efficient corresponding algorithm. In particular, the minute detail of each representation system, and the minute detail of the transfer from one system to the other, was not emphasised in the low isomorphism index teaching approach.

Chapter 8

Results and Discussion

This chapter reports on the analyses of the quantitative data gathered from subtraction tests and measures of cognitive processing. Results are first presented concerning the procedural analogy theory, then concerning dimensions of cognitive processing, and finally, on the interactions of dimensions of cognitive processing with the procedural analogy theory. This sequence of presentation reflects the order of the research questions posed in chapter six.

Using Concrete materials

An important goal of mathematics education is for students to develop understanding of what they learn in school mathematics, that is, to move beyond the answer to a problem to some higher level of understanding. And there are considerable beliefs in the mathematics education community that the use of concrete materials assists students to develop such understandings. Frequently, concrete materials have been recommended in teacher education programs, mathematics education textbooks, school curriculums and academic papers with considerable conviction. But it is commonly the case that anecdotal accounts, systematic observation of classroom practices, and research in general, have been unable to explain the value and role of concrete materials. It seems that classroom teachers sometimes expect that the mathematical ideas embedded in these materials, and in actions on them, will be absorbed by students as if learners were porous and infinitely receptive.

Mathematics educators appear to have set aside questions concerning the roles of concrete materials in learning, perhaps in the belief that they have been answered, or perhaps because there are more fundamental questions that need to be addressed. Arguably, the explanation for ignoring some research possibilities while adopting others seems to lie in pursuing interesting ideas and endeavouring to keep current and fashionable than attempting to find answers to persistent educational problems requiring pragmatic solutions. Whatever the reason, research in mathematics education has been unable to answer significant questions about the use of concrete materials in the teaching and learning of school mathematics. In particular, questions such as the roles of concrete materials in helping students learn mathematical concepts and skills, in assisting students to develop richly-connected cognitive networks, and in helping them to make meaning, generally remain unanswered. The results of the present study relate to these issues.

Research questions concerning the procedural analogy theory

The procedural analogy theory is an attempt to provide a model of instruction that assists teachers to use concrete materials efficiently with learners, particularly through the calculation of an isomorphism index prior to teaching. There are two research questions relating to the procedural analogy theory, and to the application of the isomorphism index. These questions are:

> can the procedural analogy theory be applied effectively in the teaching of subtraction to elementary grade students; and
>
> will participants in a high isomorphism index treatment gain greater scores on subtraction achievement tests than participants in a low isomorphism index treatment?

Comparing the two high isomorphism index treatments

The two high isomorphism indices[1] teaching approaches had many similarities. The main difference was that in teaching approach 1 the initial movement from concrete materials to written symbols used expanded numerals as an intermediate representation, prior to representing materials as contracted numerals. Teaching approach 2 moved from concrete representations immediately to symbolic representations as contracted numerals. One may expect that the learning outcomes of students in each of these two teaching approaches would be similar, particularly the results on subtraction achievement tests, but the equivalence of these two groups cannot be taken for granted, so it was necessary to compare the test scores of students in these two treatment groups.

Data in Table 8.1 shows the results of a multivariate analysis of covariance (MANCOVA) based on the scores of students in teaching approaches 1 and 2; with treatment group, simultaneous and successive processing as the independent variables, subtraction posttest and retention tests scores as dependent variables, and with subtraction pretest scores as the covariate. These data indicate that there are differences between teaching approaches 1 and 2 ($F = 2.6327$, $p = .042$), with the stepdown analysis showing that these differences are accounted for by differences in scores in the algorithm component of the posttest ($F = 7.6460$, $p = .007$), where the mean score of 82.7 by students in teaching approach 2, was higher than the mean score (69.4) by students in teaching approach 1.

[1] The three teaching approaches used in this research were described in detail previously, in Chapter 7.

Table 8.1
Multivariate analysis of covariance with treatment group, simultaneous and successive processing as the independent variables, subtraction posttest and retention tests scores as dependent variables, and with subtraction pretest scores as covariate.

Treatment	Wilks' Lambda	Hypoth df	Error df	F	Prob
Multivariate					
1 v 2	.8659	4	68	2.6327	.042
Associated Univariate					
Post algorithm		1	71	7.6460	.007
Post problem solv		1	71	.0069	.934
Retention algorithm		1	71	.0937	.760
Retention prob solv		1	71	.0456	.831
Associated Stepdown					
Post algorithm		1	71	7.6460	.007
Post problem solv		1	70	1.1590	.285
Retention algorithm		1	69	1.6044	.210
Retention prob solv		1	68	.1564	.694

Interpretation of these results is somewhat problematic since these were both high isomorphism treatments. However, one possible explanation for this difference is that during the initial teaching, students in treatment group 1 learnt subtraction with expanded notation before using a contracted notation. This meant that when first moving from MAB representations to written symbols, the written representation used expanded notation, so that 165 would be written in its place value position and according to its representation by MABs as 100 + 60 + 5. This teaching approach began with concrete representations, moved to expanded symbolic representations, then to contracted numerals as they would be written in everyday life (for example, 165). This is likely to have required additional learning for students, beyond that required in teaching approach 2 where the movement from concrete representations was directly to contracted written symbols, without expanded numerals as an intermediary step. So at the time of the completion of instruction, students in teaching approach 2 (where they did not have to learn the intermediary step) were more able to successfully completing the final target (contracted) algorithm than were students in teaching approach 1.

This explanation is given some additional support by the fact that the differences between scores of students in teaching approaches 1 and 2, on the algorithm component no longer existed at the time of the retention test. At that time, students from teaching approach 1 actually increased their scores (to a mean of 74.7), so they may have continued cognitive processing of expanded and contracted numerals after the completion of the teaching sequence. Of course, there is no way of being certain of this, but the possibility that increased skill with contracted and expanded notation by students in teaching approach 1, together with the more normally anticipated loss of scores between post and retention tests by teaching approach 2 students, moved students' scores closer together (means 74.7 and 76.8 respectively). The

Table 8.2
Adjusted means from a multivariate analysis of covariance with treatment group, simultaneous and successive processing as the independent variables, subtraction posttest and retention tests scores as dependent variables, and with subtraction pretest scores as covariate.

Treatment	Algorithm post	Algorithm retention	Problem solving post	Problem solving retention
1	69.4	74.7	19.4	19.3
2	82.7	75.8	19.6	19.7

equivalence of the treatments, and of learning outcomes, is also supported by scores on the problem solving components of the tests. At the time of the posttest, the mean scores on the problem solving component for teaching approaches 1 and 2 were 19.4 and 19.6 respectively; and 19.3 and 19.7 at the time of the retention test. These data are indicated in Table 8.2, which shows adjusted means from a multivariate analysis of covariance with treatment group, simultaneous and successive processing as the independent variables, subtraction posttest and retention tests scores as dependent variables, and with subtraction pretest scores as the covariate.

Comparing high and low isomorphism index treatments

Since the previous MANCOVA analysis indicated that students' scores in teaching approach 1 differed from students' scores in teaching approach 2, the comparison of high isomorphism index treatments with low isomorphism index treatment indicates a need for a comparison of teaching approach 1 with teaching approach 3, together with a comparison of teaching approach 2 with teaching approach 3. These comparisons are shown in Table 8.3, where the multivariate analysis contrasted each high isomorphism index treatment group in turn with the low isomorphism treatment group. Table 8.3 presents results from a MANCOVA with treatment group, simultaneous and successive processing as the independent variables, subtraction posttest and retention tests scores as dependent variables, and with subtraction pretest scores as the covariate.

In comparing teaching approaches 1 and 3, the MANCOVA analysis suggests that there are no statistically significant differences between these two groups, though there is a trend for such a difference ($F = 2.2716$, $p = .070$). But the associated univariate and stepdown analyses indicate differences between these groups in the algorithm component of the retention test ($F = 5.6690$, $p = .020$). Scores by these two treatment groups on the algorithm component of the posttest were similar, with a mean of 69.4 for teaching approach 1 and 65.6 for teaching approach 3 (refer to Table 8.4[2]), but by the time of the retention tests, scores on the algorithm component of students in teaching approach 1 (mean 74.7) were higher than scores of those in teaching approach 3 (mean 58.6).

Table 8.3
Multivariate analysis of covariance with treatment group, simultaneous and successive processing as the independent variables, subtraction posttest and retention tests scores as dependent variables, and with subtraction pretest scores as covariate.

Treatment contrast	Wilks' lambda	Hypoth df	Error df	F	Prob
Multivariate					
1 v 3	.8821	4	68	2.2716	.070
Associated Univariate					
Post algorithm		1	71	.6888	.409
Post problem solv		1	71	.0552	.815
Retention algorithm		1	71	5.6690	.020
Retention prob solv		1	71	3.4429	.068
Associated Stepdown					
Post algorithm		1	71	.6888	.409
Post problem solv		1	70	.3840	.538
Retention algorithm		1	69	5.6363	.020
Retention prob solv		1	68	2.1954	.143
Multivariate					
2 v 3	.7804	4	68	4.7848	.002
Associated Univariate					
Post algorithm		1	71	12.265	.001
Post problem solv		1	71	.0207	.886
Retention algorithm		1	71	6.6664	.012
Retention prob solv		1	71	3.9506	.051
Associated Stepdown					
Post algorithm		1	71	12.265	.001
Post problem solv		1	70	2.4814	.120
Retention algorithm		1	69	.7864	.378
Retention prob solv		1	68	2.9512	.090

The proposed explanation for differences between algorithm scores by teaching approaches 1 and 2 is relevant to this situation too. At the completion of the instruction sequence, the learning difficulties associated with the intermediary step of expanded notation in teaching approach 1, and the lack of isomorphism between concrete materials and symbolic representations in teaching approach 3, suggested neither group maximised their learning potential. But by the time of the retention test, scores by students in teaching approach 1 had increased (mean 74.7), presumably through continued cognitive processing as was argued before, but the lower isomorphism teaching approach for students in teaching approach 3 led to a decrease in scores on the algorithm component (mean 58.6). These differences in the

[2] Table 8.4 is a modification of the data in Table 8.2.

algorithm component of the tests did not occur in the problem solving component. Results from the MANCOVA analysis shown in Table 8.3, together with the mean scores shown in Table 8.4, suggest there was little difference between treatments 1 and 3 in terms of scores on the problem solving components of the posttest and retention test.

Table 8.4

Adjusted means from a multivariate analysis of covariance with treatment group, simultaneous and successive processing as the independent variables, subtraction posttest and retention tests scores as dependent variables, and with subtraction pretest scores as covariate.

Treatment	Algorithm post	Algorithm retention	Problem solving post	Problem solving retention
1	69.4	74.7	19.4	19.3
2	82.7	76.8	19.6	19.7
3	65.6	58.6	19.9	15.7

Data in Table 8.3 indicate that there are statistically significant differences between teaching approaches 2 and 3 ($F = 4.7848$, $p = .002$). The associated univariate analyses suggest that the algorithm component of the posttest ($F = 12.265$, $p = .001$), together with both the algorithm component ($F = 6.6664$, $p = .012$) and the problem solving component ($F = 3.9506$, $p = .051$) of the retention test indicate statistical differences between the two treatment groups, but the subsequent stepdown analysis suggests that the differences between these two groups occur in the algorithm component of the posttest ($F = 12.265$, $p = .001$) and not in the other components of the subtraction tests. Mean scores by these two treatment groups, shown in Table 8.4, indicate that students in teaching approach 2 scored more highly on the algorithm component of the posttest, with a mean of 82.7, than did students in teaching approach 3, with a mean of 65.6. These differences might be accounted for by the higher isomorphism teaching approach of treatment 2 over the teaching approach used for students in treatment group 3. At the time of the retention tests, scores on the algorithm component by students in teaching approach 2 had decreased (mean 76.8) but remained higher than scores by those in teaching approach 3 where the mean score (58.6) had also decreased. These scores appear to support the procedural analogy theory, that a teaching approach with the higher isomorphism index leads to greater learning outcomes than a one with a lower isomorphism index.

These data give some support to each of the expected outcomes posed in the research questions at the beginning of this analysis. That is, the procedural analogy theory can be applied effectively in the teaching of subtraction to elementary grade students, and participants in a high isomorphism index treatment gain greater scores on subtraction achievement tests than participants in a low isomorphism index treatment However, there are exceptions, and the pattern of scores is not this simple. For example, students in teaching approach 2 score significantly better than students in teaching approach 3, but students in teaching approach 1 are not significantly better than those from teaching approach 3 though there is a trend in this direction; and there are differences in scores by students in teaching approaches 1 and 2, the two high isomorphism index treatments.

These results are promising, but not conclusive. They reflect the data reported in the literature, where results from the many studies on concrete representations differ from one another (Berlin & White, 1986; Bobis, 1992; Boulton-Lewis, 1992; Dufour-Janvier, Bednarz & Belanger, 1987; Fennema, 1972; Fuson & Briars, 1990; Fuson & Willis, 1989; Hart, 1989; Hiebert & Carpenter, 1992; Labinowicz, 1985; Lesh, Post & Behr, 1987; Resnick & Omanson, 1987; Sowell, 1989; Suydam & Higgins, 1977; Wearne & Hiebert, 1988). In the present study, the topic chosen for teaching may partly explain the lack of differences between the scores. That is, subtraction was not a new topic for any of the students involved in this study, all participants had experienced subtraction algorithms before, and this may have limited the variation in scores. It would be interesting to see the results where the topic taught was new to all students.

Fennema (1972) lists a number of studies from the 1950s and 60s where results achieved through the use of concrete materials in teaching were inconclusive. Of the 15 studies reported seven showed no significant differences between manipulative and non-manipulative treatments, four favoured the manipulative groups, three showed mixed results and one favoured the non-manipulative group. Suydam and Higgins (1976) surveyed 40 studies on the use and effects of manipulative materials in teaching, and reported 24 as showing positive effects on student achievement. Freidman (1978) noted that of the 18 doctoral dissertations in the period 1970-1977 investigating the use of manipulatives in the teaching of mathematics, only four showed significant differences favouring the manipulative groups over the non-manipulative groups. Labinowicz (1985) reported little gain in third grade students using MAB to develop computational skills, whereas Fuson and Briars (1990) reported using MAB materials to gain high levels of skills in addition and subtraction algorithms, and Wearne and Hiebert (1988) reported fourth, fifth and sixth grade students showed some gain in decimal numeration, addition and subtraction from the use of MABs. Fuson (1986) also studied young children taught addition and subtraction algorithms with MAB who were interviewed after making errors, and reported that most children "were able to use a mental representation of the blocks" to self-correct errors (p.183). Given these reports there appears to be reason to support Raphael and Wahlstrom (1989) in their claim that "relatively little detailed work has examined the use of instructional aids and their effects on students' achievement" (p.173), and Post's (1980) scepticism about claims concerning the way in which mathematics educators are building a persuasive body of literature that supports the use of manipulative materials in teaching mathematics. These quantitative results of the present study may add to the uncertainty of the literature, though a more positive interpretation may be possible, in the light of the detail of the instruction used here. It seems that where students use concrete materials some teaching approaches are likely to be more effective if the procedural analogy theory is applied. In the present study, there are some cases where the teaching approaches used have produced different outcomes, but the data are far from conclusive. The analysis of the qualitative data, to be presented in the next chapter, will be shown to support this link between teaching approach and learning outcomes. For the present, a realistic interpretation of the results would seem to be that the procedural analogy theory has some pragmatic educational value in planning for instruction in that the way teachers instruct students in the use of concrete materials seems to have an impact on student learning, and that it seems the case that different teaching approaches may lead to different learning outcomes both as tested immediately after the completion of instruction, and as tested as some later date.

Fennema (1972) also reported the inconclusive nature of findings contrasting the use of concrete materials in teaching with other, traditional approaches. Interestingly, she reported a number of studies where the delayed posttest resulted in superior scores for the concrete materials approach over the traditional approach, where there had previously been no significant differences at the time of the posttest (Ekman, 1966; Howard, 1950; Norman, 1955). That is, these studies had a similar pattern to some of the findings of the current research. All the same, given that we know little of the actual detail of these studies, certainly about teachers or teaching approaches, their findings must be interpreted cautiously.

The study reported in the literature most similar to the present research, is that of Resnick and Omanson (1987) who examined the relationship between performing arithmetic and understanding it, especially by illustrating procedural learning with "well-grounded mathematical principles". I have previously mentioned this study, but some points about it bear repetition here. The researchers developed a *mapping instruction* which was intended to maintained a step-by-step correspondence between the manipulation of blocks and written symbols. They had 80 fourth, fifth and sixth grade students perform tasks, both written and using MAB materials, where representations of numbers were constructed and decomposed, and where activities involved addition with carrying, and subtraction with decomposition. After a period of instruction, posttest scores showed children taught with the mapping instruction did not differ significantly from children in the comparison group, but in a delayed posttest carried out by Resnick and Omanson the mapping instruction group gained higher scores. In looking at Resnick and Omanson s study it is possible to calculate an isomorphism index based on Ohlsson and Hall's (1990) procedural analogy theory. This calculation led to a low isomorphism index of .46 (Ohlsson and Hall, 1990). That is, even though Resnick and Omanson believed they used a high analogy mapping instruction, their analogy appears to have been relatively low, and this may have been a factor in the results they achieved. Results from the present investigation suggest a more positive view about the value of concrete materials, particularly where the instruction is highly detailed, than did Resnick and Omanson's study.

Resnick and Omanson expressed disappointment at children's levels of achievement in their research, even though the mapping instruction group gained higher scores than the comparison group on the retention test. They concluded that the mapping instruction was not effective in curing subtraction bugs. My findings do not support this interpretation. Certainly students in the low isomorphism index teaching approach achieved lower scores on the retention test than did students from the high isomorphism index teaching approach, but this is to be expected simply through lack of practice and loss of memory over time. And while the scores of students in the low isomorphism index teaching approach on the retention test were lower than their posttest scores, their scores on the retention test were higher than their pretest scores. That is, they had learned during the teaching sessions, and they did remember much of it. While there were some who returned to the subtraction bugs shown at the time of the pretest, the majority had not. More importantly, those students in the high isomorphism index teaching approaches lost little in terms of achievement scores between the posttest and the retention test. Indeed students in treatment group 1 actually gained in achievement between the posttest and the retention test, on the algorithm and problem solving components.

Possible interpretations of these outcomes include the view that low isomorphism index teaching approaches may illustrate procedures, but they appear to provide students with too little insight into the use and purpose of manipulating concrete materials, and their role in making mathematical meaning. Such use of materials is likely to be superficial, to lead to incomplete conceptual maps and to cognitive structures missing important network links. It may be that students involved in teaching approaches with high isomorphism indices develop more meaningful understandings of the relevance of concrete materials and manipulations on them. They may also be more likely to understand the relationship between manipulation and the algorithm, to have greater conceptual understanding and to have more fully interwoven cognitive networks. Further, since some of these students gained higher retention test scores than posttest scores, it seems that for some students high isomorphism index teaching approaches encourage cognitive restructuring beyond the end of the period of instruction. Additional research is required to investigate these possibilities.

Fuson and Briars (1990) studied the use of MAB materials in the addition and subtraction of four-digit numbers by first and second grade pupils. This study was reported in considerable detail, and made explicit many aspects of teacher preparation, teaching approaches and the novel lesson content which would not normally be covered until later in elementary school when children were some years older. They used a number of strategies to emphasise the link between action on the blocks and written symbols. For example, action with the blocks was immediately followed by the equivalent written symbol, and there was much verbalisation about the blocks, in everyday English and in base ten terms. Blocks were placed on large cardboard calculating sheets, where both sets of numbers were arranged for both addition and subtraction. They reported children performed these calculations at a level well above what is normally expected of their age group, showed little of the-small-digit-from-the-large error, labelled digits in their place values, changed word names to numerals and vice versa, and selected the correct digits in trading which they were able to describe in terms of its place value. Results from the present investigation are consistent with those of Fuson and Briars: that is, highly detailed teaching approaches lead to more superior learning outcomes than do less detailed teaching approaches. Many of the activities described in the Fuson and Briars (1990) investigation are reflected in the present research. In particular, the high isomorphism index teaching approaches emphasised the link between actions on blocks and written symbols, and emphasised repetitious use of blocks, and actions and words associated with their use, especially place value aspects. That is, there was a focus on both understanding and procedural competence, in both the Fuson and Briars study and in the present research. There is also some support here for my earlier comment regarding the topic of subtraction as 'already known' by these students, and that a topic not previously studied may show clearer differences between treatment groups.

Findings from the present research do not support Sowell's (1989) conclusions from her meta-analysis of 60 studies, previously reported in this thesis. She claimed that treatment lasting a school year or longer favoured the manipulative groups over non-manipulative approaches, and that treatments involving shorter periods of instruction showed no differences between manipulative and non-manipulative groups, on either posttest or delayed posttest scores. As Sowell herself indicated, analysis of these studies was complicated by a general lack of detail as to who did the teaching, what training they undertook relevant to the study, and what the teaching treatments actually involved. This is consistent with an earlier study by Fennema (1972) who noted that studies involving concrete materials were often inconclusive, and referred to experimental and control groups "often defined no better than this" (p. 636), and with Scott and Neufeld (1976) who noted that many studies gave little indication of the nature of the "concreteness" used. The point to note here is that the present study involved very specific uses of concrete materials, emphasised specific relationships between the action on these materials and written symbols, took place over a four week period, and provided better learning outcomes both immediately and in the longer term for one form of instruction over another.

The procedural analogy theory, then, appears to provide a tool for planning teaching in considerable detail, and one that allows teachers a good deal of flexibility in their planning. Having decided upon an instructional approach, application of the theory helps teachers to encourage students to move towards algorithms that are conceptually consistent with materials used. This does not imply that all students will have the same cognitive structure, and that there is only one efficient algorithm. But it does suggest that learning outcomes, through the use of the procedural analogy theory, will be superior to the outcomes from a range of other possible teaching approaches. At the same time, it needs to be recognised that whole class teaching approaches are only one determinant of learning outcomes. Another particularly important determinant concerns individual differences in cognitive processing. It may be that individual differences in cognitive processing lead to differences in learning outcomes, and it my that there are interactions between particular teaching approaches and particular cognitive processing abilities. Therefore, the present results concerning the procedural analogy theory have to be interpreted cautiously, and to wait until the later analysis in this chapter where the impact of dimensions of simultaneous and successive processing on mathematics learning, and the interaction between these dimensions of cognitive processing with specific teaching approaches, are reported.

From the results of the present investigation examined so far, it appeared that teaching approaches with high isomorphism indices allowed students to remember what had been taught, and to manipulate the learned concepts and skills more effectively than a lower isomorphism teaching approach. This suggests that high isomorphism index teaching approaches encourage and support the making of meaning, and the development of understanding. It may be that this occurs through assisting the development of a more highly integrated cognitive structure than is the case in a teaching approach using a lower isomorphism index, but there is much room for debate here. Perhaps these more interconnected and complete cognitive networks, in part, enable students to retain for longer, and to more effectively recall, earlier learned concepts and skills; and to better apply this knowledge in novel problem solving situations. Results of the present study also suggest that the use of concrete materials, together with specific pedagogies, assist the development of an effective cognitive structure; one in which concepts and skills are stored in a meaningful and efficient manner, where they can be remembered, recalled and reconstructed as necessary.

The trend for students from teaching approach 1, a treatment with a high isomorphism index, to increase scores from the posttest to the retention test, may be interpreted as a positive finding in terms of the procedural analogy theory, but clearly needs further investigation. It may be that for a range of teaching approaches students will not score significantly differently on posttests if the test is administered immediately after completion of the topic, and that differences in test scores resulting from different teaching approaches become evident only over time after instruction has finished. Further, it may be that learners' cognitive networks of the topic being taught continue to alter after completion of the teaching unit, and that this alteration is more likely to occur in particular teaching approaches, or as will be considered later, that this development is an outcome of particular cognitive processing abilities.

Boulton-Lewis (1992) argued that concrete materials were useful in teaching only if learners recognised "the correspondence between the structure of the materials and the structure of the concept" (p.10). She explained the failure of concrete materials to improve student learning as a function of the cognitive processing loads required in their use. Unfamiliarity with the materials or analogies used, analogies that were inappropriate and a lack of declarative or procedural knowledge all increased processing load. This appears to be a sensible view per se, in that it provides indirect support for the procedural analogy theory, and reflects the common classroom occurrence of children trying to solve an algorithm where the procedure is interrupted as the child counts fingers in order to answer some elementary number combination. In such finger-counting cases, the child loses the sequence of the procedure, presumably because the cognitive load to perform both the calculation and to remember the procedural sequence is too great. These kinds of limits on school students' information processing capacities have been widely investigated (Bennett, Desforges, Ferrara & Campione, 1983; Hegland, 1991; Hiebert, Carpenter & Moser, 1982; Sweller, 1988). Boulton-Lewis suggested that using a range of materials to overcome learner boredom may have the positive effect of showing children that numbers can be represented in many ways, but may also have the negative effect of not ensuring children have made strong mappings between a set of materials and place value concepts. For example, when children come to use materials to solve questions involving place value, their lack of declarative and procedural knowledge of any materials being used will increase their processing load as they deal with numbers unit by unit, rather than recognising and using the place value format. This work, together with that of Bobis (1992), and a point made by Lesh, Post and Behr (1987b) about perceptually compelling but misleading cues, show that teacher introduced representations, aimed at assisting learning, may actually make learning more problematic. It appears then, that these researchers' views are consistent with my argument for using teaching approaches with high isomorphism indices. That is, concrete materials have to be used with care, consistency and order during instruction, but this necessity for proceduralisation has to be balanced with opportunity for exploration and making meaning. Without freedom for learners to explore, concrete materials are likely to become another barrier between learners and their construction of meaningful mathematical knowledge.

Simultaneous and successive processing

This section of the results chapter reports findings concerning dimensions of cognitive processing, and the interactions of these dimensions with mathematics achievement. Simultaneous processing is concerned with the requirement of processing a number of variables together, and is frequently associated with spatial skills. In the context of learning mathematics, it seems a reasonable expectation that simultaneous processing is necessary to understand the links between the manipulation of concrete materials, and the manner in which this manipulation leads to a corresponding symbolic algorithm. It also seems reasonable that simultaneous processing will be required in problem solving activities where a strategy has to be developed through consideration and trialling of a range of alternatives. Successive processing is concerned with considering one item at a time, and is typically associated with sequential events such as handwriting. In the context of mathematics learning, it seems reasonable to link successive processing with the completion of algorithms, which by their very nature are completed one step at a time. But these appeals to reasonableness need to be empirically supported. Indeed, an argument suggesting that effective algorithm completion requires both simultaneous and successive processing could be mounted. For example, selecting the appropriate algorithm to apply in a given situation requires insights into all the conditions relevant to the situation, together with a selection from competing possibilities, and these appear to be situations requiring simultaneous processing. When it comes to completing the algorithm, it may be short-sighted to suggest this requires only successive processing, since an understanding of the structure of the algorithm, and of recognising why a particular step is necessary, and where it leads to, may also require simultaneous processing. The earlier literature review indicated the lack of certainty regarding the contribution of simultaneous and successive processing to the completion of mathematical activities. The major research question related to these dimensions of cognitive processing only was:

> will there be differences in scores on subtraction tests between students with different patterns of competence on the two dimensions of cognitive processing?

This led to other, more specific, research questions concerned with subtraction test scores and dimensions of cognitive processing. For example:

> will there be differences in subtraction test scores between high simultaneous-high successive processing students and low simultaneous-low successive processing students; and
>
> will there be differences in subtraction test scores between high simultaneous-low successive processing students and low simultaneous-high successive processing students?

A multivariate analysis of covariance (MANCOVA) was conducted in order to answer these research questions. The most likely combination of cognitive processing dimensions that would exhibit differences in subtraction test scores was a comparison of high simultaneous-high successive processing students with low simultaneous-low successive processing students. Here, the MANCOVA involved simultaneous and successive processing as the independent variable, subtraction posttest and retention tests scores as dependent variables, and with subtraction pretest scores as the covariate. The results of this analysis are shown in the top half of Table 8.5, in the section headed "multivariate, HH v LL". The data in this section of Table 8.5 suggest that there are no significant differences between scores on subtraction tests by these two groups of students (F = 1.3980, p = .244). At the same time, the univariate and stepdown analysis reported in Table 8.5, suggest that the two groups differ in their scores on the algorithm component of the posttest (F = 4.0503, p = .048), with the high simultaneous-high successive group averaging 77.4, and the low simultaneous-low successive group averaging 65.9. This lends some support to the notion that both simultaneous and successive processing are required for successful completion of arithmetic algorithms. The data cannot help to decide if simultaneous processing is more important than successive processing, but it suggests that having high levels of these processing dimensions will lead to better learning outcomes than lower levels of these cognitive processing dimensions.

Table 8.5
Multivariate analysis of covariance with simultaneous and successive processing as the independent variable, subtraction posttest and retention tests scores as dependent variables, and with subtraction pretest scores as covariate.

Sim/succ contrasts	Wilks' lambda	Hypoth df	Error df	F	Prob
Multivariate					
HH v LL	.9240	4	68	1.3980	.244
Associated Univariate					
Post algorithm		1	71	4.0503	.048
Post problem solv		1	71	2.5193	.117
Retention algorithm		1	71	1.0403	.311
Retention prob solv		1	71	2.8172	.098
Associated Stepdown					
Post algorithm		1	71	4.0503	.048
Post problem solv		1	70	.6596	.419
Retention algorithm		1	69	.0593	.808
Retention prob solv		1	68	.9112	.343
Multivariate					
HL v LH	.9727	4	68	.4777	.752
Associated Univariate					
Post algorithm		1	71	.3444	.559
Post problem solv		1	71	.5694	.453
Retention algorithm		1	71	.6552	.421
Retention prob solv		1	71	1.9231	.170
Associated Stepdown					
Post algorithm		1	71	.3444	.559
Post problem solv		1	70	.3128	.578
Retention algorithm		1	69	.2463	.621
Retention prob solv		1	68	1.0174	.317

A MANCOVA was conducted on a second combination of cognitive processing dimensions to see if there were differences in subtraction test scores. Here high simultaneous-low successive processing students were contrasted with low simultaneous-high successive processing students, to see if there were differences between their subtraction test scores. Again, the MANCOVA involved simultaneous and successive processing as the independent variable, subtraction posttest and retention tests scores as dependent variables, and with subtraction pretest scores as the covariate. The results of this analysis are shown in the lower half of Table 8.5, in the section headed "multivariate, HL v LH". It is clear from the data in Table 8.5 that there are no statistical differences in subtraction test scores between high simultaneous-low successive processing students and low simultaneous-high successive processing students ($F = .4777$, $p = .752$). That is, quantitative data from the present research did not demonstrate differences between these two groups, nor to assist in any argument concerning the importance of simultaneous processing compared with the importance of successive processing in the successful completion of subtraction algorithms. The later qualitative analysis, in the next chapter, will suggest that simultaneous and successive processing do, indeed, play different roles in the successful completion of subtraction algorithms.

Data in Table 8.6 show the adjusted mean scores from a multivariate analysis of covariance with treatment group, simultaneous and successive processing as the independent variables, subtraction posttest and retention tests scores as dependent variables, and with subtraction pretest scores as the covariate. For example, the average score on the algorithm component of the retention test, for high simultaneous-high successive processing students was 74.0, and the average score on the problem solving component of the posttest, for low simultaneous-low successive processing students was 17.0.

Table 8.6

Adjusted means from a multivariate analysis of covariance with treatment group, simultaneous and successive processing as the independent variables, subtraction posttest and retention tests scores as dependent variables, and with subtraction pretest scores as covariate.

Interaction	Algorithm post	Algorithm retention	Problem solving post	Problem solving retention
HH	77.4	74.0	21.0	19.9
HL	75.3	73.0	19.9	19.6
LH	70.0	66.5	18.1	16.4
LL	65.9	65.6	17.0	16.0

The MANCOVA analysis, reported in Table 8.5, showed few statistically significant differences between students with different patterns of competence on the two dimensions of cognitive processing. This analysis suggested that the only statistical difference favoured high simultaneous-high successive processing students over low simultaneous-low successive processing students on the algorithm component of the posttest. The data in Table 8.5 and 8.6 are unable to clarify the comparative roles of simultaneous and successive processing in mathematical achievement.

Das (1988), Leong, Cheng and Das (1984) and Mwamwenda, Das and Das (1985) reported that levels of simultaneous processing were good predictors of mathematics achievement and that successive processing was unrelated to mathematics achievement. And Garofalo (1986) reported that simultaneous processing was found to be important for mathematical problem solving, but that simultaneous and successive processing were equally important for computational arithmetic. Das, Naglieri and Kirby (1994) list numbers of studies where successive processing and mathematics achievement were found to be correlated, and other studies where simultaneous processing was found to be important in mathematics achievement. Consequently, the actual roles of simultaneous and successive processing in solving algorithms, and in mathematics in general, appear to be potentially fruitful areas for further investigation.

In terms of the present research, it appears that students with high levels of both simultaneous and successive processing are likely to outscore students with low levels on both dimensions, but this does not clarify the issue as to whether it is simultaneous processing or successive processing that makes the difference. Both dimensions of cognitive processing could be important, or each dimension could be important at different times or with different kinds of mathematical questions. The lack of statistical differences in subtraction test scores between the high simultaneous-low successive processing students and the low simultaneous-high successive processing students suggests that both are capable of achieving in mathematics. It may be that students in each of these groups solve mathematical questions differently, so that their methods may differ, but their overall results may be the same. This is an attractive explanation, but the data here cannot be used to argue this case, so further research is necessary. The qualitative analysis presented in the next chapter will consider this issue of differences in mathematical knowledge and diffemces in levels of cognitive processing abilities.

The possibilities that simultaneous and successive processing both impact on mathematical learning, but in differing ways, has a precedent in language learning. Kirby and Robinson (1987) investigated the possibility that different patterns of competence on the two dimensions of cognitive processing were relevant to different aspects of language learning. They based their study on the premise that reading skills required both decoding (successive processing) and comprehension (simultaneous processing). In the same way, the syntax of a piece of language requires successive processing, but semantic understanding will require simultaneous processing. This premise may have an analogy in the completion of arithmetic algorithms, where successive processing may be required to complete the algorithm correctly, but where simultaneous processing may be required to understand the structure, logic and complexity of the algorithm. Simultaneous processing may be necessary for understanding why rather than how, and for the development of relational rather than instrumental understanding. Consequently, successive processing may lead to a correct answer, but it may be simultaneous processing that is the more truly mathematical activity. Again, this is an area worth further research.

More recently, Das, Naglieri and Kirby (1994) have developed an approach to intelligence based on what they named the PASS (planning, attention, simultaneous, successive) model, which they claim as a theory that can be applied not only to measure intelligence, but also to cognition in general. The PASS theory of intelligence has its background in the years of research in which the authors have investigated simultaneous and successive processing, and other aspects of cognitive function such as planning, attention and speed. An important element of the PASS model is that these aspects of cognitive processing can be enhanced through training, and the authors claim to have developed an effective training program. That is, individuals' levels of simultaneous and successive processing can be altered through training. Solan (1987) also claimed that both simultaneous and successive processing skills were able to be increased through training, and Das (1988) both supported this and referred to studies by Brailsford (1982), Kaufman (1978) and Krywaniuk (1974), showing such capabilities. Das (1984) had previously reported on training guidelines, and Leasak (1982) had reported success in an intervention program designed to increase simultaneous and successive processing by fourth grade students. If it is possible to teach class-sized groups of students to increase their skills in simultaneous and successive processing, this could have important educational implications. Clearly, further research is needed.

The studies mentioned above, together with others (Crawford, 1987; Merritt & McCallum, 1984; Molloy & Das, 1980; Tulloch, 1986) indicate that there are still many unanswered questions concerning dimensions of cognitive processing in general, and the manner in which these dimensions of cognitive processing relate to mathematical achievement. In the context of mathematical learning, some studies suggest the importance of simultaneous processing, others suggest this is only the case for problem solving, and yet others claim that successive processing is important too. That is, the picture from the broader literature is inconsistent. Indeed Warrick, Gensheft and Naglieri (1992) reported the relationship between these cognitive processing dimensions, and mathematics achievement, altered from grade to grade. The results in the present study reflect these uncertainties.

Simultaneous and successive processing has been investigated for some years now, more frequently with learning disabled students than with those in typical classrooms, and more often in terms of language learning rather than mathematics learning. There is, then, need for further research into the roles of simultaneous and successive processing with typical students in typical classrooms: and in mathematics, not just arithmetic. And there is need for investigations into the way relationships between simultaneous and successive processing, and mathematics achievement, alter as students move through their years of formal schooling.

Such investigations could hope to identify teaching strategies where individual teachers are able to recognise and make use of students' simultaneous and successive processing capabilities.

This section of the results chapter has considered only differences in scores on subtraction tests in relation to dimensions of cognitive processing. There may be some interactions between dimensions of cognitive processing and high or low isomorphism index teaching approaches, with regard to scores on subtraction tests. This is explored in the next section of this chapter.

Interactions between dimensions of cognitive processing and teaching approaches, and their impact on subtraction test scores

This section investigates the interactions between dimensions of cognitive processing, and teaching approaches based on different levels of isomorphism indices developed through application of the procedural analogy theory. That is, it explores ways in which scores on subtraction tests may differ according to dimensions of cognitive processing and teaching approach employed.

In considering dimensions of cognitive processing used in this investigation, the variables were dimensions of simultaneous or successive processing; and in terms of teaching approaches, there were high and low isomorphism indices. The interactions of these variables, and their joint impact on subtraction test scores led to the research question:

> will students with different patterns of competence on the two dimensions of cognitive processing react differently to high or low isomorphism index teaching approaches, so as to produce differences in subtraction test scores according to dimensions of cognitive processing and teaching approach?

There are subsidiary questions following from this research question, for example:

> will subtraction tests scores by high simultaneous-high successive processing students differ from those by low simultaneous-low successive processing students on subtraction test scores, according to whether they participate in a high isomorphism index or a low isomorphism index teaching approach; and

> will subtraction tests scores by high simultaneous-low successive processing students differ from those by low simultaneous-high successive processing students on subtraction test scores, according to whether they participate in a high isomorphism index or a low isomorphism index teaching approach?

In order to answer the research questions outlined, a MANCOVA was conducted with treatment group, simultaneous and successive processing as the independent variables, subtraction posttest and retention test scores as dependent variables, and with subtraction pretest scores as the covariate. Resul.s from this analysis are shown in Tables 8.8.

Table 8.7 (Part 1) shows results of a MANCOVA contrasting high simultaneous-high successive processors with low simultaneous-low successive processors, and taking into account both high and low isomorphism teaching approaches. These data indicate that there are no statistically significant differences in subtraction test scores between these various groups where the joint effect of dimensions of cognitive processing and teaching approach is taken into account. For example, contrasting teaching approach 1, a high isomorphism index treatment, with teaching approach 3, a low isomorphism treatment, and simultaneously comparing high simultaneous-high successive processors subtraction test scores with those of low simultaneous-low successive processors leads to a multivariate F = 1.2725, with *p* = .289.

Table 8.7 (Part 1)
Multivariate analysis of covariance with treatment group, simultaneous and successive processing as the independent variables, subtraction posttest and retention tests scores as dependent variables, and with subtraction pretest scores as the covariate.

Interaction	Wilks' lambda	Hypoth df	Error df	F	Prob
Multivariate					
1 v 3 x HH v LL	.9304	4	68	1.2725	.289
Associated Univariate					
Post algorithm		1	71	.2335	.630
Post problem solv		1	71	.9570	.331
Retention algorithm		1	71	.0314	.860
Retention prob solv		1	71	1.5159	.222
Associated Stepdown					
Post algorithm		1	71	.2335	.630
Post problem solv		1	70	.7195	.399
Retention algorithm		1	69	.4291	.515
Retention prob solv		1	68	3.6638	.060
Multivariate					
2 v 3 x HH v LL	.9250	4	68	1.3793	.250
Associated Univariate					
Post algorithm		1	71	.8070	.372
Post problem solv		1	71	.0022	.963
Retention algorithm		1	71	.0760	.784
Retention prob solv		1	71	3.0957	.083
Associated Stepdown					
Post algorithm		1	71	.8070	.372
Post problem solv		1	70	.1187	.732
Retention algorithm		1	69	.7844	.379
Retention prob solv		1	68	3.7525	.057

Associated univariate and stepdown analyses provided no statistically significant differences in scores between the various groups, though the problem solving component of the retention test, with stepdown F = 3.6638 and $p = .060$, suggests a trend towards a difference in scores. Results for the comparison of teaching approach 2, a high isomorphism index treatment, with teaching approach 3, are very similar to that which has just been reported. For example, in this case the multivariate F = 1.3793, with $p = .250$ is still not significant, and the univariate stepdown analysis indicates a trend towards differences in scores in the problem solving component of the retention test (F = 3.7525, $p = .057$).

In the comparison shown in Table 8.7 (Part 1) there are no clear patterns of scores on the various subtraction tests. For example, looking at the data in Table 8.8, both high simultaneous-high successive and low simultaneous-low successive students in teaching approach 1 increased their algorithm component scores from the posttest to the retention test; and this was the case for low simultaneous-low successive students in teaching approach 3, but not for high simultaneous-high successive processors. Table 8.7 (Part 1) and Table 8.8 indicate that there are no significant interactions between teaching approaches 1 and 3, and high simultaneous-high successive and low simultaneous-low successive processing students.

Table 8.7 (Part 2) presents results of a MANCOVA contrasting high simultaneous-low successive processors with low simultaneous-high successive processors, and taking into account both high and low isomorphism teaching approaches. These data indicate that there are statistically significant differences in subtraction test scores with respect to the interaction between high simultaneous-low successive processors versus low simultaneous-high successive processors and the contrast of teaching approaches 1 versus 3 (F = 4.7526, $p = .002$), but that the interaction is not significant with respect to the contrast of teaching approaches 2 versus 3 (F = .9921, $p = .418$).

Table 8.7 (Part 2)

Multivariate analysis of covariance with treatment group, simultaneous and successive processing as the independent variables, subtraction posttest and retention tests scores as dependent variables, and with subtraction pretest scores as the covariate.

Interaction	Wilks' lambda	Hypoth df	Error df	F	Prob
Multivariate					
1 v 3 x HL v LH	.7815	4	68	4.7526	.002
Associated Univariate					
Post algorithm		1	71	2.0317	.158
Post problem solv		1	71	6.0290	.017
Retention algorithm		1	71	2.7309	.103
Retention prob solv		1	71	.9122	.343
Associated Stepdown					
Post algorithm		1	71	2.0317	.158
Post problem solv		1	70	10.5854	.002
Retention algorithm		1	69	1.8691	.176
Retention prob solv		1	68	3.5403	.064

Table 8.7 (Part 2 - continued)
Multivariate analysis of covariance with treatment group, simultaneous and successive processing as the independent variables, subtraction posttest and retention tests scores as dependent variables, and with subtraction pretest scores as the covariate.

Interaction	Wilks' lambda	Hypoth df	Error df	F	Prob
Multivariate					
2 v 3 x HL v LH	.9449	4	68	.9921	.418
Associated Univariate					
Post algorithm		1	71	.9140	.342
Post problem solv		1	71	.5199	.473
Retention algorithm		1	71	1.3039	.257
Retention prob solv		1	71	.5084	.478
Associated Stepdown					
Post algorithm		1	71	.9140	.342
Post problem solv		1	70	1.4378	.235
Retention algorithm		1	69	.7783	.381
Retention prob solv		1	68	.8471	.361

In the context of teaching approaches 1 and 3, the univariate stepdown analysis indicates that differences in scores are largely accounted for by differences in scores in the problem solving component of the posttest (F = 10.5854, $p = .002$), though there remains a trend towards differences in scores in the same component at the time of the retention test (F = 3.5403, $p = .064$).

Table 8.8 presents adjusted mean scores based on a multivariate analysis of covariance with treatment group, simultaneous and successive processing as the independent variables, subtraction posttest and retention tests scores as dependent variables, and with subtraction pretest scores as the covariate. In the calculation of these adjusted means, the posttest means are corrected for pretest scores, whereas the retention test means are corrected for both pretest and posttest scores so as to reflect the stepdown procedures of the analysis. The table indicates, for example, that high simultaneous-high successive processing students averaged 69.4 on the algorithm component of the posttest if they participated in teaching approach 1, 79.9 if they were in teaching approach 2 and 68.5 if they were in teaching approach 3. From the table it can also be seen that low simultaneous-high successive processors in teaching approaches 1, 2 and 3 scored 16.3, 19.5 and 18.8 respectively on the problem solving component of the retention test.

Table 8.8

Adjusted means from a multivariate analysis of covariance with treatment group, simultaneous and successive processing as the independent variables, subtraction posttest and retention tests scores as dependent variables, and with subtraction pretest scores as covariate*.

Interaction	Algorithm post	Algorithm retention	Problem solving post	Problem solving retention
Treatment 1				
HH	69.4	78.1	17.7	20.8
HL	74.0	83.0	15.1	22.6
LH	64.9	71.2	23.8	16.3
LL	69.5	76.1	21.2	18.1
Treatment 2				
HH	79.9	72.3	20.4	21.2
HL	83.8	70.2	20.7	19.3
LH	81.6	68.6	18.5	19.5
LL	85.5	66.4	18.8	17.7
Treatment 3				
HH	68.5	59.7	20.9	12.6
HL	60.0	56.9	23.2	12.7
LH	71.2	70.4	16.7	18.8
LL	62.6	67.6	19.0	18.9

* So as to reflect the stepdown procedure, posttest means are corrected for pretest scores, whereas retention means are corrected for both pretest and posttest scores.

Table 8.9 is a modified version of Table 8.8, so as to highlight the mean scores of the statistically significant interaction described immediately above. Table 8.9 contains adjusted mean scores based on a multivariate analysis of covariance with treatment group, simultaneous and successive processing as the independent variables, subtraction posttest and retention tests scores as dependent variables, and with subtraction pretest scores as the covariate; but only for high simultaneous-low successive and low simultaneous-high successive processors who participated in treatments 1 and 3. In the calculation of these adjusted means, as was the case in Table 8.7, the posttest means are corrected for pretest, whereas the retention test means are corrected for both pretest and posttest so as to reflect the stepdown procedures of the analysis.

Table 8.9

Adjusted means from a multivariate analysis of covariance with treatment group, simultaneous and successive processing as the independent variables, subtraction posttest and retention tests scores as dependent variables, and with subtraction pretest scores as covariate*.

Interaction	Algorithm post	Algorithm retention	Problem solving post	Problem solving retention
Treatment 1				
HL	74.0	83.0	15.1	22.6
LH	64.9	71.2	23.8	16.3
Treatment 3				
HL	60.0	56.9	23.2	12.7
LH	71.2	70.4	16.7	18.8

* So as to reflect the stepdown procedures, posttest means are corrected for pretest scores, whereas retention means are corrected for both pretest and posttest scores.

Results in Table 8.9, together with the MANCOVA data in Table 8.8 (Part 2), suggest that teaching approach 1 helps both high simultaneous-low successive and low simultaneous-high successive processors increase their scores between the time of the posttest and that of the retention test, but this is not the case for students in teaching approach 3; and this result was indicated previously, in Table 8.3, as a trend. Equally, in the problem solving component of the tests, there is a pattern where the high isomorphism index treatment helps high simultaneous-low successive processors to increase their scores from posttest to retention test, but low simultaneous-high successive processors have a lower retention test score than posttest score. But the opposite is true for the low isomorphism index treatment, where there is a decrease in problem solving scores from the posttest to the retention test by high simultaneous-low successive processors, but an increase for low simultaneous-high successive processors. This pattern gives some support to the idea that a high isomorphism index treatment both requires, and supports, high simultaneous processors and that a low isomorphism index treatment ignores the potential of high simultaneous processors, and so is more suitable for high successive processors. Of course, this interpretation is supported by only some of the data here, and clearly needs further investigation.

In considering the research questions here, results of the present investigation, thus far, do not give a clear picture as to whether students with different patterns of competence on the two dimensions of cognitive processing react differently to high or low isomorphism index teaching approaches, so as to produce differences in subtraction test scores according to dimensions of cognitive processing and teaching approach. While three of the multivariate analyses involving interactions are not significant, the case of comparing high simultaneous-low successive and low simultaneous-high successive from teaching approaches 1 and 3, do lead to significant differences in scores on subtraction tests. That is, there do not seem to be differences in subtraction tests scores when there is an interaction involving high simultaneous-high successive processing students, and low simultaneous-low successive processors, together with high and low isomorphism index teaching approaches. But there do appear to be some differences in subtraction test scores involving high simultaneous-low successive processing students, and high simultaneous-low successive processors, together with high and low isomorphism index teaching approaches. Further investigation of all these interactions are needed before one could be confident either of establishing differences between various groups, or interpreting the meaning of such data for educational purposes. The qualitative analysis to be reported in the next chapter will also elaborate on these interactions.

Results summary

The data reported in this chapter sought to answer three sets of research questions. The first research questions were concerned with the possibility of applying the procedural analogy theory to mathematics instruction. The findings give some, though equivocal, support to the value of using the procedural analogy theory when planning instruction. Results indicated cases of a high isomorphism teaching approach leading to significantly higher learning outcomes than a low isomorphism index teaching approach. While there were exceptions to this finding, and these exceptions could be explained, it is still the case that further investigation of the procedural analogy theory is necessary. All the same, the theory does provide a supportable theoretical position, together with a pragmatic, systematic and worthwhile guide for planning instruction in mathematics, when teachers intend students to use concrete materials.

The second set of research questions were concerned with the possibility that differences in individuals' patterns of competence of cognitive processing could lead to differences in scores on subtraction tests. The results from this investigation gives limited support to this possibility. For example, the comparison between high simultaneous-high successive processing students and low simultaneous-low successive would be expected to maximise any differences, but results from the multivariate analysis were not significant, though a univariate analysis showed significant differences in the algorithm scores of the posttest favouring the high simultaneous-high successive group. Comparisons between high simultaneous-low successive processors and low simultaneous-high successive indicated no differences in subtraction test scores. The quantitative results from the present research are unable to clarify the roles of simultaneous and successive processing in subtraction. It may be that simultaneous and successive processing lead to different kinds of learning outcomes, for example, high levels of successive processing may lead to correct algorithms, but high levels of simultaneous processing may lead to a deeper understanding of the process. Or it may be that both simultaneous and successive processing are needed, but at different times, for example, successive when learning a new procedure, but simultaneous when relating this procedure to other mathematical knowledge. Results presented so far do not give strong support to any of these views, there is clearly a need for further investigation of the possibilities described here.

The final set of research questions were related to the possibility that scores on subtraction tests would alter according to both the students' patterns of competence of cognitive processing and the teaching approach in which they participated. That is, that there was an interaction between cognitive processing and teaching approach. More particularly, the research questions were concerned with the extent to which different patterns of competence of cognitive processing would interact differently with high or low isomorphism index teaching approaches, so as to produce differences in subtraction test scores. The results of the present investigation indicated that there were generally not differences in the various interactions studied, but there were exceptions. For example, in interactions involving high simultaneous-high successive students and low simultaneous-low successive, and high and low isomorphism index treatments, the univariate stepdown analysis showed trends towards differences in the problem solving component at the time of the retention test, favouring the high simultaneous-high successive processors in the high isomorphism teaching approach. The most significant outcomes though concerned the interactions of high simultaneous-low successive processors with low simultaneous-high successive processors, with the contrast of teaching approaches 1 and 3, where algorithm scores appeared to favour the high isomorphism index treatment. However, in the problem solving component of the retention test, high simultaneous processors did well in the high isomorphism treatment, but not in the low isomorphism treatment. These findings may be interpreted to mean that high isomorphism teaching approaches favour, or require, high simultaneous processing, and that high simultaneous processors are unable to use this capability in a low isomorphism teaching approach. However, the present data does not unambiguously suggest such interpretations, they are only possibilities, though they do help to identify questions for further research.

Taken as a whole, these results give some support to the value of the procedural analogy theory as a model upon which to design instruction, give some clues as to the importance of simultaneous and successive processing in mathematics learning, and indicate that there are some interactions between these variables. While it would be useful for further research to gather additional quantitative data to answer the research questions posed here, an analysis of text data gathered in the course of the present study may provide additional insights into these questions. This analysis is reported in the next chapter.