INTRODUCTION

This study focused on adult learners' responses to fraction questions. Previous research by Hart (1981) and Kerslake (1986) indicated that younger students had considerable difficulty with fraction questions. They also identified several problems that young children face. However, there was little evidence to indicate if adult learners experienced similar problems noted in the child studies, or whether there were fundamental differences between adults' perceptions of fractions and those of younger children.

Plausible explanations for childrens' difficulties with fractions were inconclusive. For example, there were several, often conflicting, interpretations and hypotheses with respect to students' understandings of fractions. Some authors, such as Streefland (1991), suggested that this was due to a lack of 'connectedness' between the historical development of fractions and the methodologies used to teach fractions in the classroom. Hunting (1986) argued that students' difficulties with fractions could be traced to a delayed introduction to them. In contrast, other writers (Dickson, Brown & Gibson, 1984; Freudenthal, 1973; Hart, 1981; Kerslake, 1986) argued that fractions should not be introduced too early, but should be delayed until the child is cognitively ready. None of these approaches addressed adult learners' interpretations of fraction questions.

Given this, it became important to extend the range of theoretical frameworks within which a plausible explanation of adult learners' responses to fraction questions could be interpreted. Clearly, any such framework should be age independent. As a result, a framework, which focused on the students' responses within an acknowledged theoretical position, was deemed to be an essential part of the design. One such framework was the SOLO (Structure of the Observed Learning Outcome) Taxonomy of Biggs and Collis (1982, 1991). This framework together with the literature surrounding childrens' difficulties with respect to fraction understanding are discussed in Chapters One and Two.

Chapter Three describes the design for the first investigation of the study. Initially, a test was administered to 103 adult learners in a TAFE (Technical and Further Education) college at the start of the academic year. This study consisted of a sub-set of items from the Kerslake (1986) study, and it was divided into three themes. These themes were: (i) models of fractions; (ii) fractions as numbers; and, (iii) equivalence of fractions.

Chapter Four outlines the data analysis plan for the main study. This involved the development of a fraction quiz which was administered to 106 TAFE students at the start of the academic year. The students were divided into two broad groups, Associate Diploma (AD) and Tertiary Preparation (TP) classes, because it was a convenient platform with which to administer the quiz. The questions on the fraction quiz were divided into four main research themes. These were: understanding fractions, comparison of fractions, operations on fractions and description of fractions. In addition, the questions relating to each theme were presented in two ways, namely, typical school textbook questions (referred to as *context-free* problems) were used, and, second, fractions placed into familiar situations (referred to as *in-context* problems). Each of the themes are analysed throughout Chapters Five to Eight.

The understanding fractions theme focused on adult learners' responses to fraction questions that involved equivalence and sharing techniques. These two concepts are deemed to be fundamental to understanding fractions. Traditional textbook questions were presented to provide students with the opportunity of addressing equivalence questions in ways they may have seen previously. These questions form the basis of the context-free questions. In addition, students were given several similar situations in which pizzas (and cakes or watermelons) were to be shared between a certain number of people. These questions formed the basis of the in-context questions.

The comparison of fractions theme asked students to rank two or three fractions. All fractions chosen would be expected to be found in traditional mathematics textbooks, and one question contained fractions with numerators equal to '1'. These questions are referred to as the context-free questions. The in-context questions asked students to compare fractions by placing them into two real-life situations. The first situation asked students to compare the various strengths of two different drink recipes. The second situation required students to compare the amount of money two people, who have different salaries, are able to save.

The operations on fractions theme required students to perform the four basic operations (+, -, x, ÷) on fraction questions. Again, this provided students with the opportunity of addressing this theme using traditional textbook examples, i.e., by addressing the context-free questions. In addition, each of the four operations were also placed into a context. In some cases, the context may have been unfamiliar and non-routine to the students.

The description of fractions theme was designed to provide students with an opportunity to describe fractions. The research indicated that, perhaps apart from Kerslake (1986), students are given very little opportunity of describing fractions.

Finally, an overview of fractions is provided in Chapter Nine. The aim of this chapter is to provide an holistic description to the learning of fractions.

Throughout all of the above five chapters, the adult learners' responses to the questions on the fraction quiz are analysed both qualitatively and quantitatively. Typical responses and summary tables are presented on a question-by-question basis. The Quest package, a computer program designed to perform Rasch analysis, (Adams & Khoo, 1993) has been utilised for the quantitative analysis. In the Quest package, Thurstonian Threshold values enable comparisons to be made between different items and cases (students). Overall Difficulty and Step Difficulties can be calculated using the Tau option and these provide measures of (i) the overall complexity of a question, and (ii) the comparative difficulties involved in reaching each response category of a complex problem. Where appropriate, more specific information has been included in Appendices. An overview for each question is presented, while a more detailed discussion, which draws together any similarities across various questions within a theme, is included.

Finally, Chapter Ten discusses the main conclusions of the study and presents a comparison between the responses to one of the questions and a similar question noted in the recent work of Watson *et al.* (in press).

CHAPTER ONE

A REVIEW OF THE LITERATURE

> ***A common error for these students was the so-called freshman error of adding numerators and adding denominators. For example, when asked to find the sum of 1/2 + 1/3, these students commonly answered 2/5 [obtained as (1 + 1)/(2 + 3)]. Although this error has a great deal of intuitive appeal and is common among students who are first learning the subject, it is striking that the error had persisted for so long with these students, despite many attempts by teachers to eradicate the error. ... The usual explanations for this error seemed inadequate in the face of the fact that for at least 5 school years these young adults had received instruction designed to correct the error.***
>
> **Silver (1986, p. 189)**

INTRODUCTION AND ORGANISATION OF THE CHAPTER

The above quote says quite a lot about how adults perceive fractions, and, in particular, how some adults persist with errors long after their original introduction to fractions. Often this is despite repeated intervention programs. Although the reasons for this remain unclear, the difficulty of trying to reconcile and incorporate new ideas with ones that a student has previously established may be a major contribution to the confusion identified (Payne, 1976). Clearly, "certain 'standard' wrong answers given by many different people - should certainly be explained by any adequate theory of human mathematical thought" (Davis, 1984, p. 97). Notwithstanding this, very little research appears to have been carried out investigating mature-age students' understandings of mathematical concepts. Such research would seem to have many benefits; two stand out. First, it would allow a perspective on what understandings are retained after formal schooling. Second, it would provide a firmer basis on which to structure bridging or support programs to help older students in their move to tertiary education.

However, there is considerable evidence regarding younger learners, and their understandings of fractions (Behr, Harel, Post, & Lesh, 1992; Dickson, Brown, & Gibson, 1984; Hart, 1981; Hunting, 1984, 1986; Kerslake, 1986; Post, 1988;

Streefland, 1982; Watson, Campbell, & Collis, 1993; Watson, Collis, & Campbell, 1991ab, 1992ab). The findings indicate that fractions is a difficult topic to teach, and a difficult topic to learn.

A major investigation into fractions was undertaken as part of *The Concepts in Secondary Mathematics and Science (CSMS)* project (Hart, 1981). This study was carried out between 1974 and 1979 where almost 10 000 children between the ages of 12-to-15 years were tested. Approximately 30 students were also interviewed, typically for one hour (Hart, 1981, pp. 1-2). In the case of the fractions topic, 246 first-year (12 years), 309 second-year (13 years), 308 third-year (14 years) and 215 fourth-year (15 years) students were surveyed (Hart, 1981, p. 1). Part of its sequel, the *Strategies and Errors in Secondary School Mathematics* (SESM) research, also examined fractions. The findings of this investigation were presented in *Fractions: Children's Strategies and Errors* (Kerslake, 1986). This latter study consisted of two phases of testing and interviewing, followed by a teaching experiment. In the first phase, 23 students aged from 12-to-14 years were interviewed. The interviews, which were taped, lasted for approximately 30 minutes (Kerslake, 1986, pp. 8-9). In the second phase, fourteen 13-year-olds were interviewed for between 30-to-40 minutes. Fifty-nine students completed all sessions of the teaching experiment, and they were used in the overall evaluation of the program. Subsequently, class trials were administered in six schools on 81 students across a similar age range.

Results consistent with the research carried out by the above authors has also been noted by other researchers in different countries, e.g., Bourke, Mills, Stanyon and Holzer (1981) in Australia; Hasemann (1981) in Germany; Post (1988), as part of the *National Assessment of Educational Progress (NAEP),* in the United States; and Streefland (1991) in The Netherlands. In general, the findings indicate that many students have considerable difficulty with fractions, primarily because they do not understand basic fraction concepts.

The main aim of this chapter is to investigate the literature regarding fractions and the different approaches used by learners. The following four questions serve to structure the analysis:

(i) What evidence is there that students have difficulty manipulating fractions?

(ii) What similarities or differences are there among different studies or different age groups?

(iii) What reasons, if any, does the literature suggest for student errors?

(iv) Is there evidence to suggest a cognitive hierarchy?

To provide a focus for the existing work, this chapter is divided into two broad sections which relate to the foundations of fractional understanding and the treatment of fractions as numbers. The rationale for this decision was based loosely on evidence accumulated throughout the literature search. For example, both Hart (1981) and Kerslake (1986) identified three main problem areas of fractions. These were: fractions as numbers, the use of diagrams to illustrate fractions, and equivalence of fractions. However, to encompass contributions from other authors, such as the historical insights of Kieren (1988) and Streefland (1991), the decision to divide the literature review into two broad sections appeared feasible and desirable.

FOUNDATIONS TO UNDERSTANDING FRACTIONS

This section of the work investigates four main areas which relate to understanding fractions. First, an historical perspective on fractions is presented. This establishes the basis on which the need for fractions developed. This is followed by a discussion of typical models used in schools to demonstrate the part-whole construct. Finally, since the fraction one-half is the dominate fraction used by many (including adult) learners, special attention has also been given to this.

HISTORICAL PERSPECTIVES ON FRACTIONS

> An old Arab, Anwar his name, decreed before he died that his eldest son inherit one-half, his second son one-fourth and his youngest son one-fifth of all his camels. He died leaving 19 camels and his three sons could not agree on how to divide them. A dervish - passing by on his camel - observed the disagreement, dismounted and stated helpfully: 'I will loan you my camel.' Each son now took his share of the 20 camels. The dervish then remounted his beast of burden and continued along his way, leaving all three heirs contented. And so did come to pass the last will of Anwar.
>
> Streefland (1991, p. 5)

While humanity may not have always used fraction arithmetic as we know it today, fractions, in some form, are found among the earliest permanent records known to exist (Kieren, 1976; National Council of Teachers of Mathematics (NCTM), 1964; Streefland, 1991). Fractions were invented when the need arose to ensure equality and fairness in agriculture and business transactions between early traders and merchants. In addition, there is some evidence to suggest that early scholars also pursued them for their intellectual enjoyment, e.g., the above problem and the Rhind (Ahmes) papyrus (c. 1700 BC). The ancient Babylonians (c. 2000 BC), for example, devised a sexagesimal system which enabled fractions to be expressed as a mixture of sexagesimal and decimal parts. It is worth noting that this system is still used in expressions, such as 167 degrees, 35 minutes and 15.5 seconds (Kieren, 1976, p. 101).

The best example of ancient 'success' with manipulation of fractions can be seen in the Rhind (Ahmes) papyrus of ancient Egypt which contains "a table of quotients resulting when 2 is divided by an odd number greater than 1 and less than 103" (NCTM, 1964, p. 215). Today these fractions would be expressed as 2/3, 2/5, 2/7, etc. However, the ancient Egyptians only had symbols for unit fractions, such as 1/2, 1/3, 1/4, etc. The only notable exception to this was the symbol for 2/3. Despite this, fractions with numerators greater than one, were able to be written as a sum of the unit fractions (Kieren, 1976; NCTM, 1964). For example, 47/60 could be expressed as 1/3 + 1/4 + 1/5.

The next major contribution to fraction arithmetic was not until the sixteenth century, when "an anonymous author, probably Christianus van Varenbrajen, deals with first the arithmetic theory and thereafter with its applications" (Streefland, 1991, p. 7). It is worth noting that this approach is still the dominant method of instruction in many common textbooks. This is despite the advent of the pocket calculator and other technological advancements which have minimised the use of fractions as a basic mathematical skill required by many adults. However, both Kieren (1976) and Streefland (1991) argued that students have lost touch with the 'primitive' or 'realistic' approach to fractions that our ancestors sought. This leads to a lack of fundamental understanding of fractions, such as the part-whole comparison or associated models.

THE PART-WHOLE COMPARISON

The part-whole comparison is the most fundamental concept of a fraction and forms the basis of all further work on fractions (Behr & Post, 1988; Davis, 1984). The part-whole construct is that aspect of fractions that deals with geometric representations of fractions. The notion that a whole can be divided at all is critical to the development of fractional understanding, e.g., some two-year-old children refuse to cut shapes at all (Piaget, Inhelder, & Szeminska, 1960, p. 277). However, once the idea that the whole can be sub-divided into any number of parts is established, there would appear to be two subsequent, but distinct, conditions which are necessary for further development of fractional understanding. These are: (i) the idea that the parts have to be equal; and, (ii) that the sum of all the parts equals the whole.

However, the literature does not support the notion that these two conditions necessarily develop logically nor simultaneously. Research undertaken by Davis and

Hunting (1990) suggested that the act of 'partitioning' did not occur naturally for four-year-old children sharing jelly beans in the absence of a teacher, i.e., partitioning is a "learned mechanism" (Kieren, 198 , p. 71). The findings of another experiment, by Watson *et al.* (1991a, p. 18) would also appear to confirm the above assertion. In their experiment, chocolate bar wrappers were glued to pieces of cardboard, and then some cut lengthways, while others were cut widthways. Although some children in the study recognised that the 'halves' were fair, many respondents indicated that this was not the case, or requested a re-arrangement of the pieces before agreeing that the two halves were equal. The important point here is that it is the synergy of the above two conditions that is crucial to further partitioning, since the new parts then become wholes, and so the process of subdivision can be continued. For example, 1/6 could be obtained by dividing 1/3 into half. These premises have been referred to by many authors including Piaget, Inhelder and Szeminska (1960, p. 277), who included them as a set of criteria for understanding the part-whole aspect of fractions.

Since it has been "hypothesised that one of the higher level concepts associated with dealing with fractions is the recognition of the necessity to know the whole in relation to the fractional part being considered" (Watson *et al.*, 1991a, p. 16), then it seems plausible that there should exist broad well-structured models and manipulatives which teachers should be able to apply to teaching fractions. However, it is worth noting that in addition to fulfilling both of the above conditions, there is the complication of relating the physical act of partitioning to its 'geometrical' or 'logical' pencil and paper representation. To date, there are only three major representations of the part-whole construct, namely, area diagrams, such as with circles or rectangles; subset of a whole representation; and, the use of the number line. While the area model has achieved widespread acceptance, the literature indicates that there are important concerns with each of the models. For example, it has been suggested that "counting the number of shaded parts and the total number of parts and reporting this pair of numbers ... does not relate the act of partitioning to the fractional number" (Kieren, 1980, p. 74). Further research into this area would seem crucial, since it has been postulated that it is not possible for students to undertake effective problem solving until they relate their informal partitioning knowledge to formal symbols (Mack, 1990).

MODELS OF THE PART-WHOLE CONSTRUCT

There are three main models which describe the part-whole construct associated with the geometric representation of fractions. These are: the area model; the subset of a set of discrete-objects model; and the number line model. A discussion of each model is now presented.

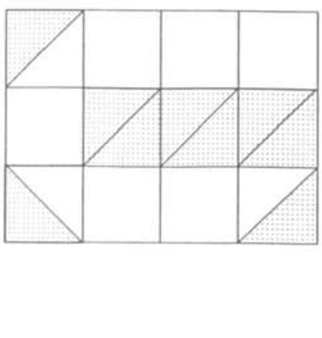
The area model

Of all the part-whole models, the area model has become synonymous with the part-whole construct. The area model has received disproportionate attention in the literature. It is the most prevalent model found in textbooks, and is often the sole model presented to demonstrate fractions. Although the reasons for the widespread use of the area model are difficult to determine, the main attribute of the part-whole model is that it attempts to enable learners to identify the whole and hence conceptualise fractions as parts of the whole. Hence, it seems plausible that area diagrams should provide obvious clues to use to describe fractions. However, several writers (Davis, 1984; Kieren, 1991; Watson *et al.*, 1991a) acknowledged that this is not always apparent to students. For example, the choice of whether the shaded area or the unshaded area should represent the fraction is often not consistent between teachers or textbooks, and frequently it is left to the learner to decipher, perhaps erroneously, the 'real' meaning of the model.

In addition, the area model is limited by both the choice of shape and the variety of fractions that can be represented adequately. Circles and rectangles are usually the only shapes used to represent fractions that occur in textbooks or as typical manipulatives, and the number of different fractions that can be adequately represented by such shapes is limited to only simple fractions, such as 1/2 and 1/4.

In addition, both Hart (1981, p. 69) and Watson *et al.* (1992a) noted that many students appeared to be misled or easily confused by perceptual distracters in representations of the part-whole model. For example, when Hart (1981, p. 74) administered the following question:

(P20) I am putting tiles on the floor; they are shown shaded. What fraction of the floor has been tiled? A diagram similar to that shown was provided.



the results indicated that many students preferred to use "a square rather than a triangle as the unit for counting parts" (Hart, 1981, p. 74). Table 1.1 summarises the results to this question. The table indicates that a sizeable number of students across a wide age group had difficulties with interpreting and addressing the question.

TABLE 1.1

Percentage of replies to P20 in Hart (1981, p. 74)

Student Ages	Replies to P20	
	9/24 or 3/8	4½/12
12 years	30.5	18.7
13 years	28.8	22.0
14 years	37.0	21.4
15 years	42.3	20.5

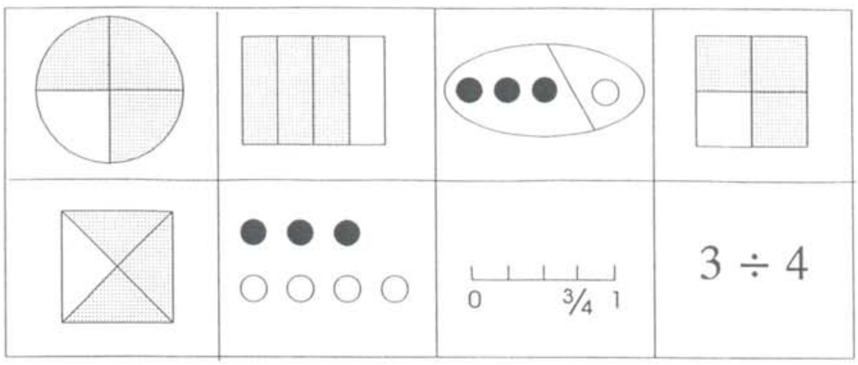
Some authors (Dickson *et al.*, 1984; Kieren, 1980) have suggested that this model may limit the development of the idea that fractions can be greater than one, i.e., the difficulty of representing improper fractions using geometric shapes. For example, many children deciphered a similar diagram to that shown in Figure 1.1, below, as 7/10 rather than 7/5 (Dickson *et al.*, 1984, p. 279).

FIGURE 1.1

Representation of part-whole area model of 7/5

Despite these misgivings, Kerslake (1986, p. 89) reported that subjects in her study accepted the part-whole model without reservation. For example, when Kerslake (1986, p. 9) asked the following question:

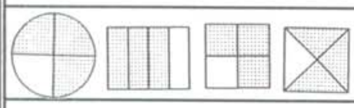
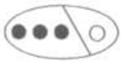
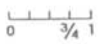
Which of the following cards would help someone to understand what the fraction 3/4 is?



the results indicated a high acceptance of the area model, and, unlike other representations of the part-whole model, no rejection of it. The findings of this question are summarised in Table 1.2. However, it is unclear if this widespread acceptance of the area model is due to its 'familiarity', frequent occurrence in textbooks, or if it was the 'easiest' model to choose. In either case, the model would appear to be of limited use, since it does not adequately provide a basis which enables further exploration of fractions (Dickson *et al.*, 1984, p. 279), including the four operations. For example, 3/5 + 2/7 is not enhanced by a diagram indicating these two fractions.

TABLE 1.2

Representation of children's choice of models of the fraction 3/4 (Kerslake, 1986, p. 12)

Model	Accepted by	Rejected by
	23	0
	15	8
● ● ● ○ ○ ○ ○	8	15
	19	4
3 ÷ 4	3	20

The subset of a set of discrete-objects model

The 'subset of a set of discrete-objects' model is similar to the area model and has the same deficiencies as the area model. For example, 7/5 would be represented by the diagram in Fig. 1.2. Here, the left hand shape represents 5/5 or one whole, and the right hand shape indicates 2/5. When viewed together, these two structures represent 7/5.

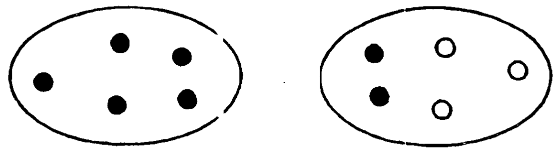


FIGURE 1.2

Representation of subset of discrete-objects model of 7/5

Investigations into the subset of a discrete-objects model (Novillis, 1976; Payne, 1976) were inconclusive. For example, Novillis (1976) found no significant difference between this structure and the area model. However, Dickson *et al.* (1984, p. 280) noted that "it appeared to cause confusion among children in their understanding of the other models" and was removed from the teaching course as a result of the Payne study. However, caution must be exercised in drawing conclusions from this report since Payne (1976) dealt with younger children than Novillis (1976), and this could explain the increased difficulty identified. A typical problem associated with this model is described in Post (1988, p. 200):

> for the discrete set of 12 apples, each equal-sized part (equivalent subset) consists of four separate, nonconnected objects. Nevertheless, in the process of partitioning and conceptualizing 2/3 of 12 apples, the child must conceptually think of twelve apples as one whole unit. That is, the 12 objects must become a conceptual entity. Similarly, it is difficult for the child because each of the three parts has four objects, so now the child must mentally think of four objects as one part. ... Some children will pick out two of twelve, thinking that that's what the numerator means, two parts. It is difficult for some children to understand initially that each part has four subparts in it.

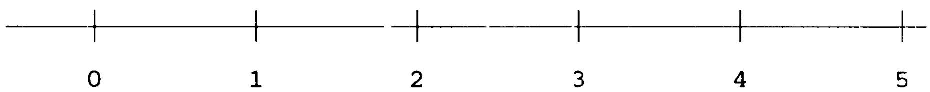
The advantage of the subset of a discrete-objects model is that it relates well to the idea of a ratio and percentage, which is better than the traditional, geometric-area model. However, the 'area' model is considered to be a better model than either the subset model or the number line representation, since it provides a more reasonable explanation for multiplication of fractions by using the area of a rectangle to justify the process.

The number line model

The use of the number line as an adequate representation of a fraction presented students with difficulties (Bright, Behr, Post, & Wachsmuth, 1988; Kerslake, 1986; Novillis-Larson, 1980). The students' problems were exacerbated if the number line was not of unit length or if the length was not divided into exactly the same proportions as the denominator of the given fraction. For example, some students could not locate 3/8 if the number line was divided into quarters, halves or sixteenths.

Kerslake (1986) noted problems that many children would have when they were asked to locate 3/5 on a number line. She posed the question:

Where would the number 4 go on this number line? And the number 3/5? And the number 1 1/5?



Kerslake (1986, p. 33) concluded: "how much more successful the children were with 1 1/5 than 3/5. In the case of the mixed number, the interpretation of finding that fraction of the whole line was abandoned, and the children appeared to switch methods depending on whether the fraction was more or less than one". This would indicate that '1' is some sort of a yardstick which the children need in order to draw together all the other information in order to make a comparison.

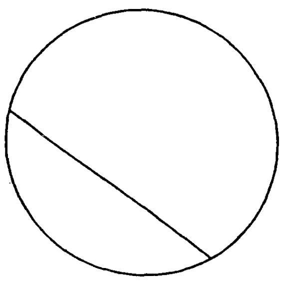
Similar difficulties were confirmed by the work of Novillis (1976) and Payne (1976). Furthermore, Dickson *et al.* (1984, p. 282) noted that "unlike the two other representations, the number line does not incorporate the notion that a fraction can be thought of as a part of a concrete object, or as a part of a set of objects, but reduces it to an abstract number". This is an important point. Until the utilisation of the number line approach, fractions are identified as parts of areas and are not necessarily viewed as numbers.

Streefland (1978, p. 63) and Kerslake (1986, p. 91) noted that many children found it difficult to accept fractions as numbers. This may explain why many children became confused over what constituted the 'unit' in the previous studies. Although Dickson *et al.* (1984) argued that children were not as familiar with the number-line approach as other part-whole models, they still needed more experience in reading scales involving fractions, such as rulers or barometers or thermometers, before moving on to this model. However, much of this type of equipment is becoming digitised, and the traditional 'reading of scales' apparatus soon may be no longer readily available.

Irrespective of which model is preferred, the simplicity, familiarity and dominance of one fraction over all others makes it worthy of special mention in a section devoted to the fundamentals of understanding fractions. This fraction is, of course, one-half.

ONE-HALF

The simplest subdivision of all is 1/2, followed by successive halving, such as 1/4, 1/8, etc (Post, 1988, p. 199). However, the term 'a half' as used in everyday language is not always the mathematical equivalent to 1/2. To be able to say 'one-half' is very different to being able to work with the symbol '½' (Gunderson & Gunderson, 1961, p. 250). For example, the first encounter with the part-whole notion for many children is via '1/2 a glass of drink' or they ask for the 'biggest 1/2' of an edible item. In many everyday terms, a half is merely somewhere between a whole and nothing, e.g., half way home. A further example of students misconceptions of 1/2 can be seen by the following question, first posed by Hart in 1981.



Has this circle been split into 2 halves?................
Why do you think this?

Dickson *et al.* (1984, p. 277), when commenting on this item, reported that: "While 89 per cent answered correctly that the circle had not been split into halve because the pieces were unequal, 6 per cent thought that the circle had been split into halves because there were 'two pieces'". This observation, and others (e.g., Watson *et al.*, 1991a), confirms that children's perceptions of 'sharing' do not equate to the adult notion of 'equal sharing'. Instead, the idea of 'fairness' appears to be acquired separately and at some later time in their development (Watson *et al.*, 1991a). Recall, that it is this concept of fairness that is subsumed in the notion of equal sharing that forms the first step towards understanding fractions.

Historically, the need to extract a 'fair deal' developed in precisely the same manner. Currency exchanges between traders and merchants sanctioned the need for the formalisation of equal shares (Streefland, 1991), and, hence, hastened the formation of fractions. However, according to Streefland (1991), this rudimentary requirement of fractions has been progressively and systematically eroded through the generations - to be replaced by theoretical perspectives and out of context problems. As a consequence, Streefland (1991) argued, many of the problems associated with fractions could be minimised, if not avoided altogether, if a return to 'realistic' mathematics was sanctioned.

Notwithstanding the above, the notion of one-half, once established, was considerably "easier to deal with than any other fraction" (Hart, 1981, p. 69). One-half is the predominant fraction used by many individuals (Streefland, 1978) well into adult life (Kieren, 1981, p. 71). Other fractions such as 1/3 were not 'self-evident' (Streefland, 1978, p. 53). Lesh, Behr and Post (1987, pp. 53-54) have "shown 'halfness' to be a cognitively primitive concept; youngsters typically show earlier proficiency for tasks involving one-half than for any other fraction items (Kieren, 1976)". Hunting (1984a), for example, noted that one student (Sean), in an attempt to demonstrate 3/5 of a ribbon, used a strategy that involved 'halving'. The student folded the ribbon in half and then in half again to make four sections. Although the student was not totally satisfied with the result he did not attempt an alternative strategy. In another experiment, Hunting (1986, p. 53) concluded that a nine-year-old girl (Rachel) was operating "with two distinct schemas for interpreting fractions. One was a partition-based scheme restricted mainly to the fraction 1/2. The other was an intuitive scheme whereby fraction information was assimilated into already well-established whole number structures". Interestingly, the repeated use of the 'halving strategy' was also the way ancient Egyptians solved many 'fraction' problems.

SUMMARY

There are three main findings from this section of the work. First, historically, fractions were invented (discovered?) to aid trade and other merchandising negotiations. Without them it would have been difficult to convert different monetary values and hence to achieve agreement between traders of what was fair and equal. For example, "One el of linen costs 18 stuivers. A pound of sugar costs 3 stuivers. How much sugar will one get for 43 els of linen?" (Kool, (1984 part 2, p. 63 and p. 183) as quoted in Streefland (1991, p. 13)).

Second, the answer to the question: "what is the fundamental notion of a fraction?" is difficult to determine. Clearly, it relies on the notion that a whole can be sub-divided, and encompasses the idea that such partitions are equal and contribute to make up the whole. However, evidence from the literature suggests that neither of these conditions may occur spontaneously and synchronously. In addition, some authors (e.g., Kieren, 1984) have argued that the act of partitioning, and the pen and paper representations that often occur in the classroom, may actually mislead or confuse students new to understanding fractions. Notwithstanding this, these perceptions of fractions must be firmly entrenched into the learners' cognisance before further developments in fraction understanding can occur.

Finally, fraction arithmetic may also be inhibited until learners become aware of the limitations of natural numbers, i.e they must be able to conceptualise the need for arithmetic other than that applied to natural numbers. This may invoke a 'need' in many students to resort to other methods, such as learning by rote, rather than learning for understanding. This would be particularly prominent when fractions must be treated as numbers. This issue is now taken up in the next section.

FRACTIONS AS NUMBERS

> Rational numbers are the first set of numbers children experience that are not based on a counting algorithm of some type. To this point, counting in one form or another (forward, backward, skip, combination) could be used to solve all of the problems encountered. Now with the introduction of rational numbers the counting algorithm falters (that is, there is no next rational number, fractions are added differently, and so forth). This shift in thinking causes difficulty for many students.
>
> Post (1988, p. 190)

The German word for fraction (*Bruchzahl*) means "broken number" (Kieren, 1980, p. 74). However, treating fractions as numbers is fundamentally and qualitatively different to the part-whole construct as noted in the previous section. For example, there are some aspects of fractions which necessitate the treatment of fractions as possessors of number properties. These include decimals, percentages, ratios, an operator approach, and an indicated division. In addition, fractions can be ordered and compared (by utilising equivalent fractions), and operated upon. Each of these issues is now discussed.

DECIMALS

> The relationship between fractions and decimal fractions is not unlike that between the system of Roman numerals and the Arabic system which we normally use; in both cases the underlying concepts, whether of whole numbers or rational numbers, are the same. The essential difference between the systems lies in the particular conventions according to which the numbers are recorded
>
> Dickson *et al.* (1984, p. 284)

Clearly, there is value in converting fractions to decimals to aid in solving a lengthy or complex problem, a calculation that is particularly tedious or repetitive, or a question which involves the use of a calculator or computer. Given this, and the increased use of technology in society, decimals have become the dominant method by which fraction problems are solved This does not, however, mean that learners find decimals any easier to interpret or understand. For example, some learners often perceive that the longer a number is, the greater its value. For example, Hart (1981, p. 52) asked students which number was the bigger of 0.75 or 0.8. The results are presented in Table 1.3.

TABLE 1.3

Results comparing 0.75 and 0.8 in Hart (1981, p. 52)

Student Ages	12	13	14	15
Facility (%)	57	65	69	75

When interviewed, Hart noted that one student (Jane, aged 12, 2nd year) selected 0.75 because: "This is nothing before and seventy-five; this is nothing before and just eight" (Hart, 1981, p. 52), possibly indicating that the student did not relate place value concepts to decimals, and therefore chose the *longest* number to mean the *biggest* number, as if dealing with whole numbers.

Other learners apparently believe that the tenths and hundredths columns after the decimal point are repetitions or mirror images of the units and tens columns, i.e., some children think that the number after the decimal point is a 'different' number, which also contains tens and hundreds (Hart, 1981, p. 51). For example, Hart (1981, p. 52), asked students to say the number 0.29. The results are presented in Table 1.4.

TABLE 1.4

Verbalisation of 0.29 in Hart (1981, p. 52)

Student Ages	12 years	13 years	14 years	15 years
(Nought) point two nine	26	32	41	41
(Nought) point twenty-nine	25	32	30	27
twenty-nine	19	13	8	10

The table indicates that some children do not understand the significance of place value with respect to decimals. This confusion may in part be due to digital clocks and the decimal system of money where it is acceptable to read 6.15 as six fifteen.

'Tenths' were significantly easier to work with than subsequent decimals. For example, the findings of Brown (1981b) and Dickson *et al.* (1984, p. 290) indicated that while 65 percent of 12 year olds and 85 percent of 15 year olds can work with 'tenths' in a similar way to the fraction models presented previously; approximately 25-to-30 percent of 12 year olds and 55-to-70 percent of 15 year olds were able to deal with hundredths in a concrete way. As Hart (1981, p. 64) wrote: "It was clear that many children still needed visual models of tenths, hundredths and so on, to bring out the relationships in a more concrete way".

Finally, the use of calculators may not necessarily overcome children's misunderstanding of decimals (Hart, 1981, p. 64). Many children "who did rely blindly on rules more often misapplied them than not" (Hart, 1981, p. 64). It appears that calculators will not significantly promote understanding of decimals by these children, and "without careful structuring of the work it [a calculator] may just be used to produce meaningless answers which can be copied down faithfully to eight decimal places" (Hart, 1981, p. 64).

PERCENTAGES

Although considered a third method of representing fractions, the only difference between this representation and the above method, is that it is based on the concept of hundredths. Dickson *et al.* (1984, p. 291) claimed that conversion of decimals to percentages or vice versa, "in the case of familiar numbers only, was carried out correctly by about 50 percent of 11 year olds". A plausible explanation is that the more common conversions could be easily 'memorised' or 'rote learned'. This could at least, in part, explain the persistence of the common error of many students who write 1/3 as 30% consistently.

RATIOS

A ratio is said to convey the notion of relative magnitude, and, as such, is the first time that rational numbers are used in absence of a 'whole' for reference. Novillis (1976) found that this aspect was one of the last to be developed in students aged 10-to-12 years, and implies that this model of fractions may be acquired at a much later stage than any of the previous models.

This observation would appear to be partially confirmed by other authors, such as Piaget *et al.* (1960), Karplus, Karplus and Wollman (1974) and Hart (1981), who have shown that children rarely use the rule a/b = c/d. Instead, many children resort to 'primitive' adding or doubling (or halving) techniques when dealing with ratios almost irrespective of the ratio. Hart (1981, p. 97) reported that many students still select the adding strategy when asked to enlarge a rectangle in the ratio 3:5, even when the final figure showed little similarity to the original, and could not be compared to it.

There are several problems associated with expressing fractions as ratios. The major difficulty is that, while all fractions are ratios, not all ratios are fractions. For example, ratios can express different units, such as four passionfruits for 70 cents etc. Other examples include liquidity ratios and the Consumer Price Index. It is also acceptable for ratios to have 0 in the second place. For example, it is possible, to represent 5 lollies to zero chocolates as a ratio.

Fractions can only be used to represent parts of one object, e.g., 2/3 is two out of three parts. Rates can also be expressed as ratios, although the term is frequently used to compare two unrelated or different quantities, e.g., 5 laps/24 minutes, (Heller, Post, Behr, & Lesh, 1990) and usually only require a single number. For example, although 60 kmph is really representing two units - the 60 km and the 1 hr, it is redundant to express this relationship as a fraction.

However, in dealing with the rate aspect of fractions, it is possible to ask *directional questions* to "determine the qualitative direction of change in the value of a fraction or rate, given specified qualitative changes in the numerator/denominator" (Heller *et al.*, 1990, p. 390). For example, what will happen to a fraction if the numerator is increased (or the denominator is decreased)?

Ratios are not always rational. For example, π represents the ratio of the circumference of a circle to its diameter and is an irrational number. However, it is used to represent the division of two numbers (circumference and diameter).

Similarly, "in a square, the ratio of the side length to the diagonal is 1: $\sqrt{2}$, and since $\sqrt{2}$ is not equal to the ratio of two integers, it is [also] not a rational number. Fractions, on the other hand, by their very nature are rational numbers because every fraction is equal to the division of two integers" (Hoffer, 1988, p. 289).

Ratios and fractions are not always dealt with consistently or combined in the same way. For example, 1 mark out of 2 on a test is frequently written as 1/2. If this score is then combined with a further 3 out of 4 marks (3/4) then the total score 4 is out of 6, i.e., 1/2 + 3/4 = 4/6 is implied. However, the addition of the fractions 1/2 + 3/4 $\neq$ 4/6. A further example also helps to illustrate this point. If there are 17 girls in a class of 30 students, the corresponding fraction is written as 17/30. This may seem confusing to many students who use ratios as a comparison and would expect to compare the number of females to the number of males, e.g., 17:13. This can be further complicated when students are taught to incorrectly write 1:3 = 1/3, when there are really 4 parts present in the original statement (1:3 means 1 part compared to 3 parts, i.e., there are 4 parts altogether).

Discrepancies with the ratio aspect of fractions are further exacerbated by the use of d*y*/d*x* terminology in the Chain Rule in differentiation of composite functions, e.g., despite the fact that d*y*/d*x* does not represent a fraction, many high school children are presented with the following 'proof': d*y*/d*x* = d*y*/~~d*u*~~ x ~~d*u*~~/d*x*.

AN OPERATOR APPROACH

This is a relatively new (c. 1966) aspect of fractions (Streefland, 1982, p. 234). Fractions are treated as an 'operator' so that when applied to a particular problem, a transformation occurs (Behr *et al.*, 1992). Fractions are functions that are capable of shrinking or stretching the original construct or number. For example, the numerator part of 2/3 is considered to multiply the original construct, followed by the application of 3, which is considered to divide it, i.e., the numerator stretches and the denominator contracts. The advantages of this approach are that the order does not change the original problem, and that "no special status is given to fractional numbers less than one; a '2/3 operator' and a '3/2 operator' function in the same way" (Kieren, 1981, p. 73). This mode can also be applied equally well to both discrete and continuous quantities. The main disadvantage with this approach is that it may require students to have been exposed to the notion of operators prior to their introduction to fractions.

AN INDICATED DIVISION (QUOTIENT)

This aspect of fractions deals with the equivalent statements a/b = a ÷ b and appeared to be the most difficult aspect for learners to assimilate. This is despite the fact that division is clearly related to partitioning (Dickson *et al.*, 1984; Post, 1988). Kerslake (1986) reported that this model was the one that was most strongly rejected, even among the teachers of the students interviewed.

This is not an isolated incident. A significant number of respondents in many studies when faced with situations when division is unavoidable, often chose the biggest number first, i.e, the biggest number was also chosen to be the numerator (Dickson *et al.*, 1984, p. 283; Hart, 1981, p. 68; Kerslake, 1986, p. 91). One plausible explanation for this observation may be the mistaken idea that 'you can't divide by a bigger number'. For example, in a similar item regarding 3 ÷ 4 in the Kerslake study, many children believed that "'you always divide the larger number by the smaller number' or 'fours into 3 won't go so you bring down the nought'" (Kerslake, 1986, p. 91). It is also worth noting that even when this strategy was used, many children still preferred to acknowledge a remainder rather than give the remainder in fraction form. "It seems that some children fail to appreciate that although these general 'principles' were acceptable when dealing only with integers, they are no longer valid when working with the set of rationals" (Kerslake, 1986, p. 91). When students were asked to cut a piece of ribbon 17 cm long into 4 equal pieces, many children preferred to select the answer "4 cm remainder 1 cm" from a short list, i.e., they chose an answer with two whole numbers in it in preference to an answer containing fractions, such as 4¼ (Hart, 1981, p. 68). Dickson *et al.* (1984, p. 283) concluded: "only one third of children in the first two years of secondary school have appreciated that any whole number can be divided by any other to give an exact result expressible as a fraction".

EQUIVALENT FRACTIONS

The notion of understanding equivalence is essential to being able to add, subtract and compare fractions. The first notion of equivalence is typically represented by the part-whole model. As highlighted in the earlier sections of this work, it is the link between the physical act of partitioning and the diagrammatic representation that is important to understanding fractions, rather than the use of the part-whole construct *per se*. In the case of equivalent fractions, in particular, it is of paramount importance that any diagrammatic representations and algorithmic utilisation also be linked if fractions are to be understood in terms of their number properties. If this idea has not been integrated in the learner's mind, then it is unlikely that further developments (such as operations) in fractions will be forthcoming or understood. For example, Dickson *et al.* (1984, p. 306) concluded that "many children find the idea of equivalence difficult unless a concrete context (e.g., a diagram) was provided. Many children may therefore be taught the procedures for adding fractions by rote, with little basis of understanding". This result would appear to be exacerbated when combined with the findings of the chocolate wrapper experiment of Watson *et al.* (1991a, p. 18) and experiments of Hart (1981, p. 69), which indicated that some children were distracted or confused by erroneous information when presented with 'diagrammatically equivalent' shapes.

In essence, it is this lack of 'connectedness' (Streefland, 1982) in many learners understandings of fraction concepts (Kerslake, 1986; Streefland, 1982) that contribute to the learning of certain standard algorithms by rote by students. This is usually evident by the continual occurrence of typical standard errors (Dickson *et al.*, 1984, p. 306). For example, Kerslake (1986, p. 94) noted that many children could write equivalent fractions, but were unable to utilise them in any form whatsoever. This included the ability to add even simple fractions. Many children in the Kerslake study were aware that 2/3 and 10/15 were the 'same', but also maintained that because 2/3 was 'multiplied by 5' that 10/15 was now 'bigger' than 2/3. Kerslake's explanation for this inconsistency was:

> the idea that successive multiplication and division, ... leaves the value of the fraction unaltered is quite sophisticated, and it seems likely that, for many children, it is obscured by the algorithmic approach of 'multiply top and bottom' ... . The algorithm, having no basis in meaning, leads children to believe that the whole fraction has been multiplied
>
> Kerslake (1986, p. 93)

However, the literature is largely inconclusive with respect to how equivalent fractions should be taught. For example, it has been argued that the "isolated use of models and patterns, ... never seems to serve the processes of algorithmization or mathematization" (Streefland, 1982, p. 135). However, to teach children, to an acceptable level of proficiency, multiple approaches to fractions, would require an inordinate amount of class time to consolidate children's understanding of equivalence. In addition, many teaching strategies do not take into account the pre-existing, and often robust, schema of learners. One attempt to address this issue and develop the notion of equivalent fractions utilised situations which were familiar to children. Streefland (1982, p. 244) devised a process by which children progressively developed equivalent fractions based on a patterning approach that relied almost exclusively on the ratio construct. For example, consider 3/4 of a pancake. A typical pattern response may resemble that provided in Table 1.5.

TABLE 1.5

Patterns showing 3/4 of a pancake in Streefland (1982, p. 244)

Pancakes	3	6	9	12						
Children	4	8	12	16						

Streefland (1982, p. 244) argued that this approach "does justice to the children's inclination to algorithmizing and their need for building on personal algorithms". Eventually, Streefland (1982, p. 244) stated, children could successfully answer questions, such as "16 pancakes are ordered for 24 children. Does this example fit?" Streefland (1982) also acknowledged that students could invent 'shortcuts' by comparing tables directly, such as locating 24 children in the other table first. This approach now made subtraction possible. However, there are problems inherent with this approach. For example, minor calculational errors meant that a child may continue to reproduce redundant and repetitive solutions via tables or give up completely.

The literature also noted that some students did not take up standard or acceptable patterns when dealing with equivalence of fractions. Instead, they invented different approaches which appeared to work, although in some cases, this was only on an intermittent basis. For example, Hunting (1984a) identified seven different strategies children use in completing equivalence tasks. In essence, all of these strategies relied on 'competing a pattern' concepts. The strategies were:

1. Common factorisation, e.g., 2/6 = /3 can be 'solved' by dividing 6 by two to get 3 and then by dividing 2 by two to get 1. This approach works if one denominator is a factor of the other.

2. Cross multiplication, e.g., 3/12 = /8 can be 'solved' by first multiplying 3 by 8 to get 24, and then by dividing 24 by 12 to get the numerator.

3. Recalled knowledge, e.g., students 'know' that two quarters equals a half because they have remembered it.

4. Invented algorithm, e.g., Hunting (1984a, p. 27) reported that 2/6 = /3 could be 'solved' by multiplying the numerator of the first fraction with its denominator to obtain 12. The missing numerator could then be calculated by dividing 12 by 3.

5. Use of ratios, e.g., 1/4 = /8 could be 'solved' by arguing that because 8 had increased by 4 (the original denominator), then the unknown numerator should also be obtained by increasing the original numerator by 1 to give 2.

6. Intermediate fraction, e.g., 1/4 = /8 is solved by arguing that: "four-eighths is one-half and one-forth is one-half of one-half" Hunting (1984a, p. 27).

7. Guess and see, e.g., a particular number is chosen, and then accepted or rejected when additional information either confirms or contradicts the original choice.

In general, the various approaches used by students to solve problems involving equivalent fractions, can be divided into two main categories (Dickson *et al.*, 1984; Hart, 1981; Kerslake, 1986; Streefland, 1982). As Streefland (1978, p. 56) wrote: "One has to distinguish 'the concept of subdivision into equivalent parts' and its technical performance". This implies that unless students are fully aware of the implications of the symbols associated with fractions, then they may be unaware of any 'apparent' conflict in their answers. This would appear to be particularly so in the operations performed on fractions.

OPERATIONS ON FRACTIONS

Operations on fractions includes addition, subtraction, multiplication and division, and, like equivalence, can be divided into two main categories (Dickson *et al.*, 1984; Hart, 1981; Kerslake, 1986; Streefland, 1982). Dickson *et al.* (1984) differentiated between the two approaches as 'meaning' and 'computation'. However, the identification or separation of these two different approaches to fractions may be too simplistic as the following example from Hasemann (1981, p. 79) illustrates. Students were given a circle diagram divided into 12 equal sectors and asked to shade 1/4 and then shade 1/6. They were then asked how much of the circle was shaded in altogether, i.e., the students were asked to represent 1/4 + 1/6 = 5/12. One student drew a diagram (see Fig. 1.3).

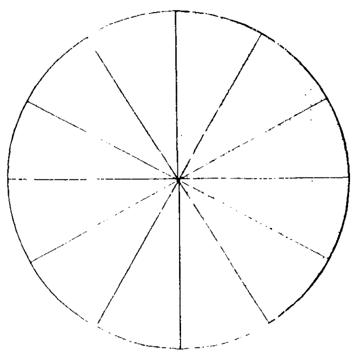


FIGURE 1.3

Circle showing five-twelfths

The student then wrote 1/10 next to the diagram. This would indicate that the student could represent 1/4 and 1/6 but could not relate the addition of the two to 5/12, preferring instead to apply whole number arithmetic to the written or symbolic version of 1/4 + 1/6. Hasemann (1981, p. 79) commented that many of the students were: "not unduly concerned at getting different answers through using different ways of solving problems (as here by calculation and by a diagram)".

Despite the above observation, addition and subtraction of fractions is best related to the area or geometric model discussed in the first section of this paper. Typical problems, such as 1/2 hour + 1/4 hour, can be shown comparatively easily using circles and rectangles. However, other operations on fractions, in particular division, which do not rely on equivalent fractions, are not adequately explained by use of the same model. Instead, multiplication and division are more closely aligned to the operator aspect, such as 1/2 of 1/3 This implies that if the only model to be used is based on the part-whole construct to establish 'connectedness' between meaning and computation of addition and subtraction problems, then it may be inadequate to explain multiplication and division. Hart (1981, p. 81) concluded:

> Concrete embodiments used when multiplication of fractions is introduced are usually restricted to the area of rectangles when the dimensions are no more complicated than wholes and halves because the diagrams become so complicated when further divisions are made. Very soon the rule is introduced but probably not in the context of problems and so the child does not recognise the need for its use.

As a result, rote learned algorithms may become the dominant method used to solve multiplication and division of fractions. The danger with this approach is that techniques performed by rote are unlikely to be consistent, accurate, or retained. Without recourse to earlier understanding of the fundamentals of these algorithms, successful problem solving cannot usually occur. For example, a typical error in response to the question, 10 ÷ ½, is 5. The literature (Hart, 1981; Kerslake, 1986) revealed that many students, when faced with a multiplication or division of fractions that required thought, and not just computational accuracy, could not answer the problem successfully. For example, Hart (1981, p. 73) asked: "1/17 ÷ 2/5 = 5/34 then 1/17 ÷ 4/5 is: (a) twice 5/34 (b) 4/5 of 5/34 (c) 5/4 of 5/34 (d) half of 5/34" and reported that approximately only 20% of her sample of 14 and 15 year old students selected the correct answer. Further research (Bourke, Mills, Stanyon, & Holzer, 1981; Hart, 1981; Hunting, 1984; Post, 1988) confirmed that many students use a variety of often inappropriate methods in seeking solutions to problems involving fractions. For example, some students apply the rule of common denominators, which suits addition and subtraction, to multiplication.

SUMMARY

There are three main findings from this section of the work. First, fractions, decimals, percentages, ratios, operators and division are all interchangeable. However, despite this repertoire of alternatives, students appear to have difficulties operating both within and between all the different constructs of fractions as numbers. In many cases, operating in an alternative context only serves to compound existing errors. In addition, when faced with a different construct, many children refrain from using fractions, preferring instead to treat the numbers as if they were dealing with whole numbers, e.g., place value is confused in decimals, and studies reveal that students would prefer to have a whole number remainder rather than a fractional part when treating fractions as an indicated division.

Second, the research indicated that both students and some teachers found the concept of $a/b = a \div b$ to be the most difficult to assimilate. The reasons for this are unclear, especially since the act of partitioning would appear to readily relate to this aspect of fractions.

Finally, there is substantial evidence (Hart, 1981; Kerslake, 1986; Streefland, 1991; Watson *et al.*, 1991a) to indicate that when learners do not understand the fundamentals of fractions (the lack of 'connectedness' (Streefland, 1982)), a plethora of predicable, standard errors flourish. For example, students will state that one fraction is larger (or smaller) than its equivalent fraction and proceed to solve operations on fractions questions as if dealing with whole numbers.

CONCLUSION

To be able to understand and operate with fractions appears to relate to four main premises. These are:

i. wholes can be partitioned into subparts;
ii. the subparts are equal;
iii. the subparts add to generate the whole; and,
iv. fractions are numbers.

However, there is sufficient evidence to suggest that each of these conditions may not occur spontaneously, simultaneously or naturally. If one of these constraints is absent, fundamental understanding of fractions cannot yet be said to have taken place. Despite this, the development of the first three points is typically presented to students as a *fait accompli* and is clearly well suited to the geometric model of the part-whole construct. This observation may explain the prevalence of this model in mainstream education, although the reasons for this are blurred. For example, although the literature acknowledged its limitations, Silver (1981, p. 156) noted that, in a study involving young adults, 15 out of 20 students, when asked to think about the fraction 3/4, reported a "'pie' or circle subdivided into four congruent parts, with three shaded parts". This image of a fraction was so strong that one subject is reported as saying: "I just keep seeing that pie in four pieces. I can't shake that picture", when asked to "think about it in a different way from the way you first 'saw' it" (Silver, 1981, p. 156).

Clearly, a majority of early developmental work in fractions relies on the substantiation of the first three points mentioned above. However, the realisation that fractions are numbers (point (iv) above) involves a considerable paradigm shift on behalf of the learner, and marks an important step to mathematical maturity with respect to fractional understanding This is an important point, since all fractional numbers can now be considered. Prior to this stage, only very simple fractions could be compared, ordered or operated upon. Although it is unclear if the above points occur chronologically, the research suggested that many students and some adults do not treat fractions as numbers, or, when they do, rely almost exclusively on only one fraction, namely one-half. The literature indicated that many children (and some adults) appeared to avoid fractions altogether if given a choice, preferring instead to depend on whole number concepts. This was particularly evident in the aspects of fractions that treated fractions as numbers, e.g., decimals, percentages and division. For example, Kerslake (1986, p. 9[illegible]) found that only one-quarter of the children in the sample, when asked to express their views about fractions, made any reference to numbers. The children in her study frequently referred to fractions as 'broken up numbers' or 'not quite whole numbers' or 'split from other numbers' or 'two numbers just put on top of each other'. Of greater concern, is that one in eight teachers in the study, said they thought of a fraction as 'not a number at all' when given the option between 'one number' or 'two numbers' or 'not a number at all'.

Given the above, understanding fractions would appear to involve considerably more than the ability to describe all the different aspects of fractions as noted by a majority of the literature. Despite this, there is little evidence in the literature which suggests that learners have been asked to describe a fraction. The most notable exception to this occurred in the Kerslake study (1986) who asked the following question:

How would you explain to someone, who didn't know, what a fraction was?

The results are presented in Table [illegible].6

TABLE 1.6

Children's definition of a fraction in Kerslake (1986, p. 11)

Choice of Model	Number of children
Part of a whole	10
Part of a number	3
One number over another	6
Don't know, or couldn't say	4

Typical responses in the first category included examples, such as cakes or pies, as well as references to "'whole ones' or 'whole things'" Kerslake (1986, p. 11). There were no examples presented for the second classification and the third category consisted of responses, such as that offered by one student (GP) who said: "I'd probably say it was two numbers with a line through the middle" (Kerslake, 1986, p. 11). As Kerslake concluded:

> It can be seen that many of the responses consisted of giving instructions for a procedure - either of splitting up a whole, or writing one number over another. Some children found it difficult to produce any explanation, ... the only model produced was the 'part of a whole' one, although the response 'part of a whole' might suggest that some of those children see a connection between fractions and numbers. Those who said 'one number over another number' show that they recognize the way fractions are written, but do not give any evidence as to whether they understand what they represent.
>
> Kerslake (1986, p. 11)

In many cases, the use of the word 'number' has become interchanged with that of 'whole number'. As Kerslake (1986, p. 92) wrote: "it appears that they are not able to detach themselves from the idea of numbers as counting numbers which describe actual groups of objects and so to move to the more abstract mathematical system of which counting numbers are but a part". Other authors, such as Behr and Post (1988), Post (1988), and Ohlsson (1988) have compared and contrasted the rational numbers with whole number concepts. Clearly, a rational number, such as 2/3, can be represented concretely in many ways. In contrast, the whole numbers are primarily used for counting (Behr & Post, 1988), and are not, in general, ambiguous. As Ohlsson (1988, p. 53) wrote: "How is the meaning of 2 combined with the meaning of 3 to generate a meaning for 2/3?". This issue is compounded when fractions are combined with operations. For example, many students habitually answer 2/3 + 3/5 as 5/8, i.e., (2 + 3 = 5)/(3 + 5 = 8) as in whole number arithmetic.

WHY DO STUDENTS HAVE DIFFICULTIES WITH FRACTIONS?

The literature is divided on the issue of why fractions cause such concern to the general student population. However, in general, there were three main attempts to explain students' difficulties with fractions.

The first reason revolves about the perceived value of fractions in modern society (van Hiele 1986), and the subsequent loss of 'connectedness' between the historical development of fractions and the methodologies used to teach fractions in the classroom (Streefland, 1982). With the increased use of calculators and computers, there is little scope for the traditional use of fractions. Despite this, fractions remain an integral part of most school curricula. This is contrary to the fact that "there are not many realistic contexts where any except very simple fractions are used" (Queensland Education Department Year 7 Sourcebook, 1990, p. 59). Other authors, such as van Hiele (1986, p. 211), argued for the abandonment of fractions altogether in schools, believing that they have no practical use, and that this is the main reason many adult learners forget the techniques learned in childhood.

In addition, aspects of 'connectedness' also raise the issue of the validity of placing questions in-context and context-free situations. Although some authors (Heller *et al.*, 1990, p. 389) denote the word *fraction* as a context-free situation whose elements are integers (e.g., 3/12), there is considerable debate regarding placing fraction questions into both in-school and out-of-school situations. Some authors (Hart, 1981; Mack, 1990) saw this distinction as having severe implications with respect to fraction understanding. For example, Hart (1981, p. 68) noted that "between 25 and 30 per cent of each year divided the smaller number into the larger in some way" when given a problem free of context. More detailed information is provided in Table 1.7. Here the results of dividing 3 by 5 is provided for 12 and 13 year old students. Significantly only 35% of 12 year olds and 31% of 13 year olds were correct (percentages do not add to 100%).

TABLE 1.7

Percentages of responses to 3 ÷ 5 in Hart (1981, p. 68)

Student Ages	3/5 or .6	1 2/5	1 rem 2	5/3 or 1 2/3
12 years	35	5.3	18.3	3.3
13 years	31	9.4	17.5	8.7

Notwithstanding all of the above, fractions **are** an important tool in developing mathematical maturity. A thorough understanding of fractions would seem to be vital for students who wish to pursue further studies in Mathematics or related areas.

A second reason identified by some writers (e.g., Hunting, 1986) is that the difficulties students face with fractions can be traced to a delayed introduction to them. This argument has some validity, since, if the teaching of fractions is delayed for too long, then whole-number arithmetic becomes the dominant method of problem solving. However, Behr and Post (1988, p. 190) argued that children have difficulties with fractions since it is the first time that simple 'counting on' or 'complete the pattern' type problem solving strategies do not always lead to successful problem solution. For example, "counting in one form or another (forward, backward, skip, combination) could be used to solve all of the problems encountered. Now with the introduction of rational numbers the counting algorithm falters (that is, there is no next rational number, fractions are added differently ...)".

Finally, the third reason is in a way the opposite of the second. Many authors (Dickson *et al.*, 1984; Freudenthal, 1973; Hart, 1981; Kerslake, 1986) argued that fractions should not be introduced too early. This means that the topic should not be attempted until the child is cognitively 'ready', i.e, the computational algorithms for manipulating fractions had been introduced before the student had gained a sufficient understanding, at least at a concrete level, of fractions (Dickson *et al.*, 1984, p. 304). For example, "about one-third of children may have no clear concept of a fraction, even in a very concrete sense, at entry to secondary school" (Dickson *et al.*, 1984, p. 280). Kieren (1984) argued that unless the fundamental notion of a fraction, which relies on partitioning or the notion that the whole can be subdivided at all, is firmly entrenched, then it is unlikely that further developments in fraction arithmetic can occur successfully. For example:

> teaching an algorithm such as a/b = c/d is of little value unless the child understands the need for it and is capable of using it. Children who are not at a level suitable to the understanding of a/b = c/d will just forget the formula
>
> Hart (1981, p. 101)

This would imply that it is not until a student is aware of the limitations of the natural numbers, that they may be able to conceptualise the need for arithmetic other than that applied to natural numbers. This may invoke a 'need' in many students to resort to other methods, such as learning by rote, rather than learning for understanding. Dickson's interpretation of Freudenthal (1973) views argued that "fractions should not be introduced until children are able to view the rational numbers abstractly as an example of a formal number system with specific algebraic properties; it seems likely that this would involve a delay until late in the secondary school for most children, and many might not reach it at all" (Dickson *et al.*, 1984, p. 305).

Clearly, the effects of these conclusions must be monitored, and appropriate intervention programmes instigated if these problems are to be addressed and rectified. Of interest, is the question of how many of these learners view or understand fractions in later life - particularly if their problems in this area have not been dealt with in previous schooling.

Finally, although the literature reported above investigated specific details of instruction and methods of dealing with fractions, there did not appear to be a global approach to the understanding of fractions. Given this, it is prudent to discuss the established mathematical hierarchies since they may be able to enhance the understanding of the learning of fractions from a philosophical point of view. Several authors, Hart (1981), Novillis (1976), Ohlsson (1988) and Skemp (1986), as well as Piaget, have proposed mathematical hierarchies with respect to fractions. However, there would appear to be a need to broaden the literature review to encompass more general approaches to mathematical hierarchies which are readily applicable to the learning of fractions, although they may not have been previously applied to adult learning (Hart, 1981; Kieren, 1991, 1992; Skemp, 1986), or fractions specifically (Biggs & Collis, 1982; van Hiele 1986). One model of particular interest is that postulated by Watson *et al.* (1991ab, 1992ab, 1993). These issues are developed in the next chapter.

CHAPTER TWO

THEORETICAL FOUNDATIONS

> ***... the research found that children frequently tackle mathematics problems with little or nothing to do with what has been taught. This may be because mathematics teaching is often seen as an initiation into rules and procedures which, though very powerful (and therefore attractive to teachers), are often seen by children as meaningless. It follows that children's methods and their levels of understanding need to be taken into account, however difficult this may be in practice.***
>
> **Küchemann (1981, p. 118)**

INTRODUCTION AND ORGANISATION OF THE CHAPTER

One of the major conclusions reached in the previous chapter, and one that is echoed in the above quote, is that many children perform fraction tasks with little if any understanding. While some observations noted in Chapter One may be explained by students learning rules by 'rote', rather than understanding, it is clear that a number of children's errors cannot be explained this easily. Clearly, a sizeable number of students' errors are consistent, predictable and appear to be confined to a comparatively limited selection of answers or 'groupings'. However, despite these observations, the research was unable to provide reasons for, or rectification of, childrens' errors. In addition, there did not appear to be a global approach to understanding fractions. Instead, only a variety of, often conflicting, reasons were suggested for the difficulties children have with fractions. These suggestions ranged across the spectrum of possibilities - from an earlier introduction, to delaying their introduction.

There has been little research which has focused on adults' understandings of fractions. It is feasible that as children mature into adulthood, their conceptions of fractions may evolve or alter with their intellectual maturity and experience. Of course, it is also possible that certain 'rote learned' responses may corrupt or impinge upon further mathematical development if such misinterpretations are not identified, acknowledged and rectified by providers of adult education. Given these conclusions, it is relevant to discuss established mathematical hierarchies since they may be able to enhance the understanding of the learning of fractions.

To aid in the analysis, the aims of this chapter are to:

(i) investigate the attributes of mathematical hierarchies with respect to fractions;

(ii) determine the feasibility of interpreting adult responses in terms of an existing theoretical framework.

Several authors, such as Hart (1981, 1985), Kieren (1988), Novillis (1976) and Skemp (1976, 1986), as well as Piaget, have proposed mathematical hierarchies with respect to fractions. A discussion of each of these hierarchies forms the basis of the first part of this chapter. However, any theoretical framework that is to be applied to adult learners, must offer more to adult learners than an interpretation of fractions, i.e., any theoretical framework that has the potential to be applied to adult learners must not be specifically dependent upon age. One way to address this issue is to move the focus of attention away from the person and focus on the response. For this reason, particular attention is drawn to the SOLO (Structure of the Observed Learning Outcome) Taxonomy by Biggs and Collis (1982, 1991). This framework is presented in detail in the second section of the chapter.

FRACTION HIERARCHIES

The first part of this chapter investigates several authors' attempts to classify fractions into a mathematical hierarchy. The authors selected postulate mathematical hierarchies from a variety of viewpoints. For example, some authors have based their conclusions on empirical information while others have taken a more psychological approach to the establishment of a fraction hierarchy. Where possible, typical examples at each level are presented, however, it was sometimes difficult to determine if the authors had classified students' responses, the items themselves, individual students, or if they have presented theoretical possibilities.

THE HART (1981) HIERARCHY

As previously stated, Hart's work in *Children's Understanding of Mathematics: 11-16* (1981), was based on the results of five years of research into children's understanding of mathematics. This involved testing and interviewing over 10 000 children across ten different mathematical topics, including fractions. The results indicated that several 'levels of understanding' could be identified throughout the various topics. The selection criteria which differentiated between levels was based on: "Attainment at any level was defined as successfully solving about two-thirds of the items in that level and about two-thirds or more of the items in all easier levels.

The two-thirds criterion was arbitrary and in some cases the nearest fraction was taken" (Hart, 1985, p. 2). In general, the levels were allocated values 1 to 4. However, those students who did not obtain the pass criteria were assigned Level 0. It was assumed that the students in the sample were familiar with numbers up to 20, could recognise numbers up to 1000 and could find the length or area of a simple object. As part of the Chelsea Diagnostic Mathematics Tests, two fraction tests - Incorporating Fractions 1 (Computation) and Incorporating Fractions 2 (Computation) (Hart, 1985) - were designed which enabled teachers to ascertain children's understanding of fractions as designated by the Hart hierarchy.

When combined across topics, the levels formed four stages and produced a more general Mathematical hierarchy. Major attributes of the four stages with particular reference to fractions are presented below.

Stage 1 fraction questions involved shading part-whole area constructs, simple equivalent fractions that could be obtained easily by doubling and the addition of two fractions with the same denominator (Hart, 1985), e.g., "In a bakers shop 3/8 of the flour is used for bread and 2/8 of the flour is used for cakes. What fraction of the flour has been used?" (Hart, 1985, p. 189).

Stage 2 was concerned primarily with the application of equivalent fractions, e.g., simple equivalent fraction questions such as "2/3 = ?/15 or ... A relay race is run in stages of 1/8 km each. Each runner runs one stage. How many runners would be required to run a total distance of 3/4 km?" (Hart, 1981, p. 195).

Stage 3 is where abstraction and the introduction of problem-solving strategies first appeared, e.g., "questions are not always tied to a diagram or the child is asked to hypothesise about situations which are not shown. ... in Fractions the child has to cope with 2/7 = ?/14 = 10/?? and see that 10/?? is connected to the 2/7 as well as 4/14" (Hart, 1981, p. 199). Students in stage 3 should also be able to calculate compound problems involving tiling floors or calculating how much carpet is required to cover a room. Hart, however, concluded: "The hardest Stage 3 items are successfully solved by 20-30 percent of the sample. If mathematics teaching is designed to enable a child to face a new problem and invent a method of solution then a majority of secondary school children have evidently not reached this stage" (Hart, 1981, p. 200).

Stage 4 responses noted the continued use of abstraction but also included evidence that respondents were accessing other knowledge from outside the immediate problem to obtain the correct solution, e.g, "In Decimals and Fractions the questions require the child to appreciate the nature of these new numbers and not be firmly fixed with the set of whole numbers" (Hart, 1981, p. 203). To be operating at stage 4, a student must have an overview of fractions. They must be able to see that there is an answer to questions such as 15 ÷ 20, and not be restricted to the idea of having only a whole number answer.

In general, "Stage 1 and 2 items in Fractions are very often of the type 'what name do we give to this', Stage 3 items require rather more, in that the connection between elements is needed. Stage 4 items need a departure from concrete referents, in particular, they need a recognition that multiplication or division is needed and an ability to carry out the operation" (Hart, 1981, p. 204). In addition, Hart (1981, p. 190) also related the use of the language of Mathematics to the stages. This provides an important clue to the level at which a student may be responding, since it is difficult to use and interpret correctly language at a higher level if the child is not operating at that level. For example: "Many of the items are testing the knowledge of a new mathematical vocabulary e.g., tenths ... Children capable of solving Stage 1 items could be regarded as knowing 'meanings' in the mathematical language, but nearly thirty per cent of our child population (even at age 15) can go no further i.e., cannot apply this language" (Hart, 1981, p. 190).

A brief description of each level, and the appropriate age group to which it applies, as defined in Hart (1985), is presented in Table 2.1.

TABLE 2.1

Summary of levels of fractions
(adapted from Hart, 1985, p. 26 & p. 38)

LEVEL	FRACTIONS 1 (AGE 11+ TO 13+)	FRACTIONS 2 (AGE 13+ TO 15+)
1	The meaning of a fraction using pieces 1/2, 1/5, 2/3	The meaning of fraction, seen as part of a whole, no equivalence needed. Equivalent fractions obtained by doubling. Addition of fractions with same denominator.
2	The meaning of a fraction as a subset of a set, or naming given configuration of pieces. Equivalent fractions obtained by doubling. Addition of two fractions with the same denominator.	Equivalent fractions not obtained by doubling. Using equivalence to name parts, with familiar fractions or when diagram provided. Ordering unit fractions.
3	Using equivalence to name parts, with familiar fractions. or when diagram provided. Equivalent fractions not obtained by doubling or less familiar fractions, e.g., 2/7 = ?/14. Ordering unit fractions.	Questions where more than one operation is required, e.g., equivalence followed by addition or subtraction.
4	Questions where more than one operation is needed, e.g., an equivalence followed by addition or subtraction.	Division and multiplication of fractions. Generalisation.

Although Hart (1985, p. 2) stated that the above levels were 'levels of understanding', O'Reilly, (1991, p. 86) queried whether the 'hierarchy of understanding' was a "hierarchy of questions or a cognitive hierarchy of children". This issue is further confused, since, in some cases, it is clear that the classification relates to the item and not the response. Irrespective of this, the findings of the CSMS have achieved widespread acceptance, and formed the basis of the follow-up study by Kerslake (1986) as part of the SESM (Strategies and Errors in Secondary School Mathematics) project. Despite the foundations formed by Hart (1981), and, although Kerslake (1986) reported on several aspects of fractions, which clearly indicated a mathematical hierarchy, little interpretation with respect to a mathematical hierarchy was indicated in the sequel.

THE SKEMP (1971, 1986) HIERARCHY

In his book *The Psychology of Learning Mathematics* (1971, 1986 (2nd ed.)), Skemp attempted to explain how students learn mathematics from a psychological point of view. Skemp developed a schema approach which distinguishes between 'everyday' or 'intuitive' learning and 'reflective' learning, i.e., "being able to do something is one thing; knowing how one does it is quite another" (Skemp, 1986, p. 53).

The main tenet of Skemp's work is that almost all learning depends on the existence and maintenance of conceptual structures or schemas (Skemp, 1986, p. 37). Initial schemas develop from infancy when children are conditioned into defining things from a very early age. For example, provided a child already has an existing schema for a 'chair', then it is possible to identify a 'peanut chair' as a chair even if it does not immediately look like one. Skemp argued that this transference of knowledge is possible since human beings possess an innate ability to 'abstract' ideas and to 'classify' things. This ability to abstract and classify are the foundations of concept formation, as Skemp (1986, p. 21) wrote:

> **Abstracting** is an activity by which we become aware of similarities (in the everyday, not the mathematical, sense) among our experiences. **Classifying** means collecting together our experiences on the basis of these similarities. An **abstraction** is some kind of lasting mental change, the result of abstracting, which enables us to recognize new experiences as having the similarities of an already formed class. ... to distinguish between abstracting as an activity and an abstraction as its end-product, we shall hereafter call the latter a **concept**.

Since concept formation requires "a number of experiences which have something in common" (Skemp, 1986, p. 21), intellectual growth is dependent upon "knowing something else already" (Skemp, 1986, p. 38). This implies that subsequent schemas either complement or supplement existing schemas, i.e, new material is assimilated into an existing schema, or a new schema containing new information comes into existence. However, both of these conditions rely on the consistency and proficiency of existing schema remaining reliable and valid. If the existing schema fails under the new schema's premises, then the existing schema must be altered or abandoned altogether in order to acquire or assimilate new concepts. Contradictions or minor inadequacies in a pre-existing schema cannot be avoided or overlooked if learners are to grow cognitively. For example, Skemp (1986) argued that fractions should be seen as an extension of the natural number system, since this schema should already be established. By building on the similarities that already exist between fractions and natural numbers means that the same numerals are used for both systems and "the same methods for adding as those which we have already learnt for natural numbers" (Skemp, 1986, p. 174). Figure 2.1 represents the Skemp hierarchy for fractions.

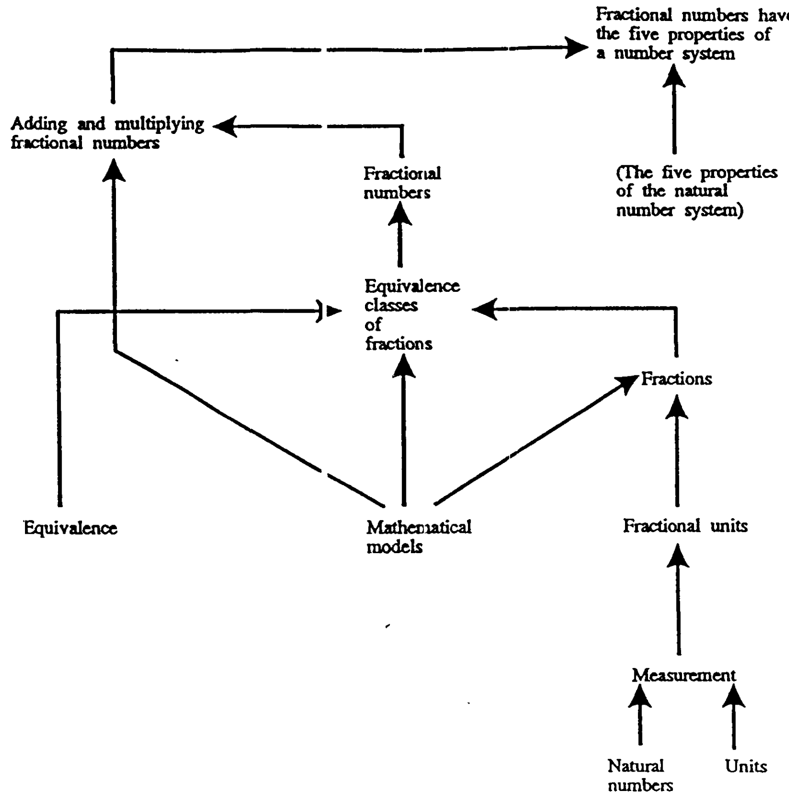


FIGURE 2.1

Skemp's hierarchy for fractions
(Skemp, 1986, p. 283)

It is worth noting that Skemp's hierarchy is based almost entirely on the geometric part-whole construct (pp. 174-182) Hence, equivalence (p. 177) and addition of fractions (p. 179) are presented to the reader via this model. The previous chapter has already acknowledged the deficiencies with this approach. Skemp also utilises a 'new' symbol (⊕) for addition of fractions as a reminder "that adding does not mean quite the same for fractional numbers as for natural numbers" (Skemp, 1986, p. 179). The development of an addition of fractions schema, based on a previous addition of whole numbers schema, could therefore develop as follows:

i. the learner becomes aware of the need for an additional schema (addition of fractions)
ii. the learner must then alter or abandon a now faulty schema (e.g., you don't add fractions in the same way that whole numbers are added)
iii. the learner then assimilates the new schema in conjunction with the pre-existing one (e.g., fractions are added in a certain way, and whole numbers in a different way)

Although Skemp (1986) acknowledged "just how we are able to make deliberate changes in our schemas ... is still unknown" (p. 54), the author claimed that "we can certainly do so" (p. 54). Clearly, this does not happen easily for some students. This suggests that the mechanics of the schema-adapting and schema-adopting approach advocated by Skemp is a difficult and complex process. For example, a learner must first acknowledge that an existing schema must be altered or abandoned, and subsequently replaced with a more appropriate schema. Evidence from the previous chapter would suggest that this may be an unrealistic expectation for some learners, e.g., some learners did not realise that an additional (fraction) schema was required since some students did not relate fractions to numbers at all. In addition, an experiment conducted by Hunting (1986) indicated that an existing, although incorrect, schema was proving to be extremely robust.

> Rachel showed time and again that she preferred her intuitive scheme to the schemes which the investigators sought to establish. The investigators were in fact engaged in a process of precipitating scheme displacement of one scheme by another. In this experiment the beginning of that replacement process was initiated, though not completed.
>
> Hunting (1986, p. 63)

Finally, additional schemas did not always appear to be consistent or permanent. Clearly, learners can be in a transition stage in which two conflicting schemas appear to occur simultaneously. For example, some children are aware that two equivalent fractions are that 'same', such as 2/3 and 10/15, but have a "feeling that the multiplication by 5 had made 10/15 bigger than 2/3" (Kerslake, 1986, p. 93).

THE NOVILLIS (1976) MODELS HIERARCHY

Novillis (1976), basing her work on that of Gagné (1965), developed a hierarchy of the various models associated with fractions. This was based on the notion that complex concepts could be related to a number of dependent subconcepts. A simplified version of the fractions hierarchy is presented in Table 2.2.

TABLE 2.2

Representation of Novillis (1976) hierarchy of fractions

LEVEL	DESCRIPTION
1	the student associates a/b in terms of one of the part-whole constructs, e.g., part-whole geometric diagrams or subsets of sets, provided that the objects are congruent.
2	the student associates a/b in terms of the part-whole constructs, however, the objects do not need to be congruent. The number line is also located at this level.
3	the student compares two or more fractions or correctly associates two or more models representing the fractions, i.e., equivalence is partially observed, however, re-arrangements of the parts or objects may confuse the student.
4	the student compares two or more fractions, however, the visual re-arrangement of the parts or objects does not confuse the student, i.e., equivalence is possible with the part-whole and subset of a set aspects.
5	the student associates equivalence with a number line.
6	the student associates a/b with many models, e.g., part-whole, subsets of a set, number line or comparison. The models may be both visual and non-visual.

Novillis (1976) also identified a series of subordinate subconcepts. For example, although level 1 is associated with the part-whole model, level 1a specifies the subset of a set model, while level 1b is associated with the geometric model. This hierarchy then formed the basis of the Fraction Concept Test which was administered to a total of 279 students across grades four to six. The criterion level was established at 75% for each subconcept.

The findings from the study partially confirmed the above notional hierarchy. For example, Novillis (1976, p. 143) concluded that certain subconcepts were prerequisites to other concepts, e.g., the part-whole models precede both the ability to associate fractions on a number line (level 1-2) and the ability to compare fractions involving the same model (level 2-3). However, the results from the investigation were inconclusive. Further work by Payne (1976) indicated that the sets model was considerably more difficult than the part-whole construct. This implies that understanding the connections between the different aspects of fractions is as important as understanding the different aspects independently and subsequently. In addition, this new insight implies that understanding fractions may be more complex than Table 2.2 suggests. The present research did not consider ratios or division as other aspects of fractions. Currently, there is no evidence, and little research, to suggest that the removal of any aspect of fractions may not effect the further understanding of fractions, i.e., "Are all the sub-constructs necessary? Do some subsume others?" (Kieren, 1980, p. 146). Finally, the current model does not take into account operations on fractions, which are an essential part of students' fractional maturity.

THE KIEREN (1988) MODEL OF KNOWLEDGE BUILDING

Kieren (1988) postulated a model of intuitive mathematical knowledge which attempted to "account for personal and mathematical features of such knowledge building" (Kieren, 1988, p. 162). Rather than a series of discrete linear levels, Kieren postulated a series of layers (usually represented as circles or semi-circles) much like the growth rings associated with tree development. A simplified version of the model is presented in Figure 2.2.

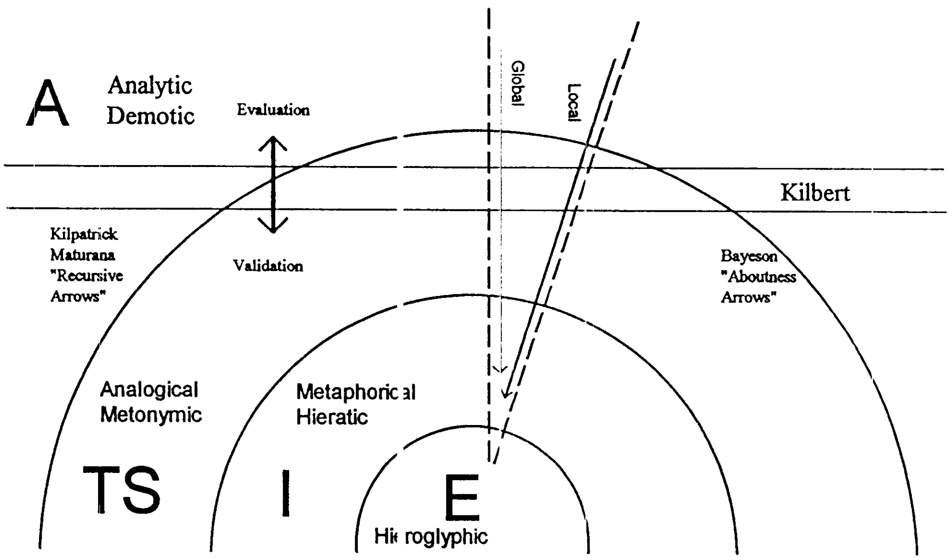


FIGURE 2.2

Kieren's (1988) model of knowledge building

A brief description of the features of this model, including the meanings of the layers is provided below.

The core of this model consists of mathematics typically associated with **ethnomathematical knowledge** (designated E in the above diagram), i.e., "knowledge, quantitative, spatial, and/or pattern oriented in nature, which one builds up because one lives in a particular environment ... this is not 'schooled' knowledge" (Kieren, 1988, p. 169). 'Fractions' at this level barely exist and consist only of a half, quarter or third. A typical response at this level to a 'pizza' question would be: "Cut the pizza into pieces and leave the rest".

The **Intuitive Level (I)** is associated with "schooled or taught knowledge which involves the deliberate conjoining of image, thought tool, and informal use of language" (Kieren, 1988, p. 170). Kieren claimed that: "patterns of actions which generalize real-life sharing and allow a person to develop mental objects and actions (hundredths, thousandths, or repeated division by 2 or 10) which are dependent of, although analogous to, real-life actions" (p. 170) occurred at this level. Fractions at this level are based in 'everyday' situations, such as partitioning a pizza, although the distribution may not always be a mathematically fair and equitable distribution, e.g., "Person A gets a third and a quarter of a quarter" (Kieren, 1988, p. 172). Typical fractions are more abstract than the previous level and include fifths and sixths.

Technical Symbolic (TS) knowledge is associated with "standard language, notations, and algorithms ... it should be testable against some form of 'reality' or represent a local logical sequence which can be evaluated in terms of axioms for rational numbers (Kieren, 1988, p. 170). Typical fractions at this level involved the use of standard language and symbolism.

Axiomatic (A) knowledge is the formalisation of rational number knowledge including the "relationships about rational numbers, seen and described ... at inner levels, in the axiomatic structure" (Kieren, 1988, p. 170).

In addition, Kieren (1988) acknowledged that the boundaries between the layers of the above model become blurred and he adapted the model to include 'transition' levels which were designated EI (e.g., "each gets a third and a bite more" (p. 172)) or ITS (e.g., symbols were used but always treated the situation as if dealing with real pizzas (p. 193)).

These ideas were developed further in later work by Kieren and Pirie (1989). They produced a more general model of knowledge which incorporated the notion of understanding "'in many ways at once' and at the dynamics of how that comes to be and how it grows over various periods of time" (Kieren, 1991, p. 170). The modified model consisted of eight layers of understanding. Figure 2.3 represents the new model.

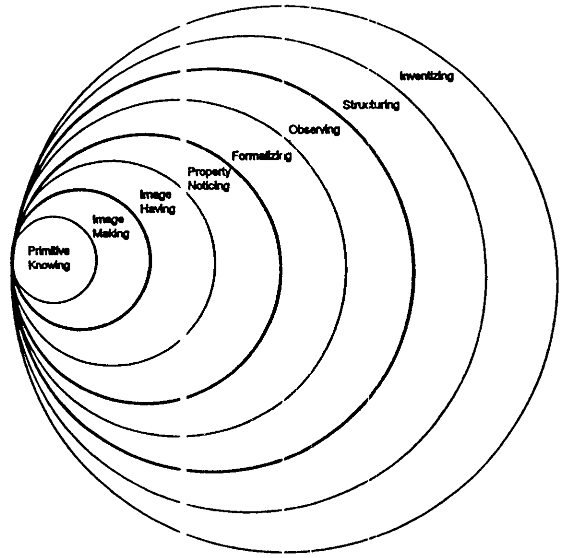


FIGURE 2.3

Kieren and Pirie's model of knowledge

A brief description of five of the eight layers of the new model, as it relates to fractions, is now presented. Typical examples are based on the analysis of one 12-year-old girl (Tanya) presented in Kieren (1992).

Primitive Knowing marks the beginning of fraction understanding, i.e., fraction 'language' is first used.

Image Making consists of single, independent 'image making' activities and discovery of the relationships between the individual pieces of information, such as physically combining fraction shapes and describing the results (Kieren, 1992). Addition of fractions at this level is therefore possible, if the result can be arranged to cover a known fraction shape. Correct results for addition problems can only be obtained if all fractions involved can be represented by concrete objects, such as fraction shapes (Kieren, 1992).

This level, like all the remaining levels require both *action* and *expression*, i.e., record keeping, reflection and description of the activities give rise to entrance into the next higher level.

Image Having concerns the generalisation of the above level, i.e., any two or more 'fractions' are seen to be equal if they cover the same amount of space. However, there is a fundamental shift from the previous level in that learners are now able to imagine the fraction shapes, and they do not necessarily require the physical shapes to be present to perform the calculation. It is enough that they can imagine them. Learners actively 'seek out' the required 'fractions' they need in order to solve a problem. This may require 'folding back' (Kieren, 1992) to the context-bound level for additional information not obtained previously.

Property noticing is the 'discovery' of patterns of numbers when dealing with fraction pieces. It marks the entrance to the next level in which fractions can be dealt with independently of fraction shapes.

Formalizing consists of the deliberate and appropriate selection of a common denominator, independent of concrete shapes - either real or imaged as in the previous levels. For example, sevenths are able to be dealt with at this level. Fractions should now be able to be perceived in their equivalent symbolic forms.

Little information is available with respect to the remaining 3 layers. These are seen to pave the way for the outermost level in which questions are asked that 'break away' from the traditional view of fractions. This tends to create 'new' areas of mathematics. It is not envisaged that a majority of students would need to progress to this degree of fraction understanding.

A major innovation of the new model was that student understanding was not seen as a direct, linear progression from one level to the next. Instead, learners were seen to 'weave' through the various levels as they constructed their own knowledge base of fractions. 'Folding back' or revising and re-visiting existing knowledge in lower levels, was viewed as an essential part of growth, i.e, knowledge discovered in one level could require similar knowledge-type skills that were encountered previously at an earlier level. This process changes both the dynamics and didactics of both the learner and the levels re-visited; and perpetuates both 'growth' and movement in both learner and levels. Students may return time and again to earlier levels as they conceptualise their own personal understanding of fractions. As a consequence of this 'folding back', teachers were encouraged to take an active part in their students' paths of understandings by making appropriate interventions as required, i.e., "the answer determines the question" (Kieren & Pirie, 1992, pp. 2-8).

SUMMARY

In this section, five models have been presented. Clearly, there is a degree of commonality between the models. For example, all acknowledge that the foundations of fraction understanding is related to partitioning, such as sharing pizzas, or its representation, such as the geometric part-whole model. In addition, the individual models promulgate the overall existence and advantages of acknowledging learning as a hierarchy. A summary table indicating the main attributes, and advantages and disadvantages is presented in Table 2.3

TABLE 2.3

Summary of models of hierarchies of fractions

Hierarchy Author	Source/criteria (i.e., how the hierarchy was established	Number of levels	Measures	Advantages	Disadvantages
Hart	Testing and interviewing Chelsea Diagnostic Mathematics Tests (2/3 successful for each level)	4	Students or student responses or items?	Use of language is related to level	Difficult to determine if hierarchy is measuring understanding or students Age dependent?
Skemp	Hypothetical	Schema approach	Learning of concepts	Knowing how and knowing why	Schema's may need to be abandoned and supplanted
Novillis	Gagne (1965) Fraction Concept Test (75% for each subconcept level)	6	Models of fractions	Partially identified subordinate subconcepts prerequisites for total concept	Contradictory and conflicting findings and limited to models of fractions
Kieren Kieren and Pirie	Hypothetical?	4 8 (only 5 identified)	Knowledge	'Transition levels' folding back	'Layers' were not discrete linear levels examples based on only one 12-year old student

While some of the models presented have strong empirical bases, others have approached fractions from a more philosophical point of view. Ideally, any theoretical interpretation that has the potential to be applied to adult learners and fractions must have the following attributes:

i. the hierarchy must be independent of age;
ii. the hierarchy must attempt to explain all major aspects of fractions that adult learners are likely to have encountered, including operations on fractions; and
iii. from the researcher's philosophical viewpoint, the hierarchy must attempt to analyse the students' responses.

Given these constraints, it is necessary to broaden the search to encompass other more general theories of mathematical hierarchies. In addition, it is hoped that some of the individual and independent advantages offered by some of the above hierarchies may be encompassed into a broader mathematical hierarchy. The Structure of the Observed Learning Outcome (SOLO) Taxonomy of Biggs and Collis (1986, 1991) is one such hierarchy which appears to have the potential to meet the above criteria. Although the SOLO Taxonomy has achieved widespread acceptance and credibility in the previous decade, it has only recently been applied to adult learners in the area of algebra (Coady, 1994), and to fractions (Watson *et al.*, 1991ab, 1992ab, 1993). It is therefore appropriate to discuss the major attributes of the SOLO Taxonomy in detail. Where relevant, typical examples related to interpreting fractions using this structure have been included.

THE SOLO TAXONOMY

The SOLO Taxonomy of Biggs and Collis (1982) arose out of dissatisfaction with the accepted Piagetian paradigm when conflict arose between theory and practice. Early research undertaken by Biggs and Collis observed that the fundamental assumption of Piaget's décalage was found to occur, not rarely as Piaget postulated, but frequently in the classroom, i.e., in a classroom situation students did not respond in characteristic ways within or across subjects. "There are 'natural' stages in the growth of learning any complex material or skill and that in certain important respects these stages are similar to, but not identical with, the development stages in thinking described by Piaget and his co-workers" (Biggs & Collis, 1982, p. 15). A further major difference between the work of Piaget and the SOLO Taxonomy was that Biggs and Collis' work focused on analysing and classifying students' responses to stimuli. The SOLO Taxonomy recognises that a student, when re-tested at a later date, may respond at a higher or lower level, i.e., "SOLO levels are equivalent to test results; they describe a particular performance at a particular time. They are not meant as labels or to tag students ... It carries with it a warning not to overgeneralise" (Biggs & Collis, 1982, p. 23). As a consequence, this attribute makes the SOLO Taxonomy a viable framework with which to interpret adult learners' responses to fraction questions.

One of the main strengths of the SOLO Taxonomy is that it accounts for both the type of thinking the student demonstrates (the modes) and the extent of understanding (the levels). A brief discussion of each of these two aspects is now presented.

MODES OF LEARNING

Collis and Biggs (1986) used the term mode to distinguish between fundamentally different types of cognitive development. They identified five modes. The names of the modes (which are Piagetian in nature) are Sensory Motor, Ikonic, Concrete Symbolic, Formal and Post-formal. The modes exhibit one major difference to the work of Piaget. Piaget argued that each stage of learning was subsumed in the prior stage. In the SOLO model, learning in each mode is not **replaced** by the next mode, instead modes co-exist. The SOLO model acknowledges that different types of thinking could exist concurrently and simultaneously in two or more modes. This issue is taken up in more detail in a later section on multi-modal learning. A summary of the characteristics of the different modes is provided below.

Sensory motor This mode is characterised by the ability to co-ordinate motor skills. This is learning of the most fundamental type, and would appear to occur 'spontaneously' from early childhood, although adults continue to develop in this mode throughout their lives. This is the only mode in which very young children, up until about eighteen months, are able to utilise. Knowledge in this mode is best described as 'tacit'.

Ikonic This mode is characterised by the ability to internalise a picture or 'ikon', and marks the first abstraction from actual 'doing'. Imaging and imagining occur in this mode. Knowledge in this mode is best described as intuitive.

Concrete symbolic This mode is based on representing and symbolising concrete experiences, e.g., numerals represent numbers and pronumerals represent algebraic expressions. Typical school mathematics occurs in this mode. Knowledge in this mode is best described as declarative.

Formal This mode is based on the generalisation of principles and the meaning of theories.

Post-formal This mode marks the challenging of existing theories and the generation of new ones. Knowledge in this mode is best described as theoretical.

Clearly, mature-age learners have a wider variety of, and greater exposure to, life experiences, and hence, modes. Adult students should therefore be able to access and utilise additional modes which may have been unavailable to them as children (Biggs & Collis, 1989, p. 10). However, there is little evidence to suggest that adult learners 'take advantage' of this situation, and utilise other modes available to them. Nor is there any evidence to suggest that learners always operate at the highest possible level available to them, e.g., "the latest developing stage represents a current ceiling to abstraction, <u>not</u> the level to which all performances must conform" (Biggs & Collis, 1989, p. 8).

If the ability to think differently characterises the differences between the modes, then the increasing ability to deal with shifts in structural complexity indicates progress or growth within a mode.

> It is not just a matter of getting students to give the 'correct' response to an item: many responses can be 'correct' in some sense, but some responses are clearly of better <u>quality</u> than others ... the better response has a more complex <u>structure</u>.
>
> Collis and Biggs (1979, pp. 13-14)

This notion leads to the question of how is development within modes best characterised. The SOLO Taxonomy uses the term 'levels' to address this issue.

LEVELS OF LEARNING

Although the existence of 'levels' has found general agreement within the literature, there is considerable debate with respect to both the number of sub-stages and the definitions associated with each level (Biggs & Collis, 1982, 1986, 1989, 1991; Case, 1985; Fischer & Silvern, 1985). Early work on the SOLO Taxonomy, for example, suggested that there were five broad levels. A brief description of each is now presented.

Prestructural In the case of the target mode being concrete-symbolic then a typical response simply repeats or re-phrases the question. Responses classified as prestructural indicate that the response is below the current or target mode under investigation.

Unistructural Responses in which the subject has focused on one aspect or piece of information typify this level. Responses at this level have not utilised any further information.

Multistructural Although responses at this level now take into account more than one aspect of a problem, there is no overview evident in the response. Typical responses usually consist of 'lists' of several different aspects of a problem, but do not yet indicate any relationships or links between the different properties listed. Individual elements of a problem are seen to operate independently.

Relational The main component of this level is that of 'control'. All aspects of the problem that are within the real-world referent are under the control and direction of the problem solver. A response at this level indicates that the student has an overview of the situation. This concept is lacking in the earlier levels. Elements that were stated in the multistructural level are now seen as crucially interrelated and interdependent at this level, i.e., the solution to the problem is presented as an integrated package which acknowledges all the relationships between the relevant properties presented in the problem. Solutions are the result of a determined and deliberate act.

Extended abstract A response at this level indicates that the student questions the original assumption of the situation, i.e., the student goes 'outside' the original problem. In the case of the target mode being concrete-symbolic then an extended abstract response sees a general principle.

The criteria for determining how to classify a particular response is based on several factors. Of these, the use of language plays a most important role in determining the classification of a response. A response which either 'closes' too soon or provides a quick response to a question, would be ranked lower than a response which provided greater depth and complexity in addressing the question. As Biggs and Collis (1980, p. 20) stated: "Lower level responses are either dogmatically 'closed', or indecisive. At the highest end, extended abstract responses are often qualified, leaving room for different interpretations and mitigating circumstances". Responses which have an overview of integrated information and/or suggest possible alternatives or refinements, not originally included in a question are classified at high SOLO levels. It has been suggested (Biggs & Collis, 1982, p. 67) that such information can be used successfully to trace a child's thinking. However, caution should be exercised in applying this interpretation since the amount of available working memory (that part of the consciousness that is directly working or thinking at a given time) a student has available, can have a significant impact on a student's responses. In addition, experience and familiarity **may** enable "more space [to become] available for higher level responses. Experience is then necessary but not sufficient condition of high SOLO's" (Collis & Biggs, 1979, p. 37).

Of significance is that these levels are seen to re-occur in each mode. This implies the same structure to development (i.e., that unistructural, multistructural, and relational growth) occurs within each mode. Hence, in the case of the sensory motor mode, a unistructural response may contain a single physical aspect, a multistructural response a series of independent aspects, while a relational response indicates some coordination of the different relevant aspects. Figure 2.4 takes up this point and illustrates how the unistructural-multistructural-relational (UMR) cycle can be seen to re-occur in each mode.

Since a question frequently has a particular mode as the target mode, two types of level descriptors (addressed above) may be redundant. For example, a pre-structural response in the concrete symbolic mode may in reality represent a different type of thinking to concrete symbolic. This implies the answer is better suited to a classification in the ikonic mode. Similarly, an extended abstract answer in the concrete symbolic mode, may be more representative of growth in the next mode, i.e., unistructural in the formal mode. For these reasons, the labels 'prestructural' and 'extended abstract' are frequently omitted from analyses involving the SOLO Taxonomy, since a greater understanding of knowledge seeking is better obtained by classifying responses in their most appropriate mode as well as level.

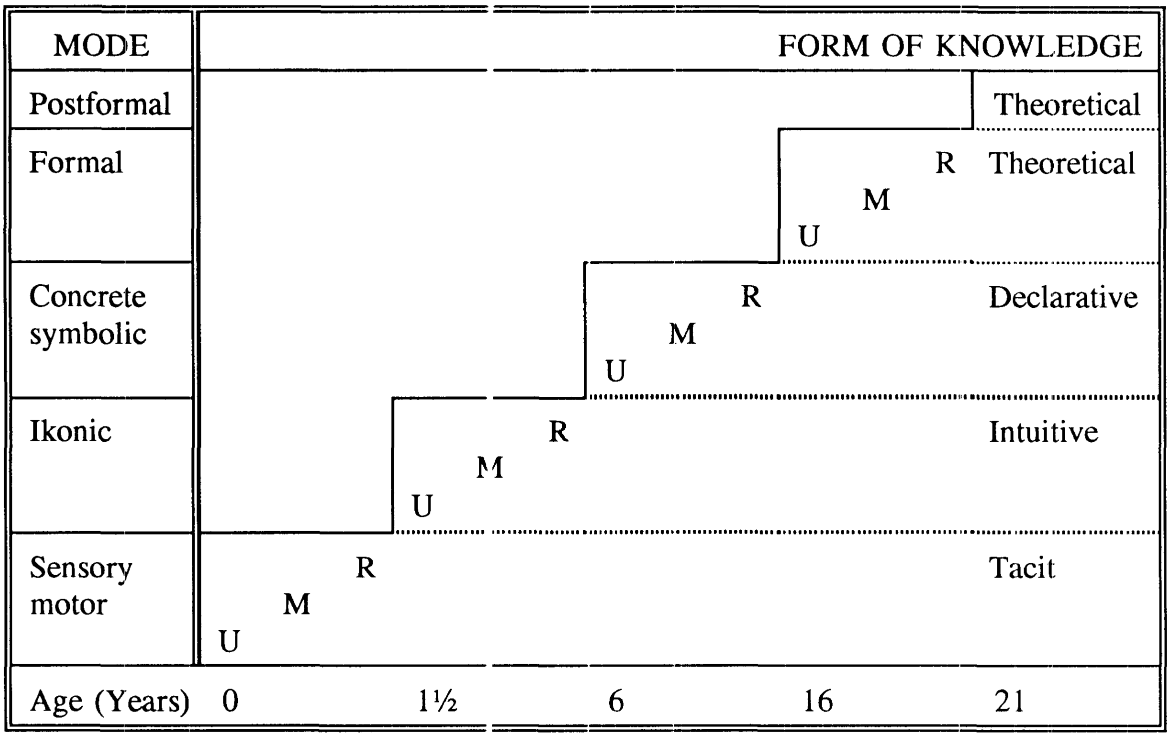


FIGURE 2.4

The SOLO Taxonomy
(adapted from Biggs & Collis, 1991)

The above table demonstrates an important feature of the SOLO Taxonomy, i.e., typical growth develops via progressing through a unistructural, multistructural and relational (UMR) cycle before moving on to a similar cycle within a different mode. However, human nature is unlikely to progress quite so uniformly and linearly, i.e., as the major diagonal across the table suggests. This is where the SOLO Taxonomy has a unique advantage with respect to mathematical hierarchies. It has several additional 'in-built' features which have the potential to account for a greater variety of interpretations of responses without jeopardising the basic integrity of the model. The additional attributes of the SOLO Taxonomy have particular appeal when adult learners are under investigation. For example, Biggs and Collis (1982) acknowledged the possibility of transition levels. Transition stages occur between levels where students appear to be operating somewhere between two distinct levels (Biggs & Collis, 1982). Transition levels are unstable, temporary and transient. However, the greatest assets to interpretation of adult learners' responses would appear to rely in two comparatively new additions to the Taxonomy. They are cycles of levels within modes and multi-modal functioning. These issues are now explored in detail.

MULTIPLE UMR CYCLES WITHIN MODES

An extension to the single cycle of levels in a mode model was suggested by several writers (Campbell, Watson, & Collis, 1992; Pegg, 1992; Watson *et al.*, 1992a). In particular, they have identified two cycles of levels within the concrete symbolic mode. The cycles, which consist of unistructural, multistructural and relational levels, interact in such a way that the first cycle becomes a fundamental element at the unistructural level in the next cycle, while still remaining an integral part of the concrete symbolic mode. A simple analogy to help better explain the cycles is illustrated by considering chaos theory and fractals. The ability to describe a 'fractal' will depend on the size of the 'microscope' in use. While still retaining essentially the same features, the greater the magnification, the more detail is present. Conversely, the further you stand back, the less detail is seen, however, a better overview is gained. As Pegg (1992, p. 384) wrote:

> the identification of cycles of learning allows a clearer perspective of the useful skills needed for students to grow. While these skills might be known in a general way by teachers, there would be few teachers who could state them emphatically. ... Knowing when student responses, are either in early or late cycles, is an important skill for teachers: it not only helps the formative assessments that teachers need to make but it allows the focus on instruction to be directed more accurately at the students' needs.

While current research has found only two cycles within a mode, there is nothing intrinsic in the SOLO model to suggest that the number of cycles be confined to only two. "Further research is required to establish whether the number of such sequential UMR learning cycles discovered within a single mode is determined by the size of the microscope used to analyse the individual components of skill acquisition." (Watson *et al.*, 1992a, p. 16).

In their study on fractions, Watson *et al.* (1992a) hypothesised a two cycle UMR approach based in the concrete symbolic mode. This conclusion was reached after testing and interviewing over five-hundred school-age children from infant school (Prep) to high school (Year 10). A summary of the two-cycles within the concrete symbolic mode, including examples from the study, is presented below.

Unistructural 1 This marks the start of fraction understanding. At this level, children's understanding of 'sharing' is based on the idea that an equal number of pieces is enough to satisfy the condition of 'fairness'. Watson *et al.* (1992a, p. 14) gave the example of a child (Yvonne) who divided a pancake into four to distribute it equally between three dolls. The child gave two pieces to one doll and one piece each to the other two dolls. When she realised that this was not equable, she split the single pieces into two. This meant that each doll now had two pieces, irrespective of size. The child, however, according to the study, was satisfied that this was fair.

Multistructural 1 This is where the idea that the same number of parts is not enough to establish fairness. However, this notion is limited. For example, Watson *et al.* (1992a, p. 15) stated that when students were asked to distribute twice as many dog biscuits to one dog, many gave two more and not twice as many. However, it is plausible that the students may not have understood the use of the word 'twice' appropriately.

Relational 1 This is characterised by "a more complex notion of sharing (i.e., fractional parts)" (Watson *et al.*, 1992a, p. 15). Students were "now aware that both the number of parts and their equivalence were significant, at least in a practical situation" (Watson *et al.*, 1992a, p. 15). Although it is the highest level in the first UMR cycle, it is firmly entrenched with reality. "The internalisation of the common fraction construct as developed in UMR cycle 1 became the unistructural element from which responses to more complex tasks were built for the next UMR cycle" (Watson *et al.*, 1992a, p. 16).

The second cycle is dominated by the relationship between equivalence and area diagrams. Children who had not yet been exposed to area problems would not usually be able to provide relational level responses.

Unistructural 2 This is characterised by students closing on an answer without fully considering all aspects of a problem. Watson *et al.* (1992a, p. 17) identified this level as where the focus on the denominator becomes prevalent. For example, Watson *et al.* (1992a, 17) quoted one of the subjects in their study as saying: "3/4's is one piece less than a whole and 7/9's is two less than a whole". Learners at this level would be able to compare successfully two simple fractions, but only if the denominators were the same.

Multistructural 2 This is similar to the above level, except that students were able to focus on more than one aspect of the problem. However, learners were easily misled by re-arrangements. For example, in the chocolate bar question, in which the bar was cut lengthwise or widthwise, the students whose responses indicated that they had to physically rearrange the chocolate bar to test for equality were classified into this level.

Relational 2 This is where students would not be tricked or mislead by visual distracters.

Watson *et al.* (1992a, p.20) concluded:

> In the present study, the relational responses at the end of the first UMR cycle (involving an accurate understanding of fractional parts and their equivalences) would not provide a sufficient precursor for entry into the formal mode. However, completion of the second UMR learning cycle involving work on ratios, and flexible performance of the four operations on common fractions probably requires sufficient dissociation from the empirical to pave the way for formal modes of thought.

Clearly, the above classification system provides considerable detail in describing students' growth in fractions. However, it is feasible that some of the descriptions may not be observed with respect to adult learners' understandings of fractions. For example, it would seem that most of the first cycle, which depends on the development of a notion of sharing, may not be applicable with respect to adult learners.

In the second cycle, it would be valuable to have further evidence to help establish, more broadly, the level descriptions. As they stand, the jump between the multistructural 2 and relational 2 levels seems quite large.

Finally, the cycles approach does not offer a complete explanation to some of the results to the experiments, such as the one involving the chocolate bar. Later work by the same authors has attempted to consider the two UMR cycles within the context of multi-modal learning. This issue is now taken up in more detail.

MULTI-MODAL FUNCTIONING

Another new feature which has extended the 1982 model of SOLO concerns multi-modal functioning. This is the ability to access a different type of learning from another mode to support or enhance learning that is usually considered to be based primarily in the current or target mode. As the number of modes available to a learner accrues, such as would be expected on reaching adulthood, the more likely the availability of multi-modal learning (Collis & Romberg, 1991, p. 87). Although multi-modal functioning is desirable in certain situations, it is essential in many problem-solving activities. For example, it is feasible that a concrete symbolic response may require visual support. However, such images are generally considered to be indicative of the ikonic mode. Collis and Romberg (1991, p. 103) have mapped a detailed schema which is presented in Table 2.4.

TABLE 2.4

Multi-modal interactions and decision making points
(adapted from Collis & Romberg, 1991, p. 103)

THE STUDENT READS THE QUESTION AND DECIDES TO CHOOSE:			
	L		R
A	Translation to ikonic mode		Translation to concrete symbolic mode
B	Creation of images or intuitions		Creation of statements or symbols
C	Precedes in absence of relevant information pertaining to the problem	Precedes using 'everyday' mathematics, which may be inadequate for the problem	Precedes with typical concrete symbolic manipulation. This may involve utilising other modes, particularly the ikonic mode
D	Solution is irrelevant	The solution may be reinterpreted with respect to the current problem, however, the steps are usually developed ad hoc.	The solution will be reflected in the steps developed by the student.

Students may progress directly down the left hand column (L) or the right had column (R). Typically, the right hand column is associated with problem-solving skills. However, the research indicated that this 'streamlined' approach could disadvantage students, particularly when asked to solve new or unfamiliar problems. Instead, the research suggested that it was advantageous for students to move sideways (probably at row B or C or both), irrespective of which route (L or R) was the original starting point (Collis & Romberg, 1991, p. 103).

Watson *et al.* (1992b, p. 6) adapted the above table and directly related the left column to the ikonic mode and the right column to the concrete symbolic mode. The authors then produced an elaborate structure that combined both the two UMR cycles noted previously and the multi-modal effects advocated above. This is best demonstrated by an example.

An experiment was designed (Watson *et al.*, 1992b, p. 10) in which children were asked to share a large number of silver, cake-decoration ballbearings between two dolls and a student. The results showed that young children (below Year 3) were unable to complete the task. For example, one child responded with "a triangle". The authors coded this I_A which indicated an ikonic response, i.e., one that was based on intuition, but prestructural in the concrete symbolic mode since this response did not contribute to the solution of the problem. By contrast, another child in the experiment, chose to give one ballbearing to one doll, three to another, and keep the remainder for herself. The authors coded this response I_B in the ikonic mode, and U_1 in the concrete symbolic mode, since it indicated that the child had only a unistructural notion of fractions in the concrete symbolic mode, although the child could conserve number. An I_C, M_1 response was noted by one student who could approximate both 1/2 and 1/4 of the cake decorations. However, the child could only answer 'not much' when asked to identify what part was left, indicating that the child was using visual support to decide the answer. Finally, the authors noted that there were two different approaches used by students who identified 1/4 as the remaining ballbearings after the initial distribution. In the first case, the student said they "saw" the cake decorations "in their heads and imagined the sharing to obtain what was left" (Watson *et al.*, 1992b, p. 11). This was coded I_D, R_1 since it relied on imagery to visually re-arrange the ballbearings. The students who just used mental arithmetic, i.e., in absence of the context that the problem was presented in, were coded R_1 since no imagery was required. A summary of this information is presented in Table 2.5.

TABLE 2.5

Summary of codings for ballbearings problem
(Watson *et al.*, 1992b, p. 11)

Type of Ikonic response	Level of response Ikonic	Concrete-symbolic
Intuition based on an inappropriate image	I_A ('a triangle')	Prestructural
Intuition based on patterns	I_B (1-3-remainder)	U_1
Intuition based on visual appearance	I_C ('not much')	M_1
Correct visual imagery manipulation	I_D (imagined correct sharing)	R_1
No Ikonic support	-	R_1

Once students had mastered the calculations required in the concrete symbolic mode, the use of the ikonic mode was no longer needed and fell away accordingly (Watson *et al.* 1992, p. 12). For example, students were required to solve 1/2 + 1/3 and accompany their solutions by diagrams. Although this type of problem was aimed at the R_1 level, results indicated that over half of the students attempted to draw some sort of diagram. However, this may be an overestimate, due to the obvious prompting in the question, and caution should be exercised in interpreting the significance of this result. Table 2.6 presents a summary of typical responses and classifications to 1/2 + 1/3 problem.

TABLE 2.6

Summary of codings for 1/2 + 1/3 problem
(Watson *et al.*, 1992b, p. 17)

Type of Ikonic response	Level of response Ikonic	 Concrete-symbolic
Incorrect drawing of 1/2 and 1/3	I_A (drew two triangles)	U_1
Correct drawing of 1/2 and 1/3, no combining	I_B (drew boxes of raisins indicating 1/2 and 1/3)	U_2
Correct visualisation of approximate combination	I_C (estimated answer to be between 3/4 and 3/3)	U_2
Correct diagram	I_D (divided rectangle into sixths)	R_2
No ikonic support	-	U_2
No ikonic support	-	M_2
No ikonic support	-	R_2

Clearly, this type of interpretation offers an intricate explanation to some complex issues raised by responses to the experiments. It attempts to isolate ikonic and concrete symbolic aspects within a response and highlights that more than one type of thinking, or mode, can be assessed by a student and provided in a response. Nevertheless, the approach seems unwieldy in its current formulation and difficult to accurately apply in practice.

IMPLICATIONS

There are several implications from the above findings to both this study and the SOLO Taxonomy. For example, evidence from both the Watson study and the Collis and Romberg study indicated that many students 'compartmentalise' their studies, i.e., they do not take school learning into their out-of-school environment. This raises many philosophical and ethical questions, such as:

i. Should strategies be instigated to enable learners to harness the usefulness of multi-modal learning, i.e., is multi-modal learning 'teachable'?

ii. What are these strategies and how have they been discovered or assessed, i.e., what does the research suggest with respect to encouraging multi-modal learning?

It is difficult to address either of these issues. Current research would suggest that there exists a complex interaction between the ikonic and concrete symbolic modes, with each supporting and mutually 'feeding' into one another, and that expert problem solving may be hindered without such interaction. For example:

> mathematically competent children link mathematical symbols and rules to the concepts to which they refer, making an effort to make sense of the connection. In contrast, weaker mathematics learners allow those rules and symbols to become dissociated from corresponding referential concepts.
>
> Watson *et al.* (1992b, p. 24)

A similar observation to the above has been noted with respect to adult learners (Collis & Romberg, 1991, p. 10[illegible]). Unfortunately, current research suggests that development between modes is haphazard and non synchronous, and therefore difficult to map. In addition, true and precise multi-modal functioning may only be possible for "expert learners", or, at the very least, those with a wide variety of modes with which to offer possible interaction.

CONCLUSION

All the mathematical hierarchies discussed above appear to have many strengths and similarities. However, it is the SOLO Taxonomy that would appear to have the greatest general applicability to interpreting adult learners' responses to fraction questions. There are three main reasons for this decision:

(i) The theory offers a systematic method of classifying students' responses that is largely independent of the age of students.

(ii) The model allows the classification of students' responses and does not attempt to discriminate or label students in any permanent way. Instead, 'hope' is offered to both teachers and struggling students as the model has the potential to present possible pathways of intellectual growth that are diagnostic and prescriptive.

(iii) The work of Watson *et al*. (1991ab, 1992ab) has dealt with the issues of applying some of the more novel aspects of the SOLO Taxonomy (such as multiple UMR cycles) to describe childrens' responses to fraction questions.

All of these attributes pave the way for interpreting adult learners' responses to fraction questions using the SOLO Taxonomy as a theoretical framework. One of the main strengths of the SOLO Taxonomy lies in its simplicity. The two-dimensional nature of the model makes it possible to locate a student's response in both scope (modal reference) and depth (level reference). As an evolving model, it also has 'built-in' contingencies, such as multiple UMR cycles. One of the advantages of incorporating multiple cycles into the framework would seem to be that a more detailed progression of learning can be 'mapped'. This in turn should provide greater depth to understanding how learning takes place.