CHAPTER SEVEN

RESEARCH THEME III: OPERATIONS ON FRACTIONS

> ***The ability to solve addition and subtraction computation declines as the child gets older. The ability to solve the problem does not decrease with the age and one is left with the hypothesis that the problems are solved without recourse to the computational algorithms. Many children do not in fact seem to connect the algorithm with the problem solving and use their own methods.***
>
> **Hart (1981, p. 79)**

INTRODUCTION AND ORGANISATION OF THE CHAPTER

This chapter reports the results to the third research theme, namely, how students perform the four standard operations (+, -, x, ÷) on fraction questions. Once again, the questions were divided into context-free and in-context items. In general, the research questions identified in Chapter Five provide similar aims for this chapter, except the focus is now on the *methods* adult learners use to *operate* on fractions. They are:

(i) Is there an identifiable hierarchy of responses to the questions posed?

(ii) Does placing a fraction problem into a familiar context elicit from adult learners different types of responses, i.e., do some adults spontaneously use fractions to solve problems or do they resort to more primitive responses, such as 'guessing'?

(iii) Do adult learners find it easier to respond to in-context or context-free questions regarding operations on fractions, i.e., are responses in this sample equally spread irrespective of context, or do familiar contexts elicit responses indicative of only a few main techniques?

(iv) Do adult learners respond to similar questions with similar responses irrespective of the context?

(v) Can responses be interpreted with the SOLO Taxonomy?

Again, the chapter is divided into three main parts and follows a similar sequence to that established in the previous chapters. This has been summarised in Table 7.1.

TABLE 7.1

Structure of the analysis for research theme III: Operations on Fractions

<table>
<tr><td colspan="3">OPERATIONS ON FRACTIONS
Q10a, Q10b, Q10c, Q10d, Q10e, Q10f, Q11, Q12, Q15, Q16</td></tr>
<tr><td rowspan="2">Part 1</td><td colspan="2">Qualitative Analysis</td></tr>
<tr><td>Section 1: Context-free
Q10a, Q10b, Q10c,
Q10d, Q10e, Q10f</td><td>Section 2: In-context
Q11, Q12, Q15, Q16</td></tr>
<tr><td>Part 2</td><td colspan="2">Quantitative Analysis</td></tr>
<tr><td>Part 3</td><td colspan="2">SOLO Taxonomy Interpretation</td></tr>
</table>

QUALITATIVE ANALYSIS

The qualitative analysis follows a similar style to that of the previous chapters. In addition, each context-free problem has at least one equivalent operation expressed in the 'in-context' form. The relationships between the parts of Q10 and Q11, Q12, Q15 and Q16 are represented Table 7.2.

TABLE 7.2

Structure of the Distribution of Questions between In-context and Context-free items

OPERATION	Context-free	In-context
ADDITION (+)	Q10 part (a) and (b)	Q12
SUBTRACTION (-)	Q10 part (c)	Q11
MULTIPLICATION (x)	Q10 part (d)	Q15
DIVISION (÷)	Q10 part (e) and (f)	Q16

CONTEXT-FREE QUESTIONS - Q10a, Q10b, Q10c, Q10d, Q10e, Q10f

The context-free questions for this part of the work were all typical text-book style questions involving the four operations. All fractions were strictly less than one, and, hence, there was no expectation that students would need to convert mixed numbers to improper fractions at the outset of the task.

QUESTION 10a

This question was the first part of Question 10 and asked students to add two familiar fractions (1/2 and 1/4) together. In many ways, this question was a 'warm up' question to enable students to focus on operations of fractions.

Q10 Complete the following $\frac{1}{2} + \frac{1}{4}$

The four categories identified are presented in Table 7.3.

TABLE 7.3

Summary of adult learners responses to Q10a on the Fraction Quiz

RESPONSE	NUMBER OF STUDENTS AD	TP
No response	1	2
2/6 = 1/3 or 1/2	3	6
Provided unusual answers	0	5
3/4		
. converted to 6/8 then 3/4	3	1
. used LCD's	28	26
. no working	20	12

Typical responses

Typical responses, such as 2/6, indicated that the students had treated the numerator and denominator as if they were adding separate whole numbers. Some students cancelled this number down to 1/3. Some students incorrectly cancelled to 1/2. In both cases, students failed to register that their final answer was smaller or the same size as one of the fractions in the question.

The unusual responses classification consisted of numbers, such as 2/4, 3/2, 1 1/2. There was no clear indication of the methods students used to obtain these answers. However, students in this category attempted to manipulate the fractions as if they were trying to remember a technique or plan of action that they could apply to enable the fractions to be added. For example, there was some evidence to suggest that some students tried to use common denominators, but then did not know what to do with them.

The final category consisted of correct responses. However, a majority of students in this last category used either common denominators or lowest common denominators to arrive at their answers. Only 32 students (<30%) answered this simple fraction question without showing any working.

Overview

In general, the responses to Q10a indicate that a majority of adult learners in the sample could successfully add two simple fractions, such as 1/2 and 1/4. However, there was a small number who either did not attempt the question or treated the fractions as if they were manipulating separate whole numbers.

Of those students who could arrive at the correct answer, some answers were clearly of better quality. For example, a sizeable number of students used common denominators, and although a correct procedure, would seem to be comparatively inefficient approach for a question of this degree of difficulty. It is plausible that the instruction to show all working may have influenced a number of students who did show working, when they may not have usually done so.

There was a significant difference between the Associate Diploma and Tertiary Preparation students and their performance on this item (χ^2=7.81, d.f.=3, $p<0.05$). The Tertiary Preparation students had more difficulty than the Associate Diploma students. For example, five of the Tertiary Preparation students 'played' with the fractions as if trying to remember a technique to enable the addition of fractions. There were no Associate Diploma students which responded with this type of answer.

QUESTION 10b

This question asked students to add two fractions which both had different, (and non-unitary) numerators and denominators.

Q10b Complete the following: $\frac{3}{5} + \frac{2}{7}$

There were four broad categories of responses identified for this question which mirrored the categories of the previous question. These are shown in Table 7.4.

TABLE 7.4

Summary of adult learners responses to Q10b on the Fraction Quiz

RESPONSE	NUMBER OF STUDENTS AD	TP
No response	1	6
5/12	1	4
Provided unusual answers	4	2
31/35 (correct) . used LCD's . no working	 39 10	 37 3

Typical responses

Typical responses in the 5/12 category again indicated that students treated numerators and denominators as if they were adding separate whole numbers.

Responses in the unusual answers category consisted of responses, such as 31/7, 31/37, 17/35, 35/35 = 1. Although it was difficult to determine the reasons for many of these responses, it appeared that students attempted to apply rules from their past, concerning finding common denominators and 'cross multiplying'. Unfortunately for some students in the sample, many of these rules had been only partially remembered.

It was not until the final category, that students were able to solve the problem successfully.

Overview

Overall, a majority of adult learners in the sample were able to complete Q10b successfully. In addition, the number of students in each category did not increase significantly when compared to the previous question, although the same broad variety of responses was exhibited. There was no significant difference between the two groups of students ($\chi^2=6.76$, d.f.=3, $p<0.08$). Although there was a trend for the Tertiary Preparation students to provide either no response or answer 5/12, compared to the Associate Diploma students.

QUESTION 10c

This part of Question 10 asked students to subtract two fractions with different numerators and denominators.

Q10c Complete the following: $\frac{3}{4} - \frac{2}{3}$

Once again, the responses to part (c) of the question were similar to parts (a) and (b). Table 7.5 summarises the responses

TABLE 7.5

Summary of adult learners' responses to Q10c on the Fraction Quiz

RESPONSE	NUMBER OF STUDENTS AD	TP
No response	1	9
1/1 = 1	0	2
Provided unusual answers	1	1
1/12		
. used LCD's	44	36
. no working	9	4

Typical responses

Again, a small number of students treated fractions as if they were dealing with separate whole numbers. Some students who arrived at 1/1 'cancelled' to 1. Students in this category remained unaware that their final answer to a subtraction problem was larger than either of the two fractions in the original question.

Unusual responses consisted of answers, such as 2 and 2/4. There was usually little, if any, indication of the strategies that the students employed. However, it again appeared as if students in this category realised that treating fractions as whole numbers was inappropriate, but could not remember how to obtain common denominators, or what to do with them, if they could.

Almost all students in the sample completed this question successfully.

Overview

A majority of students in the sample could successfully subtract two fractions. In general, there was an overall decrease in the number of students who treated the fractions as if they were separate whole numbers or gave unusual answers, when compared to the two previous questions. The absurdity of the answer of 1/1 may have prompted some students to rethink their approach. Alternatively, it is plausible that some students may had 'warmed up' by this stage and remembered how to work with common denominators. It is also possible that, particularly for the Tertiary Preparation students, that some were simply starting to opt out, since there was a slight increase in the 'No response' category for these students and there was a significant difference between the two groups of students ($\chi^2=9.84$, d.f.$=3$, $p<0.02$). In particular, the Tertiary Preparation students appear to be splitting into two broad groups. Some students seem to be answering the questions correctly by routinely using common denominators, while others appear to have chosen not to respond.

QUESTION 10d

This part of Question 10 asked students to multiply two fractions. Students were also given opportunities to cancel fractions before solving the problem.

Complete the following $\frac{5}{9} \times \frac{3}{5}$

Table 7.6 presents the five categories for responses to Q10 part (d).

TABLE 7.6

Summary of adult learners' responses to Q10d on the Fraction Quiz

RESPONSE	NUMBER OF STUDENTS AD	TP
No response	4	6
Provided unusual answers	5	6
Employed common denominators	9	3
Did not cancel fractions and obtained answers, such as 3/9, 15/45	21	14
1/3		
. cancelled only one common factor first (i.e., intermediate step of 3/9's etc)	0	5
. cancelled both common factors first	8	16
. no working	8	2

Typical responses

Typical responses in the unusual answers category consisted of fractions, such as 27/25 or 1 2/25. These numbers were generated by a confused use of 'cross multiplication'. Some students acknowledged that they had forgotten how to multiply fractions and had tried (unsuccessfully) to remember the rules. However, there was a collection of students who routinely applied this procedure and acknowledged the process as a 'rule', e.g., one student (BK) called it the "the cross division rule". It is worth noting that this student applied the method consistently throughout the paper.

The next category consisted of responses in which both fractions were converted to common denominators, such as would be employed in solving typical addition and subtraction problems. For example, one student (BW), wrote:

$$\frac{25+27}{45}$$
$$\frac{52}{45}$$
$$1\frac{7}{45}$$

There were variations on the common denominator approach. For example, some strategies involved a complex multistage approach which could yield the correct answer, provided it was followed exactly and no mistakes were made, e.g., one student (OL) wrote "5/9 x 3/5 = 25/45 x 27/45 = 675/45 = 15/45 = 3/9".

Only in the last two categories were more traditional methods for multiplication of fractions observed. Again, these were also of varying quality, and this usually depended on the manipulation skills of the student to arrive (eventually) at the correct answer. For example, many students did not cancel the individual fractions before multiplying. As one student (MV) explained during an interview: "My understanding is that multiplication is pretty straight forward. So I multiplied the top numbers and the bottom numbers together. Then I reduced down". The next stage consisted of students who would cancel only one common factor, (and not both), before multiplying. Some of these students would then reduce their answers to lowest terms while others did not.

Thirty-nine students (36%) obtained 1/3 as the correct answer to Question 10d.

Overview

There were a considerable number (approximately 2/3) of students in this sample who could not multiply two fractions together. Apart from the students who did not respond, there was a number of students who attempted to manipulate the fractions with little success. Some students tried to convert the fractions to common denominators, a technique which is more appropriate to the addition and subtraction of fractions. In addition, and despite arriving at the correct answer, a number of inefficient strategies were employed, such as failing to cancel either or both fractions before multiplication proceeded. There were no significant differences identified between the responses of the two groups of students (χ^2=4.61, d.f.=4,p<0.33).

QUESTION 10e

This part of Question 10 asked students to divide a half by a quarter. However, it was written in a traditional textbook style.

Q10e Complete the following $\frac{1}{2} \div \frac{1}{4}$

There were four categories for the responses to this question. Again, these responses were similar to those observed with the previous parts to this question. These are shown in Table 7.7.

TABLE 7.7

Summary of adult learners' responses to Q10e on the Fraction Quiz

RESPONSE	NUMBER OF STUDENTS AD	TP
No response	5	9
Provided unusual answers	15	8
Employed common denominators	9	2
2		
. 1/2 × 4/1 = 2	15	4
. no working	11	29

Typical responses

Unusual answers for this question were based on trying to deal with the fractions as whole numbers or trying to adapt a procedure used in the previous question on multiplication. For example, common answers, such as '2/4' or '1/2', arose out of the 'cross division' technique noted in the previous question.

Again, the use of common denominators was observed. This was despite the fact that the question could have been solved by literally interpreting the question as "given one half, how many quarters?". For example, a common answer of '1/4' could be obtained by converting both fractions to quarters (or eighths) and then subtracting the two numerators, but leaving denominator the same. Other responses in this category, such as '1/16' or '1/8' were obtained by using common denominators in conjunction with standard addition and subtraction techniques applied to fractions.

It was only in the final category that the correct answer of '2' was obtained. Some students employed the usual calculations for division of fractions, while other students did not show any working.

Overview

The responses to this question indicate that there were fewer students who could complete this successfully, when compared to the previous four parts of this question, e.g., there was a substantial increase in the number of unusual answers. Again, the use of common denominators was observed, although no student who used this strategy obtained the correct answer. Again, there was a sizeable number of Tertiary Preparation students who obtained the correct answer, but did not show any working. There was a significant difference between the two groups ($\chi^2=8.31$, d.f.=3, $p<0.04$). More Associate Diploma students were incorrect and used inappropriate methods while more Tertiary Preparation students were able to carry out the question correctly.

QUESTION 10f

This was the last part of Question 10 and asked students to divide two fractions. This question represented a more difficult question than Q10e.

Q10f Complete the following $\frac{2}{3} \div \frac{5}{9}$

Once again four classifications were identified. These are detailed in Table 7.8

TABLE 7.8

Summary of adult learners' responses to Q10f on the Fraction Quiz

RESPONSE	NUMBER OF STUDENTS AD	TP
No response	8	12
Provided unusual answers	8	7
Employed common denominators	12	2
6/5 or 1 1/5		
. did not cancel, i.e., 18/15	10	22
. cancelled 2/3 x 9/5	10	5
. no working	7	4

Typical responses

The unusual answers section became increasingly more varied for this question. For example, responses included: 27/15, 15/18, 1/9, 1/5. Again, the most common technique applied involved the use of 'cross division'.

The use of common denominators was again observed, although the techniques became more difficult to follow.

It was only in the final section that the correct answer of 1 1/5 was obtained. The question was answered correctly by 54% of the students. Of these, some students did not cancel their final answers down to simplest form. Overall, 26 (24%) could make efficient use of cancelling techniques or did not show working.

Overview

The techniques used for this question part were similar to the ones observed for all other parts of this question. However, there was a considerable increase in the number of students who did not complete this question successfully, when compared to the previous parts. It is plausible that students were not as familiar with division problems as they were with the other operations associated with fractions. It is also plausible that the students found division of fractions more difficult than the other three operations. There was a significant difference between the two groups of students ($\chi^2=8.31$, d.f.=3, $p<0.04$). The main difference (and this occurred with Q10e) was that fewer Tertiary Preparation students accessed the inappropriate common denominator technique.

DISCUSSION

A majority of students could complete problems involving addition, subtraction, multiplication and division of fractions successfully. However, the number of correct responses steadily decreased as indicated in Table 7.9.

TABLE 7.9

Summary of number of correct responses to all six parts in Q10

Question number	Number of correct responses
10a (+)	90
10b (+)	89
10c (-)	93
10d (x)	74
10e (÷)	59
10f (÷)	58

The data indicate that there were a greater number of students who could answer addition and subtraction problems correctly. Fewer students could succeed at multiplication, and only about half of the sample could divide two fractions successfully. There may be many reasons for this. For example, some students may have forgotten how to multiply and divide fractions. Despite this, it is interesting to note that addition and subtraction - and the use of common denominators in general - has continued to be remembered over multiplication and division. Also, there did not appear to be a great difference in asking an 'easy' and 'difficult' version of the addition and division problems, although this issue is analysed more deeply in the next section.

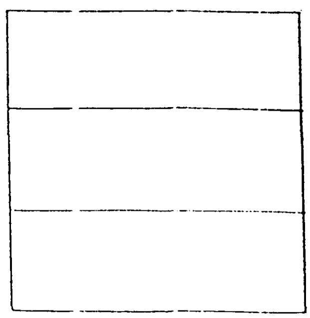
In general, the incorrect procedures students used to operate on fractions could be classified into three main groups. In many cases, the strategy used was to operate on the fractions in a similar manner irrespective of the operation in the question. For example, the first process produced answers, such as 2/6 (Q10a), 5/12 (Q10b) and 1 (Q10c). All of these responses indicate that, at least for addition and subtraction questions, many students treated the numerator and denominator as separate numbers, and tried to apply arithmetic rules which were suited to whole numbers. Responses in these categories did not attempt to form common denominators. It is as if the notion of equivalence was not yet available to students who provided these responses.

The second type of response consisted of unusual answers. Responses in this category could be divided into two main groups. The first groups consisted of responses in which some students had simply forgotten how to work with fractions as one student (AD) explained during an interview:

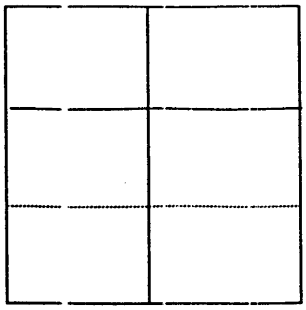
> *AD:* *I wasn't sure if I had to cross divide or I'd forgotten how to do those ones - multiplication and division.*
>
> *I:* *Do you want to have a go at one?*
>
> *AD:* *I sort of work it out. I can sort of remember how to do cross dividing, I don't know if it was for fractions or something else* (multiplies 5 by 5 to get a denominator of 25 and multiplies 9 by 3 to get a numerator of 27). *But I don't think that was right. They could have been the other way around* (pause) *or you could divide them.*

The second group of students in this category continued to apply incorrect methods rigidly and consistently throughout the quiz and during subsequent interviews. In some cases, the (incorrect) method was so entrenched that the student would become confused, and argue with the interviewer in support of their method, although they could not substantiate it. For example, one student (BK), applied the following technique to all multiplication problems:

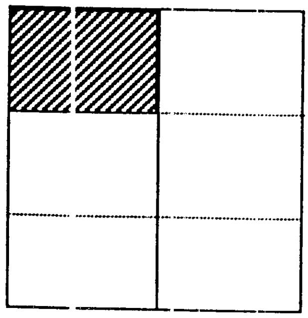
> *I:* *How did you multiply the fractions? You tell me.*
>
> *BK:* *OK, I crossed them over. I multiplied the 9 by 3, to get 27 and the 5 by 5 to get 25. So 27 over 25 equals 1 and 2 twenty fifths.*
>
> *I:* *Have you always multiplied fractions like this?*
>
> *BK:* *Yes!*
>
> *I:* *Have you?*
>
> *BK:* *You mean there's another way?* (Interviewer now explains multiplication using concrete example)
>
> *I:* (long pause) *OK. Suppose you have 1/3 of a cake left.* (Draws this)



But suppose I'm on a diet, so I'm not going to eat the 1/3 of the cake, I'm going to only eat 1/2 of the piece. 1/2 of the 1/3? (Draws this)



So how much am I going to eat? I'm going to eat this much. (Draws this)



BK: *One sixth!*

I: *So if you carry that across to this problem (5/9 x 3/5) you'll get 15/45 which in fact cancels down to one third.*

BK: *NO!*

I: *Why is that?*

BK: *Whoa! NO! NO! NO! NO! NO! NO! I didn't multiply it sideways. I multiplied it.* (repeats his crossing technique)

The above type of rote-learned response was not just confined to multiplication. For example, and despite a particularly high 'No Response' rate for the two division problems (Q10e and f), there were a number of students who attempted to modify the 'rules', such as those that had been 'established' in the previous multiplication example, e.g., one student (JM) wrote: "1/2 ÷ 1/4 = 1/2 x 4/1 = 1/8".

The third category consisted of students who applied common denominators to all four operations on fraction problems systematically, irrespective of how inappropriate or inefficient this may prove to be. For example, in responding to Q10e, one student (SG) wrote: " 1/2 ÷ 1/4 = 2/4 ÷ 1/4 = 2/1 = 2", which is the correct answer. However, the student could not reproduce this method successfully for Q10f.

A variation of the above technique was also observed for division. For example, common denominators were employed, but the denominator was just left to stand, as the following student (NC), in completing part (e) and part (f) wrote:

part (e)	part (f)
= 2/4 ÷ 1/4	= 6/9 ÷ 5/9
= 2/4	= 1.1/9
= 1/2	

The use of common denominators and subtraction were also combined, as one student (MA) wrote:

part (e)	part (f)
= 2/4 ÷ 1/4	= 6/9 ÷ 5/9
= 1/4	= 1/9

Finally, it was not until the fina category, that responses indicated that students selected both the correct operation (+, -, x, ÷) and the corresponding appropriate technique, e.g., the use of common denominators for addition and subtraction. However, it is worth noting that the selection of a common denominator of a particular problem was not necessarily the lowest. For example, some students selected eighths when asked to add 1/2 to 1/4 (Q10a).

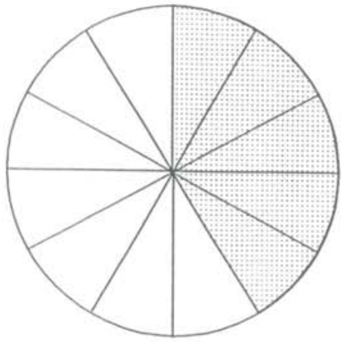
IN-CONTEXT PROBLEMS - Q12(+), Q11(-), Q15(x) AND Q16(÷)

This part of the qualitative analysis investigates the responses to the in-context questions relating to the theme operations on fractions. Again, summary tables are presented followed by typical responses, and a discussion section at the end of the questions is presented.

QUESTION 12

Question 12 provided students with a situation in which another student had correctly answered the addition question posed on a diagram, but had incorrectly answered it using fraction symbols. This question was originally cited in the literature review.

Q12 A student was asked to add 1/6 to 1/4. She drew:



and then concluded that: $\frac{1}{6} + \frac{1}{4} = \frac{1}{10}$

Discuss her conclusion.

Six categories of responses were identified and they are presented in Table 7.10.

TABLE 7.10

Summary of adult learners' responses to Q12 on the Fraction Quiz

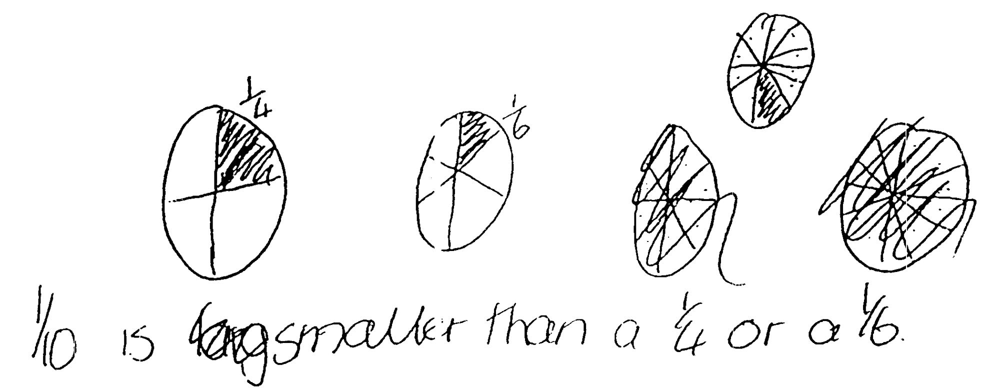
RESPONSE	NUMBER OF STUDENTS AD	TP
No response	10	15
Responses that require an interview for clarification	3	5
Focused on 'picturing' 1/6 and/or 1/4 and argued that the answer of 1/10 was wrong because it was < 1/4 (or 1/6)	6	0
Focused on 1/6 + 1/4, but did not obtain 5/12 and became confused by the diagram	1	4
Focused on the use of common denominators to solve 1/6 + 1/4 and ignored the diagram	32	21
Focused on the correctness of both the diagram and the written answer	3	7

Typical responses

Some students were unable to attempt an explanation to this question. For example, one student (DS) wrote: "Don't know, can't think about it".

One of the responses (OL) to the 'responses that require an interview' category wrote: "1/6 + 1/4 = ". It is unknown if this student was searching for a technique to perform this addition, or simply repeated the question.

The third category consisted of a small number of responses (n=6) in which students focused on the individual fractions in the question and drew diagrams of either 1/4, 1/6, or 1/10. One student (SG) seemed confused by the concept of addition of 1/6 and 1/4 and wrote: "One sixth of something is smaller than one quarter of something. You cannot divide one thing into sixth's and quarters". In general, responses in this category concluded that 1/10 was less than 1/6 (or 1/4). However, their reasons for this conclusion were based on the relative sizes of each fraction diagram. For example, one student (TS) drew:



Another student (LT), who used this approach, concluded that "the answer 1/10 is wrong but the diagram shows the correct answer". This response indicated that the student could visualise the addition of 1/6 and 1/4 to obtain 5/12 correctly. None of the responses in this category solved this problem by using common denominators in the usual way.

The fourth category, consisted of responses which focused on the written statement "1/6 + 1/4 = 1/10" but did not (usually) obtain 5/12, or became easily confused by the diagram and were unable to reconcile both aspects of the problem. For example, one student (BW) wrote: "The ans of 1/10 is wrong because of the sum 1/6 + 1/4 = 3/12 + 2/12 = 1/2". The diagram caused confusion for many students, as one student (JM) indicated when he wrote: "I can't from the picture but my working out is 6/20 = 4/20 = 10/20 = 1/2". In this category, many students stated that the written answer was incorrect, but did not, or could not, provide reasons. It is as if these students had become confused by the diagram and wanted to access a systematic method of solution for the addition of fractions, such as common denominators. For example, one student (JW) appeared to acknowledge this when she wrote: "My answer would not be 1/10. Surely the denominator would have to be 12 or 24. He has obviously got it wrong. The diagram confuses me completely".

The fifth category consisted of responses which focused on the use of common denominators for solving the written statement '1/6 + 1/4 = 1/10'. In addition, the diagram was ignored in this category. It is as if the diagram did not form any part of these students' problem-solving strategies, as one student (OS) stated: "The student didn't find a common denominator and the diagram has nothing to do with the sum". Typical responses performed standard addition of 1/6 and 1/4 to obtain 5/12 correctly, but did not comment on the correctness of the diagram. It is difficult to determine if it was because they did not feel the need to justify their conclusions any further, or if they simply did not realise the relationship between the diagram and the written conclusion of 5/12. Some responses gave 'advice' on how to add fractions, for example, one student (DD) wrote: "you cannot add 1/6 and 1/4 as they are two different things. They need to have a common denominator in order for it to work out". Other responses consisted of a list of 'instructions' on how to add two fractions. For example, one student (GL) wrote: "find the lowest common denominator and then add the top numbers", while another (KL) stated: "You can't do that. You have to have common denominators. Then add the numerators".

Only a few responses focused on both the diagram and the written response, and were able to reconcile the issues. These responses stand out since students who responded this way were able to relate both aspects of the problem. For example, one student (RC) wrote: "Absolute rubbish. 1/6 + 1/4 = 2/12 + 3/12 = 5/12. The diagram is correct". Another student (MD) stated: "The student didn't use the diagram to arrive at the conclusion. He only added the denominators. 1/6 + 1/4 = 2/12 + 3/12 = 5/12".

Overview

As indicated above, very few responses to this question focused adequately on both the diagram and the written form of the question. Although a considerable number of students knew that the written statement was incorrect, a small number of students could not provide a viable alternative. A larger number of students could only correct the written statement. Of the second group, many responses indicated that the student (in the question) should have used common denominators and followed a standard procedure, as if following a recipe in a cook book.

Overall, there was a significant difference between the two groups of students ($\chi^2=10.03$, d.f.=4, $p<0.04$). In particular, there were no Tertiary Preparation students who focused on drawing diagrams for 1/6 or 1/4 and then compared 1/10 to 1/6 or 1/4. In contrast, there was a higher number of Tertiary Preparation students who became confused by the diagram or did not add 1/6 and 1/4 successfully.

QUESTION 11

This question placed the subtraction of two fractions into a familiar context of a the volume of a drink decreasing at a party.

> ***Q11 At a recent function, the punch bowl was 2/5 full. At the end of the function, the bowl was 3/8 full. How much punch was consumed during the evening?***

The five categories of responses to this question are presented in Table 7.11.

TABLE 7.11

Summary of adult learners' responses to Q11 on the Fraction Quiz

RESPONSE	NUMBER OF STUDENTS AD	TP
No response	2	11
Responses that require an interview for clarification	0	3
Incorrect application (used + or x)	3	4
Correct choice of process (subtraction)/incorrect application, e.g., obtained 1/3	7	2
Correct process (subtraction) and correct manipulation of fractions to obtain the correct answer (1/40)	43	32

Typical responses

There were a relatively large number of non-responses from the Tertiary Preparation students and there was a small number of students whose scripts could not be interpreted without an interview, e.g., three Tertiary Preparation students repeated the question but did not provide any further information.

Seven students did not realise that a subtraction was required and selected an inappropriate operation, such as addition for the two fractions. Some students appeared to guess an answer. For example, one student (RK) wrote "1/5", while another student (KR) drew a diagram and wrote "1/8".

Nine students indicated that a subtraction was required to calculate the remaining punch, but were unable to manipulate the fractions successfully. Some of these were of a better quality than others. For example, one student (TF) wrote: "3/8 - 2/5 = 1/3", while another student (DR) appeared to experiment with the fractions as if trying to remember how to produce common denominators. The student wrote: "2/5 - 3/8 = 16/40 - 24/40 = 8/40 = 1/5. 1/5 was consumed". Finally, another student (AB) converted both fractions to common denominators (correctly) but could not complete the calculation successfully.

It was not until the final category that students converted correctly both individual fractions and obtained 1/40 as the correct answer by manipulation of fractions.

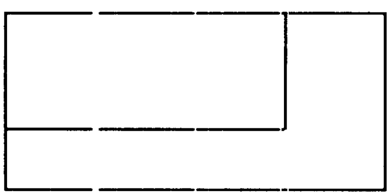
Overview

Overall, a majority of students could answer this question successfully. There were also two small groups of students who either did not associate subtraction with the problem or selected the appropriate process (subtraction) but did not subtract the fractions. There was a significant difference between the two groups of students ($\chi^2=13.28$, d.f.=4, $p<0.01$). There were more Tertiary Preparation students who failed to attempt the question and this is mirrored in the lower success rate when compared to Associate Diploma students.

QUESTION 15

This question had an abstract nature by requiring students to find the area of a room. While at its basis it required multiplication of two fractions, the lack of actual measurements could cause problems for some students.

Q15 A carpet piece is placed in the corner of a room as shown. The carpet is found to go along 4/5 on one wall, and 2/3 of the other wall. What fraction of the floor does the carpet cover?



There were six categories to this question and these are listed in Table 7.12.

TABLE 7.12

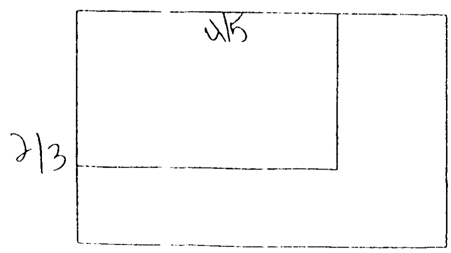
Summary of adult learners' responses to Q15 on the Fraction Quiz

RESPONSE	NUMBER OF STUDENTS AD	TP
No response	12	15
Responses that require an interview for clarification	4	2
Wrote 2/3 and 4/5 on diagram (correctly)	4	2
Added fractions		
. incorrectly	2	5
. correctly (22/15 = 1 7/15)	7	6
Added 1/5 to 1/3 to obtain 8/15	2	2
Applied correct process 4/5 × 2/3		
. incorrect solution	2	7
. obtained 7/15 (i.e., area uncovered)	2	0
. = 8/15 (correct)	20	13

Typical responses

This question had a particularly high non-response rate. It is plausible that this was because students were required to use knowledge of area of a rectangle, in a non-standard form, in their solution. Some responses in the 'require an interview for clarification' category appeared to be confused by the use of fractions as measurements. For example, one student (AS) wrote: "I don't know because for problems of area I need some form of measurement (e.g., m, cm) so I can use decimals and a calculator". This response indicates that the student may have been able to calculate the area of a rectangle, but needed a more familiar diagram which included actual length measurements.

A small number of students placed fractions on the diagram. They did not continue the calculations as the following example from a student (MA) highlighted:



One student (TF) wrote "3/4" as the answer since this is what the area of carpet 'looked like' on the diagram.

The next two categories consisted of responses that dealt with fractions by adding them together. However, a small number of students selected 1/5 and 1/3 to add together which, if correctly completed, gave the correct answer (8/15), although for all the wrong reasons. Other students selected the appropriate fractions correctly, but were unsuccessful at addition. For example, one student (JM) wrote: "4/5 + 2/3 = (7 + 7)/15 = 14/15", which indicated both an incorrect operation and an incorrect procedure for that operation. Other students performed the addition calculation correctly, but did not address the question. For example, one student (HS) just wrote "4/5 + 2/3 = 22/15", but did not acknowledge that this was an inappropriate answer, given the question.

By contrast, the final category consisted of responses which indicated that the students knew that the area of a rectangle could be obtained by multiplying the length by the width, irrespective of the measurements being in fractions. As one student (BE) wrote: "4/5 x 2/3 = 6/8 = 3/4 of the floor is covered". This response indicated that the student knew how to calculate the area of a rectangle, without the need of specific measurements, but was unable to manipulate the fractions successfully. It is worth noting that the student was not aware of how incorrect the answer was since "3/4" is what the area of the carpet "looks like". Some students manipulated the fractions successfully, and then subtracted their answers from 1 to obtain the area of the floor left uncovered (7/15). The last responses consisted of students who multiplied successfully 4/5 x 2/3 to obtain 8/15.

Overview

In general, there were a variety of methods used to address this question - many of them inappropriate or incorrect. However, it is plausible that some students did not know how to calculate the area of a rectangle within the constraints imposed by the question, irrespective of whether the question contained fractions or not. There were no significant differences in responses between the Associate Diploma and Tertiary Preparation students (χ^2=2.14, d.f. =5, $p<0.83$).

QUESTION 16

This question asked students to divide two fractions in a non-routine or non-traditional way.

Q16 What does $\frac{3/2}{5/3}$ mean?

The six categories of responses are tabulated in Table 7.13.

TABLE 7.13

Summary of adult learners' responses to Q16 on the Fraction Quiz

RESPONSE	NUMBER OF STUDENTS AD	TP
No response	13	13
Responses that require an interview for clarification	2	2
1½/(5/3)	3	1
3/2 ÷ 5/3	13	21
3/2 × 3/5		
. no further working	2	0
. unsuccessful solution	2	0
9/10		
. manipulated 3/2 x 3/5	18	14
. no working	2	1

Typical responses

There were many students who did not attempt this question. It is also feasible that many had not seen this type of question previously or one written in this form. A small number of students could only express the top fraction (3/2) as 1 1/2, but could not proceed with any further calculation.

The last three categories consisted of responses in which students attempted to address the question. However, these responses differed in quality and could be divided into three main groups. The first group consisted of thirty-four students who expressed the fraction in the question as 3/2 ÷ 5/3. This response indicated that the students could interpret the question but chose not to perform any further calculations involving the fractions. It is possible that some of these students were not clear what was required, e.g., 3/2 ÷ 5/3 is correct, it is just that it is possible to simplify this statement further. A small number of students were observed in the next category in which some students wrote 3/2 ÷ 5/3 = 3/2 x 3/5. Again, some students did not, or could not, attempt any further manipulation of the fractions. Two students did attempt the calculation but were unsuccessful. Only in the final category did students manipulate successfully the fractions to arrive at 9/10. A small number of students also wrote 9/10 but did not show any working. In all, 35 students (approximately 33%) were able to answer this question successfully.

Overview

Overall, the responses to this question indicated that although a number of students could decipher fraction questions, very few were able to interpret, structure and manipulate the fractions successfully to obtain a correct answer. It is difficult to determine the reasons why some students did not attempt the question when they clearly understood its meaning. Once again, there were no significant differences observed between the two groups of students ($\chi^2=7.44$, d.f.=5, $p<0.19$).

DISCUSSION

All questions produced a wide variety of responses. In general, these responses could be divided into three broad groups. The first group consisted or responses in which students who did not know how to work within the framework of the question. For example, students either did not reply or selected an inappropriate method of solution, such as addition or multiplication for Q11 (which required subtraction), or addition for Q15 (which required multiplication). The second group of students identified the properties intrinsic to the question, but did not demonstrate mastery over the fractions to achieve success. For example, some students selected subtraction for Q11, but did not perform fraction subtraction correctly, or wrote 3/2 ÷ 5/3 as their final answer to Q16, apparently unaware that this answer should be simplified further. The final category consisted of answers in which both the situation into which the fractions had been placed became subsumed within the notion of how to deal with the fractions at hand. Only then was the correct answer obtained, i.e., by applying fraction arithmetic appropriately and correctly.

GENERAL CONCLUSIONS

The main finding from this section of the work, is that the worded problems yielded a higher non-response rate compared to the context-free questions. There may be many reasons for this, e.g., students may have required additional information that was unavailable in the question. For example, students needed to know how to calculate the area of a rectangle, without actual length measurements, to answer Q15 successfully.

In general, the responses across all ten questions could be placed into four main groupings. The first category of responses focused on treating fractions as if dealing with separate whole numbers, e.g., this type of response was seen in the questions that related to addition and subtraction of fractions, such as: Q10 part (a), (b) and (c), and Q11 and Q12 of the in-context problems. There was also a small number of students who focused on the look of a fraction, such as the early responses to Q12 and Q15. The responses to the in-context questions indicated that the responses in this category could choose the correct operation, but could not manipulate the symbols correctly.

The second category consisted of responses which appeared to be based on remembering procedures, and also relating which rule was appropriate for which problem. For example, the 'cross division' rule was very popular, although it was based on little more than the routine application of a 'completing the pattern approach' noted in the previous chapters. This process was continued in this category by the responses which selected an inappropriate operation to address the question. Although there was usually some attempt to manipulate the fractions, it also relied on completing patterns or was dependent on rules.

A minor variation of the above has been classified as the third major category since it was so prominent. This category consisted of responses which focused on common denominators, however inappropriate they may have been for the particular problem being addressed, e.g., multiplication of fractions. In many cases, students could write equivalent fractions, but seemed to overvalue their purposes and applied them haphazardly.

The final classification consisted of responses which indicated that the student could extract the vital pieces of information from the in-context questions, ignore the extraneous details, and know enough about fractions to manipulate them appropriately to reach a successful conclusion. These students were able to reconcile both the diagrammatic and written components of the question.

QUANTITATIVE ANALYSIS

The ten tables provided earlier in Table 6.3 to 6.8 and 6.10 to 6.13 have now had the categories condensed to four, labelled 0, 1, 2, and 3, in order to facilitate the quantitative analyses. Response category 0 is associated with the no response category. Although few modifications were necessary for this theme, compared to the previous themes, the modified tables can be found in Appendix H.

This section of the work again focuses on the fit of these data, for the Operations on Fractions theme, to the Rasch model. For this theme, the Infit Mean Square value was 0.99 with a standard deviation of 0.19. The Infit-t value was 0.04 with a standard deviation 1.33. These results indicate that model is appropriate to use with respect to the above data. Individual Infit Mean Squares can be calculated for each of the above ten questions, and are presented in graphical form (Figure 7.1) in the Item Fit map. There were two items which fell outside the vertical lines, i.e., they lack a mean square which was more than 30% above or below the expected value. This means that there is a tendency for more students to answer these items correctly but respond incorrectly to easier items and vice versa.

```
--------------------------------------------------------------------------
INFIT
MNSQ           .63       .71        .83       1.00      1.20      1.40
--------------+----------+----------+----------+----------+----------+-------
    item 10a                .          *  |              .
    item 10b                .           * |              .
    item 10c                .   *         |              .
    item 10d                .             |    *         .
    item 10e                . *           |              .
    item 10f        *       .             |              .
    item 11                 .             |       *      .
    item 12                 .             |       *      .
    item 15                 .             |              .*
    item 16                 .             | *            .
==========================================================================
```

FIGURE 7.1

Map of Item Fit for the Operations on Fractions theme

The two items were Q10f and Q15. Q10f involved dividing two fractions and Q15 involved the multiplication of two fractions within a context that required additional knowledge of the area of a rectangle. It would appear that these items were seen by the students in the sample to contain some different aspect from the other eight items. Nevertheless, these different values are sufficiently small not to be of a concern. All items are considered in the resulting analysis.

THRESHOLD VALUES

As in previous chapters, the 'threshold' value of a particular item response or student (case) estimate can be read from the logit scale on the far left of Figure 7.2 (exact values can be found in Appendix I). The response categories are found on the right-hand side of Fig. 7.2. Once again, only the top three response categories (1, 2 and 3) are given. The left-hand side of Fig. 7.2 illustrates the distribution of the students' performance over the logit scale. The student (X) has a 50% chance of being able to provide the response category of an item located at the same logit score.

```
------------------------------------------------------------------------
Logit Scale  Student Estimates        Item  Response  Category  Estimates
------------------------------------------------------------------------
                                |  12.3 (2.80)
                       XXXXXX   |
                                |
   2.0                          |
                                |
                  XXXXXXXXXXX   |
                                |
                     XXXXXXXX   |    16.3
                                |
                                |    16.2
                    XXXXXXXXX   |
                                |    15.3
                                |
                       XXXXXX   |
   1.0                          |
                     XXXXXXXX   |
                                |   10f.3
                     XXXXXXXX   |   10e.3
                                |
                          XXX   |
                                |
                      XXXXXXX   |    15.2
                                |    12.2
                      XXXXXXX   |   10e.2
                                |   10f.2
                                |    15.1
                           XX   |    11.3   12.1
                                |   10d.3
                           XX   |
                            X   |
                                |    16.1
    .0                          |    11.2
                           XX   |
                           XX   |
                                |   10d.2
                            X   |
                                |   10f.1
                          XXX   |
                                |    11.1
                                |   10b.3
                                |   10c.3
                            X   |
                                |   10c.2
                            X   |   10b.2
                                |   10a.3   10c.1   10e.1
                                |
  -1.0                          |   10d.1
                           XX   |
                                |
                                |   10b.1
                                |   10a.2
                           XX   |
                                |
                                |
                            X   |   10a.1 (-2.63)
------------------------------------------------------------------------
Each X represents    1 student
========================================================================
```

FIGURE 7.2

Map of Thresholds fc r the Operations on Fractions theme

Initially, the codings for the responses to this theme seem more widely distributed than for the themes described in previous chapters. However, on closer inspection, trends can be observed for the four operations. In particular, there was a clear distinction between the responses to the context-free and in-context questions. In general, the students in this sample found the context-free questions to be easier than the ones placed in-context. This suggests that placing operations on fractions into a context, actually increases the complexity of the problem.

Overall, the first response categories (1's) for the context-free questions (Q10a, Q10b and Q10c) were the easiest to reach. Apart from Q10a, all the first response categories for Q10b, Q10c, Q10d, Q10e occur in close proximity. This is possibly because many students had previously encountered these types of questions as part of their formal schooling. In the case of Q10a, all three response categories for this question were considerably lower than any of the other response categories for the other questions. This is possibly because the fractions 1/2 and 1/4 are used more frequently than other fractions. All of the responses focused on treating fractions as if dealing with separate whole numbers. In contrast, response category 1 for Q10f did not occur until considerably later in the logit scale. This can be attributed to the large number in the sample who, when faced with a division of two not so common fractions, chose not to reply to the question.

The next group of response categories was 10b.2 to 16.1. In general, this was the category in which students treated the fractions as if completing a pattern. This is consistent with the first response category to Q16 (16.1) in which students could write 3/2 = 1½ only. For Q11, some students found it difficult to attempt to perform a subtraction of two fractions when the fractions were placed into a context. However, once accepted, students could reach response category 2 successfully.

Finally, a majority of the remaining responses consisted of in-context questions which indicated that the students knew enough about fractions to be able to work with them in both in-context and context-free situations. There were three exceptions to this. First, the results to Q15 suggest that the calculation of the area of a rectangle, when combined with fractions made the question difficult to start. Second, the results to Q16 suggest that learners find division of fractions, when not placed into a typical textbook style, to be unfamiliar and non-routine and, hence, considerably more difficult to address. Finally, the results to Q12 suggest that many students did not have a clear overview of the problem. As discussed previously, a number of students who calculated the answer correctly to the written part of the problem, ignored or did not comment on the correctness of the diagram.

DIFFICULTY VALUES

The following table contains the overall difficulty and step difficulties for both the context-free questions (Q10a, Q10b, Q10c, Q10d, Q10e and Q10f) and the in-context questions (Q11, Q12, Q15, Q16).

TABLE 7.14

Overall Difficulty and Step Difficulties for Questions in the Operations on Fractions theme

	Question	Overall Difficulty	Step Difficulties 1	2	3
Context-free	10a (+)	-1.60	-.78	.60	.19
	10b (+)	-.81	.11	.72	-.83
	10c (-)	-.63	1.32	.72	-2.03
	10d (x)	-.33	-.27	.62	-.35
	10e (÷)	.12	-.69	1.07	-.39
	10f (÷)	.29	-.01	.35	-.34
In-context	12 (+)	1.41	.44	-2.49	2.06
	11 (-)	-.06	.58	.28	-.86
	15 (x)	.70	1.09	-1.15	.06
	16 (÷)	.90	-.73	2.05	-1.32

Overall, students found Q10a to be the easiest question (-1.60). However, students found the other addition question (Q10b) and subtraction problem (Q10c) to be more difficult (-0.81 and -0.63 respectively), with the multiplication item (Q10d) (-0.33) more difficult again. However, the most difficult context-free questions, by a comparatively large margin, were the division problems (Q10e and Q10f)(.12 and .29 respectively), with Q10f the most difficult. It is also worth noting that although Q10b and Q10c were addition and subtraction questions, students found both of these questions to be of comparable difficulty, but considerably more difficult than Q10a, which was an addition of two small and common fractions. This implies that a hierarchy of operations for this particular sample would be: addition, subtraction, multiplication and division.

The Step Difficulties provide further insight into students' performances on a question by question basis. In the case of Q10a, which required students to add one half to one quarter, students found starting this question comparatively easy (Step Difficulty 1 was -.78). This step is associated with moving from not addressing the question (response category 0) to adding fractions as if dealing with four separate whole numbers, i.e., 1/2 + 1/4 = 2/6 (response category 1). In contrast, students found starting the other addition question (Q10b) and the subtraction question (Q10c) to be difficult (Step Difficulty 1 was .11 and 1.32, respectively). This implies that the fractions 1/2 and 1/4 may have been so familiar that many students simply by-passed this response category. However, students found reaching response category 2 (Step Difficulty 2 was .60, .72, .72 and .62, respectively) of all four questions to be difficult. This step was associated with moving from response category 1 and reaching response category 2. This is associated with the realisation that the response category 1 technique, (for example, adding fractions as if they consisted of separate whole numbers), is inadequate for manipulating fractions and students begin to experiment with other techniques, including common denominators (response category 2). Although this is an inappropriate and inefficient technique for the multiplication of fractions (Q10d), it does demonstrate a maturity on the part of students to attempt to use a more systematic technique that is related to operations on fractions in some way. This technique would appear to pave the way for more appropriate techniques associated with the final response category. As the table suggests, once this category has been reached, it is comparatively straight forward to reach the final response category for a majority of the questions (Step Difficulty 3 was -.83, -2.03 and -.35 for Q10b, Q10c and Q10d).

In general, students found the in-context questions for the Operations on Fractions theme, to be more difficult than the context-free ones. There was only one in-context question which students found easier to deal with (-.06) than the context-free division problem (Q10e). This was Q11, and it was the easiest in-context question by a wide margin. Although students had difficulty in starting Question 11 (Step 1 was .58), the results indicated that students found the question became comparatively easier as they moved through the solution process For example, Step 1 was associated with moving from response category 0 (no response) to response category 1, i.e., subtraction was not selected to solve this problem. However, once students established which operation to use, i.e., subtraction, the question then became comparatively easy to address. Step 2 is associated with moving from response category 1 and reaching response category 2. This category consisted of responses which indicated that students attempted to manipulate the fractions, although they did not obtain a correct answer. However, once this response category was reached, then the final response category was forthcoming, i.e, Step Difficulty 3 was -.86.

Students found all the division problems (Q10e, Q10f and Q16) comparatively easy to start, irrespective of the context, i.e., Step Difficulty 1 was -.69, -.01 and -.73, respectively. This is associated with moving from not addressing the question (response category 0) to writing unusual answers (Q10e and Q10f) or to writing fractions in the more traditional (horizontal) form compared to how they were originally presented in Q16. In all cases, this suggests that students must have known enough about fractions to recognise what the question meant, i.e., division of two fractions, even recognising that two fractions that had been written vertically were, in fact, asking the students to divide the two fractions. However, the table indicates that students had considerably difficulty in moving on from this point, i.e. Step Difficulty 2 was 1.07, .35 and 2.05, respectively. Response category 2 is associated with common denominators, and although incorrect and inappropriate for division of fractions, does indicate that students knew enough about fractions to use them. In the case of Q16, response category 2 is associated with knowing enough about fractions to write "3/2 x 3/5". In all cases, it seems as if once students arrived at the second response category, it was comparatively easy (Step Difficulty 3 was -.39, -.34 and -1.32, respectively) to arrive at the correct solution (response category 3) in which students divided fractions appropriately.

The most difficult question on the quiz involved the addition of two comparatively simple fractions (Q12). However, on closer investigation, it is plausible that many students had not been asked this type of question before, and simply did not know how to address it. Question 12 was the question that placed the adult learners in the teacher's shoes and asked them to explain how a students could draw correctly a diagram showing 5/12 with a written response of 1/4 + 1/6 = 1/10. This scenario may have made the question a lot more difficult to solve. The table indicates that students had difficulty in attempting this question, i.e., Step Difficulty 1 was .44. This step is associated with moving from response category 0 (no response) to response category 1, which is associated with focusing on drawing diagrams of 1/6 and 1/4 and stating that 1/10 was always smaller than 1/6 or 1/4. However, Step Difficulty 2 (-2.49) suggests that many students found it easier to move from response category 1, which focused on the written response as incorrect, and suggested that the use of common denominators would solve the problem (response category 2). It was only at the very highest level (response category 3) that both aspects of this problem could be reconciled, and as the above table indicates, students found this response category (Step Difficulty 3 was 2.06) extremely difficult to reach.

In general, the results of the Rasch analysis indicate that students found operations on fractions to be more difficult if the fractions were not presented in traditional textbook style, i.e., the context-free questions were easier to deal with than the in-context items. In addition, students also found it difficult to divide two fractions, irrespective of whether they were presented in typical textbook style or not. Finally, a hierarchy of operations on fractions was identified, with addition and subtraction easier than multiplication, and division the most difficult.

GENERAL CONCLUSIONS

There were two main findings from the quantitative analysis. First, the results of placing questions in-context and context-free did not appear to influence the number of categories of responses, nor the style and content of each category. Overall, adult learners in this sample found division to be the most difficult operation, irrespective of the context. In general, there was evidence to suggest that a possible hierarchy of operations was possible, i.e., addition, subtraction, multiplication and division. However, there would appear to be need to clarify this situation. For example, the addition of 1/2 and 1/4 (Q10a) was too easy and almost all students could answer this question correctly. In contrast, Q12 which was found to be the most difficult question overall, indicated that very few students in this sample could reconcile both a diagrammatic representation of the addition 1/6 and 1/4, with the written expression '1/6 + 1/4 = ', typically found in many traditional textbooks.

Second, there were differences between placing questions into a context or a context-free situation for the theme of operations on fractions. For example, students experienced greater success at the earlier levels of the operations on fraction questions provided that the questions were presented in traditional textbook style. Questions placed in-context, or unfamiliar (Q16), appeared to become considerably more difficult for students to handle, particularly at the highest categories of response. This is in contrast to the two previous chapters, in which students found the lower levels of the in-context questions comparatively easier to deal with. This issue is addressed more fully in the following section in which the responses from this theme are interpreted within the framework of the SOLO Taxonomy.

A SOLO INTERPRETATION

Previous chapters have advocated a two-cycle UMR approach in the concrete symbolic mode. Evidence presented throughout this chapter would appear to confirm that most of the responses involving operations on fractions primarily occur in the second cycle. Descriptions and appropriate responses are now presented for each SOLO level.

TARGET MODE RESPONSES

The following responses attempted to address the question asked and were all classified in the concrete symbolic mode. The first cycle has been associated previously with a reliance on a concrete objects or diagrammatic representations. However, in the Operations on Fractions theme, there were limited opportunities to enable students to treat fractions as anything other than numbers. As a consequence, there were very few responses observed that could be classified in the first cycle, since it did not deal with fractions as numbers. For example, there were no responses observed that were classified as unistructural 1. Responses that were placed into the first cycle tended to apply arithmetic more suitable to whole numbers than to fractions. In contrast, the second cycle is characterised by the use of fractions as numbers.

THE FIRST UMR CYCLE

Multistructural 1 responses were the lowest responses observed for this theme. Typical responses treated fractions as if dealing with separate whole numbers. For example, "1/2 + 1/4 = 2/6" (Q10a), "3/5 + 2/7 = 5/12" (Q10b) for addition; and, "3/4 - 2/3 = 1" (Q10c) for subtraction. This is because, at this level, operations on fractions do not have the accepted mathematical meaning.

Responses to the in-context questions were similar to the above examples. For example, responses in which students selected the correct operation, but incorrectly operated on the fractions, would be classified at this level. For example, 3/8 - 2/5 = 1/3 (Q11). Responses to Q12 suggest that students focused on drawing individual fractions for 1/6, 1/4 and 1/10. Any conclusions reached were based on comparing the diagrams, i.e., 1/10 < 1/6.

Relational 1 responses at this level have an overview of fractions as parts of objects, and it is feasible that, provided that all three fractions, including the solution fraction, can be visualised at one time; that simple addition, such as 1/2 + 1/4 = 3/4 may be able to be correct at this level. This observation is partly confirmed since 10 students answered this part of Q10 successfully, but then lapsed into treating fractions as if they consisted of separate whole numbers for subsequent questions. It is plausible that the fractions in the other questions were too difficult to visualise and add all at once, whereas 1/2 + 1/4 is considerably easier as the following transcript demonstrates:

I:	*How did you do it* (part (a))*?*
MV:	*... in my head.*
I:	*So how did you?*
MV:	*Because I know there are 2 quarters in a half. So I added the half plus a quarter together.*
I:	*So how did you know?*
MV:	*... I think I related it to pizzas. I know what a quarter pizza looks like. Cos, that's an easy fraction, so I just added them together.*

The example used for the question in Q12 would be typical of a response at this level, i.e, responses at this level may be able to reproduce or understand the diagram as a consequence of adding the two fractions pictorially (diagrammatic equivalence). However, to name the subsequent fraction and relate all the components simultaneously to the symbols would appear to be beyond this level. Instead, responses at this level are typically satisfied with the diagram, and do not yet realise that there is conflict between the diagram, showing 5/12, and the fraction expression '1/6 + 1/4 = 1/10'.

The was only one student (LT) in the sample who appeared to be operating just beyond this level. The student wrote: "the answer 1/10 is wrong but the diagram shows the correct answer". The student did not attempt to work with the written symbols and did not provide an alternative means of solution.

THE SECOND UMR CYCLE

Unistructural 2 responses treat the number properties of fractions as a collection of number patterns. This includes procedures useful for operating on fractions, in the absence of objects. As a consequence, this is an experimental level for discovering useful patterns, such as common denominators. At this level, responses indicated that students rejected both (i) the notion of addition and subtraction of fractions as if dealing with whole numbers (M1), and (ii) the option of 'guessing' an answer on the basis of what the fraction looks like diagrammatically (R1). For example, there was a small number of students (n = 5) who were confused by the diagram in Q12, and attempted to manipulate the fractions 1/6 + 1/4, but did not obtain the correct answer of 5/12.

Common denominators are relevant here, and are seen independently of diagrams, but they have not been defined, and their intrinsic use has yet to be established. They are merely one of many different pattern combinations which may be no more useful to a particular problem than any other type of number pattern, i.e., they are not discernible for any specific reason. As a consequence, responses indicated that students focused on one pattern and habitually applied it, however incorrect or inappropriate it may be in general, e.g., the 'cross division' rule used by several students to multiply fractions.

Responses at this level can lead to some correct answers, if the student selects a pattern that happens to coincide with the meaning of the question. For example, a unistructural 2 response to Q16 in which students were asked to divide two fractions in a non-routine setting, would consist of those responses in which students focused on only one aspect of the problem and wrote 3/2 = 1½.

Multistructural 2 is the first leve at which common denominators are consistently and correctly established. Typical responses indicated that students recognise the need for a systematic method of comparing fractions and can generate common denominators on demand. However, responses at this level do not yet indicate the 'power' such a scheme offers of how to combine fractions. For example, it was not uncommon to see students use common denominators for multiplication problems. In addition, responses at this level may still make calculational errors in operations on fraction questions, e.g., one student (SG) in addressing a follow-up question wrote: "3/8 + 4/5 = 15/40 + 32/40 = 47/80".

Other responses classified at this level include responses to Q11, in which an appropriate operation was followed by the correct conversion of the two fractions involved to common denominators, but an incorrect result was still obtained. For example, one student (AH), in addressing Q11, became confused and subtracted the two fractions in the reverse order, hence producing a negative number, e.g., 15/40 - 16/40 = -1/40. In general, responses at this level were able to manipulate fractions to a certain extent, e.g., obtain common denominators which were correct, but the responses did not indicate that the students realised that their answers were not correct or they did not relate the answer back to the question.

Students who responded at the multistructural 2 level, unlike the previous level, know that the written answer of 1/10 (to Q12) is wrong, but only by operating with common denominators with respect to the written expression '1/6 + 1/4'. The diagram is ignored. For example, one student (EP) wrote: "the student added the written fractions incorrectly. ... basically he didn't grasp it". Responses at this level convert each fraction to common denominators and may arrive at a different answer to 1/10, but it also may not necessarily be correct. It is not until the next level, that a consistent and permanent approach to operations is possible. Responses at this level, although able to calculate correct common denominators consistently, do not show an overview of when it may be inappropriate or inefficient to use them, e.g., for example, in multiplication or division problems.

Relational 2 responses occur wher the algorithms for the operations on fractions are used appropriately, independently and reliably. Responses at this level have a clear overview of common denominators and know where and when to apply them advantageously, i.e., they are useful for addition and subtraction. This implies that this is the first time that it is possible to fully understand both aspects of Q12, i.e., the diagram (first cycle) and the written algorithm (second cycle). It is only at this level that Q12 can be understood, explained and corrected. Since it is only at this level that a total overview of both cycles is possible. For example, one student (SG) wrote:

The conclusion is easily shown to be incorrect as $\frac{1}{6}$ and $\frac{1}{4}$ added together would have to make a larger amount whereas $\frac{1}{10}$ is smaller than both.

$\frac{1}{6} + \frac{1}{4}$

$= \frac{2}{12} + \frac{3}{12}$

$= \frac{5}{12}$

A summary table is presented below indicating typical responses at appropriate UMR levels.

TABLE 7.15

Summary of adult learners' responses to Questions for the Operations on Fractions theme

UMR	CONTEXT-FREE (Q10a, Q10b, Q10c Q10d, Q10e, 10f)	IN-CONTEXT (Q11, Q12, Q15, Q16)
<U1 (CS)		
U1 (CS)	bigger denominator/smaller fraction rule	Q15 - wrote fractions on diagram
M1 (CS)	treated fractions as if dealing with separate whole numbers, e.g, 1/2 + 1/4 = 2/6 (Q10a), 3/5 + 2/7 = 5/12 (Q10b), and 3/4 - 2/3 = 1 (Q10c)	Q12 - drew diagrams of 1/4, 1/6 or 1/10, and compared the diagrams Q11 - treated fractions as if dealing with separate whole numbers, e.g., 3/8 - 2/5 = 1/3 Q15 - selected an incorrect operation, e.g., 4/5 + 2/3, instead of multiplication, and did not manipulate the fractions correctly
R1 (CS)	Q10a - 1/2 + 1/4 = 3/4 (because it looks like it)	Q12 - stated that the answer '1/10' was wrong and the diagram (5/12) was correct, but no further details were provided Q11 - 1/8 or 1/5 or 1/2 Q15 - 3/4 because it looked like it

UMR	CONTEXT-FREE (Q10a, Q10b, Q10c, Q10d, Q10e, 10f)	IN-CONTEXT (Q11, Q12, Q15, Q16)
U2 (CS)	Q10 attempted to use fraction 'rules' (e.g., common denominators) in absence of object - plays with numbers as if completing a pattern, e g., selects inappropriate processes such as the 'cross division' rule for multiplication (Q10d)	Q12 - focused on the written expression '1/6 + 1/4 =', but did not obtain the correct answer, and became confused by the diagram in the question Q11 - 2/5 - 3/8 = 24/15 = 8/5 NB. the answer obtained is 'unreasonable' Q15 - rules are applied incorrectly, e.g., 4/5 x 2/3 = 6/8 = 3/4 and the operation may also be incorrect, e.g., 4/5 + 2/3 = 13/15 Q16 - wrote 3/2 = 1½
M2 (CS)	Used common denominators correctly, but made calculational errors, e.g., inappropriate use of common denominators for multiplication and division of fractions (10def); 3/8 + 4/5 = 15/40 + 32/40 = 47/**80**	Q12 - focused on common denominators to solve '1/6 + 1/4 =', and ignored the diagram of 5/12 Q11 - focused on common denominators, but may have confused the order of the fractions and did not relate the answer to the question, e.g., 15/40 - 16/40 = -1/40. NB. near correct answers are observed here, since the common denominators are correct, but the final answer is (usually) unable to be obtained correctly Q15 - focused on common denominators, e.g., 4/5 x 2/3 = 12/15 x 10/15 = 12/3 x 2/15 = 24/45 Q16 - attempted to manipulate 3/2 ÷ 5/3 unsuccessfully
R2 (CS)	all operations on fractions correct	correct choice of operation and correct process (Q12, Q11, Q15, Q16)

CONCLUSION

There were four main findings for this chapter. First, responses to fraction questions could be placed into a notional hierarchy which could be interpreted within the framework of the SOLO Taxonomy

Second, it was apparent that the ability to succeed at operations on fractions in either context-free or in-context situations lies within the relational level (coded as R2) in the concrete symbolic mode. There is, however, one major exception to this, and it concerns the fraction 1/2. It is clear from the responses to the quiz, that many students who had difficulty with a majority of questions could still answer 1/2 + 1/4 correctly. This observation implies that addition of these two fractions probably occurs much earlier than addition of other fractions. This may be because it is comparatively easy to add 1/2 + 1/4 diagrammatically, or that adult learners may have had a number of experiences with these very common fractions in other parts of their lives.

Third, there were significant differences between the two groups of students and their performances on operations of fractions. In general, there is an emerging trend which suggests that the Associate Diploma students tend to address the questions better and produce answers at higher levels compared to Tertiary Preparation students. For example, there were more Associate Diploma students prepared to use common denominators to answer operation on fractions questions. Although this was an inappropriate strategy for some operations on fraction questions, it does indicate a willingness on behalf of these students to incorporate concepts related to fractions in some form. In contrast, there were more Tertiary Preparation students who did not respond to the operation on fraction's questions. This suggests that Associate Diploma students, most of whom were recent school leavers still had retained parts of rules and a general familiarity with fraction question operations than the adult learners who have absent from school for some time. What we are seeing are the consequences of forgetting material that is usually not accessed in everyday life.

Finally, some responses indicated that a number of students relied on rote learning rules for their survival. In doing this they actually did themselves a greater injustice. For example, rote learning rules, such as the 'bigger denominator/smaller fraction' (U1) and 'cross multiplication' (U2) which have both been classified as unistructural responses, appeared to set up a resistance to using strategies that would enable them to respond at higher levels. To compound the problem some of these survival rules often give a correct response to a question thus providing further reinforcement for their retention.

CHAPTER EIGHT

RESEARCH THEME IV: DESCRIPTION OF FRACTIONS

> ***... when fractions and rational numbers as applied to real-world problems are looked at from a pedagogical point of view, they take on numerous "personalities". From the perspective of research and curriculum development, the problem is to describe these personalities in sufficient depth and clarity so that the organization of learning experiences for children will have a firm theoretical foundation.***
>
> **Behr *et al.* (1992, p. 296)**

INTRODUCTION AND ORGANISATION OF THE CHAPTER

This chapter investigates the responses to the final research theme, namely, descriptions of fractions. This theme is based solely on the first question in the fraction quiz. The research questions which structure the analysis for the chapter are:

(i) Is there an identifiable hierarchy of responses to the question posed?, and;

(ii) Can responses be interpreted within the SOLO Taxonomy?

The chapter is divided into two main sections - a qualitative analysis, and a SOLO interpretation. Unlike the previous chapters, a quantitative analysis is not included, since it is not appropriate to perform a Rasch analysis on only one question. The first section details a qualitative analysis. As in previous chapters, students' responses are grouped according to similarity of response and a summary table presented accordingly. Typical responses including relevant sections of student interviews are presented. The second section details an interpretation of the results of Question 1 in terms of the SOLO Taxonomy.

QUALITATIVE ANALYSIS

QUESTION 1

This question asked students to describe a fraction.

> ***Q1 Imagine you are writing a dictionary of Mathematical terms. Explain, giving as many details as possible, how you would describe what a fraction is.***

The eight categories of responses are presented in Table 8.1

TABLE 8.1

Summary of adult learners' responses to Q1 on the Fraction Quiz

RESPONSE	NUMBER OF STUDENTS AD	TP
No response	5	5
Responses that require an interview for clarification	3	4
Described the look of a fraction as 'one number over another'	2	1
Focused on a fraction as an object		
. part of a whole	4	14
. part of a whole and defines a number in terms of objects, such as cakes, pies, etc	9	12
Described a fraction as part of a whole or number	3	5
Focused on a fraction as a number		
. part of a number	7	2
. part of a number and gives an example to support this definition, e.g., 1/2	2	1
Listed several aspects to do with fractions as numbers, such as decimals or percentages. Usually responses also stated 'not a whole number'.	14	5
Fractions are numbers and related several aspects to do with fractions, e.g., 1 ÷ 2 = 1/2	6	3

Typical Responses

In order to clarify the table, typical responses for each major category are now provided.

Responses in the 'part of a whole category included students who wrote "part of a whole" and added little else. This is exemplified by one student (BK) who during an interview could not add any more, e.g.,

I:	*You wrote: "Fraction - is a breakdown of parts of an item". Would you like to add anything to your definition?*
BK:	*Not really. That's basically how I think what a fraction is.*
I:	*So you're still happy with that?*
BK:	*Yes. Still happy with that.* (Repeats his original definition).

Some students would add other statements or draw diagrams to explain their description. Examples of one half or one quarter of a 'pie' were common. For these students, the diagrams appeared to be the fractions. For example, one student (DM) wrote: "the fraction representing a piece of pie". Another student (JE) wrote: "A number used to describe a part of a whole", while yet another student (PZ) wrote: "A fraction can tell you how much pizza is left". One example that highlighted the 'concreteness' of thinking came from a student (BW) who wrote: "Fraction is a easy way to break a whole thing into sizeable pieces". Presumably, this student related fractions to concrete objects, since it would be very difficult to 'break' a number into 'sizeable pieces'.

The most common characteristic added to this definition of a fraction was that the 'bits add up to the whole'. For example, one student (TC) wrote: "A fraction is a whole piece of an object of any kind that can be divided up into pieces, e.g., A cake. The object can be divided up into 4 pieces. If I take away one of the pieces you are left with 3 quarters, which is a fraction. A fraction is not a whole, it is a part of it".

The next group of responses indicated that students were considering the 'number' aspect of fractions, since these responses stated that fractions were 'part of a whole or a number'. However, the acceptance of fractions as numbers did not become evident until the next classification. It was not until this stage that fractions were treated as numbers. Most responses in this section identified fractions as numbers and some gave examples, such as '1/2'. These responses differ to the above categories since the need to equate fractions with objects is not necessary for the respondents in this section. Usually 'part of a number' was all that was written.

The next classification consisted of lists of several number aspects of fractions. For example, one student (MA) wrote:

> - *part of a number*
> - *a fraction can never be negative must be positive*
> - *not a whole number*
> - *has a numerator and a denominator*
> *e.g., 1/2*

Two students (TL and DM) wrote: "fraction can change to decimal, percentage" and "good for converting to and from decimal amounts to fractions", respectively. Many of these type of responses also stated that fractions were "not a whole number". The responses in this category indicated that fractions could be expressed in many ways but failed to link up the various aspects.

The final classification stood out, since these responses related and integrated, rather than just listed, the different aspects of fractions, e.g., '1/2 or 50%'

Overview

There were two major ways in which students described a fraction. One group of students stated that fractions were labels for concrete objects. The other relied on the number properties of fractions, and, in general, did not draw objects and ignored any specific reference to an object. This observation has also been noted in the previous chapters. Overall, there was no significant difference ($\chi^2=13.49$, d.f.=7, $p<0.06$) between the Associate Diploma and Tertiary Preparation groups. However, there did appear to be a trend for more Associate Diploma students to provide more number-based descriptions than the Tertiary Preparation students.

DISCUSSION

There were two broad approaches used to describe fractions. Clearly, a majority of adult learners entering TAFE in the study related fractions to tangible objects. In general, this group stated that fractions were "part of a whole", although some students elaborated on this statement. However, these responses indicated that the students associated fractions with concrete objects, such as cakes or pies, e.g., students indicated that diagrams and the concept of a fraction were interchangeable. There were more Tertiary Preparation students who described a fraction in terms of cakes and pies, although, both groups of students exhibited a similar range of responses.

The Associate Diploma students described fractions in terms of properties usually associated with numbers. It was clear that these students saw fractions as numbers, although some answers contained more information than others. For example, responses in this category stated that fractions were 'part of a number' or 'must be less than one'. Some responses also included details which indicated that fractions could be converted into decimals or percentages. However, it was only in the final category that the responses indicated that fractions were related to these other number properties, such as division of numbers.

A SOLO INTERPRETATION

Findings from the previous chapters indicate that there are two main approaches used by adult learners in dealing with fraction problems. In general, the reliance on a concrete object or diagrammatic representation formed the first UMR cycle. By contrast, the second cycle was characterised by the use of fractions as decimals and percentages. Similar observations were also noted for the responses to Question 1. For example, one approach described fractions in terms of diagrams or related fractions to cakes and pies. The other method described fractions in terms of number properties. As a consequence, a two-cycle UMR structure within the concrete symbolic mode appears suitable for the responses to Question 1.

TARGET MODE RESPONSES

The following responses attempted to address the question asked and were all classified in the concrete symbolic mode. The first cycle has been associated previously with a reliance on concrete objects, such as cakes or pies. As a consequence, responses which indicated a dependence on real-world examples or diagrams were placed into the first cycle. In contrast, the second cycle is characterised by descriptions of fractions as possessors of number properties.

THE FIRST UMR CYCLE

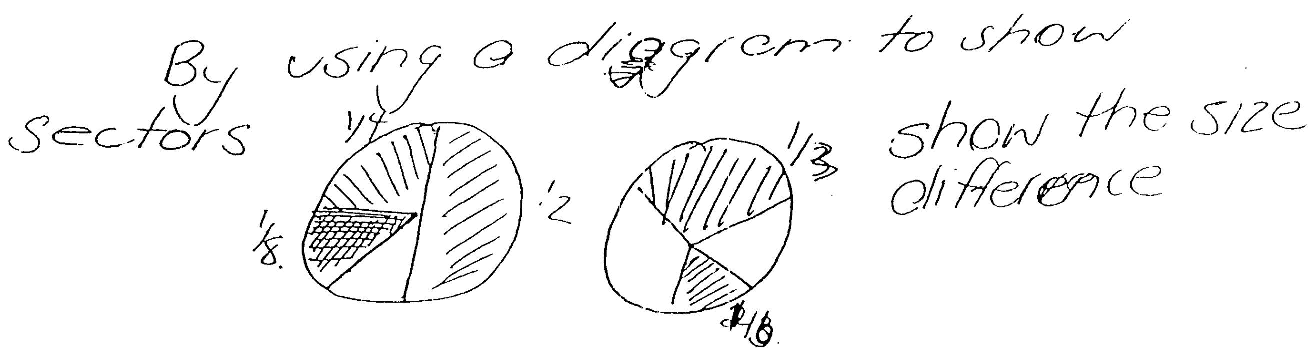
Unistructural 1 responses consisted of 'part of a whole' statements and little else. The dependence on tangible objects is typified by one student (BW) who wrote: "fraction is a easy way of break [sic] a whole thing into sizeable pieces". Here, the student related fractions to concrete objects, since it would be very difficult to break a number into sizeable pieces. Other typical responses included: "A fraction is a representation of a part of a whole unit", "the fraction representing a piece of a pie", "A fraction can tell you how much pizza is left", "It is part of a complete thing", "It's a portion of a whole object" and "One object can be broken up into smaller parts".

At this level, fractions were treated as symbols, or a type of shorthand, used to represent parts of objects. The fraction symbol itself had no independent meaning - it could not be used without reference to an object.

Multistructural 1 responses still depended on the use of diagrams to describe fractions, except the responses added characteristics to the diagrams not found in the previous level. Again, cakes and pies were usually given as examples, but the phrase "bits add up to the whole" was usually added. There was also an implied 'equality' that the bits had to be the same size For example, one student (TC) wrote:

> *A fraction is a whole piece of an object of any kind that can be divided up into pieces. e.g., A cake the object can be divided up into 4 even pieces. If I take away one of the pieces you are left with 3 quarters, which is a fraction. A fraction is not a whole, it is a part of it.*

Basic rearrangements of the subparts of the whole, such as eating one piece of pie, were also noted at this level, e.g., as one student (TB) wrote:



Responses at this level usually contained the phrase "not a whole". It is plausible that this is because it is unlikely that more than a whole cake would be eaten.

Relational 1 responses demonstrated an overview of fractions with respect to diagrams, and treated numbers as if they were representatives of these objects. For example, one student (AD) wrote: "a fraction could be described as a part of or a fraction of a whole number ... if there was one orange and two people wanted [it] the whole number (or orange) could be divided to provide for both people". While another student (SS) wrote: "A fraction is a number of parts smaller than a whole number". In addition, a small number of responses (n=8) acknowledged that fractions could be 'parts of a whole or parts of a number'. Responses similar to this are characteristic of the complementary nature of the relational1/unistructural 2 levels and the paradigm shift associated with leaving one cycle to build upon the next.

THE SECOND UMR CYCLE

Unistructural 2 responses regarded fractions as numbers, independently of objects. Responses at this level tended to be brief, leaving many questions with respect to the number properties of fractions unanswered, i.e., some students just wrote 'part of a number' and could add no more, even when prompted during an interview.

Multistructural 2 responses treated fractions as numbers, but placed restrictions on, or limited the choice of, the sort of numbers fractions could be. For example, responses usually stated that fractions must be strictly 'less than one'. Responses indicated that fractions could be converted to decimals or percentages via a 'conversion process', rather than as an intrinsic definition of a fraction. For example, typical responses included: "fraction can change to decimal, percentage", "good for converting to and from decimal amounts to fractions", "not always whole, expressed in divisible form of whole numbers", "similar to decimal", or "is another way of expressing numbers other than in decimal".

Relational 2 responses described, and not just listed (as in the previous level), the various attributes of fractions as a fully integrated package of number properties. This was the first time that division of integers, for example, could be associated with fractions. However, only one student (IS) (as part of the pilot study) acknowledged that there was a proviso on the value that the denominator could take. The student wrote: "A fraction a/b and b $\neq$ 0 is a division (a is divided by b). It's also a ratio". It is worth noting that the responses in this category did not 'convert' fractions to division, ratios or decimals as in the previous level. Instead all aspects of fractions were considered to be equal and interchangeable.

DISCUSSION

A majority of responses to Q1 could be placed within a developmental path which was interpreted using the existing framework of the SOLO Taxonomy. In general, there were two main approaches used by adult learners to describe fractions, and these two broad groups formed the basis of two discrete cycles which co-exist within the concrete symbolic mode. In addition, there was evidence to suggest an increasing complexity within both cycles. Analysis of this complexity yielded subsequent UMR levels within each cycle. The results of this analysis have been summarised in Table 8.4.

TABLE 8.2

Summary of adult learners' responses to Question 1
for the Description of Fractions theme

UMR	KEY RESPONSE A fraction is ...
U1 (CS)	a part of a whole
M1 (CS)	a part of a whole and the bits add up to the whole
R1 (CS)	a part of a whole or a number
U2 (CS)	a part of a number
M2 (CS)	a part of a number and lists that it can be changed to a decimal or a percentage
R2 (CS)	a part of a number and is states that it is interchangeable with a decimal or a percentage or a division provided that the denominator $\neq$ 0.

CONCLUSION

There are two conclusions that can be seen from the analysis of Q1 responses. First, there were two main approaches used by adult learners to describe fractions. One approach was to describe fractions in terms of objects. The other approach was to describe fractions in terms of properties associated with numbers. The approach that described fractions in terms of numbers tended to be more coherent and complex in nature than those of the diagram dependent group.

Second, subsequent to the two main approaches were a series of several minor categories which could be grouped into a notional hierarchy which indicated that the description of fractions progressed from describing fractions as parts of wholes through to describing them as wholes, then numbers and resulting in their final description as an integrated package of number properties.

Given that there was only one question for this theme, and the overlapping trends noted in the three previous chapters, it is now appropriate to pursue the issue of adult learners' approaches to fraction questions on a more holistic basis. This presents an invaluable opportunity to investigate a more general view of fractions by amalgamating all the questions which were the focus of the themes into a more universal picture of adult learners' understandings of fractions. This is the focus of the next chapter.