CHAPTER FIVE

RESEARCH THEME 3: UNDERSTANDING FRACTIONS

Fractions are not just an easy step from whole numbers. Their use introduces considerable problems for the child - new rules and new possibilities.

Hart (1981, p. 81)

INTRODUCTION AND ORGANISATION OF THE CHAPTER

This chapter investigates the theme concerning adult learners' understandings of fractions as it concerns the two concepts, equivalence and sharing. Equivalence consists of the more mathematical notions associated with fractions, such as would be expected to be found in many traditional mathematics textbooks. The notion of sharing is associated primarily with equality and fairness. When this is applied to fractional understanding, this means that there must be the same number of parts, and that each sub-part is the same size.

The focus of this study is the methods used by the adult learners on fraction questions. In general, five main research questions structure the analysis of this chapter. They are:

(i) Is there an identifiable hierarchy of responses to the questions posed?

(ii) Does placing a fraction problem into a familiar context elicit from adult learners different types of responses, i.e., do some adults spontaneously use equivalent fractions or do they resort to more primitive 'sharing' type responses?

(iii) Do adult learners find it easier to respond to in-context or context-free questions regarding equivalent fractions, i.e., are responses in this sample equally spread irrespective of the context, or do familiar contexts elicit responses indicative of only a few main techniques?

(iv) Do adult learners respond to similar questions with similar responses irrespective of context?

(v) Can responses be interpreted within the SOLO Taxonomy?

The chapter is divided into three main sections. The first part of the chapter is qualitative in nature and is concerned primarily with the collation of responses to the questions that address understanding of equivalence. However, since an aim of this work is to compare fraction problems both in-context and context-free situations, the questions have been divided into these two groupings and analysed separately. The second part of the chapter describes the quantitative analysis using the Rasch model of partial credit. Finally, an interpretation involving the SOLO Taxonomy is presented. Table 5.1 provides a summary of the chapter's structure.

TABLE 5.1

Structure of the analysis for research theme 1: Understanding Fractions

<table>
<tr><td colspan="3">UNDERSTANDING FRACTIONS
Q4, Q5, Q6a, Q6b, Q8, Q13, Q14</td></tr>
<tr><td rowspan="2">Part 1</td><td colspan="2">Qualitative Analysis</td></tr>
<tr><td>Section 1: Equivalence
(context-free)
Q5, Q13, Q14</td><td>Section 2: Sharing
(in-context)
Q4, Q6a, Q6b, Q8</td></tr>
<tr><td>Part 2</td><td colspan="2">Quantitative analysis</td></tr>
<tr><td>Part 3</td><td colspan="2">SOLO Taxonomy interpretation</td></tr>
</table>

QUALITATIVE ANALYSIS

The first part of this section consists of two sub-sections. The first sub-section involves analysis of the equivalence (or context-free) questions. The second sub-section deals with the sharing (or in-context) questions. Both groupings have been analysed separately before a quantitative analysis was undertaken.

EQUIVALENCE (CONTEXT-FREE) QUESTIONS - Q5, Q13, Q14

For the qualitative analysis, each question is presented followed by a summary table. The summary tables present the responses to each question and have been arranged to nominally reflect greater complexity in response down the left hand column. Sample student responses and relevant sections of the student interviews are presented.

Student initials are given in brackets to maintain anonymity. In the transcripts of student interviews, 'I' refers to the interviewer and the students initials are then given whenever the student responds. Non responses are shown accordingly. Responses that were 'illegible' or did not appear to be similar to any other responses were coded as 'Responses that require an interview for clarification'. A brief discussion concludes each group of questions.

QUESTION 5

This question asked students to find the equivalent fraction for 14/16 provided that the denominator was 24.

Q5 Complete the following: $\frac{14}{16} = \frac{\_}{24}$

Table 5.2 presents the five broad categories identified for the responses to this question.

TABLE 5.2

Summary of adult learners' responses to Q5 on the Fraction Quiz

RESPONSE	NUMBER OF STUDENTS AD	TP
No response	10	13
Responses that require an interview for clarification	2	4
22/24		
. by adding 8	7	3
. no working shown	4	2
21/24		
. by patterns (e.g., add '1/2')	12	14
. by multiplying by 1.5, 1.5	5	3
21/24		
. by cancelling to 7/8 first	12	9
. by algebra techniques	2	1
. no working shown	1	3

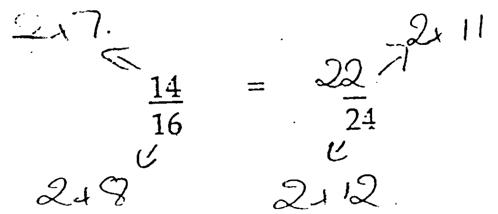
Typical Responses

To clarify Table 5.2, typical responses for each major category are now presented.

Responses in the "22/24" category do not involve concepts of equivalent fractions. Students who arrived at 22/24 obtained the answer by adding 8 to 14, i.e., treating the numerators and denominators as individual whole numbers. For example, one student (DS) wrote:

$$\frac{14}{16} = \frac{22}{24}$$ added 8 to 14 because 16 added onto 8 = 24

Other responses in this classification used a 'complete the pattern' approach to obtain their answer. For example, another student (TC) wrote:

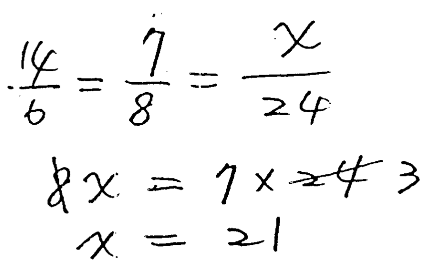


The next category consisted of "21/24" responses, and again relied on a patterning approach used in the previous category, except that students were now able to obtain the 'correct' answer. There were two main types of responses observed in this category. The first response consisted of students who added a half of 14 (to get 7) to 14 to get 21 in the numerator and then added a half of 16 (to get 8) to 24 in the denominator. The second method indicated that students multiplied both the numerator by 1.5 and the denominator by 1.5. Both of these types of solutions in this category showed that correct answers, without further investigation can mislead both students and teachers. For example, one student (MS), who appeared to use appropriate fraction language, e.g., "whatever you do to the top, you do to the bottom" later revealed that the student had very little concept of what this phrase actually meant. The student's working indicated that she had applied a 'patterning' approach and was unaware of the relationship between two equivalent fractions, e.g., the pattern obtained by ÷2, × 3 gives 21, which is the equivalent fraction of 14/16. The statement would appear to be the result of learning by rote, rather than for understanding, and, when this failed, the student turned to more primitive methods of dealing with the situation, such as completing a pattern to obtain an answer. This observation was confirmed by another student (MV), who, during an interview, had little understanding of equivalent fractions, but could match patterns very successfully. The student stated:

> *MV:* *(wrote 21/24). My first indication was to multiply the two together. And end up with a goggle [sic]. But I looked at it again and found that 8 would divide into these two [16 and 24]. So it was reasonable that this one here [14] was 7 and it was twice 7. 8 went into that [16] twice and 7 went into that [14] twice. This one seemed a multiple of that by 3. So it zipped across there.*

The final category consisted of the correct answer (21/24) and three successful techniques that were considered mathematically sound. The first technique reduced 14/16 to lowest terms (7/8) and then multiplied by 3/3 to gain 21/24. The second technique, involved the use of algebra to solve the problem. For example, one student (DC) wrote:

$$\frac{14}{16} = \frac{x}{24}$$



The four students in the last category did not indicate any working and it is unclear what methods were used.

Overview

In general, a majority of adult learners provided the correct answer to this question, although their reasons for doing so, were often dubious. For example, some students relied upon a patterning approach with which to find an equivalent fraction. Sometimes this strategy was successful and sometimes it was not. However, Table 5.2 indicates that very few adult learners (approximately 25%) in this study could legitimately claim to understand the concept of equivalence based on this question. There were no significant differences between either group of students or their performance on this question (χ^2=1 25, d.f.=4, $p<0.87$).

QUESTION 13

Question 13 asked students to increase both the numerator and the denominator of a fraction (1/5) by 2. In addition, students were also prompted to investigate the possibilities of incrementing fractions in this way.

> ***Q13 If I add 2 to both the top number (numerator) and bottom number (denominator) of 1/5, describe, in detail, what will happen. [I'm looking for more than just the answer].***

Table 5.3 presents the ten broad categories of responses observed for Question 13.

TABLE 5.3

Summary of adult learners' responses to Q13 on the Fraction Quiz

RESPONSE	NUMBER OF STUDENTS AD	TP
No response	5	9
Responses that require an interview for clarification	3	6
The fraction stays the same	1	4
3/8 or 2/7	6	1
Added 1/5 + 2/2		
. unsuccessful	1	3
. obtained 6/5	5	3
3/7	7	9
3/7 and the fraction decreases	1	0
3/7 and the fraction changes (but does not describe how)	10	3
3/7		
. the fraction is larger	13	11
. used diagrams	1	3
3/7		
. concluded that the fraction gets closer to 1	2	0

Typical responses

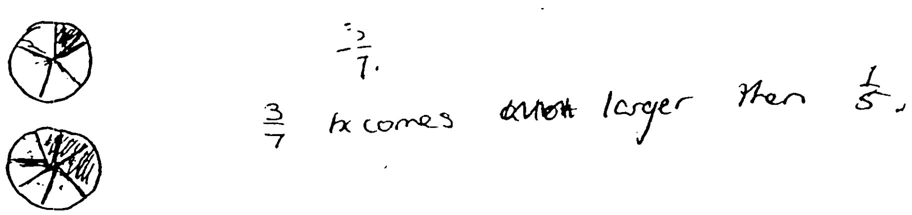
Responses, such as "the fraction remains the same", indicated that students did not realise that adding the same number to the numerator and the denominator of a fraction would alter the fraction.

In general, responses, such as "3/8" or "2/7" suggested that students had made a calculational error. However, one student (IS) indicated that 2/7 was the equivalent fraction, i.e., he wrote: "You will have the same amount left, it will just look different = 2/7".

Some students attempted to calculate 1/5 + 2/2. These students misinterpreted the question and proceeded to add the two fractions. One student (TB) who did this stated: "I can do the problems, just can't explain why".

The last four categories all provided the same 'answer' of "3/7". However, the responses varied according to the complexity of the answer. For example, sixteen students just wrote "3/7", and despite the obvious prompting included in the question, did not provide any further details. Thirteen students went a step further and concluded that the fraction had changed, but they did not describe how.

The next category consisted of twenty-eight students who commented on the fraction getting larger. Four of these students added diagrams to their answers, and some of them 'visualised' the relevant fractions in order to facilitate the comparison. For example, one student (DH) wrote: "You get a larger fraction. A larger portion of a whole". Another student (NK) also appeared to depend on the diagram in order to reach a decision. The student wrote:



By contrast, other students in this category reached their decision by converting the fraction to common denominators or percentages. For example, one student (DC) wrote: "3/7 = 15/35. 1/5 = 7/35. Therefore 3/7 > 1/5".

The final classification consisted of only two students who both concluded that the fraction would not only get bigger, but if the process were to continue, then the resulting fractions would tend towards 1. One of the students did this by first converting 1/5 and 3/7 to percentages. The other student (LP) wrote: "the fraction becomes larger in terms of ---> it is getting closer to a whole (1). 1/5 ---> 3/7 ---> 5/9 ---> 7/11 ---> 9/13 ---> 11/15 ---> 13/17 ---> 15/19 ---> 17/21 ---> 19/23 etc".

Overview

Overall, Table 5.3 indicated that a majority of adult learners in this study could interpret the question correctly and arrive at 3/7 for their answer. However, the table also indicates that many adult learners did not describe the effect of adding 2 to both the numerator and denominator. Some students drew diagrams to support their answers, while other students used decimals or percentages to determine how, and if, the fraction had altered. Only two students suggested that if the process were to continue, the progressions would eventually approach one. Both of these students came from the Associate Diploma class. There were no significant differences in responses between the two groups of students or their performance on this question (χ^2=14.46, d.f.=9, p<0.11).

QUESTION 14

Question 14 asked students to determine how the fraction 2/3 had 'changed' if the denominator and numerator were multiplied by 2. The question was deliberately placed to follow Question 13 in the quiz, to determine if some students treated both questions similarly, irrespective of the different content.

> ***Q14 If I have the fraction 2/3 and I double the numerator (top number) and the denominator (bottom number), describe the effect this will have on the 2/3. Why?***

Table 5.4 presents the six broad categories for the responses to this question.

TABLE 5.4

Summary of adult learners' responses to Q14 on the Fraction Quiz

RESPONSE	NUMBER OF STUDENTS AD	TP
No response	3	9
Responses that require an interview for clarification	2	0
The fraction changes/increases or decreases	11	3
2/3 = 4/9	3	2
Multiplied by 2 to get 4/3 = 1 1/3	0	3
4/6	1	9
. the fractions are the same	35	25
. drew diagrams	0	1

Typical responses

Two responses were not able to be coded. One student (NW) wrote: "The first fraction [2/3] the numbers are even. When doubled they are both even". Another student (SG) wrote: "You will only change the fraction". It is possible that these students did not understand the intent of the question.

The fraction "increases" category consisted of responses, such as one student (HK), wrote: "2/3 = 4/6. Doubling the size of the portion, making the fraction larger". Responses that indicated that the fraction would "decrease" seemed to be relying on the incorrect assumption that the 'bigger the denominator, the smaller the fraction'. As one student (DS) wrote: "It will become smaller because it will then be 4/6 which is smaller than 2/3".

The students who wrote "4/9" misread the question and squared the original fraction. Similarly, some students multiplied by 2 and obtained 4/3 or 1 1/3 for their answers.

Ten students wrote "4/6" as their only answer. Other students attempted to substantiate their claim of equality. Again, these responses were also of varying quality. For example, one student (DM) wrote: "done to the top must be done to the bottom". A second technique was observed in which students related the two fractions to a concrete property. Although the use of diagrams was not generally observed in answering this question some students still appeared to need to relate the fractions to a concrete entity. For example, one student (DC) stated: "2/3 and 4/6 but you end up with the same amount because 2/3 of something equals 4/6 of something". Another student (DH) wrote: "the pieces or parts would in fact look the same when viewed". The responses appeared to be context-free and relied solely on number properties. As one student (AE) wrote: "2/3 is the simplified form of 4/6. Therefore they are equivalent".

Overview

In general, a majority of TAFE students in this sample could successfully answer this question. However, there was still a sizeable number (approximately 10%) who either did not attempt the question, or failed to state that the two fractions did not change, i.e., they were equivalent. It is worth noting that some students still referred to the fractions in a visual way when describing their equivalency, e.g., "the pieces look the same". There was a significant difference between the way the two different groups solved this problem ($\chi^2=12.73$, d.f.=5, $p<0.03$). This occurred because more TP students did not respond and more AD students thought that the fraction changed.

DISCUSSION

The above three questions gave a clear indication that many of the students in this sample did not understand the underlying assumptions behind the notion of equivalence. There were three main approaches used by students to deal with questions involving equivalent fractions. The first technique involved a 'complete the pattern' approach in order to answer questions involving equivalence (such as Question 5). In general, the patterning strategy was popular with adult learners, and was generally effective at obtaining correct' answers, provided the student selected an appropriate pattern and did not make a calculational error. The important point here is that a focus on answers can be misleading in identifying students' thinking, as inappropriate strategies can sometimes give correct responses in certain cases. In some cases, there was no alternative strategy employed or available if the pattern approach failed. For example, during an interview, one student (JG) stated:

JG: *Well, I wasn't too sure what you had to do, but two times eight are sixteen, and three times eight are twenty-four. Two times seven is fourteen and two times eight is sixteen. So that's worked out all right.*
I: *Can you think of another way?*
JG: *No.*

However, another student (LT) appeared to want a mystery number to solve her problems. She selected '4'. However, when asked about this choice she experienced difficulty in justifying her preference. The student claimed that she "couldn't think of anything else to do".

I: *What did you try?*
LT: *I multiplied by 4 and then I realised 16 times 4 doesn't give you 24. So I just worked it out - everything. But I think what you do is probably multiply this* (points to 14/16) *with another fraction if it gives you a common denominator you can divide it into (?) to come up with 24.*
I: *Do you want to try that?*
LT: *It will take me a long time to do it.*
I: *Oh well.*
LT: *No, I mean if I have to multiply 16 by something which the value of the number gives me 24.*
I: *Oh well, just show me what you would have done.*
LT: *16 times ... I don't know. Four becomes four ... (pause) gives me ... ? This number gives me ... something, no, thirty-two?*

The second strategy observed showed that students could write equivalent fractions, but had little understanding of what the equivalent fraction meant, even in simplest terms. For example, a majority of responses to Q14 indicated that students had no difficulty in reaching 4/6 as the correct solution. However, fourteen students (approximately 14%) believed that the fraction had changed or increased or decreased in some way. These students did not understand that the size of a fraction remained constant even when it was represented in a different form. For example, one student (JG) stated that there was "twice as much ... you've got two lots".

The third strategy indicated that legitimate approaches, such as finding common denominators or converting to percentages, were used by students to calculate equivalence. However, as the results to Q13 indicate, they were often inappropriate or unrelated to the question. For example, a majority of students (n = 60) could obtain 3/7 as their answer. However, only a comparatively small number of students could determine how this number had changed. For example, one student (LT), when asked to compare the results of Q13 and Q14 stated that "even if you add two or multiply by two, you end up with a smaller number".

In general, there were no significant differences between the two groups of students in responding to the three questions. The only exception to this was Q14 in which more Tertiary Preparation students did not respond and more Associate Diploma students stated that the fraction had changed.

SHARING (IN-CONTEXT) QUESTIONS - Q4, Q6a, Q6b, Q8

The in-context questions are all similar versions of the same question and involve the equal distribution of pizza or cake or watermelon. Summary tables and analyses are presented in a similar way to the previous section.

QUESTION 4

This question asked students to distribute a relatively large number of pizzas (9) to a larger number of people (15).

> ***Q4 You have 9 pizzas to be shared between 15 people. Describe how you would do this.***

The nine categories for this question are displayed in Table 5.5.

TABLE 5.5

Summary of adult learners' responses to Q4 on the Fraction Quiz

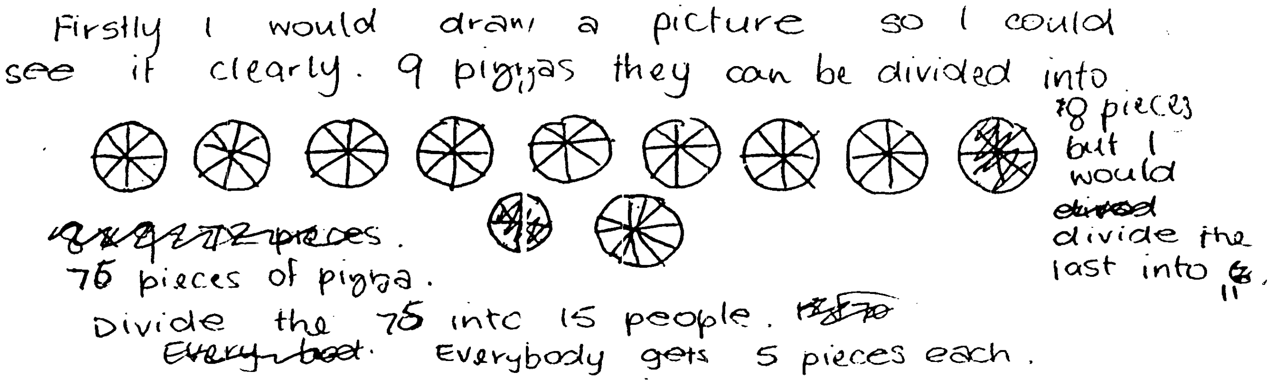
RESPONSES	NUMBER OF STUDENTS AD	TP
No response	2	4
Responses that require an interview for clarification	1	1
Cut into pieces and distribute until none left	3	5
Drew 9 pizzas . unsuccessful solution (½ or ¼'s) . successful solution (1/5's)	 2 1	 2 3
Cut each pizza into: . 1/2's . 1/4's then distributed evenly	 3 1	 1 0
Stated that they would need to multiply the number of pieces by 9 and divide by 15	2	0
Cut each pizza into: . 1/10's . 1/15's . 1/5's then distributed evenly	 2 5 6	 1 3 9
15/9 or 5/3 or 1 2/3	5	3
9/15 or 3/5	22	20

Typical responses

Responses that require an interview included those students who chose to be 'humorous'. For/ example, one student (DK) wrote: "Open pizza box, first in first serv [sic]".

Responses in the "Cut into pieces and distribute until none left" category indicated that the student was unaware of the inequality in doing this. Their choice of number of pieces and piece size seemed irrelevant to them. A slightly better answer (by student GS) was: "I would divide the pizzas into groups of three and the people into groups of 5 give each of 5 three pizzas and let them work it out!"

The next category consisted of students drawing 'pizzas' and then selecting a seemingly arbitrary fraction, such as "1/2 or 1/4". Their choice of a fraction had little to do with either the nine pizzas or the 15 people. The students in this category not only drew diagrams of 'pizzas', but treated the diagrams as if they were physically dividing up real pizzas. For example, one student (UK) wrote:



There were a small number of people who, although they did not draw diagrams, selected inappropriate fractions, such as 1/2 or 1/4, and could not complete the problem successfully.

A small number of people appeared to be in a transition stage in which they wanted to access an appropriate fraction, but were unable to think of one. For example, one student (OS) wrote: "Cut the pizzas into equal parts and divide the number of parts by the number of people".

In general, the major difference between the above classifications and the next category was the choice of an appropriate fraction, such as 1/5 or 1/10, and the absence of diagrams. However, in some cases, it was difficult to determine if the selection of a fraction was related to the question or if the student was selecting the fraction with which it was comparatively easy to manipulate. For example, some students selected 1/10. Although this response seems more 'mathematical' than the previous category, there may have been little mathematics behind the decision to use this fraction. For example, one student (BE) wrote: "I would divide each pizza in a specific no. of slices - probably 10 or 12. After this I would 'deal' out the slices somewhat like playing cards".

Students who responded with 1/15 or 1/5 attempted to take into account both aspects of the question, i.e., the 9 pizzas and the 15 people. It is as if these people were looking for a multiple of 9 and 15 - as a few students stated. For example, one student (LP) provided intermediary steps in her calculation:

Find a number divisible by 9 & 15.
I chose 45.
45/9 · 9/45 = 5 so cut pizzas into 5 pieces (equal)
45 pieces in all to be divided amongst 15 people
45/15 · 15/45 = 3
So each person will get 3 pieces.

It was not until the final two categories that the problem of the pizzas was dealt with in a more general form. This was a fundamentally different way of thinking about the problem when compared to the previous classifications. In general, these students did not draw diagrams, select 'arbitrary' fractions or include a transition stage in which they added up all the pieces. However, some responses were the inverse of the correct answer, i.e., 15/9 or 5/3 rather than 9/15 or 3/5.

Overview

The responses to this question indicated that some students focused on the reality of the problem, and attempted to solve it as if dealing with real pizzas. Some students did not consider the possibility of selecting fractions that were appropriate to the setting of the problem. Other students had a much broader view of the situation and chose fractions that related to the problem. In some cases, students appeared to cross-over this boundary, i.e., they selected inappropriate fractions, such as a half, but then selected a more appropriate fraction such as a fifth, once the initial halving had been accomplished. There were no significant differences between the two groups of students ($\chi^2=5.71$, d.f.=7, $p<0.57$).

QUESTION 6a

This question was the first part of Question 6, and asked students to distribute three cakes between 5 people. This question was the mathematical equivalent of Question 4.

> ***Q6a You have 3 cakes to be shared between 5 people. Describe how you would do this.***

The eight categories are shown in Table 5.6.

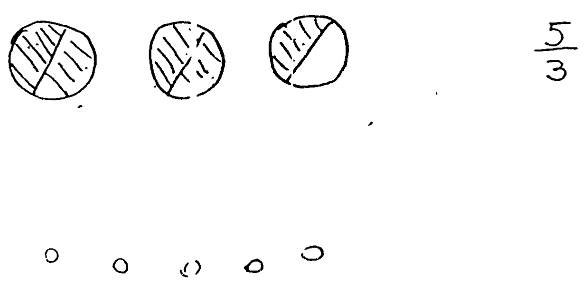
TABLE 5.6

Summary of adult learners responses to Q6a on the Fraction Quiz

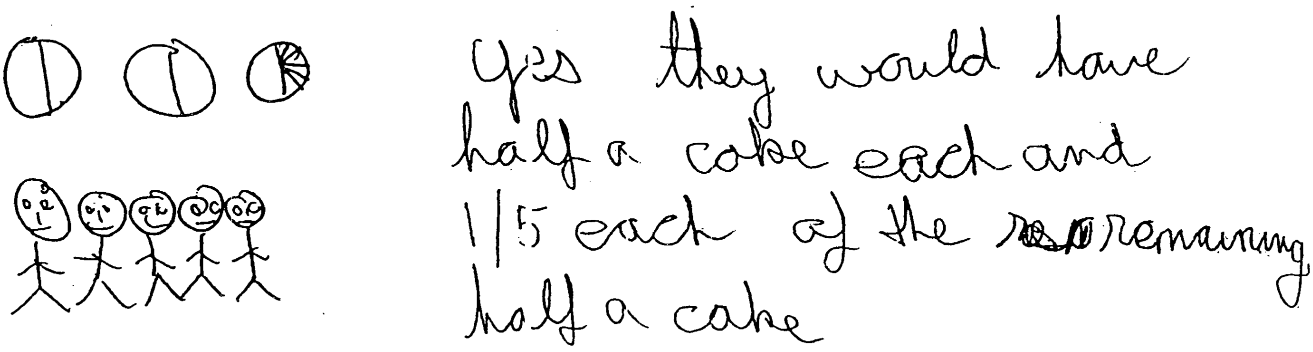
RESPONSES	NUMBER OF STUDENTS AD	TP
No response	3	3
Responses that require an interview for clarification	1	2
Drew 3 cakes		
. wrote 3 ÷ 5	0	1
. divided into 1/2's to distribute	1	2
. divided into 1/4's to distribute	1	0
. divided into 1/5's to distribute	1	3
Diagram independent		
. cut into 1/2's to distribute	3	0
Cut the cakes into even portions and gave each person an even amount	2	0
Diagram independent		
. cut into 1/10's to distribute	1	2
. cut into 1/15's to distribute	0	2
. cut into 1/5's to distribute	14	15
5/3 or 1 2/3	1	3
6/10 or 0.6 or 3/5	27	19

Typical Responses

Similar responses to those of Question 4 were observed for this question, with the added simplicity of smaller numbers. For example, some students who drew diagrams, still appeared to work with them as if dealing with real cakes. Again, some students appeared to select arbitrary fractions to divide the cakes. For example, the selection of 1/2, although inappropriate, was still popular, as the following student (MP) indicated:



In general, the responses in the above category relied on selecting a common fraction, such as 1/2 or 1/4, and sharing the pieces until there was a piece remaining. Some students were able to take this procedure a step further by treating the final piece as a new 'whole', and repeating the process, usually by selecting a more 'user-friendly' fraction, such as 1/5. For example, one student (RM) wrote:



As observed in the previous question (Q4), there were a number of students who went through an intermediate stage of counting up the number of pieces before distribution could take place. For example, one student (JW) wrote: "Divide each cake into 5 pieces. How many pieces altogether? Then give each person one piece from each cake. (1/5 three times)."

The final two categories of responses occurred in the absence of the real-world context. A few students provided the inverse and wrote '5/3' rather than '3/5'. Others related their answers back to the context of the question and wrote '3 pieces'.

Interestingly, only two students, one student from each group, commented that this question was the same as question 4

Overview

The spread of responses for this question was similar to those of the previous question (Q4). For some students the real-world context was important and the practicalities of actually dividing up cakes were an important aspect. For others the context appeared of secondary value and the students focused on the numbers (people and cakes). There were no significant differences between the two groups of students ($\chi^2=7.44$, d.f.$=6$, $p<0.28$).

QUESTION 6b

This question was the second part of Question 6 and required students to distribute two cakes between five people. Unlike Question 4 and 6a, which could be addressed by dividing the pizzas or cakes into halves, the halving strategy was inappropriate for this question.

> ***Q6b You have 2 cakes to be shared between 5 people. Describe how you would do this. (You may use diagrams if you wish).***

Table 5.7 presents the six categories for the answers to this question.

TABLE 5.7

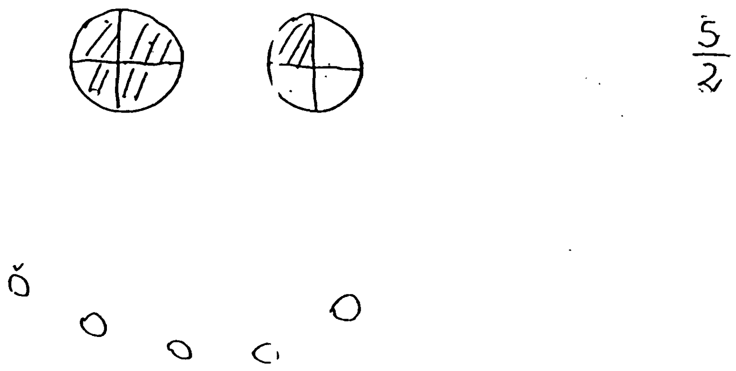
Summary of adult learners' responses to Q6b on the Fraction Quiz

RESPONSES	NUMBER OF STUDENTS AD	TP
No response	5	5
Responses that require an interview for clarification	4	2
Drew 2 cakes		
. divided into 1/2's to distribute	3	0
. divided into 1/4's to distribute	1	2
. divided into 1/8's to distribute	1	0
. divided into 1/3's to distribute	0	1
. divided into 1/10's to distribute	4	0
. divided into 1/5's to distribute and incorrect solution	0	2
. divided into 1/5's to distribute and correct solution	6	5
Diagram independent		
. cut into 1/10's to distribute	2	4
. cut into 1/5's to distribute	16	11
5/2 or 2 1/2	2	3
4/10 or 0.4 or 2/5	11	17

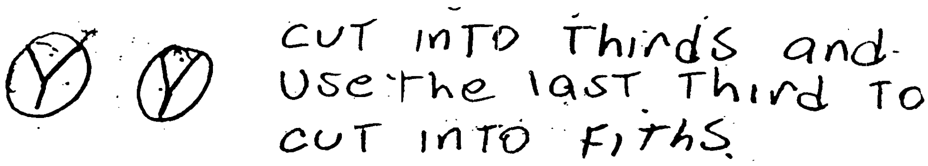
Typical responses

Similar responses to the other two questions (Q4 and Q6a) were observed for this question. The difficulty of having more people than cakes has shown up in the number of different sub-divisions that were tried. Interestingly, a few students still used the halving strategy with some modifying it to include 1/4 or 1/8. For example, one student (TS) wrote: "If there were four people they would get 1/2 a cake each. Each person takes a little of there [sic] half to give to the fifth person".

An example of a student (MP) who modified the 'halving strategy' by halving and halving again to produce quarters wrote:



Another student (IS), who apparently realised that the half strategy would not work, (there is evidence of erasing in his script), changed fractions and adapted the technique to thirds. He drew:



Overview

Although the main objective of setting this question was to 'test out' the acceptability of the halving strategy noted in the previous questions, a few students still opted to stay with the strategy even when they ran out of pieces. In addition, it is worth noting that the number of students in the first two categories increased for this question. Finally, there was a much broader selection of fractions used for dividing cakes for this question, when compared to previous questions There was no significant difference between the two groups of students in terms of the numbers responding in each category (χ^2=3.34, d.f.=4, p<0.50).

QUESTION 8

This question was the logical reverse of Q6a, i.e., this time there were more objects (watermelons) to distribute than people.

> ***Q8 You have 5 watermelons to be shared between 3 people. Describe how you would do this.***

Table 5.8 displays the seven categories of responses identified for this question. Two students (identified by *) were unable to complete the conversion to decimals successfully, however, their scripts indicated their desire to do the calculation this way.

TABLE 5.8

Summary of adult learners responses to Q8 on the Fraction Quiz

RESPONSE	NUMBER OF STUDENTS AD	TP
No response	2	5
Responses that require an interview	2	0
Cut into pieces, divided by the number of people	2	0
Gave each person one watermelon. Leaves 2 between 3, 1/2 both then 1/2 again	1	4
Gave each person one watermelon. Leaves 2 between 3, each is then divided into 1/3's, . with diagrams	13 3	9 1
Cut each watermelon into 3, then divided the remaining 1/3's	11	9
1 2/3 or 5 ÷ 3 or 5/3 or 1.6	21	24*

Typical responses

As the logical reverse of Q6a, the responses reflected those noted in the previous questions. However, some students still preferred to 'opt out' of the fraction game by just referring vaguely to a solution. For example, one student (KL) wrote: "Divide the watermelons up evenly and equally. Then make sure each person got the same amount of equal portions".

A variation of this response involved distributing the watermelons with 2 remaining. Some students would subdivide the remaining two watermelons. Again, typical fractions used included 1/2. However, a wider selection of fractions, such as 1/10 or 1/6, were also noted.

The next category of responses was a modification of the previous one. In it, each person was given a watermelon and then the remaining two watermelons were divided into thirds before being distributed. There was no shortage or wastage observed with this strategy.

An alternative strategy to the above was noted in the next category of responses in which all the watermelons were first divided into thirds and then distributed to the people.

It is worth noting that, unlike the previous question, there were no inverted answers noted for this question, e.g., 3/5 rather than 5/3. It is possible that this is because this question, unlike the previous ones, provided more watermelons than people. It is also feasible that some students habitually selected the larger number as the numerator and the smaller number as the denominator, in a similar way that some people always subtract a smaller number from a larger number, even when it is inappropriate to do so.

Overview

There was considerable similarity in responses identified compared to the other three questions. Again, some students selected the '1/2' strategy with which to work, even though this is inappropriate. Other students focused on the reality of the situation, even when the problem was solved successfully, e.g., "one and a third watermelons is a lot to eat". There were no significant differences between the responses of the two groups of students ($\chi^2=8.79$, d.f.$=5$, $p<0.19$).

DISCUSSION

There were four main findings from this section of the work. First, there was a group of responses which not only involved diagrams, to arrive at the answer, but appeared to rely on the diagrams, as if dealing with real pizzas or cakes. There was a considerable increase in the use of diagrams in Q6b. However, it is feasible that this was due to the 'prompt' in the question which suggested that students draw diagrams. This was partially confirmed during an interview with one student (DM) who answered all four questions correctly, but also included a diagram for Q6b only. The student stated:

I: *Why did you draw a diagram here (b) and not here?*
DM: *Because here (points to part b) it said you may use diagrams.*

In cases similar to this, it would appear that some students drew diagrams, not as an aid to their solutions, but because the question requested it.

Secondly, there was a considerable number of responses which selected comparatively easy fractions with which to work such as 1/2 or 1/4, even when their use was inappropriate, e.g., Q6b. Diagrams may have accompanied these strategies. In some cases, students' diagrams appeared to be vital to their solution, i.e., it appeared as if the students were dealing with real pizzas or cakes. For example, one student (MS) stated:

> *MS:* *Well, on a separate sheet, I did it the way I would have, had I had the 9 pizzas and 15 people. I cut them all up halves.* (Re-drew 9 pizzas). *So I got to 14 then I've still got 2½ left. But I've still got 15 people. So what I did then was break it down into quarters. And quarters again. ... But I came up with 1/2 and ...* (loses track of her argument). *It came out even, but I can't remember what I did.* (Repeats procedure, but does not solve the problem).

A refinement of the above technique occurred when students re-distributed any outstanding 'halves'. For example, one student (LT), in response to Q4, wrote: "I will divide the pizzas in 2 and I'll end up with 18 pieces. After everyone has one (piece), I will have 3 more pieces left which will be divided for everyone", although she did not state how she would do this. During an interview, one student (LT) was asked if she wished to alter her strategy. She did not reply, but was concerned about the three remaining pieces from her original answer. She decided:

> *LT:* *Cut them into five.*
> *I:* *Cut them into five?*
> *LT:* *That gives you another 15. So everyone gets one part* (meaning the original 1/2) *and a bit.*

This type of response appeared to be taking on the nature of the problem, at least as far as selecting a fraction (1/5) that was more appropriate to the context of the question.

Responses in the final category formed the basis of the third approach used by students to solve the above four questions. In this category, responses indicated that students selected appropriate fractions from the outset of the problem.

Finally, there were no significant differences noted between the two groups of students and their responses to the above three questions.

GENERAL CONCLUSIONS

Results from analysis of the context-free (Q5, Q13, Q14) and in-context (Q4, Q6a, Q6b, Q8) questions reveal that responses to fraction questions by adult learners can be placed into a notional hierarchy. The early levels of such a hierarchy would appear to consist of responses which focus on treating contextual problems as if dealing with real-life situations, such as dividing pizzas or cakes between groups of people. In these situations it is acceptable to have remaining portions or an inequitable distribution of parts. However, from a mathematical standpoint, these types of responses do not address the mathematics inherent in the problem. Selecting familiar fractions, such as 1/2 or 1/4, does not really address the mathematical issues raised by the in-context questions, although some students appeared to select more appropriate fractions, such as 1/5, after some initial subdivisions had already occurred. When this happened, it appeared to be an afterthought or sudden discovery that was not already present in reading the problem initially. Only in the final stages of these questions did responses indicate that the students selected appropriate fractions, such as 1/5, and provided answers that were, in general, independent of the context.

QUANTITATIVE ANALYSIS

The seven tables provided earlier in Tables 5.2 to 5.8 have had the categories condensed to four in order to facilitate the quantitative analyses. In each case, the compression of the categories was addressed so as to maximise the direct evidence. Response category 0 consisted of non responses or 'Responses that require an interview for clarification'. Other response categories were coded 1, 2 and 3, respectively. These tables can be found in Appendix D. The following sections consider the application of the Quest package.

RASCH ANALYSIS

This section of the work focuses on the fit of these data involving the responses to questions, concerning the theme: understanding fractions, to the Rasch model. For this theme, the Infit Mean Square value was 0.98 with a standard deviation of 0.12. The Infit-t value was -0.08 with a standard deviation of 0.94. These results indicate that model is appropriate to use with respect to the above data. Individual Infit Mean Squares can be calculated for each of the above seven questions. These values are presented in graphical form (Figure 5.1) and indicate the degree of proximity to perfect association (i.e., 1). Statistics that lie within the two vertical dotted lines are considered acceptable (Masters, 1982).

```
------------------------------------------------------------------------
INFIT
MNSQ          .63       .71       .83      1.00      1.20      1.40
--------------+---------+---------+---------+---------+---------+-------
    item 4                   .            * |              .
    item 5                   .              |        *     .
    item 6a                  .          *   |              .
    item 6b                  .     *        |              .
    item 8                   .       *      |              .
    item 13                  .              |  *           .
    item 14                  .              | *            .
========================================================================
```

FIGURE 5.1

Map of Item Fit for the Understanding Fractions theme

All the questions within the Understanding Fractions theme fell within acceptable limits. This suggests that the data fit the model and can therefore be used to investigate the placement of categories in the proposed hierarchy. This involves interpreting Thurston Threshold values and Difficulty values using the Tau option.

THRESHOLD VALUES

The Quest output also displays data in the form of a variable map (see Fig. 5.2). This provides information on item difficulties as well as student estimates on a standard logit scale. The 'threshold' value of a particular item response or student (referred to in the package as a case) estimate can be read from the logit scale on the far left of Figure 5.2 (exact values for both these measures can be found in Appendix E). On the right hand side of Fig. 5.2 are the response categories. For each question the top three response categories are given, e.g., in the case of Q4 there is 4.3, 4.2 and 4.1. A feature of the Quest package is that the lowest response category (in this case 4.0) is not provided as it serves as baseline data. From Fig. 5.2 it can be seen that response category 4.3, which has a threshold score of .74, is one of the more difficult questions for students to respond at whereas response category 4.1, which has a threshold of -1.22, is the easiest.

Each X represents a particular student from the sample. The left hand side of Fig. 5.2 illustrates the distribution of the students' performance over the logit scale. The student (represented by a particular X) has a 50% chance of being able to provide the response category of an item located at the same logit score, and, consequently, a better than 50% chance of reaching a response category for answers below this score.

```
--------------------------------------------------------------------------
Logit Scale  Student Estimates        Item Response Category Estimates
--------------------------------------------------------------------------
  3.0                             |
                                  |
                                  |
                                  |
                                  |
                                  |
                       XXXXXXXX   |
                                  |
                                  |
                                  |
  2.0                             |
                                  |
                                  |
                       XXXXXXXX   |
                                  |
                                  |
                                  |
                                  |     6b.3
                         XXXXXX   |
                                  |
                                  |      5.3
  1.0                   XXXXXXX   |
                                  |      8.3
               XXXXXXXXXXXXXXX    |
                                  |      4.3    6a.3
                            XXX   |
                                  |
                        XXXXXXX   |     13.3
                                  |      5.2
                  XXXXXXXXXXXXX   |     6b.2
                      XXXXXXXXX   |
                                  |     14.3
   .0                      XXXX   |
                          XXXXX   |     13.2    14.2
                             XX   |      4.2     5.1
                                  |     6a.2    13.1
                       XXXXXXXX   |
                              X   |
                                  |      8.2
                                  |     14.1
                                  |     6b.1
                             XX   |
                                  |      8.1
 -1.0                             |     6a.1
                                  |
                                  |      4.1
                             XX   |
                                  |
                                  |
                                  |
                              X   |
                                  |
                                  |
                                  |
 -2.0                             |
--------------------------------------------------------------------------
  Each X represents    1 student
==========================================================================
```

FIGURE 5.2

Map of Thresholds for the Understanding Fractions theme

There appeared to be three main groupings associated with these data, and these are consistent with clusters surrounding the three separate steps of each question response, e.g., .1, .2 and .3 category responses. The lowest group of question responses (4.1 to 8.2) consisted of mainly the first category responses of context-free questions. Clearly, the easiest response was to provide an initial attempt to Question 4 which asked students to distribute 9 pizzas by 15 people. This response was equated with drawing nine pizzas and sub-dividing them into halves and quarters. In general, most responses in this grouping consisted or diagrams and/or the selection of 1/2 and 1/4.

The middle group of responses (6a.2, 13.1 to 13.3) were a mixture of both in-context and context-free questions. Responses to the in-context questions indicated that there was a gradual increase in the range of fractions selected, e.g., 1/5 was also noted although this was often after 1/2 had initially been selected. Responses to the context-free questions indicated that students were attempting to deal with fractions as numbers, although their responses suggested little underlying fraction understanding. For example, the response to 5.1 equates to students adding '8' to both the numerator and denominator to obtain 22/24, which is not the equivalent fraction of 7/8. The response to 13.1 demonstrates that students stated that if '2' was added to both the numerator and the denominator of 1/5, that the resultant fraction, 3/7, had not changed.

The highest group of responses (4.3 to 6b.3) consisted of the final steps to a majority of in-context questions. This implies that very few students were able to answer the in-context questions with a context-free answer. For example, 4.3, 6a.3 and 8.3 all related to stated 3/5, 3/5 or 1 2/3 respectively, and independently of the situation. In addition, 5.3 required students to provide 21/24 as the equivalent fraction for 7/8. Overall, the most difficult response was to provide a correct answer to Question 6b, which asked students to divided 2 pizzas by 5 people. Approximately only one student in five (20% of the sample) had a 50% chance, or better, of getting either of these last two questions correct.

DIFFICULTY VALUES

Table 5.9 contains the overall difficulty values, and step difficulty values for both the context-free questions (Q5, Q13, Q14) and the in-context questions (Q4, Q6a, Q6b, Q8).

TABLE 5.9

Overall Difficulty and Step Difficulties for Questions
in the Understanding Fractions theme

	Question	Overall Difficulty	Step Difficulties 1	2	3
Context-free	5	.46	.17	-.43	.26
	13	.06	1.23	-.91	-.32
	14	-.22	.19	.83	-1.02
In-context	4	-.20	-.72	.21	.50
	6a	-.20	-.29	-.32	.62
	6b	.27	-.61	-.12	.73
	8	-.17	.25	-1.21	.96

The easiest question was Q14 (-.22) which required students to write the equivalent fraction for 2/3. (This question was equivalent in difficulty with a majority (three of the four) of the in-context questions). The most difficult move in the responses to this question was that associated with Step 2 (.83), i.e., the movement from response category 1 to response category 2 This is the step associated with moving from stating that the fraction increases or decreases to rejecting this and operating with fractions as numbers. At this stage an incorrect conclusion is reached. However, as the above table suggests, once students begin to operate with fractions as numbers, correct conclusions become considerably easier, i.e., the move from response category 2 to response category 3 was the easiest (-1.02). This is the move that required students to build upon the number patterns established in the previous step to state that 2/3 and 4/6 were equivalent fractions.

Most students found a majority of the in-context questions to be of about equal difficulty and comparatively easy (Q4, Q6a, Q8). However, the above table indicates that many adult learners found Question 6b (.27) to be a considerably more difficult question to address. This was the question that involved dividing 2 pizzas among 5 people. The table suggests that many students found Step 1 (i.e., the move from response category 0 to response category 1) of this question comparatively easy. This is possibly because of the prompting in the question to draw diagrams. This Step involved moving from a no response strategy to that of drawing diagrams and dividing them into 1/2's, 1/4's etc. The next step, from response category 1 to response category 2, was associated with streamlining the fraction selection and choosing more appropriate fractions, such as 1/5 or 1/10. The most difficult move was that associated with Step 3 (.73), i.e, transferring between response category 2 and response category 3. This required students to move away from the context and to focus on the numbers to arrive at 2/5, independently of the context.

Of the other three in-context questions, Q4 showed that students had difficulty transferring from drawing diagrams and dividing them by an inappropriate fraction, such as 1/2 or 1/4; to selecting fractions that were more appropriate to the problem, such as 1/10's or 1/5's, (moving from response category 1 to response category 2). However, this was not observed in the other three in-context questions. It is possible that the added complexity of the larger numbers (9 pizzas and 15 people) may have increased the question's difficulty over the other three in-context questions. In general, all four in-context questions indicated that students found Step 3 to be the most difficult. This is associated with moving from an appropriate selection of a fraction to providing an answer which was independent of the context. This was particularly pronounced in Question 8. Of all four similar questions, Step 3 was the most difficult and is associated with going from treating the watermelons as objects and dividing them between three people (response category 2), to providing an answer independently of the context (response category 3). This implies that students in this sample found improper fractions or mixed numbers to be more difficult than proper fractions, when fraction problems are placed in context.

Of the remaining two questions, students found Q13 to be considerably more difficult (.06) than many of the previous questions, with Question 5 the most difficult (.46) question of all. In Question 13, the most difficult step was the first (1.23), i.e., students found it difficult to start this question. This is associated with moving from not addressing the question (response category 0) to stating that no change had occurred to the fraction (response category 1). However, once it was accepted that the fraction could change, it became easier to describe a change, i.e., Step 2 was comparatively more straight forward (-.91). This equates to stating that no change has occurred (response category 1) to a fraction which has had 2 added to both the numerator and denominator (a non-mathematical statement) to applying a more mathematical interpretation, such as adding two fractions (response category 2). Although this was an incorrect procedure for this problem, it appeared to pave the way for a more reasonable and correct mathematical interpretation in response category 3. This is confirmed by the comparatively small difficulty score (-.32) associated with Step 3.

Question 5 was the question in which students were asked to write the equivalent fraction for 14/16 with a denominator of 24. Responses indicated that students had difficulty both starting (Step 1) and finishing this question (Step 3). Step 1 indicates that students found moving from not addressing the question (response category 0) to answering 22/24 (response category 1) to be difficult (.17). This step is associated with treating fractions as numbers, and disregards the 'fractionness' of the fractions, i.e., fractions were treated as whole numbers and '8' was added to both the numerator and the denominator. However, once started, responses indicated that Step 2 became comparatively easier. This was the move from response category 1 to response category 2 (-.43), and is associated with rejecting the 'whole numbers' approach in favour of a more streamlined version. Although incorrect at this stage, there is some evidence that, although the numerator and denominator were still treated as separate numbers, students attempted to seek a relationship between all four 'separate' numbers that was not observed in the previous level. However, this strategy was based largely on completing a pattern, and there is a considerable lapse, as Step 3 (.26) indicates, between this approach and the final one which required students to write the equivalent fraction 21/24 for 7/8 (response category 3). This final step equates to moving from an answer that is obtained by completing a pattern (such as dividing both the numerator and denominator by 2 and then multiplying by 3) to one that involves acknowledging the relationship between two equivalent fractions, e.g., students would reduce 14/16 to lowest terms or apply algebra. In both cases, responses indicated that students knew enough about equivalent fractions to let this concept guide their working.

GENERAL CONCLUSIONS

In general, the results imply that adult learners have few difficulties working with relatively common and comparatively simple fractions, such as 1/2 and 1/4, i.e., the earlier response categories of all problems. However, responses in these categories indicated that fractions were linked to the context of the problem, rather than the focus of the problem itself. As a result, students were often misled and produced inappropriate answers. For example, some students still selected 1/2 when attempting Q6b, which required students to divide 2 pizzas by 5 people. In general, the results from this section of the work indicated that many students select fractions, such as 1/2 or 1/4, almost irrespective of the context of the problem and the fractions inherent in the problem.

Responses to questions that were placed in a familiar context, produced a wider variety of responses, although there were fewer students' answers to be observed at the highest levels, i.e., responses free of the context. For example, Q6b was found to be one of the most difficult questions on the quiz, and only a few students could reach the final response category of Q6b, i.e., to answer the question independently of the context.

With respect to the context-free questions, there was some evidence to suggest that once a problem can be solved using a 'patterns' approach it is a comparatively minor step to generalise such an approach, and solve these types of problems by knowing enough about fractions to focus on the underlying relationship of producing an equivalent fraction. For example, once response category 2 was reached it was a comparatively minor step to reach response category 3 as observed in the responses to both Q13 and Q14. In both of these questions, the earlier response categories equated to 'playing with numbers', whereas the final response categories required students to focus on the production of equivalent fractions.

A SOLO INTERPRETATION

Results from both the qualitative and quantitative sections indicate that there were two main approaches used by adults in dealing with fractions observed throughout this work. In general, one approach related fractions to objects, such as cakes and pies, and the other method described fractions in terms of number properties. The former corresponded to the earlier response categories in all questions in the Rasch analyses, while the latter corresponded to the later response categories. This implies that a single UMR structure in one mode does not adequately explain such phenomena. Neither does the supposition of the diagrammatic dependence into the ikonic mode serve as an appropriate model of understanding, since any question that requires a concrete-symbolic response is clearly not aimed at the ikonic mode as its target mode. Rather the smaller changes in complexity, i.e., Step 2 and Step 3 would be expected to indicate differences between levels in two different UMR cycles. This implies that there are to be at least two distinct cycles in the concrete symbolic mode with respect to fractions. This is consistent with the implications of Pegg (1991, p. 383) and Campbell *et al.* (1992). In general, the first cycle would appear to be diagram dependent, while the second cycle would appear to treat fractions more like numbers and less like concrete objects. The following diagram indicates the relationships between the different levels in the two distinct UMR cycles. Throughout this analysis, U1 will represent a Unistructural response in the first cycle, M1 will represent a multistructural response in the first cycle, and so on. A diagrammatic representation showing the two cycles in the concrete symbolic mode is presented in Figure 5.3

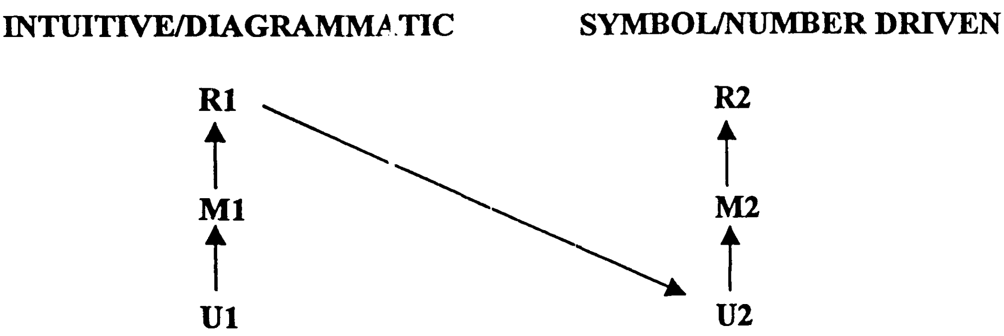


FIGURE 5.3

Diagrammatic representation of two learning cycles within the one mode

RESPONSES PRIOR TO THE TARGET MODE

Some responses, which indicated that students focused on the reality of the problem, such as dividing pizzas into pieces and distributing these pieces, were considered to be outside the concrete symbolic or target mode. There was no relationship between the size of the piece or the number of pieces or the number of people, e.g., "cut into pieces and give to each until none left". A small number of students drew an appropriate number or pizzas but did not add any further information and did not attempt to work with the diagrams. Such responses are consistent with belonging to the ikonic mode. They have not attempted to address the mathematics of the situation but relied on intuitive ideas of what feels an appropriate thing to do.

TARGET MODE RESPONSES

The remainder of the responses did attempt to address the question asked and most were classified in the concrete symbolic mode. The first group of these relied on a concrete object or diagrammatic representation. By contrast, the second group is characterised by the description of fractions as numbers, such as decimals and percentages. A description, with appropriate examples of each of the SOLO levels, spread across the two cycles is now presented.

THE FIRST UMR CYCLE

Unistructural 1 responses focused on one aspect of the problem, such as just dividing pizzas into pieces using a simple familiar fraction. Typical responses at this level drew a number of pizzas and then divided the pizzas into 1/2's or 1/4's but did not deal with any remaining pieces. The choice of fraction had little to do with either the number of pizzas or the number of people. Students who selected these fractions did not appear to be aware of the unsuitability of the chosen fractions to the problem, i.e., their repertoire of fraction choice was very limited. The responses to Q6b, in which students were required to divide 2 pizzas by 5 people, indicated that, at this level, there was very little understanding of any other fraction, other than a half or a quarter. Respondents who selected 1/2 could not share the cakes equally, since they would run out of pieces after serving only four people.

Multistructural 1 responses focused on more than one aspect of the problem, but only by dealing with the situation one step at a time, i.e., the repertoire of fractions is beginning to expand. In general, students who responded at this level did not relate the choice of fraction to the situation (like the previous level) until the final stage.

The most typical response at the multistructural 1 level was the situation in which students would select 1/2 or 1/4, divide up the pieces and then deal with any remaining pieces as if dealing with a new whole. The final stage in this process usually involved selecting a new fraction other than one half. However, at the entry (or lowest) point to this level, the new fraction choice was not necessarily a more appropriate fraction, as the following example indicates. During an interview, about her response to Q6a (3 cakes, 5 people), one student (LT) stated: "if you cut this into 1/2, just sees 1/2 and everybody gets one half. And the other half [remaining half] is divided into 5, no 3, no 5!". The more sophisticated responses at this level selected more appropriate fractions, such as 1/5, to divide the remaining 1/2.

Responses which counted the total number of pieces before distribution have also been classified at this level, since they did not indicate that the student had an overview of the problem. It is feasible that some students sought security or confirmation in their (often) correct answers by counting the total number of pieces prior to distribution.

A minor transition level was observed between multistructural 1 and relational 1 in this cycle. Responses in which students indicated that they wanted to select an appropriate fraction, but were unable to do so would seem to occur between these two levels. For example, a typical response (for Q4) was "pieces in each x 9, divide by 15".

Clearly, the context of a fraction problem would have a particularly important role in classifying responses in this cycle. Fraction questions which enabled or encouraged students to equate fractions with objects would receive more attention in this cycle. As a consequence, questions that were posed out of context (Q5, Q13 and Q14) and tested the more mathematical concepts traditionally associated with fractions would cause responses to appear illogical. For example, responses to Q5 (which asked 14/16 = ?/24), such as 22/24, treated the numerator and the denominator as separate whole numbers. This is because without any contextual reference, such as an object, the fractions become meaningless symbols.

Relational 1 responses indicated that students had an overview of the problem with respect to the number of pizzas and the number of people. Although the fractions would still be equated to slices of pizza and dealt with the pieces "like dealing out playing cards", the responses indicated that the student had an overview of the problem with respect to selecting an appropriate fraction, e.g., 1/5, 1/10 or 1/15. However, the problem was still context-bound and responses indicated that some students counted the number of pieces 'just to make sure there were enough'. Students chose from a wider range of fractions.

The end of the first cycle is marked by the ability to select or reject a particular fraction, e.g., 1/2, from a wide variety of fractions. For example, responses at this level automatically reject inappropriate fractions, such as 1/2 and 1/4, for the pizza questions, and exhibit preference for selecting more appropriate fractions, such as 1/5, for particular problems, e.g., Q6b. However, it is not until the next cycle that the problems of the pizzas could be dealt with independently of the scenario in which the question is placed.

THE SECOND UMR CYCLE

Transference from the first cycle to the second (relational 1 to unistructural 2) is best characterised as the notion that equivalence can be described as the separation of the dependence on diagrams to the awareness (if not yet complete mastery) of fractions as numbers. In particular, equivalent fractions that were associated with diagrams (of equivalent fractions) in the first cycle, start to be seen independent of diagrams in the form of number patterns. Eventually, the number patterns take on a life of their own and can subsequently be used to generate multiple and reliable 'reproductions' of themselves (multistructural 2).

The departure from the first cycle would appear to coincide with the arrival of the second cycle, i.e., there is a synchrony between the relational 1/unistructural 2 levels. For example, there is a limit to the number of fractions that diagrams are useful for, and some equivalent fractions are just too difficult to conceptualise as diagrams. When this stage is reached, equivalent fractions may be seen to be numbers, but only because of the patterns between the numbers and not as equivalent fractions *per se*.

Unistructural 2 responses are the first type of responses that appear to be independent of the context. Responses at this level can be deceptive, i.e., in the absence of a contextual situation, responses at this level may appear to be more 'mathematical' than the previous ones. However, on detailed examination, responses at this stage do not yet have an overview of equivalence without the aid of a diagram, although they may appear to operate, or function, independently of diagrams. Typical responses, which may be correct, are usually based on 'patterns', and do not exhibit any underlying understanding. For example, some of the responses to Q5 obtained the correct answer (21/24), by 'adding 1/2 [of 14]' or by multiplying both numerator and denominator by 1.5. In addition, there was little indication, either from scripts or interviews, that students were aware that they had reached a correct solution.

Unfamiliar, or non-routine questions, such as Q13, would cause confusion at this level. Responses at this level could not interpret the question satisfactorily, or could not determine the effect of adding 2 to both the numerator and denominator. As one student (GH) stated at the interview "do you mean 2 the fraction or 2 the number?". Without an effective diagram or a clear pattern to follow, many respondents could only guess an answer, such as "the fraction stays the same".

Multistructural 2 responses indicate that the number 'patterns', established in the first level, consistently and reliably produce equivalent fractions, irrespective of the context or the presence of 'distracters' in the problem (e.g., Q5), i.e., fractions are independent of diagrams and synonymous with numbers. This is the first level in which either in-context or context-free questions are truly dealt with independently of the context. Responses at this level show that students do not, in general draw diagrams, count individual pieces or treat any of the in-context questions as if dealing with real pizzas. Typical answers are 9/15 or 3/5.

Typical responses at this level could generate correct equivalent fractions *ad infinitum*, and would obtain the correct answers to Q5 (21/24) and Q14 (the fractions are the same) for mathematical reasons. However, unfamiliar or non-routine problems, such as Q13, may still cause confusion, although the student is aware that the fraction has altered in some way. For example some students knew the fraction had changed, but could not describe how.

Relational 2 responses indicated that the student had an overview of the problem and could work with fractions as numbers. Fractions can be generated independently of diagrams and irrespective of the context. In addition, distracters do not, in general, hinder progress, and students are able to establish a more general rule, such as that stated by two students in response to Q13. The students stated that if the process was continued, then the progression would get closer and closer to 1.

Table 5.10 summarises the descriptions above and indicates typical responses at UMR levels across the two cycles.

TABLE 5.10

Summary of adult learners' responses to Questions for the Understanding Fractions theme

UMR	EQUIVALENCE (CONTEXT-FREE Q5, Q13, Q14)	SHARING (IN-CONTEXT Q4, Q6a, Q6b, Q8)
<U1 (CS)		divide the pizzas by pieces
U1 (CS)		1/2 or 1/4 selected
M1 (CS)	Q5 - 22/24, 'add 8'	1/2 or 1/4 and later 1/5 selected
R1 (CS)	Q5 - 22/24	1/15, 1/10 or 1/5 selected from the outset, still context dependent pizza pieces were counted
U2 (CS)	Q5 - 21/24 by patterns only Q13 - fraction stays the same	1/15, 1/10 or 1/5 selected from the outset, context independent
M2 (CS)	Q5 - 21/24 by using equivalence of fractions Q13 - fraction changes Q14 - equivalent fractions	correct answer for pizza problem (context-free) undertaken in a sequential manner
R2 (CS)	Q13 - by continually adding 2 to the numerator and the denominator of a fraction, the fractions will progress to 1	

CONCLUSION

There were four main findings for this chapter. First, the responses to fraction questions involving sharing and equivalence formed a notional hierarchy. Furthermore, these responses could be interpreted into the existing framework of the SOLO Taxonomy.

Second, two main approaches were identified which were adopted by students in answering fraction questions with respect to understanding fractions. These can be described as object related or number related. The major difference in classifications across all questions was that some students could only deal with the problems by directly relating the problem to a concrete situation. In particular, some students drew diagrams and divided the diagrams as if they were actually dealing with 'cakes', 'pizzas' or 'watermelons'. It was not until the final stages of all questions, that students could provide answers, irrespective of the context, and independently of any diagrams.

Third, there were no significant differences between the two groups of students and their approaches to answering fraction questions. The only exception to this concerned the responses to Q14. This is the question that required students to double both the numerator and the denominator of 2/3, and comment on any changes. Table 5.4 indicates that there were more Tertiary Preparation students who did not respond to the question and there were more Associate Diploma students who thought that the fraction had changed. Although it is feasible that both groups were confused by the wording of the problem, the results suggest that more Tertiary Preparation students chose not to attempt the question. It may be that they felt intimidated by the question or simply preferred not to guess due to their unfamiliarity with the concept. In contrast, the Associate Diploma students, many of whom had only recently completed their secondary schooling, felt confident enough, or at least comparatively more familiar with the concept in the question to attempt an answer.

Finally, there was some evidence, although tentative, to distinguish between responses to in-context questions and those that were context-free. For example, the in-context problems appear to be slightly easier at the lower levels, while the context-free questions appear to be slightly easier at the highest levels. However, it was clear that the in-context questions drew a greater variety of responses. In particular, there was more use of diagrams or references to physical objects. However, caution should be exercised in this interpretation, for example, Question 6b prompted students to use diagrams. It is plausible that many of the students felt obliged to demonstrate their solution by referring to objects indicating that many of these students could answer the question by using either the number or the object approach.

CHAPTER SIX

RESEARCH THEME II: COMPARISON OF FRACTIONS

> ***... evidence is beginning to accumulate that suggests students bring to instruction a rich store of informal knowledge of fractions.***
>
> **Mack (1990, p. 17)**

INTRODUCTION AND ORGANISATION OF THE CHAPTER

This chapter investigates the responses to the second research theme, namely, the strategies employed by students to compare fractions. Again, the questions relating to this theme have been presented in two ways. First, typical school textbook questions (referred to as context-free problems) are described, and second, fractions are placed into familiar situations in which students need to determine methods to enable the fractions to be compared (referred to as in-context problems). In general, the research questions identified in the previous chapter provide a similar aim for this chapter. They are:

(i) Is there an identifiable hierarchy of responses to the questions posed?

(ii) Does placing a fraction problem into a familiar context elicit from adult learners different types of responses, i.e., do some adults spontaneously use fractions to solve problems or do they resort to more primitive responses, such as 'ratio'?

(iii) Do adult learners find it easier to respond to in-context or context-free questions regarding comparing fractions, i.e., are responses in this sample equally spread irrespective of context, or do familiar contexts elicit responses indicative of only a few main techniques?

(iv) Do adult learners respond to similar questions with similar responses irrespective of the context?

(v) Can responses be interpreted within the SOLO Taxonomy?

Again, the chapter is divided into three main parts and follows a similar sequence to that established in the previous chapter. This is summarised in Table 6.1.

TABLE 6.1

Structure of the analysis for research theme II: Comparison of Fractions

COMPARISON OF FRACTIONS Q2a, Q2b, Q3a, Q3b, Q7, Q9		
Part 1	Qualitative Analysis	
	Section 1: Context-free Q2a, Q2b, Q3a, Q3b	Section 2: In-context Q7, Q9
Part 2	Quantitative Analysis	
Part 3	SOLO Taxonomy Interpretation	

QUALITATIVE ANALYSIS

CONTEXT-FREE QUESTIONS - Q2a, Q2b, Q3a, Q3b

The qualitative analysis follows a similar style to that of the previous chapter. Summary tables are presented followed by typical responses for each question part. The discussion section provides an overview of the context-free and in-context questions. Interview data have been included to develop further the trends noted in the written test.

QUESTION 2a

This question was the first part of Question 2 and asked students to compare two fractions. One of the fractions was the reciprocal of the other, and this meant that one fraction was less than 1 and one fraction was greater than 1.

Q2a Compare and contrast what is meant by $\frac{5}{7}$ ***and*** $\frac{7}{5}$***?***

Table 6.2 presents the seven broad categories identified for the responses to this question.

TABLE 6.2

Summary of adult learners' responses to Q2a on the Fraction Quiz

RESPONSE	NUMBER OF STUDENTS AD	TP
No response	1	10
Responses that require an interview for clarification	6	5
Wrote seven over five or seven fifths	15	10
Drew diagrams as illustrations of fractions	2	8
Compared each fraction to the number 1	15	10
Converted before comparing		
. used percentages	3	2
. used common denominators	4	4
Stated 5/7 is smaller or 7/5 is larger (no working)	9	3

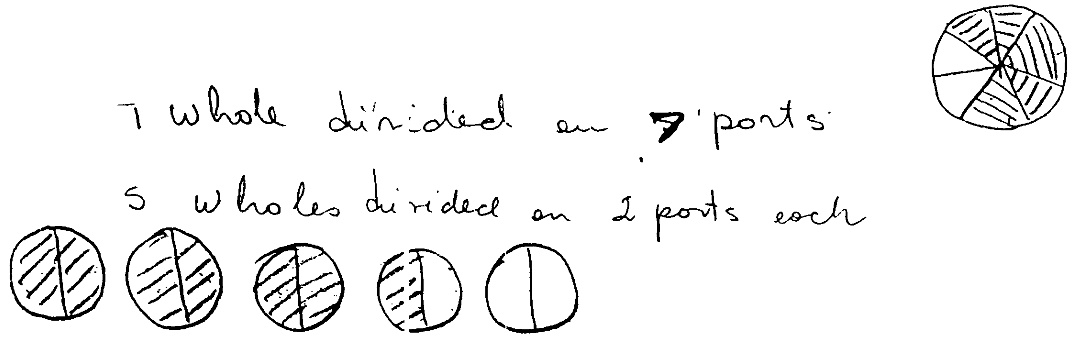
Typical responses

In order to clarify the table, typical responses in each category are now provided, commencing with category 2.

There were a small number of responses (n=11) which did not address the question asked. For example, one student (MS) wrote: "improper fraction", while two other students (MS and MP) added the two fractions together. It is unclear from this written information whether the task was unable to be completed or whether the students had misinterpreted the question.

The 'seven over five or seven fifths' responses did not usually contain any further information, although two students (OS and KL) acknowledged that the two fractions were reciprocals since, as KL wrote: "one is [the] inverted version of [the] other", and a further three students described 5/7 as "5 divided by 7". However, none of these students in this classification drew a conclusion. It is feasible that the words 'compare and contrast' may have caused confusion for these students.

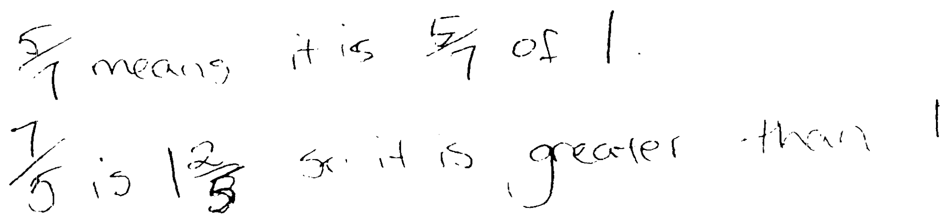
The next classification consisted of responses in which students chose to draw diagrams in an effort to compare the two fractions. This strategy often led to confusion as one student (MP) indicated:



A further example in this category is provided by another student (DB) who drew the following diagram. The student indicated the relationship between fractions and 'concrete' objects, such as pies:

This visualisation of fractions was a theme continued throughout this classification, and at least one student (SJ) 'saw' the fractions as representative of cakes. The student wrote: "5/7 is like cutting a cake into 7 slices and taking 5. 7/5 is cutting two cakes into 5 slices and taking 1 whole cake and 2 slices". It is also worth noting that none of the above students were able to complete part (b) of this question successfully.

The responses classified as 'compared each fraction to the number 1' used a multiple stepwise approach. Individual fractions were first compared to a whole. Decisions were then made based on how much of the whole each fraction covered individually. Finally, the two original fractions were then ranked. For example, one student (GS) wrote:



For some students, this multi-step process, caused problems. They found it difficult to keep track of the steps and would invariably become lost, as the following example (by student HP) indicates:

$\frac{5}{7}$ = is an part of 7
7/5 there is not anough

A similar strategy was employed in the preliminary study in Question 5 where students were asked to plot 3/5 and 1 1/5 on a number line.

The second last category consisted of a comparatively small number of students (n=13) who accessed alternative strategies involving conversion to percentages, decimals or common denominators before comparing the original two fractions. For example, one student (RS) wrote:

%100 ÷ 7 × 5 = 71.5% or 0.715
100 ÷ 5 × 7 = 140.0% or 1.40
the difference 's value can clearly be seen when transferred to percentage or decimals

It was only in the last category ('stated 5/7 is smaller or 7/5 is larger') that the responses indicated that a direct comparison between the two numbers was possible. The responses in this category stand out due to the absence of any intermediary steps.

Overview

There were two main approaches used by adult learners to compare the fractions 5/7 and 7/5. Some students drew diagrams or conjured up images of 'wholes' which enabled them to compare the two fractions. Other students related fractions to numbers, such as percentages or decimals. Only a very few students (n=12) could provide a direct answer to this question, i.e., 7/5 was bigger than 5/7. It is possible that the words 'compare and contrast' may have caused confusion for some students.

There was strong evidence ($\chi^2=16.81$, d.f.=6, $p<0.01$) that there was an association between the categories of responses and the group to which a student belongs. The AD students performed better as a group than those who attempted the TP program. This implies that older students or students who have been absent from school for longer periods of time may have more difficulty interpreting and responding to questions of this nature.

QUESTION 2b

This was the second part of question 2 and again asked students to compare two fractions. However, this part of the question included fractions which were both less than 1.

Q2b Compare and contrast what is meant by $\frac{2}{3}$ and $\frac{3}{5}$?

Table 6.3 presents eight broad categories of responses to Question 2b.

TABLE 6.3

Summary of adult learners' responses to Q2b on the Fraction Quiz

RESPONSE	NUMBER OF STUDENTS AD	TP
No response	9	15
Responses that require an interview for clarification	3	3
Wrote two parts out of three	14	7
Drew a diagram to represent 2/3 and 3/5	2	9
Compared two fractions to a whole and noted that both are less than a whole	2	4
Compared both fractions to a half		
. unsuccessful conclusion	2	1
. successful conclusion	1	0
Converted to a common denominator or to a percentage	15	10
Stated 2/3 is larger than 3/5 (no working)	7	3

Typical Responses

The nature and style of the responses to the question were very similar to that in the previous question. To avoid repetition, only brief comments are given where they offer a different or clearer perspective to that provided previously.

There were a number of responses (n=30) which were unable to be placed into any major classifications. This was a slight increase from the equivalent part of the previous question. For example, one student (JP) wrote: "The fraction 2/3 is greater even though the addition of 2 and 3 is less than the addition of 3 and 5".

Again, some students just wrote 'two parts out of three', while a small number of students still attempted to sketch the two fractions in order to compare them. In this case, this was not a particularly successful strategy since both fractions 'look' to take up similar areas when represented diagrammatically.

Another unsuccessful strategy was that used by six students who attempted to compare the fractions with the number 1. This 'whole comparison' was also noted in part (a), and this enabled correct decisions to be made. However, in part (b), since both fractions were less than 1, this strategy was not very useful. Some students appeared to modify this strategy, and compared the individual fractions to a 'half', before drawing a conclusion. This strategy was successful on only one occasion, i.e., one student (GS) wrote: "3/5 is the dividing line between 1/2 and 2/3. 2/3 is the dividing line between 1/2 and 3/4".

Twenty-five students used standard techniques, such as common denominators or percentages, to arrive at the correct answer. There was only one person who used common denominators and also drew a diagram. However, it appeared that the diagram was superfluous to the outcome of the problem. Instead, it looked as if the student had drawn the diagram to exemplify the fraction just as a teacher might do in a typical classroom lesson. Only in the last classification was there no working shown.

Overview

In general, the two different approaches used by adult learners were noted in responses to this question. However, there was also the additional category in which students compared the given fractions in the question to the number 1/2. It is feasible that this was a modification of the previous category in which students compared the two given fractions to the number one. It is feasible that students who used this technique realised that, when compared to a whole, both fractions appeared to cover approximately the same amount of area, and was therefore not a very useful strategy to employ.

There were no differences observed between the two different groups of students ($\chi^2=13.84$, d.f.$=8$, $p<0.086$), although there is a trend favouring the students in the AD group. It is worth noting that there was an increase in numbers for both groups of students in the 'no response' category, indicating that both groups of students found this question to be more difficult than its companion question (Q2a).

QUESTION 3a

Question 3a was the first part of Question 3 which asked students to rank three fractions from smallest to biggest. All of the fractions in this part of the question had 1 for a numerator.

Q3a Place in order from smallest to biggest $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{4}$

There were six broad categories observed for the responses to Q3a. These are tabulated in Table 6.4

TABLE 6.4

Summary of adult learners' responses to Q3a on the Fraction Quiz

RESPONSES	NUMBER OF STUDENTS AD	TP
No response	7	2
Responses that require an interview for clarification	8	0
Wrote one part out of etc.	2	0
Drew diagrams		
. stated fractions were different	2	0
. ranked in correct order	2	0
Focused on one fraction only e.g., 1/2 is the largest	5	0
Ranked in correct order		
. converted to percentages, decimals or common denominators	13	8
. no reason given	16	42

Typical Responses

Once again there was a degree of similarity in the type of categories identified in Q3a compared to Q2a and b. For example, some students still focused on the look of a fraction and described fractions as 'one part out of ...'. Some students still sketched diagrams of fractions although this category was not as popular as in the two parts of the previous question. Clearly, a majority of students could rank the three fractions successfully.

Overview

The added difficulty of arranging three fractions, rather than two, did not make this question more difficult. It is possible that having a 1 in the numerator was the telling factor.

There were significant differences between the responses provided by the two different groups of students (χ^2=27.29, d.f.=5, p<0.00). For example, there was a much broader spread of answers observed from the Associate Diploma group compared to the Tertiary Preparation students. In addition, a majority of Tertiary Preparation students answered this question successfully without supplying any reasons. It is possible that some students obtained the correct answer to this question without utilising sound mathematical procedures, simply because the question was a comparatively simple one. (This issue is addressed later in the chapter).

QUESTION 3b

This was the second part of Question 3 and asked students to rank three fractions from smallest to biggest. However, this part of the question contained fractions with numerators other than 1.

> ***Q3b Place in order from smallest to biggest*** $\frac{2}{3}, \frac{5}{7}, \frac{3}{4}$

There were seven broad categories for the responses to part (b) of Question 3. These are tabulated in Table 6.5

TABLE 6.5

Summary of adult learners' responses to Q3b on the Fraction Quiz

RESPONSES	NUMBER OF STUDENTS AD	TP
No response	18	7
Responses that require an interview for clarification	8	0
Wrote two out of three	1	0
Drew diagrams	2	1
. and stated fractions were different	1	0
. and ranked in correct order	1	0
Focused on only one fraction, e.g., 3/4 is the largest	4	0
Stated an unusual order, e.g , 5/7, 3/4, 2/3	6	24
Ranked in correct order, e.g., 2/3, 5/7, 3/4		
. converted two of the three fractions and then compared the third to one of the other two	3	2
. converted to percentages/decimals or common denominators	9	7
. no reason given	2	11

Typical Responses

Again, typical responses confirmed many of the observations made in the previous questions. However, there was the additional 'unusual order' category, in which a majority of the responses to this category ranked the fractions on the basis of the size of the denominator, e.g., the bigger the denominator, the smaller the fraction. This category was not singled out in the previous question since the fractions involved all had 1 in the numerator, and hence would have yielded correct answers if this strategy had been used and not stated.

Some students, who used decimals and percentages, still applied a multi-stage approach to comparing the fractions, e.g., converting two of the three fractions and then comparing the third fraction to each of the others.

Overview

Overall, this question provided valuable information with respect to adult learners strategies for comparing three fractions. In particular, the overall number of adult learners who could rank in correct order three fractions, which did not contain 1 in the numerator, diminished considerably between Q3a and Q3b. It is possible that students who used the 'bigger denominator, smaller fraction rule' ranked the three fractions successfully in the first part of the question without explaining their reasoning.

The responses to this question also indicated that there was a strong association ($\chi^2=31.44$, d.f.=6, $p<0.00$) between the type of response received and the group to which the student belonged. This implies that students who have been absent from formal study for several years may have forgotten appropriate techniques, such as using common denominators, for comparing fractions that have numerators other than 1. It could also explain the apparent success many Tertiary Preparation students had with Q3a, but not Q3b.

DISCUSSION

In general, there were a number of similar strategies observed in the responses for the questions (Q2a, Q2b, Q3a, Q3b) in this theme. Some strategies, even when they became inadequate or deficient continued to be selected by the students. This implies that there were only a limited number of approaches from which individual adult learners were able to select. In general, these strategies appeared to fall into three distinct groupings. First, there were the responses that included diagrams as if they were the only source available to the student to answer the problem. For example, one student (MV), who drew diagrams to compare 2/3 and 3/5, could not suggest an alternative method that did not involve or depend on a 'visualisation' of a concrete reference in some form. The following extended transcript demonstrates this. Key phrases have been shown in boldface print.

MV: *2/3 is bigger, visually* (drew two circles).
I: *How do you know?*
MV: ***It seems bigger. Visually.***
I: *What if you didn't have the diagrams, could you tell then?*
MV: *No! It would be very difficult for me.*
I: *What would you do?*
MV: *OK.* (pause). ***No, outside from actually drawing the circles and visualising them that way?*** *Unless I had something else to compare them to.*
I: *What do you mean else? What else would you want to compare them to?*

MV: *That fraction* (points to 2/3) *it seems on the surface to be very close* (points to 3/5). *Let me think on this one. Two-thirds. Three fifths. Now it would be difficult.* ***I would need to visualise it.*** *Another way? If my life depended on it?* (long pause). *I'd measure a line! Once again, it would be visually. Actually measuring a line. That's the way I think I'd do it. Measure a line. I would need to get some sort of visual comparison like that. Just* ***numbers on paper would be difficult for me to visualise.***

I: *Could you give me another way?*

MV: ***Outside of visually?*** (Interviewer nods). (pause). *Just off the top of my head* ***I don't think I could.***

The above student experienced difficulty with treating fractions solely as numbers. However, this was not an isolated incident. One other student interviewed (KL) also experienced considerable difficulties in completing part (b) of Question 3. This was primarily because the student based her solution on the diagrams she drew. Unfortunately, her ability to sketch accurate diagrams was limited and, hence, the diagrams impeded, rather than supplemented, her progress to a solution. Despite this, during the interview, the student continued to focus on the (now multiple versions of the) misleading diagrams, as if this strategy was the only one available to her. At no stage did the student suggest the use of common denominators, decimals or percentages to solve the problem. She depended on the diagrams and, in this case, they did not support her.

The second type of response appeared to rely on a rote learned 'rule', which was applied across all four questions, although it was more prominent in the responses to Q3b, since this was the first time it produced wrong answers in the Quiz. As one student (AB) explained, with respect to Q3a: "It could easily be believed that 1/4 is bigger than 1/2, as we are brought up knowing 4 is bigger than 2. But in fractions the reverse is correct for the denominator". While another student (AM) wrote: "These numbers are sometimes confusing. Looking at them, one may be tempted to think that 1/4 is the largest number when in fact it is 1/2. The large lower numbers can confuse". Despite the concern shown, the widespread application of the 'bigger denominator, smaller fraction rule' could result in many successful answers. For example, during an interview, one student (LT) applied this 'rule' which enabled her to obtain a large degree of success for the first few problems.

I: *Which is bigger of these two* (points to Q2, part (b)).

LT: *This one is bigger* (points to 2/3).

I: *Do you know why?*

LT: *Well I've been told it is in a fraction the bigger the number in on the underneath number, the value is smaller. The value goes smaller. Then I gauge. This one* (2/3) *is bigger than this one* (3/5).

However, this method does not provide an appropriate order for the fractions in which the numerator is not 1, e.g., Q3b; and was the most common 'unusual order' response noted to the question, i.e., 2/3, 3/4, 5/7. Unfortunately, the above student (LT) continued to apply the 'rule' routinely. At no stage, during the interview, did the student become aware of the inappropriateness of the rule in certain circumstances. Instead, the student had complete faith in it, as the following brief transcript indicates:

> *LT:* (points to her answer in Q3b). *The higher the number* (denominator) *it is, the value is less. That's exactly correct!*
> *I:* *Can you think of another way you might work that out?*
> *LT:* *Yeah, you make a diagram. And you can divide that into seven parts.*
> *I:* *Do you want to have a go?*
> *LT:* *Not really.*
> *I:* *But you could make a diagram?* (to confirm the above?)
> *LT:* *Yeah, you could make a diagram and you would see* (indicating that there was no doubt that the diagram would confirm the 'rule').

Clearly, the student did not know why the rule was 'correct'. The important conclusion to be made from this observation is that some students have absolute 'faith' that success with a strategy in one situation can be generalised to others. It is plausible that this approach has been reinforced by certain examples and types of questions. For example, the bigger denominator, smaller fraction rule, when applied to fractions with 1 in the numerator guarantees success, irrespective of the number of fractions involved. Ranking unitary fractions, however, takes up a comparatively small part of a more general process which enables all fractions to be compared. At least in LT's case, this broadening of scope has not happened. The student did not doubt her ranking of fractions and was convinced that any other method, e.g., diagrams, would only confirm her rule.

Third, there was a small number of responses which compared fractions, but only by introducing a third (or fourth) number, such as 1 or 1/2, with which to compare the fractions in the problems. Of particular interest, are the students who compared each fraction to 1 in Q2a; and continued to do so for the other question. Although this was an acceptable strategy for Q2a, it was inadequate for the other problem.

Finally, the use of diagrams decreased with the complexity of the problem. Responses which used common denominators, if not consistently, at least for Q3b, were not usually accompanied by diagrams. There is evidence that at least one student wanted another method other than diagrams, such as common denominators, to solve Q3b, but was not able to remember how to calculate them. She wrote: "I can't picture which is larger and have no idea how to change them to find out. Do I find the common denominator which would be a factor of 7? What is it?".

IN-CONTEXT QUESTIONS Q7 AND Q9

This part of the qualitative analysis investigates the responses to the in-context questions related to the theme comparison of fractions. Again, summary tables are presented followed by typical responses, and a discussion section at the end of both questions is presented.

QUESTION 7

Question 7 provided students with a situation in which employing fractions would be an appropriate technique to use.

> ***Q7 You have two recipes to choose from to make a drink of punch for a party. One recipe calls for 3 bottles of sherry and 6 bottles of soda water. The other calls for 2 bottles of sherry and 5 bottles of soda water. Which is the stronger drink. Why?***

There were six categories identified and these are displayed in Table 6.6

TABLE 6.6

Summary of adult learners' responses to Q7 on the Fraction Quiz

RESPONSE	NUMBER OF STUDENTS AD	TP
No response	2	3
Responses that require an interview for clarification	0	4
Stated:		
. 2 bottles stronger	2	3
. they are equal	4	2
. first recipe (or 3 bottles, no reason given)	11	6
Concluded that three bottles are stronger:		
. compared 1:2 and 1:2.5	6	4
. compared 3/6 = 1/2 to 2/5 which is < 1/2	16	17
Concluded that three bottles are stronger		
. compared 3/6 to 2/5 by using common denominators	10	10
. compared 3/6 to 2/7	1	1
. compared 3/9 to 2/7 by using common denominators or percentages	3	2

Typical responses

Although this question placed fractions in a familiar context, a considerable number of students chose not to utilise fractions when given the choice, or could not determine the fractions appropriately. In some cases, students concluded that there was no change. For example, one student (KL) wrote: "They would be the same because the sherry has been decreased by one but so has the water". This student chose not to consider the different contributions from the sherry and the soda water, i.e., the student appeared to focus on only one aspect of the problem and could not take any more elements of the situation into account.

Seventeen students selected the first recipe because they only considered the extra bottle of sherry. For example, one student (DS) wrote: "Three bottles of sherry and six bottles of soda water is stronger because there is an extra bottle of sherry which makes the alcoholic percent larger and one bottle of soda water more won't make much difference compared to a bottle of sherry". This implies that some students may have arrived at the correct answer by guessing, arbitrarily selecting the first recipe, or relying on experience.

The next category consisted of evidence which suggested that students preferred to use ratio strategies, and not fractions, to solve this problem. Although this is an acceptable strategy, it was rather unusual in a 'test' on fractions. This was despite the fact that students where informed that the quiz was investigating fraction understanding. The most common answer consisted of comparing 1:2 to 1:2.5 and concluding that the first recipe was stronger. When fractions were used, a common approach was to compare 3/6 to 1/2 and then 2/5 to 1/2. For example, one student (SK) wrote: "3 bottles of sherry to 6 bottles of soda would give you half strength. 2 bottles of soda to 5 bottles of soda water would give you less than half strength. 3/6 = 1/2, 2/5 = 4/10". This 'halving strategy' was also observed in other questions noted previously.

The final category consisted of responses that did use fractions, (usually with common denominators), although some students had difficulty considering the contribution of each partial volume to the total volume. Frequently, only the total volume for one recipe was considered. For example, one student (DC) wrote: "3/6 = 15/30, 2/5 = 12/30 make common denominator. Recipe with 3 bottles of sherry". It was not until this category that students took into account the correct contribution of the total volumes of the drinks.

Overview

The responses to this questions revealed an important point with respect to fractions - some students avoid them when given the opportunity. This was despite the obvious setting of the question into a quiz based on the understanding of fractions. Clearly, a sizeable number of students preferred to utilise ratios rather than fractions to address this question. Several possibilities present themselves, namely, it is feasible that some adults were more familiar with ratios than fractions, students found ratios conceptually easier to deal with than fractions, or students preference was to avoid the use of fractions. There were no obvious differences between the two groups of students (χ^2=5.50, d.f.=4, $p<0.24$) and their strategy of response.

QUESTION 9

Question 9 provided students with a situation in which two fractions (1/5 and 1/3) could be compared in a non-traditional (non-textbook) way.

Q9 Two people, who have different occupations, save a certain part of their salaries each week. The first person saves 1/5 of their salary. The second person saves 1/3. Is it possible for each to save the same amount? Give details.

There were ten broad categories associated with this question, and these are presented in Table 6.7

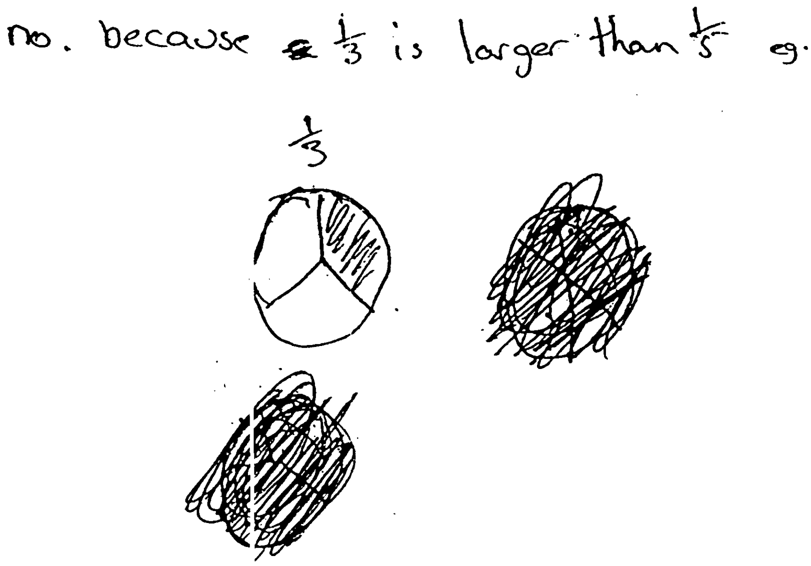
TABLE 6.7

Summary of adult learners' responses to Q9 on the Fraction Quiz

RESPONSE	NUMBER OF STUDENTS AD	TP
No response	4	9
Responses that require an interview for clarification	3	7
Answered 'No' and assumed 1/3 > 1/5 always	6	6
Answered 'No' and assumed the wages were the same, but calculated (using LCD's = 15's) that first wage earner needed to increase their contribution by 2/15	6	2
Stated it was possible, but did not provide reasons	8	3
Stated it was possible, if the wages were different, but did not provide reasons for how different	4	0
Stated it was possible, if the first wage earner > second wage earner	7	7
Stated the above, but also provided an example to indicate the two respective wages, i.e., to indicate that the student had an overview of the problem	9	12
Stated it was possible if 1/5 of the first person's salary = 1/3 of the second person's salary	5	4
Stated it was possible if second's salary is 5/3 times the first's salary	3	2

Typical responses

The category that chose to respond by writing "No. 1/3 is > 1/5" appeared to have a very strict and rigid concept of what 1/3 and 1/5 'looked like'. For example, one student (DS) drew the following diagrams, in which he attempted to represent the two fractions. The student wanted to sketch 1/5 (in the scratched out diagram), but could not internalise the structure of 1/5 well enough to reproduce it on paper.



The next category consisted of responses which indicated that it was 'possible, but assumed the wages were the same'. These responses attempted to counter the apparent inequalities of this statement by calculating the difference (2/15) and stating that the first person must save more. A small number of students 'corrected' the inequity and suggested that the difference could be made up if the first person "saved over a longer period of time". These responses indicate a lack of unawareness of the implications of this approach, presumed that the other would need to work less, or lose their job temporarily in order to let the first person 'catch up' to seek real-world practical solutions.

The next two categories consisted of responses that indicated that the people could save the same amount, but did not provide reasons. This may suggest that these students could have been guessing. In contrast, the second type of response indicated that the students attempted to prove their decisions. The responses indicated that the students wanted to know the wages of the people but, since this information was not available to them, they were unable to proceed without this information. For example, one student (CF) wrote: "Yes it is possible to save the same amount and express it as a fraction but I would need to know the gross salary to work out which fraction of one would be equal to what fraction of the other".

Responses in the next category stated it was possible for two people to save the same amount, and provided (their own) examples to demonstrate this. For example, "1/5 of $150 = $30 and 1/3 of $90 = $30". All the students in this category concluded that it was possible for the two people to save the same amount on the basis of their examples. Only one student selected an inappropriate example, but she was still able to conclude that it was possible.

Responses such as "1/3 of A's salary = 1/5 of B's salary" (student TA) were typical responses in the second last classification, with the algebraic equivalent of this statement occurring in the final section. These last two categories were very different from the others in that the students did not need any further information from what was already contained in the question. In other words, they could provide an 'abstract generalisation' as their response to the question.

Overview

In general, the responses to this question indicated that some students had a notion that 1/5 and 1/3 were of a 'fixed' size that could never vary. Other students wanted knowledge of the different wages. Some of these students appeared to believe that since this information was not given, it was not possible to solve the problem. It is worth noting that some students were able to reach a conclusion by generating their own wages, and, in general, the wages selected by the students did not appear to be random. They were selected on the basis that 1/3 of one wage would equal 1/5 of the other. It was not until the final category that responses indicated this relationship completely in the absence of specific wages. There were no differences observed between the two groups of students ($\chi^2 = 12.43$, d.f. = 9, $p < 0.19$) and their method of response.

DISCUSSION

Both of these in-context questions (Q7, Q9) revealed that, when fractions were employed, students frequently had difficulty taking into account all aspects of the problem. For example, it was not until the second last category in the responses to Q7 that the contribution to the total volume of each partial volume was considered. A possible reason for these responses may be found in the difficulties students face in manipulating the fractions (comparing 3/9 to 2/7 in Q7). However, a more plausible explanation may be that students selected the most appropriate strategy with which to solve this problem, i.e, ratio. This would imply that students find ratio is a more useful concept in their everyday lives than fractions, and, hence, requiring a lighter cognitive load. As a consequence, this question was very difficult to analyse in terms of fraction concepts due to the use of ratio techniques by many students.

However, the results to Question 9 produced several distinct levels of responses, which gradually increased in complexity. Responses indicated that it was not possible for two people to save the same amount, were closed to the idea that 1/5 could be greater than, or equal to, 1/3, i.e., they did not consider that two different salaries could yield two equal sub-parts, or they assumed that both the salaries were the same. For example, one student (MS) stated the following during an interview:

I: *What did you decide?*
MS: *No, they couldn't.*
I: *Why couldn't they?*
MS: *One third is taking one third of a whole wage and this* (points to 1/5) *would be less.*
I: *Always?*
MS: *Well, yeah, because if it was $200 and you're taking one third of it, the one fifth would be less.*

This approach continued in the following category, although the responses suggested that students were starting to question the original assumption and were investigating various methods that would bring about the possibility that the two people could save the same amount. There is some evidence, provided by one student (LP) during an interview, which indicated that the student appeared to be in a transition between acknowledging that it was possible, but did not draw all the information together to defend her intuitive response. Instead, the student became confused and regressed to the earlier category. The student stated:

LP: *I don't think that's right* (her original answer was 'yes' but no details were provided).
I: *Why?*
LP: *Because it doesn't make sense. And I don't know why. Because I worked out that's what he's saving. I want the problem right in my head again. First - OK that person's saving one fifth and that person's saving one third. So that's one less than the other. But I mean they can save the same amount. Yes, I agree with that. But I can't work out how much more. I don't think that's right. In fact, I'm pretty sure that it's not right. But I can't arrive at the way my logic is for saying yes.*

However, the acknowledgment that two people could save the same amount was a feature of the next few categories, with students becoming more aware of the possibility and the underlying mathematical reasons for this. One student (MV), during an interview, indicated that, provided he could invent appropriate wages, he could reach the same conclusion, almost irrespective of the fractions provided. The student's technique involved working backwards, to arrive at appropriate wages for the two people. This was the only method the student had to test the original context of the question. Without the provision of wages, the student may not have been able to address the question as the following transcript indicates:

MV: *What I did was I just took a number there, 50, it could have been 10, it could have been anything and then I multiplied by 5 and the second one I multiplied by 3.*
I: *So you worked backwards?*
MV: *Yes. I don't know how I would have worked that out another way. Unless I had a set figure of course, then it would have been a bit different.*

It was only in the last two categories that the responses no longer seemed limited in any form to the context of the problem. These students were able to generalise, irrespective of the selection of fractions involved.

GENERAL CONCLUSIONS

In general, the combined results from the context-free (Q2a, Q2b, Q3a, Q3b) and in-context questions (Q7, Q9), indicated that adult learners' responses to fraction questions can be placed into a notional hierarchy. Once again, some students were observed to treat fractions as concrete objects. However, these types of responses were more prevalent for the easier fraction questions, such as Q2a and Q3a. By the time students had reached the in-context questions (Q7 and Q9), the use of diagrams was starting to decrease, however, this was accompanied by a corresponding increase in the number of non-responses. It is, of course, possible that some students were tailoring their response to suit the level of the question. However, this does not seem to apply for all students who responded in this way, i.e., there were still some students who continued to rely on diagrams irrespective of the problem, and even when this technique was unlikely to bring success because the diagrams became too difficult to interpret. In addition, there was the application of the smaller denominator, bigger fraction rule, which appeared to replace or supplant the use of diagrams in Q3a and Q3b. Finally, when faced with a choice, many students chose not to use fractions at all, e.g., Q7.

QUANTITATIVE ANALYSIS

In order to facilitate the quantitative analyses, the responses summarised earlier in Tables 6.2 to 6.7 have been restructured so that the number of categories was condensed to four, and coded 0 (for the lowest response category) through to 3 (for the highest response category). In each case the compression of categories was addressed so as to combine equivalent categories, and, hence, maximise diversity. These modified tables, which indicate what categories were combined, can be found in Appendix F. This procedure was necessary so that the question responses were in an appropriate form for the application of the QUEST package.

RASCH ANALYSIS

This section of the work focuses on the fit of these data to the Rasch model. For this theme, the Infit Mean Square value was found to be 1.01 with a standard deviation of 0.17. The Infit-t was 0.09 with a standard deviation of 1.30. These results imply that the model is appropriate to use with respect to the above data. Individual Infit Mean Squares can be calculated for each of the above six questions. These values are presented in graphical form (Figure 6.1) and indicate the degree of proximity to perfect association (i.e., 1). Item estimates that lie within the two vertical dotted lines are considered acceptable (Masters, 1982).

```
------------------------------------------------------------------------
INFIT

MNSQ          .63       .71       .83      1.00      1.20      1.40
--------------+---------+---------+---------+---------+---------+-------
    item 2a                  .        *     |              .
    item 2b                  .     *        |              .
    item 3a                  .              |     *        .
    item 3b                  .            * |              .
    item 7                   .            * |              .
    item 9                   .              |              *
========================================================================
```

FIGURE 6.1

Map of Item Fit for the Comparison of Fractions theme

All of the six questions fell within acceptable limits. Overall the data fit the model and can therefore be used to investigate the placement of categories in the proposed hierarchy. This involves interpreting Thurston Threshold values and Difficulty values using the Tau option.

THRESHOLD VALUES

In the Quest package the 'threshold' value of a particular item response or student (case) estimate can be read from the logit scale on the far left of Figure 6.2 (exact values for both these measures can be found in Appendix G.) On the right hand side of Fig. 6.2 are the response categories. As in the previous chapter, only the top three response categories are given, e.g.. in the case of Q7 there is 7.3, 7.2 and 7.1. The lowest response category (in this case 7.0) is not provided as it serves as baseline data.

Each X represents a particular student from the sample. The left hand side of Fig. 6.2 illustrates the distribution of the students' performance over the logit scale. The student (represented by a particular X) has a 50% chance of being able to provide the response category of an item located at the same logit score, and, consequently, a better than 50% chance of reaching a response category for answers below this score.

```
-----------------------------------------------------------------------
Logit Scale  Student Estimates        Item Response Category Estimates
-----------------------------------------------------------------------
                       XXXXXXXX  |
                                 |
  2.0                            |
                                 |
                                 |
                                 |
                                 |
                           XXXX  |
                                 |
                                 |
                                 |
                                 |     7.3
                        XXXXXXX  |    2a.3
                                 |
  1.0                            |
                      XXXXXXXXX  |
                                 |    3b.3   9.3
                                 |    2b.3
                          XXXXX  |
                                 |
                         XXXXXX  |    2b.2
                       XXXXXXXX  |    2a.2
                                 |
                    XXXXXXXXXXX  |
   .0                            |
                     XXXXXXXXXX  |     9.2
                                 |    3b.2
                          XXXXX  |
                                 |     7.2
                                 |    3a.3   3b.1
                   XXXXXXXXXXXX  |
                                 |    3a.2
                      XXXXXXXXX  |    2b.1
                                 |
                           XXXX  |    3a.1    9.1
                                 |
                                 |    2a.1
                            XXX  |
 -1.0                            |
                                 |
                              X  |
                                 |
                                 |
                                 |
                              X  |
                                 |   7.1
                                 |
-----------------------------------------------------------------------
Each X represents 1 student
=======================================================================
```

FIGURE 6.2

Map of Thresholds for the Comparison of Fractions theme

The data indicate that the coding of each question, i.e., the allocation of response categories was consistent. This can be seen by the bunching of .3 category responses and .2 and .1. Since the original coding was on the quality of the response, it implies that there is some underlying structure to the groupings. This would appear to support the notion of levels. There were no clear distinctions between the responses to context-free and in-context questions.

The only item not to conform to this was item 3a. From Fig. 6.2 it can be seen that there is little change between response category 3a.1 and 3a.3 and that the top category of response is much easier to achieve than the top response category for other questions. These data provide further confirmation of the anomaly associated with this question, identified previously. The graph provides clear evidence of the impact of students applying an incorrect rule (larger denominator, smaller fraction) to obtain the correct answer.

Overall, the response to Question 7 (response category 1) was the easiest to reach. This is possibly due to the fact that many students could simply guess an answer, such as "two bottles are the stronger". In general, it was felt that this type of response was outside the range expected from adult learners since it does not appear to relate the answer to the question. The next small collection of response category 1's (2a.1 to 3a.1 and 9.1) appeared to be focused on diagrams, or provided answers which treated fractions as if they were of a fixed size e.g., 1/3 > 1/5 as the response to Q9.

Following this stage, was a collection of response category 2's (2b.1 to 9.2). These type of responses indicated that fractions were becoming more complex, i.e., there was a mixture of responses in which diagrams were starting to decrease, but other strategies, such as the 'bigger denominator, smaller fraction' rule were on the increase. In addition, some adult learners were beginning to focus on only one fraction, when two or more were presented as part of the problem, or would compare the given fractions to a constructed 'whole' (2a.2 and 2b.2).

It was only in the final stages (response category 3's) of all of the questions (2b.3 to 7.3) that fractions were treated as numbers, e.g., common denominators, percentages or decimals were used in answering the questions.

DIFFICULTY VALUES

Table 6.8 contains the overall difficulty and step difficulty values for both the context-free questions (Q2a, Q2b, Q3a and Q3b) and the in-context questions (Q7 and Q9).

TABLE 6.8

Overall Difficulty and Step Difficulties for Questions in the Comparison of Fractions theme

	Question	Overall Difficulty	Step Difficulties 1	2	3
Context-free	2a	.23	-.78	.28	.50
	2b	.22	-.40	.95	-.54
	3a	-.51	1.01	.51	-1.52
	3b	.18	1.04	-1.50	.45
In-context	7	-.17	-1.11	-.12	1.22
	9	.05	-.15	-.26	.41

The easiest question was Q3a (-.51) which asked students to rank 1/2, 1/3 and 1/4. However, the table suggests that students had difficulty starting (Step Difficulty 1 was 1.01) and continuing (Step Difficulty 2 was .51) this question. Step 1 was associated with moving from no response (response category 0) to drawing diagrams of all three fractions (response category 1). Step 2 was associated with moving from the diagrams to focusing on only one of the fractions, such as 1/2 (response category 2). It is almost as if the extra encumbrance of having to deal with three fractions made these 'simple' techniques unwieldy. Instead, it appears as if many students have opted to utilise other methods, e.g., percentages or decimals or just stated the answer. Such techniques are associated with reaching the final response category and, as the table shows, indicates that very few students had difficulty in reaching the final response category (Step Difficulty 3 was -1.52) in this particular problem. This suggests that fractions that contain 1 in the numerator or simple numbers, such as 2, 3 or 4 in the denominator are comparatively easier to deal with than other fractions.

The next easiest questions were the in-context questions (Q7 and Q9), although students found Question 9 more difficult than Question 7. This is possibly because Question 7 could be answered correctly by using ratios. The table suggests that many students may have guessed an answer to Q7 (response category 1) (Step Difficulty 1 was -1.11) without considering the wider issues surrounding the context, such as an increase in the overall volume. Step 2 (Step Difficulty 2 was -.12), or reaching response category 2, indicates movement towards solving Q7, ratios were employed, although the 'new' total volume was often still not taken fully into consideration. In contrast, very few students could reach the final response category of this calculation (Step Difficulty 3 was 1.22). This is associated with having an overview of the problem and required students to treat fractions as numbers and to take into account other aspects of the problem such as partial volumes.

In Q9, students also found starting (Step Difficulty 1 was -.15) this question and continuing this question (Step Difficulty 2 was -.26) relatively easy. Step 1 equates to moving from not addressing the question (response category 0) to stating that two people could never save the same amount if they were saving two different fractions of their respective wages (response category 1). Step 2 develops from this response category to acknowledging that it was possible, even if they did not state how (response category 2). In contrast, Step 3 (Step Difficulty 3 was .41) indicates that very few students could state how it was possible that two people could save the same amount (response category 3). It is worth noting that this response is similar to the final response category in Q7 which required students to take into account the comparative volumes with respect to two different recipes.

The other three context-free questions were all of approximately equal difficulty. In addition, the considerable gap between Q3a and the rest of the questions would appear to confirm the observations suggested by the analysis of Q3a above. For example, students found it comparatively more difficult to compare two fractions, if the numerators were not 1 (Q2a and Q2b) (.23 and .22, respectively). Step Difficulty 1 (-.78 and -.40, respectively) for both of these questions indicates that students found it comparatively easy to move from not addressing the question (response category 0) to sketching diagrams of the fractions (response category 1), i.e., more students chose to sketch diagrams for two fractions (even if they did not contain 1 in the numerator). In contrast, the results (Step Difficulty 2 was .28 and .95, respectively) suggest that there is a considerable gap between response category 1 and the next response level, i.e., students found it comparatively more difficult to reach response category 2 in either of these questions. Response category 2 was associated with comparing the fractions to an intermediary number, such as 1 or 1/2. In the case of Q2a, students found it difficult to move to the final response category (Step Difficulty 3 was .50). This suggests that once an intermediary number is chosen, other options, such as percentages and decimals (response category 3) are not usually forthcoming. However, Step Difficulty 3 (-.54) for Q2b, appears to contradict this. It is possible that some students who selected '1' for the intermediary number in Q2a rejected this notion as inappropriate for Q2b and moved directly to percentages and decimals, i.e., response category 3.

In contrast to Q3a, students undertaking Q3b had considerably more difficulty in ranking three fractions (.18) that did not contain numerators equal to 1. In particular, students had difficulty moving from response category 0 (no response) to response category 1, i.e., Step Difficulty 1 was 1.04. Response category 1 is associated with drawing diagrams. It is as if many students realised that the sketching of diagrams was a cumbersome and ineffective technique for this particular problem. It is worth noting that students found reaching response category 2 comparatively easy (Step Difficulty 2 was -1.50). Response category 2 consisted of responses which either focused on only one fraction or provided an unusual order. These results suggest that many students simply gave up trying to conceptualise the three different fractions diagrammatically (response category 1) and simply guessed an (incorrect) ranking (response category 2). This would seem to be confirmed since students found reaching the final response category difficult, i.e., Step Difficulty 3 was .45. Response category 3 consisted of responses which ranked all three fractions in correct order.

GENERAL CONCLUSIONS

There were four main findings from this section. First, there was considerable evidence to suggest that students compare fractions with 1 in the numerator 'differently' to fractions that do not possess this feature. For example, the overall difficulty of Q3a indicates that students found it easy to rank 1/4, 1/3 and 1/2 in order. However, their choice of procedures on three other similar questions (Q2a, Q2b and Q3b) suggest that Q3a was treated fundamentally differently to the other three questions - all of which were of comparable overall difficulty. It is possible that three fractions chosen in Q3a were so familiar and easy to compare, that complex calculations were not required. This implies that students find fractions which have '1' in the numerator, or a small number, such as 2, 3 or 4 in the denominator, to be considerably easier to deal with than other fractions.

Second, students used different strategies to compare two fractions, in contrast to comparing three. This was observed even when the numerator did not contain '1'. For example, some students stayed with sketching diagrams or focusing on one fraction, irrespective of the number of fractions present. However, the step difficulties suggest that students found these approaches to be considerably more difficult (inappropriate) when dealing with three fractions, even when the fractions contained '1' in the numerator (Q3a). For example, students had difficulty reaching response category 1 in Q3b. This implies that some students abandoned this approach and simply guessed a ranking. For example, the most common unusual answer for Q3b, and which could have also explained the results to Q3a, was that some students applied the 'smaller denominator, bigger fraction' rule.

Third, there was some evidence to distinguish between in-context and context-free questions. In general, students found it easier to reach the final response categories of the context-free questions. [This is the same as the previous chapter]. It is also worth noting that the responses to Q7 indicated that, when faced with a familiar context, many students prefer to avoid fractions altogether.

Finally, the fourth finding suggests that responses to some questions, and Q9 in particular, indicate that there was a considerable gap to be bridged between treating fractions as objects and the ability to work with them as numbers. In addition, the results suggest that students do not have a clear understanding of how to compare fractions when they are placed into a context. These issues are now dealt with as an interpretation of the data, using the SOLO Taxonomy, is conducted.

A SOLO INTERPRETATION

The previous chapter has postulated the notion that there may be more than one cycle of responses in the concrete symbolic mode within which to interpret the adults' responses to fraction questions. This would again appear to be plausible with respect to the current theme of comparison of fractions.

RESPONSES PRIOR TO THE TARGET MODE

There were a number of responses which were unaware of the relevance of fractions to the question. For example, some responses to the drink-mix question (Q7) selected the recipe containing the two bottles of sherry without considering any other information. Other typical responses consisted of students writing 'one part out of four' to describe 1/4. Such responses are consisted with belonging to the ikonic mode, and do not belong to the target mode of the question (concrete symbolic).

TARGET MODE RESPONSES

The next group of responses did attempt to address the question asked and are classified in the concrete symbolic mode. Again, the first group of responses relied on a concrete object or diagrammatic representation with which to compare fractions. By contrast, the second group is characterised by the use of fractions. A description, with appropriate examples of each of the SOLO levels, spread across the two cycles is now presented.

THE FIRST UMR CYCLE

Unistructural 1 responses focused on only one aspect of the problem. For example, all the context-free questions had responses in which students concentrated only on the denominators, or focused upon one familiar fraction, e.g., '1/2 is the largest' (Q3a). Responses at this level are able to comprehend 1/2, 1/4, although there is some evidence that 1/3 was developed considerably later than 1/2 and 1/4, i.e., the logit scores for 3a.1 and 9.1 were much higher. Given this, it is feasible that other common fractions with a numerator of 1 were also envisioned at this level, e.g., 1/5 (from Q9). However, responses suggested that students had difficulties working with more than one fraction at a time, particularly if the denominators were not the same. It is plausible that students could appear to be proficient at these types of fractions because it is comparatively easy to visualise such fractions. Simple, and obvious comparisons, such as between 1/4 and 1/2 would be available at this level. Hence, students achieve success at comparing 1/3 and 1/5, since 1/3 will always be larger than 1/5, i.e., no student who responded at this level in answer to Q9 doubted that 1/3 was ever able to be smaller than 1/5. This is because, at this level, comparisons between 1/3 and 1/5 become 'fixed', and hence 1/3 is always larger than 1/5 as it is only concerned with a fraction of the same object.

This level is the first at which the 'bigger the denominator, the smaller the fraction' rule occurs. This is because, this level is concerned with the influence of the denominator on the fraction, and, hence, the numerator is usually 1. As a consequence, students who use the above rule, for questions at this level, are usually successful.

Multistructural 1 responses consisted of a multi-stage procedure, which involved the introduction of an already stable and fixed value, such as 1 or 1/2, e.g., Q2a and Q2b. Early responses in this level could only compare two fractions by first comparing each fraction, separately and independently to a whole or one, e.g., some responses to Q2a in which 5/7 was compared to 7/5 would compare each fraction to 1, before a decision was made. Later responses in this level indicated that the above procedure had been modified to enable two different fractions to be compared to an already familiar fraction, such as a half. No other intermediary numbers were noted for this technique in this work. For example, some responses in Q3b compared 2/3 and 3/5 to 1/2 before reaching a conclusion.

Responses at this level still focused on a visual representation of fractions. However, it is feasible that two fractions, with a numerator greater than 1 could be compared by the above method, provided that the fractions were visually different when represented on diagrams. This implies that only very simple fractions, such as 1/2 and 3/4, could be compared.

Responses at this level appeared confused and tended to focus on the wrong cue in the question. For example, some responses to Q9 stated that one person would have to increase their contribution to save the same amount, i.e., they are seeking diagrammatic equivalence, but cannot provide the name, only describe the technique. This would then enable the person saving 1/5 of their salary to 'catch up' to the person saving 1/3 of their salary. The process was based on the assumption that 1/5 and 1/3 can never represent the same amount, i.e., as in comparing these two fractions using same sized diagrams.

Relational 1 responses had an overview of how to compare fractions, but only diagrammatically. For example, some students were able to draw diagrams to represent difficult fractions, such as 5/7 compared to 2/3 (KL). However, typical responses at this level were more likely to avoid drawing tedious diagrams, preferring instead to approximate or guess at an answer, such as, '2/3, because it seems the biggest'. This level marks the start of students 'questioning' the dependence on diagrams and opens up the possibility of working with fractions as numbers which are independent of diagrams. This is because this is the first level to recognise, at least implicitly, that a traditional diagrammatic approach to this problem was inadequate, e.g., such as that posed in Q9. This equates with the notion that, for the first time, the wages did not necessarily have to be the same for both people. A typical response at this level (student KL) stated that it was possible, but, even during an interview, was unable to say how.

THE SECOND UMR CYCLE

Unistructural 2 responses were the first responses to reject the notion that 1/3 was always greater than 1/5, and provide a mathematical reason for their choice. For the first time, typical responses, which compared the wages of the two people, indicated that the wages had to be different for the two people, e.g., "is possible, but only if the wages are different". Responses at this level did not say how different.

Multistructural 2 responses consisted of students accessing common denominators and comparing two or more fractions (Q2a, Q2b, Q3a, Q3b). Typical multistructural 2 responses to Q9 assumed that the wages were different and attempt to quantify this statement by stating: "if the first wage earner earns more than the second wage earner, then it is possible". It is as if these responses wanted to know the wages of the people, but since this information was not stated in the problem, they were unable to proceed any further.

Relational 2 responses had a clear overview of the problem, and could work with fractions as numbers. In the case of Question 9, responses at this level could provide appropriate examples to solve the problem, e.g., they gave the two people appropriate wages that enabled them to answer the question correctly. The most common examples were $500 and $300 or $150 and $90, since the calculation of 1/5 and 1/3 of these numbers was relatively simple. All of the responses in this section 'knew' which numbers to chose for examples, unlike the previous level. This was the last level in which fractions were number dependent, or number constrained.

An important characteristic of the second cycle is that fractions are treated more like numbers and less like representatives of objects, i.e., responses focus on the manipulation of fractions as numbers and do not depend on diagrams. However, generality, in absence of numerical values, or the dependence on numerical values does not occur until the next mode, i.e., the formal mode.

RESPONSES BEYOND THE TARGET MODE

There were a small number of responses which were more sophisticated than those discussed above since they did not need actual numbers for the wages. In these responses students chose to use algebra as a tool to address the question. There was an identifiable structure to these responses which could be interpreted by unistructural, multistructural and relational levels within the first cycle of the formal (F) mode. These codings are consistent with those of Coady and Pegg (1994) when they explored students' responses to more difficult senior-secondary algebra questions.

Unistructural 1F responses were qualitatively better than those from the concrete symbolic mode, since they could address the question without recourse to providing actual wages. For example, one student (TF) wrote: "Yes. 1/5 of Mr A's wage could = 1/3 of Mr B's wage if Mr A earns more". This statement has spontaneously chosen to use algebraic concepts, but has not yet come to grips with the algebra.

Multistructural 1F responses, such as: "1/3 a = 1/5 b", indicated that some students, had been exposed to more algebra and could treat the problem in more traditional and abstract algebraic terms. Nevertheless, the students did not provide an overriding statement for the answer.

Relational 1F was the final category identified in the responses for this section of the work. It is only at this level, that responses could develop an overriding principle that incorporated the generalisation established in the previous level, e.g., students could apply algebra (which lay outside the question) to the problem and could conclude: "The salary of the first person should be 5/3 salary of the second person" as one student (DC) wrote.

Table 6.9 summarises the descriptions above and indicates typical responses at UMR levels across the two modes. Since the majority of responses to Q7 used ratio this question has not been included in the table.

TABLE 6.9

Summary of adult learners' responses to Questions
for the Comparison of Fractions theme

UMR	CONTEXT-FREE (Q2a, Q2b, Q3a, Q3b)	IN-CONTEXT (Q9)
<U1 (CS)	one part out of 4	
U1 (CS)	can order simple fractions with 1 in numerator, e.g., Q3a biggest denominator, smallest fraction 'rule'	Q9 - not possible, assumes wages are the same, i.e., 1/5 > 1/3 always (earliest possible level)
M1 (CS)	can order simple fractions with >1 in numerator, e.g., Q2a, compares 5/7 and 7/5 to 1 or 1/2	Q9 - is possible, assumes wages are same, i.e., person A saves longer
R1 (CS)	can compare more difficult fractions diagrammatically such as 2/3 and 3/5 (Q2b), but will prefer to avoid difficult diagrams and 'guess', e.g., 3/4 is the biggest because it seems that way (Q3b)	Q9 - is possible, but cannot provide details
U2 (CS)	attempts to 'play with numbers' as if completing a pattern	Q9 - is possible, states that the wages must be different, but unable to say how
M2 (CS)	equivalent fractions	Q9 - is possible, if 1st wage earner earns > 2nd wage earner
R2 (CS)		Q9 - is possible, if A earns $150 and B earns $90
U1 (F)		Q9 - is possible, if 1/5 of 1st wage earner = 1/3 of 2nd wage earner
M1 (F)		Q9 - is possible, if 1/5 a = 1/3 b
R1 (F)		Q9 - is possible, if second's salary is 5/3 times the first's salary

CONCLUSION

There were six main findings for this chapter. First, the responses to fraction questions involving comparing fractions formed a notional hierarchy, which could be interpreted using the SOLO Taxonomy.

Second, there were two main approaches adopted by students in answering fraction questions with respect to comparing fractions. The first approach consisted of responses in which students appeared to guess answers as if visualising the relative sizes of fractions. In some cases, fractions, such as 1/2, were selected irrespective of the fractions indicated in the problem. This familiarity with 1/2 showed up in it being used as a tool to solve problems. Other fractions that were commonly selected were 1/4, 1/3 and 1/5. In all of these cases, the responses implied that the fractions appeared to be of a 'fixed size' as if the respondents were looking at a diagram of such fractions. In some cases, responses used divided diagrams to indicate the fractions. It was as if the diagrams were the fractions to these students. Other responses included inappropriate rules, rather than work with fractions, in an attempt to explore answers to the respective problems. In contrast, the second approach focused on treating fractions more 'mathematically'. For example, common denominators were used to compare or rank fractions, irrespective of the numerator or denominator. Other responses, for example to Q9, suggested that responses now had an added flexibility and students could accept that 1/5 could be greater than 1/3 provided that two people had different wages.

Third, both qualitative and quantitative analysis suggest that many students avoid the use of fractions whenever possible. For example, the responses to Q7 indicated that many of the adult learners in this sample selected a ratio strategy rather than approach the question from a fraction perspective. It is plausible that these students were more familiar with ratios or that they simply did not like fractions. However, students were informed that the quiz was investigating fractions.

Fourth, the responses to three questions (Q2a, Q3a, Q3b) from this theme revealed significant differences between the two groups of students investigated. All of these questions were context-free and required students to compare fractions. Throughout these questions, there were fewer Tertiary Preparation students in the higher categories when compared to the Associate Diploma students. In general, Tertiary Preparation students either did not answer or produced a wider variety of incorrect rankings. In many cases, responses indicated that some students demonstrated a reliance on, and application of, inappropriate rules, such as the 'smaller denominator, bigger fraction' rule.

Fifth, a two cycle UMR approach in the concrete symbolic mode of the SOLO Taxonomy has been proposed for interpreting adult learners' responses to fraction questions with respect to the comparing fractions theme. The first group of responses which focused on a concrete object or diagrammatic representation were placed into the first UMR cycle. By contrast, the second group of responses, which characterised fractions as numbers, were classified into the second UMR cycle. In addition, there was also a small number of responses which could be identified within an early cycle in the formal mode. All of these responses came from the results to Q9, wherein students spontaneously chose algebra to address the question.

Finally, the trends noted in the previous chapter have continued throughout this chapter. Students had more difficulty (Step Difficulties 2 and 3) reaching response categories in which fractions were treated as numbers, than earlier response categories (Steps Difficulties 1 and 2), when the question was placed into a familiar context. This is consistent with the findings of the previous chapter. It implies that students find working at the highest levels in a familiar context question and their ability to work with fractions, to be one of the most difficult areas in which to achieve success.