INTRODUCTION

This study was an investigation of students' understanding of indices (exponents). In particular the emphasis was on persistent errors students make when working with indices, and on the reasons which explain their performance.

An examination of the literature relating to errors in index work showed little has been done in this area. A number of researchers (DeVincenzo, 1980; Childers, 1987) catalogued numerous errors and compared errors in arithmetic with errors in algebra. No real attempt was made to provide general explanations for these errors. Wilson (1985) explored students' understanding of indices within a framework developed from the theories of Bruner, Skemp, Krutetsksi and Gagne. He found that students, when working with indices, made errors because of a lack of genuine understanding of the rules they had acquired. From an examination of students' written work, Shevarev (1946) drew conclusions about the processes students use in arriving at answers to questions involving indices. He focused, particularly, on the influence teachers and textbooks have on the kind of thinking students apply.

While there is extensive evidence of the problems students have in operating with indices, little has been done to explore the reasons. To provide focus for such an examination, frequent errors were first identified. This was done both through a review of relevant literature and from an analysis of errors in a large public examination conducted annually in N.S.W. From these errors, themes, relating directly to the problems students have, were identified for exploration.

A purpose of this research was to provide teachers with direction so that they might better cater for learning needs of students in this area. Accordingly, the educational context in which students acquire their understanding was given a high profile throughout the research. This was done in terms of both: the influence the context has on understanding; and, the implications the research has for classroom practices.

Additional themes, relating to theories of learning which may explain the understanding being applied, were developed for exploration. The aim of this was to place the research in a wider context, and to provide further direction for teachers. A pilot study was carried out prior to the main study. This was done to clarify issues concerning both the instrument to be used and the appropriateness of the themes.

Both quantitative and qualitative analysis were employed in the research. Persistent errors, and the students making them, were identified by testing. Students were then interviewed to ascertain the kinds of understanding they were applying. Statistical analysis was also used to explore levels of understanding being exhibited by students.

Chapter 1 provides a background to this research by examining relevant issues in the teaching and learning of algebra. Indices are largely applied within algebra and the understandings students have in algebra influence their understanding of indices. The nature of algebra, and its teaching, are first discussed, then problems students have in algebra are broadly examined. Following this, consideration is given to implications certain research in the field of algebra has for this present project.

In Chapter 2, the research is focused on the central issue, namely, students' errors and understandings in their work with indices. It begins by defining what is meant by indices and describes the position that indices occupy within the curriculum. This provides a structure within which errors can later be classified. Following this, research in the field of indices is reviewed. While mention has been made of how limited this is, several significant projects have been undertaken in this area by Shevarev, DeVincenzo, Childers and Wilson. An overview of the research is given and, then, these particular projects are considered in detail. A number of research issues, relating to students' understandings, are identified from the literature. Finally, consideration is given to implications the literature has for the way in which errors are to be classified.

A comprehensive analysis of student responses to questions involving indices in the N.S.W. School Certificate Moderator examination is carried out in Chapter 3. Nineteen categories are used to classify errors made over a period of 11 years. From the persistent errors identified in this analysis, and from the literature reviewed in the previous chapter, themes for investigation are developed.

Theories of learning, which may offer explanation for errors students make in index questions, are examined in Chapter 4. When selecting the theories, issues raised earlier concerning algebra and indices were taken into account, together with results of the analysis in Chapter 3. From the examination of the theories, three themes which may explain the kinds of understandings students apply in index work are then developed for research.

Chapter 5 describes the Pilot Study. It begins with a synthesis of themes and issues raised in the preceding chapters and a discussion of the purpose of the Pilot Study. The remainder of the chapter is devoted to the design and analysis of the study and to implications the analysis has for the Main Study.

The design of the Main Study is presented in Chapter 6, together with an overview of its results. At the end of this chapter, specific research questions are posed within each of the content related themes that emerged from the literature review in Chapter 2 and the analysis of Chapter 3. Chapters 7 and 8 provide a full analysis and discussion of those research questions, within the framework of each theme.

In Chapter 9, the first of the three themes relating to theories which may explain the kinds of understandings students apply in index work is addressed. This theme was concerned with levels of response and, in particular, the SOLO Taxonomy as a framework within which to view student responses to questions involving indices. In this chapter, further statistical analysis of the data, directed at identifying whether levels of learning can be discerned in student responses, is undertaken. Following that analysis, the qualitative data is examined to ascertain how it relates to, and elaborates on, findings emerging from the quantitative analysis.

Chapter 10 focuses further on the question of a theoretical framework. The two remaining themes, relating to theories which may explain the understandings students apply in index work, are investigated. The first of these uses Skemp's concept of 'relational' and 'instrumental' understanding to explore the kind of understanding being applied. The second theme examines whether particular persistent errors can be explained in terms of the cognitive constructs of 'connections', as postulated by Shevarev, and 'frames', as put forward by Davis. Finally, a synthesis is undertaken of the findings from all three themes concerned with theories of learning.

The final chapter brings together, and concludes upon, the issues raised throughout the research The results of the study are summarised, following a discussion of the limitation and constraints. Implications the research has for teaching practices, and for further research, are then considered.

Chapter 1

ALGEBRA - THE SETTING FOR THE RESEARCH

Introduction to Chapter

Students' understandings of indices develop, and are applied, within their knowledge bases of arithmetic and algebra. Skills in arithmetic should, for the most part, pose few difficulties to those students pursuing courses of study in which work involving indices has a significant position. However, the same cannot be said of algebra, where there is extensive evidence that students have widespread problems.

Algebra sets the context for this research in two ways. Firstly, the nature and purpose of algebra should determine the nature of understanding of indices. Indices are a tool of algebra and these understandings should be such as to facilitate success in that field. Secondly, misunderstandings in algebra influence, interact with and recur in students' work with indices. While neither issue can be explored in depth here, it is important to highlight the most relevant points.

The way in which algebra is taught influences students' understandings of indices. A section in this chapter considers some implications, pertinent to this research, of the different approaches to teaching algebra.

Finally in this chapter, two research projects are discussed from a perspective of the contributions their methodologies and findings might make to this research. The first of these is the 'Concepts in Secondary Mathematics and Science' project conducted in England between 1974 and 1979. The second, which developed from the first, is the 'Strategies and Errors in Secondary Mathematics' project conducted from 1980 to 1983.

THE NATURE AND PURPOSE OF ALGEBRA

If judgments are to be made on the quality of students' understandings in algebra it is first necessary to consider what writers in this field understand by the term 'algebra', and how its major role in the secondary mathematics curriculum is justified. It is worthwhile observing that the literature certainly does not see algebra as simply the manipulation of symbols with little apparent purpose and involving little depth of understanding. Algebra can be seen to have three roles. These are described below.

Firstly, algebra is generalised arithmetic. It is a precise and powerful language within which we can see and express general statements. The Mathematical Association (Boy's Schools Committee1934, p.6) reported that "the symbolisation involves, or brings to light, generalisation; it helps us to realise the wide applicability of a single statement or that a multitude of single facts are included in one general rule". In *Routes to/Roots of Algebra*, when focusing on algebra's capacity to express generalisations, it is stated

> Generalising is not restricted to numerical situations. It is a much more fundamental and wide-ranging activity. It can be argued that all human learning involves the distillation of individual experiences into broader general principles ... As teachers, it seems appropriate that we should help pupils to express their own generalisations more clearly and efficiently, and encourage pupils to check for validity.
>
> (Mason, Graham, Pimm and Gowar 1985, p.4)

Secondly, algebra facilitates the solving of problems in number and measurement. It does this through the transformation of algebraic statements, using procedures whereby general statements may be manipulated to provide alternative but equivalent general statements. In this use of algebra "the key instructions are *simplify* and *solve* ... (and) variables are either *unknowns* or *constants*" (Usiskin 1988, p.13). Traditionally, the justification for algebra in the curriculum has been in terms of its practical applications in the sciences through its ability to facilitate both calculation and problem solving.

Thirdly, algebra is the study of relationships among quantities with a central concept being that of a *function*. Usiskin observed "the crucial distinction between this and the previous conceptions is that, here, variables *vary*" (1988, p.13). The concept of a function is central to much of the content of the courses followed by students in New South Wales schools. Without a genuine understanding of variable much of the work students do, when graphing functions, answering question in differential and integral calculus and working with series, becomes quite meaningless.

DIFFICULTIES IN THE LEARNING OF ALGEBRA

The strengths of algebra are also the features which make it difficult to both teach and learn. It is a precise and compact language but because of its succinctness "is open to incomprehension" (Mason 1985, p.1).

While the problems students have with algebra manifest themselves in diverse ways, recent articles frame the difficulties within the bounds of: (a) establishing meaning for letters; (b) knowing what is expected of the answer; (c) understanding the nature and purpose of generalisations; and, (d) working with algebraic notation.

Problems of establishing meaning for letters

In discussing this, Milton stated "some children do not see that a letter is used to stand for a generalised number, whilst still others think that the 'size of a number' varies with the 'position of the letter'; meaning that *Y* is worth more than *A*" (1988, p.6) *Routes to/Roots of Algebra* (p. 5) contains material which identifies the confusion which can occur in translating an English statement. This can result in students treating letters as units of measure or as representing objects, not as generalised numbers.

The importance of establishing meanings for letters is reinforced by Booth's statement that "one could say that until a student does appreciate the use of letters as variables, or at least as 'generalised numbers' then algebra can have little meaning" (1986, p.3).

Problems in knowing what is expected of an answer

Problems which centre on the difficulties students have in deciding what is expected in the way of an answer have their basis in the experiences students typically have in arithmetic. Here the student is usually expected to arrive at some 'specific numerical answer' so that

> Many students assume the same is true in algebra, even in cases where no specific answer is desirable. ... Consequently, children will resort to various strategies, in order to derive a numerical answer. ... Even when the possibility of a non-numerical answer is accepted, there is a tendency for students to assume that at least what is required is a single term.
>
> (Booth 1986, p.3)

Further to this point, students also have problems knowing just when they have the answer. In *Routes to/Roots of Algebra* it is said that

> Many children find it difficult to know when they have arrived at an answer in an algebra question. The implicit rules of the game in algebra are that if the question contains an equation, you solve it, if you are given an expression, you simplify it. Even then, there is still the problem of what constitutes a final answer.
>
> (1985, p.5)

Problems with generalisation

This is closely tied in with problems of establishing meaning for letters, but it goes further. In this case, the development of the concept of a generalised number proceeds to a point where the notion of a 'variable' has genuine meaning. Milton reported

> Even quite able children do not understand the nature and purpose of generalisation. At the heart of generalisation is structure and form - or pattern, if you like. To capture and express pattern with number operations is the point and purpose of algebra and algebraic thinking, especially at the outset of algebraic study.
>
> (1988, p.6)

Chalouh and Herscovics (1988) highlighted the fact that teachers can contribute to problems that students face. They commented

> Quite often, algebraic expressions are introduced by stating that they involve variables and that a "variable is a letter that stands for one or more numbers." Such formal definitions may be adequate for mathematics teachers but they often fail to provide meaning for the beginning student.
>
> (p.33)

Can students be expected to develop meanings for variable if they are not provided with the experiences needed in looking at patterns and the time in which to let these experiences take effect?

Problems in working with algebraic notation

Difficulties students have with algebraic notation have received considerable comment from researchers in recent times. Kieran (1989) observed that "many high school algebra students appear to be experiencing serious obstacles in their ability to recognise and use the structure of algebra" (p.52). Booth (1989a) believed that "without an understanding of the semantics of algebra, the mere manipulation of symbols becomes a fairly arbitrary exercise in symbol gymnastics, sometimes performed correctly and sometimes not, but in either case with little sense of purpose" (p.58).

The introduction of a raised number into an expression can only compound problems students are already having with the complexities of notation.

THE TEACHING OF ALGEBRA

In conducting research in the field of algebra an important consideration is the impact teaching has on students' understandings. Teaching approaches may account for many difficulties students have with indices. In light of what has gone before, effective teaching approaches in algebra are ones which: develop abilities to generalise; minimise difficulties in working with symbolism; and, develop skills in using algebra as a tool and a vehicle for the creative solution of problems.

Teachers bring to their classrooms a variety of strong influences on their teaching. Perhaps the strongest are: those features that were evident in their own teachers; contact with associates; and, 'the textbook'. Recent, and not so recent articles in the research literature indicate that, in the area of algebra, the outcomes these influences have generated are not particularly satisfactory for many students. While traditional teaching strategies and the role of algebra are being widely questioned by researchers, it seems that there has been little impact in the classroom. Thorpe commented that

> The teaching of algebra in the schools is not significantly different today from what it was fifty years ago ... meanwhile mathematics and its applications have changed dramatically.
> (1989, p.11)

In *Routes to/Roots of Algebra* it is suggested that algebra is "generally perceived as both hard and pointless" (1985, p.1) while Fey stated that "the agenda for high school algebra is overloaded with skills that prove difficult for many students ... as a consequence, there are usually substantial differences among the intended, implemented and achieved algebra curricula" (1989, p.199). In an effort to satisfy requirements placed on them by the syllabus and examiners, teachers and textbook writers have resorted to strategies designed to enhance success through rote learning. Such approaches are counterproductive to the development of understanding, and do not support the purpose of algebra.

As far back as 1934 in Britain the Mathematical Association Report, *The Teaching of Algebra in Schools* (Boys' Schools Committee of the Mathematical Association), raised the issue of the gap between algebra as it was, and undoubtedly still is, taught, and what the writers saw as the essence of algebra. The need for teachers to re-evaluate their role was reflected in the remark that "the teacher must get out of his head that juggling with *X* and *Y* has any special merit of its own" (p.9). The situation appears to be much the same now as indicated by Booth's comment that, for many students, "algebra amounts to little more than a set of fairly arbitrary manipulative techniques which seem to have little, if any, purpose" (1986, p.2).

In discussing the great emphasis which has been placed on students acquiring manipulative skills, Booth reported

> one of the problems with the teaching of algebra has been that there often was a very unclear picture of what the goals of learning algebra might be. Failing a clear and convincing analysis in this regard, teachers and textbooks fell back on the surface features of algebra and contented themselves with teaching manipulative skills and the routine application of a few standard algorithmic procedures.
>
> (1989b, p.238)

Researchers have identified students' first experiences of algebra as the source of much of the difficulties. The introduction provided by teachers appears to be quite crucial in setting the ground work for students' development. In commenting on the introduction of algebra, the Mathematical Association Report (1934) stated that, "Historically, algebra grew out of arithmetic, and it ought so to grow afresh for each individual" (p.5). While recent research throws new light on the way this may occur, it remains the underlying principle on which students' understanding of algebra needs to be built. Booth, in writing about an approach in which algebra does not grow out of arithmetic, stated

> Such an introduction may not serve appropriately to reveal the power and purpose of algebra. What is perhaps required, is an introduction which focuses on this latter aspect, leaving issues of simplification and manipulation until the need for these is recognised by the students.
>
> (1986, p.4)

MacGregor (1986), in discussing the commonly adopted teaching and textbook strategy of introducing algebra through a "letter as an object" approach, e.g., "$2a+5a=7a$ because two apples plus five apples gives seven apples", commented

> Advice to manipulate symbols without concern for their denotation is surely dangerously misleading. The student learns that algebra is a meaningless and useless game played with the letters of the alphabet.
>
> (p.10)

It appears that what is needed is an introduction which focuses on structure in number giving "experiences which lead from concrete arithmetic situations to algebraic generalisations over sufficient time and with sufficient variety of problem situations to allow for growth and maturity to occur" (Briggs, Demana and Osbourne 1986, p.5).

Pegg and Redden (1990), in their article *Procedures for and Experiences in Introducing Algebra in NSW*, looked closely at the benefits, and implications, of such an approach. Benefits they listed are: the avoidance of early manipulation of symbols; the assistance it gives students to see algebra as more than just a series of abstract rules; the highlighting of the benefits of learning algebra; and, the leading of students to more readily realise that symbols stand for numbers (rather than objects) (p.1). Such an approach also emphasises 'procedure' and could assist students overcome the problem of wishing to 'close' on a simple answer.

The writers of *Routes to/Roots of Algebra* advocated avoiding premature use of symbols and believed "more time can profitably be spent with the prior aspects of 'seeing' and 'saying', even if such activities seem not to be explicitly algebraic in nature" (p.8). From this 'seeing and saying' the teacher needs to use appropriate language and communication to guide the student to appreciate the need for a concise symbolism and assist the student to put meaning into this symbolism.

Such an approach is far removed from the one under which most mathematics teachers acquired their skills, and this in itself is a hindrance to change. If the move in this direction is to be pursued successfully, and with conviction, it needs to be supported by convincing evidence from researchers.

The placement of algebra within the curriculum is also a consideration. Some problems may result from the introduction of algebra at too early a stage. Students' first experiences typically occur around 12 years of age. Jesson (1983), in considering what students might be expected to be capable of at various ages, commented "a number of studies have shown that teachers consistently overestimate children's ability or, equally underestimate the difficulty of the mathematics involved" (p.125). Students vary greatly in their mathematical performance on entering secondary school and, in view of the difficulties experienced by many, it would seem that the timing and sequencing of algebra should be looked at.

Algebra is an integral part of our secondary mathematics curriculum and, given that all is not well in the teaching of it, educators must seek strategies to improve the outcomes for students. MacGregor in addressing this issue stated

> If we accept that the misconceptions of elementary algebra are caused, at least in part, by the teaching strategies we are using, it may be beneficial to consider other ways of presentation.
> (1986, p.11)

Indices are an integral part of algebra and the comments above, while applied to algebra generally, substantially apply also to indices in particular. If the topic is to be taught for understanding, then the educational researcher needs to consider classroom practices and to help teachers decide which of them encourage achievement of the aims, which of them do not, and what changes might be appropriate.

DISCUSSION OF TWO RESEARCH PROJECTS IN ALGEBRA WHICH ARE OF RELEVANCE TO THIS RESEARCH

In this section two related research projects, which look at difficulties students have with algebra, are discussed. These are the 'Concepts in Secondary Mathematics and Science' and the 'Strategies and Errors in Secondary Mathematics' research projects.

The first of these projects, the algebra section of the Concepts in Secondary Mathematics and Science (CSMS) project, was conducted in England between 1974 and 1979. Children aged 13-to-15 years were tested for understanding of basic algebraic concepts, and the results were used to identify and examine the types of errors made. The project was reported by Küchemann in *Children's Understanding of Mathematics* (Hart (ed.) 1981)

From the CSMS project grew the second research project, Strategies and Errors in Secondary Mathematics (SESM), which was carried out between 1980 and 1983. It sought "to investigate the reasons underlying particular errors in generalised arithmetic which the earlier CSMS project had shown to be widely prevalent among 2nd to 4th year students in English secondary schools" (Booth 1984, p.1). The examination of errors was directed toward establishing the strategies used. The project was reported in Booth's book, *Algebra: Children's Strategies and Errors*. Since the CSMS led directly into the SESM project, the projects are treated as one in the discussion.

The following discussion has a two-fold purpose. Firstly, it provides information on the research methodology that these researchers found to be appropriate in examining students' understanding in algebra. This offers guidance to the methodology of this present research. Secondly, it adds to information on problems students appear to be having in algebra generally, problems we might expect to occur also in questions which focus on indices.

Implications of the Methodology Used in this Research

There are five main aspects of the methodology used in the CSMS and SESM projects which seem particularly relevant to this current research. These are now discussed.

1. The Researchers Chose to Focus on Underlying Themes

In devising their test, the researchers began by looking to sample a wide range of typical high-school algebraic activities but quickly realised that "such activities, singly and in combination were far too numerous to be investigated thoroughly by one test, and that other, far more fundamental criteria had to be found". Because of this, the scope of testing was restricted to algebra as generalised arithmetic, and the researchers decided to focus on two themes only. These themes were the effect of **structural complexity of the items** and the **meanings given to letters**. The approach of looking for underlying threads running across activities, rather than focusing on the nature of the activities themselves, is more likely to provide practical and general strategies for the classroom teacher.

2. The Researchers Sought to Explain Errors Using Cognitive Theory

The research attempted to identify the quality of the thinking used. Items in the CSMS project were sorted into four levels according to the complexity of the item and the nature of their elements. These levels were:

Level 1. Items of simple structure which could be solved using lower level interpretations of letters.

Level 2. Items of some increased complexity but still only requiring letters to be evaluated or used as an object. Children at Level 2 showed a growing acceptance of answers which were not closed.

Level 3. Children at this level were able to use letters as specific unknowns if the item structure was simple.

Level 4. Children were able to cope with items requiring specific unknowns and having a complex structure.

Küchemann (p.117) suggested it can be argued that there is the following correspondence between these levels and the Piagetian sub-stages:

Level 1	Below late concrete
Level 2	Late concrete
Level 3	Early-formal
Level 4	Late-formal

In the SESM report, Booth said that "the emphasis on analysing children's errors is based upon the Piagetian view that a consistently made error to a given problem reflects a way of viewing that problem or handling its solution, which is consonant with the child's cognitive structure" (1984, p.7). However, Booth commented later that there were inconsistencies "mitigating against the unqualified acceptance of the 'unified stage' view of cognition which characterises the Piagetian formulation" (1984, p.95). The research does not reach firm conclusions about the relationship between the nature of a child's cognition and their performance in algebra, but does signal it as an important area for research. The work of Biggs and Collis, focusing on students' responses in specific tasks, is referred to in the text. It is suggested that their Structure of Observed Learning Outcomes (SOLO) Taxonomy may be of significance to future research in light of the need to elaborate on "the mechanisms by which task-specific and task-independent abilities are coordinated" (1984, p.95).

3. The Researchers Made Use of Both Testing and Interviews

The CSMS testing and analysis provided valuable information on common errors students make in algebra. However, while some evidence of the strategies students use resides in the answers themselves, errors may occur at several points along the way. In light of this, the use of interviews in this kind of research would seem not simply appropriate but quite essential. Their value was clearly demonstrated in the SESM research. As well as clarifying the processes students were using, interviews can be used to identify the meanings students attach to the input they receive and the output they produce. Often these are different from what might be expected.

4. Students Selected for Interview Were from a Targeted Group

The CSMS Algebra test was administered to 3550 students aged from 13-to-15 years across the full spectrum of abilities. The SESM project started by using the CSMS items on approximately 240 students from five high schools with a view to eventually interviewing approximately 50 students. In schools, where classes were graded, the students were selected from the middle ability groups, while in schools with ungraded classes, students were chosen at random. This was because it was felt that the more able students would make fewer of the identified errors while the less able would not be as confident in explaining their methods and ideas.

The compromise taken by sampling from the middle stream in just several schools does place some limitations on generalising from the results. Nevertheless, the sample used focuses almost certainly on the group where problems with algebra have most impact, in that middle-stream students obviously have trouble in meeting the demands of the present curriculum.

5. Students Were Interviewed on Items with Which They Had Difficulty

Children were selected for interview on the basis of the errors they made in the test. The interviews focused on identifying the reasons for those errors. This does limit the inferences which can be made about the wider population. However, the loss of generality caused by the methodology is far outweighed by the fact that important issues were given the in-depth examination which they warranted using what seemed the best approach to get to the heart of the research questions.

Findings from the two Projects which are of Relevance to this Research

From the CSMS project, nine items were selected for further investigation in the algebra section of the SESM research. These item were ones in which student responses exhibited a high incidence of systematic errors. The questions involved operating with, and interpreting, algebraic expressions.

The errors, as listed by Booth (1984, p.3), seemed to result from: the treating of letters as objects (as indicated by the conjoining of numerical and algebraic elements); the ignoring of letters (as indicated by certain numerical answers); the unjustifiable numerical substitution for letters (as indicated by certain numerical answers); and, by the viewing of letters as specific unknowns rather than generalised numbers.

The SESM research continued to confirm the importance of the interpretation students use for letters. It clarified some previously identified issues and raised a number of new ones. The following six points stand out as being of relevance, possibly, to research in the field of indices,

1. Students Often Misinterpret a Letter and this May Lead to Errors

Six different interpretations of a letter were identified in the CSMS project and these were further investigated in SESM project. These were: assigning it a numerical value; not using it; regarding it as an object; treating it as a specific unknown; treating it as a generalised number; and, treating it as a variable. The first three, lower level interpretations, are ways "by which children avoid having to operate on a specific unknown" (Küchemann 1981, p.105). They are incorrect interpretations but do not lead necessarily to incorrect answers. The latter three, higher level interpretations, are correct interpretations. However, their use does not guarantee success as noted by Booth (1984) in the comment that "children who recognise letters as representing number may still produce the erroneous answer" (p.29).

2. Items May be Solved in Unexpected Ways

The different meanings students may give to letters does "affect item difficulty in that some items might be solved in unexpected ways" (Küchemann 1981, p.103). Booth (1984) said that interviews indicated that, for some commonly asked types of questions in algebra (such as 'add 3 to $5y$'), "the level of letter interpretation may bear little relation to success" (p.29). No doubt this makes the job of researchers more difficult, in that, as 'mathematicians' they have preconceptions as to appropriate solutions and find it difficult to account for less usual approaches.

3. Notation and Symbolism Cause Considerable Difficulties

Booth (1984) reported that "the problems children have with symbolisation must not be underestimated. Children do not readily assimilate the meaning of abstract representations" (p.94). In many cases, students treated $2e$ as meaning "the sum of 2 and e" while in other cases they accepted $2e$ as meaning a product but then translated 27 as meaning 2×7 (Booth 1984, p.30). The role of the addition sign in algebra as opposed to arithmetic also caused problems. In arithmetic '+' indicates the need to operate and this leads the child to view an expression with '+' in it as requiring something more to be done. As Booth (1984) stated "This distinction between what constitutes a question or statement of method, and what constitutes an answer, appears to underlie much of children's difficulty in algebra" (p.29).

4. Many Students Fail to Understand the Significance of Brackets
Booth (1984) found that

> A number of children considered expressions with or without brackets to be equivalent, maintaining that 'you can put the brackets in if you want, it's just the same' and a similar number exclude the bracket expressions altogether, either because they considered their presence unnecessary, or more rarely, because they did not know what the brackets meant.
>
> (p.31)

This was not restricted to the less able students, "children in the top ability groups also appeared to ignore the need for brackets" (p.54). Difficulties with notation not only cause errors but result in erroneous answers following correct reasoning. In a particular question there was "a proportion of children (five out of fourteen) whose otherwise correct answer was rendered incorrect because of their apparent non-recognition of the need to use brackets" (p.20).

5. Informal Methods Developed in Arithmetic can Lead to Errors in Algebra
Students may successfully use their own informal rules in arithmetic. However, algebra involves the writing of general statements using the rules of arithmetic and so "the possibility of non-use or non-recognition of these rules and structures in arithmetic may have considerable consequence for the child's performance in algebra" (Booth 1984, p.6). "These methods are also characterised by being strongly adhered to and reluctantly abandoned by the child, possibly because of previously successful usage" (p.37).

6. Delineating Between Problems Resulting from Teaching/Learning Experiences and Those Grounded in the Child's Cognitive Development Can be Difficult
While considerable reference is made to cognitive development of students through Booth's text, few definitive statements are made about its impact. Observations are very much of a general nature such as "whilst some of the misconceptions that children have may be due to inadequacies in the

teaching-learning situation ... some of the difficulty appears to be related more to a 'cognitive readiness factor' " (Booth 1984, p.87).

This present research is seeking to identify and explain systematic errors students make in an area where both letter interpretation and the use of notation are central concerns. The projects discussed in this section have provided valuable information on both these issues. In addition, the methodology used by these researchers provides guidance as to appropriate strategies for investigating children's understanding of indices.

CONCLUSION

Proficiency in algebra is an important skill especially for those wishing to pursue a higher level of mathematics in the senior secondary school. It is of considerable concern that many students find great difficulty in understanding algebra and have come to see its study as having little value.

In examining difficulties students have in learning algebra, clearly it is not sufficient to look only at answers, as these disguise many misconceptions involving the meanings of letters, operations with letters, notations and conventions, problems of knowing what constitutes an answer and so on. Correct answers obtained through incorrect reasoning and incorrect answers obtained from correct reasoning both appear to be frequent occurrences. Interviews need to be an integral part of the research if students' understanding is to be examined closely.

In that much of students' work with indices is within the general framework of algebra, the findings of the SESM project provide guidance to issues which may be of relevance to this research. The extent to which difficulties students have with questions involving indices reflects difficulties in learning algebra generally is reflected upon in the concluding chapter.

Reasons for students' understanding, or lack of understanding, in algebra are composed of a complex interweaving of issues concerned with both the cognitive development of each student and their classroom experiences. Like the two projects reviewed in this chapter, this current research focuses on a number of key themes which can be probed in detail. From this it would be of great value to teachers and researchers if a theoretical framework could be found to describe the thinking used by students.

Chapter 2

INDICES - THE SUBJECT OF THIS RESEARCH

> The construction of the exponential function is far from a trivial task (Goldin and Herscovics 1991, p.70).

Introduction to Chapter

Chapter 1 was devoted to establishing the broad framework for this research by examining the nature of algebra and the problems posed in both its teaching and learning. This chapter focuses on the central issue of the research, that is, **students' understanding of indices**. The position indices occupy within the mathematics curriculum is examined, existing research in the field of indices is considered and research issues arising from this review of the literature are noted. Finally, consideration is given to any implications the research has for the categorisation of errors.

It should be noted that research into students' problems of understanding in the field of indices appears limited. Students' strategies and errors in index work seem not to have come under close scrutiny and in depth probing to the same extent as other topics in arithmetic and algebra. There are several researchers, however, namely, Shevarev, DeVincenzo, Childers and Wilson, whose work does relate closely to this current project, and their research is examined in some detail in this chapter.

INDICES IN THE CURRICULUM

While mathematicians see indices as forming a complete and coherent system, students learn of the different aspects of indices in a developmental sequence. Questions posed to students tend to be readily identifiable with steps in this sequence. In this section, an **International Overview** is provided which shows the position of indices in the mathematics curriculum in the United Kingdom, United States and Australia. This is then followed by a **New South Wales Overview** which focuses on the specific syllabuses applying to students in Year 9 to Year 12 where indices are developed and applied. It shows both the content and the typical sequence of development to which they have been exposed. However, before these overviews are provided, it is important to clarify what is meant by the term 'index' in the context of this research.

What are Indices?

The sense in which 'index' is used in this work is in the traditional mathematical sense. An index is first thought of as a positive integer indicating a repeated factor, and is then allowed to take values other than positive integers in such a way that is coherent both with this starting point and with mathematics generally. In mathematics the terms 'power' and 'exponent' are frequently used instead of 'index' and these terms are regarded as meaning exactly the same throughout the text.

The *Universal Encyclopedia of Mathematics* defines an index as "a product of equal factors, e.g., $4^3 = 4\times4\times4$, $a^5 = a\times a\times a\times a\times a$" (Newman 1976, p.334) and goes on to discuss the various operations which can be carried out with such expressions. The *World Book Dictionary* describes an index as "a small number written above and to the right of a symbol or quantity to show how many times the symbol or quantity is to be used as a factor, for example $2^3a^2= 2\times2\times2\times a\times a$" (Barnhart & Barnhart 1986, p.751). While the concept of powers "was already known in antiquity ... our notation for powers essentially goes back to René Descartes (1596–1650)" who used it "for integral exponents greater than 2" (Gellert, Kustner, Hellwich & Kastner (eds) 1975, p.47).

It is common for syllabuses to not explicitly state a definition of an index, but imply it by listing content and skills objectives which evolve from the definitions given above.

International Overview

In 1989 the United Kingdom with its document *Mathematics in the National Curriculum* and the United States with its *Curriculum and Evaluation Standards for School Mathematics* made statements about what students should be achieving at various stages of their education. In the same year, Australia, with its *Mapping the Australian Curriculum*, examined the content of the mathematics curriculum being provided for students in each of the

Australian States. The position indices occupies in these documents is now considered.

In the United Kingdom document, *Mathematics in the National Curriculum*, a sequence of levels is used to indicate expected student attainment. A student preparing to pursue mathematics at a high level in the later years of schooling would, on completing the year in which most of the age cohort turn 16, be expected to be able to:

- use index notation to express powers of whole numbers
- express a positive integer as a product of primes, e.g., $147 = 3 \times 7^2$
- express numbers in standard index form using positive and negative powers of 10
- use index notation to represent powers and roots
- use the rules of indices for positive integer values, e.g., simplify $2x^2+3x^2$, $2x^2 \times 3x^3$ and $(3x^2)^3$
- use the rules of indices for negative and fractional values, e.g., $x^0=1$, $y^{-3} = 1/y^3$, $x^2/x^3 = 1/x = x^{-1}$

(Dept. of Ed. and Science and the Welsh Office 1989, p.8, p.18)

Curriculum and Evaluation Standards for School Mathematics provides a set of standards for the mathematics curricula in United States Schools (K-12). It is "a vision of what the mathematics curriculum should include in terms of content priority and emphasis" (Commission on Standards for School Mathematics 1989, v.). The document focuses on what students ought to be able to do. Standards are indicated across grades and within content areas. For Grades 5-8 it is stated that the curriculum should be such as to enable students to "understand, represent and use numbers in a variety of equivalent forms (integer, fraction, decimal, percent, **exponential and scientific notation**) in real-world and mathematical problem situations" (p.87). While very little direct comment is made on indices, topics discussed in the section on Grades 9-12 (such as Algebra, Functions, Probability and Calculus) imply that students require the same skills in indices as described in the UK and Australian documents.

Mapping the Australian Curriculum (Australian Education Council 1989, p.64) is similar to the UK document in the pattern it describes for students intending to study senior-school tertiary-entrance mathematics courses. In seven of the eight Australian States or Territories (all except Queensland) the topics taken prior to the end of Year 10 contain the following work with indices:

- operating with exponentials having numerical bases
- graphing exponential expressions

- using the index laws for the multiplication, division and the raising to a power of exponential expressions
- working with zero, negative and fractional indices
- applying the laws to numerical and algebraic problems
- solving simple indicial equations.

New South Wales, Victoria, the Australian Capital Territory and the Northern Territory also include work on indicial equations with unlike bases. Details are now provided of the position of indices in the mathematics curriculum of New South Wales.

New South Wales Overview

The mathematics curriculum for N.S.W. is determined by syllabuses relating to stages of schooling and, in high school, to the ability and interests of students.

At the primary level a single syllabus applies. This covers the school years from Kindergarten to Year 6 and, commonly, the ages of 5 to 12 years. Students enter high school at around age 12 and, if progressing through to Year 12, leave near the age of 18. During Years 7 and 8 students follow a common syllabus. In Years 9 and 10 three distinct courses are available. The Advanced Course is taken by the more capable mathematics students, representing about 36% of the age cohort. The Intermediate Course is followed by middle-ability students who constitute approximately 43% of the cohort. The General Course is for the less able students. By the end of Year 10 (16 years of age), students should have acquired the index skills needed for the senior courses. No new skills are introduced in the syllabuses for Years 11 and 12.

The 2 Unit and 3 Unit Mathematics Courses for the N.S.W. Higher School Certificate are the senior courses in which facility in operating with indices is of considerable importance. The 2 Unit course is designed to prepare students to undertake tertiary study of mathematics as a minor discipline while the 3 Unit course can lead to the study of mathematics as a major discipline. The syllabus for those two courses makes direct reference to index skills only in terms of revising previous work (Board of Senior School Studies 1982, p.8).

Typically, a student's experience of indices in the N.S.W. Education System follows the pattern described below and occurs at about the times mentioned.

(i) First experiences of indices occur around Year 3 (8 years of age) with the use of a raised ^{2} to indicate squaring. In Year 5 (10 years of age) indices are used for place value, e.g., $274=2\times10^2+7\times10+4$. In both cases it might be expected that students would see indices as a name for a specific operation rather than as a general indicator of a repeated factor.

(ii) Indices are then used as a more general indicator of the number of times a numerical factor repeats, e.g., $2^3 = 2\times2\times2$, and are applied in simple numerical expression. This usually occurs in Year 6 (11 years of age) but is extended considerably in Year 7 (12 years of age), the first year of secondary school.

(iii) Towards the end of Year 7, or some time in Year 8 (13 years of age), indices are used as an indicator of the number of times an algebraic factor repeats. This is coupled with first ideas on the product of algebraic terms.

(iv) The collection of like terms involving indices is then addressed.

(v) At approximately the same time, students multiply, divide and raise to integral powers algebraic expressions which involve coefficients and integral powers of a pronumeral, e.g., $3a^2\times5a^4$, $8m^7\div 4m^3$ and $(5b^2)^3$. Those who later pursue the General Course are not expected to go beyond this.

(vi) The meaning of a zero index is developed.

(vii) Students are then introduced to negative indices to indicate rational numbers, e.g., $4^{-3}=1/4^3$. Advanced Course students do this near the end of Year 9. It is optional for Intermediate Course students but most would come across the concept in Year 10.

(viii) The use of the fractional index as an indication of a root of a number, e.g., $8^{1/3}=\sqrt[3]{8}$ is introduced to the Advanced Course students in Years 9 or 10.

(ix) Experience is given in simplifications involving a combination of the above concepts, and in substituting in more complex expressions. Changing the subject of more involved formulae is addressed. The degree of difficulty of questions varies with the ability level of the students.

(x) Scientific notation is introduced to Advanced and Intermediate students in Years 9 or 10.

The specific references to indices given in the documents of the relevant syllabuses are provided in Appendix A.

A point to note is that literature on students' learning in algebra, as discussed in Chapter 1, indicates that notation can pose considerable difficulties. The questions generated from the syllabus content listed above often involve the added complexities of brackets and fraction bars as well as raised notation and radical signs. This provides students with extensive opportunities to make notation-related errors.

Another important consideration is the position that calculators occupy in students' work with indices. The calculator is much more than a tool with which to more easily obtain correct answers. Besides expanding the scope of questions which can be posed, the variety of ways in which different calculators display an index has introduced to students a new range of likely answers, both correct and incorrect. Calculators do have implications for both the teaching and assessment of index work.

Despite the fact that calculators have been widely available since the mid 1970s their position in the N.S.W. Mathematics Curriculum is one which is neither stable nor clearly understood. For that reason the development of index related calculator skills cannot be readily placed in the sequence above. Attitudes vary greatly on the role of calculators and on the point at which students should be introduced to them. Some students have extensive access to scientific calculators at the beginning of Year 7, while others only use them on a limited basis until Year 9. Certain students have used calculators while learning indices, others are only introduced to the index application of calculators after they have covered all of the concepts.

It appears from the documentation that indices occupy a similar position in the N.S.W. curriculum as they do in the curriculums of the U.K., U.S.A. and other Australian States. This being the case it is likely that findings of research undertaken within N.S.W. will find application within the other contexts.

RESEARCH IN THE FIELD OF INDICES: AN OVERVIEW

As mentioned above, there appears to have been little research into students' problems of understanding in the field of indices. In order to gain direction for this current research, four significant projects carried out in this area are examined in some detail in later sections of this chapter.

The first of these projects was a psychological analysis of algebraic errors first published in 1946 by the Soviet psychologist Shevarev. In this publication, Shevarev offered explanations for certain errors students make in index questions. Shevarev's work was one of a number of Soviet publications translated into English in a joint project by the School of Mathematics Study Group at Stanford University, the Department of Mathematics Education at the University of Georgia, and the Survey of Recent East European Mathematical Literature at the University of Chicago. The translations were published in 1975.

The second project was a Ph.D. dissertation by DeVincenzo, published in 1980, in which she compared errors made by students when answering similarly structured problems in arithmetic and algebra. Many of the problems which were the subject of her research involved indices.

The third project was a Ph.D. dissertation by Childers, published in 1987. In this she examined the effect on error patterns in index problems as bases and exponents changed from being constants to being variables or polynomials. Like DeVincenzo, Childers tested hypotheses about the number of errors in various types of problems and whether the errors were alike or different. Both researchers drew on the work of Shevarev in structuring their research.

The final project was a Ph.D. dissertation by Wilson, published in 1985. Wilson examined the understanding of exponents by Remedial Algebra Students at a Four Year College. He interviewed students and qualitatively classified their understanding of the concept of exponent within a theoretical framework developed from the theories of Bruner, Skemp, Krutetskii and Gagne.

Before considering these research projects, there are a number of other authors who provide some research information on students' understanding of indices and relevant points from their publications warrant examination.

Bernander and Clement (1985) investigated errors in basic arithmetic and included ten errors which students make with indices. The source of the data are given as "In-classroom observations of tutors and instructors in remedial level mathematics courses at the University of Massachusetts" together with "The retrospective reports of tutors and instructors" (p.6). The more substantial research projects, yet to be discussed, show that the errors listed by Bernander and Clement indicate only a small number of those that students make. However, two aspects of index work which are given little attention elsewhere do figure in their list of errors. These are the interpretations given to negative indices and to the zero index. The errors identified were:

(i) ignoring the negative sign completely ($5^{-2} = 25$)

(ii) treating a negative index as though it is positive but placing the negative sign in front of the answer ($5^{-2} = -25$)

(iii) obtaining a correct fractional value but making the answer negative ($5^{-2} = -1/25$)

(iv) treating the zero index as generating an answer of zero.

(p.24)

Other errors mentioned include multiplying the base and the power ($2^3=6$) and multiplying powers when multiplying numbers expressed with an exponent ($a^3 \times a^5 = a^{15}$). These are also mentioned by other researchers.

Operations with exponents are given brief mention in the National Assessment of Educational Progress (NAEP). In 1981 the NAEP found that 90% of students, with two years of algebra, correctly answered expressions of the form $a^3 \times a^4$ but that only 43% could correctly answer a^4/a^{20}. In the later question there was "a strong tendency to perform a fraction simplification on the exponents alone" (Corbitt 1981, p.64) and to arrive at an answer of $1/a^5$ (25%) or even $1/5$ (4%). Students also had difficulty in taking the square root of an algebraic expression involving an exponent. For the question $\sqrt{a^{36}}$ only 26% obtained the correct answer of a^{18} while 47% chose the answer of a^6 by taking the square root of the exponent. Each of these questions were in multiple-choice format. In 1989 the NAEP found that in open ended questions only one in four students with 2 years of algebra could correctly answer $\sqrt{x^{12}}$ (Lindquist 1989, p.60).

Vinner (1977) asked the question "do students look at exponentiation the same way some of their teachers tried so hard to teach them?" (p.17). The research addressed whether students, studying mathematics at university, could "identify the defining formulae of exponentiation, $a^m = a.a.a........a$ (m times), $a^{-m} = 1/a^m$ and $a^{m/n} = \sqrt[n]{a^m}$ " (p.18). He examined these issues with 195 mathematics freshmen and 56 students at the upper level of the mathematics undergraduate studies. Vinner used a multiple-choice questionnaire containing these three formulae among others whose task was a "masking role". He asked the students to identify each formulae as being either (A) a theorem, (B) a law, (C) a fact about numbers, (D) a definition or (E) an axiom. Even these capable students of mathematics found considerable difficulty with this task and only one fourth of the freshmen and one half of the other undergraduates identified all three defining formulae. Vinner attributed the problems to students being taught through a "definitional approach" at a stage when they were not ready for it. He said that "Mathematical maturity as well as intellectual development needs both time and experience ... to teach the definitional approach before the student is at the suitable intellectual stage is just useless" (p.24). Vinner referred to the work of Piaget and advocated teaching approaches which take account of the intellectual stage of the student.

Much of the literature on understanding of indices tends to concern itself with discussing techniques that might be appropriate to some highly specific issue in the teaching of indices. It seems that observations, on the implications of such techniques, are made with little effort to justify their reality in the classroom situation. Examples of these subjective observations are "exponents other than whole numbers are quite natural when calculators are used" (Comstock & Demana 1987, p.48) and "very quickly, the student will find procedural shortcuts when larger exponents are involved (e.g. $2^6=2^3\times2^3=8\times8$)" (Goldin & Herscovics 1991, p.69). Evidence from the research projects now to be examined suggest that such remarks gloss over the myriad of problems that students have in index work and hardly provide a structure for development of sound programming or of more appropriate teaching methods.

RESEARCH OF P.A.SHEVAREV

In the publication *An Experiment in the Psychological Analysis of Algebraic Errors* (1975), Shevarev offered a theoretical basis to explain errors students make in algebra. He exemplified his theory with errors made in index questions. Both the errors and the theory are of considerable relevance to this current research. While Chapter 4 of this thesis is devoted to considering several theoretical frameworks, which may be of value in explaining students' understanding of indices, it is neither desirable nor particularly manageable to separate Shevarev's discussion of errors from his theory. For that reason both are addressed in this chapter. Important aspects of the theory are briefly highlighted again in Chapter 4 and research questions relating to his theory are listed there.

Shevarev's psychological analysis was based on written answers in workbooks of students of two eighth-grade classes in a Moscow secondary school. Also, textbooks used by the students were examined in order to ascertain how their structure influenced students' answers.

Shevarev categorised errors in algebra into those which occur when a student: does not know the required 'rule'; cannot apply a 'rule'; and, "knows the rule, is able to apply it, but nevertheless acts contrary to it" (Shevarev 1975, p.1). His discussion focused on this last type of errors, and he examined them within the context of the errors students make when working with indices.

Shevarev does not define, unfortunately, the term 'rule' as he uses it. However, the implication is that a rule for a particular problem is one whereby the student is able to clearly identify the operation to be carried out and then carries out the operation with genuine understanding. For instance, in questions of the type $A^M \times A^N$ the rule would be that A^M means M factors of A multiplied together and A^N means N factors of A multiplied together, so we have $M+N$ factors of A which can be written A^{M+N}.

Shevarev suggested that when students solve problems, they are unaware, almost always, of rules because the recalling of a rule disappears "during the repeated solving of problems of the same type" (p.3). Students can still act in accordance with rules by using "connections" they have developed. These connections being a special combination of mental processes made up of two components. The first component is the recognition of the "general features" of a specific part of an algebraic expression and the second component is the kind of operation that is performed in response to that recognition. In this way, by using only selected features of an algebraic expression to identify the operation to perform, the student is able to reduce a complex problem to a routine process. If the selected features clearly identify the operation to be performed then the connection will be a correct one and give a correct result.

Problems occur when the connections, developed to replace rules, are not formed correctly. Such incorrect connections are formed and reinforced while the student is "correctly solving problems of a definite type" (p.58). They are related to the nature of the teaching and the structure of the textbook.

Two of the three types of incorrect connections discussed by Shevarev relate to students' recognition of general features of a specific part of the expression with which they are working. The first of these is now discussed in considerable detail since it may offer an explanation for errors students make in a variety of index questions. The second was one of the issues examined by Childers whose research is discussed later in this chapter.

In the first type of incorrect connection, students omit some essential feature when recognising the general features of the expression. An example is the mistake of the type $(a^M)^N = a^{M+N}$. Shevarev argued that it is clear from the sequence of teaching, the written answers and the textbook structure that students have "somehow confused the expression $(a^M)^N$ with $a^M \times a^N$ and, therefore, perceiving the first expression, performed an operation pertaining to the second" (p.6). Shevarev, in discussing student responses to $a^M \times a^N$ and $(a^M)^N$, said that features common to both expressions were "the absence of plus and minus signs on the base line and the presence of two exponents ... the specific feature of the first expression is the presence of two identical letters (bases) on the line; the specific feature of the second is the presence of only one such letter" (p.9). The connection formed by the students for multiplying omitted the important specific features. Further, because multiplication of expressions had been more strongly reinforced by more questions, and at an earlier stage, students added indices. As a result students employing this connection obtained correct answers when doing questions involving a product but incorrect answers when raising to a power.

Shevarev continued his argument by saying that this particular error could not be perceived as accidental as it was made by eight of the students in the two classes and was, in fact, the only error. Nor could the error be viewed as forgetting the correct rule and proceeding from an incorrect rule since "in the same assignment the pupils had to formulate, in writing, the rule for elevating a power to a new one, and all pupils in question formulated it correctly" (p.5). This showed that the students certainly understood the rule and yet did not apply it correctly.

He pointed out that the same eight students did, however, obtain correct answers to a similar question of the type $(a^M)^2$, giving an answer of the form a^{2M} (p.9). This he attributed to the fact that "squaring occupies a special place in an elementary algebra course" and "pupils are acquainted with this operation much earlier than elevating a power to a new one" (p.10). This suggests that squaring, and possibly cubing, should be treated as separate cases of raising to a power.

Two other incorrect connections of this first type were also analysed in detail by Shevarev. One of these occurred, as part of another question, when students gave the answer $-x^7 = 1/x^7$. Here they have obviously used a connection which would be correct for questions of the type x^{-n} (pp.18-19). The other was where students "reduced" the expression a^8b^{12}/a^6b^{10} and obtained a^4b^6/a^3b^5. Shevarev noted that those pupils who made this last mentioned error did obtain a correct answer for b^{y-1}/b^{y-2c}. Obviously, they knew the rule for this situation but did not always apply it. Shevarev believed this incorrect connection "arose while the pupils were still practising the reduction of arithmetic fractions ... awareness that both members of the fraction were products and that one number stood in the numerator and the other in the denominator was sufficient basis for reducing the number" (p.21). He said that students saw a^6/a^3 in the same light as $6a/3b$. A point he does not address is how students who make these errors respond to questions where there is both a coefficient and exponent in the numerator and the denominator, e.g., $8a^6/4a^3$.

In the second type of incorrect connection identified by Shevarev, the first component, namely, recognition of the "general features" of a specific part of an algebraic expression, 'contains something which should not have entered into it and which narrows its scope" (p.26). Shevarev found that students who could simplify a^9y^5/ay^{10} could not simplify $a^4(x - y)^5/a(x - y)^{15}$ and explained this by saying "the feature that the power's base is a letter, and not a polynomial enclosed in parentheses, entered into the first component of the connection . . this attribute, of course, does not enter into a correct connection" (p.27). He noted that the reason this occurred could be found in the fact that, in doing such questions, students were given very little data of the type involving a polynomial.

The key issue arising from the above discussion is that students using connections, either correctly or incorrectly, do know the 'rule' to answer the question correctly but normally they do not use it. Shevarev found that "a definite rule is recalled only when a special orientation for recalling rules exists" (p.59) and that this normally arises when dealing with a problem which is new or when working with a new type of data.

Shevarev addressed also the question of why incorrect connections arise in only some students. He suggested some students "have a general orientation, habit, or custom of solving algebraic problems consciously ... to clearly recognise all essential features of the data even when this situation does not demand this awareness of them. Hence the correct connection arises in them" (p.17).

Shevarev carefully, and thoroughly, analysed the errors he identified.

While he did not interview the students who made the errors, it would appear that there could be considerable merit in his explanations of the circumstances leading to these errors. There would seem to be value in teachers adjusting their strategies to take his findings into account.

RESEARCH OF M.A.R.DEVINCENZO

DeVincenzo, in her Ph.D. dissertation *An Investigation of the Relation Between Elementary Algebra Students' Errors in Arithmetic and Algebra in Selected Types of Problems*, looked at the question of whether errors "in arithmetic may transfer to algebraic problems which demand similar skills" (1980, p.3). "Transfer" meant the proposition that "students apply what they previously learnt in mathematics to new mathematical situations" (p.3). Students might therefore tend to make errors in algebra which are essentially the same as their errors in arithmetic. The subjects of the study were students enrolled in an elementary algebra course in 14 Public and Parochial schools in New York City.

DeVincenzo's findings are of special relevance to this present research in that three of the eight types of mathematical problems she examined involved indices. A description of the types and the actual questions used are now given. The letters (D), (E) and (F) used below are the letters which DeVincenzo attached to these particular questions. They are:

(D) Multiplying two numbers in exponential form with the same base

Arithmetic questions	5×5^2	$2^3\times2^2$	$7^6\times7^5$
Algebraic questions	$b\times b^2$	$b^3\times b^2$	$b^m\times b^n$

(E) Raising a number in exponential form to a positive integral power

Arithmetic questions	$(2^3)^2$	$(5^2)^4$	$(3^6)^3$
Algebraic questions	$(a^3)^2$	$(a^2)^4$	$(a^m)^n$

(F) Dividing two numbers in exponential form with the same base

Arithmetic questions	$2^{\text{[illegible]}}/2$	$3^8/3$	$5^{12}/5^3$
Algebraic Questions	$a^{\text{[illegible]}}/a$	a^8/a^6	a^m/a^n

For these questions, DeVincenzo tested the hypothesis that

> For Multiplication, Division, and Simplification of Exponential Problems of this study, there are more instances where errors are different for algebra and arithmetic than instances where

errors are the same for both forms.

(p.14)

In putting forward this hypothesis, DeVincenzo suggested that "one must be aware of the existence of alternate techniques for solving these problems in arithmetic, ways which cannot be used to solve the same sort of problem in algebra" (p.15). This, of course, is because numerical expressions may be correctly evaluated in various ways such as converting a number expressed exponentially into an integer before using it to multiply. For these reasons, DeVincenzo argued that it was not unlikely for students to react differently to algebraic and arithmetic forms. It was also pointed out by DeVincenzo that "results of the NAEP (1975 p.255) indicate that students are not very successful at solving problems with exponents in arithmetic, whatever technique is used" (p.16).

In carrying out her research DeVincenzo designed the problems in arithmetic and algebra "so that corresponding problems would have the same instructions, same written format, same operations, and some of the same numbers ... similar skills" (p.4). Errors were regarded as being consistent if two of the three problems, which were used for each type, were solved in the same incorrect way. An example of the same error in arithmetic and algebra is answering 5^6 for $(5^2)^4$ and a^6 for $(a^2)^4$. A total of 1122 students were given both an arithmetic and algebraic diagnostic test and from these 93 were selected for Observation-Interview sessions.

From the statistical analysis of the written tests it was found, as hypothesised, that there were significantly more instances of different errors than the same errors for the arithmetic and the algebraic forms of the index questions considered in the research. Of those students who made an error in both the arithmetic and algebraic forms in the Type (D) questions, only 12% made the same error, while for Type (E) it was again 12% and for (F) it was 21% (p. 68). Taken as a percentage of the total population, 0.17% made the same error in algebra and arithmetic in (D), 0.71% in (E) and 0.5% in (F) (p.66).

DeVincenzo identified 15 types of errors for the type (D) questions, 32 for the type (E) questions and 20 for the type (F) questions (p.223). When one looks at the few elements, which are part of the type (E) questions, it is remarkable that students have managed to generate so many consistent error types. That this applies also to the other two types of questions clearly demonstrates the extensive difficulties students have with indices.

The error types listed indicate that, when answering an index question, many students were very confused as to the mathematical operation they should apply, and to which elements of the expressions. For example, errors listed for type (D) questions included "performs multiplication and then uses this as a base with sum of exponents for new exponent ... multiplies base times exponent and then divides instead of multiplying" (p.223). The frequency of error types is not provided in the report except for occasional and incidental references to the frequency with which certain errors occurred in the interviews.

The Observation-Interview sessions were designed to "establish the reliability of categorising errors from written solutions and to understand more deeply the nature of specific errors" (p.175). Students were selected at random from each of three categories which had been identified using the written tests. These categories were: (a) students who made the same errors in both the arithmetic and algebraic forms; (b) students who made different errors in each form; and, (c) students who made consistent errors only in algebra and no errors in arithmetic (p.51). While the total of students interviewed was 93, the way they were distributed between categories was not indicated.

From DeVincenzo's discussion of the interviews emerge a number of issues of relevance to this research, and they are given in point form below.

- The transcripts of interviews given in the dissertation confirmed that students have extensive problems in answering index questions.

- In Type (D) problems, i.e., multiplying two numbers in exponential form with the same base, there is a strong tendency to multiply somewhere in the question. The interviews "indicated that certain students adhered to a designated rule for the exponents in this problem type and were guided by the times sign for the bases" (p.127) leading to the result $2^3 \times 2^2 = 4^5$. Multiplying exponents also occurred 23 times in the interviews ($7^6 \times 7^5 = 7^{30}$) and again students indicated in interview that it was because of the multiplication sign. Other errors included a combination of these two, that is, $2^3 \times 2^2 = 4^6$.

- Errors for Type (F) questions, i.e., dividing two numbers in exponential form with the same base, mirrored, to some extent, those for Type (D) questions. Common errors appear to have been: correctly subtracting indices but dividing bases (corresponding to adding indices and multiplying bases in Type (D) ; dividing indices while not changing bases; multiplying base times exponent before dividing; and, dividing both the bases and the exponents (pp.141-145).

- Students do have problems in adapting the additional notation of a raised index to other mathematical conventions. For example, a student giving an answer of 36 to $(3^2)^4$, having multiplied the result of 3^2 by the power of 4, said "They don't have any sign here and usually when the numbers are together and there's no sign, it's usually times" (p.135). Similar errors occurred in Type (D) questions giving results such as $7^6 \times 7^5 = 42 \times 35$ (p.129).

- Some students recognised "the exponential format" and correctly "performed computations associated with exponents", but not in a correct order. For example $(2^3)^2 = 2^9$ where they have raised the 3 to the power of 2 (p.133).

- Students who lack basic understanding in algebra obviously have considerable difficulty with index questions. For example there was evidence that some students, who gave the answer b^{mn} for $b^m \times b^n$, regarded mn as meaning $m+n$ (p.129)

- "Rote activities seemed to be the key to how many students applied correct as well as incorrect algorithms" (p.188). Here DeVincenzo is implying that students are not working from genuine understanding but are applying rules they have developed. This supports Shevarev's argument that students are using connections. It may be that this application of an 'incorrect algorithm' is the result of an incorrect connection.

- "The acquisition of mathematical jargon without much understanding was frequently demonstrated during the interviews" (p.189). As with the previous point, this indicates that students are not applying genuine understanding.

- "Discussion with some teachers in the participating schools verified that many topics were 'covered' but not taught with mathematical meaning and certainly not individualised for particular students" (p.189).

- DeVincenzo observed that "several discussions with students during the Observation-Interview sessions confirmed the tendency for students to experience difficulties in only the algebraic forms of these problem types" (p.82). It is surprising there was no attempt to substantiate this important point using evidence of the testing. Even more so when one takes into account that her discussion of errors is more about arithmetic problems than algebraic problems.

The fact that the frequency of errors was not included in the research report makes it difficult to isolate which were the common errors. Despite DeVincenzo's observation that problems were more common with the algebraic form than the numerical form, her discussion does indicate that difficulties students have in operating with numerical bases is an issue which should be pursued. DeVincenzo's research highlights the fact that errors are extensive and that student strategies are not particularly obvious. She used the interviews to "understand more deeply the nature of specific errors" (p.175) but did not attempt to integrate the findings for specific errors into more general explanations of why they are made. This is obviously an area which needs to be addressed in this current research.

RESEARCH OF G.F.CHILDERS

In 1987 Childers published her Ph.D. dissertation, *An Error Analysis of Algebra Problems Containing Exponents when Constants, Variables, and Algebraic Expressions are Interchanged in the Laws with Implications for Cognitive Style Theory and Latent Trait Test-Scoring Methods*. This research investigated systematic errors made by students when using index laws.

The following laws for exponents were investigated:

1) $a^m.a^n = a^{m+n}$
2) $a^m/a^n = a^{m-n}$
3) $(a^m)^n = a^{mn}$
4) $(ab)^m = a^mb^m$

(Childers 1987, p.12)

Childers examined the differences in error patterns which occurred in problems using these laws when: (i) the exponents in the expression were interchanged between being constants and variables; and, (ii) the bases were interchanged between being constants, variables and algebraic expressions. Childers generated six types of problems using various combinations of base and exponent. These types were:

TYPE 1 Problems involving powers in which the bases are variables and the exponents are constants representing non-zero integers, for example, $a^3.a^2$, a^3/a^2, and $(a^3)^2$.

TYPE 2 Problems involving powers in which the bases are constants representing positive integers and the exponents are variables, for example, $3^m.3^n$, $3^m/3^n$, and $(3^m)^n$.

TYPE 3 Problems involving powers in which both the bases and the exponents are variables, for example, $a^m.a^n$, a^m/a^n, and $(a^m)^n$.

TYPE 4 Problems involving powers in which the bases are binomial expressions and the exponents are constants representing non-zero integers, for example, $(x+y)^2(x+y)^3$, $(x+y)^2/(x+y)^3$, $[(x+y)^2]^3$.

TYPE 5 Problems involving powers in which the bases are constants representing positive integers and the exponents are constants representing non-zero integers with at least one negative exponent in each problem, for example, $2^{-4}.2^8$, $2^{-4}/2^8$, and $(2^{-4})^8$.

TYPE 6 Problems involving powers of a term made up of a coefficient, a variable to an integral power (greater than 1) and a variable with an unwritten exponent of 1, for example, $(3x^2y)^{-2}$.

(pp.12-13)

Childers made considerable reference to the work of DeVincenzo, whose thesis was discussed previously and used it for guidance as to the types of errors students make in index questions. Apart from Type 5, which involved a negative index, Childers chose not to examine errors occurring where the base and the exponent were both constants. The reason given was that "DeVincenzo found that more errors and more different errors than same errors were made in algebraic exponent problems than in exponent problems involving bases and exponents which were both constants" (p.37).

The subjects of this study were 52 students enrolled in Basic Algebra I at the J.Sargeant Reynolds Community College in Richmond Virginia. Students were tested for field-independent and field-dependent cognitive style and then given a test containing sixteen tasks, each task having five repeated measures. A systematic error was regarded as being the same error for three of the five repeated measures. The tasks and Childers findings are discussed below.

Fifteen of the tasks were problems applying the first three laws listed above to each of the expressions of Types 1 to 5. For Law 1 there were five questions of the form $a^3.a^4$ and similarly for the forms $3^m.3^n$, $a^m.a^n$,

$(x + y)^3.(x + y)^2$ and $2^{-4}.2^8$. The same was done for Laws 2 and 3. Childers found "there was a significant difference in the number of systematic errors in Types 1 to 5" using the first three laws of exponents and "students quite often make different types of errors on exponent problems when constants, variables, and algebraic expressions are interchanged" (pp.135-136).

The sixteenth task was simplifying expressions of the form $(2y^3z)^4$. This was to determine how students applied the exponent outside the brackets to each of the three components of the product inside (in effect treating each question as three problems). The components were a constant coefficient, a variable to an integral power (neither zero nor one) and a variable to the power of one (the power being understood, not written). She found there was a significant difference in the errors students make when applying the exponent to each of these components (Childers 1987, pp.121-122).

Childers also tested the relationship between the number of systematic errors made and the field-independent or field-dependent cognitive style of the subjects. She referred to the work of Shevarev and stated that "of special interest to this study is Shevarev's conclusion of the reason for the error made by four students who got the problem a^9y^5/ay^{10} and $x^{A+1}.A$ / $x.A^{1-y}$ correct but missed $A^4(x - y)^5/A(x - y)^{15}$" (p.28). Shevarev's explanation of this error was discussed earlier in this chapter. He said that in such questions "the feature that the power's base is a letter, and not a polynomial enclosed in parentheses, entered into the first component of the connection" (Shevarev, p.27). This meant that students looked on the case with a polynomial base as being quite different to the one where the base was a letter, and so did not bring the appropriate strategy to bear.

Within her Type 4 classification of questions, Childers investigated students' responses to questions involving polynomial bases. She "felt that the reason or reasons why many students seemed puzzled by Type 4 problems ... was either related to the students' spatial abilities, abstract abilities, or previous mathematical training" (p.51). Childers proposed that "one theory which seemed to connect all of these constructs was Herman A.Witkin's (and associates) theory of field-dependent and field-independent cognitive styles" (p.51). However, in her research, she found that this theory did not explain the errors and that "Systematic errors in exponent problems were not significantly related to the cognitive style of a subject" (p.136).

Childers listed sixty-nine error types which had been identified by other researchers (mainly DeVincenzo) in these types of questions. In addition, she identified another forty six through testing and ten additional error types through audio-taped interviews (p.91). Brief mention only is made of the interviews, this being that "Ten audio-taped interviews with students taking a shortened version of this test were also used to locate errors and to confirm other errors already found" (p.138).

Childers provided a table indicating the proportion correct for each of the items (pp.208-209). Besides giving an indication of the relative difficulty for students of each of the individual items it can also be used to indicate the relative difficulty of each Type within each Law. Examination of Childers' table showed that for Law 1/Type 1 questions, the percentages correct were: $a^3.a^4$ (77%); $x^{-5}.x^2$ (60%); $y^4.y^7$ (87%); $z^9.z^{-2}$ (75%); and, $b^6.b^{-1}$ (81%).

Since all candidates were given all questions, these results can be averaged to show that, in general, 76% of all answers to these questions were correct. A similar analysis by this present researcher, of each of the combinations of Laws and Types presented in Childer's data provided the results listed in Table 2.1.

Table 2.1. A Summary of Student Performance on Index Questions Developed from Childers' Data

Category	Example	% Correct
Law 1, Type 1	$a^3.a^4$	76%
Law 1, Type 2	$3^m.3^n$	59%
Law 1, Type 3	$a^m.a^n$	77%
Law 1, Type 4	$(x + y)^m.(x + y)^n$	41%
Law 1, Type 5	$2^{-4}.2^8$	61%
Law 2, Type 1	x^7 / x^2	72%
Law 2, Type 2	$2^a / 2^b$	58%
Law 2, Type 3	c^m / c^n	62%
Law 2, Type 4	$(x + 4)^6 / (x + 4)^3$	45%
Law 2, Type 5	$4^{-5} / 4^6$	54%
Law 3, Type 1	$(a^2)^3$	84%
Law 3, Type 2	$(7^m)^A$	92%
Law 3, Type 3	$(x^y)^m$	90%
Law 3, Type 4	$[(x + y)^7]^3$	35%
Law 3, Type 5	$(2^{-3})^4$	72%

Table 2.1 shows that students had most difficulty when dealing with a base which was a polynomial. However, it is also apparent, from the table, that the error rate for Laws 1 and 2 is noticeably higher where the base is a constant than when it is a single variable. The fact that the same did not apply for Law 3 with Type 2 questions may be because students' options were limited by the fact that both the exponents were variables. This observation is supported by the higher error rate for Law 3, Type 5 questions where the indices are constants.

Type 6 questions were concerned only with Law 4 and were all of the form $(4A^5B)^3$. The overall success rate with these questions was 41% which indicates that students had considerable difficulty with them.

RESEARCH OF F.J.WILSON

In 1985 Wilson published his Ph.D. dissertation, *A Clinical Investigation of the Understanding of Exponents by Remedial Algebra Students at a Four Year College.* In this research Wilson investigated the following questions:

1. Do remedial algebra students have a relational, instrumental or no understanding of the prerequisites conjectured as necessary for success in dealing with the concept of exponent?
2. Do remedial algebra students have a relational, instrumental or no understanding of the concept of exponent?
3. Do remedial algebra students have the ability to generalise the various properties of exponents?
4. What types of imagery do students use when working with the concept of exponent?
5. Do successful students differ from unsuccessful students with respect to the four questions above?

(Wilson 1985, p.7)

In his investigation, Wilson used a theoretical framework which was "developed by combining critical features of each of the theories of Bruner, Skemp, Krutetskii and Gagne" (p.9). Wilson interviewed students and classified, qualitatively, their understanding of the concept of exponent as belonging to one of the cells of the matrix provided in Figure 2.1.

		UNDERSTANDING	
		Instrumental	Relational
MODE	Iconic		
	Symbolic		

Figure 2.1. Wilson's Framework for Categorising Understanding

In this matrix, 'instrumental' and 'relational' refer to Skemp's classification of understanding while 'iconic' and 'symbolic' refer to Bruner's modes of internal representation of information.

The subjects of the research were students from the Ferris State College in Michigan. Students from two courses, a beginning algebra course (comparable to a ninth-grade algebra course) and an intermediate algebra course, were selected for interview on the basis of a diagnostic examination delivered after the completion of a unit on exponents. It is relevant to note that, in the diagnostic examination, the most commonly missed problems were x^0, 2^0, $8^{2/3}$, $4^2/2^2$, $2^3 + 2^2$, $x^3 + x^2$, $(xy)^2$ and $(x^3)^2$ (p.140). Wilson mentions also that, in unit exams, twenty-seven students out of one hundred and fifty-nine beginning students missed $3^4.3^2$, while twenty-three and twenty-eight out of thirty-two intermediate students missed $2^5.5^5$ and $14^7/7^5$ respectively (p.140).

Fourteen students across the two groups were interviewed. Students were asked to explain their thinking, while working through forty-nine problems covering basic operations involving indices. The interviewer "was free to offer hints of encouragement, challenge and contradict, or to present related problems in an effort to 'draw out' the thought processes of the students" (p.136).

Wilson found that "beginning and intermediate algebra students, both successful and unsuccessful, have a relational understanding of positive integer exponent ... it is with the use of the various exponential properties that instrumental understanding replaces relational understanding" (p.229). In discussing his results, Wilson expressed considerable concern that students have little relational understanding of indices and carried this through to a more general observation on algebra by saying that "one must question the educational priorities of a system which enables a student to continually move up the educational ladder by continually memorising material" (p.240).

Most of the students interviewed worked within the symbolic mode and only used the iconic mode when input from the interviewer stimulated such a response. Wilson observed that "The primary difference in imagery usage between the relational thinkers and instrumental thinkers appeared to be the fact that the students with relational understanding recognised that they could use numerical imagery" (p.233).

Wilson's comment on the failure of some students, "to notice all the relevant features of a problem ... frequently home in on one surface feature that matches a feature of an algorithm available to them" (p.232), is very interesting in the light of Shevarev's findings that certain students are using 'connections' when working with indices.

Wilson's research confirms that there are widespread problems existing in the understanding of indices and that interviewing is a valuable tool for obtaining insight into the difficulties students have.

RESEARCH ISSUES ARISING FROM THE LITERATURE

The literature does indicate that students make many and varied errors in questions involving indices. The diversity of errors is no doubt related to the many combinations of operations available to students in dealing with the components of such questions. A remarkable number of erroneous answers come from the simplification of $(3a^2)^4$. How many more errors can be made by Year 8 students when confronted, in a typical popular textbook in New South Wales, by the simplification of "$(3a^2)^2 \times 5a^3/9(a^3)^2$" (McSeveny, Conway and Wilkes 1988, p.122)?

In making sense of students' difficulties in this area, there seems little point in simply continuing to catalogue errors. A more productive strategy is likely to be one which focuses on students' thinking when answering index questions and, in particular, when making the more common errors. Under such an approach an examination of the extensive range of questions generated by the literature gives rise to four issues described below.

Issue 1

The research into errors students make while undertaking index questions has been carried out in other countries.

Issue 2

In questions which require multiplication or division of expressions involving exponents: what is the thinking behind students' tendency to multiply or divide numerical bases when they do not do this if the bases are variables?

Issue 3

What relationships do students see as holding between the components of a numerical expression involving an index and a power to which that expression is raised? What impact does the inclusion of variables/constants have on their perception?

Issue 4

The concept of connections was used by Shevarev to explain inconsistencies in student responses across similar questions.

(a) Is there evidence to support this hypothesis?

(b) Are there other instances in the topic of Indices where this concept seems to apply?

The first of these research issues is resolved in the following chapter while the others are considered later in the thesis.

ISSUES IN CATEGORISING ERRORS

In determining an appropriate classification of errors for this current research it is of value to consider how errors were classified by the four researchers whose work has been discussed.

Shevarev chose to categorise errors according to the type of incorrect connection he saw as generating them. There are two such types which seem most relevant. Firstly, there are those where the first component of the connection, the recognition of the 'general features' of a specific part of an algebraic expression, is missing some essential feature. Secondly, there are those where the first component contains something unessential which narrows the connection's scope. The sequencing of course content is a vital aspect of the development of connections. Students having a commonality of experiences, given by following a particular learning sequence, are likely to develop the same connections and, if the connections are incorrect ones, make the same errors. Shevarev considered only errors made in algebra but it is possible his theory has applications in many other fields.

DeVincenzo classified errors by the index law being used, and then looked at whether the same errors were being made in arithmetic and algebraic forms. The three laws she considered were: multiplying two numbers in exponential form with the same base; raising a number in exponential form to a positive integral power; and, dividing two numbers in exponential form with the same base. She found that students made different errors in the algebraic forms to those made in the arithmetic forms.

Childers classified errors according to the nature of the bases and indices in a particular problem. She used six categories of problems generated from various combinations of having a constant or variable for an index and a constant, variable or algebraic expression for the base.

Wilson classified responses according to the mode of internal representation of the information and whether relational or instrumental understanding was demonstrated.

It is apparent, from the experiences of these researchers, that it would be of benefit to have a classification of errors which takes into consideration three particular issues: (1) the important role the sequence of instruction may play in generating errors; (2) the distinction between algebraic and arithmetic types of problems; and, (3) the great variety of errors which do occur. Accordingly, a categorisation of errors which accounts for both the content and sequence of the syllabus is indicated. Earlier in this chapter the order in which students' experiences commonly develop was described. From that sequence, broad categories for classifying errors can be identified. These are errors in:

(A) Evaluating a Numerical Expression.
(B) Interpreting an Algebraic Expression.
(C) Simplifying an Algebraic Expression.
(D) Substitution and Evaluation.
(E) Changing a Subject.
(F) Scientific Notation.

In the following chapter an extensive examination of the errors students make is undertaken. The categories listed above provide a useful overall framework for that examination. Within each of the six categories, there will need to be sub-categories to take account of the great variety of errors.

CONCLUSION

Although the research in this field is limited, it is sufficient to indicate that many students do have difficulties in answering index questions and that numerous, systematic errors are being made. The research consistently supports the proposition that many students, even when giving a correct answer, are not applying a genuine understanding of the meaning of an index but are using other strategies. An important point, which needs to be addressed, is the extent of the use of other strategies.

While researchers agree that many students are not applying an understanding of index rules they appear to disagree on whether or not students do have that understanding available and could use it if circumstances encouraged them to do so. In reporting her interviews, DeVincenzo implied that errors were simply a result of students not understanding the concepts. Shevarev, on the other hand, believed many errors result when a student knows a rule "but nevertheless acts contrary to it" (Shevarev 1975, p.1). The explanations offered by Shevarev are thought provoking and bear further research. The clarification of this issue is obviously a crucial one for research in this area. This requires a close examination, not only of the errors being made, but of the understandings students have developed, and are applying.

The curriculum context within which students in N.S.W. gain their understanding of indices has been discussed. Mathematical performance, stemming from that curriculum is assessed near the end of Year 10 using an external examination called the N.S.W. School Certificate Moderator. By that stage, index skills needed to support students' work in mathematics in the two senior years of schooling should have been acquired. An analysis of responses to index questions contained in that public examination is the subject of Chapter 3.

The analysis of the School Certificate data will address the first of the issues listed above, namely, that previous research into errors students make while undertaking index questions has been carried out in other countries. The analysis will determine whether Australian students are making the same sorts of errors as identified in the literature. Additionally, it is an important source of evidence on how significant the other three issues are likely to be in this examination of students' understanding of indices. Finally, from that analysis, new issues which warrant further research will undoubtedly emerge.

Chapter 3

AN ANALYSIS OF ERRORS MADE IN INDEX QUESTIONS BY N.S.W. SCHOOL CERTIFICATE CANDIDATES, 1981-91

Introduction to Chapter

In Chapter 2 the position of indices in the Mathematics Curriculum was considered and existing research in the field of indices examined. It appears that work in this area is limited and there is little to indicate a strong direction for future research.

It is fortunate that there already exists a substantial data source covering an extensive range of questions involving indices. This source is the subject of the present chapter. It is in the form of student responses to questions in the objective answer section of the School Certificate Moderator, an examination undertaken by students throughout N.S.W. near the end of Year 10. The percentage of students choosing particular responses to questions in these examinations are published each year. There has been no previous research in which data from the questions involving indices have been collated, and the nature of student responses explored. In this chapter the source of the data is discussed, the way in which the errors are categorised is explained, and the data are analysed. From the analysis, research themes concerning students' understandings of indices are drawn.

THE SOURCE OF THE DATA - THE N.S.W. SCHOOL CERTIFICATE

The New South Wales School Certificate Moderator examination is administered each July to Year 10 students of N.S.W. schools, both Government and Private. Since 1983 it has been set at three levels, Advanced, Intermediate and General with the candidature for each being as described in Chapter 2. The data are drawn from the 1981 and 1982 papers, which covered all levels of ability, and from the Advanced and Intermediate papers from 1983 to 1991.

Questions discussed below are from the Part A section of each paper, the section for which statistics are readily available. This is made up of 50 multiple-choice questions which count half the total value of the paper.

There are limitations to such data when researching students' understanding, these being the constraints posed to answers by the limited number of possible responses, and the fact that students' thought processes are not being explored. However, the data are drawn from a remarkably extensive and representative sample (i.e., approximately 80 000 students each year), and, as such, provide a wealth of information on which to establish directions for research.

CATEGORISATION OF ERROR TYPES

As discussed in the preceding chapter, there is benefit in categorising errors in a way which accounts for: the sequence of instruction; the distinction between algebraic and arithmetic types of problems; and, the great variety of errors which occur.

This research categorises the occurrence of errors according to both the content and sequence of the syllabus under six main headings, previously identified, from the course of study followed by students in N.S.W. schools. These are hierarchical in nature, beginning with errors in arithmetic then moving into problems of understanding, and operating with, algebraic expressions. Arithmetic and algebra come together in questions involving substitution. Operations with indices and equation-solving skills both come into play when dealing with a change of subject. Finally, a specialist application of indices in the form of scientific notation is considered. To take account of the great variety of question types, the categories are further subdivided into a total of nineteen sub-categories. These are also hierarchical in that, within the six areas, the sub-categories are arranged in the typical order in which the content is taught.

The categories for classification of error types are:

(A) Evaluating a Numerical Expression.
  (a) Multiplying Numbers with the Same Base.

(b) Dividing Numbers with the Same Base.
(c) Raising a Term to Another Power or Taking a Root of a Term.
(d) Operations Involving a Zero or Fractional Index.

(B) Interpreting an Algebraic Expression.
(a) Writing an Expression Involving Repeated Factors.
(b) Giving the Expanded Form of an Expression.
(c) Questions Involving Negative and/or Fractional Indices.

(C) Simplifying an Algebraic Expression.
(a) Collecting Like Terms Involving Indices.
(b) Finding the Product of Algebraic Terms.
(c) Finding the Quotient of Algebraic Terms.
(d) Finding Powers of Algebraic Terms.
(e) Simplifying Algebraic Fractions of Greater Complexity.
(f) Taking the Square Root of an Algebraic Term.
(g) Simplifying Expressions Involving a Zero Index.

(D) Substitution and Evaluation.

(E) Changing a Subject.

(F) Scientific Notation.
(a) Writing a Decimal Number in Scientific Notation.
(b) Converting a Number in Scientific Notation to a Decimal.
(c) Written Problems Involving Scientific Notation

The N.S.W. School Certificate Moderator examination is designed to assess students' success at the end of Year 10 and, as such, covers all the content areas under consideration. Although the test is wide ranging, the questions tend to be clearly identifiable with particular parts of the syllabus, and are readily classified. From the test papers considered, all questions which fall clearly within each category have been selected for analysis. It should be noted that links between the concepts in different content areas mean, almost certainly the analysis will generate research questions which are relevant across categories.

ANALYSIS OF THE DATA

In the analysis, the following information is provided for ease of reference. For each content area, questions are given a sequential number and then the year, level and number of the question in the examination paper, from which it came, is identified. The terms "Advanced" and "Intermediate" are abbreviated to "Adv." and "Int.". Questions from Advanced papers are listed, followed by Intermediate questions and then questions from 1982 and 1981, years in which the papers covered all the levels of ability. Within these groupings, the questions are ordered from most recent to least recent. The percentage of students, giving each response, is provided in parentheses after the response (percentages are rounded and may not total exactly 100%). The percentage population making the correct response is underlined.

Following each set of questions, there is an analysis section within which frequent errors are examined. Where relevant, comparisons are made concerning how different candidatures performed on similar questions, and how the same candidature performed on different questions. Any relationship that seems to exist between performance in the different categories is commented upon. Issues appearing to warrant further research are drawn from the points made in the analysis and are listed under the heading 'For Investigation'.

It should be noted that where distractors are referred to as 'corresponding' this means they are arrived at using the same operations. For example, the response of 3^6 for $3^3 \times 3^2$ corresponds to 5^{12} for $5^3 \times 5^4$ since, in both, the bases were left unchanged and the indices multiplied.

(A) Evaluating a Numerical Expression

(a) Multiplying Numbers with Numerical Bases

There were four questions in this category. One of these was asked of both the Advanced and Intermediate candidature in the same year (1984).

[1] $3^3 \times 3^2 = ?$ (1984 Adv. Course Question 1)

(A) 3^5 (68%) (B) 3^6 (2%) (C) 9^5 (28%) (D) 9^6 (2%)

[2] $5^3 \times 5^4 = ?$ (1986 Int. Course Question 1)

(A) 5^7 (48%) (B) 5^{12} (3%) (C) 25^7 (45%) (D) 25^{12} (4%)

[3] $3^3 \times 3^2 = ?$ (1984 Int. Course Question 1)

(A) 3^5 (44%) (B) 3^6 (3%) (C) 9^5 (47%) (D) 9^6 (5%)

[4] $2^3 \times 3^2 = ?$ (1982 Question 2)

(A) 6^5 (66%) (B) 6^6 (5%) (C) 36 (1%) (D) 72 (28%)

While many students realised the indices needed to be added in [1], [2] and [3], there was a great temptation to multiply bases. This shows many candidates did not have, or were not using, genuine understanding. Such responses would be precluded if students worked from a position of sustained awareness that an index meant a repeated factor. Items [1] and [3], being the same question, but for different candidatures, showed that the error of multiplying bases occurred less frequently with Advanced students. Nevertheless, the error rate was still high with almost a third of this able group (approximately the top 36% of the age cohort) choosing that option.

Very few students multiplied indices. That examiners persisted with these distractors perhaps supports the view that wrong responses 'mathematicians' expect from students often are not the ones that actually occur.

It is possible that students look for an answer involving indices. This is shown in [4] where the correct numerical response ((D)) is not particularly popular compared to the correct responses in the other questions.

A consistency across the same level candidature in different years is demonstrated in the results for Items (2) and (3). The questions are very alike and an examination of the responses shows remarkably similar percentages choosing corresponding distractors (which, from left to right, happen to be in the same order).

For Investigation: (i) What meanings do students attach to integral indices?

(ii) When multiplying numbers with the same base, why do some students multiply bases as well as adding the indices?

(b) Dividing Numbers with Numerical Bases

One question, only, required students to divide expressions with numerical bases. The base in both expressions was 2 and it may be that the equality, or otherwise, of bases is an issue affecting responses.

[1] $2^{12} \div 2^{2} = ?$ (1983 Int. Course Question 16)

(A) 1^{6} (10%) (B) 2^{6} (6%) (C) 1^{10} (40%) (D) 2^{10} (<u>44</u>%)

This question appeared in the 1983 Intermediate paper which was directed at approximately the middle 43% of the age cohort. The majority of students chose the correct option with the indices ((C) or (D)), as they did for multiplication. Still, almost half of these also chose incorrectly to divide the bases ((C)). This again shows lack of application of genuine understanding of the meaning of an index.

The fact that students were satisfied with an answer of 1^{10}, an answer which implies the numbers being divided were equal, shows the answer was either not put under scrutiny by the students or not viewed with understanding. As for multiplication, it appears that many students are not working from the basis of an integral index as indicating a repeated factor.

Half of the candidates (responses (A) and (C)) chose to divide bases in this question. This closely mirrors the proportion of Intermediate candidates who operated on bases in the multiplication questions discussed in the previous section (49% for Item 1 and 52% for Item 3).

For Investigation: (i) When dividing numbers with numerical bases, why do students divide bases as well as subtract indices?

(c) Raising a Term to Another Power or Taking a Root of a Term

Three questions involved the application of a power to a numerical expression either by the use of an index outside parentheses or a radical sign. The success rate was not high.

[1] $\sqrt{16^8} =$? (1987 Adv.Course Question 10)

(A) 4^4 (43%) (B) 4^8 ($\underline{51}$%) (C) 8^4 (3%) (D) 8^8 (1%)

[2] $(4\sqrt{3})^2 =$? (1982 Question 21)

(A) 12 (10%) (B) $16\sqrt{3}$ (38%) (C) 36 (12%) (D) 48 ($\underline{40}$%)

[3] $(2^3)^2 =$? (1981 Question 9)

(A) 2^5 (19%) (B) 2^6 ($\underline{52}$%) (C) 2^9 (4%) (D) 4^6 (24%)

Many students found difficulty in determining whether the base, the exponent or both were altered. Responses [1] (A), [2] (B) and [3] (D) seem to indicate that students want to apply the familiar operations of squaring or finding the square root to the more obvious number only.

It seems many students confused raising to a power with finding the product of terms which have the same base (see response [3] (A)). Shevarev used an example of this type when explaining his first incorrect connection.

Questions [2] and [3] both required students to square a numerical expression. Despite the operation being the same, there is no obvious similarity in errors between the two questions. While this is largely a function of the distractors provided, it would appear the nature of the term being squared largely determines the kinds of errors made.

For Investigation: (i) What relationship do students see existing between a root or power and the term to which it is being applied?

(ii) Do students confuse raising to a power with finding the product of two terms with the same base?

(d) Operations Involving a Zero or Fractional Index

There were two questions in this category, the second of which required students to have an understanding, also, of the index of a half.

[1] $8 \times 3^0 = ?$ (1983 Int.Course Question 5)

(A) 0 (10%) (B) 1 (3%) (C) 8 (62%) (D) 24 (25%)

[2] $16^0 + 16^{1/2}$? (1982 Question 40)

(A) 4 (15%) (B) 5 (42%) (C) 8 (27%) (D) 9 (16%)

Responses seem to indicate a different interpretation of the zero index between the two candidatures. In [1], 10% of candidates appear to have multiplied the base and index or interpreted a zero index as generating a zero answer in some fashion. The second question, having been examined in 1982 when the candidatures were combined, covers a broader range of abilities than the first. In [2], 42% of students ((A) and (C)) appear to have done the same thing. While the candidatures differ, the results are surprising. Perhaps this is related to the fact that in [2] there was no option to treat 16^0 as being simply 16. That type of option attracted 25% of the candidature to select (D) in [1] and it may be that many of these could be those choosing (A) and (C) in [2]. Whatever the explanation, it is evident that a significant source of error results from treating the zero index as giving an answer of zero or as being able to be ignored.

For Investigation: (i) What meanings do students attach to the zero index?

B) Interpretation of an Algebraic Expression

(a) Writing an Expression Involving Repeated Factors

The two items in this category came from papers for the Advanced course. Both questions were framed in words.

[1] In a certain area of Australia the rabbit population doubles each year. At the beginning of one year there were x rabbits in the area. How many rabbits will there be in the area after n years? (1987 Adv.Course Question 24)

(A) $2x^{n-1}$ (7%) (B) $2x^n$ (61%) (C) $2^{n-1}x$ (8%) (D) $2^n x$ (<u>24</u>%)

[2] Which expression shows the product of p factors, each of which is m? (1985 Adv.Course Question 10)

(A) pm (52%) B) p^m (20%) (C) m^p (<u>24</u>%) (D) $p+m$ (3%)

These questions show that even capable students have great difficulty describing situations requiring the use of indices as a shorthand for repeated factors. Comprehension of a written question may have been a factor in the error rate of [1] and, perhaps, [2]. This is something which needs to be kept in mind when considering responses to any questions posed in words.

The success of distractor (B) in [1] may indicate a preference for applying an index to a pronumeral, rather than to a number, where such an option exists. Textbooks rarely use an expression of the form $2^n x$ and students may be conditioned not to expect such an answer.

The popularity of response (A) in [2] indicates students may have a strong preference for multiplication in situations where the word 'factor' is used. In [2], those who realised that an index would describe the repeated factor were equally divided over which pronumeral was the base and which was the power. Is this a result of the wording, or of a lack of genuine understanding?

For Investigation: (i) What are the difficulties students have when translating cases, where a factor repeats, into algebraic notation?

(ii) Is there a particular problem with students accepting that a pronumeral can be the index?

(b) Giving the Expanded Form of an Expression

The two questions in this section were similar in structure and had corresponding distractors. Answers for the first question were all written in fully expanded form. Multiplication signs, between the unknowns a and b, were omitted from two distractors of the second.

[1] $\frac{1}{2}at^2 = ?$ (1985 Int.Course Question 23)

(A) $\frac{1}{2}\times a\times t\times t$ (68%) (B) $\frac{1}{2}\times a\times t\times 2$ (4%) (C) $\frac{1}{2}\times a\times a\times t\times t$ (16%) (D) $\frac{1}{2}\times a\times t\times\frac{1}{2}\times a\times t$ (11%)

[2] $3ab^2 = ?$ (1984 Int.Course Question 15)

(A) $3ab\times 3ab$ (30%) (B) $3\times ab\times ab$ (14%) (C) $3\times a\times b\times 2$ (2%) (D) $3\times a\times b\times b$ (54%)

Students had considerable success, though the main error again appeared to be applying the index to all or several of the bases preceding it. That students were less inclined to do this in the first question may relate to the coefficient being a fraction. However, it may also be that the omission of the multiplication sign, between the coefficient and bases, in (A) of [2] made it more attractive than was its corresponding distractor, (D) of [1].

For Investigation: (i) What relationships do students see existing between an index and numbers or pronumerals preceding it?

(c) Questions Involving Negative and/or Fractional Indices

All eight items in this group were from Advanced papers. Each required students to convert an expression to, or from, index notation.

[1] Which expression is equivalent $4x^{-1}$ (1991 Adv.Course Question 39)

(A) $-4x$ (5%) (B) $\frac{1}{4x}$ (61%) (C) $\frac{x}{4}$ (5%) (D) $\frac{4}{x}$ (28%)

[2] Which is equivalent to $64a^{2/3}$? (1990 Adv.Course Question 21)

(A) $16\sqrt[3]{a^2}$ (16%) (B) $64\sqrt{a^3}$ (14%) (C) $64\sqrt[3]{a^2}$ (67%) (D) $512\sqrt{a^3}$ (3%)

[3] Which expression is equivalent to x^{-2}? (1990 Adv.Course Question 42)

(A) $-\sqrt{x}$ (7%) (B) $-x^2$ (7%) (C) $\frac{1}{\sqrt{x}}$ (16%) (D) $\frac{1}{x^2}$ (69%)

[4] $x^{-1}y^{1/4} = ?$ (1988 Adv.Course Question 7)

(A) $-x\sqrt[4]{y}$ (15%) (B) $\frac{-x}{y^4}$ (10%) (C) $\frac{1}{xy^4}$ (26%) (D) $\frac{\sqrt[4]{y}}{x}$ (49%)

[5] $2x^{-1/2} = ?$ (1987 Adv.Course Question 30)

(A) $\frac{2}{\sqrt{x}}$ (29%) (B) $\sqrt{\frac{2}{x}}$ (3%) (C) $\frac{1}{2\sqrt{x}}$ (23%) (D) $\frac{1}{\sqrt{2x}}$ (40%)

[6] $\dfrac{1}{2x^3} = ?$ (1985 Adv.Course Question 11)

(A) $2x^{1/3}$ (12%) (B) $2x^{-3}$ (61%) (C) $\frac{1}{2}x^{1/3}$ (8%) (D) $\frac{1}{2}x^{-3}$ (19%)

[7] $5x^{-1/2} = ?$ (1984 Adv.Course Question 9)

(A) $\dfrac{-1}{\sqrt{5x}}$ (28%) (B) $\dfrac{5}{x^2}$ (13%) (C) $\dfrac{1}{5\sqrt{x}}$ (33%) (D) $\dfrac{5}{\sqrt{x}}$ (25%)

[8] $3a^{-2} = ?$ (1983 Adv.Course Question 5)

(A) $\dfrac{1}{3a^2}$ (62%) (B) $\dfrac{3}{a^2}$ (28%) (C) $\dfrac{-1}{3a^2}$ (7%) (D) $\dfrac{-3}{a^2}$ (3%)

The success rate for these questions was very low despite the candidature being the most able (Advanced) students. Applying the index to both the coefficient as well as the base was the main source of error. In each case, where the index was a negative integer and the term included a coefficient (Items [1], [6] and [8]), this error was made by more than 60% of candidates. Where the index was a negative fractional index (Items [5] and [7]) the error was less, though still high. Distractor (A) in (7) may have proved more popular were it not for the attachment of a negative sign to the numerator. Despite the high error rate, it is clear, from the responses across the eight questions, that the great majority of students understand a negative index means reciprocal and an index of $\frac{1}{2}$ means square root.

Though appearing as structurally complex as the other questions, [2] and [4] were completed much more successfully. In [2], this could be because students were encouraged to think more critically by the larger coefficient or because there was no option where 64 was included within the radical sign. In [4], the lack of a numerical coefficient, or the fact that each base had an index attached, may have made the question easier.

The simple interpretation of a negative index, in [3], was well answered. This supports the proposition that the problem is not so much in understanding the meaning of a negative or fractional index but in applying it in less straight-forward situations. It appears that the error, which Shevarev identified, of associating $1/x^7$ with $-x^7$ does not occur frequently if the question is simple. However, where questions have been more complex, such as in [1] and [7], one in four students have chosen to interpret a negative index as leading to a negative expression of some sort. Hence, when the cognitive load associated with a question is higher, students are more likely to make an error with the interpretation of the negative sign. Could the theory of Shevarev explain why students seem to have understanding and yet not apply it correctly?

For Investigation: (i) What relationships do students see existing between an index and numbers or pronumerals preceding it?

(ii) Why do students find less problems where each or several of the bases have an index attached?

(iii) Why do students have less problems working with pronumeral bases than with numerical bases?

(C) Simplifying an Algebraic Expression

(a) Collecting Like Terms Involving Indices

Three of the four questions in this section came from Intermediate papers. The question from the combined paper of 1982 had the apparent added complication of involving operations with fractions.

[1] Simplify m^3+m^3 (1991 Int. Course Question 2)

(A) $2m^3$ (32%) (B) m^6 (35%) (C) $2m^6$ (22%) (D) m^9 (12%)

[2] $3y^2-2y+5y+4y^2 = \ ?$ (1988 Int.Course Question 18)

(A) $7y^2+3y$ (51%) (B) $7y^2-7y$ (13%) (C) $7y^4+3y$ (26%) (D) $7y^4-7y$ (10%)

[3] $a^3+a^3 = \ ?$ (1984 Int. Course Question 9)

(A) a^6 (34%) (B) a^9 (15%) (C) $2a^3$ (27%) (D) $2a^6$ (25%)

[4] $\frac{x^2}{5}+\frac{2x^2}{5} = \ ?$ (1982 Question 46)

(A) $\frac{3x^2}{5}$ (43%) (B) $\frac{3x^2}{10}$ (9%) (C) $\frac{3x^4}{5}$ (32%) (D) $\frac{3x^4}{10}$ (15%)

Problems students have with directed number concepts and fractions have confused the issues in [2] and [4], respectively. However, when taken with [1] and [3], responses to these questions show many students do not have the understanding needed to avoid adding indices when adding like terms.

A number of students wish to multiply indices when the operation is the addition of terms (see [1] and [3]). Perhaps the action of adding indices when multiplying terms has developed an association with the operations of multiplication and addition when working with indices.

Items [1] and [3] are identical, except for the letter used as the base and the order of the distractors. The percentage selecting corresponding responses differs little across the two candidatures. This consistency would seem to indicate the range of understandings being applied by Intermediate candidates has remained largely unchanged over 8 years, from 1984 to 1991.

For Investigation: (i) What problems do students have in collecting like terms involving indices? In particular, why do they add or multiply the indices?

(b) Finding the Product of Algebraic Terms

Relatively few questions have been posed on what might be expected as a common operation in the mathematics classroom. None of the questions were from Advanced papers.

[1] Simplify $-6a^5 \times (-2a^2)$ (1990 Int. Course Question 2)
(A) $-12a^7$ (10%) (B) $-12a^{10}$ (4%) (C) $12a^7$ (<u>75</u>%) (D) $12a^{10}$ (11%)

[2] $2a \times 3a \times 4a$ = ? (1987 Int. Course Question 2)
(A) $9a$ (1%) (B) $9a^3$ (2%) (C) $24a$ (21%) (D) $24a^3$ (<u>76</u>%)

[3] $(ab)^2 \times b^3$ = ? (1981 Question 27)
(A) a^2b^5 (<u>71</u>%) (B) ab^5 (13%) (C) a^2b^6 (10%) (D) ab^6 (7%)

It appears that questions of this type pose few problems. The option to add the coefficients in [2] (A) provided almost no attraction. This indicates there is little substance in examiner s expectation that students might look to add indices but, on not finding them, would instead add coefficients.

A substantial proportion of candidates correctly multiplied the coefficients but did not insert the index in question [2] (21%). Was this because there were no indices in the original question to provide a cue, or was it because there was some confusion with the collecting of like terms where the index does not change?

Questions [1] and [3] show that only a small proportion of students are tempted to multiply indices when finding the product of algebraic terms.

For Investigation: (i) Does the absence of a unit index result in a failure to trigger the normal index operations?
(ii) What would those students, who did not insert the index, have done if the terms had indices attached?

(c) Finding the Quotient of Algebraic Terms

In this group, the Advanced, Intermediate and combined candidatures each were represented by one item. A fraction bar was used to indicate division in the Advanced question. The base was a known constant and the index an unknown. The other items were similar in structure to each other. They used the conventional division sign and involved terms containing a coefficient and an unknown base.

[1] $\dfrac{2^{n+1}}{2^{n-1}} = ?$ (1983 Adv. Course Question 8)

(A) 2^{-1} (26%) (B) 2^{2} ($\underline{22}$%) (C) 2^{2n} (17%) (D) $2^{n+1/n-1}$ (32%)

[2] $12m^{6} \div 4m^{3} = ?$ (1984 Int. Course Question 3)

(A) $3m^{3}$ ($\underline{85}$%) (B) $3m^{2}$ 12%) (C) $8m^{3}$ (3%) (D) $8m^{2}$ (0%)

[3] $8m^{6} \div 2m^{2} = ?$ (1981 Question 28)

(A) $6m^{4}$ (5%) (B) $6m^{3}$ 1%) (C) $4m^{4}$ ($\underline{79}$%) (D) $4m^{3}$ (15%)

Despite being answered by the most able candidature, Item [1] was by far the most poorly done. Using the fraction bar to denote division may have contributed to the error rate, especially when the popular distractor (D) is considered. Working from the real meaning of index could yield nothing like that answer. Students, having not seen the fraction bar as indicating division, may have selected (D) on the basis of appearance.

Problems in working with the base as a number and the index as a pronumeral could also have been a factor in Item [1]. This difficulty was noted previously in questions requiring students to write expressions involving repeated factors.

Students were very successful in Items [2] and [3]. This may reflect frequent exposure to such questions in the classroom and through textbooks. The items contained corresponding distractors which attracted similar percentages of students. Students found little attraction to subtracting coefficients. However, some students did divide, rather than subtract, the indices.

For Investigation:

(i) Do students have greater problems manipulating expressions when the base is a number and the index a pronumeral? (note the relationship of this with (B).(a).(ii)).

(ii) What understandings influence students to divide indices when dividing algebraic terms?

(iii) What problems are posed by using a fraction bar to denote division rather than a division sign?

(d) Finding Powers of Algebraic Terms

There were five questions requiring students to raise to a power a term which contained a coefficient and an unknown base with an integral index.

[1] $(2a^3)^3 = ?$ (1983 Adv.Course Question 6)

(A) $6a^6$ (3%) (B) $6a^9$ (6%) (C) $8a^6$ (24%) (D) $8a^9$ (67%)

[2] $(2x^2)^3 = ?$ (1988 Int. Course Question 2)

(A) $6x^5$ (20%) (B) $6x^6$ (19%) (C) $8x^5$ (24%) (D) $8x^6$ (37%)

[3] $(4x^3)^2 = ?$ (1987 Int. Course Question 6)

(A) $4x^5$ (17%) (B) $4x^6$ (21%) (C) $16x^5$ (22%) (D) $16x^6$ (40%)

[4] $(2x^3)^2 = ?$ (1985 Int. Course Question 6)

(A) $4x^5$ (27%) (B) $4x^6$ (50%) (C) $64x^5$ (14%) (D) $6x^6$ (8%)

[5] $(3m^2)^3 = ?$ (1982 Question 23)

(A) $27m^6$ (43%) (B) $27m^5$ (17%) (C) $9m^6$ (21%) (D) $9m^5$ (19%)

The Advanced candidates experienced a relatively high level of success in Item 1 and were noticeably more successful than Intermediate students in this type of question. It is apparent from the responses in [2], [4] and [5] that many students are tempted to multiply both the index and the coefficient by the power to which the term is being raised. This means that many students who answered [4] correctly may, in fact, have multiplied the coefficient by the index of 2 rather than squared it.

The problem of students adding indices in such questions is clearly evident. More than a third chose to do this in all but one of the above questions. This is one of the errors explained by Shevarev in terms of his first 'incorrect connection'. It supports a positive response to the question of whether errors students made in the School Certificate are the same sorts of errors as found in research elsewhere.

The only item in which the available solutions allowed students to leave the coefficient unchanged, was [3]. Brackets should have given a clear signal to apply the index, in some fashion at least, to the coefficient. The fact that 38% of candidates (responses (A) and (B)) chose not to, shows clearly that the meaning given by students to parentheses does warrant further investigation.

For Investigation: [i] When a term containing a coefficient and index is raised to a power, why do students often multiply the power and the coefficient?

(ii) When a term containing a coefficient and index is raised to a power, why do students often add the index and the power?

(iii) When available, is the option of not applying the index to the coefficient attractive to students and, if so, why?

(e) Simplifying Algebraic Fractions of Greater Complexity

There were eight questions in this section. These consisted of algebraic fractions in which numerators and denominators contained more than one unknown base or required some simplification prior to dividing numerator and denominator by a common factor. They came from across the range of candidatures. Items [1] and [4] are the same question but for different candidatures in the same year (1991).

[1] Simplify $\frac{a^6 b}{a^2 b^2}$ (1991 Adv.Course Question 32)

(A) $\frac{a^3}{b}$ (20%) (B) $\frac{a^4}{b}$ (65%) (C) $a^3 b$ (5%) (D) $a^4 b$ (9%)

[2] Simplify $\frac{m^{24}}{(m^2)^4}$ (1990 Adv. Course Question 6)

(A) m^3 (32%) (B) m^4 (12%) (C) m^{16} (49%) (D) m^{18} (8%)

[3] $\frac{(a^4)^4}{a^2} = ?$ (1988 Adv. Course Question 2)

(A) a^4 (10%) (B) a^6 (21%) (C) a^8 (24%) (D) a^{14} (45%)

[4] Simplify $\frac{a^6 b}{a^2 b^2}$ (1991 Int.Course Question 36)

(A) $\frac{a^3}{b}$ (23%) (B) $\frac{a^4}{b}$ (45%) (C) $a^3 b$ (16%) (D) $a^4 b$ (17%)

[5] Simplify $\left(\frac{m^6}{m^2}\right)^2$ (1990 Int.Course Question 10)

(A) m^5 (9%) (B) m^6 (35%) (C) m^8 (39%) (D) m^9 (17%)

[6] $\frac{6a^2}{2ab} = ?$ (1986 Int. Course Question 7)

(A) $3ab$ (32%) (B) $\frac{3a}{b}$ (62%) (C) $6b$ (3%) (D) $\frac{6}{b}$ (3%)

[7] $\frac{x^4 \times x^6}{x^2} = ?$ (1983 Int. Course Question 8)

(A) x^5 (46%) (B) x^8 39%) (C) x^{12} (14%) (D) x^{22} (1%)

[8] $\dfrac{ab^8}{(ab)^2} = ?$ (1982 Question 36)

(A) b^4 (9%) (B) b^6 (15%) (C) $\dfrac{b^4}{a}$ (23%) (D) $\dfrac{b^6}{a}$ (53%)

These questions were poorly answered across the range of candidatures. A major source of error was dividing the indices. Responses [1] (A), [2] (A), [3] (C), [4] (A), [5] (B) and [7] (A) indicate that many students who correctly raise a term to a power, or successfully multiply two terms, are attracted to carrying out a division of indices, if the division is stated using a fraction bar. Other distractors show, as may be expected, many students making initial errors in simplifying the numerator or denominator also continue on to divide indices. It is again evident that School Certificate students are making errors of a type identified by other researchers.

While question [6] was relatively well done, it could be that students have used an elimination process to narrow answers down to the first two options. Coefficients do not seem to have caused significant problems.

There is again some evidence in [3] (responses (A) and (B)) that students confuse raising to a power with multiplying terms in that they have added the indices.

In the common question of 1991, Items [1] and [4], the order in which the distractors attracted students was the same, though the Advanced students were noticeably more successful. The tendency to divide indices has been noted previously. However, it is surprising that, in distractor (A) the students who have divided indices for base a, have, in effect, subtracted indices for base b. This probably relates to the issue of students treating an unwritten unit index differently to a written index,

Again, in Item [1], it is somewhat surprising that 15% of this able candidature chose distractors (A) or (B) in which the factor b appeared in the numerator. Application of the basic concept of a fraction would seem to preclude such a response.

For Investigation: (i) What problems are posed by using a fraction bar to denote division rather than a division sign?

(ii) Do students confuse raising to a power with finding the product of two terms with the same base?

(f) Taking the Square Root of an Algebraic Term

This operation was only examined once. The question came from an Advanced paper.

[1] $\sqrt{16x^{16}}$ = ? (1986 Adv.Course Question 1)

(A) $4x^4$ (59%) (B) $4x^8$ (39%) (C) $8x^4$ (1%) (D) $8x^8$ (1%)

Again the success rate of Advanced students was relatively low with 59% choosing to take the square root of both the base and the index. This they would not do if they were applying a meaningful understanding of indices. It must be a great temptation to take the square root of any perfect square that lies within a square root sign, especially when that solution is one of the responses. It would be of interest to see the response to a question of the form $\sqrt{16x^{10}}$. It would also be interesting to see what distractors the examiners would use if they did set such a question.

For Investigation: (i) What relationship do students see existing between a square root sign and the term to which it is being applied?

(g) Simplifying Expressions Involving a Zero Index

Only two of the seven items came from Advanced papers. This was in contrast with the questions involving negative or fractional indices of which all were from Advanced papers.

[1] $2a^0 + a^{-1}$ = ? (1986 Adv.Course Question 17)

(A) $\frac{1}{a}$ (6%) (B) $2 + \frac{1}{a}$ (68%) (C) $1 + \frac{1}{a}$ (20%) (D) $2 - a$ (5%)

[2] $2 - 3x^0$ = ? (1983 Adv.Course Question 2)

(A) -1 (72%) (B) 0 (2%) (C) 1 (18%) (D) 2 (8%)

[3] Simplify $2a^0b^0$ (1991 Int.Course Question 17)

(A) 0 (8%) (B) 1 (12%) (C) 2 (43%) (D) $2ab$ (38%)

[4] The expression $7k^0$ is equivalent to: (1990 Int. Course Question 31)

(A) 0 (11%) (B) 1 (24%) (C) 7 (35%) (D) $7k$ (30%)

[5] $(3x)^0 + 2y^0$ = ? (1987 Int.Course Question 17)

(A) 0 (26%) (B) 2 (34%) (C) 3 (14%) (D) 5 (27%)

[6] $1 + x + x^0$ = ? (1986 Int.Course Question 28)

(A) $1 + x$ (24%) (B) $1 + 2x$ (30%) (C) $2 + x$ (41%) (D) 3 (4%)

[7] $3a^0$ = ? (1985 Int.Course Question 19)

(A) $3a$ (25%) (B) 0 (15%) (C) 1 (35%) (D) 3 (25%)

Answers selected by the great majority of students indicate they are aware a zero index, in some way, generates a result of 1. Despite this, the correct answer in [5] was the least popular option. This exemplifies the fact that many students fail to see the relevance of brackets. In this question, 61% of candidates treated the two terms as being identical in structure by either applying the index to both the base and the coefficient (response (B)) or to the base only (response (D)).

One alternative, to interpreting a zero index as generating an answer of 1, was simply to rewrite the term without the index. That choice was available in Intermediate questions only (see ([3] (D), [4] (D), [6] (B) and [7] (A)). This option of ignoring the index, or consider it as having no effect, was chosen, in each instance, by at least 25% of candidates,

Another alternative, available in all questions, was to treat the zero index as giving an answer of zero. Only a small proportion of students were attracted to this, except in item [5] where more than a quarter of Intermediate students selected (A). This question had two terms with a zero index, one of which involved parentheses. It may be that students felt an answer of zero was a way of dealing consistently with the two terms.

The Advanced students had a relatively high degree of success with their two questions. Their errors came, almost exclusively, from incorrectly applying the index to the coefficient. This problem of applying the index to both the base and the coefficient, and arriving at an answer of 1, accounts for a high proportion of errors throughout the questions (see responses [1] (C), [2] (C), [3] (B), [4] (B), [5] (B) and [7] (C)).

For Investigation: (i) What meaning do students attach to the zero index?
(ii) What do students see as the significance of brackets?

(D) Substitution and Evaluation

There were no sub-categories in this section. Three of the four questions came from Intermediate papers and each required the substitution of a negative number. The fourth was from a combined candidature paper and involved the substitution of positive values for two unknowns.

[1] When $x = -1$, $3x^2 - x =$? (1988 Int.Course Question 15)

(A) 2 (13%) (B) 4 (<u>49</u>%) (C) 8 (14%) (D) 10 (23%)

[2] When $x = -3$, $x^2 + 5x =$? (1986 Int.Course Question 21)

(A) -24 (8%) (B) -6 (<u>73</u>%) (C) 6 (6%) (D) 24 (9%)

[3] If $a = -3$ then $4a^2 =$? (1984 Int.Course Question 23)

(A) -36 (6%) (B) 36 (<u>59</u>%) (C) -144 (8%) (D) 144 (27%)

[4] If $a = 2$, $t = 6$ then $\frac{1}{2}at^2 =$? (1982 Question 18)

(A) 72 (12%) (B) 36 (<u>75</u>%) (C) 12 (8%) (D) 9 (5%)

Responses to these questions show a relatively high success rate compared to other types. One reason might be that the act of substitution makes students apply genuine understanding of the meaning of an index. Answers to Items [1], [2] and [3] indicate that substituting a negative number does not pose as great a problem as might be expected.

It is evident that having the option of applying the index to both the base and the coefficient contributed to the lower success rate in Items [1] and [3] (see responses [1] (C) and (D), [3] (D)). This has already been mentioned as a major source of error in other situations.

For Investigation: (i) What relationships do students see existing between an index and numbers or pronumerals preceding it?

(E) Change of Subject.

There were only two questions in this group. Both came from Advanced papers.

[1] Given that $s = \frac{1}{2}at^2$ then $a =$? (1988 Adv.Course Question 14)

(A) $\frac{t^2}{2s}$ (18%) (B) $\frac{2s}{t^2}$ (<u>43</u>%) (C) $\frac{s}{2t^2}$ (22%) (D) $\frac{\sqrt{2s}}{t}$ (15%)

[2] If $\sqrt{A} = n$ then $2A =$? (1987 Adv.Course Question 48)

(A) $2\sqrt{n}$ (21%) (B) $\sqrt{2n}$ (13%) (C) $2n^2$ (<u>59</u>%) (D) $4n^2$ (7%)

The complexity of these items is probably responsible for the way in which student support is spread widely across the range of distractors. The results indicate an uncertainty between the relationship of indices to the operations of squaring and taking the square root. This is the same concern as raised in (A) part (c).

For Investigation: (i) What relationship do students see existing between a root or power and the term to which it applies?

(F) Scientific Notation

(a) Writing a Decimal Number in Scientific Notation

There were five items in this section. Questions in this section were structured with the decimal point correctly placed behind the first significant figure. This meant that students were only required to determine the index to attach to the 10. Scientific notation has not been examined in the Advanced papers under consideration and that candidature is not represented in this, or the following two, sub-categories.

[1] Write in scientific notation, -0.23 (1990 Int.Course Question 42)
(A) -2.3×10^{-1} (39%) (B) -2.3×10^{1} (28%) (C) 2.3×10^{-1} (28%) (D) 2.3×10^{1} (5%)

[2] In scientific notation 0.000852 = ? (1988 Int. Course Question 5)
(A) 8.52×10^{3} (8%) (B) 8.52×10^{4} (19%) (C) 8.52×10^{-3} (13%) (D) 8.52×10^{-4} (60%)

[3] $0.00057 = 5.7\times10^{n}$. The value of n is: (1987 Int.Course Question 22)
(A) 4 (25%) (B) 3 (10%) (C) -3 (13%) (D) -4 (51%)

[4] 0.0002 = ? (1985 Int. Course Question 9)
(A) 2×10^{-4} (56%) (B) 2×10^{4} (17%) (C) 2×10^{-3} (19%) (D) 2×10^{3} (7%)

[5] $17\ 000\ 000 = 1.70\times10^{n}$, $n = ?$ (1983 Int. Course Question 1)
(A) 5 (11%) (B) 6 (17%) (C) 7 (68%) (D) 8 (4%)

Two formats were used in posing these questions: one where the possible answers were written in scientific notation; and, another where the correct index needed to be selected.

Apart from the format of [3], Items [2], [3] and [4] were very similar, with each containing a negative power of 10. An examination of the responses to these questions shows the different format had no apparent influence. In these items, errors were divided relatively equally between those giving the index an incorrect direction and those where the value assigned to the index was obtained by counting the number of zeros.

Item [1] involved a negative number and a negative index. It was the least well answered of the questions. All three distractors that contained a negative sign proved attractive to students

In Item [5] the direction sign of the index was not an issue and results did improve. However, 32% is still a large proportion of students giving an incorrect response to a very straight-forward question.

For Investigation: (i) Are strategies used in establishing the index of the 10 related to a genuine understanding of an index?

(b) Converting a Number in Scientific Notation to a Decimal

There were two questions which required students to convert a number presented in scientific notation to a decimal number.

[1] $1.2 \times 10^4 = ?$ (1984 Int. Course Question 2)

(A) 48 (3%) (B) 12 000 (<u>86</u>%) (C) 20736 (1%) (D) 120000 (10%)

[2] $1.2 \times 10^{-2} = ?$ (1982 Question 13)

(A) 120 (22%) (B) 0.0144 (6%) (C) 0.0012 (16%) (D) 0.012 (<u>56</u>%)

Students had considerably more success in Item [1] than in Item [2] where the index was negative. This had also occurred, though not as noticeably, in the previous section. The larger difference here may relate to Item [1] being more readily answered using a calculator than is Item [5] from the previous category (students have had access to calculators in the School Certificate Moderator examination from 1983).

In Item [2], students could choose to apply the index to both the base and coefficient in distractor (B). This option, despite being a significant source of error elsewhere, did not prove attractive to students. That it did not, may relate to the coefficient and base both being numerical and separated by a multiplication sign.

For Investigation: (i) Are strategies used in establishing the index of the 10 related to a genuine understanding of an index?

(c) Written Problems Involving Scientific Notation

The two questions in this section came from Intermediate papers of successive years.

[1] The budget deficit is $3800 million. The number 3800 million expressed in scientific notation is: (1986 Int.Course Question 18)

(A) 3.8×10^3 (36%) (B) 3.8×10^6 (15%) (C) 3.8×10^8 (17%) (D) 3.8×10^9 (32%)

[2] Radio waves travel at 3×10^8 metres each second. How many kilometres will they travel each second ? (1985 Int.Course Question 22)

(A) 3×10^5 (36%) (B) 3×10^6 (23%) (C) 3×19^{10} (17%) (D) 3×10^{11} (23%)

The fact that these questions are framed largely in words adds an extra dimension of complexity to them. It appears that this, coupled with the 'sameness' of the distractors, has contributed to students' indecision and to a lack of any particular pattern in responses. There is some evidence that, in grappling with Item [1], students have avoided complexity by focusing on an obvious number, either '3800' or the 'million', and used that number to determine the index.

It seems unusual for examiners to have included distractor (C) for Item [2], in that it contained 19 as the base. The fact that it attracted 17% of candidates gives further indication that students found this a complex question.

For Investigation: (i) To what extent does the complexity of an item cause students to focus on one aspect of a question involving indices and, in so doing, miss other relevant points.

IDENTIFICATION OF RESEARCH THEMES

This analysis of the School Certificate data has generated thirty-one questions under the heading of 'For Investigation'. A number of these are common to several of the content areas, and so the number of different questions totals twenty-two. This large number is a function of the many errors being made.

As indicated before, simply identifying and examining more individual errors does little to assist us understand students' thinking when answering these questions. What is needed is a limited number of issues which can be examined in detail and which, if they can be explained, will provide insight into the thought processes applied in working with indices.

An examination of the twenty-two questions arising in this chapter shows there are a number of themes which run through them and within which questions can be grouped. The School Certificate results confirmed the problems students have when multiplying and dividing expressions with numerical bases, as opposed to variable bases. It also confirmed students' confusion over the relationship of the index to the base and coefficients. These two issues were also raised by the literature review and are the first two themes listed for research. The other themes are concerned with several specific aspects which seem to pose problems to students when working with indices, namely, students' interpretation of the fraction bar, interpretation of the radical sign and interpretation of the zero index.

The themes are outlined below and are worthy of being the basis for research. The references given in brackets, at the end of each theme, refer to related questions identified as 'For Investigation' in each subsection of the data analysis in Chapter 3.

Theme 1: Integral Bases Versus Variable Bases

DeVincenzo and Childers both identified the strong tendency for students to multiply and divide bases in questions where these bases were numerical (e.g., $3^4 \times 3^2$, $3^4 \div 3^2$, $2^3 \times 3^5$). Similar questions with variable bases (e.g., $a^3 \times a^4$) pose few problems. The School Certificate data strongly support these results. What is the thinking that makes such a high proportion of students multiply and divide numerical bases?

{ (A) (a) (ii); (A) (b) (i); (B) (c) (iii); (C) (c) (i) }

Theme 2: The Relationship of Indices to Bases and Coefficients

The errors listed by DeVincenzo and Childers (Childers 1987, pp.184-207) include a high proportion where the index has been applied incorrectly to coefficients and bases. Errors included: multiplying bases by the index; multiplying coefficients by the index; and, applying indices to bases and coefficients to which they do not relate. Where an expression involving an index is raised to a power, the two indices are often added, not multiplied. Many of the errors are associated with the treatment of the expression $3ab^2$ as though it were $(3ab)^2$ and vice versa. The School Certificate analysis confirmed the frequency of such errors. In 1985, 61% of the Advanced students applied the index to the coefficient and gave the answer of $2x^{-3}$ for $1/2x^3$ while only 19% gave the correct answer of $\frac{1}{2}x^{-3}$. Why is it that students misunderstand such basic conventions of notation?

{ (A) (a) (i); (A) (c) (i); (A) (c) (ii); (B) (a) (i); (B) (a) (ii); (B) (b) (i); (B) (c) (i); (B) (c) (ii); (C) (a) (i); (C) (b) (ii); (C) (d) (i); (C) (d) (ii); (C) (d) (iii); (C) (e) (ii); (C) (g) (ii); (D) (i);(E) (i)}

Theme 3: Interpretation of the Fraction Bar

The tendency for students to divide indices when simplifying algebraic fractions (e.g., $a^{10}/a^2 = a^5$) has received mention by Shevarev and also in the 1981 *National Assessment of Educational Progress Report.* A high incidence of this error appears in Childers data and in the results of the School Certificate analysis. What is the thinking behind students dividing indices? Is it concerned with the fraction bar, or would students make the same error when the question is written as a division?

{ (C) (c) (iii); (C) (e) (i) }

Theme 4: Interpretation of the Radical Sign

In 1981, the *National Assessment of Educational Progress Report* found that for the question $\sqrt{a^{36}}$ only 26% of students obtained the correct answer of a^{18} while 47% chose the answer of a^6, by taking the square root of the exponent (Corbitt 1981, p.65). In 1989 the NAEP found that, in open ended questions, only one in four students with two years of algebra could correctly answer $\sqrt{x^{12}}$ (Lindquist 1989, p.60). In the 1986 School Certificate Moderator examination for the Advanced Course 59% of candidates responded with $4x^4$ for the simplification of $\sqrt{16x^{16}}$ while only 39% gave the correct answer of $4x^8$. Why are students responding this way, and how will they answer questions where the index is not a perfect square (e.g., $\sqrt{25x^8}$)?

{ (A) (c) (i); (C) (f) (i); (E) (i) }

Theme 5: Interpretation of the Zero Index

The zero index has received little attention from researchers but the School Certificate analysis shows the success rate in such questions is not high. In 1985 only 35% of Intermediate candidates could correctly answer $3a^0 = ?$. In 1987, for that same course, only 14% correctly simplified $(3x)^0 + 2y^0$. Incorrect thinking commonly manifests itself in answers which treat the index of zero as giving an answer of zero (e.g., $a^0 = 0$) or as having no effect (e.g., $a^0 = a$). While the issue of the zero index is a relatively specific one, the difficulties students exhibit do demand that it be addressed. It may be that insights, gained here, relate also to Theme 2, above.

{ (A) (d) (i); (C) (b) (i); (C) (g) (i) }

The first of the research issues, raised from the literature in Chapter 2, asked whether data gathered from the N.S.W. School Certificate

Moderator examination indicates that students from within Australia are making the same sorts of errors identified by the research carried out in other countries. Given the School Certificate findings, as discussed above, the answer to this question is an emphatic yes. This is reflected in the five research themes, generated from eighteen of the twenty-two questions listed for further investigation in this chapter. It can be seen that these themes closely relate to the broad categories of errors described in the research discussed in Chapter 2. Given that the same sorts of errors are identified, it would seem that Australian students, as would be expected, use similar thinking in index questions to that used by students elsewhere.

CONCLUSION

The high error rate for index questions in the School Certificate Moderator examination does give cause for concern. This is especially so if we regard the ability to operate with indices as a basic skill expected of students pursuing higher levels of mathematics. This research needs to determine whether students in the senior secondary years of schooling face the same problems as those in Year 10. Another concern is that the School Certificate items may not be testing genuine understanding of indices but testing strategies which students have developed personally while answering drill type questions.

Despite the research which has been carried out, there is no clear vision of how students think when answering index questions. The work of DeVincenzo and Childers throws little light on this issue. Wilson found the subjects of his study relied extensively on instrumental understanding when answering index questions. As discussed in the previous chapter, Shevarev used students' written responses to provide strong evidence for the wide spread use of 'connections' when answering index questions. He did not attempt to substantiate these arguments with data from interviews. It may be that student interviews would support his findings but they could point also to quite different forms of thinking by students.

In seeking a better understanding of students' strategies, research should employ both written responses and interviews. Testing allows the identification of students making systematic errors. Students need then to be interviewed with the aim of ascertaining the understandings they have of indices and the thought processes which have led to their answers.

It would be of great benefit for teachers if a theoretical framework could be provided which explains the thinking behind students' errors in index work. Such a theoretical context would give teachers guidance in determining appropriate programming and lesson strategies directed at improving students' learning of index concepts. The next chapter considers a number of theories which may provide such a theoretical framework.