## Chapter 10

# THEORETICAL PERSPECTIVES: FORMS OF UNDERSTANDING and 'CONNECTIONS'OR 'FRAMES' (Themes 7 and 8)

### Introduction to Chapter

The preceding chapter addressed the first of three themes relating to theories of learning which may explain errors students make in questions involving indices. This theme was concerned with the SOLO Taxonomy as a model for classifying and examining student responses to such questions. In this chapter, the two remaining themes are considered. They are concerned, respectively, with **forms of understanding** and **'connections' or 'frames'**. The nature of the understandings being applied by students was a consideration throughout the analysis of the content related themes in Chapters 7 and 8. Findings which emerged there are now reviewed from a perspective of how they contribute to explaining, within the more general framework of the two themes, students' learning in the area of indices. The themes, as identified in Chapter 4, and their associated research questions are:

**Theme 7: Use of Relational and/or Instrumental Understanding**

- *To what extent do those students making errors in questions involving indices obtain answers by applying 'instrumental' strategies as opposed to using 'relational' understanding?*
- *In questions involving indices, do students view their responses meaningfully and in what way do they reconcile inconsistencies?*

**Theme 8: Use of Connections or Frames**

- *How do the constructs of 'connections' or 'frames' relate to the way students work with indices?*
- *What common errors, if any, result from the incorrect application of connections or frames ?*

There are three sections in this chapter. The first two are devoted to Theme 7 and Theme 8 respectively. The final section provides a synthesis of the findings from all three themes concerned with theoretical frameworks, i.e., the two addressed in this chapter and the theme relating to the SOLO Taxonomy.

## USE OF RELATIONAL AND/OR INSTRUMENTAL UNDERSTANDING (Theme 7)

The conceptual and procedural dimensions of the thinking involved, when solving questions of the type used in this research, can be appropriately described in terms of Skemp's constructs of relational and instrumental understanding. The importance of students being able to integrate these two aspects of understanding was emphasised in the work of Goldin and Herscovics. Issues for this research are: the extent to which students rely on either, or both of, relational or instrumental understanding; and, how students view and accept answers when these understandings are not coordinated. Two research questions were posed for this theme and they are now addressed.

*To what extent do those students making errors in questions involving indices obtain answers by applying 'instrumental' strategies as opposed to using 'relational' understanding?*

Interview evidence and written responses showed little sign of students using relational thinking in questions involving indices. Instrumental strategies dominated the approaches taken, as was demonstrated in the interviews reported in Chapters 7 and 8. Some comments exemplifying this were: "indices ... you just add em"; "It's just the rule I guess"; "I said 10*m* divided by 2*m* is 5*m*"; "well I just sort of just took the square root of everything"; and, "I think it must just have been an automatic thing cause they were the numbers there". While students selected for interview were those making particular systematic errors, the consistent way in which they described their thinking in instrumental terms suggests such thinking may predominate with the full range of students.

The reasons students rely strongly on instrumental strategies appear to be twofold. First, they find it convenient and efficient to use instrumental strategies in such questions and, as recognised by Skemp, "even relational mathematicians often use instrumental thinking" (Skemp 1978, p.12). Second, it is apparent that many students have a relational understanding of only the simpler concepts, such as the meaning of integral indices, and can only apply that understanding in straightforward contexts. The structure of index questions is such as to encourage these two factors to work together in that instrumental strategies readily provide a response to questions which are difficult to approach relationally.

The use of instrumenta strategies, coupled with the fact that students tend not to place answers under scrutiny, can make it appear that they have little understanding of even basic conventions. However, there are times where it seems the conventions are understood but not brought in to play. For instance, the results o' this present research offer strong support for the finding that able students seem to be ignoring brackets (e.g., by responding with answers such as 8 for $3p^0+(5q)^0$) but would not support the proposition that this reflects a failure to understand their significance. This apparently contradictory situation exists because the instrumental approaches taken preclude the need to apply genuine understanding of brackets. The cases where even less successful students described how a power outside parentheses meant the term inside was a repeated factor, indicates that the grouping role of brackets is able to be explained even though it is often not applied.

There were similar instances, in each of the content themes, when the answers given seemed to preclude students having genuine understanding whereas under interview it was demonstrated to the contrary. Further evidence for this comes from the results for questions which involve a substitution. For multiplication and division of expressions with numerical bases, the action of substituting in an expression triggers a relational, rather than instrumental, response in that many students correctly evaluate the numerical expression deriving from the substitution. This was despite the fact that, when confronted by the same expression without the need to substitute, many made the error of multiplying or dividing bases.

The few cases where students described their thinking in relational terms tended to relate to multiplying or dividing expressions with numerical bases and indices. Linda's interview showed, as might be expected, a relational response was discouraged where the index was a variable. She obtained the answer of $2^7$ for $2^3 \times 2^4$ by using relational understanding, however, her response of $9^{x+y}$ for $3^x \times 3^y$ indicated the use of instrumental strategies.

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| Interviewer: | How did you arrive at your answer? |
| Linda: | I said ... Oh no this is different to what I did before. |
| Interviewer: | Different to which question? |

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| Linda: | Different to $3^x \times 3^y$ where I multiplied 3 by 3 to get 9. |
| Interviewer: | Which approach do you think is correct? |
| Linda: | Probably $2^3 \times 2^4 = 2^7$ I'd say. |
| Interviewer: | Why do you think you approached the two questions differently? |
| Linda: | Probably because of the numbers at the top. Instead of $x$ and $y$ it was the 3 and the 4. |
| Interviewer: | And yet you carried out the same operation with the 3 and the 4 and the $x$ and the $y$ didn't you. |
| Linda: | I think with the numbers up here I think first of all I found (the value of) $2^3$ and then I found (the value of) $2^7$ and then I multiplied them together and found it was 128 and I knew that was $2^7$. |
| Interviewer: | Oh so that's the order you worked things out in. What answer do you think you may have given if you didn't work them out individually? |
| Linda: | Probably $4^7$. |

It appears that, while there are occasions where relational strategies are used, it is inevitable students will resort to instrumental strategies and teachers need to recognise this fact and take it into account.

*In questions involving indices, do students view their responses meaningfully and in what way do they reconcile inconsistencies?*

It is evident students, making errors in these questions, use superficial approaches predominantly, and are not looking to apply an understanding of underlying concepts. Given such processing, it is not surprising that students: view answers with little meaning; have little concern for their correctness or otherwise; and. are not aware of contradictions to be reconciled.

What is surprising, perhaps, is there appears to be little transfer of thinking from situations where students are encouraged to look at the underlying concepts to those which are mathematically equivalent but lack context. Students were forced to take a deeper view of questions in a variety of circumstances. Examples of such cases were where: a question came from an application; a substitution was required; different notations were used; and, the numbers contained in the question were ones which 'less obviously' fitted the operation. An example of this last situation was where the index under the radical sign was not a perfect square. These circumstances did result in students achieving greater success, however, these situations did not encourage or facilitate students placing other questions under similar scrutiny.

There have been many instances where, in discussing a response to a question, students were given the opportunity to place other answers under scrutiny, yet did not do so, when it seemed they should have. Such was the case where students believed, when index notation is used to denote a square root, everything is times by a half, but when the operation is denoted by a radical sign, the square root is taken of all numbers within it. There were also the instances where, in applying a radical sign, students took the root of the index when it was a perfect square, but halved it when it was not. Similarly, students were more successful where the requirement to take a root of an expression emerged from an application, and yet, interviews demonstrated this did not influence their responses to questions involving the radical sign. Again, there were the cases where students operated on numerical bases as though they were coefficients yet responded correctly when such expressions came to them through a substitution. It was also evident that, because students were not viewing the zero index with genuine understanding, they were not concerned with such issues as whether they had accounted correctly for parentheses.

The few cases where students did review their answers tended to occur in the more straightforward situations. An example was where students, in simplifying an algebraic fraction realised that dividing indices contradicted the approach they had taken to similar questions in which the operation was denoted by a division sign. Another case was where students decided they were incorrect to operate on numerical bases because to do so disagreed with how they had responded where bases were variables. However, such review of answers usually did not involve students in accessing underpinning understandings. Rather, it was associated with their becoming convinced an alternative instrumental approach was the correct one.

It is apparent that students are not viewing their responses meaningfully, in the sense of understanding them relationally. Therefore, when they do make errors, they are not seeing any inconsistencies to reconcile.

**Summary of Theme 7**

Students making errors in questions which are the subject of this research use instrumental strategies extensively. While they may have a basic understanding of the underpinning concepts, they have trouble applying these in other than simple situations. Because of this, they work through 'rules'. Such rules often lead to errors. However, because answers have not emerged from relational thinking, responses are not viewed meaningfully or placed under scrutiny. Students take an essentially pragmatic approach characterised by an attitude of 'this is how you do it and here is your answer'.

## USE OF 'CONNECTIONS' OR 'FRAMES' (Theme 8)

Shevarev and Davis argued, in the learning of mathematics, students develop 'connections' or 'frames' by which they carry out mathematical processes. Errors can result when the connection or frame is not sufficiently well developed, and the student fails to identify all relevant aspects of a question. This results in their applying incorrect operations which would, however, be correct for a different question of similar appearance to the one being undertaken.

Extensive use of notation is made when posing questions involving indices. Students need to operate with brackets, raised numbers, coefficients, and bases, together with the other symbols and conventions of basic number work and algebra. Items students are working with frequently share several features of notation yet differ in other significant ways. Accordingly, there is abundant scope for students to focus on certain aspects of a question and neglect other pertinent issues contained in the visual symbols. In order to explore the role connections or frames may have in explaining errors in index items, two research questions were posed and these are now addressed.

*How do the constructs of 'connections' or 'frames' relate to the way students work with indices?*

The test results and interviews confirmed that, in items involving indices, students frequently do not take account of all relevant aspects of the question.

An example of this is where students treat items without an unknown as though they contained one, e.g., $2^3 \times 3^5 = 6^8$. This response would be correct if an unknown was inserted as the base, i.e., $2a^3 \times 3a^5 = 6a^8$. The interviews demonstrated clearly that the thinking used has the characteristics of an incorrect connection of the first type discussed by Shevarev. There, a specific feature (such as the existence of the pronumeral) did not enter into the connection.

The issue of why students operate on numerical bases was touched on in interviews conducted by both DeVincenzo and Wilson. In discussing the answer of $4^5$ for $2^3 \times 2^2$ DeVincenzo stated "(interviews) indicated that certain students adhered to a designated rule for the exponents in this problem type and were guided by the times sign for the bases" (DeVincenzo 1980, p.127). Wilson attributed the problem to students operating in an Instrumental-Symbolic mode. He described one such student as having an understanding of exponents which "is for the most part an instrumental one" and "even though he does have rules, he is not quite sure of these" (Wilson 1985, p.162). While both identified that students were taking a routine approach to such questions neither offered any real explanation as to how this resulted in errors.

This present research shows the cause of the more common errors is the incorrect use of an operation applied successfully in other questions of different structure but similar appearance, e.g., students would answer $2^3 \times 3^5$ as though it were $2a^3 \times 3a^5$. Interviews demonstrated that students making this error either identified no difference between the questions or attached little significance to the variable. This was evident in the student responses discussed in detail in Chapter 7. Comments included:

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|---|---|
| Malcolm: | I timesed the front numbers ... it just looked the same ... it just had something times something else to the power of something. |
| Bianca: | Cause it was nearly the same question ... well you've got your 2s, same base and ... just indices ... you just add em. |
| Mark: | That's the rule ... you add for multiplication ... you times the one out the front. |
| Brian: | Its the same as if the m wasn't there ... I'd probably just times the 5 and the 2 and leave the 3 there, so $10^3$ (In response to "what do you see as the role of the m in $5m^3 \times 2m$ ?"). |
| Kirsty: | It doesn't really make any difference if there is an *a* there or not. |
| Tim: | Well in Item 4 the *a* has the index and ... eh I don't think I really took it into account actually. |
| Philip: | They are the same I think .. except this one has an *a*. Oh ... not a lot I don't think, its just a variable type thing (In reply to "what difference does the *a* make?"). |

While this kind of thinking has features of Shevarev's 'connections', (or Davis' 'frames') the assertion, by Shevarev, that such students know the rules (in the sense of understanding them) but do not apply them is open to question. Shevarev suggested, on the basis of written responses, that students using incorrect connections could, if circumstance required, recall the rule and answer the question correctly. Interviews show this is usually not the case as students rely strongly on routine strategies and find it difficult to exercise their understanding of underlying concepts in other than simple situations.

Certainly, there were circumstances where students changed from an incorrect strategy to a correct one. Usually, however, this did not involve activation of the student's understanding of underpinning concepts but was simply a change from one routine strategy to another. This can perhaps be explained in terms of Davis' theory as discussed in Chapter 4. Davis stated that errors occur due to the selection of the wrong frame for the circumstance. This happens when the preliminary selection frame, which differentiates between a frame used for a previously learnt concept and a competing one formed for a new concept, is not developed sufficiently strongly. A student having used an incorrect frame may, under the stimulus of the interview situation, recall the preliminary selection frame and correct their answer. Such would appear to be the situation with Linda when she had second thoughts after describing how she had arrived at an answer of $x^5$ in simplifying $\frac{x^4 \times x^6}{x^2}$ (Item 22).

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|---|---|
| Linda: | I did $x^{10}$ divided by $x^2$. |
| Interviewer: | Why did you divide 10 by 2? |
| Linda: | Oh it should be take away shouldn't it ... should it? |
| Interviewer: | What do you think? |
| Linda: | I think it should be 10 take away 2 to give $x^8$. |

It can be argued that Linda, using a frame developed in early work on fractions, had divided indices. However, in the interview, she brought into effect a preliminary selection frame which distinguished between a coefficient and an index. She then applied the correct frame and subtracted indices.

It is evident that, in students' work with indices, connections or frames operate substantially as described by Shevarev and Davis and provide a useful way of viewing errors in this field.

*What common errors, if any, result from the incorrect application of connections or frames ?*

A number of persistent errors considered within the content themes occur when students adopt approaches which, while correct for certain other items of similar appearance, ignore important aspects of the question. Five incorrect applications of connections or frames are given below. The first three cases of incorrect application of connections or frames relate to a failure to attach significance to the omission of an unknown. In the last two cases, students, when working with fraction bar and radical sign notations, respectively, operate on indices as though they are coefficients.

*1. Treating multiplication questions without an unknown as though they contain one and, as a result, the bases are multiplied.*

This situation has been discussed already in the preceding research question.

*2. Treating division questions without an unknown as though they contain one and, as a result, the bases are divided.*

This was shown in the popular responses of 5 for $10^4 \div 2^3$ and 1 for $5^6 \div 5^2$ and in comments, such as made by Mark where he said "we just divide the numbers out the front". That many students attached little significance to the presence, or otherwise, of an unknown base was apparent when similar questions, with and without an unknown base, were compared. These comparisons gave rise to comments such as "It is the same as if the *m* wasn't there" (Brian), "it doesn't really make any difference if there is an *a* there or not" (Kirsty), and "not a lot (of difference having an unknown base) I don't think, its just a variable type thing" (Philip).

*3. When raising a numerical expression to a power, treating that expression as though it contained an unknown and, as a result, the base is raised to the power.*

Here a high proportion of students operated upon the base as though it were a coefficient and gave responses such as $27^6$ for $(3^2)^3$. Innes explained that answer by saying "I cubed 3 and timesed the power of 2 by 3 because it was inside the brackets". When asked about his answer of $8m^6$ for $(2m^2)^3$ he replied "It was the same way, I times the 2 ... I cubed the 2 ... sorry ... and got $8m$ and then the power of 2 I times by 3 to get 6". As detailed in Chapter 7, a number of other students gave similar responses to Innes indicating they attached little significance to the variable.

*4. Failing to distinguish between coefficients and indices when simplifying algebraic fractions and, as a result, the indices are divided.*

As discussed in Chapter 8, students' failure to distinguish between coefficients and indices results in a high error rate in questions requiring the simplification of algebraic fractions. Errors occur despite the fact that students know the fraction bar indicates division and are able to divide successfully when the conventional division sign is used. Interviews showed it was the fraction bar notation which encouraged students to divide indices. Tim commented that his reason for dividing indices was "simply that it was written that way I think with the line between the denominator and numerator there". He went on to show that he associated division with subtraction when he said "I think it was only because I was dividing and I must have automatically just divided the indices there. Just looking at it now I thought that if you were going to divide it you should have perhaps minussed them". Other students confirmed the influence of the notation including Linda who commented "because I'm used to numbers, a top number divided by a bottom number so I just divided ... without even thinking". When asked to clarify what she meant by "used to numbers" she responded "sort of like maybe 10 over 2, so that's 5".

5. *Not taking account of whether the numbers under a radical sign are coefficients or indices and so attempting to take the square root of all numbers within the sign, including the index.*

Interviews show that the radical sign causes many students to 'think square root' and proceed to take the square root of both coefficients and indices. For Item 36 ($\sqrt{16x^{16}}$), 33% of students obtained an answer of $4x^4$. Kirsty commented "Oh Yeah well I just sort of just took the square root of everything" while Bianca explained "I just square rooted everything I could ... so you get $4x^4$". While the tendency to take the square root of the index was much lower when the index was not a perfect square, it was clear that students were tempted initially to take the square root of the index but, on finding that difficult, either resorted to different strategies or did not answer the question. Christine, commenting on her failure to respond to the simplification of $\sqrt{25x^8}$, said "I was going to ... like ... to square root the 25 but the 8 doesn't have a square root so I gave it a miss". Innes took the square root of the index in $\sqrt{16x^{16}}$ but then correctly answered $\sqrt{25x^8}$. He showed, through his remark, 'I wasn't sure on this question because I couldn't get a square root of 8 so I divided it by two", that his first choice would have been to take the square root.

**Summary of Theme 8**

There is evidence that students often use incorrect 'connections' or 'frames'. This leads to their selecting operations which are correct for a particular circumstance and incorrect in other situations. However, the assertion by Shevarev that students making such errors 'know the rules' but do not apply them does not reflect closely the situation identified in this study. Usually, students begin by using routine strategies and, when placed under pressure to substantiate a response, look for other superficial strategies rather than applying a deeper understanding.

## SYNTHESIS AND CONCLUSION

Each of the three theoretical frameworks examined in this research has something to contribute in explaining the errors students make in questions involving indices.

The theory of 'relational' and 'instrumental' understanding does provide a useful structure within which to view the quality of thinking being used by students answering questions involving indices. Interviews show that students making systematic errors in questions of the type being investigated almost exclusively use instrumental strategies, or 'rules without reasons'. Irrespective of whether they understand the underlying concepts, students do not attempt to bring genuine understanding into play.

The construct of 'connections' or 'frames' explains the tendency for some students, in an unquestioning way, to focus on, and respond to, a limited number of aspects of a question and neglect other important issues. It is evident from interviews that this happens within the framework of instrumental thinking. Shevarev and Davis see students inevitably developing their own strategies as they answer questions. These strategies allow certain questions to be answered correctly but can lead to errors in similar questions which have some features in common. This occurs when students do not take account of the features which are not common. When this happens, the aspects which these strategies address are ones which received either earlier or greater attention in the student's educational experiences. The commonality of students' experiences would seem to explain the fact that many errors being made are systematic ones.

Responses generated by the use of incorrect 'connections' or 'frames' are of a multistructural SOLO level in that they do not account for all aspects of the question. The statistical analysis shows that there is a consistency of level of performance across both the subjects of this research and across the items used.

To exemplify the way in which these theories interrelate we can examine how students respond with an answer of 1 when simplifying $5^6 \div 5^2$. Interviews show that students, in answering questions of the form $5m^6 \div 5m^2$ develop their own instrumental strategies. Some build into their rule all relevant information including the importance of the variable base. Those operating at a multistructural level do not integrate all relevant aspects and develop a rule to 'divide the bases and subtract the indices'. This gives success without any need to consider the role of the unknown. This 'connection' or 'frame' may then be incorrectly applied in the less frequently encountered situation of $5^6 \div 5^2$. Having not applied genuine understanding means students see no need to place their answer of 1 under scrutiny despite it being patently incorrect.

In summary, students making the errors which are the subject of this research use, almost invariably, instrumental strategies. In using these strategies they often provide a multistructural level SOLO response which correctly accounts for many of the aspects of the question, but not all. The consistency of the incorrect responses, and the fact that certain aspects are not accounted for, can be explained in terms of 'connections' or 'frames'.

## Chapter 11

# SUMMARY AND CONCLUSIONS

### Introduction to Chapter

This final chapter draws together, and concludes upon, the many issues raised throughout this research relating to students' understandings when working with indices. To provide a context within which to reflect on these issues, the limitations and constraints of the study are first discussed. Results of the research are then summarised. To locate the findings within the wider mathematical setting, a consideration is undertaken of how they compare to those of the two research projects, in algebra, discussed in Chapter 1. The remaining sections of the chapter are devoted to a consideration of implications the findings have for teaching practice and for future research.

### LIMITATIONS AND CONSTRAINTS OF THE STUDY

The subjects of this study were forty students from a large comprehensive high school in rural N.S.W. Thirty students were enrolled in the 2 Unit mathematics course and ten in the 3 Unit mathematics course. This was the only sample available for the project because of the personal circumstances of the researcher, a teacher at the school. Results of the study are limited to this group and the generalisability of the results needs to be considered within these constraints. However, since the 2 Unit and 3 Unit courses are examined state-wide for the N.S.W. Higher School Certificate, comparisons can be made with the large body of students (approximately 20,000 for 2 Unit and 10,000 for 3 Unit) who were candidates for these courses. On the basis of that comparison, the overall mathematical performance of the subject group was quite representative of the range of candidates who sit for the state-wide examination. As such, there is a strong likelihood that the results apply generally to students' work with indices in the senior secondary years of schooling in N.S.W.

There could have been benefit in interviewing some students who were subjects of the Pilot Study. This would have allowed further trialing of questions and, perhaps, the identification of other issues for investigation. It was not possible to carry out such a program of interviews because of the school's policy to minimise disruption to senior students as they prepared for external examination.

An issue for research was the change in students' understandings as they progressed from Year 10 into the senior secondary school. There would have been advantage in being able to directly compare the performance of the senior students, on questions involving indices, with the results obtained by Year 10 students on the state-wide School Certificate Moderator examination. A number of factors made such a direct comparison impossible. Firstly, the Year 10 data were drawn from examinations at three different levels, one for all levels in 1981 and 1982, and separate examinations for Advanced and Intermediate students in subsequent years. Secondly, the Year 10 data covered examinations over ten years. Finally, the pattern of student entry for the senior courses does not correspond to that for Year 10 courses, i.e., two students following the same course in Year 10 may pursue different courses in Years 11 and 12. Therefore, only in the broadest sense was it possible to assess the overall change in students' understandings.

## SUMMARY OF RESULTS

This exploration of students' understanding of indices began with an examination of the literature relating to student understanding in the field of algebra. It identified possible causes of error and highlighted the fact that many students find algebra difficult, and see its study as having little relevance. The examination provided guidance as to appropriate methodology and established that interviewing students was needed if understandings were to be closely investigated.

The review of the literature relating to students' understanding indicated that numerous, systematic errors are being made in questions involving indices and that, even when responding correctly, many students are not applying a genuine understanding of the meaning of an index. The effect, on responses, of the base being either a numerical constant or an expression involving an unknown, was identified as an important issue for research. Operating on numerical bases, when multiplying and dividing expressions, and when raising expressions to a power, was identified as a significant source of error.

The review of the literature also established that researchers, while identifying many errors, had not placed the thinking which generated those errors under close scrutiny.

That students in N.S.W. are making the same systematic errors as found in research carried out in other countries was established through the analysis of responses to the School Certificate Moderator examination. The analysis provided a comprehensive source of data which, when taken with the review of the literature, allowed the identification of themes which run through errors students make in questions involving indices. These themes encompass a large proportion of the errors, and relate, respectively, to: the tendency to multiply and divide bases in questions where bases are numerical; the incorrect application of indices to coefficients and bases; the interpretation of the fraction bar in questions involving indices; the interpretation of the radical sign in questions involving indices; and, the interpretation of the zero index.

The Pilot Study established that students in the senior school do have considerable difficulty with questions involving indices and are making the same errors as those made in the School Certificate. It confirmed that the themes previously identified provide an appropriate structure within which to examine the understanding of indices being applied by students in their senior years of schooling. It also showed that in the free-response situation students make the same errors, and with similar frequency, as they do with multiple-choice questions.

The Main Study found, through interviewing, that students making systematic errors are predominantly using routine approaches which often give success, but frequently lead to error. Students were found to understand the underlying concepts relating to positive integral indices and to know the meaning of the index of a half and the zero index. However, they tended not to access such understanding and instead use instrumental approaches. Such approaches can give success, providing all aspects or the question are accounted for, although, they also lead to systematic errors for many students.

It was found that students making errors in items involving indices take an uncritical view of their responses and fail to see the need for consistency across questions. This occurs even to the point where students can believe two differing approaches to a question are both correct even though the respective answers generated are patently different. The uncritical view students take of these questions, and of their responses, manifests itself also in their attaching little significance to pronumerals. Where a substitution or, to a lesser extent, an application focuses their attention on the pronumeral a higher success rate results.

When explaining the reasoning behind their responses, different students making the same systematic error showed a consistency in the thinking which generated the error. This occurred within each of the five content related themes.

For multiplication and division questions involving numerical bases, the high error rate resulted from students perceiving such bases as being the same as a coefficient followed by an unknown base, i.e., $2^3 \times 3^2$ is the same as $2a^3 \times 3a^2$. Students respond to $2a^3 \times 3a^2$ by saying $2a$ times $3a$ is $6a$ and then adding indices. While the reasoning is incorrect, it does lead to a correct response. This treating of the base and coefficient as an entity then carries over to students responding to $2^3 \times 3^2$ in what they see as the same way, i.e., multiplying the 2 and 3 and adding indices to obtain the incorrect answer of $6^5$.

As for multiplication and division, when applying an index to an expression many students view the coefficient and unknown base as a single entity. In similar fashion to above, they arrived at responses such as $8^9$ for $(2^3)^3$. Other students, in cases involving a coefficient and an unknown base, treat the coefficient as though it were a numerical base and arrive at answers such as $2m^6$ for $(2m^2)^3$.

The problem of students treating a coefficient and base as an entity arises also in questions involving the zero index. That students uncritically respond with '1', both for $7m^0$ and $(7m)^0$ is a result of a failure to separate, in their thinking, the base and the coefficient. Such thinking makes the role of the parentheses irrelevant.

Also, students fail to take account of all relevant issues when simplifying algebraic fractions and when taking the square root of an expression involving an index. When simplifying algebraic fractions, many students associate the fraction bar with the division of numbers above and below the bar. They do not discriminate between indices and coefficients and, as a result, divide indices rather than subtract. This occurs despite the fact that students readily simplify such expressions when they are written with division signs. Similarly, when taking the square root of an expression involving an index, students do not discriminate between coefficients and indices and take the square root of both.

A purpose of the research was to provide teachers with a theoretical framework to guide them in determining appropriate strategies directed at improving students' learning of index concepts. Three themes were developed relating respectively to: the usefulness of the SOLO Taxonomy in explaining errors made in questions involving indices: students' use of Relational and/or Instrumental Understanding; and, the use of 'Connections' or 'Frames'.

It was found, with regard to the first of these themes, that student responses to the questions used in this study could be classified according to SOLO type levels on the basis of the number of errors made. The twenty-four most difficult items were able to be coded according to the following classification: correct response - relational; response with one error - high multistructural; and, response containing more than one error - lower multistructural or unistructural.

Subsequent quantitative analysis of the data, using a partial credit form of the Rasch Model, found a cognitive factor to be influencing the quality of student responses across the content areas. A remarkably clear separation was found between the level of difficulty, as measured by threshold values, of achieving a high multistructural level response as against a relational level response. Students found a relational level response, for almost any item, harder to achieve than was a multistructural response for any other item. Within each item, the Category Delta estimates showed that, for all except three the step to a relational level response was more difficult to achieve than was the step to a high multistructural level one. Since coding made no attempt to identify the order in which aspects of the question were addressed, this increase in difficulty relates to the increased cognitive load of successfully addressing all aspects of the question rather than to the difficulty of any one aspect.

The analysis also provided strong evidence that individual students are responding consistently at one of the three levels. The Infit Mean for Cases was 1.01 and the Infit t value was 0.04 indicating that, to a substantial degree, students who achieved success for the harder item/steps also successfully achieved the easier item/steps, while those who were unsuccessful with the easier item/steps were unsuccessful also with the harder ones.

Students' use of Instrumental and/or Relational Understanding was the subject of the second theme relating to theories which may help explain errors. As indicated previously, students making errors in these questions were found to use instrumental strategies extensively. Students apply 'rules' they have developed and, despite having a basic understanding of underlying concepts, rarely bring relational understanding into play.

The final theme was concerned with the use of 'connections' or 'frames'. This study shows that these constructs appropriately describe the thinking leading to a number of consistent errors students make in the questions which are the subject of this study. As discussed previously in this section, some systematic errors result when students do not incorporate all the relevant information into the 'rules' they apply. Such is the case where students have not built a role for the variable into the 'connection' or 'frame' they have developed for responding to $2a^3 \times 3a^2$, and so readily arrive at an answer of $6^5$ for $2^3 \times 3^2$.

Other systematic errors which can be explained using the constructs of 'connections' or 'frames' are the tendency for students to: divide numerical bases; apply a power to a numerical base; divide indices when simplifying algebraic fractions; and, take the square root of an index inside a radical sign. The extensive use of notation in questions involving indices, coupled with the the instrumental approaches which students widely adopt, would appear supportive of the development of incorrect 'connections' or inadequate 'frames'

The findings with regard to students' use of 'connections' or 'frames' are compatible with the findings of the other two themes related to theories which may explain errors. In applying instrumental strategies some students are unable to integrate all the relevant information and respond at a multistructural level. "Connections' or 'frames' explain the systematic nature of such responses, and also the particular aspects for which students have not taken account.

## INDICES AS RELATED TO THE GENERAL ALGEBRAIC FRAMEWORK

Algebra provides the context for much of students' work with indices. Difficulties students have in learning algebra were discussed in Chapter 1. The extent to which students' difficulties with indices reflects the difficulties they have in algebra can now be concluded upon.

The discussion below is carried out under six headings relating to reasons for errors in algebra. These were identified in Chapter 1 and derive from findings of the 'Strategies and Errors in Secondary Mathematics' (SESM) research project, which had, in turn, grown out of the earlier Concepts in Secondary Mathematics and Science (CSMS) project. How students' errors in indices, as identified in this research, fit within those six reasons for errors in algebra, is now considered.

*1. Students Often Misinterpret a Letter and this May Lead to Errors*

Six different interpretations of a letter were identified in the projects discussed in Chapter 1. They were: assigning the letter a numerical value; not using the letter; regarding the letter as an object; treating the letter as a specific unknown; treating the letter as a generalised number; and, treating the letter as a variable.

These interpretations of a letter were of interest to this present research from a view point of whether particular interpretations were influencing students' responses to the questions being examined. It was found that students making errors in questions involving indices rarely used an understanding of letters as representing numbers to support them in their work. This was despite the fact that questions requiring substitution were answered very successfully. While it is apparent that students in this study accept letters can represent numbers, there is a strong tendency to work with letters more as objects. This interpretation of a letter as an object is, however, somewhat different to that discussed in Chapter 1. Here the understanding that the letter can represent a number exists, but is not being applied. Nevertheless, although not being viewed meaningfully, letters are perceived as being able to be operated on in some routine fashion.

The treatment of letters as objects is evident in interview responses where students treat bases and coefficients as entities. This is shown in comments such as "I divided the $2m$ into the $10m$ to get the $5m$", "I timesed the $2a$ and $3a$ and got $6a$", and "I took $2m$ to the power of 3 and got $8m$ and then timesed the indices". The failure to attach real significance to the unknown is further illustrated in remarks such as: "It is the same as if the m wasn't there"; "I dunno ... they are just ... the $a$ sort of ... it doesn't really make any difference if there is an $a$ there or not"; "I don't think I really took it into account actually"; "its just a variable type thing"; and, "I didn't think it had much to do with it".

In questions involving indices, students fail to make clear links between operations they carry out and the concepts which underpin those operations. Consequently, they are not applying, or seeing any need to apply, appropriate interpretations of the letters.

*2. Items May be Solved in Unexpected Ways*

Results of this present research show students are using instrumental strategies extensively when responding to questions involving indices. As a result, many students are arriving at answers using mathematically 'naive' strategies which, nevertheless, often provide correct answers. Indeed, such strategies are developed while students are responding correctly to particular questions. While these strategies may seem unexpected to a mathematician, this research has identified a number of such 'connections' or 'frames' which are commonly held. Further research, by identifying other 'connections' or 'frames', may increase the range of answers 'expected' from students, and, conversely, reduce the number of inappropriate answers.

*3. Notation and Symbolism Cause Considerable Difficulties*

The CSMS and SESM projects found many students have difficulty working with the abstract representation which constitutes the written language of mathematics. Extensive use of notation is made in questions involving indices. Students are required to work with: variables; raised notation; fraction bars; brackets; and, radical signs. In addition, they must observe the other common mathematical conventions. These complexities of the written language provide ample opportunity for notation related errors to be made.

Difficulties students confront, in appropriately interpreting the notations listed above, have been identified frequently in this research. The influence the symbolic structure has on responses was evidenced in situations where the use of alternative notation caused students to approach, essentially identical, questions in quite different ways. Such was the case where algebraic expressions were divided correctly yet, when the fraction bar notation was used, indices were divided. Similarly, some students responded with $8a^8$ for $(16a^{16})^{1/2}$ and $4x^4$ for $\sqrt{16x^{16}}$ and maintained the correctness of the two answers, even after realising the questions were, in essence, the same.

Notation is clearly posing difficulties to students in their work with indices.

*4. Many Students Fail to Understand the Significance of Brackets*

Students' perceptions of the role of brackets, in questions which are the subject of this research, have been discussed earlier in this chapter. The projects discussed in Chapter 1 found many students felt the presence of brackets was unnecessary. In this present research, interview evidence showed the subjects did see the relevance of brackets and had a basic understanding of their grouping role. However, the use of routine strategies meant that brackets were often not accounted for correctly.

*5. Informal Methods Developed in Arithmetic can Lead to Errors in Algebra*

Routine strategies, developed in arithmetic, do lead to errors in algebraic questions involving indices. Such is the case where students divide the numerator and denominator of a fraction without any real sense of 'simplification'. This carries through to students dividing indices when simplifying algebraic fractions. Again, students taking the square root of all numbers under the radical sign has its basis in routine strategies developed in arithmetic. However, the converse is also true, in that routine strategies, developed in algebra, can result in errors when students 'step back' to arithmetic. This is evidenced in the response of $4^8$ for $2^3 \times 2^5$ as though the question contained a variable, i.e., as though responding with $4a^8$ for $2a^3 \times 2a^5$.

The fact that such errors are prevalent can be explained in terms of the constructs of 'connections' or 'frames'. Features of these constructs are in evidence in Booth's comment that informal methods are "strongly adhered to and reluctantly abandoned by the child, possibly because of previously successful usage" (1984, p.37).

*6. Delineating Between Problems Resulting from Teaching/Learning Experiences and Those Grounded in the Child's Cognitive Development Can be Difficult*

'Cognitive readiness' was identified, in the research projects discussed in Chapter 1, as being a factor in students' success in algebra. The projects did not explore this issue closely or arrive at definitive statements concerning its effect on student responses, however, they did indicate this was an important area for research.

This present research has shown that, where it is possible to identify different levels of success in questions involving indices, it is possible also to identify students whose responses show a consistency in terms of those levels. Particular students are limited in their ability to integrate all aspects of the more difficult questions involving indices. As discussed in Chapter 9, it is not the difficulty of any one particular aspect of the question which poses problems, rather it is the difficulty in ensuring all aspects are addressed. Therefore, it appears that, for such students, a cognitive factor, rather than their learning experiences, is limiting the level at which they can respond.

While further research will need to be done to explore this issue, it does appear possible to more closely delineate between problems resulting from teaching/learning experiences of indices and those grounded in the student's cognitive development. This could be achieved through identifying students' levels of response and examining errors made by those performing at higher levels.

In conclusion, reasons for errors in algebra, identified in the research discussed in Chapter 1, provide an appropriate framework within which to view students' errors in questions involving indices. However, this present research project brings to those reasons a number of significant new perspectives. These relate, essentially, to: how certain unexpected responses may be explained; how students who understand underlying concepts, frequently do not apply them; and, the fact that levels of cognitive development are apparent in student responses.

## IMPLICATIONS FOR TEACHING PRACTICES

The effective teaching of indices appears to be a complex and difficult task. Even deciding what constitutes 'effective teaching' poses its problems. Is it having students acquire a relational understanding of indices? If so, the evidence from the sample tested points to a decided lack of success in this area. An examination of the numerous errors made shows the great majority could not have been generated by students seeking to apply genuine understanding. Interviews demonstrate that students, even when they understand the underlying concepts, rarely apply relational understanding and then only in relatively simple circumstances. 'Effective teaching' might also be viewed as teaching which gives students success in answering questions involving indices? However, success is a questionable criteria for such questions in that many correct answers are achieved by routine strategies and are viewed with little understanding by the students providing them.

Strong arguments can be made for students to have both a relational understanding of their work, and to have success in answering index questions. This is a challenging task in that efforts to address one will not necessarily be supportive of addressing the other. Another challenge is that teachers, themselves, have a bias towards relational or instrumental understanding and catering for both involves "great psychological difficulty for teachers of accommodating (restructuring) their existing schema (conceptual structure)" (Skemp 1978, p. 13).

The starting point is to recognise that instrumental thinking is extensively used in questions of the kind investigated. Possibly, the nature of teaching which uses rules to develop ideas followed by extended practice compounds the problem. In Chapter 1 the difficulty students have in working meaningfully with algebraic notation was discussed. This, when taken with the increase in structural complexity which comes with index notation, would seem to dictate that students use instrumental strategies both for convenience and through necessity. Even when a teacher has relational understanding as their aim, students will, in practising their work, develop their own 'connections' or 'frames'. This allows them to answer questions without needing to bring genuine understanding into play. While this does not necessarily mean that such understanding does not exist, if it does, it is not brought into action.

If we accept that success and relational understanding are both important, and that instrumental thinking is widely used, then instrumental and relational techniques each have merit and should be taught appropriately. After first experiences in relationally understanding a 'rule' it would seem sensible that, when discussing applications of the rule, teachers take an approach along the following lines - "The reasoning behind it is ... (relational), you may find it convenient to ... (instrumental), however try to keep in mind what doing that achieves ... (referring back to the relational), and be careful you are using the right rule ... (check the important features)". The appropriateness of such an integrated approach is supported by an observation made by Goldin and Herscovics (1991) when proposing their model of understanding which was discussed earlier. They stated that "cognitive obstacles can occur when form and content do not develop in a synchronised fashion--the content may be difficult to comprehend if it lacks an adequate formal representation; or, more often, the mathematical form becomes devoid of meaning for the learner" (p.70).

An approach of the kind suggested above is not supported by requiring students, in their earlier experiences with indices, to answer questions such as $(3a)^2 \times 5a^3/9(a)^3$ (referred to earlier as coming from a Year 8 textbook). It is difficult to envisage a teacher providing a suitable relational or instrumental explanation to a student having difficulty with such a question let alone an explanation which gives attention to both kinds of understanding. Many such students would be able to provide, at best, a multistructural level response and find it impossible to link together all the concepts involved. The need is for much simpler questions where the two aspects of understanding can be discussed and integrated.

Teachers have to accept that students will acquire and use connections or frames, and work to ensure that all relevant components are incorporated into these. To do this successfully they need to be aware of some of the common incorrect connections, or inadequate preliminary selection frames, which lead to error. Examples include those which result in: multiplication and division of numerical bases; dividing indices when divisions are written using a fraction bar; taking the square root of indices within radical signs; and, adding indices when raising an expression (containing an index) to a power

Earlier, a difference was pointed out, between the theories of Shevarev and Davis. Shevarev focused on students building into their thinking some inappropriate signal to carry out a particular operation. For example, in responding to questions of the kind $a^2 \times a^3$, raised numbers are added. Later, this leads to responses such as $a^5$ for $(a^2)^3$. Davis, however, talked about inadequate preliminary selection frames whereby students, when introduced to $(a^2)^3$ are not sufficiently made aware of the two alternatives and are unable to effectively distinguish between them. As previously indicated, this is only a difference in emphasis. Shevarev's research demonstrates that incorrect 'connections' will happen and need to be addressed, while Davis' 'frames' give guidance to pre-empting some of the problems which occur.

The approach of setting easier questions of different types, rather than harder questions of the same type, would seem to be highly appropriate for assisting students to form correct connections or frames. By requiring students to use all relevant information teachers will help build connections and frames which incorporate all the important features and allow distinctions, such as between $a^2 \times a^3$ and $(a^2)^3$, to be made. Also, if easier questions are used, students can more readily associate the instrumental strategy with the relational explanation. Using more diverse but less difficult questions would not only assist students to make correct connections and build understanding but would support them in moving towards responding at the relational SOLO level. While students would be required to take account of aspects which varied between questions they would also have greater success in responding to those aspects and in recognising when all have been accounted for. This would encourage students to give responses that address all elements of the question.

What is meant by 'questions of different types', as used above, requires some clarification. The 'types' used need to be such as to encourage students to build into their connections all relevant features. Depending on the particular concept being developed 'different types' may mean: different operations; the same operation but with different notations; or, the same operation but with some questions having variables and others having constants. Some examples are: when dividing algebraic fractions, students should be experiencing both division and fraction notation; when the index of a half is introduced, question sets should contain cases where the radical notation is retained; when multiplying algebraic expressions, questions containing numerical bases and others involving substitutions should be incorporated; and, when raising expressions to a power, questions involving multiplication of expressions should be included.

Teachers also need to be aware that students carry other connections or frames with them which do not relate directly to index work but affect students' thinking. A number of misconceptions manifested themselves in students actually rethinking questions and arriving at correct answers. Such was the case with $\sqrt{25x^8}$ where students expected only to be asked to take the square root of a perfect square and, on encountering $\sqrt{8}$, had second thoughts and responded correctly. This highlights the point that teachers, in choosing numbers which are 'tidy', may reinforce incorrect connections such as 'you only take square roots of numbers which are perfect squares'.

The question of understanding of variables has not been the subject of this research but consideration was given to the impact of varying bases and indices, i.e., using constants and variables. Interviews showed some students saw variables as having little relevance. The fact that certain students viewed a coefficient and base as an entity shows the concept of the base as a variable was far from their minds. Efforts to address the general problem of establishing meaning for letters must assist students not only to understand concepts relationally but also to take account of all relevant features when using instrumental strategies.

Hart (1981) discussed implications that findings from the CSMS project had for the teaching of algebra. There is value in indicating here some of the points raised. It was observed that "mathematics is a *very difficult* subject for most children" (p.209). The project found, as did this research, that "to a great extent children adapt the algorithms they are taught or replace them by their own methods" (p.212). Also, the project identified consistent errors occurring across many different schools. Hart commented, "if these errors are widespread as seems likely then when teachers present a topic they should be aware of what is likely to occur" (p.214). Students' pragmatic approach to answers, identified in this present research, was referred to in the comment on a student's uncritical acceptance of answers with the attitude that "it is so because it is so" (p.215). Such findings are compatible with the approach advocated above whereby: account is taken of the fact that students will develop their own strategies; certain incorrect answers are expected and addressed; and, relational and instrumental understandings are both fostered.

In summary, discussion in this section has highlighted the fact that effectively teaching index concepts is not a simple process. Some suggestions have been made as to how students' needs in this area may be met. These can form a basis for the practitioner to further develop strategies suited to their particular circumstances. The task, though not easy, is one which teachers can and must take steps to address.

## IMPLICATIONS FOR FUTURE RESEARCH

This research project has been wide ranging in terms of the content of index work covered. The many issues raised in Chapters 2, 3 an 4 were condensed into five content based themes and three themes relating to theoretical frameworks. While significant findings have been made with regard to these, the area of indices is an extensive one and offers a rich field for the researcher. Some particular directions for further exploration are discussed below.

It would be of significant interest to examine in greater detail the understandings applied by successful students, and to know to what extent they use relational or instrumental strategies. The analysis of levels of responses, in Chapter 9, identified a group of students consistently performing at the relational SOLO level and gaining success through accounting for all elements of each question. While findings of this research may be used to make some inferences about the understandings of such students, the focus has been on the thinking used by students making consistent errors. If, as seems likely, instrumental strategies are widely used, then what is it that enables certain students to give responses which control and account for all the elements?

A possible development of the study would be to undertake an extensive SOLO analysis in terms of cycles in modes and levels. This lay outside the focus of this investigation and would require an extensive testing, interviewing, and analysis procedure in addition to what was carried out here. While the SOLO categories in this present study were broad, three categories being used, the work was consistent with early formulations of SOLO and provided sufficient discrimination to offer worthwhile perceptions on the nature of students' responses. In light of this, investigation within the elaborated SOLO structure could well produce further significant insights.

The act of substituting at the start of a question increased greatly students' chances of success in items involving indices. It would be of interest to examine more closely the thinking which generated such disparities in performance between questions which did and did not involve substitution.

The understanding students have of negative and fractional indices was not investigated closely. The impact of using an index of a half to indicate a square root was looked at and several test items also included negative indices. Results for questions involving these concepts show senior students do have considerable difficulties with them. The literature and the analysis of the School Certificate Moderator examination point to these as being important areas for future research.

Calculators are normally available to the students, however, in order to focus on mental strategies, subjects of this research were required not to use them. It would be of interest to know to what extent students would use calculators, if available, and what impact this would have on responses. Some questions to ask are: how does the use, or otherwise, of a calculator, affect students' success in questions with numerical bases; what kinds of errors arise from the misuse of calculator conventions or the misreading of displays; and, how successfully are questions involving scientific notation dealt with using a calculator?

The SOLO Taxonomy has been shown to provide a framework within which to view students' responses to questions involving indices. Items used in this research could form the basis of an instrument to identify students performing at different SOLO levels in questions involving indices. Given the nature of the questions, it is also likely that the performance of students in items used in this research reflects, to an extent, their performance in algebra. It would be of great interest to see: (1) if levels can be identified for student performance in other areas of algebra: and, (2) to what extent students are performing at the same level across different content areas in algebra. If differences were identified it would be valuable to find the reasons for this occurrence.

Other directions for research can be identified using the extensive data resource available in the form of the School Certificate Moderator examination reported on in Chapter 3. Present research in the field of students' understandings of indices is limited and there are many issues to be explored beyond those discussed above.

## CONCLUSION

The purpose of this research has been to explore students' understanding of indices and so give direction for developing strategies to improve learning outcomes in this area. For this to occur requires the identification of general, and manageable, principles on which to base teaching approaches rather than a detailed analysis of the many errors students make.

In order to move in that direction, five content related themes were identified and students' understandings within each were explored. Following that, three themes concerned with theories of learning, which may explain errors students make in questions involving indices, were examined.

It was established that the SOLO Taxonomy provides a useful framework for assessing students' responses to questions involving indices. Further, it was shown that the instrumental strategies, which students use predominantly, lead to persistent errors, and that the constructs of 'connections' or 'frames' can explain many of these errors.

From the findings of the research, suggestions have been made concerning the teaching of indices. Its hoped that in time they will be picked up by teachers, examined, and as appropriate, applied. Hopefully, these suggestions will lead to fewer student responses such as: "indices ... you just add em"; "I have no idea how I did that"; "I think it must just have been an automatic thing"; "a negative I thought was a fraction and a half was the square root so I just threw it all together"; " I couldn't get a square root of 8 so I divided it by two"; "I was just going on a Year 10 exam I did ages ago"; "I don't like these *p*s and these zeros"; and, "its just a variable type thing". Such comments, coming from relatively able students of mathematics, do give considerable cause for concern.

Little research has been carried out into understandings being applied when working with indices. Students, internationally, are facing similar problems, and using similar approaches to those identified in this research. This is evident from the findings of other researchers, and the commonality of curriculum experience identified between countries. Routine strategies are being used extensively with varying degrees of success. Further examination of those strategies, not just of the less successful, but also of the more successful, will assist in improving outcomes for all.

## REFERENCES


Adams, R.J. and Khoo, S. (1993). *Quest: The Interactive Test Analysis System.* Australian Council for Educational Research, Hawthorn, Victoria.

Australian Education Council (1989). *Mapping the Australian Curriculum vol.2: The Mathematics Curriculum.* Department of Employment, Education and Training, Canberra.

Barnhart, C.L. and Barnhart, R.K. (eds.) (1986). *The World Book Dictionary.* Doubleday and Company, Chicago.

Bernander, L. and Clement, J. (1985). *Catalogue of Error Patterns Observed in Courses on Basic Mathematics.* (Working Draft).

Biggs, J.B. and Collis, K.F. (1980). The Solo Taxonomy, *Education News*, 17, (5), 19-23.

Biggs, J.B. and Collis, K.F. (1982). *Evaluating the Quality of Learning: The SOLO Taxonomy.* Academic Press, New York.

Biggs, J.B. and Collis, K.F. (1989). Towards a Model of School-Based Curriculum Development and Assessment Using the Solo Taxonomy, *Australian Journal of Education*, 33, (2), 151-163.

Biggs, J.B. and Collis, K.F. (1991). Multimodal Learning and the Quality of Intelligent Behaviour. In Rowe H. (ed.) *Intelligence, Re-Conceptualization and Measurement*, pp.57 - 76. Hillsdale, New Jersey; Erlbaum.

Board of Senior School Studies (1982). *Mathematics Syllabus 3 Unit and 2 Unit Courses Year 11 and 12.* Sydney.

Booth, L.R. (1984). *Algebra: Children's Strategies and Errors.* Nfer-Nelson, Windsor.

Booth, L.R. (1986). Difficulties in Algebra, *The Australian Mathematics Teacher*, 42, (3), 2-4.

Booth, L.R. (1989a). A Question of Structure. In Wagner S. and Kieran C. (eds.) *Research Issues in the Learning and Teaching of Algebra*, vol. 4, pp.57-59. National Council of Teachers of Mathematics, Virginia.

Booth, L.R. (1989b). The Research Agenda in Algebra: A Mathematics Education Perspective. In Wagner S. and Kieran C. (eds.) *Research Issues in the Learning and Teaching of Algebra*, vol. 4, pp.238-246. National Council of Teachers of Mathematics, Virginia.

Boys' Schools Committee of the Mathematical Association (1934). *The Teaching of Algebra in Schools*. G. Bell and Sons, Ltd., London.

Briggs, J., Demana, F. and Osborn, A. (1986). Moving into Algebra: Developing the Concepts of Variable and Function, *The Australian Mathematics Teacher*, 42, (3), 5-8.

Chalouh, L. and Herscovics, N. (1988). Teaching Algebraic Expressions in a Meaningful Way. In Coxford, A.F. and Shult, A.P. (eds.) *The Ideas of Algebra, K-12, 1988 Yearbook*. pp.33-42. National Council of Teachers of Mathematics, Virginia.

Childers, G.F. (1987). *An Error Analysis of Algebra Problems Containing Exponents when Constants, Variables, and Algebraic Expressions are Interchanged in the Laws with Implications for Cognitive Style Theory and Latent Trait Test-Scoring Methods*. Ph.D. dissertation, University of Virginia. University Microfilms International, 8819810.

Clements, M.A. (1980). Analyzing Children's Errors On Written Mathematical Tasks. In Bishop, A.J. (ed.) *Educational Studies in Mathematics*, vol 11, no.1, February 1980, pp.1-21. D.Reidel Publishing Company, The Netherlands.

Collis, K.F. (1988). *Multi-Modal Functioning and School Mathematics*. (A paper prepared for the Symposium 'Psychology of Mathematics Learning', International Congress of Psychology, Sydney, August 1988.)

Collis, K.F. and Biggs, J.B. (1991). Developmental Determinants of Qualitative Aspects of School Learning. In Evans, G., (ed.), *Learning and Teaching Cognitive Skills*. pp.185-207. Australian Council for Educational Research, Victoria.

Collis, K.F. and Romberg, T.A. (1991). Assessment of Mathematical Performance: An Analysis of Open-Ended Test Items, In Wittrock, M.C. and Baker, E. (eds.), *Testing and Cognition*, pp.82-130. Prentice-Hall, New Jersey.

Collis, K.F. and Watson, J.M. (1991). A Mapping Procedure for Analysing Responses in Mathematics. *Journal of Structural Learning*, 11, 65-87.

Commission on Standards for School Mathematics (1989). *Curriculum and Evaluation Standards for School Mathematics*. National Council of Teachers of Mathematics, Virginia.

Comstock, M. and Demana, F. 1987). The Calculator is a Problem-solving Concept Developer, *The Arithmetic Teacher*, February, 48-51.

Corbitt, M.K. (ed.) (1981). *Results from the Second Mathematics Assessment of the National Assessment of Educational Progress*. National Council of Teachers of Mathematics, Virginia.

Davis, R.B. (1984). *Learning Mathematics-The Cognitive Science Approach to Mathematics Education.* Croom Helm Ltd., Kent.

Davis, R.B. (1989). Three Ways of Improving Cognitive Studies in Algebra. In Wagner S. and Kieran C. (eds.) *Research Issues in the Learning and Teaching of Algebra,* vol. 4 pp.115-119. National Council of Teachers of Mathematics, Virginia.

Department of Education and Science and The Welsh Office (1989). *Mathematics in the National Curriculum.* London.

DeVincenzo, M.A.R. (1980). *An Investigation of the Relation Between Elementary Algebra Students' Errors in Arithmetic and Algebra in Selected Types of Problems.* Ph.D. dissertation, New York University. University Microfilms International, 80-17494, 1992.

Fey, J.T. (1989). School Algebra for the Year 2000. In Wagner S. and Kieran C. (eds.) *Research Issues in the Learning and Teaching of Algebra,* vol. 4, pp.199-213. National Council of Teachers of Mathematics, Virginia.

Gellert, W., Kustner, H., Hellwich, M. and Kastner, H. (eds.) (1975). *Mathematics at a Glance.* VEB Bibliographisches Institut, Leipzig.

Goldin, J.A. and Herscovics, N. (1991). Towards a Conceptual-Representational Analysis of the Exponential Function. *Proceedings: 15th PME Conference,* vol I, International Group for the Psychology of Mathematics Education, pp.64-71.

Hart, K.M. (ed.) (1981). *Children's Understanding of Mathematics: 11-16,* John Murray, London.

Herscovics, N. (1989). Cognitive Obstacles Encountered in the Learning of Algebra. In Wagner, S. and Kieran, C. (eds.) *Research Issues in the Learning and Teaching of Algebra,* vol. 4, pp.60-86. National Council of Teachers of Mathematics, Virginia.

Hiebert, J. and Lefevre, P (1986). Conceptual and Procedural Knowledge in Mathematics: An Introductory Analysis. In Hiebert, J. (ed.) *Conceptual and Procedural Knowledge: The Case of Mathematics.* Lawrence Erlbaum Associates, New Jersey.

Jesson, D.F. (1983). Levels of Understanding in Mathematics and Their Application to an Algebra Test, *Educational Research,* 25, (2), 125-135.

Kieran, C. (1989). The Early Learning of Algebra: A Structural Perspective. In Wagner S. and Kieran C. (eds.) *Research Issues in the Learning and Teaching of Algebra,* vol. 4, pp.33-56. National Council of Teachers of Mathematics, Virginia.

Kücheman, D.E., (1981). Algebra. In Hart, K.M. (ed.), *Children's Understanding of Mathematics*. pp.102 - 119. London: Murray.

Lindquist, M.M. (ed.) (1989). *Results from the Fourth Mathematics Assessment of the National Assessment of Educational Progress.* National Council of Teachers of Mathematics, Virginia.

MacGregor, M.E. (1986). A Fresh Look at Fruit Salad Algebra, *The Australian Mathematics Teacher*, 42, (3), 9-11.

McSeveny, A., Conway, R. and Wilkes, S. (1988). *Signpost Mathematics Year 8.* Longman Cheshire, Sydney.

Mason, J., Graham, A., Pimm, D. and Gowar, N. (1985). *Routes to/Roots of Algebra.* The Open University Press, Great Britain.

Masters, G.N. (1982). A Rasch Model for Partial Credit Scoring, *Psychometrika*, 47, (2), June 1982, 149-174.

Milton, K. (1988). Getting the Teaching of Algebra Right, *The Australian Mathematics Teacher*, 44, 5-8.

Newman, J.R. (ed.) (1976). *The Universal Encyclopedia of Mathematics.* Pan Books, London.

New South Wales Dept.of Education (1982-1991).*Statistics for the NSW School Certificate Examinations.* Sydney.

Pegg, J. (1992) Assessing Students' Understanding at the Primary and Secondary Level in the Mathematical Sciences. In Stephens, M. and Izard, J. (eds.) *Reshaping Assessment Practices: Assessment in the Mathematical Sciences Under Challenge.* pp.368 - 385. Australian Council for Educational Research, Victoria.

Pegg, J. and Redden, T. (1990) Procedures for and Experiences in Introducing Algebra in NSW. *Mathematics Teacher* 83, 386-391.

Secondary Schools Board, (1983). *Syllabus in Mathematics Years 9 and 10.* Sydney.

Shevarev, P.A. (1975). An Experiment in the Psychological Analysis of Algebraic Errors. A translation of the original Soviet publication. In Kilpatrick, J., Begle, E.G., Wirszup, I. and Wilson, J.W. (eds.). *Soviet Studies in the Psychology of Learning and Teaching Mathematics*, vol.XII, pp.1-60. National Council of Teachers of Mathematics, Virginia.

Skemp, R.R. (1978). Relational Understanding and Instrumental Understanding, *Arithmetic Teacher.* November 1978, 9-15.

Skemp, R.R. (1979). *Intelligence, Learning and Action: A Foundation for Theory and Practice in Education.* John Wiley and Sons, Chichester.

Thorpe, J.A. (1989). Algebra: What Should We Teach and How Should We Teach It?. In Wagner, S. and Kieran, C. (eds.) *Research Issues in the Learning and Teaching of Algebra*, vol. 4, pp.11-24. National Council of Teachers of Mathematics, Virginia.

Usiskin, Z. (1988). Conceptions of School Algebra and Uses of Variable. In Coxford, A.F. and Shult, A.P. (eds.) *The Ideas of Algebra, K-12, 1988 Yearbook*, pp.8-19. National Council of Teachers of Mathematics, Virginia.

Vinner, S. (1977). The Concept of Exponentiation at the Undergraduate Level and the Definitional Approach, *Educational Studies in Mathematics 8.* D.Reidel Publishing Co., Dordrecht-Holland.

Wagner, S. and Kieran, C. (1989). An Agenda for Research on the Learning and Teaching of Algebra. In Wagner S. and Kieran C. (eds.) *Research Issues in the Learning and Teaching of Algebra,* vol. 4, pp.220-237. National Council of Teachers of Mathematics, Virginia.

Wilson, F.J. (1985). *A Clinical Investigation of the Understanding of Exponents by Remedial Algebra Students at a Four Year College.* Michigan State University. University Microfilms International. 86-03497.

# APPENDIX A

## REFERENCES TO INDICES IN SYLLABUSES FOR THE N.S.W. SCHOOL CERTIFICATE

**Years 7 and 8**

N4.2 Multiples, Factors and Index Notation
- . list multiples of a given number
- . classify numbers as composite or prime
- . determine and list factors of a number
- . use index notation to find HCF and LCM

N4.3 Square and Cube Roots
- . estimate and find (where possible) square roots and cube roots.
- . use square root and cube root notation

(Board of Secondary Ed. N.S.W., 1989, p.87)

A8 Index Notation (integral values only for the index)
- . simplify expressions of the types

$a \times a \times a \times a = a^4$
$a^2 \times a^3 = a^5$
$a^4/a = a^3$
$(a^3)^2 = a^6$

(Board of Secondary Ed. N.S.W., 1989, p.114)

**Years 9 and 10 Advanced**

5.4 Indices
- i. Index properties for positive integral values
- ii. meaning of $a^0$
- iii. meaning of negative integral indices and their use with index properties and standard or scientific notation
- iv. meaning of $x^{1/2}$, $x^{1/3}$, $x^{p/q}$

(Secondary Schools Board, 1983, p.24)

**Years 9 and 10 Intermediate**

4.3 Revision of index properties for positive integral indices. The meaning of $a^0$. Standard or scientific notation. Negative indices as required for use in calculators

(Secondary Schools Board, 1983, p.16)

Lobe - Surds and indices
- . meaning of negative integral indices and their use with index properties and scientific notation
- . meaning of $x^{1/2}$, $x^{1/3}$

(Secondary Schools Board, 1983, p.21)

# APPENDIX B

## PILOT STUDY - MULTIPLE CHOICE QUESTIONS AND OPTIONS

Please circle the answer of your choice in pencil. If you wish to reselect, erase your first choice then circle your new choice.

[1] $3^3 \times 3^2 = ?$ (1984 Adv.&Int Courses Question1)
(A) $3^5$ (B) $3^6$ (C) $9^5$ (D) $9^6$

[2] $2^3 \times 3^2 = ?$ (1982 Question 2)
(A) $6^5$ (B) $6^6$ (C) 36 (D) 72

[3] $2^a \times 2^b = ?$ (New)
(A) $2^{ab}$ (B) $2^{a+b}$ (C) $4^{ab}$ (D) $4^{a+b}$

[4] $4a^6 \times 6a^3 = ?$ (New)
(A) $24a^9$ (B) $24a^{18}$ (C) $10a^9$ (D) $10a^{18}$

[5] $2a \times 3a \times 4a = ?$ (1987 Int. Course Question 2)
(A) $9a$ (B) $9a^3$ (C) $24a$ (D) $24a^3$

[6] $2b^4 \times 3b = ?$ (New)
(A) $6b^5$ (B) $5b^4$ (C) $5b^5$ (D) $6b^4$

[7] $a^m \times a^n = ?$ (New)
(A) $a^{m+n}$ (B) $2a^{m+n}$ (C) $a^{mn}$ (D) $2a^{mn}$

[8] $2^{12} \div 2^2 = ?$ (1983 Int. Course Question 16)
(A) $1^6$ (B) $2^6$ (C) $1^{10}$ (D) $2^{10}$

[9] $12m^6 \div 4m^3 = ?$ (1984 Int. Course Question 3)
(A) $3m^3$ (B) $3n^2$ (C) $8m^3$ (D) $8m^2$

[10] $8a^{12} \div 2a^8 = ?$ (New)
(A) $4a^{1\ 1/2}$ (B) $6a^4$ (C) $6a^{1\ 1/2}$ (D) $4a^4$

[11] $a^3 + a^3 = ?$ (1984 Int. Course Question 9)
(A) $a^6$ (B) $a^9$ (C) $2a^3$ (D) $2a^6$

[12] $5x^{-1/2} = ?$ (1984 Adv.Course Question 9)
(A) $\frac{-1}{\sqrt{5x}}$ (B) $\frac{5}{x^2}$ (C) $\frac{1}{5\sqrt{x}}$ (D) $\frac{5}{\sqrt{x}}$

[13] $\frac{1}{2x^3} = ?$ (1985Adv.Course Question11)
(A) $2x^{1/3}$ (B) $2x^3$ (C) $\frac{1}{2}x^{1/3}$ (D) $\frac{1}{2}x^{-3}$

[14] $3a^{-2} = ?$ (1983 Adv.Course Question 5)
(A) $\frac{1}{3a^2}$ (B) $\frac{3}{a^2}$ (C) $\frac{-1}{3a^2}$ (D) $\frac{-3}{a^2}$

[15] If $a = -3$ then $4a^2 = ?$ (1984 Int.Course Question 23)

(A) -36 (B) 36 (C) -144 (D) 144

[16] $3ab^2 = ?$ (1984 Int.Course Question 15)

(A) $3ab \times 3ab$ (B) $3 \times ab \times ab$ (C) $3 \times a \times b \times 2$ (D) $3 \times a \times b \times b$

[17] $(2^3)^2 = ?$ (1981 Question 9)

(A) $2^5$ (B) $2^6$ (C) $2^9$ (D) $4^6$

[18] $(3m^2)^3 = ?$ (1982 Question 23)

(A) $27m^6$ (B) $27n^5$ (C) $9m^6$ (D) $9m^5$

[19] $(a^2)^3 = ?$ (New)

(A) $3a^6$ (B) $a^9$ (C) $a^6$ (D) $a^5$

[20] $(a^b)^c = ?$ (New)

(A) $a^{b+c}$ (B) $abc$ (C) $a^{bc}$ (D) $a^c b^c$

[21] $8 \times 3^0 = ?$ (1983 Int.Course Question 5)

(A) 0 (B) 1 (C) 8 (D) 24

[22] $3a^0 = ?$ (1985 Int.Course Question19)

(A) $3a$ (B) 0 (C) 1 (D) 3

[23] $(a^6)^0 = ?$ (New)

(A) 0 (B) 1 (C) $a^6$ (D) $a^{60}$

[24] $16^0 + 16^{1/2}$ ? (1982 Question 40)

(A) 4 (B) 5 (C) 8 (D) 9

[25] $(3x)^0 + 2y^0 = ?$ (1987 Int.Course Question17)

(A) 0 (B) 2 (C) 3 (D) 5

[26] $1 + x + x^0 = ?$ (1986 Int.Course Question 28)

(A) $1 + x$ (B) $1 + 2x$ (C) $2 + x$ (D) 3

[27] $2a^0 + a^{-1} = ?$ (1986 Adv.Course Question17)

(A) $\frac{1}{a}$ (B) $2 + \frac{1}{a}$ (C) $1 + \frac{1}{a}$ (D) $2 - a$

[28] $\frac{6a^2}{2ab} = ?$ (1986 Int. Course Question 7)

(A) $3ab$ (B) $\frac{3a}{b}$ (C) $6b$ (D) $\frac{6}{b}$

[29] $\frac{x^4 \times x^6}{x^2} = ?$ (1983 Int. Course Question 8)

(A) $x^5$ (B) $x^8$ (C) $x^{12}$ (D) $x^{22}$

[30] $\frac{(a^4)^4}{a^2} = ?$ (1988 Adv. Course Question2)

(A) $a^4$ (B) $a^6$ (C) $a^8$ (D) $a^{14}$

# APPENDIX C

# STATISTICS FOR THE MAIN STUDY

| | |
|---|---|
| Mean Test Score | 31.92 |
| Standard Deviation | 10.12 |
| Internal Consistency | 0.92 |

**Summary of Item Estimates** **Main Study**

| | |
|---|---|
| Mean | 0.00 |
| SD | 1.23 |
| SD (adjusted) | 1.15 |
| Reliability of estimate | .87 |

**Fit Statistics**

| Infit Mean Square | | Outfit Mean Square | |
|---|---|---|---|
| Mean | .99 | Mean | 1.13 |
| SD | .19 | SD | .63 |

| Infit t | | Outfit t | |
|---|---|---|---|
| Mean | -.04 | Mean | .15 |
| SD | .97 | SD | 1.09 |

0 items with zero scores
1 items with perfect scores

## Item Estimates: Thresholds Main Study

| Item Name | Score | Thresholds<br>Mean Thresh (St.Error) |
|---|---|---|
| Item 1 | 23 | 0.44 (0.37) |
| Item 2 | 16 | 1.38 (0.38) |
| Item 3 | 16 | 1.38 (0.38) |
| Item 4 | 40 | perfect score |
| Item 5 | 33 | -2.69 (0.75) |
| Item 6 | 33 | -2.69 (0.75) |
| Item 7 | 29 | -0.42 (0.40) |
| Item 8 | 34 | -1.35 (0.48) |
| Item 9 | 21 | 0.70 (0.37) |
| Item 10 | 37 | -2.23 (0.63) |
| Item 11 | 21 | 0.70 (0.37) |
| Item 12 | 13 | 1.83 (0.40) |
| Item 13 | 11 | 2.15 (0.42) |
| Item 14 | 37 | -2.23 (0.63) |
| Item 15 | 36 | -1.88 (0.56) |
| Item 16 | 30 | -0.58 (0.41) |
| Item 17 | 36 | -1.88 (0.56) |
| Item 18 | 23 | 0.44 (0.37) |
| Item 19 | 27 | -0.12 (0.39) |
| Item 20 | 28 | -0.27 (0.39) |
| Item 21 | 29 | -0.42 (0.40) |
| Item 22 | 22 | 0.57 (0.37) |
| Item 23 | 24 | 0.30 (0.37) |
| Item 24 | 24 | 0.30 (0.37) |
| Item 25 | 31 | -0.75 (0.42) |
| Item 26 | 26 | 0.02 (0.38) |
| Item 27 | 28 | -0.27 (0.39) |
| Item 28 | 22 | 0.57 (0.37) |
| Item 29 | 34 | -1.35 (0.48) |
| Item 30 | 32 | -0.94 (0.44) |
| Item 31 | 27 | -0.12 (0.39) |
| Item 32 | 21 | 0.70 (0.37) |
| Item 33 | 19 | 0.97 (0.37) |
| Item 34 | 27 | -0.12 (0.39) |
| Item 35 | 16 | 1.38 (0.38) |
| Item 36 | 12 | 1.99 (0.41) |
| Item 37 | 21 | 0.70 (0.37) |
| Item 38 | 21 | 0.70 (0.37) |
| Item 39 | 28 | -0.27 (0.39) |
| Item 40 | 11 | 2.15 (0.42) |
| Item 41 | 21 | 0.70 (0.37) |
| Item 42 | 20 | 0.84 (0.37) |
| Item 43 | 22 | 0.57 (0.37) |
| Item 44 | 35 | -1.60 (0.52) |
| Item 45 | 28 | -0.27 (0.39) |
| Item 46 | 22 | 0.57 (0.37) |
| Item 47 | 18 | 1.11 (0.37) |
| Item 48 | 31 | -0.75 (0.42) |
| Item 49 | 19 | 0.97 (0.37) |
| Item 50 | 32 | -0.94 (0.44) |
| Mean | | 0.00 |
| SD | | 01.23 |

# APPENDIX D

# STATISTICS FOR THE SOLO STUDY

| | |
|---|---|
| Mean Test Score | 31.45 |
| Standard Deviation | 9.56 |
| Internal Consistency | 0.90 |

**Summary of Item Estimates** **SOLO Study**

| | |
|---|---|
| Mean | 0.00 |
| SD | .56 |
| SD (adjusted) | .00 |
| Reliability of estimate | .00 |

**Fit Statistics**

| Infit Mean Square | | Outfit Mean Square | |
|---|---|---|---|
| Mean | .99 | Mean | 1.03 |
| SD | .18 | SD | .27 |

| Infit t | | Outfit t | |
|---|---|---|---|
| Mean | -.03 | Mean | .10 |
| SD | .87 | SD | .99 |

0 items with zero score
0 item with perfect score

**Item Estimates: Thresholds** **SOLO Study**

| Ref.No. | Item Name | Thresholds | |
|---|---|---|---|
| | | 1 | 2 |
| | | Mean Thresh (St.Error) | Mean Thresh (St.Error) |
| 1 | Item 2 | -2.25 (1.09) | 1.39 (0.70) |
| 2 | Item 3 | -0.53 (0.69) | 1.30 (0,66) |
| 3 | Item 9 | -1.31 (0.84) | 0.76 (0.66) |
| 4 | Item 11 | -1.31 (0.84) | 0.76 (0.66) |
| 5 | Item 12 | -2.31 (1.13) | 1.80 (0.75) |
| 6 | Item 13 | -2.34 (1.13) | 2.10 (0.79) |
| 7 | Item 18 | -1.59 (0.94) | 0.54 (0.69) |
| 8 | Item 22 | -2.09 (1.06) | 0.63 (0.74) |
| 9 | Item 23 | -2.03 (1.06) | 0.40 (0.74) |
| 10 | Item 24 | -1.56 (0.94) | 0.42 (0.71) |
| 11 | Item 28 | -0.78 (0.75) | 0.66 (0.66) |
| 12 | Item 32 | -0.28 (0.67) | 0.75 (0.62) |
| 13 | Item 33 | -0.16 (0.66) | 0.96 (0.63) |
| 14 | Item 35 | -0.53 (0.69) | 1.30 (0.66) |
| 15 | Item 36 | -0.44 (0.69) | 1.84 (0.68) |
| 16 | Item 37 | -0.59 (0.69) | 0.75 (0.64) |
| 17 | Item 38 | -0.28 (0.67) | 0.75 (0.62) |
| 18 | Item 40 | 0.89 (0.64) | 1.81 (0.66) |
| 19 | Item 41 | -1.03 (0.78) | 0.77 (0.66) |
| 20 | Item 42 | -0.03 (0.66) | 0.85 (0.61) |
| 21 | Item 43 | -0.59 (0.72) | 0.65 (0.65) |
| 22 | Item 46 | -2.84 (1.38) | 0.63 (0.75) |
| 23 | Item 47 | -0.19 (0.66) | 1.06 (0.64) |
| 24 | Item 49 | 0.31 (0.63) | 0.94 (0.60) |
| Mean | | 0.00 | |
| SD | | 0.56 | |

| Ref.No. | Item Name | Deltas 1 Mean Delta (St.Error) | Deltas 2 Mean Delta (St.Error) |
|---|---|---|---|
| 1 | Item 2 | -2.21 (0.84) | 1.35 (0.49) |
| 2 | Item 3 | -0.35 (0.52) | 1.14 (0,46) |
| 3 | Item 9 | -1.17 (0.65) | 0.62 (0.46) |
| 4 | Item 11 | -1.17 (0.65) | 0.62 (0.46) |
| 5 | Item 12 | -2.29 (0.84) | 1.77 (0.52) |
| 6 | Item 13 | -2.33 (0.85) | 2.09 (0.55) |
| 7 | Item 18 | -1.46 (0.73) | 0.40 (0.47) |
| 8 | Item 22 | -2.02 (0.84) | 0.58 (0.48) |
| 9 | Item 23 | -1.93 (0.85) | 0.32 (0.48) |
| 10 | Item 24 | -1.41 (0.73) | 0.27 (0.47) |
| 11 | Item 28 | -0.50 (0.57) | 0.37 (0.45) |
| 12 | Item 32 | 0.17 (0.51) | 0.31 (0.45) |
| 13 | Item 33 | 0.23 (0.49) | 0.56 (0.45) |
| 14 | Item 35 | -0.35 (0.52) | 1.14 (0.46) |
| 15 | Item 36 | -0.33 (0.51) | 1.73 (0.49) |
| 16 | Item 37 | -0.31 (0.54) | 0.45 (0.45) |
| 17 | Item 38 | 0.17 (0.51) | 0.31 (0.45) |
| 18 | Item 40 | 1.40 (0.45) | 1.30 (0.51) |
| 19 | Item 41 | -0.85 (0.60) | 0.57 (0.46) |
| 20 | Item 42 | 0.52 (0.48) | 0.30 (0.45) |
| 21 | Item 43 | -0.24 (0.54) | 0.30 (0.45) |
| 22 | Item 46 | -2.81 (1.11) | 0.62 (0.49) |
| 23 | Item 47 | 0.16 (0.49) | 0.72 (0.45) |
| 24 | Item 49 | 1.08 (0.46) | 0.17 (0.45) |
| Mean | | 0.00 | |
| SD | | 0.55 | |