Chapter 7

INTEGRAL BASES VERSUS VARIABLE BASES and RELATING INDICES TO BASES AND COEFFICIENTS (Themes 1 and 2)

Introduction to Chapter

In the preceding chapter specific research questions, relating to the content related themes identified in Chapter 3 were posed. Answers to these questions are now sought. Of the five content related themes, two, which are of a more general nature, are addressed in this chapter while the three relating to more specific content areas are considered in the following chapter. Test results and interview data are used in the analysis. The remaining three themes, relating to theoretical frameworks discussed in Chapter 4, are addressed in Chapters 9 and 10. That consideration of the theoretical frameworks draws on information obtained from the analysis in this, and the following, chapter.

The two themes addressed here are concerned with issues which reach across a range of questions involving indices. They are the tendency for many students: to multiply and divide numerical bases; and, to use incorrect relationships between indices, bases and coefficients. These are referred to as **Integral Bases Versus Variable Bases** (Theme 1) and **The Relationship of Indices to Bases and Coefficients** (Theme 2), respectively. Prior to considering these themes, the following section explains how the analysis is structured and provides an overview of the items used.

STRUCTURE OF THE ANALYSIS AND OVERVIEW OF ITEMS

In this chapter, and the following one, the analysis is carried out within the themes nominated. The research questions, as identified in Chapter 6, are considered in sequence. Questions relating to the written responses of students are dealt with first, the analysis being based upon test results for the relevant items. For each theme, the later questions relate to the understandings students apply in answering items involving indices. For such questions a qualitative analysis, based upon the data gathered during the student interviews, is used. These data contain responses students gave when discussing their written answers. Also, they contain relevant data gathered from the five questions directed at ascertaining students' understanding of several concepts basic to their work with indices. These were posed at the end of each interview. The research questions relating to the themes which are the subject of this chapter are:

Theme 1: Integral Bases Versus Variable Bases

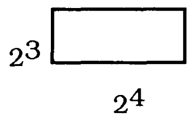
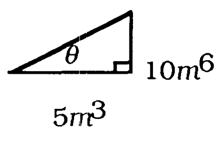
- *What is the effect on students' responses of having the base as a constant as opposed to having the base as an unknown?*
- *With multiplication or division of constant bases, what is the effect of the bases being the same constant as opposed to their being different?*
- *What is the effect on students' responses of having a substitution as the first step in such questions?*
- *What is the effect on students' responses of having the operation arise out of an application?*
- *What do interviews reveal about the understandings students apply when multiplying or dividing in index questions involving numerical bases?*

Theme 2: The Relationship of Indices to Bases and Coefficients

- *When a term containing an index is raised to a power, what is the effect on student responses of that term having: a constant base; an unknown base; or, an unknown base with a coefficient?*
- *What is the effect on responses of having a substitution as the first step in questions of the kind examined in this theme?*
- *What do interviews reveal about the understandings students use when applying an integral power to a term involving an index as that term varies between having: a constant base; an unknown base; or, an unknown base with a coefficient?*
- *What do interviews reveal about the understandings students have of how a non-integral index relates to its base and the coefficient in expressions such as* $5x^{-1/2}$*?*

Before proceeding to the analysis it is of value to review the scope of the items used for the themes addressed in this chapter. As indicated previously, the nature of the bases and indices is a central consideration for these two themes. For the Main Study, additional questions have been added to cover a wider range of situations than were addressed in the Pilot Study. This includes questions involving: a numerical base; a pronumeral base; a pronumeral base and index; a substitution; and, an application. Table 7.1 shows how the items address these additional contexts. Certain items relate to several of the five content related themes (a full list of test items is provided in Table 6.5). Items common to the Pilot Study and Main Study are, in most cases, numbered differently for each because new questions have been added.

Table 7.1. Test Items Relating to Theme 1 and Theme 2

	THEME 1 Integral Bases Versus Variable Bases (a) Multiplication	(b) Division	**THEME 2 Relating Powers to Bases and Coefficients**
Numerical Base	[1] $2^3 \times 2^4$ [2] $5^2 \times 2^3$ [3] $3^x \times 3^y$ *	[11] $5^6 \div 5^2$ [12] $6^3 \div 2^2$ ** [13] $10^4 \div 2^3$ **	[26] $(3^2)^3$ [27] $(2^3)^3$ ** [34] $(2^3)^{-2}$ ** [41] 2×6^{-1}
Pronumeral Base	[4] $2a^3 \times 3a^2$ * [5] $3b \times 2b \times 5$ [6] $5m^3 \times 2m$ *	[14] $10m^6 \div 2m^2$ [15] $12k^{10} \div 3k^4$ *	[25] $2m^3 + m^3$ [28] $(2m^2)^3$ [29] $(a^4)^5$ * [40] $5x^{-1/2}$ **
Pronumeral Base &Index	[7] $b^x \times b^y$ *	[21] $m^a \div m^p$ **	[30] $(a^m)^n$ *
Substitution and Applications	[8] If $a = 2$, $b = 5$ then $b^2a^3 = ?$ ** [9]  Area = ? ** [10] If $m = 5$, $n = 2$ then $3mn^2 = ?$	[18] If $p = 4$, $q = 2$ then $p^3 \div q^2 = ?$ ** [19] What is the common ratio for the GP $2^3, 2^7, 2^{11}, ...?$ ** [20]  $\tan\theta = ?$ **	[31] If $m = 10$ $(m^2)^3 = ?$ ** [32] If $f(x) = x^3$ $f(2^4) = ?$ ** [33] If $f(x) = x^3$ $f(2a^3) = ?$ ** [39] If $x = 9$ $4x^{1/2} = ?$ [42] If $m = 2$ $m^{-2} = ?$ [50] If $n = -2$ $3n^2 = ?$

(** - item developed for the Main Study; * - item developed for the Pilot Study; no asterisk - item developed from SC item)

INTEGRAL BASES VERSUS VARIABLE BASES
(Theme 1)

This theme was concerned with errors made when students multiply and divide expressions involving indices. It focused particularly on the tendency for many students to multiply and divide numerical bases.

A summary of student responses is provided in the tables below. As indicated previously, the questions in this research have been grouped separately for multiplication and division to allow identification of issues specific to either, and to assist in managing the large number of questions relating to this theme. For ease of reference, two tables are provided for each of multiplication and division. Table 7.2 and Table 7.3 are both associated with multiplication (Theme 1 (a)). Items in Table 7.2 are in chronological order, together with: the correct answer; the number correct; and, a list of the other responses. In the 'Other Responses' column, numbers in brackets show how many candidates gave a particular incorrect response. Where there is no number, that answer was given by one candidate only. The letters NA indicate a question was not answered. In Table 7.3 the items are listed in order of the difficulty they posed to students. This is on the basis of the item thresholds obtained in the analysis discussed in Chapter 6. Table 7.4 and Table 7.5 provide similar information for items associated with division (Theme 1 (b)).

As mentioned previously, seven of the subjects of this research were placed in the bottom 20% of candidates at the HSC. It is important to note, that these students accounted for a high proportion of the more obscure incorrect answers appearing in tables in this, and the following, chapter. Since this research is concerned with students' thinking in index questions and with 'misunderstandings' or alternative interpretations, rather than 'lack of understanding', it is the more common errors, rather than the obscure ones, which are of interest.

Table 7.2. Theme 1 (a): Summary of Results

	Item	Answer	No.Corr.	Other Responses and (n)*
[1]	$2^3 \times 2^4$	2^7or 128	23 (58%)	4^7(17)
[2]	$5^2 \times 2^3$	200	16 (40%)	10^5(21), 33, 10^6(2)
[3]	$3^x \times 3^y$	3^{x+y}	16 (40%)	9^{x+y}(15), 3^{xy}, 9^{xy}(6), NA(2)
[4]	$2a^3 \times 3a^2$	$6a^5$	40 (100%)	

[5]	$3b \times 2b \times 5b$	$30b^3$	38 (95%)	$30b$, $10b$
[6]	$5m^3 \times 2m$	$10m^4$	38 (95%)	$10m^3$(2)
[7]	$b^x \times b^y$	b^{x+y}	29 (73%)	b^{xy}(6), $b^x \times b^y$, $2b^{xy}$, $(2b)^{xy}$, $(b^2)^{x+y}$, $(b^2)^{xy}$
[8]	a=2, b=5, b^2a^3= ?	200	34 (85%)	33(2), 10^5, 80,150, NA
[9]	2^3 2^4 Area= ?	2^7or128	21 (53%)	4^7(14), 2^{12}, 24, 4^{12}(2), 2 (2^3+2^4)
[10]	m=5,n=2,$3mn^2$= ?	60	37 (93%)	900(2), 30^2

* When only one student gave a response (n) was not include.

For the ten items relating to Theme 1(a), the subjects of this research had a mean score of 7.3 (73%) and a standard deviation of 2.00. Questions involving integral bases contributed greatly to this relatively poor result.

In the following table the items are listed in order of difficulty.

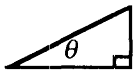
Table 7.3. Theme 1 (a): Items in Order of Difficulty

	Item	Answer	No.Corr.	Threshold
[2]	$5^2 \times 2^3$	200	16 (40%)	1.38
[3]	$3^x \times 3^y$	3^{x+y}	16 (40%)	1.38
[9]	2^3 2^4 Area= ?	2^7or128	21 (53%)	0.70
[1]	$2^3 \times 2^4$	2^7or 128	23 (58%)	0.44
[7]	$b^x \times b^y$	b^{x+y}	29 (73%)	-0.42
[8]	a=2, b=5, b^2a^3= ?	200	34 (85%)	-1.35
[10]	m=5,n=2,$3mn^2$= ?	60	37 (93%)	-2.23
[5]	$3b \times 2b \times 5b$	$30b^3$	38 (95%)	-2.69
[6]	$5m^3 \times 2m$	$10m^4$	38 (95%)	-2.69
[4]	$2a^3 \times 3a^2$	$6a^5$	40 (100%)	Perfect score

The percentage of subjects successfully answering the items varied greatly. All responded correctly to Item 4, however, only 40% answered correctly Items 2 and 3.

Following are the results for items relating to Theme 1 (b).

Table 7.4. Theme 1 (b): Summary of Results

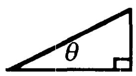
	Item	Answer	No.Corr.	Other Responses and (n)*
[11]	$5^6 \div 5^2$	5^4 or 625	21 (53%)	1(14), 5^3, 5^2, 0^4, NA(2)
[12]	$6^3 \div 2^2$	54	13 (33%)	3(25), 12, NA
[13]	$10^4 \div 2^3$	1250	11 (28%)	5(26), 12500, $\frac{5^4}{1^3}$, NA(1)
[14]	$10m^6 \div 2m^2$	$5m^4$	37 (93%)	$5m^3$(2), $5m^8$
[15]	$12k^{10} \div 3k^4$	$4k^6$	36 (90%)	$36k^{14}$, $4k^5$, $\frac{4k^5}{k^2}$, $\frac{4k^5}{3k}$
[18]	$p=4$, $q=2$, $p^3 \div q^2 =$?	16	23 (58%)	2(11), 8(3), 64(2), 4^4
[19]	Ratio of $2^3, 2^7, 2^{11}$	2^4	27 (68%)	1(4), x^4(2), 4(2), 2^{n+4}, 2^{15}, $2^{1/3}$, power increases by 4, NA
[20]	 $5m^3$ $10m^6$ $\tan\theta =$?	$2m^3$	28 (70%)	$2m^2$(5), $5m^3$(2), m^2, 30^0, $5(2m^6-m^3)$, NA(2)
[21]	$m^a \div m^p$	m^{a-p}	29 (73%)	1^{a-p}(6), $\frac{a}{p}$(2), $\frac{1^a}{1^p}$, $\frac{m^a}{m^{p}}$, NA

* When only one student gave a response (n) was not include.

The mean score achieved by subjects on these nine items was 5.60 (62%) and the standard deviation was 1.97. Again the level of success is low.

Table 7.5 lists the results for this theme in the order in which questions posed difficulties to students.

Table 7.5. Theme 1 (b): Items in Order of Difficulty

	Item	Answer	No.Corr.	Threshold
[13]	$10^4 \div 2^3$	1250	11 (28%)	2.15
[12]	$6^3 \div 2^2$	54	13 (33%)	1.83
[11]	$5^6 \div 5^2$	5^4 or 625	21 (53%)	0.70
[18]	$p=4$, $q=2$, $p^3 \div q^2 =$?	16	23 (58%)	0.44
[19]	Ratio of $2^3, 2^7, 2^{11}$	2^4	27 (68%)	-0.12
[20]	 $5m^3$ $10m^6$ $\tan\theta =$?	$2m^3$	28 (70%)	-0.27
[21]	$m^a \div m^p$	m^{a-p}	29 (73%)	-0.42
[15]	$12k^{10} \div 3k^4$	$4k^6$	36 (90%)	-1.88
[14]	$10m^6 \div 2m^2$	$5m^4$	37 (93%)	-2.23

As for multiplication, the percentage of subjects successfully answering items varied greatly. Item 14 was answered correctly by almost all students, however, only one in four, approximately, were able to respond correctly to Item 13. Students had greater difficulty with division questions than with multiplication. This is evident from a comparison of item thresholds. It is reflected also in the mean scores of 5.60 (62%) and 7.3 (73%) for the respective operations. A two-tailed Repeated Measures t test revealed a significant difference between the multiplication and division group means ($t = 6.402$, $df = 39$, $p = 0.0001$). The lack of success in the division questions was, again, a result of the strong tendency to operate on constant bases as in $5^6 \div 5^2$.

The results show the more frequent errors are the same for the two operations in the sense that they involve operating incorrectly on numerical bases. The frequency with which this occurs is somewhat higher for division than multiplication. In questions where only the indices are integers students rarely make errors in carrying out these two operations.

The greatest difference in the kinds of errors made between multiplication and division seems to occur in questions where both the base and index are unknowns. Item 7 ($b^x \times b^y$) and Item 21 ($m^a \div m^p$) were both answered correctly by 73% of the students, but an examination of the other errors shows differences between the two. In Item 7 the most frequent error was to multiply the indices, while in Item 21 it was to divide the bases. The reasoning students may have used in multiplying the indices, i.e., $b^x \times b^y = b^{xy}$, is not obvious, especially since the students making this error had all chosen correctly to add indices in questions with numerical indices. Four of the students making this error also multiplied indices in Item 3 ($3^x \times 3^y$). There they obtained an answer of 9^{xy}, having multiplied erroneously the numerical bases. Students have few problems multiplying and dividing expressions with unknown bases and integral powers such as in $5m^3 \times 2m$.

That students, pursuing the more demanding senior courses, had this level of success with the items belonging to this theme, does give cause for concern. The result mirrors the findings from the School Certificate data, the review of the literature and the Pilot Study where students found similar difficulty with such questions. As listed at the beginning of this chapter, five research questions concerning this theme have been posed. These five research questions are now addressed.

What is the effect on students' responses of having the base as a constant as opposed to having the base as an unknown?

The problem of students multiplying and dividing numerical bases, as can be seen from the tables above, was an overwhelming cause of errors in this study. For both multiplication and division, students had far greater difficulty with numerical bases than unknown bases.

The division questions, $10^4 \div 2^3$ (Item 13) and $6^3 \div 2^2$ (Item 12), were, respectively, the equally most difficult and fourth most difficult of the fifty items. The multiplication questions, $5^2 \times 2^3$ (Item 2) and $3^x \times 3^y$ (Item 3), were equally the fifth most difficult. An examination of the Other Responses column for these four questions shows that in 91% of errors (95 students out of a total number of 104), the mistake was to multiply or divide the base. In Item 7 ($b^x \times b^y$), where both the base and indices are unknowns, students were successful, relatively, although approximately 20% did choose to multiply indices. Students had negligible problems operating with expressions which contained a coefficient and an unknown base.

The effect of having the base as a constant, rather than an unknown, is to increase greatly the error rate through students operating incorrectly on the base.

With multiplication or division of constant bases, what is the effect of the bases being the same constant as opposed to their being different?

Students found greater difficulty with questions where the bases were different constants than where they were the same. However, the error rate in both cases was high. This occurred for both multiplication and division. Students who operated on bases accounted for approximately 90% of total errors in such questions.

For multiplication, only 40% of students answered $5^2 \times 2^3$ correctly, where the bases were different, while 58% answered correctly $2^3 \times 2^4$ where the bases were the same. These questions were, respectively, the 5th and the 21st most difficult of the fifty questions. In division, where the two bases were different, only 27% answered $10^4 \div 2^3$ correctly and 33% answered $6^3 \div 2^2$ correctly. These questions were the most difficult and 4th most difficult questions, respectively. Item 11 ($5^6 \div 5^2$) was the division question with the bases being the same constant. It was the 12th most difficult question and was answered successfully by 53% of students.

It may be that the higher success rate for division questions with the same base, as opposed to those with different bases, is accounted for by some students rethinking their answer on arriving at what should appear as a mathematically absurd answer of 1. However, it is possible that this is not the explanation, given that many students were prepared to accept 1 as an answer (35% did in Item 11). Also, such an explanation obviously has no counterpart in multiplication which would account for the higher success rate when finding products in cases where bases were the same as opposed to those where bases were different. Another possibility is that a student who correctly answered a question with the same bases, e.g., $5^6 \div 5^2 = 5^4$, would be encouraged to operate on the bases, when they were different, in order to obtain a single numerical base in the answer. Such students would not be amongst those interviewed since candidates for interview were selected on the basis of consistent errors.

With both multiplication and division, students do have greater success when the numerical bases are the same than they do when the bases are different.

What is the effect on students' responses of having a substitution as the first step in such questions?

Three questions, two involving multiplication and one division, required students to substitute. Having substituted, it would seem students then needed to carry out exactly the same processing as with questions having numerical bases. Despite this, substituting did greatly affect the rate of success.

When the substitution is made in Item 8 (a=2, b=5, b^2a^3=?) the question becomes $5^2 \times 2^3$ which is identical to Item 2. However, the success rate is markedly different for the two questions. On the basis of the threshold values reported in Chapter 6, Item 2 was the 5th most difficult of the fifty questions, and Item 8 was the eighth easiest. Item 2 was answered correctly by only 40%, while 85% answered Item 8 correctly. These were surprising results given that the questions, apart from the additional step of substituting in Item 8, were identical.

First thoughts on finding such a result were that students may have, in responding to Item 8, substituted 5 for b and obtained an answer of 25 for b^2 before even looking to substitute 2 for a. If they had done this they would have been confronted with answering 25×8 rather than $5^2 \times 2^3$ and so not seen the option of multiplying bases. A closer examination of students' responses showed that this was not the case. Of the 21 students answering 10^5 in Item 2, all but one gave correctly the answer 200 for Item 8. Yet 9 of these 21 students had written in the intermediate step of $5^2 \times 2^3$ making it identical to Item 2. It seems the act of substituting caused students to focus on the meaning of indices, i.e., repeated factors.

The results for Item 10 (m=5, n=2, $3mn^2$= ?) confirm that students have a high level of success when they begin by making a substitution. Three candidates multiplied the 3, 5 and 2 together before squaring but the remaining 93% of students all answered the question correctly. This item was one of the four easiest questions.

As for multiplication, the success rate in division questions was greatly increased when a substitution was involved. Students had great difficulty with Items 12 ($6^3 \div 2^2$) and 13 ($10^4 \div 2^3$). In these questions, twenty-two students made the error of dividing the bases in both questions while six answered correctly one of the questions but divided bases in the other (three for each item). When the substitution is made in Item 18 (p=4, q=2, $p^3 \div q^2$=?) it becomes $4^3 \div 2^2$ which is of very similar structure to Items 12 and 13. However, approximately twice as many students (58%) answered this question correctly compared to those on Item 12 (33%) and Item 13 (27%). On the basis of the threshold values, Items 12 and Item 13 were, respectively, the equal most difficult and 4th most difficult questions, but, in contrast, Item 18 was 21st (equal) most difficult.

Although the results for multiplication and division were similar, in the division question (Item 18) quite a few students (n=11) did persist in operating on the base after having made the substitution. Nevertheless, there were eleven students who answered this item correctly after having divided bases in both of Items 12 and 13. Again, a high proportion of these (n= 6) had written down their substitution in the form $4^3 \div 2^2$ before correctly writing 16 as their answer.

The effect on student responses of having a substitution as the first step in a question is to greatly reduce the difficulty of the question. This reduction occurs despite students writing down the substitution step and being confronted with a calculation of the type that poses so many difficulties in cases where substitution is not involved.

What is the effect on students' responses of having the operation arise out of an application?

For the main study, three questions involving an application were added to allow examination of this issue. Comparisons can be made between Items 1 and 9, Items 14 and 20, and Items 11 and 19 since the second item in each pair is an application question generating a calculation similar to, or the same as, the first item of the pair.

Finding the area of the rectangle in Item 9 required the calculation of $2^3 \times 2^4$, the same question as Item 1. Unlike the case of substituting, the application situation seems to have made little difference to students' responses. There were eleven students who obtained the answer of 4^7 for both questions by multiplying bases. While no students chose to multiply indices in Item 1, three did in Item 9. Four students multiplied bases in Item 1 and then answered correctly Item 9, three students did the reverse.

Writing the tangent ratio in Item 20 gave a very similar calculation to Item 14 ($10m^6 \div 2m^2$). Item 14 had an extremely high success rate of 93%. Although the success rate in Item 20 was somewhat lower (70%) this is largely accounted for by the number of students (n= 5) who wrote the ratio as a fraction and divided indices. This error is the issue for discussion in Theme 3 in the next chapter. Again the application situation appears to have had little impact on student responses.

When finding the ratio of the Geometric Progression in Item 19, students were required to divide expressions with the same numerical base ($2^7 \div 2^3$). This situation applied also in Item 11 ($5^6 \div 5^2$). Students were more successful in the application question. Of the fourteen students who divided bases in Item 11 only three chose to do the same in Item 19 while six, of the other eleven, answered Item 19 successfully. There is little evidence as to why there was such a large reduction in the number of students dividing bases. Only eight of the forty students showed any working for Item 19 and all eight answered it correctly. Three of the eight wrote the ratio as a fraction and simplified it correctly, one wrote it as a division and another tested their answer using multiplication. The remaining three students each converted the terms of the Geometric Progression to integers and worked successfully with these. It may be that the writing of the sequence of numbers has made students use a counting on process to arrive at their answer.

The comparisons, made above, indicate that the impact of having the question arising from an application is dependent largely on the nature of the application. Where the cognitive processing associated with the application is similar to the written question then the response rate is, to all intents and purposes, identical.

What do interviews reveal about the understandings students apply when multiplying or dividing in index questions involving numerical bases?

Interviews proved particularly informative as to how the persistent errors, made by so many students, are obtained.

What came through clearly in the interviews was that many students perceived $2^3 \times 2^5$ as being the same as $2a^3 \times 2a^5$, and $10^4 \div 2^3$ being the same as $10a^4 \div 2a^3$, that is, they were seeing the numerical bases as though they were coefficients.

Malcolm answered correctly all items with unknown bases but chose to multiply and divide all numerical bases except for the Geometric Progression question (Item 19). There he divided indices, giving an answer of $2^{1/3}$. When comparing his responses of $6a^5$ for $2a^3 \times 3a^2$, and 4^7 for $2^3 \times 2^4$, he expressed satisfaction with his answers and said the questions were "pretty well the same as they've both got something to the power". When explaining his thinking he demonstrated clearly that he looked on the numerical bases as coefficients.

Malcolm:	I timesed the front numbers in $2^3 \times 2^4$ to get 4 and I times 2 and 3 (in $2a^3 \times 3a^2$).
Interviewer:	So the numbers you times were the numbers which were?
Malcolm:	Out the front.

During the interview Malcolm showed considerable insight and, with very little probing, soon realised the error he had made in this question. When asked why he chose to divide the numerical bases his response was:

Malcolm:	Because I didn't think about it.
Interviewer:	Why do you think you did it that way then?
Malcolm:	It just looked the same.
Interviewer:	How did it look the same?
Malcolm:	Ah'm ... it just had something times something else to the power of something. The 2 in the 2 cubed looked the same as just the $2a$.
Interviewer:	And what did you choose to ignore out of that?
Malcolm:	The power of 2 just went on the a.

Bianca, like Malcolm, operated consistently on the numerical bases except for $5^6 \div 5^2$ which she answered correctly. It emerged, in the interview, that she had answered $5^6 \div 5^2$ by expanding out the question into its factored form. She explained what she had done by saying "Yeah, I wrote it all out. I got 5 times 5 times 5 times ... ... and I just cancelled it". For the two questions discussed above she arrived at the same answers as Malcolm.

Interviewer:	Compare this ($2a^3 \times 3a^2$) to the one above ($2^3 \times 2^4$), have you taken the same approach there?
Bianca:	Yeah, I think so.
Interviewer:	Why did you take the same approach?
Bianca:	Cause it was nearly the same question.
Interviewer:	In which way was it nearly the same question?
Bianca:	Well you've got your 2s, same base and ... just indices ... you just add em.

Interviewer:	The first 2 in Item 1, what does that correspond to in Item 4?
Bianca:	Eight, two cubed.
Interviewer:	And the second 2 corresponds to?
Bianca:	The 3a.

Like Malcolm, Bianca did not distinguish between a base and a coefficient.

Mark answered 10^5 for $5^2 \times 2^3$ and explained his reasoning in the following way:

Mark:	I've times the 2 and the 5 to get 10 and then added the powers when you times.
Interviewer:	Why?
Mark:	That's the rule.
Interviewer:	What is the rule?
Mark:	You add for multiplication.
Interviewer:	What is the rest of the rule?
Mark:	You times the one out the front.

In discussing his answer of 5 for $10^4 \div 2^3$, Mark also commented "we just divide the numbers out the front".

It is clear that many students making these errors attach little significance to the role of the unknown in questions involving a coefficient and an unknown base. This did not cause errors in such questions but manifested itself in students treating questions without an unknown as though they contained one. In virtually all such cases, the only sign of the error was the omission of the unknown. For example, the incorrect response of $2^3 \times 3^5 = 6^8$ becomes a correct response if an unknown is inserted as the base, i.e., $2a^3 \times 3a^5 = 6a^8$. It appears the students were not making any distinction between the base and coefficient and were really just treating them as a single entity. This was illustrated on numerous occasions in the interviews.

Interviewer:	What do you see as the role of the m in $5m^3 \times 2m$?
Brian:	It is the same as if the m wasn't there.
Interviewer:	If the m wasn't there what would you do?
Brian:	I'd probably just times the 5 and the 2 and leave the 3 there, so 10^3.

Kirsty:	I divided the $2m$ into the $10m$ to get the $5m$ and then subtracted the indices to get 4.

Kirsty:	I dunno ... they are just ... the a sort of ... it doesn't really make any difference if there is an a there or not

Linda:	I said $10m$ divided by $2m$ is $5m$.

Christine:	Yes except this one has letters in it.'
Interviewer:	Does that make any difference
Christine:	No

Christine:	I timesed the $2a$ and $3a$ and got $6a$ and then I added the indices because you add indices when multiplying.

Tim:	I timesed the $2a$ and $3a$ and added the powers.
Interviewer:	Is this the same approach as for the preceding question ($2^3 \times 2^4$) ?
Tim:	Yes, eh ... except for the a which is in there.
Interviewer:	What difference does the a make?
Tim:	Well in Item 4 the a has the index and ... eh I don't think I really took it into account actually.

Interviewer:	Is this the same sort of question as the Item above?
Philip:	They are the same I think ... except this one has an a.
Interviewer:	What difference does the a make?
Philip:	Oh ... not a lot I don't think, its just a variable type thing.

Of the students making such comments only Innes felt there was some problem with "timesing the 2a and 3a to get 6a". When asked how he answered $2a^3 \times 3a^2$ he gave the following response:

Innes:	Same as the last one which was multiplying the $2a$ ... ah'm ... sorry the 2 by the 3 to give 6 and adding the a^3 to the a^2.
Interviewer:	Is this the same sort of question as the one above?
Innes:	Yes this is the same question as for $2^3 \times 2^4$. The 2 and the 3 correspond to the 2 and the 2. (Innes now showed some reticence about it being correct - he knew something was wrong. He wanted to say $2a \times 3a = 6a^2$ but couldn't see how that would work with the other indices. He was not concerned about the question which only involved numbers).

Innes tried to think critically through his response but was unable to fully explain the process.

As indicated previously, questions were posed at the end of each interview to see if students had underlying understandings which would provide a basis for them to undertake the question. Of relevance to this particular theme were the questions relating to students' understanding of integral indices, namely, 'What do you understand four cubed to mean?' and 'What is the meaning of a squared times a cubed?'. All students described correctly the meaning of four cubed and responded correctly with a^5 for the second question. All but three explained readily the reason for adding the indices in the second question. The responses of these three students to the question of 'why?' were:

Bianca:	Cause you are multiplying
Philip:	It's just the rule I guess.
Tim:	A'hm if you multiply them you end up with a number that is much greater ... like if you've got a^2 times a^3 and you work that out and say a is a number and if you multiplied the indices you'd end up with a great larger number.

Further probing showed all three could see the justification for adding the indices.

Interviews show that persistent errors in such questions are a result of students taking an approach which is superficial and uncritical. It appears that even when students, such as Innes, understand an integral index means a repeated factor they are unable to apply that understanding in other than simple contexts.

Summary of Theme 1

The extremely high error rate in multiplication and division questions involving numerical bases is the result of students perceiving such bases as being the same as a coefficient followed by an unknown base. There are a number of possible explanations of why they do this. It may be that students, in developing their own approach to answering questions of the type $2a^3 \times 3a^2$, have not incorporated the important feature that the expressions have an unknown base. It may also be that a question of the type $2^3 \times 3^2$ requires a deeper understanding which, if present, students are not accessing. It is apparent that the act of substituting does cause students to think differently and, by correctly relating the index to the base, achieve the correct answer more often.

THE RELATIONSHIP OF INDICES TO BASES AND COEFFICIENTS (Theme 2)

The analysis of the School Certificate data showed many students had problems relating indices to bases and coefficients correctly. This occurred in questions involving parentheses and those without. In the 1988 School Certificate only 37% of Intermediate candidates answered $(2x^2)^3$ successfully. A common error was to multiply the coefficient by the index. In 1984 only 25% of Advanced students expressed $5x^{-1/2}$ correctly as $\frac{5}{\sqrt{x}}$. Many students applied the index to both the 5 and the x while others saw the negative index as making the sign in front of the term negative.

In this present study, 15 items were used to explore students' understanding of the way an exponent relates to bases and coefficients. This large number of items was required in order to cover a variety of situations. These situations included questions with and without parentheses, cases with constant bases, others with variable bases and questions requiring substitution. Table 7.1, listed previously, shows the various situations covered by these questions. Table 7.6, below, gives the results obtained by the subjects of this study on items relating to this theme.

Table 7.6. Theme 2: Summary of Results

	Item	Answer	No.Corr.	Other Responses and (n)*
[25]	$2m^3+m^3$	$3m^3$	31 (78%)	$3m^6(4)$, $2m^6(3)$, $3m^9$, $2m^3$
[26]	$(3^2)^3$	3^6 or 729	26 (65%)	$27^6(5)$, $3^5(3)$, $9^6(2)$, $9^5(2)$, 81, NA
[27]	$(2^3)^3$	2^9 or 512	28 (70%)	$8^9(5)$, $2^6(4)$, 8^6, 244, NA
[28]	$(2m^2)^3$	$8m^6$	22 (55%)	$2m^6(8)$, $8m^5(4)$, $2m^5(2)$, $8m^4$, $4m^5$, 64, 4
[29]	$(a^4)^5$	a^{20}	34 (85%)	$a^9(5)$, $5a^{20}$
[30]	$(a^m)^n$	a^{mn}	32 (80%)	$a^{m+n}(3)$, $(a^m)^n$, $(a^{m+n})^{m+n}$, na^{mn}, $a^m xmn$, NA
[31]	$m=10$, $(m^2)^3=$?	1000000	27 (68%)	$10^5(8)$, $1000^6(2)$, 10000(2), 300^6
[32]	$f(x)=x^3$, $f(2^4)=$?	2^{12}	21 (53%)	$8^{12}(8)$, 8^7, 128, $8x^{12}$, x^3, 16, 64, NA(5)
[33]	$f(x)=x^3$, $f(2a^3)=$?	$8a^9$	19 (48%)	$2a^9(9)$, $8a^6(2)$, $2a^6(4)$, NA(6)
[34]	$(2^3)^{-2}$	2^{-6} or $\frac{1}{64}$	27 (68%)	2, $2^{-6}=\frac{2}{6}$, $2^{-1}(2)$, $2^{-5}(2)$, $\frac{1}{4}^{-6}$, -4^{-6}, $\frac{1}{8}$, $\frac{1}{8^9}$, -16, -64, -4196
[39]	$x=9$, $4x^{1/2}=$?	12	28 (70%)	18(6), 6(5), $\frac{4}{x^2}$

[40]	$5x^{-1/2}$	$\frac{5}{\sqrt{x}}$	11 (28%)	$\frac{1}{\sqrt{5x}}$(7), $\frac{5}{x^2}$, $\frac{5}{-\sqrt{x}}$, $\sqrt{5x}$(4), $-\sqrt{5x}$(2), $\sqrt{-5x}$(2), $\frac{-1}{\sqrt{5x}}$, $\sqrt[3]{5x}$, $10x$, $2x^{1/2}$, $\sqrt{5x^{1/2}}$, NA(7)
[41]	2×6^{-1}	$\frac{1}{3}$	21 (53%)	-12(8), $\frac{1}{12}$(6), -72, 1, 12, $2 \times \frac{6}{1}$, NA
[42]	m=2, m^{-2}= ?	$\frac{1}{4}$	20 (50%)	-4(7), 4(2), -1(2), $\frac{1}{2}$, $(\sqrt{2})^2$, $1/\sqrt{2}$, 0.002, 0, 1, NA(3)
[50]	n=-2, $3n^2$= ?	12	32 (80%)	36(4), -12(2), -6^2, 81

* When only one student gave a response (n) was not include.

The mean score for these fifteen items was 9.35 (62%) and the standard deviation was 3.81. Table 7.7 lists the results for this theme in the order in which questions posed difficulties to students.

Table 7.7. Theme 2: Items in Order of Difficulty

	Item	Answer	No.Corr.	Threshold
[40]	$5x^{-1/2}$	$\frac{5}{\sqrt{x}}$	11 (28%)	2.15
[33]	$f(x)=x^3$, $f(2a^3)$= ?	$8a^9$	19 (48%)	0.97
[42]	m=2, m^{-2}= ?	$\frac{1}{4}$	20 (50%)	0.84
[32]	$f(x)=x^3$, $f(2^4)$= ?	2^{12}	21 (53%)	0.70
[41]	2×6^{-1}	$\frac{1}{3}$	21 (53%)	0.70
[28]	$(2m^2)^3$	$8m^6$	22 (55%)	0.57
[26]	$(3^2)^3$	3^6 or 729	26 (65%)	0.02
[31]	m=10, $(m^2)^3$= ?	1000000	27 (68%)	-0.12
[34]	$(2^3)^{-2}$	2^{-6} or $\frac{1}{64}$	27 (68%)	-0.12
[27]	$(2^3)^3$	2^9 or 512	28 (70%)	-0.27
[39]	x=9, $4x^{1/2}$	12	28 (70%)	-0.27
[25]	$2m^3+m^3$	$3m^3$	31 (78%)	-0.75
[30]	$(a^m)^n$	a^{mn}	32 (80%)	-0.94
[50]	n=-2, $3n^2$= ?	12	32 (80%)	-0.94
[29]	$(a^4)^5$	a^{20}	34 (85%)	-1.35

The four most difficult items, Items 40, 33, 42 and 32 accounted for 21 of the 25 cases where students did not answer the question. It appears that issues, which are not the subject of this research, namely, understanding of function notation and understanding of negative and fractional indices, have contributed to the high error rate. This will need to be kept in mind when drawing conclusions from the results for those questions.

In the beginning of this chapter four research questions relating to this theme, including two directed at examining students' understandings, were posed and these are addressed below.

When a term containing an index is raised to a power, what is the effect on student responses of that term having: a constant base; an unknown base; or, an unknown base with a coefficient?

The subjects of this study had more success in raising an expression containing a numerical base to a power than when multiplying or dividing expressions with numerical bases. Nevertheless, error rates were still high.

Items 26 ($(3^2)^3$) and 27 ($(2^3)^3$) were similar to each other and received almost identical responses. They were the 26th and 27th most difficult of the fifty items. For both questions, five students (three common to both) multiplied indices correctly but then, incorrectly, applied the index to the base, i.e., they responded with $(3^2)^3 = 27^6$. For Item 32 ($f(x)=x^3$, $f(2^4)=$?) there were eight students who took this approach and arrived at the answer of 8^{12}. These errors resemble closely those made in multiplication and division where the base was operated upon as though it were a coefficient.

Where the base was an unknown without a coefficient, as in $(a^4)^5$ (Items 29) and $(a^m)^n$ (Item 30), students had few problems. Both questions were amongst the ten least difficult of the fifty items. Adding indices accounted for most errors in such questions with five students (13%) answering a^9 for $(a^4)^5$ and three (8%) responding with a^{m+n} for $(a^m)^n$. This was the kind of error which Shevarev, as an example of his first type of incorrect connection, ascribed to students confusing $(a^M)^N$ with $a^M \times a^N$. Apart from this there were only six other incorrect responses for the two questions.

It might be expected, as for Theme 1, there would be a higher success rate in questions containing a coefficient and an unknown base than in those where the base was constant. This is in light of the fact that many of the errors in Theme 1 were a result of students treating bases as though they were coefficients. However, the success rate with unknown bases was not higher, with 55% of students correctly answering $(2m^2)^3$ (Item 28) as against 65% for $(3^2)^3$ and 68% for $(2^3)^3$. The error rate in answering $(2m^2)^3$ was accounted for, in part, by seven students (18%) adding the indices. However, the most frequent error was to not alter the coefficient. The mistake of operating on the index but not altering the coefficient was made by 20% of students who answered $2m^6$ for $(2m^2)^3$. This may have resulted from students treating the coefficient as though it were a numerical base, or from them ignoring the role of the parentheses.

Parentheses, given that they are so important to such questions, should be exercising considerable influence on students' thinking but this may not be the case. In the SESM research project, as discussed in Chapter 1, Booth reported that many students failed to understand the significance of brackets and considered expressions with or without brackets to be equivalent. She said this was not restricted to less able students as "children in the top ability groups also appeared to ignore the need for brackets" (p.54). In this research the higher success rates for n=-2, $3n^2$= ? (Item 50) and x=9, $4x^{1/2}$= ? (Item 39) show that students have considerably less difficulty correctly relating indices to bases and coefficients in questions not containing parentheses. However, the act of substituting may have been a factor also in students' success in these questions.

While we have some students operating on a numerical base as though it were a coefficient, and others not operating on the coefficient, it is perhaps surprising that two students made both kinds of errors by answering 27^6 for $(3^2)^3$ and $2m^6$ for $(2m^2)^3$. For this to happen would seem contingent upon these students making no real distinction at all between what constitutes a base and a coefficient.

When raising a term to a power, students frequently make the error of treating a numerical base as though it were a coefficient. Where the base is a variable, without a coefficient, there are few problems. However, when the base contains both a coefficient and variable a variety of errors are made. This is different to the previous theme where such expressions were handled very well.

What is the effect of having a substitution as the first step in questions of the kind examined in this theme?

As indicated in Table 7.1, three additional items were added in the main study in order to explore the effect of substitution on student responses. The use of function notation is common in the courses pursued by these students and, because of this, two of those questions involved that notation.

It was evident from the results for the questions, n=-2, $3n^2$=? (Item 50) and x=9, $4x^{1/2}$= ? (Item 39), that the act of substituting an integer does make students less inclined to treat the base and coefficient as an entity. Only four students multiplied the 3 by the -2 before applying the index in

Item 50, while five followed that process in Item 39.

In Item 32 ($f(x)=x^3$, $f(2^4)=?$) function notation was used to generate a situation requiring students to raise to a power a number already having an index. On substituting, the expression becomes $(2^4)^3$ which is very like Item 27 ($(2^3)^3$). The responses for the two items were quite similar. Five students operated on the base in Item 27, giving an answer of 8^9, while eight did this in Item 32. The lower success rate for Item 32 is largely a result of five students failing to give a response. There was a notable reduction in the number choosing to add the indices from five in Item 27 to just one in Item 32 where the substitution was required.

Item 31 ($m=10$, $(m^2)^3=?$) was answered more successfully than similar questions which do not involve a substitution, such as, Items 26 and 27. However, the results just discussed for Item 32 indicate that there may be some explanation for this other than involving a substitution. Perhaps the higher success rate for Item 31 reflects a greater familiarity with the powers of 10 than with other powers.

Function notation was again used in Item 33 ($f(x)=x^3$, $f(2a^3)=?$) and the resulting expression of $(2a^3)^3$ is very similar to $(2m^2)^3$ (Item 28). Students responded in a very similar fashion to the two questions apart from the fact that six students did not answer Item 33. This failure to respond largely accounts for the difference in success between the questions and, when taken with the results for Item 32, indicates that function notation is posing difficulties for some students.

As for the previous theme it appears that the act of substituting an integer improves student success in that they are less inclined to treat the base and coefficient as an entity. However, where function notation is used, or the number to be substituted is not written as an integer, there is no obvious change in the rate of student success.

What do interviews reveal about the understandings students use when applying an integral power to a term involving an index as that term varies between having: a constant base; an unknown base; or, an unknown base with a coefficient?

As for the previous theme, the interviews provided considerable insight into the understandings being applied by students. They again illustrated that many students fail to distinguish between what constitutes a coefficient and what constitutes a base.

Interviews showed that the students' thinking, when responding with the incorrect answer of 27^6 for $(3^2)^3$, was along the same lines as that which generated the high frequency errors in the multiplication and division of expressions with constant bases. This means they were treating this question in exactly the same fashion as though it contained a variable. Also, in questions which did contain a variable, it was evident that, while students arrived at a correct answer, many were again thinking of the coefficient and base as an entity

Innes gave the answer of 27^6 for $(3^2)^3$ (Item 26) and described his thinking by saying "I cubed 3 and I timesed the power of 2 by 3 because it was inside the brackets". When asked about his correct answer of $8m^6$ for $(2m^2)^3$ (Item 28) he replied "It was the same way (as Item 26), I times the 2 ... I cubed the 2 ... sorry ... and got $8m$ and then the power of 2 I times by 3 to get 6". Kirsty arrived at identical answers to Innes for these two items. For Item 26, she provided the same reasoning as that given by Innes. For Item 28 ($(2m^2)^3$), which she answered correctly, she explained:

Kirsty:	I took 2m to the power of 3 and got 8m and then timesed the indices.
Interviewer:	Are Items 28 and 26 the same (sort of) question?
Kirsty:	Yes.
Interviewer:	Any differences at all?
Kirsty:	Only that there's an m.
Interviewer:	Does that make any difference between them?
Kirsty:	No it shouldn't.

Again, we have the case of the coefficient and variable being looked on as a single entity. Linda arrived at the answers of 27^6 for $(3^2)^3$ and 8^{12} for $f(x)=x^3$, $f(2^4)=$? Her initial response in explaining her answer of $8m^6$ for $(2m^2)^3$ was to refer to the 2m as an entity, however, she then qualified this.

Linda:	Well I went $2m$ cubed and got $8m$ ... hang on what did I do? I went 2 cubed and got 8 and went m^2 multiplied by 3 and got m^6.
Interviewer:	Comparing Item 28 and Item 27, are they the same sort of question?
Linda:	It is the same only there is an *m*.
Interviewer:	What difference does that *m* make?
Linda:	I suppose with the Item 27 you can get an actual value but with Item 28 you can't because you don't know what the *m* is.
Interviewer:	What about with regard to answering the questions, is the same approach used?
Linda:	Yes.

Linda was aware that the unknown represented a number but it seems that she did not look on the 2 and the *m* as being separate entities when answering the question.

Philip had greater problems giving meaning to the unknown. When asked about the role of the m in $(2m^2)^3$, he replied "I just took it as the same thing ... as all the other ones ... just as a variable type thing like in the others, I didn't think it had much to do with it". In explaining his response of 27^6 for $(3^2)^3$, Philip demonstrated clearly that he realised the index of 3 meant a repeated factor but he then made the error which was investigated in Theme 1, namely, that of multiplying numerical bases. This was demonstrated in his response "Well I just did 3^2 times 3^2 times 3^2 and the 3 times 3 times 3 gave me 27 and I just added the 2s together which gave me 6".

As discussed in previous chapters, students do have difficulties in establishing meaning for letters in algebra and these difficulties undoubtedly contribute to the kind of error being made. It is interesting that this manifests itself in mistakes in questions that do not contain an unknown through students treating such questions as though they did contain one, i.e., as though the base was a coefficient. As for multiplication and division, the answer $(3^2)^3 = 27^6$ is not incorrect if an unknown is appropriately inserted, i.e., $(3a^2)^3 = 27a^6$.

The situation where the base was an unknown to a power but with no coefficient was explored at the end of each interview when students were asked "If I write down a squared in brackets and then cube it, like that ($(a^2)^3$), what does it mean?". The responses demonstrated that some students had a very superficial understanding while others were able to provide a more detailed explanation. Some responses showing restricted understanding were:

Bianca:	You multiply the indices so its a^6
Interviewer:	Why do you multiply the indices?
Bianca:	I have no idea. I just remember that from years ago.
Christine:	I don't know what you mean but you times the 2 by 3 and get a^6.
Interviewer:	You aren't sure what it means?
Christine:	No ... isn't it just like numbers ... ah ... no ... I don't know.
Innes:	You'd cube everything in the brackets.
Interviewer:	What would you give as your answer?
Innes:	I'd give a^6 ... that is not cubing the power of 2 but timesing it by 3.
Interviewer:	Why? (Innes was not able to explain the reason)

It is interesting that two students who began with what appears to be a reasonable explanation, were unable to reach a successful conclusion:

Tim:	That is a^2 worked out and then that answer to the power of 3 so ... ah'm ... in that case a^5 ... ah ... yeah that is the way I see it.
Philip:	a^2 times a^2 times a^2.
Interviewer:	What would that give you as your answer?
Philip:	$3a^6$.

All of the students who had difficulty with the meaning of $(a^2)^3$ did, when it was explained to them, readily understand the reason for multiplying indices.

Several students showed greater insight. Typical of the responses applying a more inclusive view of indices to this question were:

Kirsty:	a times a and all that times a times a for six a's
Linda:	A'hm a^2 by a^2 by a^2.
Malcolm:	a^2 times a^2 times a^2 which will give six a s or a^6.

It is notable that these students, when faced with working in a more complex context, left this understanding behind and regarded the coefficient and base as an entity. This occurred both for multiplication, division and for raising a term to a power, and it is of value to compare briefly the two situations. This is done in the next two paragraphs.

In multiplication and division questions with an unknown base, the treating of the base as an entity leads to few problems. For such questions students follow the kind of reasoning exhibited by Kirsty when, in explaining her correct answer of $5m^4$ for $10m^6 \div 2m^2$, she said "I divided the $2m$ into the $10m$ to get the $5m$ and then subtracted the indices to get 4". While such thinking gives correct results in questions of this form, it results in numerous errors in items with numerical bases. It appears that some students are, in effect, not taking account of the relationship of the index to either the base or the coefficient Instead they are treating the questions as having two parts, one being to operate on the raised numbers and the other to operate with the numbers or unknowns which are not raised.

A similar kind of thinking occurs when students raise to a power an expression involving an index. Here the steps are to apply, usually correctly, the index outside the parentheses to the index inside, and apply, in some fashion, the power outside to the unraised component of the term inside.

The effect of such thinking, however, is somewhat different when raising a term to a power compared to multiplication and division. Unlike in multiplication and division, it does not lead to students having greater success with terms including a coefficient and unknown base as opposed to those with a numerical base. This is because students frequently fail to apply the index correctly to the coefficient.

Interviews show that students making the kinds of errors discussed in this theme are using superficial strategies. In many case these strategies are similar to those used in questions relating to the previous theme. Again they lead to a high error rate in questions with numerical bases but, unlike in multiplication and division, they do not generate high levels of success in questions with bases involving a coefficient and a variable.

What do interviews reveal about the understandings students have of how a non-integral index relates to its base and to the coefficient in expressions such as $5x^{-1/2}$?

Interviews showed similarities between the understandings being applied here and those which were in evidence in the previous research question.

Item 40 required students to write an expression equivalent to $5x^{-1/2}$ but not involving an index. This proved to be the second most difficult of the fifty items with only 28% of candidates giving the correct answer of $\frac{5}{\sqrt{x}}$.

It also had the highest incidence of candidates (18%) choosing not to give an answer.

The interview responses showed similarities between the thinking used in applying a non-integral index to its base and coefficient and that used when raising a term to a power. Eighteen students, or 45% of the subjects of the study, treated $5x$ as a single entity in some fashion (see the list of errors). Seven of the eighteen answered $\frac{1}{\sqrt{5x}}$ showing they understood the meaning of the index and applied it correctly but saw it as relating to both the 5 and the x. Students explained their thinking in the following ways:

Malcolm:	I don't know. I thought $-\frac{1}{2}$ was 1 over square root so I put $5x$ with the square root sign and 1 over it.
Interviewer:	So negative a half suggested 1 over square root, 1 over square root of what?
Malcolm:	What ever it was to the power of $-\frac{1}{2}$, the $5x$.

Malcolm made no attempt to separate the 5 from the x and nor did Bianca in her response:

Bianca: Oh I remembered a negative I thought was a fraction and a half was the square root so I just threw it all together.

Bianca gave the impression she felt she had done enough thinking in interpreting the index and just wanted to have the question answered.

Kirsty treated the base as an entity also but interpreted the negative index as making the answer negative, i.e., $-\sqrt{5x}$. In the interview she explained "anything to the power of a half is square root and it was a negative so I just put the negative square root of $5x$". Kirsty showed the same thinking in the case where the base was numerical but demonstrated, when discussing her answer of -12 for 2×6^{-1}, that she had knowledge of the correct application of the negative index:

Kirsty: I just times the two numbers and made it a negative.
Interviewer: Why did you make it negative?
Kirsty: Because it was a negative index.
Interviewer: What does a negative index mean?
Kirsty: One over.
Interviewer: So what does that mean about your answer?
Kirsty: Its definitely wrong.

In this more complex question, Kirsty had difficulty in applying the understandings she had and in taking account of all relevant information.

It is apparent from the interviews that students have difficulty bringing real understanding into play in questions involving negative or fractional indices. The routine strategies they adopt in such situations frequently lead to errors.

Summary of Theme 2

In this theme many students looked on the coefficient and unknown base as a single entity, just as they did for multiplication and division. This resulted in errors in questions with both unknown bases and numerical bases, whereas in the previous theme the main problem was with numerical bases. Some students chose incorrectly to apply the index outside parentheses to a numerical base as though that base were a coefficient, and so answered 8^9 for $(2^3)^3$. Other students failed to apply the index outside the parentheses to the coefficient and, through treating the coefficient as though it were a numerical base, arrived at an answer of $2m^6$ for $(2m^2)^3$.

The act of substituting an integer did make students less inclined to treat the base and coefficient as an entity though the impact of substitution was not as dramatic as for the previous theme.

CONCLUSION

It is apparent that students, despite being able to explain the meaning of integral indices, predominantly use routine approaches when answering questions of the type considered in this chapter. They may be prompted to use such strategies for convenience or because they are unable to apply deeper understanding in more complex situations.

The superficial approach often gives success but also results in many errors through students not taking account of all relevant aspects. Such is the case with parentheses where students understand their basic meaning but do not have strategies which are adequate to account for the various relationships among bases, indices and coefficients.

Problems students have in establishing meaning for letters in algebra have contributed undoubtedly to the kind of errors being made. It seems that little significance is being attached to pronumerals and the fact that they represent an unknown number appears to be far from students' thoughts when doing these questions. The effect of having a simple substitution in a question is to have students refocus on the unknown as representing a number and this results in a higher success rate.

The difficulties students had were very much the same for both themes. They were associated with the use of incorrect strategies which did not adequately take account of all relevant aspects of the questions. Aspects that students had particular difficulty in accounting for were the distinctions which needed to be made between constants and unknowns, and the way in which indices related to bases and coefficients. It is of interest to see if the findings for the more specific issues show similar threads running through them. This is addressed in the next chapter.

Chapter 8

INTERPRETATION OF THE FRACTION BAR, RADICAL SIGN and ZERO INDEX

Introduction to Chapter

Five research themes relating to students' understanding of indices arose in Chapter 3. The two more general themes were addressed in Chapter 7 and the three more specific themes are now considered. **Interpretation of the Fraction Bar** (Theme 3) relates to the fact that many students divide indices when simplifying algebraic fractions. **Interpretation of the Radical Sign** (Theme 4) refers to the tendency to take the square root of indices within the radical sign. **Interpretation of the Zero Index** (Theme 5) is concerned with errors made in simplifying expressions involving the zero index. Test results and student interviews are used to answer the research questions identified for each of these themes. Before this is done, the following section explains how the analysis is structured and provides an overview of the items used.

STRUCTURE OF THE ANALYSIS AND OVERVIEW OF ITEMS

As for the preceding chapter, the analysis in this chapter is carried out within the themes and research questions, identified in Chapter 6. These are treated in sequence. The questions concerning students' written responses are dealt with first, followed by those relating to the nature of students' thinking. The research questions relating to the themes which are the subject of this chapter are:

Theme 3: Interpretation of the Fraction Bar

- *What is the effect on student responses of using different notations, namely the fraction bar or the division sign?*

- *How will students simplify a fraction containing an unwritten 1 as an index given that, in numerical fractions, they are used to having positive integers greater than 1 for both numerator and denominator?*
- *Will having coefficients, containing a common factor, make students less inclined to divide indices given they will have divided coefficients?*
- *What do interviews reveal about the thinking students use when they divide indices while simplifying algebraic fractions?*

Theme 4: Interpretation of the Radical Sign

- *When taking the square root of an expression involving an index what is the effect on responses of this index being, or not being, a perfect square?*
- *What is the effect on student responses of the use of different notations, namely the index of $\frac{1}{2}$ or the radical sign?*
- *What do interviews reveal about the understandings applied by students when, in finding the square root of an expression, they take the square root of an index?*
- *What do interviews reveal about how the use of different notations, namely the index of $\frac{1}{2}$ or the radical sign, affects students' thinking?*
- *What do interviews reveal about the understandings students apply where the finding of a square root is evoked not through the use of an index of $\frac{1}{2}$ or the radical sign but through the context of the question?*

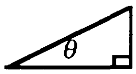
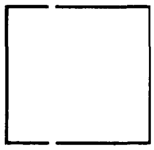
Theme 5: Interpretation of the Zero Index

- *In questions involving the zero index, what is the effect of having the base as a constant as opposed to having the base as an unknown?*
- *In questions involving the zero index, what is the effect of parentheses on students' responses?*
- *What do interviews reveal about students' understanding of the zero index?*

Before the themes are addressed an overview of the items used for the three themes is given in Table 8.1. Two asterisks indicate an item newly developed for the Main Study; one asterisk shows an item developed for the Pilot Study; and, items not asterisked were free-response items from the Pilot Study based upon an equivalent School Certificate multiple-choice item. Theme 4, Interpretation of the Radical Sign, was not part of the Pilot Study and all items for this theme were developed for the Main Study except

Item 36 which was taken directly from the School Certificate.

Table 8.1. Test Items Relating to Themes 3, 4 and 5

THEME 3 **Interpretation of the Fraction Bar**	**THEME 4** **Interpretation of the Radical Sign**	**THEME 5** **Interpretation of the Zero Index**
[14] $10m^6 \div 2m^2$	[35] $(16a^{16})^{1/2}$ **	[43] $9^{1/2} + 9^0$
[15] $12k^{10} \div 3k^4$ *	[36] $\sqrt{16x^{16}}$	[44] 4×5^0
[16] $\frac{8a^6}{4a^3}$ **	[37] $\sqrt{25x^8}$ **	[45] $7m^0$
[17] $\frac{10a^2b}{2a}$	[38]	[46] $(m^5)^0$ *
[20]	A B	[47] $3p^0 + (5q)^0$
 $10m^6$ $\tan\theta = ?$ $5m^3$ **		[48] $p^0 + 2 + p$
	D C	[49] $3p^0 + p^{-1}$
[22] $\frac{x^4 \times x^6}{x^2}$ **	ABCD is a square of area $36x^{16}$ sq. units Find the length of its sides in terms of x. **	
[23] $\frac{p^{10}}{p^3 \times p^2}$		
[24] $\frac{(y^3)^4}{y^2}$		

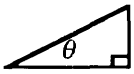
(** - item developed for the Main Study; * - item developed for the Pilot Study; no asterisk - item developed from SC item)

The first of the themes is now discussed.

INTERPRETATION OF THE FRACTION BAR
(Theme 3)

This theme focused on the tendency for students to divide indices when simplifying algebraic fractions. The results obtained by the subjects of this study on items relating to this theme are listed in the following two tables. Details included in the tables are as described previously in Chapter 7. Again, it is important to remember it is the more common errors, rather than the obscure ones, which are of interest since this research is concerned with 'misunderstandings' rather than 'lack of understanding'.

Table 8.2. Theme 3: Summary of Results

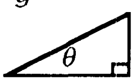
	Item	Answer	No.Corr.	Other Responses and (n)*
[14]	$10m^6 \div 2m^2$	$5m^4$	37 (93%)	$5m^3$(2), $5m^8$
[15]	$12k^{10} \div 3k^4$	$4k^6$	36 (90%)	$\frac{4k^5}{k^2}$, $36k^{14}$, $4k^5$, $\frac{4k^5}{3k}$
[16]	$\frac{8a^6}{4a^3}$	$2a^3$	30 (75%)	$2a^2$(5), $4a^3$(2), $2a^{12}$, a^3, NA
[17]	$\frac{10a^2b}{2a}$	$5ab$	36 (90%)	$10b$, $5a10b$, $\frac{10a^2}{2a}$, NA
[20]	 θ, $5m^3$, $10m^6$ $\tan\theta$=?	$2m^3$	28 (70%)	$2m^2$(5), $5m^3$(2), m^2, 30^0, $5(2m^6-m^3)$, NA(2)
[22]	$\frac{x^4 \times x^6}{x^2}$	x^8	22 (55%)	x^5(15), $2x^8$, $\frac{x^5}{x}$, x^3
[23]	$\frac{p^{10}}{p^3 x p^2}$	p^5	24 (60%)	p^2(10), p^4(4), $\frac{1}{2}p^2$, $\frac{p^5}{p}$
[24]	$\frac{(y^3)^4}{y^2}$	y^{10}	24 (60%)	y^6(9), y^5(4), $4y^6$, $\frac{y^7}{y^2}$, $\frac{y^6}{y}$

* When only one student gave a response (n) was not include.

Students did relatively well in the items used to research this theme. Their mean score on the eight items was 5.95 (74%) and the standard deviation was 1.92. Despite the degree of success, each question, apart from Item 14, attracted at least four different incorrect responses.

Table 8.3 lists the results for this theme in the order in which the items posed difficulties to students.

Table 8.3. Theme 3: Items in Order of Difficulty

	Item	Answer	No.Corr.	Threshold
[22]	$\frac{x^4 \times x^6}{x^2}$	x^8	22 (55%)	0.57
[23]	$\frac{p^{10}}{p^3 \times p^2}$	p^5	24 (60%)	0.30
[24]	$\frac{(y^3)^4}{y^2}$	y^{10}	24 (60%)	0.30
[20]	 θ, $5m^3$, $10m^6$ $\tan\theta$=?	$2m^3$	28 (70%)	-0.27
[16]	$\frac{8a^6}{4a^3}$	$2a^3$	30 (75%)	-0.58
[15])	$12k^{10} \div 3k^4$	$4k^6$	36 (90%)	-1.88

[17]	$\frac{10a^2b}{2a}$	$5ab$	36 (90%)	-1.88
[14]	10m6÷2m2	5m4	37 (93%)	-2.23

Least successfully done were Items 22, 23 and 24, which required simplification of either the numerator or denominator prior to carrying out the division. These items had no greater variety of errors than most of the others but several errors occurred very frequently. Students were highly successful with the remaining items. Items 14, 15 and 17 were all among the seven test items answered correctly by 90% or more of students.

As listed earlier in this chapter, four research questions relating to this theme were posed, the last of which relates to the understanding students apply as evidenced from their interview responses. The research questions are addressed below.

What is the effect on student responses of using different notations, namely, the fraction bar or the division sign?

While students were relatively successful with these items overall, they found some items easier than others and this difference in difficulty was closely associated with the notation used.

Students made very few errors when dividing simple algebraic expressions written using a division sign. The question $10m^6 \div 2m^2$ (Item 14) was answered correctly by 93% and $12k^{10} \div 3k^4$ (Item 15) by 90%. Of the three students who divided indices in Item 15, two had first written the expression as a fraction. The student who answered $\frac{4k^5}{3k}$ for $12k^{10} \div 3k^4$ obtained it by writing the question as a fraction and then dividing the index from the denominator into both the index and coefficient of the numerator. This student was one of those struggling with the course.

Students chose to divide indices in 28% of responses where the fraction simplification did not involve a coefficient (15 for Item 22, 10 for Item 23 and 9 for Item 24). Each of these items required a preliminary step before dividing, and, in almost every case, students clearly wrote down that step (such as putting down $\frac{x^{10}}{x^2}$ for $\frac{x^4 \times x^6}{x^2}$). It might be expected that writing the step would make students think more carefully about their answers, however, results indicate this was not the case.

The results show that the effect of the fraction bar is to lead many students to divide indices even though they answer such questions correctly when a division sign is used.

How will students simplify a fraction containing an unwritten 1 as an index, given that, in numerical fractions, they are used to having positive integers greater than 1 for both numerator and denominator?

Students had notably more success simplifying the algebraic fraction with an unwritten 1 as an index (Item 17) than they did where the indices were written integers greater than 1.

In Item 17 ($\frac{10a^2b}{2a}$) students could divide indices as well as coefficients. To do this, however, they needed to identify an unwritten index of 1 and divide it into the 2 leaving an answer of $5a^2b$. Not one student gave this response despite the fact that indices were divided by more than 25% of students in questions such as $\frac{p^{10}}{p^5}$ and by12% in $\frac{8a^6}{4a^3}$ (Item 16). While this may not be surprising it does demonstrate that many students do not look for coherency across questions nor do they place answers under great scrutiny.

It is clear that students do not see dividing indices as an option where an index is an unwritten 1, unlike in questions where they can see the indices actually written.

Will having coefficients, containing a common factor, make students less inclined to divide indices given they will have divided coefficients?

There was a noticeable difference in the level of difficulty between items which did have coefficients with common factors in the numerator and denominator and those that did not.

There were two questions where students were required to simplify a fraction containing both coefficients and indices in the numerator and denominator, and both were answered with a high degree of success by students. In Item 16 ($\frac{8a^6}{4a^3}$), 12% (n=5) of students chose to divide the indices. Although Item 20 was not written directly as a fraction, all but four students began by writing $\frac{10m^6}{5m^3}$ for $\tan\theta$; two of those four did have the correct response. Again, five students divided indices, four of these having done the same in Item 16. In the unlikely occurrence that the two students who were incorrect, and did not write the question as a fraction, would have divided if confronted by the fraction, at most 15% would have divided.

The fact that students wrote down the intermediate step for Items 22, 23 and 24 (only two out of thirty-four responses that involved dividing did not) makes these items comparable to Items 16 and 20 in that all five questions have the same structure of one simple algebraic expression over another. In the questions containing coefficients, which could be divided, namely, $\frac{8a^6}{4a^3}$ and $\frac{10m^6}{5m^3}$, only 12% of students chose to divide indices as opposed to 37% for $\frac{x^{10}}{x^2}$, 25% for $\frac{p^{10}}{p^5}$ and 23% for $\frac{y^{12}}{y^2}$.

It is evident that having coefficients, containing a common factor, does make students less inclined to divide indices when simplifying algebraic fractions.

What do interviews reveal about the thinking students use when they divide indices when simplifying algebraic fractions?

Test results had confirmed that dividing indices, when simplifying algebraic fractions, was a highly frequent error among the subjects of this study. The interviews which follow provide considerable insight into the thinking students were applying. The responses at interview were especially informative as to to the effect the use of a fraction bar, as opposed to a division sign, exerted upon students' thinking.

Linda divided indices in five of the six items which relate to this theme. She answered $\frac{10a^2b}{2a}$ (Item 17) correctly but, as discussed above, this is a somewhat different case, in that one of the indices is an unwritten 1. Her answers show dividing indices is her only mistake. When reading out questions, Linda read the fraction bar as divided by and qualified this in one case by saying "divided by or over". Clearly, she knew she was required to perform a division and had answered both $10m^6 \div 2m^2$ and $12k^{10} \div 3k^4$ correctly yet she still divided indices. She gave the response of $2a^2$ when simplifying $\frac{8a^6}{4a^3}$ (Item 16) and described her reasoning in the following fashion:

Linda:	Well I divided the $8a^6$ by the $4a^3$.
Interviewer:	And where did the 2 come from?
Linda:	The 8 divided by 4.
Interviewer:	And the index of 2.
Linda:	That came from the 6 over 3.

Linda's response shows she was well aware that a division was required. Her comment of "That came from the 6 over 3" gives a strong impression that she is thinking of the indices as the numerical fraction '$\frac{6}{3}$'. Also, Linda divided indices in simplifying $\frac{x^4 \times x^6}{x^2}$ (Item 22) to arrive at the answer of x^5. In the interview she had second thoughts.

Linda:	I did x^{10} divided by x^2.
Interviewer:	Why did you divide 10 by 2?
Linda:	Oh it should be take away shouldn't it ... should it?

Interviewer:	What do you think?
Linda:	I think it should be 10 take away 2 to give x^8.

It appears that being placed in a situation where she was encouraged to reflect on the processes made Linda go back and analyse the question more carefully. When discussing the simplification of $\frac{p^{10}}{p^3 \times p^2}$ (Item 23), she again indicated her reason for dividing was because of the fraction form.

Linda:	I multiplied the bottom out first to get p^5 then I divided p^{10} by p^5.
Interviewer:	Why?
Linda:	Because of the over, because of p^{10} was over p^5.
Interviewer:	What does over mean?
Linda:	Divide.

Despite having divided indices, Linda realised it is a division of expressions which she was required to do. In discussing her answer of y^6 for $\frac{(y^3)^4}{y^2}$ (Item 24) she confirmed further that the fraction bar caused her to divide.

Linda:	Well I went y^3 by 4 to get y^{12} and then I divided y^{12} by y^2, the 12 by 2 to give 6.
Interviewer:	Are you happy with that answer?
Linda:	Ah'm ... maybe it should be y^{10}.
Interviewer:	Why do you think you did that division?
Linda:	I thought it was y^6 cause 12 divided by 2 gives 6. But I can see now that you should just take it away, I think.
Interviewer:	I'll write down two questions for you here, y^6 divided by y^2 and y^6 over y^2. What is the answer to the first of those?
Linda:	y^4.
Interviewer:	What is the answer to the second?
Linda:	Well I would say y^3 but I think it should be y^4 as well.
Interviewer:	What makes y^3 a more attractive answer to that question than to the previous one?
Linda:	Because I'm used to numbers, a top number divided by a bottom number so I just divided the 6 by the 2 without even thinking.
Interviewer:	When you say you are used to doing that, you are used to doing that in what situation?
Linda:	Sort of like maybe 10 over 2, so that's 5.

The test results and interview demonstrate that Linda has no problems in correctly dividing such expressions and does realise that 'over' means divide, yet she chose to divide indices. Her responses in the last two items above show that the reason for her dividing the indices was the numbers being above and below the fraction bar.

Tim gave the same answers as Linda except for Item 24 where he answered y^5 for $\frac{(y^3)^4}{y^2}$.

Interviewer: How did you arrive at your answer?
Tim: That is multiplying those two, that is y^3, no its not. I have no idea how I did that. (Tim struggled without success).

Tim's written response showed that he added indices in the numerator to obtain y^7 then subtracted the index of the denominator to get y^5. This was the one case where Tim did not divide indices. It could be that 2 not being a factor of 7 prompted him to the correct operation of subtraction instead of dividing? Tim answered p^2 for $\frac{p^{10}}{p^3 \times p^2}$ and demonstrated in interview that the fraction bar caused the problem.

Tim: Ah ... I added the indices of the powers of p of the denominator and then just divided the 5 into the 10.
Interviewer: Why did you divide the 5 into the 10?
Tim: Ah ... well they were the numbers there ... yeah. Should they have been divided ... the *p* and the *p*? No ... that goes into that there.
Interviewer: Why did you divide the 5 into the 10?
Tim: I think it was only because I was dividing and I must have automatically just divided the indices there. Just looking at it now I thought that if you were going to divide it you should have perhaps minussed them.
Interviewer: You chose to divide rather than minus, why do you think you might have done that?
Tim: Ah .. I think it must just have been an automatic thing because they were the numbers there.
Interviewer: Why automatic?
Tim: Ah... (struggling)
Interviewer: Would there have been any other way to write that question?
Tim: Yes, p^{10} with a divided sign. (the question was then written down).
Interviewer: What would the answer be then?
Tim: The answer is p^5.
Interviewer: Tell me what do you think was different about the two questions which made you do a division there? (comparing way written in test to way now written down with division sign).
Tim: Simply that it was written that way I think with the line between the denominator and numerator there.

Even after the above conversation, Tim seems still to have a tendency to think division, as indicated by his use of the word "into", when discussing his answer of $2m^2$ for Item 20.

Tim:	I ended up with $2m^2$. I had $10m^6$ over $5m^3$ (Tim had written it in fraction form on his paper) ... ah ... I think I must have divided the 10 by 5 and ended up with 2 and divided the m^6 by the m^3 and ended up with m^2 but now I look at it I think I should have minussed the power of 3 into the power of 6 ... sorry ... no ... from the power of 6.

Mark demonstrated, when discussing his answer of p^2 for $\frac{p^{10}}{p^3 \times p^2}$, he knew that coefficients could be divided and felt this might carry over to indices.

Mark:	I've just timesed the bottom out and tried to cancel it through the top.
Interviewer:	When you said "tried to cancel", what did you mean.
Mark:	Well I wasn't quite sure if you could just cancel them out, but I think you can.
Interviewer:	If I let you take the 10 and the 5 and let you put them where you like is there anywhere you could put them where you would be sure about being able to cancel them?
Mark:	Yeah, if they weren't to the power they were just 'times'.

Brian shows how strong the tendency to divide is, when, on looking back, he thinks his answer of p^5 for $\frac{p^{10}}{p^3 \times p^2}$ is incorrect and that he should have divided. He then realises why he subtracted.

Brian:	Well because you times the two p's I added the indices on the bottom and then just divided it into that ... but I should have got p^2. (now thinking he is wrong)
Interviewer:	Any idea how you may have got p^2?
Brian:	No, not really ...(long pause) ... oh ... cause I was dividing I subtracted the indices.

The interviews demonstrate consistently that notation is influencing students' thinking when faced with answering such questions. There is little evidence of students applying genuine understanding irrespective of the notation, however, the success rate is much greater when a division sign is used rather than a fraction bar.

Summary of Theme 3

Interviews show it is the fraction bar which causes some students to divide indices and that this relates to their previous experiences with simplifying fractions. Clearly, the problem is with the notation and does not relate to the ability of students to carry out a division successfully. There are circumstances which make students less inclined to divide indices, such as where the index is an unwritten 1, and where a coefficient is incorporated in the numerator and denominator.

INTERPRETATION OF THE RADICAL SIGN (Theme 4)

In the questions giving rise to this issue, a radical sign was the notation used. However, students, who are the subjects of this research, frequently use the notation of a half to designate square root. In addition, they are required often to use the square root concept within questions involving an application. As detailed in Chapter 6, several questions were developed to investigate student responses in these situations. Table 8.4 gives the results obtained by the subjects of this study on the items relating to this theme.

Table 8.4. Theme 4: Summary of Results

	Item	Answer	No.Corr.	Other Responses and (n)*
[35]	$(16a^{16})^{1/2}$	$4a^8$	16 (40%)	$4a^4$(7), $16a^8$(7), $8a^8$(2), $16a^4$(2), $16a^6$, $\frac{1}{4a^{16}}$, $\sqrt[3]{16a^{12}}/4a^{16}$, $\sqrt[3]{16a^{16}}$, NA(2)
[36]	$\sqrt{16x^{16}}$	$4x^8$	12 (30%)	$4x^4$(13), $4x^{16}$(3), $16x^8$(2), $8x^8$, $16x^4$(2), $2x^{32}$, $16x^{2/16}$, 1, $16x$, NA(3)
[37]	$\sqrt{25x^8}$	$5x^4$	21 (53%)	$5x^{\sqrt{8}}$(3), $5x^8$(5), $25x^4$(3), $5x^{-8}$, $(25)^{2/8}$, $2\sqrt{5x^8}$, NA(5)
[38]	Area= $36x^{16}$, Side= ?	$6x^8$	21 (53%)	$6x^4$(8), $9x^4$(2), $36x^{16}$, $36x^{16}$x4, $36x^{14}$, $\frac{36x^{16}}{4}$, $\frac{36x^{16}}{2}$, 9^4, NA(3)

* When only one student gave a response (n) was not include.

The success rate for these four questions was very low. The mean score was 1.75 (44%) and the standard deviation was 1.35. The incorrect responses to these questions exhibited the greatest variety per question of any of the five themes. Table 8.5 lists the items in their order of difficulty.

Table 8.5. Theme 4: Items in Order of Difficulty

	Item	Answer	No.Corr.	Threshold
[36]	$\sqrt{16x^{16}}$	$4x^8$	12 (30%)	1.99
[35]	$(16a^{16})^{1/2}$	$4a^8$	16 (40%)	1.38
[37]	$\sqrt{25x^8}$	$5x^4$	21 (53%)	0.70
[38]	Area=$36x^{16}$,Side=	$6x^8$	21 (53%)	0.70

Based on the threshold estimates discussed in Chapter 6, students found Items 36 and 35 to be, respectively, the 3rd and equal 5th most difficult of the 50 items. Items 37 an 38 were also within the top one third of items in terms of difficulty.

As listed earlier in this chapter, there are five research questions relating to this theme. These include three which are directed at examining students' understandings, as evidenced from their interview responses. The questions are addressed below.

When taking the square root of an expression involving an index what is the effect on responses of this index being, or not being, a perfect square?

The two items used to examine this question, Items 36 and 37, posed considerable difficulties for students, especially Item 36 which was the 3rd (equal) most difficult of the test. While many different errors were made, there were certain persistent incorrect responses.

In Item 36 ($\sqrt{16x^{16}}$), 30% of students gave the correct answer of $4x^8$ while 33% took the square root of both the index and the coefficient to obtain an answer of $4x^4$. The success rate was considerably higher (53%) in Item 37 ($\sqrt{25x^8}$) where the index was not a perfect square. This increase reflected the fact that six of the thirteen students who took the square root of the index in Item 36 answered Item 37 correctly. Only three of the thirteen students responded in the same fashion to the two questions by persisting with taking the square root. Again, students were not looking for consistency across questions.

Where the index was not a perfect square, the number of students not giving an answer increased from three to five, and the number leaving the index unchanged increased also from three to five.

The effect of the index being a perfect square is to cause many students to take its square root. The effect of the index not being a perfect square is to increase the success rate noticeably, and also cause some students to either not operate on the index or to omit the question.

What is the effect on student responses of the use of different notations, namely the index of $\frac{1}{2}$ or the radical sign?

Error rates were high irrespective of the notation used, however, students did have more difficulty with the radical sign than with the index of $\frac{1}{2}$.

Except for the particular letter used as the unknown, the expressions which students needed to take the root of in Item 35 ($(16a^{16})^{1/2}$) and Item 36 ($\sqrt{16x^{16}}$) were identical. There was a success rate of 40% in Item 35 where the index of a half was employed to indicate square root. This rate was reduced to 30% in Item 36 when the radical sign was used.

Where an index of a half was used, the number of students taking the square root of the index in the expression inside brackets was cut by almost 50% (from n=13 to n=7) compared to the case with the radical sign. However, there was also a large increase in the number of students who did not operate on the coefficient and so answered $16a^8$ (n=7) or $16a^4$ (n=2) for $(16a^{16})^{1/2}$.

The effect of using an index of a half to indicate square root is to reduce, but by no means eliminate, the number of students taking the square root of the index in the expression. While there is a noticeable reduction in that particular error, it is balanced largely by the increase in students not operating on the coefficient.

What do interviews reveal about the thinking being used by students when, in finding the square root of an expression, they take the square root of an index?

As indicated previously the nature of the index affected significantly how students responded. Frequently, students taking the square root of an index, which was a perfect square, answered correctly where it was not. However, there were others who, when confronted with taking the square root of a number which was not a perfect square, either did not answer or left the index unchanged. Possibly, some students had run out of options when confronted by the need (in their mind) to take the square root of a number which was not a perfect square.

Christine was one of those who had taken the square root of the index in $\sqrt{16x^{16}}$ but did not answer $\sqrt{25x^8}$. She explained this by saying "I was going to ... like ... to square root the 25 but the 8 doesn't have a square root so I gave it a miss".

Innes took the square root of the index in $\sqrt{16x^{16}}$ but then correctly answered $\sqrt{25x^8}$. He showed in the interview that his first choice would have been to take the square root.

Innes: I got the square root of $25x$ ... I wasn't sure on this question because I couldn't get a square root of 8 so I divided it by 2.
Interviewer: Why did you divide it by 2?
Innes: Going on the same principle that a power of a half is the same as square root.
Interviewer: How would this fit in with Item 35?
Innes: It could have changed the answer to $4a^8$.
Interviewer: Which answer do you think is correct, $4a^4$ or $4a^8$?
Innes: $4a^8$ ... I think.
Interviewer: Why?
Innes: Ah'm ... I can't really explain it.

This again illustrates that many students are not looking for coherency across questions and are prepared to take what they see as the best option for the particular question. In many cases this best option appears to be the easiest option, that is, 'if its a perfect square take its square root, otherwise halve it'.

Kirsty took the square root of the index in both $\sqrt{16x^{16}}$ and $\sqrt{25x^8}$. In discussing answers she had no concerns with her response to the first question but demonstrated lack of confidence in $5x^{\sqrt{8}}$ as the answer to the second.

Interviewer: Did you treat this in the same fashion as for Item 36?
Kirsty: Yeah, sort of.
Interviewer: What do you mean by 'sort of'?
Kirsty: Oh well I don't know about the square root of 8.
Interviewer: Do you think you were following the same sort of process?
Kirsty: Oh Yeah.
Interviewer: Well what happened? Is there a difference between the questions?
Kirsty: Well in that you've got a square root sign around the indices (refering to the answer for Item 37).
Interviewer: Do you think the process is exactly the same?
Kirsty: Oh Yeah well I just sort of just took the square root of everything.
Interviewer: Why?
Kirsty: Cause theres a square root sign there!

Bianca explained her answer of $4x^4$ for $\sqrt{16x^{16}}$ by saying "I just square rooted everything I could ... so you get $4x^4$". She exhibited considerable confusion and lack of understanding of concepts when confronted by what she saw as the need to find the square root of 8 in simplifying $\sqrt{25x^8}$. She decided on an answer of $5x^{-8}$.

Interviewer:	How did you arrive at your answer?
Bianca:	I have no idea I put $5x$ to the power of negative eight. I didn't know the index at all.
Interviewer:	Why did you put negative eight?
Bianca:	I didn't know the square root of 8 but I remembered some times ... I couldn't remember what you do with a negative indice.
Interviewer:	What would you like that (index) to have been?
Bianca:	A 4, a 16.
Interviewer:	What was the problem with it being an 8?
Bianca:	I couldn't square root it.

Interviews show that the radical sign causes many students to 'think square root' and take the square root of all numbers inside the sign. Frequently, such students respond correctly when the index is not a perfect square. Despite this, interviews indicated that students do not have great confidence in taking that correct approach, nor do they then question taking the square root where the index is a perfect square.

What do interviews reveal about how the use of different notations, namely the index of $\frac{1}{2}$ or the radical sign, affects students' thinking?

Interview responses again provided considerable insight into students' thinking and, particularly, the difficulties they have in satisfactorily reconciling the two notations.

In responding to the question posed at the end of each interview, "What do you understand by a to the power of one half?", all replied that it meant "the square root of a". When asked to explain why, not one student could provide justification in sound mathematically terms. Typical responses were "I have no idea" (Bianca), "I'm not sure" (Philip), "Cause our teacher said so" (Kirsty), "I've no idea why" (Malcolm).

Linda attempted to justify the meaning of a to the power of one half in terms of the product of the same number but had conflict in determining that number.

Interviewer:	What do you understand by a to the power of one half?
Linda:	The square root of a.
Interviewer:	Any idea why an index of a half means square root?
Linda:	Because it is half the value times half the value to give a.
Interviewer:	What about $16^{1/2}$, what is half the value there?
Linda:	8, so it doesn't work. I'm not really sure.

Six students did take the square root of the index in simplifying both $\sqrt{16x^{16}}$ and $(16a^{16})^{1/2}$. Four of these were among those selected for interview.

Interviews showed they were not looking to use index laws to remove the parentheses but were clearly thinking 'square root' for the index of $\frac{1}{2}$. Typical of their comments were:

Innes:	I used the power of a half as a square root of, so it was the same as Item 36 ($\sqrt{16x^{16}}$).
Kirsty:	Well anything to the power of a half is square root so I just took the square root and got $4a^4$.
Tim:	I think I would have taken the total power of a half to be square root and square rooted all the way through it so I ended up with $4a^4$.
Bianca:	I kind of guessed that the power of a half is the square root of it all so I square rooted $16a^{16}$.

Of seven students, who changed from taking the square root, four gave a correct answer of $4a^8$ for $(16a^{16})^{1/2}$. One of these was Malcolm.

Malcolm:	I just halved 16 and halved a to the power of 16 ... er ... ah, the power 16 I times by the power of a half.
Interviewer:	What about the 4 (questioning the coefficient because halving 16 would have given 8)?
Malcolm:	I thought it was the square root of 16, I don't know why?
Interviewer:	Do you think you should have taken the square root?
Malcolm:	I'm not sure.

While he had arrived at a correct answer, Malcolm now felt it was wrong. He had obvious problems determining the relationship between the index of $\frac{1}{2}$ and the coefficient 16.

Linda and Christine had similar problems to Malcolm in relating correctly the index of $\frac{1}{2}$ to the 16 and in deciding whether to work through index laws or think in terms of the square root. Linda gave an answer of $(16a^{16})^{1/2} = 8a^8$ and explained her reasoning in the following terms:

Linda:	Well I said 16 to the power of a half ... or times by a half to get 8 and then I went a^{16} times by a half to get 8 (meaning a^8) but now I can see you should square root it.
Interviewer:	If you did that what would the answer be?
Linda:	$4a^4$ ... probably.

Christine gave an answer of $16a^8$ for $(16a^{16})^{1/2}$ but demonstrated in the interview that her thinking was much the same as Linda's.

Interviewer:	How did you arrive at your answer?
Christine:	I just timesed the index by a half.
Interviewer:	You timesed the index by a half but also left the 16 there. What did you think the relationship was between the 16 and the half?
Christine:	I should have times 16 by a half.
Interviewer:	So what should the answer be?
Christine:	$8a^8$ ... no because wouldn't it now be $16a^{16}$, the square root of that.
Interviewer:	And what would the answer to that be?
Christine:	$4a^4$

These interviews demonstrate a remarkable lack of coherency in thinking. Students accept that the same question can be written in two ways, but are then willing to accept that the two ways give different answers. Linda and Christine realised that both the index of a half and the radical sign meant that the square root should be taken. Despite realising this, they believed also that if index notation is used, everything is times by a half, but, where the radical sign is used, the square root is taken of all numbers within it.

It seems, in answering questions involving an index of a half, students initially opt for one of two paths. These paths, which both give the right answer when used correctly, are to 'think square root' or to look to use index laws. If they 'think square root' a substantial proportion make the error of taking the square root of the index. Just as many did the same when the question was written with a radical sign. If they take the other path, many errors are made in applying the index to the coefficient. Some students, such as Linda and Christine, multiply the coefficient by the index, but the most frequent error is to leave the coefficient unchanged (see Table 8.4).

What do interviews reveal about the understandings students apply where the finding of a square root is evoked not through the use of an index of $\frac{1}{2}$ or the radical sign but through the context of the question?
Interview responses showed that the use of an application to evoke the finding of a square root did make the question more accessible for some students. They were able to bring more considered strategies into play when there was no need to concern themselves with notation.

Item 38 (finding the side of a square of area $36x^{16}$) was used to introduce the question through an application. It provided a context where students needed to find a square root, but were not directly confronted by the notation. While the error rate was still high, students were more successful than when the radical sign was used. There was a reduction in the number of students taking the square root of both the index and coefficient from thirteen for Item 36 ($\sqrt{16x^{16}}$) to eight in Item 38.

The context did make a significant difference to Malcolm. He had answered the application question (Item 38) correctly. Following discussion of that item, he rethought his answer for $\sqrt{16x^{16}}$ and responded correctly.

Malcolm:	I took the square root of 36 because you times the sides and cause when you times you add the indices I just halved it.
Interviewer:	What do you think you are actually doing to the $36x^{16}$?
Malcolm:	Ah'm just getting the square root.
Interviewer:	How would that compare to Item 36?
Malcolm:	Mmm.. it would be $4x^8$.
Interviewer:	Why ?
Malcolm:	Because you are doing exactly the same operation.

Innes had taken the square root of the index for both $(16a^{16})^{1/2}$ and $\sqrt{16x^{16}}$. The fact he was unable to take the square root of the 8 in $\sqrt{25x^8}$ led him to use index laws successfully to answer that question. Something similar happened in Item 38 where the context resulted in his finding the correct square root of $36x^{16}$.

Innes:	That is the length of the side isn't it? (Struggled to get going). Ah because I had to times it ... ah ... we have got the area and two sides added together would have to ... I mean times together would have to give that so I worked out what would be ... ah'm ... I just square rooted the 36 because it is just timesing itself and I put $6x^8$ because they add together to give 16.

Christine, like Innes, had taken the square root of the index for both $(16a^{16})^{1/2}$ and $\sqrt{16x^{16}}$. She did not give an answer for $\sqrt{25x^8}$ but obtained a correct answer for Item 38. However, in the interview, she became focused on the index being 4 and could not see how she had arrived at the index of 8.

Christine:	I worked it out ... ah because ... I worked out that because the whole area is $36x^{16}$ and if you have to do an area it would be one side times another side so it would be $6x^4$ times $6x^4$ to give you $36x^{16}$ and then so .... that doesn't work either .... I wrote $6x^8$ .... I had 4.
Interviewer:	Why did you change it?
Christine:	I don't know (couldn't explain how she got the 8).
Interviewer:	Which do you think is the correct answer?
Christine:	$6x^4$.
Interviewer:	Why?

Christine:	I don't know because I don't know how I go the eight.

Kirsty answered Item 38 correctly despite having taken the square root of the index in each of the other three questions. She explained her answer, and was then asked if she wished to review other answers in light of her response.

Kirsty:	Well to get the area you times two sides and I suppose just the 6 times 6 gives you the 36 and you have to ... when you are timesing ... you have to add the indices so 8 and 8 gives the 16.
Interviewer:	Does that make you think at all about any of the other questions we have just been talking about?
Kirsty:	Yeah.
Interviewer:	In what way?
Kirsty:	Just adding and timesing the indices, when you are multiplying you add the indices.
Interviewer:	Would you like to have another look at Item 36?
Kirsty:	(Reads question and answer.)
Interviewer:	Would you like to do any more thinking about that?
Kirsty:	No.

It was surprising that, in the end, Kirsty still did not want to make any change to her answer of $4x^4$ for $\sqrt{16x^{16}}$. This shows the strength of her feeling that the square root should be taken of numbers inside the radical sign. Both Linda and Bianca obtained the answer of $6x^4$ for Item 38 and indicated at the interview that they were thinking of square root.

Linda:	I said $36x^{16}$ that's the four sides and to get one side you'd have to square root that.
Bianca:	Well I figured if the area is $36x^{16}$ then the side has to be the square root.

Again it seems to be that students 'think square root' and so take the square root of all numbers as though it had been written with a radical sign.

Interviews show that having the operation arise from an application does make some students think more critically about the question. This results in a noticeable increase in success, as opposed to cases where the index of the expression is also a perfect square but the context is just to apply a radical sign or index of $\frac{1}{2}$

Summary of Theme 4

The error rate for these questions was high. Many students tended to focus on, and respond successfully to, only one or several aspects of the question, but not all. Where the radical sign was used, the root of the coefficient was usually taken correctly. However, where an index of $\frac{1}{2}$ was used to indicate square root, it was the indices which were operated on successfully.

Certain circumstances encourage students to respond correctly to such questions. The success rate is higher where the index under the radical sign is not a perfect square or where the operation comes from an application and students are not confronted by the notation immediately. Interviews show that students do see the radical sign and the index of $\frac{1}{2}$ as meaning the same and being interchangeable. However, many students then respond differently to the two notations, and do not feel the need for consistency in their responses across the two forms.

INTERPRETATION OF THE ZERO INDEX (Theme 5)

Interpretation of the zero index posed many problems to the School Certificate students. However, the Pilot Study showed that Year 12 students made far fewer errors. The results obtained by the subjects of this study on items relating to this theme are listed in the tables below.

Table 8.6. Theme 5: Summary of Results

	Item	Answer	No.Corr.	Other Responses and (n)*
[43]	$9^{1/2}+9^0$	4	22 (55%)	$10^{1/2}$(6), 3(2), 12, $5\frac{1}{2}$, $1+\frac{1}{\sqrt{9}}$, $18^{1/2}$(2), $10\frac{1}{2}$(2), 18, $4\frac{1}{2}$, 13.5
[44]	4×5^0	4	35 (88%)	0(2), 20, 20^0, NA
[45]	$7m^0$	7	28 (70%)	1(5), $7m$(3), m(2), 0, NA
[46]	$(m^5)^0$	1	22 (55%)	m^5(10), m(4), 0(3), NA
[47]	$3p^0+(5q)^0$	4	18 (45%)	8(4), 2(3), $3+5q$(2), 0(2), $3p+5q$, 3, 5, $p+q$, $3p^0+q$, $3p^0+(5q)^0$, $8pq$, NA(4)
[48]	p^0+2+p	$3+p$	31 (78%)	$2p+2$(3), $2+p$(3), $3p$(2), $2p^2$
[49]	$3p^0+p^{-1}$ without indices	$3+\frac{1}{p}$	19 (48%)	$\frac{3}{p}$(2), $3+\frac{1}{\sqrt{p}}$, $3p^{-1}$, $3-p$, $3\frac{1}{9}$, $3+p-1$, 3(2),

$4p^{-1}$(2), $1+p^{-1}$, $1+\frac{p}{1}$, p^{-1}, $p(3+[1^{-1}])$,

p, $2p^{-1}$, NA(4)

* When only one student gave a response (n) was not include.

The mean score for the seven items used to research this theme was 4.35 (62%) with a standard deviation of 2.34. Item 43 includes an index of a half while Item 49 includes an index of -1. An examination of the list of errors shows that these concepts, which are not the subjects of this study, accounted for a high proportion of errors in these two questions. Responses indicate that close to 80% of students did interpret correctly 9^0 as being 1 in Item 43. Although the nature of the answers makes it hard to be precise, it is likely a similar proportion decided $3p^0$ gave 3 in Item 49. Therefore, while Items 46 and 47 did pose problems, it would seem that students were not as unsuccessful in applying the zero index as might first appear.

The following table lists the items in the order in which they posed difficulties to students.

Table 8.7. Theme 5: Items in Order of Difficulty

	Item	Answer	No.Corr.	Threshold
[47]	$3p^0+(5q)^0$	4	18 (45%)	1.11
[49]	$3p^0+p^{-1}$without indices	$3+\frac{1}{p}$	19 (48%)	0.97
[43]	$9^{1/2}+9^0$	4	22 (55%)	0.57
[46]	$(m^5)^0$	1	22 (55%)	0.57
[45]	$7m^0$	7	28 (70%)	-0.27
[48]	p^0+2+p	$3+p$	31 (78%)	-0.75
[44]	4×5^0	4	35 (88%)	-1.60

Based on the threshold estimates discussed in Chapter 6, students found most difficulty with Items 47, 49 43 and 46. While the remaining questions were relatively well done, they also attracted a range of incorrect solutions.

As listed earlier in this chapter, three research questions were posed for this theme. The last question relates to the understanding students apply as evidenced from their interview responses. The research questions are addressed below.

In questions involving the zero index, what is the effect of having the base as a constant as opposed to having the base as an unknown?

The impact on student responses of the nature of the base was a consideration in Theme 2. Here, the situation where it is the zero index being applied to the base is considered as a special case.

Item 43 ($9^{1/2}+9^0$) required students to raise a constant to the power of zero while Item 48 (p^0+2+p) required the raising of an unknown to the power of zero. Though students were far less successful in Item 43, this was substantially due to errors made in simplifying $9^{1/2}$, as indicated by responses such as $10^{1/2}$ (n=6), $5\frac{1}{2}$, $1+\frac{1}{\sqrt{9}}$, and $10\frac{1}{2}$ (n=2). In all, it appears that 32 students (80%) did correctly simplify 9^0. Depending on how the two students arrived at the answer of $3p$ for p^0+2+p, the percentage of students successfully applying the zero index in Item 48 was between 78% and 83%.

Also, Items 44 (4×5^0) and 45 ($7m^0$) required the raising of a constant and an unknown to a zero power, respectively. Both were responded to successfully by most students. The higher error rate for $7m^0$ is accounted for by the five students who applied the zero index to both the base and coefficient, and so arrived at the answer of 1.

Students have a high level of success in raising both constants and unknowns to the power of zero and where the term with the zero index contains only one element, students readily respond with the answer of 1.

In questions involving the zero index, what is the effect of parentheses on students' responses?

As in the questions relating to Theme 2, the answers students provided often did not account correctly for the parentheses and treated expressions with, or without, brackets as being equivalent. This was done despite the fact that all students who were interviewed did incorporate the brackets correctly when reading out questions involving parentheses.

In simplifying $3p^0 + (5q)^0$, nine students appear to have treated the two terms in an equivalent way by giving answers of 8, 2, $3p + 5q$ or $p + q$. As discussed in Theme 2, it seems that this does not reflect necessarily a failure to understand the grouping role of parentheses. Students are looking to apply procedures they have acquired, rather than any understanding of the basic concepts. Therefore, they do not critically consider their response, and the question of whether or not their strategies have accounted correctly for parentheses is not an issue to them.

There are other factors which also cloud the question of whether, or not, students are ignoring parentheses. For instance, students have frequently treated a coefficient and base as an entity and therefore, in their minds, the terms $5q^0$ and $(5q)^0$ are equivalent. In such a case the answer of $3p + 5q$ for $3p^0 + (5q)^0$, though an incorrect use of the zero index, would not be seen as a contradictory use of brackets.

Irrespective of whether or not students are ignoring parentheses, clearly it is evident that in such questions parentheses are not fulfilling their role. This may be because they are being ignored or it may be that the misunderstandings students have of the relationship of indices to bases and coefficients are making the parentheses appear irrelevant to the students.

What do interviews reveal about students' understanding of the zero index?
Interviews showed students realised raising a term to the power of zero gave an answer of 1. However, they showed also that students had little insight into the concept, and that many were confused as to how it was to be applied.

In order to explore students' understanding of the meaning of the zero index, the question 'What do you understand by p to the power of nought?' was posed at the end of each interview. All students other than Tim responded that it equalled 1. Tim said that the p and the nought were multiplied together to give an answer of 0. Typical of the responses were:

Bianca: I dunno why but it is 1.

Philip: I really don't know why, I only know it is 1.

Malcolm: p^0 would be ... I don't know what it means, I think it is 1 ... the teacher said so.

Linda: I just remember its 1.

Innes: I was just going on a Year 10 exam I did ages ago where there was 5 to the power of nought and I think someone said the answer was one so I was just going on that.

To probe students' understanding further, they were then asked to think of a question with an answer of p^0. A variety of hesitant suggestions were made such as "p^5 minus p^5" by Bianca and "$(2p)^0$ and $(3p)^0$" by Kirsty. Malcolm, with the help of probing, did have some success

Interviewer: How can you get an answer of p^0 from a question?
Malcolm: Ah'm ... maybe operations of where you plus or minus indices.
Interviewer: OK which one, you pick an operation?
Malcolm: Alright p squared times p to the -2.
Interviewer: What does p^{-2} mean?

Malcolm: 1 over p^2.
Interviewer: What is the answer for p^2 times $\frac{1}{p^2}$? (writing the question)
Malcolm: p squared over p squared.
Interviewer: And what is the answer to that?
Malcolm: 1.
Interviewer: So you gave me a question with answer of p^0 and what did you get when you worked it out?
Malcolm: 1.

Despite this success, Malcolm did not have an overview of why a zero index gave a result of 1.

Responses to the question above, together with results from the test, show almost all subjects of the study know raising a constant or an unknown to the power of zero does generate an answer of 1. However, lack of genuine understanding seems to result in students having difficulty applying the concept in questions which are more complex. Kirsty, who was placed in the top 10% of 2 Unit students in the HSC, knew that an index of zero gave an answer of 1. However, her confusion over how to apply this concept was demonstrated when discussing her answer of $p+q$ for $3p^0 + (5q)^0$ (Item 47).

Interviewer: How did you arrive at your answer?
Kirsty: I can see it is wrong already.
Interviewer: In what way?
Kirsty: Oh $3p^0$ is just p, no, just 3.
Interviewer: Why just 3?
Kirsty: Because anything to the power of 0 is 1. Oh I just thought p is out there.
Interviewer: What do you think the answer is to $3p^0$?
Kirsty: Probably $3p$.
Interviewer: What about $(5q)^0$?
Kirsty: Oh well that would just be q.
Interviewer: So what does an index of 0 mean?
Kirsty: Ah'm 1.
Interviewer: So 1, whereabouts is the 1? You've said $3p^0$ there and the answer to that should be?
Kirsty: Just p? $3p$? (Very uncertain)

Linda arrived at an answer of m for $7m^0$ (Item 45) and had similar problems to Kirsty in seeing just where the '1', generated by the zero index, ended up.

Linda: Well I know that anything to the power of 0 is 1 so I figured out that m was 1 and I said $7m$ multiplied by 1 and I got m ... how did I get that? That confused me ... the zero ...

Interviewer:	When you said it confused you what did you mean
Linda:	Well the $7m$ ... the $7m^0$ ... I wasn't sure, I knew the m would have to be 1 but I wasn't sure what the 7 would do to ... whether it was $7m$ or 7 or just m.
Interviewer:	Why did you decide on it being m?
Linda:	I don't know, I must have decided it was just the m it applied to.

Immediately following this, Linda was asked about her response of $3p^0+q$ for $3p^0 + (5q)^0$. She showed that given time and encouragement she could answer these questions correctly

Linda:	These questions ... I don't like these ps and these zeros ... I said that $5q$ to the power of zero was 1 but like before I just ... ah'm ... I'm not sure.
Interviewer:	Where do you think the 1 came from?
Linda:	I think now if I did it again I'd say it would be $3p^0$ that's 3 because p^0 is 1 and 3 times 1 is 3 and then $(5q)^0$, that whole thing would be 1 so that's 4.
Interviewer:	You said you thought '1' when doing the question, where do you think that 1 ended up? I mean when you said $(5q)^0$ ...1... and you wrote down the answer q ...
Linda:	I'm not sure.

The responses given by Kirsty and Linda provided insight into a concern identified in the Pilot Study, that is: why do some students simply drop the zero index where the base is a variable though they rarely do this when the base is a constant? In her interview, Kirsty gave three different responses to the question "what do you think the answer is to $3p^0$?". Firstly, she said $3p^0$ was 3, then p and finally settled on $3p$ despite having stated previously "anything to the power of nought is 1". Linda was tempted also to answer $7m$ for $7m^0$ after having said "I know that anything to the power of 0 is 1". When responding with an answer of $7m$ for $7m^0$ it seems students are not simply ignoring the zero index. They are treating the index as giving a value of "1" but, in these more complex situations, this "1" sometimes manifests itself as "1 lot of".

Many students answered either m^5 or m for $(m^5)^0$, and interviews showed they were thinking in the way just described. Bianca explained her answer of m^5 by saying "I'm not too sure, I think I just kind of went ... well to the power of zero with just one lot of it I just got one lot of m^5". Malcolm had written the answer m but then said "ah'm I had anything to the power of zero equal to 1 and that should be m^5 ... I think".

It is a matter of concern that, for many students, the learning experiences they have been provided with do not effectively support them in answering questions involving the zero index. They seem to rely on a simple rule which they attempt to reinterpret and apply when given a question which to them is non-traditional.

Summary of Theme 5

Most students cannot explain why the zero index generates an answer of 1. While many are able to work with questions involving a zero index, others have difficulty when answering items of a more complex nature. In such cases they seek to apply the concept of the zero index as, in some way, giving 1, however, this can lead to a variety of incorrect responses.

Students use of naive rules, i.e., a zero index means the answer is 1, does not support them when confronted with questions involving parentheses. It appears that some are not so much simply ignoring the existence of the parentheses, but the misunderstanding they have of the relationship of indices to bases and coefficients makes the parentheses appear irrelevant. Throughout these questions involving the zero index it seems again that students have a problem taking into account all the relevant aspects of the questions.

CONCLUSION

Predominantly, students used superficial approaches when answering questions relating to the themes discussed in this chapter. This applied also for the themes discussed in the previous chapter except that students making errors there were able, in interview, to fall back on some basic understanding, such as 2^5 being $2 \times 2 \times 2 \times 2 \times 2$. Here they were not able to do this.

Students often addressed one or several aspects of a question correctly but not all. Many students failed to see the need for consistency of responses from one question to another, and are ready to accept answers which are contradictory of each other. They seem to apply the rule that is 'most obvious' in a particular case. This overt pragmatism characterises much of the description they give of their processes in interview. Undoubtedly, this is facilitated by classroom practices which encourage many repetitions of similar question types.

The search for a theory explaining the understandings students are applying is pursued in Chapters 9 and 10 where the three themes relating to theories of learning are addressed.

Chapter 9

INDICES IN TERMS OF SOLO (Theme 6)

Introduction to Chapter

Theories of learning which may explain errors students make in questions involving indices were identified and discussed in Chapter 4. From the discussion three research themes emerged. They were related to: the SOLO Taxonomy; forms of understanding; and, connections or frames. The first of these, Theme 6, is the subject of this chapter while the remaining two are addressed in the following chapter.

The SOLO Taxonomy is concerned with categorising students' responses in terms of modes and levels. The usefulness of the SOLO Taxonomy, in explaining errors students make in questions involving indices, is dependent upon a number of factors. Firstly, levels must be able to be identified within the range of possible responses. Secondly, given this can be done, it needs to be established that there is a consistency in the levels at which particular students respond. Finally, there is the issue of whether recognisable relationships exist between the quantitative evidence and the thinking demonstrated by students at interview. Accordingly, three research questions relating to this theme were posed. The questions are:

- *Can the SOLO model provide a framework within which to view students' responses to questions involving indices?*
- *Is there statistical evidence of SOLO type levels of response in students' written answers to questions involving indices?*
- *How does the qualitative data elaborate on the findings arising from the quantitative analysis?*

The four sections in this chapter are devoted, respectively, to the three research questions and a conclusion. Before proceeding, it is important to emphasise again the distinction between 'relational level', as used in the SOLO Taxonomy, and 'relational understanding', as used by Skemp. A 'relational level' response takes account of all aspects of a question while 'relational understanding', as discussed in Chapter 4, refers to the kind of understanding existing when a student knows what to do and why.

SOLO AS A MODEL FOR RESPONSES TO INDEX QUESTIONS

With regard to the relevance of the SOLO Taxonomy, the first issue to be addressed was whether it is possible to allocate levels to the different responses students provide. The question concerning this is now considered.

Can the SOLO model provide a framework within which to view students' responses to questions involving indices?

When working with, and learning about, indices, students in this study are operating within the concrete symbolic mode. They are involved in interpreting the concrete world through a symbolic system consisting of written number and the signs and symbols of mathematics. Within this target mode the five levels of response are: prestructural - where students are unable to access the question; unistructural - where the focus is on one aspect; multistructural - where several aspects are addressed, but independently; relational - where all aspects are addressed and conflicts resolved and where the response contains an overall linking concept, which could be a rule; and, extended abstract - where data are exhaustively processed and the results compared with relevant abstract concepts not included in the data.

As explained in Chapter 4, not only does the focus of this research lie within the concrete symbolic mode, it lies within the levels of unistructural, multistructural and relational. Subjects of this research have had substantial experience with indices and, additionally, are pursuing a relatively high level of mathematics. They are able to access such questions and, by addressing at least one component, give a response above the prestructural level. At the other extreme, questions in this study do not require students to respond at an abstract level. It is possible to respond at this level by answering in terms of general principles or some theoretical position, but such an approach is not necessary for success. Exploring the issue of extended abstract responses was not part of the research design.

That the research is concerned with the middle three levels is consistent with Biggs and Collis' (1983, p.162) suggestion a student, on leaving secondary school, would be expected to function, substantially, at the multistructural or relational level. Given that students can be expected to respond within these levels, what needs to be ascertained is whether each response can be associated with a particular level.

The nature of the response is determined largely by the working memory available for the completion of the task. At the unistructural level a student has only to understand the question, relate the question and the answer, and use one concept. An example, in the case of indices, would be finding the value of 4^2 (Collis and Watson 1991). A multistructural level response requires similar ability except the student needs to be able to access a number of concepts. Collis and Watson (1991) indicated that evaluating 5^4 is a representative question at this level. For all items used in this research, the cognitive load is greater than that involved in evaluating either 4^2 or 5^4 and therefore a response at a higher level, than these two questions require, is needed for success.

The relational level response requires an overview of relevant concepts (identifying and applying a rule) while being able to monitor the process or task from beginning to end. In this way students are able to reach a logical conclusion. The example given by Collis and Watson (1991), namely, 'What is the value of 'Δ' if $(\Delta+1)^3 = 64$?', requires the overall monitoring of the equation while several tasks are performed. Although the items in this study are not equation based, the 'rule needed to enable the items to be addressed provides a similar 'relating' feature and is, for these items, equivalent to the equation noted in the Collis and Watson example. Hence, when errors occur it can be interpreted that students making them are failing to monitor the individual components within the context of the relating feature. These errors would be coded (depending on the number and the nature of the errors) as various categories within the multistructural level or, in extreme cases as unistructural level responses. Therefore, for questions used in this research, the number of errors evident in each response provides the basis for determining the level of the response.

Having established a general structure upon which to code the questions it is appropriate to address one issue that seems at variance with the need both to use an appropriate rule and to identify the relevant and irrelevant elements. In some questions, such as $2^3 \times 2^4$ (Item 1) and $6^3 \div 2^2$ (Item 11) there is the option for students to perform tasks at the multistructural level and still respond successfully. In the case of $2^3 \times 2^4$ a student could evaluate 2^3 and 2^4 and proceed to multiply. Such an approach does not require an overview of the question and the available data and the student is able to move logically along a path from the first given index to the answer without needing to know in advance how they might proceed. Indeed the answer would be unexpected until the completion of the activity. However, the interviews confirmed that students rarely choose to proceed in this fashion. While one can speculate about reasons for this, such as context for the instrument and the many rule-based activities undertaken by students in previous years, it is apparent that correct responses almost invariably result from a relational level application of the 'rule' rather than a multistructural approach of the type just described.

In obtaining a relational level response students choose not to seek individual closures (answers) before proceeding to the next step. This represents an ability to form a generalisation based on previous 'concrete' experiences with specific cases and to work with pronumerals. It is this ability to work with pronumerals which allows students to refrain from calculating particular answers for each step of a problem (a characteristic of a multistructural response), and opens the way for relationships between different concepts to be utilised.

Consistent performance at a relational level is dependent on the student being able to identify and monitor the relevant data and be aware of the solution strategy which relates the various concepts. The rule cannot be known by its separate parts, only by an overview of all the elements and the structure of the relationships can the pattern be understood. This overview allows for the reasoning at a later stage in a question to be adjusted in the light of earlier thinking. Unlike those who perform at the multistructural level, the arrival at the final answer would not come as a surprise.

In summary, all questions used belong to the relational level. Success depends on the effective use of relationships and rules which, in turn, is contingent upon the ability of the student to have an overview of the relevant elements and to form, on this basis, appropriate generalisations. Where a student does not have such an overview, and cannot appropriately generalise, a multistructural level response containing one or more errors occurs. A classification structure for responses is therefore possible. The categories are: correct response - relational; response with one error - high multistructural; and, response containing more than one error - lower multistructural or, in extreme cases, unistructural.

While it is possible to set such a structure, in order to apply it to a particular question, responses need to be in evidence across the levels. As might be expected, the answers for many easier items were either correct or contained one error. An examination of the 'Other Responses' column in the 'Summary of Results' for Theme 1(a) (Table 7.2) reveals one level of response for Item 4 (all correct) and two for Items 1, 5 and 10. Similarly, within all themes there are items with only two levels of response. Such items do not provide sufficient discrimination of levels and cannot be coded for polychotomous analysis. However, an examination of the twenty-five most difficult items (as identified by their threshold values in Table 6.5) reveals responses at more than two levels for all but one (Item 1). On this basis the classification can be applied to the remaining twenty-four items. Each of the five content related themes is represented by at least three items.

An example of how responses can be categorised according to the SOLO model is now provided for Item 28 in Figure 9.1. The number of students giving each response is shown in parentheses.

Item 28: $(2m^2)^3=?$

Level	Response		Comment
Relational	$8m^6$	(22)	Correct response
High Multistructural	$2m^6$	(8)	Coefficient not operated on
	$8m^5$	(4)	Indices added
Lower Multistructural or below	$2m^5$	(2)	Coefficient not operated on, indices added
	$8m^4$	(1)	Working shows $(2m^2)(2m^2)(2m^2)=4m^4+4m^4$
	$4m^5$	(1)	Coefficient incorrect, indices added
	64	(1)	Variable ignored, index applied to coefficient
	4	(1)	Obscure response

Figure 9.1. Sample of Coding of Responses

Almost all responses for each of the twenty-four items could be classified readily. Examination of students' working clarified questionable cases. In general there was an obvious reason for most classifications but some were not so clear. For example, three students who responded with an answer of 8 for $p=4$, $q=2$, $p^3 \div q^2=$? (Item 18) had substituted correctly in the expression but then made one index related error in simplification. Hence their answers were coded as high multistructural. Table 9.1 shows the coding of responses, in terms of SOLO, for each of the twenty-four items selected. The question numbering is as for the Main Study.

Table 9.1. SOLO Coding of the Item Responses

Item	Relational	High Multi	Lower Multi and Uni
[2] $5^2 \times 2^3$	200 (16)	10^5(21), 33	10^6(2)
[3] $3^x \times 3^y$	3^{x+y} (16)	9^{x+y}(15), 3^{xy}	9^{xy}(6), NA(2)
[9] 2^3 $\boxed{}$ 2^4 Area = ?	2^7 or 128 (21)	4^7(14), 2^{12}	24, 4^{12}(2), $2(2^3+2^4)$
[11] $5^6 \div 5^2$	5^4 or 625 (21)	1(14), 5^3	5^2, 0^4, NA(2)
[12] $6^3 \div 2^2$	54 (13)	3(25)	12, NA
[13] $10^4 \div 2^3$	1250 (11)	5(26), 12500	$\frac{5^4}{1^3}$, NA(1)
[18] $p=4$, $q=2$, $p^3 \div q^2=$	16 (23)	2(11), 8(3)	64(2), 4^4
[22] $\frac{x^4 \times x^6}{x^2}$	x^8 (22)	x^5(15), $2x^8$	$\frac{x^5}{x}$, x^3
[23] $\frac{p^{10}}{p^3 \times p^2}$	p^5 (24)	p^2(10), p^4(4)	$\frac{1}{2}p^2$, $\frac{p^5}{p}$
[24] $\frac{(y^3)^4}{y^2}$	y^{10} (24)	y^6(9), y^5(4)	$4y^6$, $\frac{y^7}{y^2}$, $\frac{y^6}{y}$
[28] $(2m^2)^3$	$8m^6$ (22)	$2m^6$(8), $8m^5$(4)	$2m^5$(2), $8m^4$, $4m^5$, 64, 4
[32] $f(x)=x^3$, $f(2^4)=$	2^{12} (21)	8^{12}(8), 8^7, 128	$8x^{12}$, x^3, 16, 64, NA(5)
[33] $f(x)=x^3$, $f(2a^3)=$	$8a^9$ (19)	$2a^9$(9), $8a^6$(2)	$2a^6$(4), NA(6)
[35] $(16a^{16})^{1/2}$	$4a^8$ (16)	$4a^4$(7), $16a^8$(7), $8a^8$(2)	$16a^4$(2), $16a^6$, $\frac{1}{4a^{16}}$, $\sqrt[3]{16a^{12}}/4a^{16}$, $\sqrt[3]{16a^{16}}$, NA(2)
[36] $\sqrt{16x^{16}}$	$4x^8$ (12)	$4x^4$(13), $4x^{16}$(3), $16x^8$(2), $8x^8$	$16x^4$(2), $2x^{32}$, $16x^{2/16}$, 1, $16x$, NA(3)
[37] $\sqrt{25x^8}$	$5x^4$ (21)	$5x^{\sqrt{8}}$(3), $5x^8$(5), $25x^4$(3), $5x^{-8}$	$(25)^{2/8}$, $2\sqrt{5x^8}$, NA(5)
[38] Area=$36x^{16}$,Side=	$6x^8$ (21)	$6x^4$(8), $9x^4$(2)	$36x^{16}$, $36x^{16}\times4$, $36x^{14}$, $\frac{36x^{16}}{4}$, $\frac{36x^{16}}{2}$, 9^4, NA(3)
[40] $5x^{-1/2}$	$\frac{5}{\sqrt{x}}$ (11)	$-\frac{1}{\sqrt{5x}}$ (7), $\frac{5}{x^2}$, $\frac{5}{-\sqrt{x}}$	$\sqrt{5x}$(4), $-\sqrt{5x}$(2), $\sqrt{-5x}$(2), $\frac{-1}{\sqrt{5x}}$, $\sqrt[3]{5x}$, $10x$, $2x^{1/2}$, $\sqrt{5x^{1/2}}$, NA(7)
[41] 2×6^{-1}	$\frac{1}{3}$ (21)	-12(8), $\frac{1}{12}$ (6)	-72, 1, 12, $2\times\frac{6}{1}$, NA
[42] $m=2$, $m^{-2}=$	$\frac{1}{4}$ (20)	-4(7), 4(2)	-1(2), $\frac{1}{2}$, $(\sqrt{2})^2$, $1/\sqrt{2}$, 0.002, 0, 1, NA(3)
[43] $9^{1/2}+9^0$	4 (22)	$10^{1/2}$(6), 3(2), 12, $5\frac{1}{2}$, $1+\frac{1}{\sqrt{9}}$	$18^{1/2}$(2), $10\frac{1}{2}$(2), 18, $4\frac{1}{2}$, 13.5
[46] $(m^5)^0$	1 (22)	m^5(10), m(4), 0(3)	NA
[47] $3p^0+(5q)^0$	4 (18)	8(4), 2(3), $3+5q$(2), 0(2), $3p+5q$	3, 5, $p+q$, $3p^0+q$, $3p^0+(5q)^0$, $8pq$, NA(4)
[49] Express $3p^0+p^{-1}$ without indices	$3+\frac{1}{p}$ (19)	$\frac{3}{p}$ (2), $3+\frac{1}{\sqrt{p}}$, $3p^{-1}$, $3-p$, $3\frac{1}{9}$, $3+p-1$	3(2), $4p^{-1}$(2), $1+p^{-1}$, $1+\frac{p}{1}$, p^{-1}, $p(3+[1^{-1}])$, p, $2p^{-1}$, NA(4)

Since the coding is based on the number of errors made, the coding makes no assumptions about the relative difficulty of addressing the different aspects of each question nor about the sequence in which they were addressed. Therefore, within this system, responses of $4a^4$ and $16a^8$, obtained when simplifying $(16a^{16})^{1/2}$ (Item 35), are of equal value and are both coded as high multistructural since they each contain one error. The key feature in both cases is that students have not incorporated all relevant elements into their responses.

Of the 960 responses made by the 40 students to the twenty-four items, 456 (47.5%) were coded as relational, 346 (36%) were high multistructural, and 158 (16.5%) were lower multistructural or unistructural. As might be expected, despite the greater number of responses being given at the higher multistructural level, there was greater variety in responses at the lower multistructural level. The higher level responses represent more consistent errors than do the lower level ones. In all, for the 40 items, 66 different responses were given at the high multistructural level as against 98 for the lower level.

In conclusion, if only the more demanding questions are considered, and if the classification is in terms of the number of errors rather than the aspects correctly addressed, it is possible to apply the SOLO model as a framework within which to view students' responses to questions involving indices. When this is done, each of the levels is represented by a substantial number of responses. The statistical examination of student responses using the SOLO model as a framework now follows.

A STATISTICAL INVESTIGATION OF LEVELS OF RESPONSES

In the previous section it was demonstrated that student responses to questions involving indices can be classified according to SOLO type levels. This section is devoted to an analysis carried out to determine statistically whether, based upon that classification, students are performing at particular SOLO levels. The research question related to this issue is now addressed.

Is there statistical evidence of Solo Type levels of response in students' written answers to questions involving indices?

From an examination of Table 9.1, there appears to be some consistency with which the group, as a whole, is providing responses at the various levels. The average number of relational level responses per question was 19 and the standard deviation 3.9. For the high multistructural level the average was 14.5 with a standard deviation of 4.8. The frequency with which students gave a lower multistructural or below level of response showed greater variation between questions. The average number of such responses was 6.6 and the standard deviation 4.4. It is possible that the overall pattern of responses can be explained by individual students providing answers which exhibit a consistency in terms of the number of errors they contain and therefore in terms of the SOLO level of response being provided across the test items.

This situation lends itself to statistical investigation using a partial credit scoring system. Partial credit scoring may be used where it is possible to identify

> several ordered levels of performance on each item and thereby award partial credit for partial success on items. The usual motive for partial credit scoring is the hope that it will lead to a more precise estimate of a person's ability than a simple pass/fail score.
>
> (Masters 1982, p.150)

The levels of response as set out in Table 9.1 provide the 'ordered levels of performance' needed for a partial credit analysis. Following is a description of the analysis which was carried out.

Firstly, the responses for each student to the twenty-four questions were examined. They were classified according to Table 9.1 and given a score based on that classification. A correct, or relational level, answer was scored as 2. Where the student had correctly accounted for all but one aspect, and so responded at a high multistructural level, a score of 1 was allocated. Responses were scored as 0 where the student had responded at a lower multistructural level or below by making more than one error.

An analysis was then conducted using the software package described in Chapter 6, namely, *Quest; The Interactive Test Analysis System*. This software allows for the measurement of the difficulty of steps within items using a Partial Credit form of the Rasch Model. It does this in the form of 'threshold values' which, for polychotomous data, measure the ability required to have a 50% chance of achieving a particular, or higher, level of response. Since these data are coded for three levels there are two threshold values generated for each item.

Results of the analysis are given in Figure 9.3 which provides a graphical representation of the distribution of step difficulties and case estimates calibrated against a logit scale (complete statistics for the analysis carried out in this section are contained in Appendix D). The following extract shown in Figure 9.2 (taken from Figure 9.3), with annotated comments in italics, is given to assist the reader in interpreting the information. In the Item/Step section, numbers in front of the decimal point identify the item while those behind signal the threshold being referred to. The higher threshold value for the item is identified by a '2' after the decimal point and the lower one by a '1'. For example, the number 28.2 locates the threshold value (0.66 on the logit scale) between a relational and a higher multistructural level response to Item 28, i.e., between responses coded as 2 and those coded as 1 Similarly, 28.1 locates the threshold value (-0.78 on the logit scale) between a higher multistructural and a lower level response, i.e., between responses coded as 1 and those coded as 0.

Logit Scale	**Case Ests.**	**Items/Steps**

```
                 XXX    |
                        |       Case estimates above the threshold have a better
                 XXXX   |       than 50% chance of answering Item 28 correctly,
    1.0           X     |       i.e., responding at a relational level
                  X     |
                 XXXX   | 28.2  Threshold of ability needed for a better than
                 XXX    |       50% chance of responding correctly to Item 28
                        |
                 XXX    |
                        |       Case estimates between thresholds have a better
     .0          XXX    |       than 50% chance of making one error or less in
                        |       in Item 28,i.e., responding at a high
                  XX    |       multistructural level or higher.
                  X     |
                        |
                  X     |
                  XX    | 28.1  Threshold of ability needed for a better than
   -1.0                 |       50% chance of making at most one error in Item 28
                        |
                        |       Case estimates below the threshold have a better
                        |       than 50% chance of making two errors or more in
                        |       Item 28, i.e., responding at a lower
                        |       multistructural level or below
                        |
   -2.0           X     |
```

Figure 9.2. Sample Item Fit Map

In providing a measure of the difficulty students have with each step within an item, threshold values also allow comparisons to be made of the difficulty of steps across items. For example, as detailed in Appendix D, the threshold value for 28.2 was 0.66 and for 23.2 was 0.40. This shows students had more difficulty in giving a relational level response to $(2m^2)^3$ (Item 28) than to $\frac{p^{10}}{p^3 x p^2}$ (Item 23).

Logit Scale	Case Estimates	Items/Steps
4.0		
	X	
3.0		
	X	
	XX	
	X	13.2
2.0	X	
		12.2 36.2 40.2
	X	
	XX	2.2
	XXX	3.2 35.2
	XXXX	47.2
1.0	X	33.2 <u>40.1</u> 49.2
	X	9.2 11.2 41.2 42.2
	XXXX	28.2 32.2 37.2 38.2 43.2
	XXX	18.2 22.2 46.2
		23.2 24.2
	XXX	49.1
0.0	XXX	
		42.1
	XX	33.1 47.1
	X	32.1 38.1
		36.1
		3.1 35.1 37.1 43.1
	X	
	XX	28.1
-1.0		
		41.1
		9.1 11.1
		18.1 24.1
-2.0	X	23.1
		22.1
		2.1
		12.1 13.1
		46.1
-3.0		

Figure 9.3. A Graphical Representation of the Threshold Parameters for each Item/Step

The statistical results generated using the polychotomously coded data provide strong evidence that levels of development of a cognitive nature are governing the quality of students' responses across the twenty-four items.

An examination of Figure 9.3 shows there was only one item (Item 40) with the lower of its two threshold values above the higher threshold value of any other item. This means that, apart from Item 40, students as a group found it more difficult to answer correctly the easiest question than to give a single error response to any other question. Moreover, from visual inspection, there appears to be a major discontinuity between the upper threshold values and the lower ones. Item 49 is the only item, other than Item 40, whose lower threshold value is close to the upper threshold values. This indicates that not only are relational level responses harder to achieve than multistructural ones, they are noticeably harder to achieve across virtually the full spectrum of questions.

The statistics also show a consistency in the levels with which individual students are responding. As discussed in Chapter 6, the measures relating to this are the Infit Mean and the Infit t values. The Infit Mean for the Cases was 1.01 and the Infit t value was 0.04. Such figures indicate that, to a substantial degree, students who achieved success for the harder item/steps also successfully achieved the easier item/steps, while those who were unsuccessful with the easier item/steps were unsuccessful also with the harder ones. This, when taken with the fact that for all items, except Item 40, a relational response was more difficult to achieve than was a multistructural response for any other item, is very strong evidence that students are responding consistently at one of the three levels.

While threshold values allow us to compare the difficulty of steps across items, the comparative difficulty of the two steps within an item is measured by the Category Delta estimates. The higher the Delta value the more difficult is the step. These figures, provided in Appendix D, show that within each item the first step was easier than the second for all except three items (Items 40, 42 and 49)

The fact that the second step is more difficult than the first in the great majority of items shows relational level responses are more than simply a level beyond multistructural responses, they represent a significant increase in cognitive load. Discussion of Item 35 draws out the issues this raises. As discussed previously, the classification of errors takes account only of how many errors are made, not the difficulty of addressing particular aspects or the order in which steps are carried out. When simplifying $(16a^{16})^{1/2}$, a relational level response of $4a^8$ was given by sixteen students while the high multistructural level responses of $4a^4$ and $16a^8$ each attracted seven students. The Category Delta estimate for the higher step in this item was 1.14 while that for the lower was -0.35. While conclusions cannot be drawn about the order in which those giving a relational level of response undertook the two operations of taking the root of the coefficient and halving the index, the operations were equally attractive to those giving a high multistructural level response. Therefore, it appears that taking the root of the coefficient was easier than halving the index, when carried out as a first step, but that the opposite was the case when the operations were approached in the reverse order. This means, the later step is not more difficult because students are coming to a more demanding operation, the difficulty resides in it being the later step.

Of the twenty-four items, the three which are exceptions to the second step being more difficult than the first are Items 40, 42 and 49. Items 40 and 49 have already been identified as the two for which students had most difficulty in achieving the first step. Item 42 is the next most difficult. These particular items each contain a negative or fractional index. Understanding negative and fractional indices was not included as a separate theme in this study, however, the School Certificate results showed even the most able students had difficulty with the less straightforward questions involving these concepts. The structural complexity of these items provides students with a range of what they see as possible answers, and this will have contributed to the variety evident in their responses (see Table 9.1). This variety gives students greater opportunity to provide answers in which more than one issue is not addressed and so respond at the lowest level. However, in these questions, where there is such a variety of possible answers, a student who has the ability to arrive at a high multistructural solution is likely to complete the question successfully.

Care was taken to ensure the test was administered under appropriate conditions and, from the responses given at interview, it appears the results reflect well the cognitive performance of students. The consistency of results indicates students were essentially giving their optimum effort and were responding at the highest level they were capable of, or that the item demanded. As discussed previously, there were no questions which required students to respond at the Extended Abstract level of the SOLO Taxonomy though there may well have been students who could have.

The original fifty items used in this research were selected on the basis of the five content related themes. All five themes were represented by three or more items in the twenty-four chosen for the polychotomous analysis. Given this, the results of the analysis, as discussed above, show that the difficulties students have in index work transcend subject content divisions and have their basis in the cognitive processing of the students.

SOLO AS RELATED TO THE QUALITATIVE DATA

The final issue for consideration, with regard to the relevance of the SOLO Taxonomy, is whether recognisable relationships exist between the quantitative evidence and the thinking demonstrated at interview. The research question concerning this is now addressed.

How does the qualitative data elaborate on the findings arising from the quantitative analysis?

Two main findings came from the quantitative analysis of student responses in terms of SOLO levels. Firstly, for these twenty-four most difficult items, students, individually, are consistently providing answers at one of the three levels across the different content areas. Secondly, for the majority of questions, students find the step from a high multistructural response to a relational level response harder to achieve than from a lower multistructural or below level to the high multistructural level.

These two issues are now reflected upon in view of the information gathered at interview.

Levels as related to individual performance across content areas

The statistical analysis demonstrated singularly that, for the twenty-four items as a whole, the levels of response are quite distinct. In Figure 9.3 there is a very obvious 'gap' between the set of upper threshold values and the set of lower threshold values and, therefore, a relational level response for a given item is clearly more difficult to attain than a high multistructural level response for virtually any other item. The statistical evidence also established that individual students are responding consistently at the particular levels. Such findings are in agreement with the evidence gathered from interviews.

Interviews showed that, across the content areas, many students were focusing on a limited number of issues and closure was taking place before all given propositions had been processed and conflicts resolved. Such was the case with Linda who, despite demonstrating an understanding of underlying concepts: multiplied and divided numerical bases as though they were coefficients; raised bases to powers, again as though they were coefficients; divided indices in algebraic fractions; and, multiplied both the base and coefficient by the index when raising an expression to the power of $\frac{1}{2}$. The difficulty Linda had in integrating all aspects was apparent when discussing her response of $8a^8$ for $(16a^{16})^{1/2}$. As reported in Chapter 8, Linda explained her reasoning in the following terms:

Linda:	Well I said 16 to the power of a half ... or times by a half to get 8 and then I went a^{16} times by a half to get 8 (meaning a^8) but now I can see you should square root it.
Interviewer:	If you did that what would the answer be?
Linda:	$4a^4$ ... probably.

Despite realising that the operations of taking the square root and raising to the power of a half were equivalent, Linda simply shifted her focus from multiplying by a $\frac{1}{2}$ to taking the square root of the coefficient and index.

This tendency to focus on a limited number of aspects came through frequently at interview in comments such as Tim's where he said "I think I would have taken the total power of a half to be square root and square rooted all the way through it so I ended up with $4a^4$". Such thinking can also explain the many cases where bases and coefficients are treated as entities. As discussed in Chapters 7 and 8, this occurred when multiplying and dividing expressions, raising expressions to integral powers, and when applying indices of - $\frac{1}{2}$ or zero. When responding to $5x^{-1/2}$, 45% of students treated the $5x$ as an entity. It was clear, at interview, that their concentration was placed on the meaning of the index and little attention was being given to an overview of the question. Malcolm commented "I thought - $\frac{1}{2}$ was 1 over square root so I put $5x$ with the square root sign and 1 over it". Bianca said "Oh I remembered a negative I thought was a fraction and a half was the square root so I just threw it all together". Similar instances were noted for the other themes dealt with in Chapters 7 and 8.

The lack of any felt need for consistency by many students was demonstrated throughout the content themes both at interview and in written answers. While there was a consistency in error patterns in questions of identical structure, a small difference in the question could result in students using contradictory strategies. An example was where six students responded with $4x^4$ for $\sqrt{16x^{16}}$ yet correctly answered $5x^4$ for $\sqrt{25x^8}$. When the inconsistency of such responses were raised at interview, students showed more concern with their correct response than the incorrect one in which they had simply applied the square root to both the coefficient and index. Another example of the acceptance of such inconsistencies is where, in simplifying algebraic fractions, students divided indices which have a common factor but answered correctly the question where the index is an unwritten '1'. There is also the case where students understood a division can be written using a division sign or fraction bar and yet chose to subtract indices in the first case and divide them in the second.

It is evident, from the interviews, that two significant factors are operating when a student consistently provides multistructural level responses. Firstly, the student is focusing on a limited number of issues and not addressing all relevant aspects. Secondly, they are accepting or ignoring any inconsistencies which such an approach generates. In this way they are able to perform consistently at a mutistructural level despite the apparently inconsistent responses which result.

The relative difficulty of steps within questions

The quantitative analysis showed that, for the great majority of questions, the step from a high multistructural response to a relational response is a harder step to take than from a lower multistructural or below response to a high multistructural response.

As the previous discussion of Item 35 indicates, the last step is not more difficult because students are coming to a more demanding operation after having addressed the easier aspects. Rather, the difficulty resides in it being the last step. Here, the problem of achieving a relational level response is associated with the ability to integrate several operations, not with the difficulty of any one particular operation.

Interviews demonstrate that students, essentially, use strategies of a routine kind and often these do not take account of all relevant aspects. When doubt is placed on the correctness of responses, students rarely refer to underlying concepts. In the few cases where they did provide alternative approaches, these, again, were usually superficial in nature. They tended to address aspects not previously accounted for but failed to address other aspects which were correct in the original approach. In this way the response remained at a multistructural level. Such was the case where students who answered $8a^8$ for $(16a^{16})^{1/2}$ then decided the answer should be $4a^4$ because the index of $\frac{1}{2}$ means find the square root.

When discussing her correct response of $6x^8$ to Item 38 (finding the side of a square of area $36x^{16}$), Kirsty demonstrated clearly the difficulty students, performing at a multistructural level, have in moving to a relational level response. Kirsty had answered this question correctly despite taking the square root of the index in each of the other three questions. She explained her answer and was asked if she wished to review the other answers in light of her response. Following is the discussion which took place.

Kirsty:	Well to get the area you times two sides and I suppose just the 6 times 6 gives you the 36 and you have to.. when you are timesing .. you have to add the indices so 8 and 8 gives the 16.
Interviewer:	Does that make you think at all about any of the other questions we have just been talking about?
Kirsty:	Yeah
Interviewer:	In what way?
Kirsty:	Just adding and timesing the indices, when you are multiplying you add the indices.
Interviewer:	Would you like to have another look at Item 36?
Kirsty:	(Read question and answer)
Interviewer:	Would you like to do any more thinking about that?
Kirsty:	No.

Kirsty realised that, in Item 38, she needed to obtain two identical terms mulitplying to give $36x^{16}$, and did so successfully. Despite this she could not shift away from taking the square root of both the coefficient and the index when answering $4x^4$ for $\sqrt{16x^{16}}$.

Throughout the interviews there was little evidence that students providing multistructural responses were, when asked to reconsider responses, able to provide relational level responses.

CONCLUSION

It is clear that, across content areas, a cognitive factor is influencing the quality of student responses. Quantitative analysis of student responses to the more difficult questions showed some were able, on a consistent basis, to integrate all aspects of each question and achieve a high level of success with their responses. Others consistently responded at a high multistructural level by frequently taking account of all but one aspect of the question. The remaining students generally responded at a lower level by often failing to correctly account for more than one aspect.

The statistical analysis, supported by data gathered at interview, demonstrates that the measuring instrument, the set of items in the test, had construct validity and that the SOLO Taxonomy provides a remarkably accurate picture of student performance in index questions. The SOLO Taxonomy, therefore, offers a highly valuable framework within which to view students' responses.