## Chapter 4

# THEORETICAL FRAMEWORKS

> There is nothing so powerful for directing one's actions in a complex situation, and coordinating one's efforts with others, as a good theory. (Skemp 1978, p.14)

### Introduction to Chapter

In the previous chapter, common errors which students make when attempting index questions were identified. This chapter considers several theories of learning which may offer explanations for the kinds of errors made. If a theoretical framework can be found to explain these errors then an informed, deliberate, and coordinated approach can be undertaken to provide guidance for teachers in programming and presenting work in ways that will assist students to avoid or overcome such errors. Also, as Wagner and Kieran pointed out when talking generally about a theoretical framework for algebra, "whether or not we succeed in developing a theoretical framework ... the quest for such a theory may itself be instructive" (1989, p.229).

This chapter is divided into six sections. In the first section, the response categorisation system put forward in the SOLO Taxonomy of Biggs and Collis (1982,1991) is discussed. The second section examines the forms of understanding which students can acquire. Here, Relational and Instrumental Understanding, as described by Skemp, and a Conceptual-Representational Analysis approach, as applied to index questions by Goldin and Herscovics (1991), are considered. In the third section, two apparently related concepts, 'connections' as described by Shevarev and 'frames' as proposed by Davis, are discussed. Because of its focus on index questions Shevarev's research has already been considered in detail in

Chapter 2. Here, the theoretical basis of that work is reviewed briefly. Davis' 'frames' appear similar to Shevarev's 'connections' and, following a discussion of Davis' work, a comparison is made of the two. The remaining sections are devoted to: issues arising from the theories; the drawing out of research themes and questions; and, a conclusion.

The theories, selected for examination in this chapter, have been chosen on the basis of their potential to explain issues arising from the literature review and the analysis of the School Certificate data. Firstly, the reason for certain students having greater success in such questions may lie in their ability to respond, consistently, at a more sophisticated level. The SOLO Taxonomy, with its levels of response, provides a framework within which the quality of students' answers can be categorised and examined. Secondly, when answering questions with indices, there is scope for students (and teachers) to access an understanding of underpinning concepts or, alternatively, to use procedural strategies. The work of Skemp, together with that of Goldin and Herscovics may provide guidance to the kind of understanding being applied and to why students often arrive, uncritically, at what appear to be mathematically 'naive' answers. Finally, the way in which students seem to 'unthinkingly', yet consistently, obtain such answers may be explained in the constructs of 'connections' or 'frames' which relate, respectively to the works of Shevarev and Davis.

It is possible that several, or all, of these theories have something to offer in explaining problems students have with index questions.

## THE SOLO TAXONOMY OF BIGGS AND COLLIS

Throughout the 1980s there was a significant growth of interest in the role that levels of thinking might play in the development of a student's understanding in mathematics. As a result, learning theories which incorporate levels or stages have come under increasing scrutiny by researchers wishing to have a framework within which to structure their findings. Within that context, the SOLO Taxonomy, first proposed, in detail, in 1982 by Biggs and Collis in the publication *Evaluating the Quality of Learning: The SOLO Taxonomy*, appears to have much to offer to the mathematics education researcher.

The Structure of Observed Learning Outcomes (SOLO) Taxonomy provides a framework for describing the quality of students' responses rather than their stage of development. Although Biggs and Collis saw Piaget's criteria as being useful as a basis for assessing a particular learning episode they suggested

> it is quite misleading to sort students into different stages of development on the basis of their responses to school learning tasks. Rather, such information tells us the more mundane, but much more useful, knowledge of how well the particular task has been learned - or taught. We are proposing a classification ... based on Piagetian developmental stages but ... we are concerned with classifying outcomes not students.
>
> (1980, p.20)

Biggs and Collis (1989) stated that "over a wide variety of tasks and particularly school-based tasks, learners display a consistent sequence, or 'learning cycle', in the way they go about learning them" (p.152) and that this 'learning cycle' holds within each of the five 'developmental modes' commonly used to describe the level of abstraction at which a human is functioning. These modes progress from

> concrete actions to abstract concepts and principles, corresponding in a large part to the developmental stages referred to by Piaget ... (they) do not successively replace each other, as is certainly suggested in classical stage theory, but each successively adds to its predecessor, so that the modal repertoire of the mature adult is considerably greater than that of the young child.
>
> (Biggs and Collis1989, p.155)

The SOLO model has been refined over time. An outline of the model, as recently described and applied by researchers, now follows.

Central to the model, as indicated above, is the concept that the level of response of an individual to an environmental cue depends upon both "the *mode* of functioning, which is determined by the level of abstraction of the elements utilized, and the complexity of the *structure* of the response within that mode" (Collis and Romberg 1991, p.87). The modes of functioning, and their approximate time of emergence are:

> *Sensori Motor* - available from birth. This mode is concerned with physical coordination and with the management of objects in the physical environment.
>
> *Ikonic* - available at approximately one and a half years of age. In this mode judgements are made based on mental images, or ikons.
>
> *Concrete symbolic* - available from around the age of 6 years. Here the concrete world is interpreted through symbolic systems

such as written number and the signs and symbols of mathematics.

*Formal* - available to students at about the age of 16 years. In this mode, abstract propositions are dealt with and relationships between them deduced.

*Post formal* - available at approximately the age of 20 years. Operation in this mode is at a higher level of abstraction, fundamental structures of theories are questioned.

Within each mode, as previously indicated, Biggs and Collis see learning as taking place in a consistent sequence, or 'learning cycle'. They stated that "change brought about by learning occurs when the student's responses become more complex within the learning cycle of that task" (1989, p.152).

Five levels of response are identified in the learning cycle. Each is characterised by: the relevance and sophistication of the response; the working memory capacity used the logical operations used; and, the felt need for consistency and closure. As described by Pegg (1992), progression through the levels involves:

1 a growing complexity in understanding;
2 an increased ability to consider more information;
3 an increased ability to accept more complexity; and,
4 an increased ability to delay in providing an answer.

(p.370)

The levels, as they relate to this research, are elaborated upon later in this section. Briefly, however, the levels, and the nature of responses within each are: prestructural - no meaningful response; unistructural - one aspect of the question focused upon; multistructural - several aspects focused upon but no relationship seen between them; relational - relationships seen and all aspects addressed; and, extended abstract - additional information can be brought to the problem from outside the question. The middle three levels, namely, unistructural, multistructural and relational, are contained within each mode. The prestructural level can be equated to a response in a preceding mode while an extended abstract response can be equated with the unistructural response in the next mode. For example, in the concrete symbolic mode a prestructural level response may be associated with levels in the ikonic mode while an extended abstract level response equates with a unistructural level in the formal mode. A summary of the model is provided in the following figure.

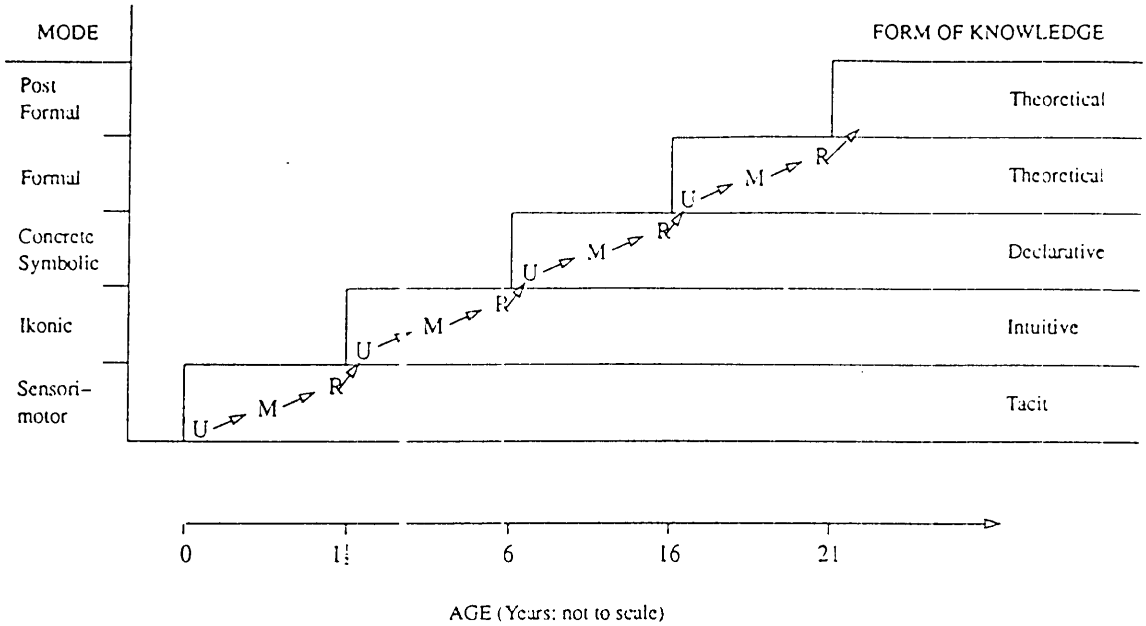


**Figure 4.1. Modes, Learning Cycles, and Forms of Knowledge of the SOLO Taxonomy**
(adapted from Biggs and Collis 1991)

An overview of the SOLO Taxonomy has been provided in order to set the context. However, the focus of this research lies essentially in the concrete symbolic mode. It is within this mode that students are mainly operating when learning about, and working with, indices. As described by Collis and Watson (1991), the particular features of each level of response relating to the concrete symbolic mode are as follows.

- **Prestructural** level responses are characterised by rapid closure using inappropriate or incorrect data, concepts, processes or strategies. Students have no felt need for consistency in their answer and closure occurs without the problem really being understood.

- **Unistructural** level responses focus on a single aspect only. They require some processing of data involving "loading and holding the question, scanning the information for a proposition, selecting it and comparing it with the question to determine the response" (p.9). They are characterised by low working memory capacity, quick closure and no felt need for consistency. "Inconsistency of response is almost guaranteed; the individual may respond to a different proposition on the next occasion the data are presented and produce a response which would be contradictory of the first" (p.70).

- **Multistructural** level responses involve similar mental processes to the unistructural level, but the working capacity required is greater because 'several propositions are monitored, while retaining the question in mind'. However, there is still considerable scope for inconsistency in that "next time the individual could select a different set of propositions to form a judgement contradictory to the first" (p.70).

- **Relational** level responses take account of all given propositions and closure only takes place after these have been processed and any conflicts resolved. This means that the response is likely to be consistent with the given information.

- **Extended abstract** level responses are characterised by the exhaustive processing of all relevant data and the comparison of results with relevant abstract concepts not included in the data. This results in consistent responses, not only to the given question but also to similar questions at a later time. The working memory capacity required is much greater than for other levels.

It might be considered that the formal mode would be of relevance also to this research since the ability to operate effectively in that mode becomes increasingly important for those pursuing the more demanding mathematics courses in Years 11 and 12. However, while success in a given mathematical situation may dictate operating in the formal mode, the index skills required are usually of a routine nature, i.e., index skills are a 'tool' to support the student while dealing, in the formal mode, with other mathematical issues. Additionally, questions of the type which are the subject of this research do not require a response at a level above relational level. It is not necessary for students to provide extended abstract responses, irrespective of whether they are capable of doing so. Because of these factors, the focus of the research lies not only within the concrete symbolic mode, but within the middle three levels of unistructural, multistructural and relational.

Collis and Watson (1991), in *A Mapping Procedure for Analysing Responses in Mathematics*, looked at applying the SOLO Taxonomy to responses in mathematics and used questions on indices to exemplify their discussion. It is of value to consider their work briefly. Collis and Watson commented that examples can be set so that a correct answer is only available if a response is made at a particular level (or any higher, but not lower, level).

(Collis and Watson 1991, p.78)

They used a format which "requires that the item has a stem which provides basic information and four questions which are designed with the criteria for the four higher levels of SOLO responses in mind" (p.71). The stem and the questions discussed by Collis and Watson (pp.71-73) were:

**Stem:** $2^3 = 2\times2\times2 = 8$ ; $3^2 = 3\times3 = 9$ ; $\Delta^2 = \Delta\times\Delta$ and $\Delta^3 = \Delta\times\Delta\times\Delta$

**Unistructural: Find the value of $4^2$**

Here only one concept is required, that is $4^2 = 4\times4$, and only minimal memory space is needed given the student's familiarity with the four operations of arithmetic.

**Multistructural: Find the value of $5^4$**

Finding the value of $5^4$ requires functioning at a multistructural level because working memory is needed to monitor the number of factors used and to perform each multiplication.

**Relational: What is the value of '$\Delta$' if $(\Delta+1)^3 = 64$?**

A likely solution would be to take the cube root of both sides and subtract 1. A relational response is required as considerable memory space must be used to monitor and integrate the involved processes of treating $\Delta + 1$ as a unit, using the concept of an inverse power and maintaining the equality. Closure must be postponed while this is done.

**Extended Abstract: If $(c+a+1)^3=512$ what pairs of whole number values can $c$ and $a$ take between 0 and 7?**

The solution to this question is similar to the relational level question above but is distinguished from it by the need for abstract concepts from outside the data whereby $a$ and $c$ are viewed as variables.

While the examples used by Collis and Watson do not closely resemble the kinds of questions being researched here, they do demonstrate that SOLO level classifications can be given to responses for certain questions involving indices. Whether this can happen for the items examined in this study is an issue for research.

The SOLO Taxonomy emphasises that, in assessing a student's understanding, we need to ensure not only that he or she has been taught the required concepts and processes but also to consider the appropriateness of the level of processing required. Errors can result from lack of understanding of concepts but also from the failure to respond at the level the question demands.

Another point of interest relating to the Solo Taxonomy is the question of multi-modal functioning. While the concrete symbolic mode is the one which is of main concern for this research the other four modes remain available and are used throughout life. With regard to this issue Collis (1988) stated

> It seems that much of the research about teaching and learning elementary mathematics has concentrated on the concrete symbolic mode of functioning. This is to be expected because this is the mode in which the 'game' known as mathematics first began to be played. But the desire to get students to learn the game has distracted teachers from the realities which apply to individuals. There seems to be little conscious attention given to earlier developed skills in the haste to push forward into 'real' mathematics which can be seen to begin in the concrete symbolic mode but which only reaches its 'proper' level in one of the formal modes.
>
> (p.7)

It may be that, by concentrating our teaching of indices within the concrete symbolic mode, we are neglecting other valuable strategies within the other modes of learning which could help students bring better understanding to the concepts involved in index work.

The SOLO theory may help explain aspects of students' thinking in answering index questions. This present research is concerned with students' understanding of, and ability to operate on, numerical and algebraic expressions involving indices. The majority of such questions, according to Collis and Watson's classification of the four index questions above, require at least a multistructural, if not relational, level of response. It could be that errors are being made as a result of students using little working memory and closing quickly without having an overview of all the elements involved. It is therefore important that this research ascertain whether errors are, in fact, resulting from students responding at lower levels than a question demands.

## FORMS OF UNDERSTANDING

The distinction between a student's understanding of underpinning concepts and their ability to carry out procedures successfully is an important issue for this research, as it is for research throughout much of the field of mathematics education. This distinction has been the subject of detailed examination in the publication *Conceptual and Procedural Knowledge: The Case of Mathematics* in which the contributing authors discussed a range of relevant theories. In the introductory chapter, Hiebert and Lefevre (1986) highlighted general issues concerned with the nature of, and relationship between, the two aspects of understanding. It is valuable to note here several of the points they made.

In discussing the place of conceptual and procedural knowledge in mathematics education, Hiebert and Lefevre stated that historically "the two kinds of knowledge have been viewed as separate entities", however, they are now seen as "distinct, but linked in critical, mutually beneficial ways" (p.2). They suggested that "students are not fully competent in mathematics if either kind of knowledge is deficient or if they both have been acquired but remain separate entities" (p.9).

Hiebert and Lefevre observed that, while the boundaries between them are not clearly defined, the substance of each of the two forms of knowledge is easily described.

**Conceptual knowledge** is the kind of knowledge being applied when, in solving a mathematical problem, a student brings into action an understanding of underlying concepts. It is knowledge which is rich in relationships and needs to be learned meaningfully. Conceptual knowledge grows as students establish links between two existing pieces of information or between existing information and new information entering the system. In index work, an example of such knowledge is where, in questions involving multiplication, a student associates the action of adding indices with the fact that an index means a repeated factor, i.e., in $a^3 \times a^5$ the three factors of $a$ and the five factors of $a$ together give eight factors which may be written as $a^8$.

**Procedural knowledge** is concerned with the mechanical processes employed when solving a mathematical problem. Hiebert and Lefevre described procedural knowledge as being made up of two distinct parts, "the formal language, or symbolic representation system, of mathematics ... (and) ... the algorithms, or rules, for completing mathematical tasks" (p.6). As opposed to conceptual knowledge, which must be learned meaningfully, procedures may or may not be learned with meaning. Through rote learning, it is quite possible that students learn to respond with $a^8$ for $a^3 \times a^5$ without associating the addition of the raised numbers with the concept of an index as meaning a repeated factor. Where procedural knowledge is the result of rote learning such knowledge "is not linked with other knowledge and therefore does not generalize to other situations; it can be accessed and applied only in those contexts that look very much like the original" (p.8). Procedural knowledge is integral to students' performance in the area of indices in that they are continually confronted with notation, and with the need to manipulate that notation using rules or algorithms. The ability to work effectively with raised numbers, radical signs, fraction bars and parentheses is an important skill for students to possess if they are to respond successfully to questions involving indices.

Hiebert and Lefevre saw the acquisition and use of procedural knowledge as being greatly advantaged when such knowledge is linked with the concepts which underpin it. Benefits they listed are: symbols have meaning for students when connected to the conceptual knowledge they represent; procedures are stored and retrieved more effectively if connected to their conceptual underpinnings; conceptual knowledge enhances problem representation; conceptual knowledge can monitor the selection of a procedure and the reasonableness of the procedural outcome; and, conceptual knowledge assists students to transfer procedures between problems that are structurally similar and, in doing so, reduces the number of procedures which must be learned.

However, Hiebert and Lefevre asserted that the potential relationships, as described in the preceding paragraph, do not represent the realities of the situation. Indeed, they observed that "examining the relationships between conceptual and procedural knowledge is a worthwhile pursuit only because students often fail to recognize or construct the relationships" (p.16). Particular problems they described include: difficulties in establishing relationships where the knowledge base is deficient; failure to encode relationships which may be obvious to adults; and, difficulties in establishing relationships because newly acquired knowledge is often context bound. This acquisition of unrelated and context-specific knowledge often manifests itself in students readily accepting quite contradictory answers. Hiebert and Lefevre commented that

> Evidence for the isolated nature of students' knowledge often is collected by observing students solve the same problem in two

> different contexts, produce conflicting solutions, and not reconcile the differences. Indeed, most investigators report that many students do not recognize that the differences must be reconciled.
>
> (p.18)

Hiebert and Lefevre emphasised the importance of describing effectively the relationship between conceptual and procedural knowledge. They stated that "If we understood more about the acquisition of these kinds of knowledge and the interplay between them in mathematical performance, we surely could unlock some doors that have until now hidden significant learning problems in mathematics" (p.22).

Both conceptual knowledge and procedural knowledge have their role to play if students are to respond with both insight and efficiency to questions involving indices. Of the various theories relating to these forms of knowledge, two appear particularly relevant to this research. These are Skemp's theory of Relational and Instrumental Understanding and Golden and Herskovic's modes of understanding. Both are now discussed in some detail.

**Relational and Instrumental Understanding as postulated by Skemp**

Skemp (1978) in his article *Relational Understanding and Instrumental Understanding* examined what is meant by the term 'understanding' as applied to students' learning in mathematics. Skemp believed there are two commonly held and quite different interpretations of the meaning of the term and that these interpretations reflect the fact that there are two different ways students can be taught, and can understand, mathematics. These he called 'relational' and 'instrumental' and described them by saying that

> Relational understanding means ... what I have always meant by understanding ... knowing both what to do and why. Instrumental understanding I would until recently not have regarded as understanding at all. It is what I have in the past described as 'rules without reasons', without realising that for many pupils *and their teachers* the possession of such a rule, and ability to use it, was what they meant by understanding.
>
> (Skemp 1978, p.9)

While advocating strongly that teaching should be directed at developing relational understanding, Skemp observed that many, and possibly the majority, of those who teach mathematics encourage instrumental understanding. He saw the major contributing factors to this being: the type of examinations used; the over-burdened syllabuses; difficulties of assessment; and, teaching traditions. The two types of understanding as described by Skemp are examined briefly below, as are the implications of the theory for research.

In describing **Instrumental Understanding**, Skemp gave a number of examples of instrumental explanations, or 'rules without reasons'. One example was "to multiply a fraction by a fraction, multiply the two numerators together to make the numerator of the product, and the two denominators to make its denominator" (p.9). He said that "other examples of instrumental explanations can be identified in abundance in many widely used texts" (p.9). Difficulties occur when a problem, involving several steps, is to be solved. Here the student is faced with applying an increasing number of rules, each of which is identified and used only when the next step in solving the problem is reached, and "there is no awareness of the overall relationship between successive stages and the final goal" (p.14).

Skemp gave several specific examples of the ways in which instrumental thinking generates errors. Each related to a student applying an inappropriate instrumental strategy in a new situation. One such example was applying the process of finding the third angle of a triangle (adding the two known angles and subtracting from $180^0$) when asked to find the third exterior angle of a triangle. This has features of Shevarev's connections whereby the important point of the angles being interior was missing from the student's thinking.

Skemp said that instrumental mathematics has some benefits in that, within its own context, it is usually easier to understand and provides a page of right answers easily and quickly. He observed that because of this "even relational mathematicians often use instrumental thinking" (p.12).

In describing **Relational Understanding** Skemp stated that "learning relational mathematics consists of building up a conceptual structure (schema) from which its possessor can (in principle) produce an unlimited number of plans for getting from any starting point within his schema to any finishing point" (p.14). Skemp saw many benefits in relational understanding. These included: being more adaptable to new tasks; being easier to remember despite being harder to learn; being effective as a goal in itself (whereas instrumental understanding requires an application to give it any purpose); and, being a positive force in motivating learners to seek understanding of new material.

There are many opportunities for teachers to teach 'rules without reasons' when teaching index work. Observation of textbooks used by the subjects of this research show that they abound with such rules. For example, McSeveny, Conway and Wilkes (1988), in their textbook *Signpost Mathematics Year 8* (currently one of the most commonly used texts in N.S.W.), provided the following rules as a preface to an exercise on simplifying algebraic expressions involving indices. The setting out is as follows:

| INDEX LAWS |
| --- |
| $a^m \times a^n = a^{m+n}$ |
| i.e. When multiplying → ADD |
| $a^m \div a^n = a^{m-n}$ |
| i.e.When dividing → SUBTRACT |
| $(a^m)^n = a^{mn}$ |
| i.e. Index to an index |
| → MULTIPLY |

(p.121)

This is immediately followed by a set of exercises with the instruction to "use index laws to simplify each expression fully" (p.122). Two of the expressions were $10xy^4 \times 2x^5 \div x^2y^2$ and $\frac{(3a)^2 \times 5a^3}{9(a)^3}$ while other questions were of both greater and lesser complexity than these. Quite obviously, answering such questions involves many skills over and above those indicated. The rules provide no guidance with regard to order of operations or the relationship of indices to the coefficients. Clearly, the rules, as written in words, are open to misinterpretation and the student is left to incorporate into them some points of crucial significance such as the requirement that the bases be the same. Not only are instrumental procedures reinforced, but, by omitting important features, circumstances are created which are conducive to the formation of incorrect connections.

Skemp makes a strong case for relational understanding, however, the relative merits of relational and instrumental understanding are of concern to this research only from the point of view of their impact on errors students make in index questions. A central issue is the extent to which students, making errors in questions involving indices, use 'relational understanding' as opposed to 'instrumental understanding'. In more demanding questions, such as $10xy^4 \times 2x^5 \div x^2y^2$ and $\frac{(3a)^2 \times 5a^3}{9(a)^3}$ , it seems highly likely that students of Year 8 would find a relational approach cumbersome and mentally demanding, and would resort to instrumental strategies. Such strategies allow different aspects of the question to be dealt with step by step whereas a relational approach requires an overview. Instrumental strategies are also likely to be extensively used in easier questions since "one can often get the right answer more quickly and reliably by instrumental thinking than relational" (Skemp 1978, p.12). Interviewing students concerning their procedures is important in determining the kind of thinking used.

A point to keep in mind, given that teachers may have a relational or instrumental emphasis to their teaching, is that problems occur when "a pupil whose goal is to understand instrumentally is taught by a teacher who wants them to understand relationally ... or the other way around" (p.10). It is not just students and teachers, but also textbooks and syllabuses which can lean towards either instrumental or relational understanding. Mismatches can occur with any combination of these four factors. Skemp commented that where the goal structures of the teacher and learner are matched, and relational, this "evokes the activity of reflective intelligence in the learner and favours its development" (1979, p.261).

**Goldin and Herscovics' Model of Understanding**

In 1991, in the publication *Towards a Conceptual-Representational Analysis of the Exponential Function*, Goldin and Herscovics discussed aspects of the understanding of the exponential function in terms of four modes of understanding. These modes are: Intuitive Understanding; Procedural Understanding; Mathematical Abstraction; and, Formalisation. The modes and their associated exponential concepts as discussed in the publication (pp.68-70) are as follows:

- Intuitive Understanding is associated with comprehension of such things as the relationship of the exponents 1, 2 and 3 with the length of a line, area of a square and volume of a cube and the understandings obtained by visualising exponential graphs.

- Procedural Understanding involves mathematical processes for evaluating exponential functions by repeated multiplication or by short cuts such as $2^6=2^3\times2^3=8\times8$.

- Mathematical Abstraction is the aspect of understanding "involved whenever the initial definition of exponentiation is expanded to a wider domain" (p.69). This includes laws for multiplying, dividing, and raising powers of exponential functions and the understanding of integral and rational exponents.

- Formalisation refers to the understanding becoming one where indices are seen as a body of content having mathematical form within which sophisticated ideas can be expressed.

From the brief discussion given by the authors about implications of this theory for learning, two points of significance to this current research emerge. The first comes from the observation that, while intuitive understanding is a preconcept or basis for fundamental mathematical ideas (such as surface being a preconcept of area), for more advanced mathematical concepts such as indices

> intuitive understanding is not a necessary prerequisite for meaningful understanding in other modes. Some procedural understandings (e.g., the interpretation of exponentiation as repeated multiplication) are likely to precede some intuitive understandings (e.g., those associated with graphical representation).
>
> (p.69)

This would indicate that growth of understanding in indices is a complex process with development of the various understandings being interwoven over time. This provides ample opportunity for teaching strategies to ignore, or not appropriately address, important aspects in the learning process associated with indices.

The second point is that "cognitive obstacles can occur when form and content do not develop in a synchronised fashion--the content may be difficult to comprehend if it lacks an adequate formal representation; or, more often, the mathematical form becomes devoid of meaning for the learner" (p.70). Given that work with indices involves extensive use of notation and complex structures there is considerable opportunity for the mathematical expressions to be meaningless to students. Again, the challenge for the teacher is one of integration of the various understandings.

## 'CONNECTIONS' AND 'FRAMES'

The two theories considered here ascribe the cause of errors to the tendency for some students to deal with selected aspects of a question while ignoring other important issues. These are the theory of 'connections' as proposed by Shevarev and 'representational structures' or 'frames' as used by Davis.

**The Theory of 'Connections' as proposed by Shevarev**

The research of Shevarev into errors students make in algebra was reviewed in Chapter Two. His examination of errors and the theoretical basis he proposed to explain those errors, were both put under scrutiny. Within the context of this chapter it is worthwhile to highlight briefly the theoretical aspects of his work.

Shevarev's analysis is based on errors made by Year 8 students of a Moscow secondary school when they were answering questions involving indices. Shevarev found that, in many cases, the students understood the rule required to correctly answer the question, but failed to apply it. Instead they adopted an incorrect strategy. He theorised that they adopted these incorrect strategies as a result of having formed 'incorrect connections' when learning the related work.

Shevarev suggested that, in repeatedly solving problems of the same type, students come to use only selected features of an algebraic expression to identify the kind of operation to be performed. This recognition, and its associated operation, he called a "connection". Through connections, students are able to reduce a complex problem to a routine process, they can obtain correct answers without having to consider all aspects of the expression.

Errors can result when the connection contains insufficient information. For instance, in the answering of a set of questions which are all of the type $a^2 \times a^3$ it is sufficient to identify that there are two raised numbers, and to associate the operation of addition with that fact. Important points, such as the base is the same and the operation is multiplication, need not come into the connection for a correct answer to be obtained for each of the questions in the set. However, if those important points are omitted from the connection the child has formed, they can respond with the answer of $a^5$ when later confronted with the question $(a^2)^3$. In this way, an incorrect connection can give correct answers for some questions and incorrect answers for others.

Errors can also result when a connection "contains something which should not have entered into it and which narrows its scope" (p.26). Such is the case when students perceive that questions of the type $\frac{a^9y}{ay^{10}}$ are answered quite differently to those which are of the form $\frac{a^4(x-y)^5}{a(x-y)^{15}}$. Here they have incorrectly adopted, as part of their connection, the idea that the base is a letter not a polynomial

Shevarev explained that correct connections arise in some students more than others because they "clearly recognise all essential features of the data even when this situation does not demand this awareness of them" (p.17). Shevarev said that while students might know the correct rule to answer a question they only recall it "when a special orientation for recalling rules exists" (p.59). This occurs when dealing with a problem which is new, or when working with a new type of data. If such circumstances do not exist, students usually apply a connection they have developed and, if it is an incorrect one, this can lead to error.

Unfortunately, Shevarev examined only a few particular types of errors. It would seem that, if his theory is substantially correct, many of the errors students are making in index work, and in algebra generally, could be attributable to the formation of incorrect connections. This present research needs to determine whether connections, as described by Shevarev, are used by students in a different education system almost fifty years later. If so, the research needs to identify also other possible connections. Shevarev did not interview students to ascertain the thought processes they used in arriving at their answers but hypothesised these from the written evidence. If we are to be confident about these thought processes then it would seem that additional evidence from interviews is essential.

**Representation Structures or 'Frames' as used by Davis**

In his book *Learning Mathematics - The Cognitive Science Approach to Mathematics Education*, Davis (1984) examined some consistent patterns in errors. He saw these errors as having their source in the knowledge representation or 'frames' which each of us build as we learn new things. These representation structures, or frames "have been postulated by many different writers" (Davis 1984, p.125) and the way in which they operate is very much the same as the connections described by Shevarev.

In explaining his ideas, Davis said that each person has a vast collection of frames. The more complex frames are built on the foundation of previous ones. Frames differ widely from one person to another. However, despite this diversity, Davis suggested that in mathematics and especially in elementary arithmetic

> there is a large commonality in what can be perceived, abstracted and generalised, and relatively little room for fundamental variations, so in beginning arithmetic we have one of our best opportunities to find frames that may exist in nearly identical forms in the minds of different people.
>
> (1984, pp.107-108)

It might be that such comments made on beginning arithmetic could equally apply to work with indices, especially considering the findings of Shevarev.

Davis (1984, pp. 108 -110) discussed a number of commonly held frames in mathematics and explained their existence in terms of three general principles of human information processing. They are:

i) that information processing systems will come to make essential discriminations but may not develop discrimination finer than necessary;

ii) that in learning we add new "productions" (a command of the form "If A is true then do B") but we do not delete old ones. Therefore contradictory information and competing alternative procedures come to exist together in our memories;

iii) 'the highest level program must run'. That is, when unable to solve a problem our whole system does not halt but alternative actions are employed. We may use different information, make assumptions or employ methods which were not our first choice. We may call into action lower level programs but will employ the highest level program which will give a result. This result may, of course, be incorrect.

Frames, when correctly applied, give correct answers. Errors arise when, in solving a problem, there are a number of frames available and an incorrect one is retrieved. Davis postulated that, as we develop competing frames in situations which have common features, we also develop *preliminary selection frames* by which we choose between them. An incorrect selection occurs if one of the possible frames has been strongly reinforced and the preliminary selection frame is not strengthened sufficiently (1984, pp.124-125). The difficulties students have with competing frames is pointed out also by Herscovics who commented "how difficult it is ... to overcome existing conceptual frameworks in order to construct new ones and how, even then, the old and the new frames of reference conflict with each other" (1989, p.63).

If we are to look to the possibility of frames offering some explanation of students' errors in index work, it is of value to examine the way in which Davis saw frames operate in a particular mathematical situation. In discussing what he called the primary-grade undifferentiated binary-operation frame, Davis quoted the following dialogue reported by David Page:

> Teacher: How much is four times four?
> Student: Eight.
> Teacher: How much is four plus four.
> Student: Oh! It should be sixteen.
>
> (1984, p.112)

Davis explained this sequence of events in the following way. When learning the addition fact, at the age of five or six, the student forms a frame for the addition of two numbers. In line with the processing principle described in i) above, the addition sign needs to play no part in this frame as the child knows no alternative operations Drill reinforces the frame and, as indicated in ii) above, while another frame is added later to differentiate between operations, the original frame persists and there is a tendency to answer 8 for 4×4. The teacher's questioning causes the student to rethink, differentiate between operations by calling up a *binary-operations selection frame*, and correct the previous error (1984, p.112).

When teaching the topic of indices, this present researcher, no doubt like most teachers of mathematics, has experienced episodes similar to that of David Page as quoted above. In the research by Childers it was apparent that students made considerably more errors in questions of the type $3^m \times 3^n$ than with those of the type $a^3 \times a^4$ (see Table 2.1). When considering this type of problem the following comes to mind as typical of conversations this researcher has had with students.

| | |
|---|---|
| Researcher: | What is the answer to $3^2 \times 3^3$? |
| Student: | It is $9^5$ |
| Researcher: | What is the answer to $a^2 \times a^3$? |
| Student: | Oh! It should be $3^5$. |

From this it would seem that students' work in indices could provide a fertile area for the incorrect application of frames.

**A Comparison of Shevarev's 'Connections' and Davis' 'Frames'**

Davis' 'frames' possess most of the features of Shevarev's 'connections'. They are formed while students are answering questions correctly, they are reinforced by drill, they are resistant to change or eradication, and they "continue to operate in the same precise way, in situations where they are appropriate and in situations where they are not" (Davis 1984, p.126). Both Shevarev and Davis saw it as inevitable that students develop connections or frames and use them quite uncritically in problems in which they see them having application. Students will not analyse questions carefully unless there is some trigger for them to do so, such as, when doing new work, as mentioned by Shevarev, or when prompted to by a teacher, as discussed by Davis.

It is worthwhile pointing out, however, that there is some difference in emphasis between the two stances. This difference is concerned with the way errors result and does have pedagogical implications. Shevarev saw a connection as being an incorrect one if it contains either insufficient information which leads to it being applied in inappropriate circumstances, or irrelevant information which leads to it not being applied when it should be. Davis, on the other hand, did not postulate incorrect frames at all but believed errors occur due to the selection of the wrong frame for the circumstance. This happens when the preliminary selection frame, which differentiates between a frame used for a previously learnt concept and a competing one formed for a new concept, is not developed sufficiently strongly. This results in the first frame being selected in circumstances where the new one should have been used. Davis, in refering to this choice between frames, stated that teachers need to "make sure that students are aware of *both* possible candidates, are aware of the likelihood of incorrect choice, and form a habit of checking to see if they have in fact chosen correctly" (1984, p.124).

An example helps clarify this issue. When first answering questions of the type 4+3 = ? a student need only identify that there are two numbers and, by a counting process, arrive at the answer of 7. The addition sign does not need to come into consideration because there is no competing operation at this stage of the student's experiences. Shevarev would see this recognition of there being two numbers and the associated process of counting as constituting an incorrect connection because the addition sign is relevant in determining when this process should be applied. Omitting the sign from the connection leads to an answer of 7 for 3×4. Davis, however, would see this as a frame which gives a correct answer in appropriate circumstances and would say that when introducing the concept of multiplication teachers need to foster a *preliminary selection frame* which uses the operation sign to distinguish between the new frame formed and the existing one for addition.

Despite the difference in the theories, as outlined above, it is almost certain that both authors would agree that it is desirable to create circumstances whereby all relevant detail is built into the original connection or frame and to also reinforce with the learner the features which make a new connection or frame different from one formed previously. Indeed, Shevarev stated "one must arrange assignments so that a correct connection pertaining to the second kind of problem will be stronger than an incorrect one that might have arisen while solving problems of the first kind" (p.57).

Given the similarities between the two, the implications of Davis' theory are as for Shevarev's, namely, can frames explain errors in index work, and, if so, what are some of the commonly held frames in this area?

## DISCUSSION OF ISSUES ARISING FROM THESE THEORIES

There are notable threads running across these theories and it might be that each has something to contribute in explaining students' understanding of index work. This section considers, in general terms, ways in which the theories may relate.

The extent to which a student has taken account of different aspects of a question, as measured by a SOLO level, may reflect the kind of understanding applied, i.e., consistent 'relational level' responses could indicate 'relational understanding'. That this is not necessarily so is a consequence of the different meaning attached to 'relational' within the separate contexts. 'Relational understanding', as used by Skemp, is concerned with the quality of students' thinking and refers to the kind of understanding which exists when a student knows what to do and why. However, as used in the SOLO Taxonomy, 'relational level' describes a response which has taken account of all given propositions and in which closure has only taken place after these propositions have been processed and any conflict resolved. No judgement is being made about the thinking used. It is entirely possible, therefore, for a student to use instrumental strategies and, by taking account of all propositions, give a 'relational level' response.

Levels of response, as described in the SOLO Taxonomy, may also relate closely to the theories of Shevarev and Davis. In each of the first type of incorrect connections, as described by Shevarev, students identified the operation to use by focusing on one feature or on several unrelated features of the question. This has characteristics of SOLO levels of response where at the unistructural level only one aspect of a question is addressed while at the multistructural level several aspects are correctly responded to. Could it be that applying an incorrect connection or, alternatively, failing to access an appropriate selection frame, is the result of the student responding at a level below that required by the question?

The tendencies for students to develop incorrect connections and to respond at lower SOLO levels would both appear to be promoted by certain classroom practices. Shevarev stated that his first type of incorrect connection arises because teachers and textbooks have grouped questions into sets which can be answered correctly through recognition of only some of the general features of the question. Connections can be formed which generate correct responses to questions of that particular type but result in incorrect responses to a different type of question that happens to possess also those specific features. An explanation of this in terms of the SOLO Taxonomy could be that teachers, in 'keeping early work simple' usually introduce concepts through questions requiring unistructural or multistructural responses. Students are able to answer these questions correctly by focusing on either the one essential feature or a set of isolated steps or routines. Later, when confronted with questions requiring a relational response (i.e., one where the student requires an overview), students performing at a unistructural or multistructural level still focus on only one feature or a limited number of features, and so arrive at an incorrect answer.

Skemp's work highlights the conflict between teaching for genuine or 'relational' understanding and teaching 'rules without reason' or 'instrumental understanding'. However, it seems, if Shevarev and Davis are correct, that even if a teacher aims for relational understanding, students will, in practising their work, develop their own connections or frames which allow them to answer questions efficiently without needing to bring genuine understanding to bear. This does not mean that the understanding is not there but just that it is not brought into action. As mentioned before, Skemp conceded that "even relational mathematicians often use instrumental thinking" (Skemp 1978, p.12).

This need for students to integrate different aspects of understanding is emphasised in the work of Goldin and Herscovics. They stated that when the development of form and content are not synchronised, as seems highly possible in learning index work, the mathematical form loses meaning. This may explain the acceptance of many mathematically obscure answers on the basis that if the meaning of the answer is not known the student accepts the possibility it could be correct.

The issue of consistency of response requires some clarification. Unistructural and multistructural levels, as proposed by Biggs and Collis, are characterised by little felt need for consistency. However, because of the limited number of responses available and the reinforcement of particular ones by drill, frames and connections are likely to lead to consistent responses. The resolution of what might appear to be a conflict lies in the term 'felt', in that students can provide consistent responses without consciously 'feeling' a need to or 'feeling' there is a problem if answers are not consistent.

## RESEARCH THEMES ARISING FROM THESE THEORIES

Consideration of the above theories, and the role they might play in explaining students' thinking when answering index questions, gives rise to the three themes listed below. Numbering of these themes, which relate to the theories of learning, continues on from the five content related themes identified previously.

### Theme 6: Indices in Terms of SOLO

The usefulness of the SOLO Taxonomy, in explaining errors made in questions involving indices, depends upon: being able to classify errors using the taxonomy; establishing there is a consistency in the levels at which particular students respond; and, being able to identify relationship between the quantitative evidence and the thinking demonstrated by students. The three research questions which emerge are:

- *Can the SOLO model provide a framework within which to view students' responses to questions involving indices?*

- *Is there statistical evidence of Solo Type levels of response in students' written answers to questions involving indices?*

- *How does the qualitative data elaborate on the findings arising from the quantitative analysis?*

**Theme 7: Use of Relational and/or Instrumental Understanding**

Relational understanding and instrumental understanding, as postulated by Skemp, provide a useful description of the conceptual and procedural aspects of the thinking involved when solving questions of the type used in this research. Goldin and Herscovics further differentiate conceptual understanding into intuitive understanding, mathematical abstraction and formalisation. Such a fine division is not appropriate in this research where the concern lies with students' use of procedural strategies as opposed to their applying an understanding of underpinning concepts. However, their work does highlight the fact that when the conceptual and procedural dimensions of understanding do not develop in a complementary fashion, mathematical expressions become either difficult to comprehend or meaningless. This raises the question of how students, who have not effectively integrated their conceptual and procedural understandings, view and accept their answers. The two research questions posed for this theme are:

- *To what extent do those students making errors in questions involving indices obtain answers by applying 'instrumental' strategies as opposed to using 'relational' understanding?*

- *In questions involving indices, do students view their responses meaningfully and in what way do they reconcile inconsistencies?*

**Theme 8: Use of Connections or Frames**

In questions involving indices, notation is used extensively, and different questions often have common features. Because of these factors, it is possible that connections or frames will explain certain errors students make in such questions. In order to explore this, two research questions are posed. They are:

- *How do the constructs of 'connections' or 'frames' relate to the way students work with indices?*

- *What common errors, if any, result from the incorrect application of connections or frames ?*

These three themes, numbered 6, 7, and 8, provide the theoretical focus for investigating the understandings students are applying when answering index questions. Their analysis occurs in the final two chapters, after the content related themes have been addressed, and draws on and integrates the results obtained for those content themes.

## CONCLUSION

As discussed in Chapter 3, evidence from the N.S.W. School Certificate Moderator examinations, and from other sources, indicate that students are making numerous errors in questions involving indices. The limited research carried out so far in this area has been largely directed to identifying errors which are being made and to examining error patterns, such as, in the research by Childers where the impact of constant and variable bases and indices was considered. Only the research of Shevarev and Wilson and the hypothesising by Collis and Watson have directly attempted to consider the thinking processes which students use in index questions.

Davis observed that "many - really most - studies focus on what a student writes and largely ignore what that student thinks. Yet what the student thinks is far more fundamental than what that student writes" (1989, p.119). If teachers are to improve the outcomes for students in this area it is insufficient to know what the numerous errors are. What is essential is that more is known about why they are being made. The remainder of this thesis addresses this idea and describes a testing and interview program directed towards establishing the thinking processes used by students when they confront index questions.

## Chapter 5

# THE PILOT STUDY

### Introduction to Chapter

The previous chapters have set the context for the research. They provided details of the position of indices in the curriculum and of the knowledge and skills which the subjects of this research are expected to acquire. A review of the literature raised issues for investigation. An analysis of responses to index questions in the N.S.W. School Certificate Moderator examinations resolved one of these issues and generated research themes which subsumed the others. Finally, several theories of learning, which may help explain students' understandings, and the reasons for their errors, have been discussed and related to themes for investigation.

The research themes provide the framework for examining students' understanding of indices. Before describing the Pilot Study it is valuable to review, and place in perspective, the research themes and issues which have emerged.

The first issue, coming from the literature review in Chapter 2, was concerned with whether the data, gathered from the N.S.W. School Certificate, indicated students from within Australia are making the same sorts of errors as found in research in other countries. This has been answered in the affirmative in Chapter 3. Issues 2 and 3, relating, respectively, to errors in multiplication and division, and to the relationship of indices to bases and coefficients, were picked up by the content related themes. Five such themes emerged in Chapter 3. Issue 4, students use of 'connections', has been taken up in one of the three themes, related to theories of learning, which emerged in Chapter 4. The eight themes, together with a brief overview of each, are:

**Theme 1: Integral Bases Versus Variable Bases.**
What is the thinking that makes such a high proportion of students multiply and divide numerical bases when similar questions with

variable bases pose few problems?

**Theme 2: The Relationship of Indices to Bases and Coefficients**
Why is it that students misunderstand basic conventions of notation and make many errors by incorrectly applying indices to coefficients and bases, ignoring brackets, adding indices when raising to a power, and the like?

**Theme 3: Interpretation of the Fraction**
What is the thinking behind students dividing indices when simplifying algebraic fractions (e.g., $\frac{a^{10}}{a^2} = a^5$)? Is it concerned with the fraction bar or would students making this error have given the same response where the question was written as a division (e.g., $a^{10} \div a^2$)?

**Theme 4: Interpretation of the Radical Sign**
In index questions involving a radical sign, why do students commonly respond by taking the square root of the index? How will they answer questions where the index is not a perfect square?

**Theme 5: Interpretation of the Zero Index**
What do students understand by the zero index given their poor rate of success in School Certificate questions which involve it?

**Theme 6: Indices in Terms of SOLO**
Does the SOLO model provide a useful framework within which to view students' responses to questions involving indices?

**Theme 7: Use of Relational and/ or Instrumental Understanding**
To what extent do students apply an understanding of underlying concepts as against routine procedures? How critically do they view their answers?

**Theme 8: Use of Connections or Frames**
Can 'connections' or 'frames' explain particular errors students are making in index questions?

The five themes from Chapter 3 are content based and provide a structure for setting questions and analysing results. By their nature they divide themselves into two distinct groups. Themes 1 and 2 are of a general nature, and are both quite basic to students' overall understanding of indices. Themes 3, 4 and 5, relating to algebraic fractions, the radical sign and the zero index, respectively, are much more specific. While it is unnecessary to distinguish between the two groups for the purposes of the Pilot Study they are separated in the analysis of the Main Study.

The Pilot Study contains questions on Themes 1, 2, 3 and 5. The interpretation of the radical sign (Theme 4) is not part of the Pilot Study but is incorporated into the Main Study. The NAEP results, reported in Chapter 2, indicate that students have great difficulty in applying correctly a radical sign to an expression with an index. This is confirmed by the fact that only 39% of the Advanced course candidates for the 1986 School Certificate correctly simplified $\sqrt{16x^{16}}$ while 59% gave the answer as $4x^4$. However, given that only one School Certificate question relates to this issue it is left until the Main Study to explore.

Themes 6, 7, and 8 relate to theories, which may explain the reasons for errors, that are addressed in the final chapters. This is done using information drawn from the Main Study analysis of the content related themes in Chapters 7 and 8.

In this chapter the purpose of the Pilot Study, and how it was conducted, is described. The results of the study are then examined.

## PURPOSE AND PLANNING OF THE PILOT STUDY

This section begins by addressing the reasons for carrying out the Pilot Study, then details are given of the design. In the subsection on design, the subjects of the study are described and the way in which the instrument was developed, and administered, is explained.

### Purpose

As indicated in Chapter 3, a central concern is that the errors, identified for students in Year 10, will persist into the senior school. This research needs to determine whether students in Years 11 and 12, for whom the ability to operate with indices is a fundamental skill, face the same problems as those in Year 10. However, students following higher-level mathematics courses in the senior secondary school have not been the subject of previous research. This raises the question of how relevant, to this group, are the content related themes developed from other research. Having students from Year 12 as subjects of a Pilot Study provides an opportunity to explore this issue and, if necessary, add further focus to the research.

The School Certificate data were gathered using a multiple-choice format. However, a free-response format is needed for the Main study. The reason for this is twofold. Firstly, the subjects of this research rarely work with indices in a multiple-choice context. Secondly, in order to examine whether the SOLO Taxonomy can provide a useful framework within which to view responses to questions involving indices, it is preferable for students to have the opportunity to give answers which lie outside a prescribed set. Accordingly, it is necessary to establish that the free-response format is appropriate for examining the themes which were identified from multiple-choice items.

A Pilot Study would establish how significant such concerns are, and provide an opportunity to gain some initial understanding of students' thinking when answering index questions. In relation to these concerns, four specific issues are to be addressed and these are described below.

- Firstly, the Pilot Study would establish if the understanding of indices is still a problem for students when they reach Year 12 (i.e., 18 year olds). It is possible that maturation and an increase in experience, combined with revision of index work, would result in Year 11 and Year 12 students making fewer or different errors than those found for Year 10 students. Should this be the case it would have implications for the directions of this research. The focus would shift from why the errors are being made to what has caused the improvement or change in students' understandings.

- Secondly, the Pilot study would determine if common errors made in the free-response situation are the same as those made in the multiple-choice type questions. When setting multiple-choice questions, examiners may not make available some answers which occur frequently in the free-response situation. It is possible also that the available options may lead students to answers they would not otherwise have contemplated. The value of using free-response questions in the Main Study has been pointed out above. It is necessary to establish that such a format can address issues identified from a multiple-choice format.

- Thirdly, it would provide the opportunity to explore some issues of relevance to the research which were not covered adequately by the School Certificate questions. For instance, with regard to the question of integral versus variable bases, the School Certificate questions did not address the commonly occuring case of multiplying two terms which have a coefficient, a variable base and a constant index (e.g., $4a^6 \times 6a^6$) although the same case was included for division ($12m^6 \div 4m^3$, 84 Int). Other cases to be covered included those where the index is a variable.

- Finally, the answers to the free-response questions could lead to the identification of other issues requiring investigation. This may result in the addition of new questions or the refining of existing questions.

**Design**

The purpose of this thesis is to explore students' understandings of indices. Skills in working with indices are especially important to students following courses which lead to tertiary studies in mathematics. Having Year 12 students, pursuing the higher level courses of mathematics, as subjects of the Pilot Study would provide an opportunity to see whether indices are still a problem at the end of the course.

**The subjects of the Pilot Study** were thirty-eight Year 12 students, twenty-four female and fourteen male, following the 2 Unit Mathematics Course for the N.S.W. Higher School Certificate (HSC) during 1991. Data were collected in June 1991 and students sat for their HSC examination in October 1991.

The students came from a comprehensive State High school in a large New South Wales country town. The HSC results for the school in this course have been consistently near the state average and this group was no exception. Across N.S.W. the scaled mean for the course is 60% with a standard deviation of approximately 12.5%. For the 1991 HSC the total entry for the 2 Unit course was 19 441 students. The mean of the scaled marks for this particular school's candidature was 59.77% and the standard deviation was 12.68%. Given that thirty eight of the school's forty-one candidates were subjects of the study the group was quite representative of the range of the performance state-wide.

**The Instrument for the Pilot Study** consisted of two tests, one was composed of multiple-choice items and the other of free-response items. Each test consisted of thirty questions selected to investigate Themes 1, 2, 3 and 5. Table 5.1, below, lists the items for both tests.

The multiple-choice test was developed using twenty-two questions from the School Certificate, together with another eight designed to explore related issues not covered by the School Certificate. For each multiple-choice item, a corresponding free-response item was developed having, as far as possible, the same structure as its multiple-choice equivalent. Items 12, 13, 14 and 16 posed particular problems in that they involved interpretation of an expression. For the free-response situation it was difficult to describe what was expected of the answer in such a question. Substitutions were used in an attempt to allow students to have similar options for solutions. However, as described later, this was not entirely successful.

**Table 5.1. Pilot Study Items**

| Item | Ques.No. | Multiple-Choice | SC Paper | Free-Response |
|---|---|---|---|---|
| **Integral Versus Variable Bases.** | | | | |
| Item 1 | (20) | $3^3 \times 3^2 = 3^5$ | (84A&I) | $2^3 \times 2^4 = 2^7$ |
| Item 2 | (17) | $2^3 \times 3^2 = 72$ | (82) | $5^2 \times 2^3 = 200$ |
| Item 3 | (26) | $2^a \times 2^b = 2^{a+b}$ | (New) | $3^x \times 3^y = 3^{x+y}$ |
| Item 4 | (21) | $4a^6 \times 6a^3 = 24a^9$ | (New) | $2a^3 \times 3a^2 = 6a^5$ |
| Item 5 | (16) | $2a \times 3a \times 4a = 24a^3$ | (87I) | $3b \times 2b \times 5b = 30b^3$ |
| Item 6 | (1) | $2b^4 \times 3b = 6b^5$ | (New) | $5m^3 \times 2m = 10m^4$ |
| Item 7 | (14) | $a^m \times a^n = a^{m+n}$ | (New) | $b^x \times b^y = b^{x+y}$ |
| Item 8 | (27) | $2^{12} \div 2^2 = 2^{10}$ | (83I) | $5^6 \div 5^2 = 5^4$ or 625 |
| Item 9 | (5) | $12m^6 \div 4m^3 = 3m^3$ | (84I) | $10m^6 \div 2m^2 = 5m^4$ |
| Item 10 | (23) | $8a^{12} \div 2a^8 = 4a^4$ | (New) | $12k^{10} \div 3k^4 = 4k^6$ |
| **The Relationship of Indices to Bases and Coefficients** | | | | |
| Item 11 | (4) | $a^3 + a^3 = 2a^3$ | (84I) | $2m^3 + m^3 = 3m^3$ |
| Item 12 | (19) | $5x^{-1/2} = \frac{5}{\sqrt{x}}$ | (84A) | $x = 9,\ 4x^{1/2} = 12$ |
| Item 13 | (22) | $\frac{1}{2x^3} = \frac{1}{2}\ x^3$ | (85A) | $2 \times 6^{-1} = \frac{1}{3}$ |
| Item 14 | (29) | $3a^{-2} = \frac{3}{a^2}$ | (83A) | $m = 2,\ m^{-2} = 1/4$ |
| Item 15 | (10) | $a = -3,\ 4a^2 = 36$ | (84I) | $n = -2,\ 3n^2 = 12$ |
| Item 16 | (12) | $3ab^2 = 3 \times a \times b \times b$ | (84I) | $m = 5,\ n = 2,\ 3mn^2 = 60$ |
| Item 17 | (24) | $(2^3)^2 = 2^6$ | (81) | $(3^2)^3 = 3^6$ or 729 |
| Item 18 | (2) | $(3m^2)^3 = 27m^6$ | (82) | $(2m^2)^3 = 8m^6$ |
| Item 19 | (9) | $(a^2)^3 = a^6$ | (New) | $(a^4)^5 = a^{20}$ |
| Item 20 | (25) | $(a^b)^c = a^{bc}$ | (New) | $(a^m)^n = a^{mn}$ |
| **The Interpretation of the Fraction Bar in Index Questions** | | | | |
| Item 28 | (8) | $\frac{6a^2}{2ab} = \frac{3a}{b}$ | (86I) | $\frac{10a^2b}{2a} = 5ab$ |
| Item 29 | (30) | $\frac{x^4 \times x^6}{x^2} = x^8$ | (83I) | $\frac{p^{10}}{p^3 \times p^2} = p^5$ |
| Item 30 | (7) | $\frac{(a^4)^4}{a^2} = a^{14}$ | (88A) | $\frac{(y^3)^4}{y^2} = y^{10}$ |
| **Interpretation of the Zero Index** | | | | |
| Item 21 | (28) | $8 \times 3^0 = 8$ | (83I) | $4 \times 5^0 = 4$ |
| Item 22 | (6) | $3a^0 = 3$ | (85I) | $7m^0 = 7$ |
| Item 23 | (11) | $(a^6)^0 = 1$ | (New) | $(m^5)^0 = 1$ |
| Item 24 | (3) | $16^0 + 16^{1/2} = 5$ | (82) | $9^{1/2} + 9^0 = 4$ |

| | | | | |
|---|---|---|---|---|
| Item 25 | (18) | $(3x)^0+2y^0 = 3$ | (87I) | $3p^0+(5q)^0 = 4$ |
| Item 26 | (13) | $1+x+x^0 = 2+x$ | (86I) | $p^0+2+p = 3+p$ |
| Item 27 | (15) | $2a^0+a^{-1} = 2 + \frac{1}{a}$ | (86A) | $3p^0+p^{-1} = 3+\frac{1}{p}$ |

In the table above, items are grouped according to the theme they addressed. However, they were administered in a randomly generated order (the same order for both tests). The number in brackets after the item number is the actual number of the question in the test. After each multiple-choice item, the School Certificate paper from which it came is indicated. Items which have been developed by the researcher are shown as 'New'.

Solutions are given also. For multiple-choice items, only the correct option is provided here but a full list of options, for each question, is provided in Appendix B. For free-response questions the instruction was to find simpler equivalent expressions for each expression given. Some free-response questions obviously had other correct solutions such as 128, instead of $2^7$, for Item 1.

The subjects completed one test immediately after the other. To account for the possibility of one test prompting the other, a random selection of half the subjects were given the multiple-choice test to do first and the other half the free-response. Students were not aware that the two tests were examining similar concepts using similar items. Subjects were not allowed to return to the first test once they had started the second.

## RESULTS OF PILOT STUDY

In this section, an overview of the results of the Pilot Study is provided and comparisons made between the two different test formats. Following this, the way in which the analysis is to be structured is described and the results analysed. The analysis is carried out within the framework of the four themes addressed in the Pilot Study (Themes 1, 2, 3 and 5).

### Overview

Student responses to the items used in the Pilot Study are summarised in Table 5.2 below. Items are ordered from those answered least successfully to those answered most successfully. Correct responses for items are provided in the table, as are the percentage of students providing that response,

**Table 5.2. Pilot Study Items in Order of Percentage Correct (n = 38)**

| MULTIPLE-CHOICE | | | %Correct | FREE-RESPONSE | | | %Correct |
|---|---|---|---|---|---|---|---|
| Item | 8 | $2^{12} \div 2^2 = 2^{10}$ | 29 | Item | 2 | $5^2 \times 2^3 = 200$ | 24 |
| Item | 3 | $2^a \times 2^b = 2^{a+b}$ | 32 | Item | 3 | $3^x \times 3^y = 3^{x+y}$ | 34 |
| Item | 13 | $\frac{1}{2x^3} = \frac{1}{2}x^{-3}$ | 34 | Item | 30 | $\frac{(y^3)^4}{y^2} = y^{10}$ | 34 |
| Item | 2 | $2^3 \times 3^2 = 72$ | 40 | Item | 29 | $\frac{p^{10}}{p^3 \times p^2} = p^5$ | 42 |
| Item | 30 | $\frac{(a^4)^4}{a^2} = a^{14}$ | 42 | Item | 8 | $5^6 \div 5^2 = 5^4$ or 625 | 47 |
| Item | 14 | $3a^{-2} = \frac{3}{a^2}$ | 45 | Item | 1 | $2^3 \times 2^4 = 2^7$ | 50 |
| Item | 18 | $(3m^2)^3 = 27m^6$ | 45 | Item | 18 | $(2m^2)^3 = 8m^6$ | 53 |
| Item | 29 | $\frac{x^4 \times x^6}{x^2} = x^8$ | 47 | Item | 25 | $3p^0 + (5q)^0 = 4$ | 61 |
| Item | 17 | $(2^3)^2 = 2^6$ | 50 | Item | 7 | $b^x \times b^y = b^{x+y}$ | 63 |
| Item | 1 | $3^3 \times 3^2 = 3^5$ | 53 | Item | 13 | $2 \times 6^{-1} = \frac{1}{3}$ | 63 |
| Item | 19 | $(a^2)^3 = a^6$ | 55 | Item | 17 | $(3^2)^3 = 3^6$ or 729 | 63 |
| Item | 7 | $a^m \times a^n = a^{m+n}$ | 53 | Item | 24 | $9^{1/2} + 9^0 = 4$ | 63 |
| Item | 12 | $5x^{-1/2} = \frac{5}{\sqrt{x}}$ | 53 | Item | 11 | $2m^3 + m^3 = 3m^3$ | 68 |
| Item | 11 | $a^3 + a^3 = 2a^3$ | 65 | Item | 12 | $x = 9,\ 4x^{1/2} = 12$ | 68 |
| Item | 20 | $(a^b)^c = a^{bc}$ | 63 | Item | 14 | $m = 2,\ m^{-2} = 1/4$ | 68 |
| Item | 24 | $16^0 + 16^{1/2} = 5$ | 74 | Item | 27 | $3p^0 + p^{-1} = 3 + \frac{1}{p}$ | 68 |
| Item | 22 | $3a^0 = 3$ | 73 | Item | 23 | $(m^5)^0 = 1$ | 71 |
| Item | 25 | $(3x)^0 + 2y^0 = 3$ | 73 | Item | 19 | $(a^4)^5 = a^{20}$ | 74 |
| Item | 27 | $2a^0 + a^{-1} = 2 + \frac{1}{a}$ | 79 | Item | 4 | $2a^3 \times 3a^2 = 6a^5$ | 76 |
| Item | 28 | $\frac{6a^2}{2ab} = \frac{3a}{b}$ | 79 | Item | 22 | $7m^0 = 7$ | 76 |
| Item | 23 | $(a^6)^0 = 1$ | 82 | Item | 5 | $3b \times 2b \times 5b = 30b^3$ | 82 |
| Item | 5 | $2a \times 3a \times 4a = 24a^3$ | 84 | Item | 6 | $5m^3 \times 2m = 10m^4$ | 82 |
| Item | 15 | $a = -3,\ 4a^2 = 36$ | 87 | Item | 20 | $(a^m)^n = a^{mn}$ | 84 |
| Item | 10 | $8a^{12} \div 2a^8 = 4a^4$ | 90 | Item | 26 | $p^0 + 2 + p = 3 + p$ | 84 |
| Item | 4 | $4a^6 \times 6a^3 = 24a^9$ | 92 | Item | 15 | $n = -2,\ 3n^2 = 12$ | 90 |
| Item | 16 | $3ab^2 = 3 \times a \times b \times b$ | 92 | Item | 21 | $4 \times 5^0 = 4$ | 90 |
| Item | 9 | $12m^6 \div 4m^3 = 3m^3$ | 95 | Item | 10 | $12k^{10} \div 3k^4 = 4k^6$ | 95 |
| Item | 21 | $8 \times 3^0 = 8$ | 97 | Item | 9 | $10m^6 \div 2m^2 = 5m^4$ | 97 |
| Item | 26 | $1 + x + x^0 = 2 + x$ | 97 | Item | 28 | $\frac{10a^2b}{2a} = 5ab$ | 97 |
| Item | 6 | $2b^4 \times 3b = 6b^5$ | 100 | Item | 16 | $m = 5,\ n = 2,\ 3mn^2 = 60$ | 100 |

The results demonstrate that understanding of indices is still a problem for students when they reach Year 12. It must be some cause for concern that, in both formats, more than half of the items were answered correctly by less than 75% of students. The error rate is very high for multiplication and division questions involving numerical bases. Questions requiring the simplification of more complex algebraic fractions were done very poorly also. However, students were highly successful in answering multiplication and division questions where the bases are unknowns.

The mean for the thirty-eight subjects on the multiple-choice section of the test was 19.18 and the standard deviation was 4.46. For the free-response questions the mean was 19.68 and the standard deviation was 5.28. A two-tailed Repeated Measures t test revealed that there was no significant difference between the multiple-choice and free-response group means ($t = 0.89$, $df = 37$, $p = 0.3791$). Given that the thirty-eight subjects, as a group, performed in an equivalent manner on the multiple-choice and free-response sections of the test it was decided that a free-response format was appropriate to adopt for the Main Study testing.

**Structure of the Analysis**

In the School Certificate analysis, index questions were identified from two different courses, namely, the Advanced and Intermediate courses. In the senior secondary school (Years 11 and 12), students have a different selection of courses from which to choose. Therefore, because the Year 12 students are not drawn uniformly from the Year 10 courses, a detailed quantitative analysis comparing performance of the two groups is not appropriate. As such, a qualitative analysis which seeks general trends is carried out.

As indicated previously, the purpose of the Pilot Study was to explore four concerns. These were:

- To establish if understanding indices is still a problem for students when they reach Year 12.
- To find if the subjects of this research make the same errors in a free-response situation as were made in the multiple-choice questions.
- To investigate other relevant issues not adequately covered by the School Certificate.
- To identify any other matters requiring investigation using the free-response answers.

With respect to the first concern, the results given in the previous section show clearly that many Year 12 students do have significant problems in answering questions involving indices. More specific comments on this, and on the other three issues, are addressed in the next section through an analysis of questions grouped according to the themes identified in Chapter 3. Five themes were identified for investigation and, as indicated previously, four were the subject of research in the Pilot Study (Interpretation of the Radical Sign being left until the Main Study). Within each, results are given for the School Certificate multiple-choice questions relating to that theme together with results obtained for the same questions by the subjects of the Pilot Study. Results for the equivalent free-response questions are given also. Table 5.3 (below) provides an example of the format used.

**Table 5.3. Sample of Format for Analysis of Questions**

| | | | | |
|---|---|---|---|---|
| **Item 1** | $3^3 \times 3^2$ = ? | | **Free-response:** | $2^3 \times 2^4$ = ? |
| 84 Adv | (A) $3^5$ (68%) | (B) $3^6$ (2%) | (C) $9^5$ (28%) | (D) $9^6$ (2%) |
| 84 Int | (A) $3^5$ (44%) | (B) $3^6$ (3%) | (C) $9^5$ (47%) | (D) $9^6$ (5%) |
| PilotMC | (A) $3^5$ (53%) | (B) $3^6$ (3%) | (C) $9^5$ (40%) | (D) $9^6$ (5%) |
| PilotFR | $2^7$or128 (50%) | $2^{12}$ (3%) | $4^7$ (37%) | $4^{12}$ (3%) |
| Other free-response answers: 208 (3%), 204 (5%) | | | | |

In this format, the first line gives: the item number; the multiple-choice question from the School Certificate; and, to the right, the equivalent free-response question (the free-response question has the same item number as the multiple-choice except where indicated otherwise). Then follows the percentage obtaining each answer in the School Certificate (this particular question was in two different test papers). PilotMC refers to the results for the multiple-choice section of the Pilot Study. PilotFR are the results for the free-response section. Answers for this section were collated and categorised according to their corresponding multiple-choice answer. For example, the answer of $4^7$ for $2^3 \times 2^4$ is equivalent to $9^5$ for $3^3 \times 3^2$ since, in both cases, the bases were multiplied and indices added. The answers are placed in the same order as their corresponding multiple-choice solution. Underlined percentages indicate the correct solution. Errors not corresponding to any options provided by the examiners are then given. This is done for all items relating to the theme and for which there was a School Certificate question.

A discussion is then undertaken, addressing any points relevant to the four issues listed above. Responses to any new items for that theme are considered in the discussion. For ease of reference, where new items are referred to the multiple-choice and free-response questions are provided in parentheses. Multiple-choice is given first, e.g., Item 1 ($3^3 \times 3^2$, $2^3 \times 2^4$).

## Results and Analysis

The analysis is carried out within each of the four themes, namely: Integral Bases Versus Variable Bases; The Relationship of Indices to Bases and Coefficients; Interpretation of the Fraction Bar in Index Questions; and, Interpretation of the Zero Index.

*Theme 1: Integral Bases Versus Variable Bases*

This theme was concerned with errors students make when multiplying and dividing expressions involving indices. Of the ten items relating to the theme, five contained questions examined at the School Certificate.

The Year 12 students fared little better on these items than did the School Certificate candidates and made similar errors. Most errors involved bases, and virtually all students knew which operation to carry out with the indices. The frequency of the various responses are given in Table 5.4.

**Table 5.4. Results for Integral Versus Variable Bases**

| | | | | |
|---|---|---|---|---|
| **Item 1** | **$3^3 \times 3^2$ = ?** | | **Free-response:** | **$2^3 \times 2^4$ = ?** |
| 84 Adv | (A) $3^5$ (68%) | (B) $3^6$ (2%) | (C) $9^5$ (28%) | (D) $9^6$ (2%) |
| 84 Int | (A) $3^5$ (44%) | (B) $3^6$ (3%) | (C) $9^5$ (47%) | (D) $9^6$ (5%) |
| PilotMC | (A) $3^5$ (53%) | (B) $3^6$ (3%) | (C) $9^5$ (40%) | (D) $9^6$ (5%) |
| PilotFR | $2^7$ or 128 (50%) | $2^{12}$ (3%) | $4^7$ (37%) | $4^{12}$ (3%) |
| Other free-response answers: 208 (3%), 204 (5%) | | | | |
| **Item 2** | **$2^3 \times 3^2$ = ?** | | **Free-response:** | **$5^2 \times 2^3$ = ?** |
| 82 | (A) $6^5$ (66%) | (B) $6^6$ (5%) | (C) 36 (1%) | (D) 72 (28%) |
| PilotMC | (A) $6^5$ (55%) | (B) $6^6$ (5%) | (C) 36 (0%) | (D) 72 (40%) |
| PilotFR | $10^5$ (63%) | $10^6$ (5%) | 60 (0%) | 200 (24%) |
| Other free-response answers: 33 (5%), 225 (3%) | | | | |
| **Item 5** | **$2a \times 3a \times 4a$ = ?** | | **Free-response:** | **$3b \times 2b \times 5b$ = ?** |
| 87 Int | (A) $9a$ (1%) | (B) $9a^3$ (2%) | (C) $24a$ (21%) | (D) $24a^3$ (76%) |
| PilotMC | (A) $9a$ (0%) | (B) $9a^3$ (3%) | (C) $24a$ (13%) | (D) $24a^3$ (84%) |
| PilotFR | $10b$ (3%) | $10b^3$ (0%) | $30b$ (11%) | $30b^3$ (82%) |
| Other free-response answers: $30b^2$ (3%), $35b^3$ (3%) | | | | |

| **Item 8** | $2^{12} \div 2^2 = ?$ | | **Free-response:** | $5^6 \div 5^2 = ?$ |
|---|---|---|---|---|
| 83 Int | (A) $1^6$ (10%) | (B) $2^6$ (6%) | (C) $1^{10}$ (40%) | (D) $2^{10}$ (<u>44</u>%) |
| PilotMC | (A) $1^6$ (5%) | (B) $2^6$ (5%) | (C) $1^{10}$ (61%) | (D) $2^{10}$ (<u>29</u>%) |
| PilotFR | $1^3$ (0%) | $5^3$ (0%) | $1^4$ (45%) | $5^4$ or 625 (<u>47</u>%) |

Other free-response answers: 1 (5%), $1^4$-1 (3%)

| **Item 9** | $12m^6 \div 4m^3 = ?$ | | **Free-response:** | $10m^6 \div 2m^2 = ?$ |
|---|---|---|---|---|
| 84 Int | (A) $3m^3$ (<u>85</u>%) | (B) $3m^2$ (12%) | (C) $8m^3$ (3%) | (D) $8m^2$ (0%) |
| PilotMC | (A) $3m^3$ (<u>95</u>%) | (B) $3m^2$ (5%) | (C) $8m^3$ (0%) | (D) $8m^2$ (0%) |
| PilotFR | $5m^4$ (<u>97</u>%) | $5m^3$ (3%) | $8m^4$ (0%) | $8m^3$ (0%) |

Other free-response answers: there were no other answers

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It is clear that students give similar responses to such questions whether they are posed in the free-response or multiple-choice format. Across these questions only 6% of free-response answers fell outside the type of options allowed for in the multiple-choice questions of the School Certificate. There is a remarkable similarity in the proportion choosing each type of option.

Responses in the Pilot Study again demonstrated the strong tendency for students to multiply and divide numerical bases (see responses Item 1 (C), Item 2 (A), Item 8 (C)). That there was an equally strong tendency to do this in the free-response situation shows that the practice is not just a result of students being led by multiple-choice distractors. Results for Item 2 indicate that in multiplication with bases that are different numbers, the proportion of students operating on them is even higher.

Questions involving multiplication or division of two expressions with variable bases and constant indices are common in class work and textbooks. However, the School Certificate contained only two items of this type (Items 5 and 9). Several items were added to investigate a number of variations of this situation. Item 4 ($4a^6 \times 6a^3$, $2a^3 \times 3a^2$) and Item 6 ($2b^4 \times 3b$, $5m^3 \times 2m$) looked at multiplication. In Item 6 one of the terms had an unwritten index of 1 and, as in Item 5 (C), there was some tendency to forget it. Results, however, indicate this may be less of a problem with the older students than it was with the School Certificate candidates. Item 10 ($8a^{12} \div 2a^8$, $12k^{10} \div 3k^4$) covered the situation in division where, unlike Item 9, the second index is not a factor of the first. Overall, these variations did not pose significant problems for students, and it can be seen from the results, in Table 5.2 that the success rate is very high in such questions.

The School Certificate did not contain questions where indices were variables. Item 3 ($2^a \times 2^b$, $3^x \times 3^y$ and Item 7 ($a^m \times a^n$, $b^x \times b^y$) were added to address this situation. Students showed an increased tendency to multiply variable indices as compared to where indices are constants. This was true, especially in free-response cases where 20% of candidates multiplied indices for each of $3^x \times 3^y$ and $b^x \times b^y$. There was still a great tendency to multiply numerical bases and 50% of students responded with a base of 9 for $3^x \times 3^y$.

The high percentage accepting a patently incorrect answer of 1 in Item 8 (C) and giving a corresponding answer in the free-response question is surprising. It shows that answers are placed under very little scrutiny.

There were no division questions with bases which were different constants. This should be addressed in the Main Study. In the one division question (Item 8) where the base was a constant the tendency to divide was even stronger than was the tendency to multiply in Item 1.

To this point, no real distinction has been made between the operations of multiplication and division with regard to Theme 1, Integral Versus Variable Bases. While it is likely that findings will be very similar for the two operations it is possible that there may be features unique to each. Splitting Theme 1 into two sections would cater for this and would facilitate analysis of the significant number of questions which are needed to fully explore this issue in the Main Study.

*Theme 2: The Relationship of Indices to Bases and Coefficients*

This theme was examined by ten items, six of which contained questions from the School Certificate. These are listed below in Table 5.5.

**Table 5.5. Results for Relationship of Indices to Bases and Coefficients**

| **Item 11** | **$a^3 + a^3$ = ?** | | **Free-response: $2m^3 + m^3$ = ?** | |
|---|---|---|---|---|
| 84 Int | (A) $a^6$ (34%) | (B) $a^9$ (15%) | (C) $2a^3$ (<u>27</u>%) | (D) $2a^6$ (25%) |
| PilotMC | (A) $a^6$ (13%) | (B) $a^9$ (3%) | (C) $2a^3$ (<u>65</u>%) | (D) $2a^6$ (20%) |
| PilotFR | $2m^6$ (15%) | $3m^9$ (3%) | $3m^3$ (<u>70</u>%) | $3m^6$ (10%) |
| Other free-response answers: | $2m^9$ (3%) | | | |

| **Item 13** | **$\frac{1}{2x^3}$ = ?** | | **No equivalent free-response question** | |
|---|---|---|---|---|
| 85 Adv | (A) $2x^{1/3}$ (12%) | (B) $2x^{-3}$ (61%) | (C) $\frac{1}{2}x^{1/3}$ (8%) | (D) $\frac{1}{2}x^{-3}$ (<u>19</u>%) |
| PilotMC | (A) $2x^{1/3}$ (13%) | (B) $2x^{-3}$ (40%) | (C) $\frac{1}{2}x^{1/3}$ (13%) | (D) $\frac{1}{2}x^{-3}$ (<u>34</u>%) |
| No equivalent free-response question | | | | |

**Item 14** $3a^{-2} = ?$ **Free-response: $2 \times 6^{-1} = ?$**

**(note this was Item 13 in the free-response)**

| | (A) | (B) | (C) | (D) |
|---|---|---|---|---|
| 83 Adv | (A) $\frac{1}{3a^2}$ (62%) | (B) $\frac{3}{a^2}$ (28%) | (C) $\frac{-1}{3a^2}$ (7%) | (D) $\frac{-3}{a^2}$ (3%) |
| PilotMC | (A) $\frac{1}{3a^2}$ (40%) | (B) $\frac{3}{a^2}$ (45%) | (C) $\frac{-1}{3a^2}$ (16%) | (D) $\frac{-3}{a^2}$ (0%) |
| PilotFR | $\frac{1}{12}$ (8%) | $\frac{1}{3}$ (68%) | $\frac{-1}{12}$ (0%) | $\frac{-1}{3}$ (3%) |

Other free-response answers: -12 (8%), 12 (8%), $\frac{13}{6}$ (3%), 1 (3%)

**Item 16** $3ab^2 = ?$ **Free-response If $m = 5$ and $n = 2$ then $3mn^2 = ?$**

| | (A) | (B) | (C) | (D) |
|---|---|---|---|---|
| 84 Int | (A) $3ab \times 3ab$ (30%) | (B) $3 \times ab \times ab$ (14%) | (C) $3 \times a \times b \times 2$ (2%) | (D) $3 \times a \times b \times b$ (54%) |
| PilotMC | (A) $3ab \times 3ab$ (3%) | (B) $3 \times ab \times ab$ (5%) | (C) $3 \times a \times b \times 2$ (0%) | (D) $3 \times a \times b \times b$ (92%) |
| PilotFR | 900 (0%) | 300 (0%) | 60 (0%) | 60 (100%) |

Other free-response answers: There were no other answers.

**Item 17** $(2^3)^2 = ?$ **Free-response: $(3^2)^3 = ?$**

| | (A) | (B) | (C) | (D) |
|---|---|---|---|---|
| 81 | (A) $2^5$ (19%) | (B) $2^6$ (52%) | (C) $2^9$ (4%) | (D) $4^6$ (24%) |
| PilotMC | (A) $2^5$ (34%) | (B) $2^6$ (50%) | (C) $2^9$ (0%) | (D) $4^6$ (16%) |
| PilotFR | $3^5$ (18%) | $3^6$ or 729 (63%) | $3^8$ (0%) | $27^6$ (0%) |

Other free-response answers: $9^6$ (11%), $9^5$ (3%), $27^5$ (3%)

**Item 18** $(3m^2)^3 = ?$ **Free-response: $(2m^2)^3 = ?$**

| | (A) | (B) | (C) | (D) |
|---|---|---|---|---|
| 82 | (A) $27m^6$ (43%) | (B) $27m^5$ (17%) | (C) $9m^6$ (21%) | (D) $9m^5$ (19%) |
| PilotMC | (A) $27m^6$ (45%) | (B) $27m^5$ (32%) | (C) $9m^6$ (18%) | (D) $9m^5$ (5%) |
| PilotFR | $8m^6$ (53%) | $8m^5$ (11%) | $4m^6$ (3%) | $4m^5$ (0%) |

Other free-response answers: $2m^5$ (13%), $2m^6$ (8%), $6m^5$ (5%), $6m^6$ (5%), $8m$ (3%)

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While the Year 12 students performed consistently better in these questions than did the School Certificate students, their error rate was still very high in all except Item 11 and Item 16. Item 11 is the one item from the School Certificate which involved the collection of like terms. It did pose difficulties to Year 10 students, however, the Year 12 students had much less difficulty with it. Senior students had few problems with substitution in Item 16. Results from the other questions, however, show students still have difficulties with: understanding basic conventions of notation; incorrectly applying indices to coefficients and bases; and, adding indices when raising to a power.

It proved difficult to obtain free-response questions equivalent to the multiple-choice questions in Items 13 and 14. This was because it is not easy, in the free-response situation, to describe what is required in the answer to such a question. Free-response Items 13 and 14 did not readily generate answers which are equivalent to their multiple-choice counterpart. It was possible to compare multiple-choice Item 14 with free-response Item 13, as the structure of these two questions is similar. However, it is unlikely that students would respond in a similar fashion since, as was discussed in Chapter 2, DeVincenzo found significantly more instances of different errors than the same errors for the arithmetic and algebraic forms of index questions. The multiple-choice questions did show that many Year 12 students do apply a negative index to both the coefficient and the base.

In Item 16 there is again the difficulty of how to phrase the free-response question and a substitution was used to obtain a similar structure. Given the difficulties that students had with numerical bases it is surprising that this one substitution question, involving neither fractional nor negative indices, was answered correctly by all students. It should be noted that the correct answer can be obtained using the strategy of one of the incorrect options (see underlined answers in Item 16 PilotFR) but this strategy attracted no Year 12 students in the multiple-choice format so the correct strategy was used almost certainly by all students. The 100% success rate for this question suggests that the action of substituting may trigger a 'back to the definition' approach and so generate a correct result. While the high success rate in the corresponding multiple-choice question does indicate that Year 12 have developed greater facility with this type of question, even without making a substitution, the issue of the effect of substituting is one which should be pursued in the Main Study.

There was evidence in the School Certificate Moderator questions that students were confusing raising to a power with finding the product of two terms having the same base. This was the first incorrect connection described by Shevarev and was evidenced by the adding of indices in a question of the form $(a^2)^3$. The Year 12 students also showed a strong tendency to add indices (see Item 17 (A) and Item 18 (B)). In the free-response situation this error was often made in combination with other errors (see the 'Other free-response answers' for Item 18). Two additional questions of this type were added to see whether students answered differently for constant and variable indices. In the free-response format, for those two questions, 25% of students added indices for $(a^4)^5$ while only 10% added indices for $(a^m)^n$. There were few other errors in these questions and certainly no other consistent error.

*Theme 3: Interpretation of the Fraction Bar in Index Questions*

This theme was examined by three items, all of which contained questions examined at the School Certificate. A summary of student responses is given in Table 5.6.

**Table 5.6. Results for Interpretation of the Fraction Bar**

| | | | | |
|---|---|---|---|---|
| **Item 28** $\frac{6a^2}{2ab} = ?$ | | | **Free-response:** $\frac{10a^2b}{2a} = ?$ | |
| 86 Int | (A) $3ab$ (32%) | (B) $\frac{3a}{b}$ (62%) | (C) $6b$ (3%) | (D) $\frac{6}{b}$ (3%) |
| PilotMC | (A) $3ab$ (21%) | (B) $\frac{3a}{b}$ (79%) | (C) $6b$ (0%) | (D) $\frac{6}{b}$ (0%) |
| PilotFR | No corresp.ans. | $5ab$ (97%) | 10b (0%) | No corresp.ans. |
| Other free-response answers: $\frac{b}{5a}$ (3%) | | | | |
| **Item 29** $\frac{x^4 \times x^6}{x^2} = ?$ | | | **Free-response:** $\frac{p^{10}}{p^3 \times p^2} = ?$ | |
| 83 Int | (A) $x^5$ (46%) | (B) $x^8$ (39%) | (C) $x^{12}$ (14%) | (D) $x^{22}$ (1%) |
| PilotMC | (A) $x^5$ (45%) | (B) $x^8$ (47%) | (C) $x^{12}$ (5%) | (D) $x^{22}$ (3%) |
| PilotFR | $p^2$ (16%) | $p^5$ (42%) | $p^{10/6}$ (0%) | $p^4$ (13%) |
| Other free-response answers: $(p^2)^5$ (3%), $\frac{p^5}{2}$ (3%), $\frac{p^5}{p}$ (5%), $\frac{p^{10}}{p^6}$ (8%), $\frac{p^{10}}{p^5}$ (5%), $p^9$ (3%), no answer (3%) | | | | |
| **Item 30** $\frac{(a^4)^4}{a^2} = ?$ | | | **Free-response:** $\frac{(y^3)^4}{y^2} = ?$ | |
| 88 Adv | (A) $a^4$ (10%) | (B) $a^6$ (21%) | (C) $a^8$ (24%) | (D) $a^{14}$ (45%) |
| PilotMC | (A) $a^4$ (18%) | (B) $a^6$ (18%) | (C) $a^8$ (21%) | (D) $a^{14}$ (42%) |
| PilotFR | $y^{3\frac{1}{2}}$ (0%) | $y^5$ (18%) | $y^6$ (26%) | $y^{10}$ (34%) |
| Other free-response answers: $\frac{4y^7}{y^2}$ (3%), $\frac{y^{12}}{y^2} = \frac{y^6}{y}$ (3%), $\frac{y^7}{y^2}$ (5%), $\frac{y^6}{y}$ (3%), $\frac{4y^{12}}{y^2} = 4y^{12} \div y^2 = 4y^{10}$ (3%), 10 (3%), $y^4$ (3%). | | | | |

Overall, the Year 12 students responded in similar fashion to the School Certificate candidates, although they did fare noticeably better in Item 28, especially in the free-response item. It may be that the free-response question was not well chosen in that the correct answer did not involve a fraction nor did the question allow responses equivalent to several of the multiple-choice options. Items 29 and 30 confirmed the strong tendency to divide indices as though they were coefficients. While this option appeared less popular in the free-response of Item 29 (only 15% choosing $p^2$) it may be that the 8% opting for $\frac{p^{10}}{p^6}$ would have divided if 6 was a factor of 10.

A situation which needs to be considered is where the algebraic fraction has both a coefficient and an index in the numerator and denominator such as for $\frac{8a^6}{4a^3}$. This is addressed in the Main Study to determine if the action of dividing coefficients affects the tendency to divide indices.

*Theme 5: Interpretation of the Zero Index*

There were seven items examining this theme, six of which contained questions from the School Certificate. The results are listed in Table 5.7.

**Table 5.7. Results for Interpretation of the Zero Index**

| | | | | |
|---|---|---|---|---|
| **Item 21** | **$8 \times 3^0$ = ?** | | **Free-response: $4 \times 5^0$ = ?** | |
| 83 Int | (A) 0 (10%) | (B) 1 (3%) | (C) 8 (<u>62</u>%) | (D) 24 (25%) |
| PilotMC | (A) 0 (0%) | (B) 1 (0%) | (C) 8 (<u>97</u>%) | (D) 24 (3%) |
| PilotFR | 0 (0%) | 1 (3%) | 4 (<u>90</u>%) | 20 (3%) |
| Other free-response answers: $20^0$ (3%), $20^3$ (3%) | | | | |
| **Item 22** | **$3a^0$ = ?** | | **Free-response: $7m^0$ = ?** | |
| 85 Int | (A) $3a$ (25%) | (B) 0 (15%) | (C) 1 (35%) | (D) 3 (<u>25</u>%) |
| PilotMC | (A) $3a$ (10%) | (B) 0 (3%) | (C) 1 (11%) | (D) 3 (<u>76</u>%) |
| PilotFR | $7m$ (11%) | 0 (0%) | 1 (8%) | 7 (<u>76</u>%) |
| Other free-response answers: $7m$ = 7 (3%), $7m^0$ (3%). | | | | |
| **Item 24** | **$16^0 + 16^{1/2}$ ?** | | **Free-response: $9^{1/2} + 9^0$ = ?** | |
| 82 | (A) 4 (15%) | (B) 5 (<u>42</u>%) | (C) 8 (27%) | (D) 9 (16%) |
| PilotMC | (A) 4 (3%) | (B) 5 (<u>74</u>%) | (C) 8 (3%) | (D) 9 (21%) |
| PilotFR | 3 (0%) | 4 (<u>63</u>%) | $4\frac{1}{2}$ (0%) | $5\frac{1}{2}$ (3%) |
| Other free-response answers: $9^{1/2}+1$ (11%), $18^{1/2}$ (11%), $10^{1/2}$ (8%), 9 (3%) | | | | |
| **Item 25** | **$(3x)^0 + 2y^0$ = ?** | | **Free-response: $3p^0 + (5q)^0$ = ?** | |
| 87 Int | (A) 0 (26%) | (B) 2 (34%) | (C) 3 (<u>14</u>%) | (D) 5 (27%) |

| | | | | |
|---|---|---|---|---|
| PilotMC | (A) 0 (3%) | (B) 2 (18%) | (C) 3 (76%) | (D) 5 (3%) |
| PilotFR | 0 (0%) | 2 (11%) | 4 (61%) | 8 (3%) |

Other free-response answers: $3p+5q$ (13%), $3p^0$ (3%), 15 (3%), 3 (3%), No answer (3%).

**Item 26** $1 + x + x^0 = ?$ **Free-response:** $p^0 + 2 + p = ?$

| | | | | |
|---|---|---|---|---|
| 86 Int | (A) $1 + x$ (24%) | (B) $1 - 2x$ (30%) | (C) $2 + x$ (41%) | (D) 3 (4%) |
| PilotMC | (A) $1 + x$ (0%) | (B) $1 - 2x$ (3%) | (C) $2 + x$ (97%) | (D) 3 (0%) |
| PilotFR | $2 + p$ (0%) | $2 - 2p$ (8%) | $3 + p$ (84%) | 4 (0%) |

Other free-response answers: $p^2 + 2$ (5%), $4 + p$ (3%), $3p$ (3%)

**Item 27** $2a^0 + a^{-1} = ?$ **Free-response:** $3p^0 + p^{-1} = ?$

| | | | | |
|---|---|---|---|---|
| 86 Adv | (A) $\frac{1}{a}$ (6%) | (B) $2 + \frac{1}{a}$ (68%) | (C) $1 + \frac{1}{a}$ (20%) | (D) $2 - a$ (5%) |
| PilotMC | (A) $\frac{1}{a}$ (3%) | (B) $2 + \frac{1}{a}$ (79%) | (C) $1 + \frac{1}{a}$ (13%) | (D) $2 - a$ (5%) |
| PilotFR | $\frac{1}{p}$ (0%) | $3 + \frac{1}{p}$ (68%) | $1 + \frac{1}{p}$ (8%) | (D) $3 - p$ (0%) |

Other free-response answers: 3 (3% , 3 (3%), $3 + p$ (3%), $4p$ (5%), $3p$ (3%), $3p + p^{-1}$ (3%), $3p - \frac{1}{p}$ (3%), $\frac{3}{p}$ (3%).

---

Of the four themes investigated in the Pilot Study, it is in the area of interpretation of the zero index that there appears to have been a notable improvement in students' rate of success as they have moved from Year 10 to Year 12. Misinterpretations which were occuring in the School Certificate appear largely to have disappeared. The main concern is that there are still students choosing to simply drop the zero index when the base is a variable though they do not do this when the base is a constant. Notably, 13% of students (5 out of 38) gave the answer of $3p+5q$ for $3p^0+(5q)^0$ in a free-response situation. Even so, this percentage was dramatically lower than for the School Certificate results.

There was no evidence that Year 12 students were inclined to interpret the zero index as giving an answer of zero despite this option having been quite popular amongst the candidates for the School Certificate. Altogether there were seven free-response questions, the above six, together with Item 23 ($(a^6)^0$, $(m^5)^0$), in which students had the opportunity to give an interpretation of the zero index. However, of the 266 answers (7 questions × 38 students) only one response indicated that the student had interpreted the zero index as giving zero.

The success rate was marginally lower in the free-response questions where a significant number of candidates did provide responses outside those expected by the examiners. Such responses showed great variety and little consistency.

Even though the Year 12 students made far fewer errors in questions involving the zero index than did the candidates for the School Certificate, the thinking of those students who chose to simply ignore the zero index does need pursuing.

**Summary of Results and Analysis**

The analysis of results shows that understanding indices is still a significant problem for many Year 12 students, despite their having had an additional 18 months of mathematical instruction since the School Certificate examination. This instruction has included time allocated to reteaching and revising indices. In addition, these students have experienced considerable opportunity to work with indices within the the topics of Series, Probability, and Calculus.

Errors observed in the free-response test mirrored closely those of the multiple-choice in terms of the answers given and, to a noticeable extent, their frequency. Certainly, in virtually all cases, popular multiple-choice answers were also popular free-response answers and similarly for unpopular answers. Because of slight differences in the way some questions were asked, an exact figure cannot be put on the proportion of free-response answers falling outside those expected by the examiners but there were approximately ninety. This represents 7.5% (90 out of 1200). Some may have come from misreading questions, and from the complexity of three particular items which between them accounted for more than 30% of the errors falling outside of those allowed for by the examiners. These questions were: $3p^0 + p^{-1} = ?$ , $\frac{p^{10}}{p^3 \times p^2} = ?$ , and $\frac{(y^4)^4}{y^2} = ?$ which had 10, 11 and 9 errors, respectively, outside of the four anticipated responses.

Within the content related themes, a number of matters requiring further investigation have been identified. These include the need: to have division questions where the bases are different constants; to develop more appropriate questions on the relationship of indices to bases; for questions where the algebraic fraction has a coefficient and an index in both the numerator and denominator: and, for questions involving substitutions. Each of these is discussed in Chapter 6 when items for the Main Study are determined.

## CONCLUSION

In general, Year 12 students are making the same errors as were made in the School Certificate, and with similar frequency. This indicates that the transition from Year 10 to Year 12 has not generated any real change in students' understandings of indices or in the thinking they bring to bear in such questions. The importance of researching the themes identified in Chapter 3 is confirmed by these findings.

Given that the subjects as a group performed in an equivalent manner on the multiple-choice and free-response sections of the test it is appropriate to use the free-response format in the Main Study to investigate errors identified from the School Certificate data. Using a free-response format addresses the fact that subjects of this study usually work within such a context. Importantly, it also provides the freedom necessary for students to give answers which can be coded and marked according to degrees of correctness. Such a coding, where partial credit is given for partial success, may lend itself to analysis in terms of SOLO levels of functioning.

There is, however, an important factor which has not been considered to this point, namely, that students in Years 11 and 12 usually answer such questions within some wider mathematical context. For example, many of the questions involving numerical bases arise from substitution in an algebraic expression. It may be that applying indices in a context, or making a substitution, triggers a different response than is given when the question is posed simply as an operation. The Main Study should take account of these possibilities by including questions where the index operation arises in the context of an application or comes from a substitution.

The research has so far been concerned with identifying errors and deriving the themes which give the focus for the research. Now, in light of the findings from the Pilot Study, the task is to develop an instrument that explores more carefully the themes, and which can be used to identify students for interview. The prime purpose of this research is to gain insights into students' understanding when attempting index questions. In addition to the testing program it is therefore important to obtain students' views on indices through interview.

Chapter 6

# THE MAIN STUDY: DESIGN AND OVERVIEW

## Introduction to Chapter

The Pilot Study, reported in Chapter 5, established that students in Year 12 make the same errors, and with similar frequency, as those made in the School Certificate. It also determined that errors made in the multiple-choice situation are similar to those made in the free-response situation. This information confirmed the need to research the five content related themes with senior students, and that employing a free-response format was appropriate. That Main Study which was then undertaken, in order to research these themes, is now described in this chapter.

The chapter is divided into three sections, Context and Design, Overview of Results, and a section titled Conclusion: Setting the Scene for the Later Chapters. In 'Context and Design' the way in which the Main Study was conducted is described. Details are given of the subjects of the research and the instrument used. The experience of the Pilot Study indicated the need to incorporate a number of new items. These are identified and reasons given for their inclusion. The interview and data analysis plan are then described. In 'Overview of Results', general results for the test are considered, the selection of candidates is discussed and some general observations are made about the interviews. To focus the analysis of the content related research themes in Chapters 7 and 8, specific research questions relating to each theme are then posed in the final section. These questions have their basis in the review of the literature, experiences gained from the Pilot Study, and in the overview of test results from the Main Study.

## CONTEXT AND DESIGN

### Context and Subjects

The subjects of the Main Study were forty Year 11 students, twenty-one females and nineteen males, from the same comprehensive State High School used in the Pilot Study. The town in which the school is located is an important educational centre for country N.S.W.. Within this town is a University, two state high schools, four private schools and a large Technical and Further Education College (TAFE). Students come from diverse backgrounds. Their socioeconomic situations cover the extensive range typical of a large town possessing both a rural and an educational focus. Academic success is valued by the school. There is considerable interaction between this school and the School of Education at the town's University. This interaction focuses on both teacher training and research.

As a result of the school's policy, not to interrupt Year 12 students' lesson and study time during their HSC year, Year 11 students were chosen as subjects for the Main Study. Although the test could be administered to all students in one session, interviews needed to be carried out over a more extended period and this would be disruptive for Year 12. Of the forty students, thirty followed the 2 Unit Mathematics Course for the HSC while ten followed the 3 Unit Course. The 3 Unit Course consists of the 2 Unit content together with a significant extension component. It has 50% more teaching time than the 2 Unit Course. For the 1992 HSC the total candidature in N.S.W. was 20 407 for the 2 Unit course (approximately 40% of the HSC candidates) and 9 891 for the 3 Unit Course (approximately 20% of the HSC candidates). The 3 Unit students sit for the same examination as the 2 Unit students but also undertake an additional examination.

The data were collected in November 1991 and students undertook their HSC examination in October 1992. As indicated previously, the scaled mean for 2 Unit candidates on the 2 Unit Course state-wide is 60% with a standard deviation of approximately 12.5%. When state-wide results for the 3 Unit candidates on the 2 Unit section of the course are included, the mean is near 65%. This figure varies slightly from year to year depending on the quality of the 3 Unit candidates. The 2 Unit candidates in this particular group had a mean of 59.21% and a standard deviation of 12.74%. Inclusion of the 3 Unit candidates brought the mean to 64.37%.

These statistics show that, as for the Pilot Study, the subject group was quite representative of the range of performance state-wide. There was some divergence from a normal distribution in that 48% of the 2 Unit students were in the top 40% of the state, while 23% were in the bottom 20%. The seven candidates in the bottom 20% found great difficulty with the course, and accounted for a high proportion of the more obscure incorrect answers.

**Instrumentation**

The Instrument for the Main Study was a test containing fifty free-response items directed at investigating the five content related themes listed in Chapter 5. Items were made up of the free-response questions used in the Pilot Study together with twenty new questions. Of these twenty, four were directed at investigating the theme relating to interpretation of the radical sign (Theme 4), which was not included in the Pilot Study. One of these four questions, Item 36, came from the School Certificate.

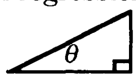
The test was designed for administration in a single one-hour session with students working through it on their own. For ease of reference, the numbering of the items in this thesis is substantially on the basis of how they group within themes. However, to avoid the possibility of prompting by this grouping, the order used in the test was generated randomly.

Until now, no distinction has been made between multiplication and division with regard to Integral Bases Versus Variable Bases. For the Main Study, Theme 1 is separated into Theme 1 (a) - Multiplication and Theme 1 (b) - Division, to allow identification of any issues specific to either, and to assist in managing the large number of questions relating to this theme.

While some reasons for introducing new items relate directly to a particular theme, two general issues arose from the Pilot Study which led to the generation of a number of items. Firstly, given that Years 11 and 12 students are commonly answering questions within a wider mathematical context, there was the need to include questions where the index operation arose from an application or came from a substitution. Secondly, interchanging bases and indices between being constants and unknowns is of vital consideration in Themes 1 and 2. Not all possible situations were addressed in the Pilot Study. To examine how the nature of the base and index impacts on student thinking, the Main Study contains items with: numerical bases; pronumeral bases; and, pronumeral bases and indices.

Under the heading of the theme to which they apply, the new items, and reasons for their inclusion are listed in the tables which follow. The correct solutions are set out to the right-hand side of the questions.

**Table 6.1. New Items for Integral Bases Versus Variable Bases**

| Item No. | Item | Answer |
|---|---|---|
| **(a) Multiplication** | | |
| Item 8 | $a=2,\ b=5,\ b^2a^3=?$ | $b^2a^3=200$ |
| Item 9 |  $2^3$ (side), $2^4$ (base) Area = ? | Area = $2^3 \times 2^4 = 2^7 = 128$ |
| **(b) Division** | | |
| Item 12 | $6^3 \div 2^2 = ?$ | $6^3 \div 2^2 = 54$ |
| Item 13 | $10^4 \div 2^3 = ?$ | $10^4 \div 2^3 = 1250$ |
| Item 18 | $p=4,\ q=2,\ p^3 \div q^2 = ?$ | $p^3 \div q^2 = 16$ |
| Item 19 | Ratio of the Geometric Progression $2^3, 2^7, 2^{11}$ is ? | Ratio = $2^4$ |
| Item 20 |  $10m^6$ (side), $5m^3$ (base) $\tan\theta = ?$ | $\tan\theta = 10m^6 \div 5m^3 = 2m^3$ |
| Item 21 | $m^a \div m^p = ?$ | $m^a \div m^p = m^{a-p}$ |

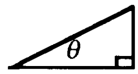
Of these new questions: Item 8 required substitution before simplifying; Item 9 involved an application; Items 12 and 13 covered the division situation where bases were different integers; Item 18 required students to substitute prior to simplifying; Items 19 and 20 involved an application; Item 21 covered the case where the base and index were both variables.

**Table 6.2. New Items for Relationship of Indices to Bases & Coefficients**

| Item No. | Item | Answer |
|---|---|---|
| Item 27 | $(2^3)^3 = ?$ | $(2^3)^3 = 2^9 = 512$ |
| Item 31 | $m = 10,\ (m^2)^3 = ?$ | $(m^2)^3 = 1\,000\,000$ |
| Item 32 | $f(x) = x^3, f(2^4) = ?$ | $f(2^4) = 2^{12}$ |
| Item 33 | $f(x) = x^3,\ f(2a^3) = ?$ | $f(2a^3) = 8a^9$ |
| Item 34 | $(2^3)^{-2} = ?$ | $(2^3)^{-2} = 2^{-6}$ or $\frac{1}{64}$ |
| Item 40 | $5x^{-1/2} = ?$ | $5x^{-1/2} = \frac{5}{\sqrt{x}}$ |

For these new items: Item 27 was added to provide a question of this type which was still relatively straight forward but did not involve squaring; Item 31 required a substitution; Items 32 and 33 used function notation to determine whether such a context affects responses; Items 34 and 40 were included to ascertain whether students would see the same relationships existing in expressions involving simple fractional or negative indices as they see existing where the indices are positive integers.

**Table 6.3. New Items for Interpretation of the Fraction Bar**

| Item No. | Item | Answer |
|---|---|---|
| Item 16 | $\frac{8a^6}{4a^3} = ?$ | $\frac{8a^6}{4a^3} = 2a^3$ |
| Item 20 |  $10m^6$ $\tan\theta = ?$ <br> $5m^3$ | $\tan\theta = 10m^6 \div 5m^3 = 2m^3$ |
| Item 22 | $\frac{x^4 \times x^6}{x^2} = ?$ | $\frac{x^4 \times x^6}{x^2} = x^8$ |

Item 16 was introduced to see if students tend to divide indices in algebraic fractions where both the numerator and denominator have a coefficient and an index, i.e., it is possible that dividing coefficients will trigger a different response for the indices; Item 20 involved an application, and is likely to be of relevance to this theme as well as to Theme 1 (a), given that students are likely to answer it by initially writing a fraction; Item 22 provided a question where students would reach the situation of dividing or subtracting indices without too many opportunities for other errors.

**Table 6.4. New Items for Interpretation of the Radical Sign**

| Item No. | Item | Answer |
|---|---|---|
| Item 35 | $(16a^{16})^{1/2} = ?$ | $(16a^{16})^{1/2} = 4a^8$ |
| Item 36 | $\sqrt{16x^{16}} = ?$ | $\sqrt{16x^{16}} = 4x^8$ |
| Item 37 | $\sqrt{25x^8} = ?$ | $\sqrt{25x^8} = 5x^4$ |
| Item 38 | ABCD is a square of area of $36x^{16}$ sq.units. Length of its side in terms of $x$ = ? | Side = $6x^8$ |

This theme was not incorporated into the Pilot Study because the School Certificate data contained only one question relating to it (Item 36 above). That the theme did require research was confirmed not only by the School Certificate results for that question but also by the results of NAEP testing.

Item 35 was introduced to examine student responses to the commonly used notation of the index of $\frac{1}{2}$; Item 37 was added to see how students would respond when the index was not a perfect square; Item 38 introduced the question through an application, and also provided a context where students needed to find a square root but where they were not directly confronted by the notation.

No new items were added for Theme 5, relating to the interpretation of the zero index. With regard to this theme, the Pilot Study had not raised additional concerns and it was felt that the existing items would adequately explore the issues.

**Interview Plan**

The purpose of this thesis is to gain insights into students' understanding of indices and, in particular, to seek to explain commonly occuring errors. The testing confirmed that subjects of this research were making common errors identified in the literature. To gain further insight into the reasons for the errors, students were selected for interview on the basis of making consistent errors in items relating to one or more of the research themes.

In the interviews, students were asked to explain the thinking they used when answering the test items. The interviewer had a copy of each student's test with answers marked and comments made. The interviewees had their unmarked, original, test to refer to. To allow students to describe accurately their thinking, interviews were completed soon after the test, within a maximum time span of one week.

An interview schedule based on the error analysis approach of Newman, as described by Clements (1980, pp.9-10), was used. Since the items were answered prior to the interviews, this modified schedule was simpler than that used by Newman. The first three questions in the schedule were strictly adhered to These are:

| | *Process* | *Question Asked (for example)* |
|---|---|---|
| 1. | Reading | Please read the question to me. |
| 2. | Encoding | Please read your answer to me. |
| 3. | Process | How did you arrive at that answer? |

The remaining questions were used, as appropriate, to explore issues of interest and probe for more information.

| | | |
|---|---|---|
| 4. | Consolidation | What does the answer mean? |
| 5. | Verification | Is there any way you can check and make sure your answer is right? |
| 6. | Similarity | Can you make up a question like that one? Which of these questions is like that one? |

Considerable use was made of the last strategy, whereby students were asked to compare certain questions and to compare responses across items. This relates especially to the comparison of the algebraic and numerical forms of particular types of questions. Occasionally, additional questions which required a student response were needed to be written down.

Interviews were conducted in a quiet and comfortable room with the interviewee and interviewer side-by-side at a large desk. Space was available for both interviewee and interviewer to place papers and write when necessary. Interviews were taped using a recorder situated on the desk at a distance. Efforts were made to put the interviewee at ease as much as possible. A period of one hour was allowed for each interview. Each session commenced with an explanation by the interviewer along the following lines:

> The purpose of this interview is to help me understand your thinking when answering questions on indices. Your answers to some questions we talk about will be correct while others will be incorrect. I will not say which as I am really only concerned with your thinking, not with how successful you have been. I will ask you to read each question aloud and then read out your answer. We will then discuss how you arrived at that answer.

From the interviews implicit, and occasionally explicit, information on students' knowledge and understanding of the index laws was forthcoming. However, in light of Shevarev's assertion that students know the rules but act contrary to them, it was important to ascertain, as accurately as possible, the understanding of each student. To ensure such information is available, a short list of questions concerning the laws was asked. The questions were:

1. What do you understand four cubed to mean?
2. What is the meaning of $a$ squared times $a$ cubed ?

3. If I write down $a$ squared in brackets and then cube it, like that, what does it mean?
4. What do you understand by $p$ to the power of nought?
5. What do you understand by $a$ to the power of one half?

Again the interviewer needed to seek clarification and more information where appropriate. Students were not asked these questions until the end of the interview to avoid the possibility of prompting students' responses when the test items were being discussed.

**Data Analysis Plan**

Quantitative analysis was used in conjunction with the qualitative data to support research findings in the following chapters. The quantitative component of the analysis made important contributions to the discussion by providing information on the relative difficulty of items, the frequency of errors, how well items fit the model and the reliability of results. This quantitative component of the analysis was carried out using a computer software package developed recently by Adams and Khoo (1993). The package is called *Quest, The Interactive Test Analysis System.* This package:

> offers a comprehensive test and questionnaire analysis environment by providing a data analyst with access to the most recent developments in Rasch measurement theory, as well as a range of traditional analysis procedures. ... Quest can be used to construct and validate variables based on both dichotomous and polychotomous observations.
>
> (Adams and Khoo 1993, p.1)

The statistical model used is of the latent trait type and, more specifically, belongs to the Rasch family of measurement models. When the data are consistent with this model "item parameters can be estimated independently of the calibrating sample and person parameters can be freed from the difficulties of the items taken" (Rasch 1960, 1977, quoted in Master 1982, p.149). Initially the data are treated as dichotomous, however, when Theme 6 was considered, the data were coded polychotomously. The *Quest* software provides for such an analysis using a Partial Credit form of the Rasch model.

The model provides estimates of item *difficulty* and respondent *ability* on a logit scale. Where the data are scored polychotomously, the model measures the difficulty of steps in levels of performance within items. These measures of step difficulties within items are reported in the form of 'thresholds'. For dichotomously scored data, each item has one threshold and that threshold describes the ability level required for a respondent to have a 50% chance of answering the item correctly. For polychotomous data the thresholds describe the ability level required by a student to have a 50% chance of responding at either the next level or a higher level still. For example, if items were coded for four levels, Levels 1, 2, 3 and 4, there would be three threshold values. The lowest threshold value would provide a measure of the ability level needed to have a 50% chance of responding at a level higher than Level 1.

The software was used to support the quantitative analysis of the data at a number of points throughout this research. First, it was used in the following section to give an overview of the results of the testing. It provided information on; the effectiveness of the test in assessing the skills; the difficulty of items; and, the fit of the items to the model. Second, in Chapters 7 and 8, it was used to provide information about the content related themes. The analysis of each theme in these chapters has a quantitative component in the form of test results and a qualitative component in the form of interview data. Finally, it was used in Chapter 9 when Theme 6 was investigated. This theme raised the question of whether the SOLO model could be used to provide a framework within which to view students' responses to questions involving indices.

To explore the issue of SOLO levels requires more than the simple classification of answers as right or wrong since students may provide answers which, although incorrect, have features indicating a certain level of response. Because of this, it is of importance to this research that the software, as indicated previously, is able to score and analyse partial credit items whereby credit is awarded according to some identifiable levels of performance. If responses can be classified in terms of levels used in the SOLO Taxonomy then the software makes it possible to identify whether there are students consistently operating at a given level across items.

**Quest output** is displayed in a number of convenient formats. The distribution of item difficulties and case estimates over the variable are illustrated graphically in the form of a *variable map* (see Figure 6.1, provided later in this chapter, as a typical example). The numbers on the extreme left of the map represent the logit scale on which both items and cases are calibrated. Figures on the right hand side of the map are the items plotted according to their difficulty. For example, taking Item 13 in Figure 6.1, the Threshold Value of 2.3 on the logit scale is the measure of the ability level required by a respondent to have a 50% chance of answering that question correctly, similarly for other items. The XXXs represent the distribution of case estimates over the logit scale. Each X represents one particular student. A student has a 50% chance of being successful in any item located on the same position as they are on the logit scale and a better than 50% chance for any items below.

The fit of the items to the model, using the infit mean square of each item, is shown using an "item fit map" (see Figure 6.2, provided later in this chapter, as a typical example). In the map, the right hand vertical dotted line indicates an item mean square 30% above its expected value and the left hand one indicates an item mean square 30% below. These measures are used as guidelines for determining the adequacy of the fit of items to the model (Adams and Khoo 1993, p.23).

The software provides several other useful statistics. Measures for the Infit Mean Square and Infit t values show the degree to which the set of items scale, i.e., fit the model. When a set is perfectly scaled and compatible with the model the expected value of the Infit Mean Square is approximately one and the expected value of the t-values is approximately zero.

Another measure provided by the package is the Reliability of Estimate which is the proportion of the observed estimate variance that is considered true. It gives a measure of the likelihood of similar results being obtained should this instrument be repeated with a similar sample.

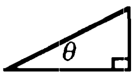
## OVERVIEW OF RESULTS

### General Results

Detailed analysis of student responses occur later within the content related themes. However, in order to provide an overview, brief details of the results for the test are now provided.

The following table (Table 6.5) lists the questions and answers for each item together with the number and percentage correct. The "Threshold", as described previously, is the parameter estimate of the difficulty of each item. The higher its value, the more difficult the item. Item 4 was answered correctly by all students and could not be incorporated in the estimates. The easiest questions, aside from Item 4, were Items 5 and 6 with a threshold value of -2.69. The equally most difficult questions were Items 13 and 40 with threshold a value of 2.15.

## Table 6.5. Results for all Items

| Item | Answer | No.Corr. | Threshold |
|---|---|---|---|
| [1] $2^3 \times 2^4$ | $2^7$ or 128 | 23 (58%) | 0.44 |
| [2] $5^2 \times 2^3$ | 200 | 16 (40%) | 1.38 |
| [3] $3^x \times 3^y$ | $3^{x+y}$ | 16 (40%) | 1.38 |
| [4] $2a^3 \times 3a^2$ | $6a^5$ | 40(100%) | Perfect Score |
| [5] $3b \times 2b \times 5b$ | $30b^3$ | 38 (95%) | -2.69 |
| [6] $5m^3 \times 2m$ | $10m^4$ | 38 (95%) | -2.69 |
| [7] $b^x \times b^y$ | $b^{x+y}$ | 29 (73%) | -0.42 |
| [8] $a$=2, $b$=5, $b^2a^3$= | 200 | 34 (85%) | -1.35 |
| [9] $2^3$  Area= $2^4$ | $2^7$or128 | 21 (53%) | 0.70 |
| [10] $m$=5,$n$=2,$3mn^2$= | 60 | 37 (93%) | -2.23 |
| [11] $5^6 \div 5^2$ | $5^4$ or 625 | 21 (53%) | 0.70 |
| [12] $6^3 \div 2^2$ | 54 | 13 (33%) | 1.83 |
| [13] $10^4 \div 2^3$ | 1250 | 11 (28%) | 2.15 |
| [14] $10m^6 \div 2m^2$ | $5m^4$ | 37 (93%) | -2.23 |
| [15] $12k^{10} \div 3k^4$ | $4k^6$ | 36 (90%) | -1.88 |
| [16] $\frac{8a^6}{4a^3}$ | $2a^3$ | 30 (75%) | -0.58 |
| [17] $\frac{10a^2b}{2a}$ | $5ab$ | 36 (90%) | -1.88 |
| [18] $p$=4, $q$=2, $p^3 \div q^2$= | 16 | 23 (58%) | 0.44 |
| [19] Ratio of $2^3, 2^7, 2^{11}$ | $2^4$ | 27 (68%) | -0.12 |
| [20]  $\theta$ $10m^6$ $\tan\theta$=? $5m^3$ | $2m^3$ | 28 (70%) | -0.27 |
| [21] $m^a \div m^p$ | $m^{a-p}$ | 29 (73%) | -0.42 |
| [22] $\frac{x^4 \times x^6}{x^2}$ | $x^8$ | 22 (55%) | 0.57 |
| [23] $\frac{p^{10}}{p^3 \times p^2}$ | $p^5$ | 24 (60%) | 0.30 |
| [24] $\frac{(y^3)^4}{y^2}$ | $y^{10}$ | 24 (60%) | 0.30 |
| [25] $2m^3+m^3$ | $3m^3$ | 31 (78%) | -0.75 |
| [26] $(3^2)^3$ | $3^6$ or 729 | 26 (65%) | 0.02 |
| [27] $(2^3)^3$ | $2^9$ or 512 | 28 (70%) | -0.27 |
| [28] $(2m^2)^3$ | $8m^6$ | 22 (55%) | 0.57 |
| [29] $(a^4)^5$ | $a^{20}$ | 34 (85%) | -1.35 |
| [30] $(a^m)^n$ | $a^{mn}$ | 32 (80%) | -0.94 |

| | | | |
|---|---|---|---|
| [31] $m$=10, $(m^2)^3$= | 1000000 | 27 (68%) | -0.12 |
| [32] $f(x)=x^3$, $f(2^4)$= | $2^{12}$ | 21 (53%) | 0.70 |
| [33] $f(x)=x^3$, $f(2a^3)$= | $8a^9$ | 19 (48%) | 0.97 |
| [34] $(2^3)^{-2}$ | $2^{-6}$ or $\frac{1}{64}$ | 27 (68%) | -0.12 |
| [35] $(16a^{16})^{1/2}$ | $4a^8$ | 16 (40%) | 1.38 |
| [36] $\sqrt{16x^{16}}$ | $4x^8$ | 12 (30%) | 1.99 |
| [37] $\sqrt{25x^8}$ | $5x^4$ | 21 (53%) | 0.70 |
| [38] Area=$36x^{16}$,Side? | $6x^8$ | 21 (53%) | 0.70 |
| [39] $x$=9, $4x^{1/2}$ | 12 | 28 (70%) | -0.27 |
| [40] $5x^{-1/2}$ | $\frac{5}{\sqrt{x}}$ | 11 (28%) | 2.15 |
| [41] $2\times6^{-1}$ | $\frac{1}{3}$ | 21 (53%) | 0.70 |
| [42] $m$=2, $m^{-2}$= | $\frac{1}{4}$ | 20 (50%) | 0.84 |
| [43] $9^{1/2}+9^0$ | 4 | 22 (55%) | 0.57 |
| [44] $4\times5^0$ | 4 | 35 (88%) | -1.60 |
| [45] $7m^0$ | 7 | 28 (70%) | -0.27 |
| [46] $(m^5)^0$ | 1 | 22 (55%) | 0.57 |
| [47] $3p^0+(5q)^0$ | 4 | 18 (45%) | 1.11 |
| [48] $p^0+2+p$ | $3+p$ | 31 (78%) | -0.75 |
| [49]$3p^0+p^{-1}$without indices | $3+\frac{1}{p}$ | 19 (48%) | 0.97 |
| [50] $n$=-2, $3n^2$= | 12 | 32 (80%) | -0.94 |

The test achieved an appropriate spread of results. The mean score was 31.92 (63.8%) and the standard deviation was 10.12. The 50 items scaled well with an Infit Mean of 0.99 and an Infit t value of -0.04. This indicates the instrument measured consistently the abilities of the students in the skills being tested in that students who answered correctly the harder items, successfully answered the easier items also, and visa versa. A Reliability of Estimate value of 0.87 indicates that were the items repeated with a similar sample there is a high expectation that similar results would be obtained.

As for the School Certificate and the Pilot Study, the success rate in many items was not high. It is evident that students again had significant difficulties with questions involving numerical bases. Questions requiring straight-forward multiplication or division of expressions containing variable bases were well represented amongst the least difficult items.

Figure 6.1 provides a graphical representation of the distribution of item difficulties and case estimates. The figures on the left represent the logit scale on which both items and cases are calibrated. When used in conjunction with Table 6.5 this figure is a convenient reference for observing how difficult students found the various types of questions.

| Logit Scale | Case Estimates | Items/Steps | | | | |
|---|---|---|---|---|---|---|
| | X | | | | | |
| | | | | | | |
| | | | | | | |
| | | | | | | |
| | X | | | | | |
| | | | | | | |
| 3.0 | | | | | | |
| | | | | | | |
| | | | | | | |
| | XX | | | | | |
| | XXX | | | | | |
| | | | | | | |
| | | | | | | |
| | X | 13 | 40 | | | |
| 2.0 | | 36 | | | | |
| | | 12 | | | | |
| | X | | | | | |
| | | | | | | |
| | XXX | | | | | |
| | | 2 | 3 | 35 | | |
| | XX | | | | | |
| | X | 47 | | | | |
| 1.0 | XXX | 33 | 49 | | | |
| | XX | 42 | | | | |
| | XXX | 9 | 11 | 32 | 37 | 38 |
| | | 22 | 28 | 43 | 46 | |
| | X | 1 | 18 | | | |
| | | 23 | 24 | | | |
| | X | | | | | |
| 0.0 | XXX | 26 | | | | |
| | XX | 19 | 31 | 34 | | |
| | XX | | | | | |
| | X | 20 | 27 | 39 | 45 | |
| | X | 7 | 21 | | | |
| | X | 16 | | | | |
| | | 25 | 48 | | | |
| | | | | | | |
| -1.0 | | 30 | 50 | | | |
| | | | | | | |
| | XX | | | | | |
| | | 8 | 29 | | | |
| | X | | | | | |
| | | 44 | | | | |
| | X | | | | | |
| | | 15 | 17 | | | |
| -2.0 | | | | | | |
| | | | | | | |
| | | 10 | 14 | | | |
| | | | | | | |
| | | | | | | |
| | | | | | | |
| | | 5 | 6 | | | |
| | | | | | | |
| -3.0 | | | | | | |

**Figure 6.1. A Graphical Representation of the Threshold Values**

Figure 6.1 demonstrates how difficult many of the students found these questions. Fourteen of the forty students (those at 0 or below on the logit scale) had a less than 50% chance of getting a correct answer in each of the 25 most difficult items. At the other end of the ability group there were seven students with a better than 50% chance of correctly answering the most difficult item. The threshold values provide an important reference for the degree of difficulty of various items in the analysis of the themes in Chapters 7 and 8.

The Item Fit Map in Figure 6.2 shows the infit mean square for each item. The eight items falling outside the vertical lines had a mean square which was more than 30% above or below their expected value. This means there is a tendency for some students to answer correctly these items but respond incorrectly to easier items, and visa versa. Items with such a reversal pattern are listed in Table 6.6.

**Table 6.6. Items Associated with Substantial Reversal Patterns**

| Item | Answer | No.Corr. | Threshold |
|---|---|---|---|
| [2] $5^2 \times 2^3$ | 200 | 16 (40%) | 1.50 |
| [3] $3^x \times 3^y$ | $3^{x+y}$ | 16 (40%) | 1.35 |
| [5] $3b \times 2b \times 5b$ | $30b^3$ | 38 (95%) | -2.69 |
| [13] $10^4 \div 2^3$ | 1250 | 11 (28%) | 2.30 |
| [19] Ratio of $2^3, 2^7, 2^{11}$ | $2^4$ | 27 (68%) | -0.16 |
| [41] $2 \times 6^{-1}$ | $\frac{1}{3}$ | 21 (53%) | 0.67 |
| [44] $4 \times 5^0$ | 4 | 35 (88%) | -1.40 |
| [48] $p^0+2+p$ | $3+p$ | 31 (78%) | -0.80 |

These items vary considerably in their level of difficulty. It is notable that three of the items involved the multiplication or division of numerical expressions with constant bases raised to an index. These particular items were all among the seven most difficult items for the test. The thinking used by students in such questions is the subject of investigation in Theme 1. While Item 5 falls into this group it should be noted that this was based upon only two students who gave an incorrect response.

Although four of the items derived from similar questions used in the School Certificate, and the remainder were not unlike other School Certificate questions, a subjective judgement would be that, apart from Item

5, these questions are not the kind asked commonly in textbooks or by teachers. They lie outside students' more common experiences of working with algebraic expressions or coming to such operations through a substitution. This could mean success in these questions depends less on learning experiences and ability in algebra than it does in other questions.

```
     Infit Mn Sq.0.63       0.71      0.83       1.00      1.20      1.40      1.60
---------------------+---------+---------+---------+---------+---------+---------+--------
         1 item 1                    .             |*             .
         2 item 2                    .             |              . *
         3 item 3                  * .             |              .
         5 item 5                *   .             |              .
         6 item 6                    .             |     *        .
         7 item 7                    .             |         *    .
         8 item 8                    .             |          *   .
         9 item 9                    .             |  *           .
        10 item 10                   .             | *            .
        11 item 11                   .         *   |              .
        12 item 12                   .             |          *   .
        13 item 13              *    .             |              .
        14 item 14                   .             |              *
        15 item 15                   .             |          *   .
        16 item 16                   .             *              .
        17 item 17                   .             |        *     .
        18 item 18                   .             |   *          .
        19 item 19                   .             |              .   *
        20 item 20                   .         *   |              .
        21 item 21                   .             |    *         .
        22 item 22                   .          *  |              .
        23 item 23                   .          *  |              .
        24 item 24                   .   *         |              .
        25 item 25                   .         *   |              .
        26 item 26                   .        *    |              .
        27 item 27                   .             |   *          .
        28 item 28                   .             |          *   .
        29 item 29                   .     *       |              .
        30 item 30                   *             |              .
        31 item 31                   .             |              *
        32 item 32                   .             |  *           .
        33 item 33                   .             |*             .
        34 item 34                   .*            |              .
        35 item 35                   .             |     *        .
        36 item 36                   .       *     |              .
        37 item 37                   .            *|              .
        38 item 38                   .             |            * .
        39 item 39                   .       *     |              .
        40 item 40                   .        *    |              .
        41 item 41         *         .             |              .
        42 item 42                   .      *      |              .
        43 item 43                   .      *      |              .
        44 item 44              *    .             |              .
        45 item 45                   .       *     |              .
        46 item 46                   . *           |              .
        47 item 47                   .         *   |              .
        48 item 48                 * .             |              .
        49 item 49                   .    *        |              .
        50 item 50                   .             |      *       .
```

**Figure 6.2** **Item Fit Map**

The test has achieved an appropriate set of results. These can be used in the quantitative analysis of the themes, and also to form the basis for selecting students from whom qualitative evidence is needed. The identification of candidates for interview is now discussed and some general observations on the interviews are made.

**Identification of candidates for interview**

Ten students, 25% of the subjects of the study, were selected for interview on the basis of having made consistent errors on items relating to one or more of the research themes. The ten selected included four girls and six boys. All were 2 Unit candidates. The 3 Unit students, in general, performed better than the 2 Unit students, however, they still made many systematic errors. Throughout the discussion which follows, pseudonyms are used to maintain confidentiality of students' identities.

It might be expected that students selected would be from those performing less well in the course. This was far from the case. The average HSC mark for the ten students was 63 whereas the state-wide average was 60. The students selected, together with their HSC percentile band in brackets were: Kirsty (90-100%), Philip and Christine (81-90%), Bianca, Malcolm and Linda (71-80%), Innes (41-50%), Mark (21-30%) and Tim and Brian (<20%). Kirsty also achieved the highest results of the school's 2 Unit candidates both for the school-based assessment of the course and the HSC examination.

Testing and interviewing was completed within a one-week period. While detailed analysis of the interviews is to occur in Chapters 7 and 8, some general observations can be made. In reading the questions and answers it was notable that, despite the number of errors being made, not one of those interviewed needed any correction of the terminology they used. Each response was completely acceptable to the interviewer and gave an appearance that the question was understood. As would be expected, where an index was a 2 or a 3 it was often referred to as "squaring" or "cubing". The most commonly used term for an index was 'power'. Despite the fact that there were times when students ignored the effect of brackets in a question, they always included them when reading the question by saying either "in brackets" or "all to the power of".

As indicated previously, at the end of each interview five questions were posed to explore students' knowledge and understanding of index laws. In many instances students responded to these questions with a simple and unexplained answer, such as, "$a^5$" for "what is the meaning of $a$ squared times $a$ cubed?" and "1" for "what do you understand by $p$ to the power of nought?". In such cases they were then asked to explain their answer.

All students interviewed, except one, gave a correct definition of each of the five concepts (Tim incorrectly defined $(a^2)^3$ as equalling $a^5$ and $p^0$ as equalling 0). Also, all students showed that they had full understanding of the first two concepts and could explain why $a^2 \times a^3 = a^5$. Half of the students could explain why $(a^2)^3$ was equal to $a^6$. However, not one student, without direct guidance, could explain the reason for a zero index giving an answer of 1 or an index of a half meaning square root.

These results mirror those of Wilson's, reported on in Chapter 2. He found the subjects of his research "have a relational understanding of the concept of positive exponent" but that "it is with the use of the various exponential properties that instrumental understanding replaces relational understanding" (Wilson, 1985, p.229). Wilson found also "the understanding of the zero exponent for all levels of students (in the study) ... to be at the instrumental level" (p.230). A closer analysis of student interview responses in Chapters 7 and 8 provides insight into the role instrumental strategies play in the errors students make

## CONCLUSION: SETTING THE SCENE FOR THE LATER CHAPTERS

In light of the content based research themes identified through the literature in Chapter 2, the analysis of Chapter 3, the experience of the Pilot Study, and, the overview of the test results for the Main Study, more specific questions are now posed.

As indicated previously, within each theme, items have been set to cover a range of relevant situations identified through the literature, the School Certificate analysis and the Pilot Study. Some ways in which items may differ are the nature of the elements, the notation used, whether the question is in the form of an application, is a substitution needed, and so on. Research questions, concerned with the influence such features have on the success of students, are addressed for each theme using a quantitative analysis of the students' written responses.

A major thrust of this research is to obtain information on the understandings which students are applying in questions involving indices. Accordingly, the final research question(s) within each theme are directed at examining this understanding using a qualitative analysis based on data gathered through interview.

The research questions for each theme, in the sequence in which they are considered in the following two chapters, are listed below.

**Theme 1: Integral Bases Versus Variable Bases.**

An initial consideration is the identification of differences or similarities associated with the operations of multiplication and division. Other central issues to this theme are the nature of the bases both with regard to their being constants or unknowns and, if constants, whether they are the same or different. The impact of students approaching these questions through a substitution or an application needs to be considered given that, for senior students, the questions usually arise in such a context. Finally, the question of the understandings students are applying, given the high rate of errors in such questions, is addressed. From these issues, the following research questions are posed.

**Written Response Analysis**

- *What is the effect on students' responses of having the base as a constant as opposed to having the base as an unknown?*
- *With multiplication or division of constant bases, what is the effect of the bases being the same constant as opposed to their being different?*
- *What is the effect on students' responses of having a substitution as the first step in such questions?*
- *What is the effect on students' responses of having the operation arise out of an application?*

**Interview Analysis**

- *What do interviews reveal about the understandings students apply when multiplying or dividing in index questions involving numerical bases?*

**Theme 2: The Relationship of Indices to Bases and Coefficients**

The Main Study confirmed that the subjects of this study have few problems in applying an integral index correctly, where the base is an unknown. Nor do they have difficulty operating with any coefficients attached to such terms. Item 10 ($m$=5,$n$=2,$3mn^2$=?) was correctly answered by 93% of students while Item 8 ($a$=2, $b$=5, $b^2a^3$=?) had 85% of students responding correctly. (These two questions are considered under Theme 1 where they have relevance to the multiplication of numerical bases). However, error rates for items requiring the raising to a power of an expression involving an index were still high. Only 55% answered $(2m^2)^3$ (Item 28) correctly and 65% responded correctly to $(3^2)^3$ (Item 26). Another issue for consideration is that students often come to these questions through a substitution.

With regard to understanding, there are two distinct issues. The first is the question of understandings that students are applying when they raise to a power a term involving an index. The second is the fact that, while few problems are found in relating an integral index to its base and coefficient, there is great difficulty in correctly interpreting this relationship in expressions such as $5x^{-1/2}$ where the index is not integral. The research questions addressing these two issues, together with those considering the written responses, are listed below.

**Written Response Analysis**

- *When a term containing an index is raised to a power, what is the effect on student responses of that term having: a constant base; an unknown base; or, an unknown base with a coefficient?*
- *What is the effect on responses of having a substitution as the first step in questions of the kind examined in this theme?*

**Interview Analysis**

- *What do interviews reveal about the understandings students use when applying an integral power to a term involving an index as that term varies between having: a constant base; an unknown base; or, an unknown base with a coefficient?*
- *What do interviews reveal about the understandings students have of how a non-integral index relates to its base and the coefficient in expressions such as $5x^{-1/2}$.*

**Theme 3: Interpretation of the Fraction Bar in Index Questions**

Issues identified as significant were: the effect of using different notations to indicate division; the effect of having an unwritten index of 1 as opposed to having a written index; whether dividing coefficients would change the way in which students operated with indices; and, the understandings that students apply, given the high rate of errors in items involving a fraction bar. From these issues the following research questions are posed.

**Written Response Analysis**

- *What is the effect on student responses of using different notations, namely the fraction bar or the division sign?*

- *How will students simplify a fraction containing an unwritten 1 as an index given that, in numerical fractions, they are used to having positive integers greater than 1 for both numerator and denominator?*

- *Will having coefficients, containing a common factor, make students less inclined to divide indices given they will have divided coefficients?*

**Interview Analysis**

- *What do interviews reveal about the thinking students use when they divide indices while simplifying algebraic fractions?*

**Theme 4: Interpretation of the Radical Sign**

Issues identified as significant were: the nature of the index within the radical sign; the effect of using different notations to indicate square root; and, the understandings that students are applying in such items. There are a number of aspects to consider with regard to the understanding being applied. Not only is there the thinking used when taking the square root of an expression involving an index but there is also the impact of the way in which students are confronted with the need to take the square root. The item may indicate square root through the notation of a radical sign or an index of $\frac{1}{2}$. Alternatively, it may be the structure of the question, not notation, which requires that a square root be taken. Such is the case where the side length of a square needs to found, given the area. The following questions were posed with regard to this theme.

**Written Response Analysis**

- *When taking the square root of an expression involving an index what is the effect on responses of this index being, or not being, a perfect square?*

- *What is the effect on student responses of the use of different notations, namely the index of $\frac{1}{2}$ or the radical sign?*

**Interview Analysis**

- *What do interviews reveal about the understandings applied by students when, in finding the square root of an expression, they take the square root of an index?*

- *What do interviews reveal about how the use of different notations, namely the index of $\frac{1}{2}$ or the radical sign, affects students' thinking?*

- *What do interviews reveal about the understandings students apply where the finding of a square root is evoked not through the use of an index of $\frac{1}{2}$ or the radical sign but through the context of the question?*

**Theme 5: Interpretation of the Zero Index**

The Pilot Study showed that Year 12 students had much greater success in applying the zero index than did the School Certificate students. However, some students were still choosing to simply drop the zero index when the base was a variable though they did not do this when it was a constant. Also, students were treating an expression such as $3p^0$ in the same way as $(3p)^0$. Accordingly the following questions are posed for this theme.

**Written Response Analysis**

- *In questions involving the zero index, what is the effect of having the base as a constant as opposed to having the base as an unknown?*
- *In questions involving the zero index, what is the effect of parentheses on students' responses?*

**Interview Analysis**

- *What do interviews reveal about students' understanding of the zero index?*

In conclusion, the test results confirmed that subjects of this study make frequent errors in questions involving indices and that these errors are the same kinds as identified in the literature. The test scaled well overall, though a small number of items with numerical bases had Infit Mean Square values that indicated they were associated with reversal patterns of response. The fact that students found these items difficult does mean that students' thinking when operating with numerical bases is worthy of further examination.

The five content based research themes are now investigated in the following two chapters using both quantitative and qualitative evidence.