1. Introduction

Multilateral comparisons of prices, real output and productivity across countries at present are of interest not only to economists and statisticians, but also to various international organisations such as the World Bank (WB), the United Nations (UN), European Union (EU), Food and Agricultural Organisation (FAO), Economic and Social Commission Asia and the Pacific (ESCAP), and Organisation for Economic Cooperation and Development (OECD). It has been a major issue that official exchange rates are not suitable for international comparisons of real per capita income, as the exchange rates do not reflect the relative purchasing power of different currencies. This issue led Gilbert and Kravis (1954) to the measurement of purchasing power parity (PPP) for the OECD countries and since then PPPs have been estimated and used for real income comparison.

Much of the work to date on PPP computation has been under the auspice of the International Comparison Programme (ICP) of the United Nation. The ICP aims at obtaining internationally comparable per capita and total gross domestic product (GDP) for different countries. It also considers the purchasing power differences of the national currencies thus providing insights in to the price levels of the different countries. Based on ICP studies, if real GDP aggregates of different countries have to be compared, calculation of PPPs is needed to convert national GDP figures into common currency unit. The purchasing power parities are essentially currency conversion rates that reflect differential price levels between countries. Once they are available GDP can be compared in real rather than in nominal terms across countries. Most of the ICP methods for international comparison to date depend on the concepts of PPP and international prices.

Two of the most widely used methods for deriving PPPs through index number formulae are the Elteto-Koves-Szulc (EKS) method and the Geary-Khamis (GK) method. Both methods produce index numbers which are transitive and base-invariant making them suitable for multilateral comparisons. All the ICP work to date has been based on the GK procedure though OECD and EU have opted to produce and disseminate results based on both of these methods. The GK method has often been criticised as a heuristic procedure with less economic theoretic foundation. Caves, Christensen and Diewert (1982) highlight this issue and then advocate the use of alternative EKS-base methods. These methods are very elegant and possess a number

of desirable properties. This brings some econometricians to look further into the use of econometric models following the EKS-base methods.

Many research studies on index numbers for multilateral comparisons to date have focused on the problem of finding aggregation methods with desirable and acceptable economic theoretic properties. A small group have been concerned with the use of econometric approach to the construction of the Theil-Tornqvist (TT) index and other indices. Introduction of stochastic approach to index-number theory (e.g. Clement and Izan 1987) and to index construction (e.g. Selvanathan 1989) gave rise to a new development in the context of multilateral comparison (e.g. Prasada Rao and Selvanathan 1991, 1992a, 1992b, 1994). A major attraction of stochastic approach is that it has the potential to provide standard errors for the price index estimates (e.g. Frisch 1936) reflecting relative price variability.

Recent developments concerning the stochastic approach to index number for multilateral comparisons are that of Prasada Rao and Selvanathan (1992a, 1992b, 1994). Prasada Rao and Selvanathan (1992a) demonstrate the scope and power of the stochastic approach to the comparison of relative levels of prices and quantities across different countries. They generalised the EKS-method for applied to a set of non-transitive indices computed using the binary TT index. The TT index is known to be exact and superlative (e.g. Diewert 1976). It possesses a number of useful statistical and economic properties necessary for multilateral comparisons (e.g. Caves, Christensen and Diewert 1982). Caves, Christensen and Diewert (1982) proposed the use of TT method in conjunction with the EKS method, thus the resulting index is referred to as the CCD index.

Following Clements and Izan (1987) work on the stochastic approach to the TT index, and the subsequent work of Selvanathan (1989), Prasada Rao and Selvanathan (1992a) provided the generalised CCD (GCCD) index for multilateral comparison. Econometric models underlying the known TT index are used to derived these indices. The CCD and GCCD indices are transitive, base-invariant and possess the usual least squares properties which are best, linear and unbiased (BLU). This class of multilateral index numbers provides a viable alternative aggregation procedure for multilateral comparisons as well as estimates of PPP with attractive properties.

1.1 Statement of the Research Problem

In view of the importance attached to the CCD multilateral index number, it could be suggested that direct computation of the general price relatives be accomplished following the stochastic approach used in Prasada Rao and Selvanathan (1994). With the use of econometric models, estimates of price relatives that are equivalent to the purchasing power parities could then be derived. These estimates will possess desirable properties necessary for multilateral comparisons.

In the context of multilateral index estimation, problems seem to arise in the use of price levels across countries. Price and quantity comparisons over space or across countries may somehow exhibit geographic ordering. Price levels of one country may be positively or negatively related with the price levels of the neighbouring country or to distant country. According to the first law of geography (Tobler 1967, p.275) 'everything is related to everything else, but near things are more related than distant things.' This phenomenon brings about the problem of spatial autocorrelation (e.g. Cliff and Ord 1972, 1973, 1981, Miron 1984, Upton and Fingleton 1985, Goodchild 1986, Griffith 1987, Odland 1988, Aten 1994). When price and quantity levels of goods and services across countries display interdependence over space, spatial autocorrelation will therefore exist. This would make the disturbance term of any econometric model involving price relatives across countries spatially autocorrelated. Consequently, this would mean that the procedures used to date in the work of Prasada and Selvanathan, are not efficient as they do not take into the spatial autocorrelation present in the efficient estimation of multilateral indices.

Aten (1994) tested for the presence of spatial autocorrelation in relative prices of goods and services across countries based on the 1985 ICP benchmark data. Aten made used three different measures of spatial autocorrelation, namely, the contiguity measure, the great circle distance between capital cities and the volume of trade between countries. It was found that relative prices are highly similar among physically close countries and spatial autocorrelation exists among estimated residuals in the cross-country demand analysis. With the presence of spatial autocorrelation in the price relatives, the econometric models used in Prasada Rao and Selvanathan (1992a) to derive CCD multilateral index numbers will lead to inefficient estimates of the multilateral indices and the PPPs associated with these indices.

The present study explores the possibilities of allowing for spatial autocorrelation in the disturbance structure specification of the CCD multilateral index model used by Prasada

Rao and Selvanathan (1992a) thus leading to more efficient estimates of the PPPs involved.

1.2 The Research Objectives

The main objective of the study is to derive estimates of purchasing power parity following the stochastic approach to multilateral index number used in Prasada Rao and Selvanathan (1992a).

The specific objectives are:

> to test the presence of spatial autocorrelation in the CCD multilateral model used for PPP estimation;

> to specify a model for the CCD multilateral index estimation that would allow spatial autocorrelation in the disturbance term;

> to construct matrices for spatial correlation based on contiguity, geographical distance and volume of trade;

> to provide an efficient estimation procedure for the derived stochastic regression model that allow for spatial autocorrelation; and,

> to examine the sensitivity of the PPP estimators to the type of spatial autocorrelation present (e.g. Aten 1994), namely,

-contiguity
-geographical distance
-volume of trade.

1.3 Outline of the Chapters

A brief review of some concepts and methods that are currently being used for multilateral comparison of prices is found in Chapter 2. In particular, this chapter defines the concept of Purchasing Power Parity (PPP) and reviews the stochastic formulation of the binary Theil-Tornqvist (TT) index. Chapter 3 details the stochastic approach developed by Clements and Izan (1987) and Selvanathan (1989), and utilised

by Prasada Rao and Selvanathan (1992a) in the construction of multilateral indices. It includes the Caves, Christensen and Diewert (CCD)'s formulation of the TT index, and the stochastic specification of the CCD multilateral index number.

The concept of spatial autocorrelation that will be used in conjunction with the regression model associated with the CCD multilateral index formula is discussed in Chapter 4. The various tools and techniques necessary to determine the presence of spatial autocorrelation among the disturbance terms of the CCD model are explained. These include several proximity measures and the Moran's *I* statistic. In Chapter 5, a generalised CCD multilateral index accounting for spatial autocorrelation is specified and estimated. The explanation of the various data used and the empirical application of the theoretical results in the study are also summarised. Finally, Chapter 6 provides a summary of what has been achieved, and looks at some implications for future research.

2. Review of Relevant Studies

This chapter provides a very brief review of some concepts and methods that are currently being used for multilateral comparisons of prices. The development of index numbers for multilateral comparisons is discussed first. Under this section, the concept of Purchasing Power Parity (PPP) and the properties and approaches to multilateral index number computations are presented. Particular attention is given to the Binary Theil-Tornqvist (TT) index number in Section 2.2. Lastly, Section 2.3 provides an introduction to the stochastic approach procedure as applied to the TT index number.

2.1 Index Number for Multilateral Comparisons

Much of the standard literature on index numbers concerns binary comparisons over time. But these methods do not possess some of the very basic requirements such as transitivity and base-invariance. This has led to considerable research, over the last three decades, on methods specially designed for spatial, in particular, on cross-country comparisons. The notation used in this and the subsequent chapters, is described below.

Suppose we are interested in price level comparisons across M countries. Let p_{ij} and q_{ij} (i=1,2,..,N and j=1,2,3,..,M) represent the price and quantity of ith commodity in jth country respectively. Clearly, N is the total number commodities for which the price and quantity informations are collected. Typically, M is greater than two. The following assumptions will be used throughout the study:

i) $p_{ij} > 0$ for all i and j ;
ii) $q_{ij} \geq 0$ for all i and j ; and,
iii) for each i, $q_{ij} > 0$ for at least one j
or for each j, $q_{ij} > 0$ for at least one i.

Given the price (p_{ij}) and quantity information (q_{ij}), we want to derive price and quantity comparisons between all pairs of countries. For a pair of countries (j,k), I_{kj} (j,k=1,2,3,...,M) represent the required multilateral index number for country j with country k as the base. Index numbers from the multilateral comparisons, involving M countries, can be arranged and presented in the form of a matrix given below:

$$I = \left[(I_{kj})\right] = \begin{bmatrix} I_{11} & I_{12} & I_{13} & \cdots & I_{1M} \\ I_{21} & I_{22} & I_{23} & \cdots & I_{2M} \\ . & . & & . & \\ . & . & & . & \\ I_{M1} & I_{M2} & I_{M3} & \cdots & I_{MM} \end{bmatrix} \qquad (2.1)$$

Various terms used in the study along with the conceptual framework underlying various methods for international comparisons are discussed below.

2.1.1 Purchasing Power Parity

Geary (1958) advocated the computation of purchasing power parities (PPP) directly from the price and quantity information and used the resulting parities to define appropriate index numbers for different pairs of countries. This framework was further refined by Khamis (1972) where he described the many interesting mathematical and statistical properties of this method. The next section will describe the initial framework for this approach.

Let Π_j represent the general price level observed in a country, then the price index, I_{kj}, for country j with country k as a base can be defined as,

$$I_{kj} = \frac{\Pi_j}{\Pi_k} \qquad (2.2)$$

Since Π_j's are interpreted as 'general price levels' in the different countries, equation (2.1) can be equivalently expressed in terms of 'purchasing powers' or implicit 'exchange rates', represented by R_j (j=1,2,3,...,M) defined as reciprocals of the general price levels. The above index numbers can then be written as

$$\Pi_{lj} = \frac{R_l}{R_j} \qquad (2.3)$$

where $R_j = 1/\Pi_j$ and represents the amount of numeraire currency equivalent in purchasing power to one unit of currency in the jth country. Thus Π_{lj} provides a measure

of the parity between currencies of country j and l. Geary (1958) defines the purchasing power currency j, denoted by PPP_j, as the reciprocal of R_j. Thus

$$PPP_j = 1/R_j \ ;$$

and

$$I_{kj} = \frac{PPP_j}{PPP_k}. \tag{2.4}$$

PPP_j in (2.4) shows the number of currency units of jth country currency equivalent in purchasing power to one unit of a reference or base country currency. Thus, if PPP of Australian dollar (A$) in terms of US ($) dollar is A$1.236=US$1.00, then this means that 1.236 Australian dollars have the same purchasing power as 1 US dollar. Thus the PPP can be used as a conversion factor that can be used in converting per capita income into US dollars in the place of the widely used exchanged rates.

From equation (2.3), it is evident that if PPP_j's can be determined, then the necessary price index numbers can be computed. Basically, the problem then is of finding method either to compute PPPs for currencies of different countries or to derive the indices in the matrix (2.1) from the observed price and quantity data. The indices/formulae used for this purpose are supposed to satisfy a number of properties. These are discussed in Section 2.1.2.

2.1.2 Properties of Multilateral Index Numbers

In the context of multilateral comparison of prices, an index number defined by equation (2.2) must satisfy two crucial properties. Ideally, we would like these comparisons to be internally consistent (i.e. satisfy the property of transitivity), and invariant to the order in which the countries are listed. These two requirements are very essential and necessary in any international comparison exercise.

Transitivity

Internal consistency requires that a direct comparison between any two countries j and k derived using a given formula should be the same as a possible indirect comparison between country j and k through a third country l. Thus this property requires that for any countries j, k and l,

$$I_{kj} = I_{kl} \times I_{lj} . \qquad (2.5)$$

This property guarantees that there is a unique index for a comparison between two countries, whether such comparison was made directly or indirectly through a link country. Thus, the condition of transitivity is an operational constraint that preserves internal consistency. When the index is in an additive form, transitivity implies

$$\Pi_{kj} = \Pi_{kl} + \Pi_{lj}, \qquad (2.6)$$

where $\Pi_{kj} = log\ I_{kj}$.

Another important result that is required for this study, comes from Prasada Rao and Selvanathan (1994, p.146) which reveals that any multilateral index number formula that is internally consistent, contains only M pieces of information. This result may be stated as follows:

Result

An index number formula Πkj, for all k and j, satisfies transitivity in log-change form if and only if there exist M real numbers $\pi_1, \pi_2,...,\pi_M$ such that $\Pi kj = \pi_j - \pi_k$, where π_j refers to *log PPPj* defined in equation (2.4).

This result will be useful in obtaining a transitive multilateral index number formula from the regression models considered in Chapter 3.

Base-invariance or country symmetry

This property holds if the country selected as base is simply a numeraire. All the inter-country comparisons must be independent of the country chosen to be the numeraire country. Base invariance guarantees that all countries are treated symmetrically in the comparisons exercise in the sense that the order in which countries are listed or introduced into computing has no effect on the results. Thus, the property of base invariance requires a multilateral index number formula to treat price and quantity data from all the M countries symmetrically. This property is also necessary in the development of our regression model in Chapter 3.

2.1.3 Two Approaches to Multilateral Index Computation

Two approaches/methodologies are available in the international comparison literature to derive index number formulae satisfying transitivity and base invariance. These two index number formulae are at present widely used by international bodies. These are the Elteto-Koves-Szulc (EKS) and the Geary-Khamis (GK) methods.

Elteto-Koves-Szulc (EKS) Method

Elteto and Koves (1964) and Szulc (1964) advocated the EKS procedure. The EKS method provides an ingenious technique that preserves characteristicity and generates transitive multilateral index numbers from a system of binary index numbers based on Fishers' index. Preserving characteristicity really means a transitive system that deviates the least from a preferred binary comparison. Their preference for the Fisher index is mainly because of the nice properties of this index number formula. Fisher index is an ideal index, exact, as well as superlative and it satisfies tests like the time/country reversal test and the factor reversal test (Diewert 1992). The EKS method defines a new index formula for a comparison between two countries k and j as

$$EKS_{kj} = \prod_{l=1}^{M} \left[F_{kl} \cdot F_{lj} \right]^{1/M} \qquad (2.7)$$

where F_{kl} -Fisher price index for country l relative to country k; and,

F_{lj} -Fisher price index for country j relative to country l.

$F_{kl} \cdot F_{lj}$ provides an *indirect* comparison between country k and j through a bridge country l. Given this equation (2.7) is a simple unweighted geometric mean of all the indirect comparisons between country k and j through link country l (l=1,2,3,...M), utilizing the Fisher index number.

The EKS index formula given in (2.7) results in multilateral comparisons that are transitive and deviate the least from the Fisher binary indices. This implies that the EKS formula leads to multilateral index numbers which are as close as feasible to the binary comparisons resulting from the use of a non-transitive binary index, but in addition satisfies transitivity. Measurement of closeness in the EKS index is based on the sum of squares of deviation in logs, i.e., $\sum_{k=1}^{M} \sum_{j=1}^{M} (\ln EKS_{kj} - \ln F_{kj})^2$, subject to the restriction that (2.7) satisfies transitivity. Thus, the EKS method preserves the characteristicity of

binary comparisons while obtaining transitive index numbers (e.g. Dreschler 1973). This aspect of the EKS index is also supported by Caves, Christensen and Diewert (1982).

Geary-Khamis(GK) Method

An alternative way of obtaining internally consistent (transitive) multilateral indices is the Geary-Khamis (GK) method. This method developed by Geary (1958) and Khamis (1972) is the most widely used aggregation or index number formula for international comparisons (e.g. Kravis, et al 1982, Prasada Rao and Selvanathan 1994). The GK method is presently being used by the UN and other international organisations in various international input, output, real GDP, and productivity comparisons. Geary (1958) provides the framework underlying this method based on the idea of the PPP of a currencyand international average prices of commodities, mentioned in Subsection 2.1.1. This framework was further refined and some analytical results concerning the existence of solution, etc. were provided in Khamis (1972). The paper described many interesting mathematical and statistical properties of the GK method.

The GK procedure defines PPP's directly using the observed price and quantity information. Moreover, this method introduces another concept known as the 'international average price' of a commodity, usually denoted by P_i , i=1,2,...,N. These international average prices are expressed in a common currency unit usually referred to as a 'numeraire currency'. In multilateral comparison exercise, GK method suggests that the observed price and quantity data on N commodities from M countries be used to determine:

i) M purchasing power parities, PPP_1, PPP_2, ... ,PPP_M, ; and,
ii) N commodity international average prices, P_1, P_2, ..., P_N .

Geary (1958) and Khamis (1972) suggest an intuitively obvious set of interelated equations to define the PPP's and the international prices. These equations formed the so called Geary-Khamis system. The system is defined below.

Suppose the PPP_j's are known, then the international price of the ith commodity (i=1,2,3,...,N) is defined as

$$P_i = \frac{\sum_{j=1}^{M} \left[p_{ij} \cdot q_{ij} / PPP_j \right]}{\sum_{j=1}^{M} q_{ij}}, \quad i = 1,2,\ldots,N \tag{2.8}$$

The denominator on the right hand side of the equation (2.8) is simply the total quantity of the ith commodity in all the M countries involved in the comparisons. The numerator is the total value of the ith commodity over all the countries, after each country's value, $p_{ij} \cdot q_{ij}$ is converted into a common currency unit using respective PPPs. The equation defined above is then repeated for all the commodities.

On the other hand, the PPP_j's are determined in the GK method using the following equation below. For country j, PPP_j is obtained by

$$PPP_j = \frac{\sum_{i=1}^{N} p_{ij} \cdot q_{ij}}{\sum_{i=1}^{N} P_i \, q_{ij}}, \quad j = 1,2,\ldots,M \tag{2.9}$$

The numerator of the PPP equation above is the total value of all the quantities in country j, expressed in the currency units of country j, while the denominator represents the value of country j's commodity bundle valued at international average prices expressed in some selected reference country (common) currency units. Thus the ratio in (2.9) provides a PPP for country j's currency.

The GK system therefore consists of the $(M+N)$ equations, (2.8) and (2.9), in the unknown entities PPP_j's (j=1,2,3,..,M) and P_i's (i=1,2,3,...,N). Further, these equations are interdependent in that values of PPP_j's depend upon international prices, P_i's, which in turn depend upon the unknown purchasing power parities, PPP_j's. Moreover, a simple method using iterative procedure is designed by Khamis (1972) to solve the GK system for the unknown PPP_j's and P_i's. The procedure is found to be useful and could lead to unique solutions (see Prasada Rao and Selvanathan 1994, pp. 167-71).

All the ICP work to date has been based on the GK procedure. Despite the many useful properties of the GK system, the GK procedure has often been criticised as a heuristic procedure with no real economic theoretic foundation. Caves, Christensen and Diewert

(1982) support this view and advocate the use of EKS-base methods. Caves, et al. preferred an EKS-type multilateral index numbers based on other binary systems with attractive theoretical and statistical properties, like the TT binary index. They demonstrate the feasibility to generate transitive indices from a set of non-transitive TT multilateral comparisons using the EKS technique. Such an application of the EKS procedure on the TT binary index, results in the CCD index which is discussed in the next chapter. It is deemed necessary to look first at the binary TT index and how stochastic approach could be applied to derive the binary TT index before dealing with the CCD index.

2.2 The Theil-Tornqvist Binary Index

This index is proposed in Tornqvist (1936) and has been introduced and discussed in the literature (e.g. Fisher 1922, Tornqvist 1936, Theil 1965, Kloek and Theil 1965). Just like the other well known index number formula, the TT index is designed to measure changes in prices or quantities over two time periods and is, therefore essentially a binary index. The TT index is known to be *exact* and *superlative* (Diewert 1976, Caves, Christensen and Diewert 1982). It has been shown that this index possessed a number of useful statistical and econometric properties necessary for multilateral comparison (Eichorn and Voeller 1983). This formula can be applied for comparisons over regions, firms or space.

Suppose we let (p_j,q_j) and (p_k,q_k) denote two pairs of price and quantity vectors of dimension N corresponding to two countries j and k (j,k=1,2,...,M; k=base country). The TT binary index in multiplicative form is given by

$$\text{(multiplicative form)} \qquad I_{kj}^{TT} = \prod_{i-1}^{N} \left[\frac{p_{ij}}{p_{ik}} \right]^{\overline{w}_{ikj}}, \tag{2.10}$$

where $\overline{w}_{ikj} = \frac{1}{2}(w_{ik} + w_{ij})$; $w_{ij} = p_{ij} q_{ij} \Big/ \sum_{l=1}^{N} p_{lj} q_{lj}$ is the value share of the ith commodity in the jth country. Expressing equation (2.10) in logarithmic form, we obtain its additive form given by

$$\text{(additive form)} \qquad \Pi_{kj}^{TT} = \sum_{i=1}^{N} \overline{w}_{ikj} Dp_{ikj} \tag{2.11}$$

where, $\Pi_{kj}^{TT} = ln(I_{kj}^{TT})$, and $Dp_{ikj} = ln(p_{ij}/p_{ik})$ is the log-change in the price of the ith commodity of country j relative o country k. Furthermore, Dp_{ikj} represents the percentage change (inflation rate) in the price of the ith commodity. So the TT index in its additive form provides an indication of the overall inflation rate or increase in prices. The weights used in defining the Theil-Tornqvist index are simple arithmetic means of the budget shares of good i for countries j and k.

The TT binary index is a well-known index number for binary comparisons of prices, real output and productivity (e.g. Kloek and Theil 1965, Diewert 1976, Dreschler 1973, Caves, Christensen and Diewert 1982, Prasada Rao and Selvanathan 1991). It is for these reason why Prasada Rao and Selvanathan (1991, 1992a, 1992b, 1994) have considered the TT binary index in the multilateral CCD index for further generalization.

2.3 Stochastic Approach to Theil-Tornqvist Binary Index

The 'stochastic approach' provides an alternative procedure to the well known 'atomistic' and 'functional' approaches in the construction of price and quantity index numbers (Frisch 1936). It was Frisch (1936) who initially considered the potential of stochastic approach to measure the central tendency from a distribution of price relatives. It is in 1981 when Clements and Izan initially made use of the stochastic approach to derive TT binary index numbers and its associated standard errors for temporal comparisons, followed by Clements and Izan (1987) where the use of stochastic approach to index number theory to estimate the rate of inflation and its standard error was explored rigorously. Their 1987 study views each price change as an independent observation on the underlying rate of inflation. Selvanathan (1989) extended Clements and Izan approach to the price indices of groups of commodities and measure the relative prices across groups. In the context of international comparisons, another procedure proposed by Summers (1973), known as the country-product dummy (CPD) method, also made use of the stochastic approach (see Prasada Rao and Selvanathan 1994, pp.199-203). Balk (1980) use stochastic specification to deal with seasonality. Much recently, stochastic approach was also applied in the study of Freeman (1992) where he incorporates commodity-specific effects on the construction of GCCD multilateral index of Prasada Rao and Selvanathan (1992a).

Under stochastic approach, each price relative is taken to be equal to the underlying price index which measures the overall price changes between two countries, plus other components which are random or non-random (Prasada Rao and Selvanathan 1994). An

advantage of the stochastic approach is that it can provide standard errors for the price index estimates reflecting price variability. It can be shown that these standard errors increase with the degree of price variability. This result is consistent with the belief that disproportionate individual price movements indicate reduced reliability in the overall inflation rate.

More recently Prasada Rao and Selvanathan (1991, 1992a, 1992b, 1994) have applied the stochastic approach in the construction of multilateral index numbers. Following Clements and Izan (1987) and Selvanathan (1989), Prasada Rao and Selvanathan derived the TT binary index given in equation (2.10) using stochastic approach, and using the regression model

$$Dp_{ikj} = \Pi_{kj} + u_{ikj}, \quad i = 1,2,\ldots,N \tag{2.12}$$

In the above model, the parameter Π_{kj} may be interpreted as a common trend in prices of all N commodities between the countries k and j.

The disturbances u_{ikj} are assumed to have the following properties:

$$\begin{aligned} &(i)\ E(u_{ikj}) = 0; \\ &(ii)\ V(u_{ikj}) = \sigma_u^2/\overline{w}_{ikj}; \quad \text{and,} \\ &(iii)\ E(u_{ikj}u_{i'k'j'}) = 0, \quad \text{for } i \neq i', k \neq k', j \neq j'. \end{aligned} \tag{2.13}$$

Applying a generalised least squares estimation procedure to the above model yields an estimator of Π_{kj} given by

$$\hat{\Pi}_{kj} = \sum_{i=1}^{N} \overline{w}_{ikj} Dp_{ikj}. \tag{2.14}$$

The above estimator is identical to the additive form of the Theil-Tornqvist index defined in equation (2.11). The estimated variance of Π_{kj}^{TT} is

$$Var(\hat{\Pi}_{kj}) = \hat{\sigma}_u^2. \tag{2.15}$$

This variance parameter σ_u^2 may be estimated unbiasedly by

$$\hat{\sigma}_u^2 = \frac{1}{N-1}\sum_{i=1}^{N} \overline{w}_{ikj}(Dp_{ikj} - \hat{\Pi}_{kj})^2 . \qquad (2.16)$$

It is interesting to note here that if there is a significant variation in the relative prices, Dp_{ikj} across i=1,2,3,...,N, the sampling variance of $\hat{\Pi}_{kj}$ will be large.

Despite the elegant properties of the TT binary index, it had no role in multilateral comparison exercises as the TT index does not satisfy the transitivity requirement when applied in the context of multilateral comparisons. The present study focuses on an alternative multilateral index formula that would satisfy the transitivity property but related to TT indices. Considering that Prasada Rao and Selvanathan (1992a, 1994) have applied the stochastic approach to the case of spatial comparisons of prices in the construction of index numbers which satisfy transitivity, this dissertation will focus on these models. Prasada Rao and Selvanathan have derived a CCD index for multilateral comparisons using the stochastic approach. This CCD multilateral index has been proven to be internally consistent in their study and may be considered a useful tool in the estimation of PPPs. This will be the focus of discussion in the next chapter.

3. The Stochastic Approach to Construction of Multilateral Index Numbers

In chapter 2 we introduced the stochastic approach and has showed how this approach can be used to derive the Theil-Tornqvist (TT) binary index number and its standard error. But despite the elegant properties of the TT index, this index has not played any significant role in the context of international comparisons. This is mainly because it fails to satisfy the transitivity requirement when applied in the context of multilateral comparisons. Caves, et al. (1982) proposed a multilateral method based on a simple averaging procedure that leads to modified TT indices that are internally consistent and base invariant. This CCD procedure can be appropriately described as a two-step procedure where the first step involves the computation of the binary TT indices and the next step involves the modification based on the simple unweighted geometric mean leading to transitive multilateral indices, referred to as the CCD indices. Two issues are of interest with regards to this procedure. First concerns the use of suitable weights that might improve the CCD multilateral indices; and the second issue is to provide proper justification for the application of this averaging procedure. Prasada Rao and Selvanathan (1992a) addressed these issues by deriving the TT and CCD multilateral indices using an econometric approach or econometric modelling procedure and provide some generalisations. These generalisations and the underlying stochastic specification form the core of this chapter.

The stochastic approaches to construction of multilateral index number to generalisation of TT indices are outlined in this chapter. Section 3.1 establishes the CCD approach to the construction of the multilateral index number satisfying the requirements of 'transitivity' and 'base-invariance', proposed by Caves, Christensen and Diewert (1982). Then, following the stochastic approach developed by Clements and Izan (1987) and Selvanathan (1989), Section 3.2 provides an econometric model specification that would result in the CCD multilateral index number, as suggested by Prasada Rao and Selvanathan (1992a). Estimation of the parameters in the econometric model that leads to the CCD multilateral index and their standard errors is included also in this section. In Section 3.3, a more generalised form of this multilateral index number that uses weights based on economic distance is presented. Section 3.4 provides the estimation of purchasing power parities and their standard errors for M-1 countries considered in multilateral comparison exercise. Concluding remarks summarise the discussions in this chapter.

3.1 The Caves,Christensen and Diewert (CCD) Multilateral Index

As mentioned above, the TT index in Section 2.3 does not satisfy transitivity. To overcome this problem, Caves, Christensen and Diewert (1982) suggested a simple averaging procedure which builds on the binary TT index number leading to a generalised TT index number that is referred to as the CCD multilateral index number. This multilateral index formula satisfies both the properties of transitivity and country symmetry.

The CCD multilateral index formula in multiplicative form is a simple geometric mean of all M indirect comparisons between country j and k, derived through a bridge country l (l=1,2,..,M) is given by

$$\text{(multiplicative form)} \qquad I_{kj}^{CCD} = \prod_{l=1}^{M} \left[I_{kl}^{TT} \cdot I_{lj}^{TT} \right]^{\frac{1}{M}} . \tag{3.1}$$

Taking the logarithm of both sides of equation (3.1), an additive form of the index can be specified as

$$\text{(additive form)} \qquad \Pi_{kj}^{CCD} = \Pi_j^* - \Pi_k^* \quad , \tag{3.2}$$

where $\Pi_k^* = (1/M)\sum_{j=1}^{M} \Pi_{kj}^{TT}$ is a simple average of the binary TT indices of all countries with country k as the base. Following the equation (2.6) in Section 2.1.2, it could be shown that

$$\begin{aligned} \Pi_{kl}^{CCD} + \Pi_{lj}^{CCD} &= (\Pi_l^* - \Pi_k^*) + (\Pi_j^* - \Pi_l^*) \\ &= \Pi_j^* - \Pi_k^* \\ &= \Pi_{kj}^{CCD}. \end{aligned} \tag{3.3}$$

Equations (3.2) and (3.3) established that CCD index satisfies transitivity and that it depends on averaging procedure. Furthermore, it can be easily shown that the CCD index satisfies base invariance or country symmetry as it is invariant to the order in which the countries are introduced in to the formula.

This CCD multilateral index numbers are now widely considered for multilateral productivity comparisons (e.g. Diewert 1992, Prasada Rao and Selvanathan 1992a, 1994). In the next section, the use of stochastic approach to derive the form of the CCD multilateral index number will be presented as well as the econometric modelling procedure proposed by Prasada Rao and Selvanathan (1992a).

3.2 Stochastic Approach to CCD Multilateral Index

Prasada Rao and Selvanathan (1992) made use of the stochastic approach in Clements and Izan (1987), and Selvanathan (1989) to derive a multilateral log-change index number formula that would yield indices which are transitive and possess some useful least squares properties. Rao and Selvanathan obtained a generalised form of an index number for multilateral comparisons by imposing the transitivity restriction in Caves, Christensen and Diewert(1982) that $\Pi_{kj} = \Pi_j - \Pi_k$.

Imposing the said linear restriction to equation (2.12) in Section 2.3, the stochastic model now becomes,

$$Dp_{ikj} = \Pi_j - \Pi_k + u_{ikj} , \tag{3.4}$$

where i = 1,2,3,...,N , k = 1,2,3,...,M-1, and j = k+1,...,M . Where $\Pi_1, \Pi_2, \ldots, \Pi_M$ are the parameters across different equations for different pairs of country j and k (see Prasada Rao and Selvanathan 1992a for more details). Again the disturbance term u_{ikj} in the above equation assumes the same properties as in (2.13) , that is

$$\begin{aligned} &(i)\ \ E(u_{ik}) = 0, \\ &(ii)\ \ V(u_{ik}) = \sigma_u^2 / \overline{w}_{ikj}; \quad \text{and,} \\ &(iii)\ E(u_{ik} u_{i'k'j'}) = 0, \ \text{for } i \neq i', k \neq k', j \neq j'. \end{aligned} \tag{3.5}$$

Following Judge, et al. (1988, p.358), the generalised least squares (GLS) estimator of the parameter vector $\Pi = [\Pi_1, \Pi_2, \ldots, \Pi_M]'$ may be obtained by applying ordinary least squares (OLS) to the transformed model

$$\sqrt{\overline{w}_{ikj}}\,Dp_{ikj} = \sqrt{\overline{w}_{ikj}}\,\Pi_j - \sqrt{\overline{w}_{ikj}}\,\Pi_k + u^*_{ikj}\,. \tag{3.6}$$

Stacking all the N observations for each pair of country j and k and for all pairs, and using the symmetry between Dp_{ikj} and Dp_{ijk}, the transformed model (3.6) may be written in matrix form

$$Y = X\Pi + u^*\,, \tag{3.7}$$

where Y is a $NM(M\text{-}1)/2$ column vector and X is a $[NM(M\text{-}1)/2 \times M]$ matrix of observations. $\Pi = [\Pi_1, \Pi_2, \ldots, \Pi_M]'$ is a vector of unknown parameters and u^* is a vector of transformed disturbances that are homoscedastic.

Application of the least squares to the transformed model yields the normal equations

$$X'X\hat{\Pi} = X'Y\,, \tag{3.8}$$

where $X'X = [M I_M - \iota_M \iota'_M]$, with I_M an identity matrix of order M and $\iota'_M = (1,1,\ldots 1)$ is an M-unit vector, and $X'Y$ is a column vector whose jth element is $\sum_{k=1}^{M}\sum_{i=1}^{N} \overline{w}_{ikj}\, ln(p_{ij}/p_{ik})$. In matrix form, it could be shown that

$$X'X_{M\,x\,M} = \begin{bmatrix} (M-1) & -1 & \ldots & -1 \\ -1 & (M-1) & \ldots & -1 \\ . & . & \ldots & . \\ . & . & \ldots & . \\ -1 & -1 & \ldots & (M-1) \end{bmatrix}$$

and

$$X'Y_{M\,x\,1} = \begin{bmatrix} \sum_{=1}^{M}\sum_{i=1}^{N} \overline{w}_{i1j}\, ln(p_{i1}/p_{ij}) \\ . \\ . \\ . \\ \sum_{=1}^{M}\sum_{i=1}^{N} \overline{w}_{iMj}\, ln(p_{iM}/p_{ij}) \end{bmatrix} \tag{3.9}$$

Following the normal equation in (3.8) the solution of $\hat{\Pi}$ will depend on the rank of $X'X$. It can be observed that $X'X$ is singular and rank($X'X$)=M-1. This indicates the presence of multicollinearity and implies that the original parameter vector Π in the equation (3.7) is not identified. In order to obtain the best linear unbiased estimator of parameter vector Π, Prasada Rao and Selvanathan (1994, p.148) made use of the following useful results.

1. All linear combinations of the form $\Pi_j - \Pi_k$ are estimable, following Schmidt (1986).
2. The best linear unbiased estimator of $\Pi_j - \Pi_k$ is given by $\hat{\Pi}_j - \hat{\Pi}_k$ where $\hat{\Pi}$ is any solution of the normal equations (3.8) and the resulting estimator is unique.
3. Since rank($X'X$) is M-1, setting $\hat{\Pi}_M = 0$, then $\hat{\Pi}_1, \hat{\Pi}_2, \ldots, \hat{\Pi}_{M-1}$ can be solved for uniquely

Prasada Rao and Selvanathan (1994) show that using the above results, the resulting estimators are given by

$$\begin{aligned}\hat{\Pi}_j &= \hat{\Pi}_j - \hat{\Pi}_M \\ &= \frac{1}{M}\left[\sum_{k=1}^{M}\sum_{i=1}^{N}\overline{w}_{iMk}\, ln(p_{ik}/p_{iM}) + \sum_{k=1}^{M}\sum_{i=1}^{N}\overline{w}_{ijk}\, ln(p_{ij}/p_{ik})\right]\end{aligned} \tag{3.10}$$

and

$$\hat{\Pi}_M = 0.$$

The solution $\hat{\Pi}_j$ may be interpreted essentially as a log-change index for country j relative to base country M. Moreover, in a multiplicative form, the index may be expressed in general, for any pair of country j and k, as

$$I_{kj}^{CCD} = \prod_{l=1}^{M}\left[\prod_{i=1}^{N}\left[\frac{p_{il}}{p_{ik}}\right]^{\overline{w}_{ikl}} \cdot \prod_{i=1}^{N}\left[\frac{p_{ij}}{p_{il}}\right]^{\overline{w}_{ilj}}\right]^{\frac{1}{M}}. \tag{3.11}$$

Following Caves, et al. (1982), Prasada Rao and Selvanathan name the index number formula I_{kj}^{CCD} as the generalised TT index or CCD index due to Caves, Christensen and Diewert. This index provides a multilateral generalisation of the TT index.

The resulting generalised TT index (3.11) clearly satisfies the transitivity and country symmetry or base invariance properties. As I_{kj}^{CCD} is obtained from a regression model, the standard errors associated with it can be computed and may as well be used in any interval estimation for the indices obtained.

Using the definition of the TT binary index from Section 2.2 we can express

$$I_{kj}^{CCD} = \prod_{l=1}^{M}\left[I_{kl}^{TT} \cdot I_{lj}^{TT}\right]^{\frac{1}{M}}. \qquad (3.12)$$

The above equation is identical to that of equation (3.1) proposed by Caves, et al. This reveals that I_{kj}^{CCD} is a simple geometric mean of M indirect comparison between country k and j, where each indirect comparison is made through a link country l using binary TT index formula. Moreover, Prasada Rao and Selvanathan (1994) noted that I_{kj}^{CCD} (for $k, j = 1,2,...,M$) provides a multilateral index that has minimum distance from the binary indices. That is, considering a problem of obtaining I_{kj} such that $\sum_{k=1}^{M}\sum_{i=1}^{N}\left[ln\ I_{kj} - ln\ I_{kj}^{TT}\right]^2$ is minimum subject to the constraint $I_{kj} = I_{kl} \cdot I_{lj}$, then the generalised TT index is the solution to this problem. Similarly this has been observed in Prasada Rao and Banerjee (1986).

Among the properties of I_{kj}^{CCD} mentioned in this section, the minimum distance property is considered as the most important. As I_{kj}^{CCD} deviates least from the binary TT index I_{kj}^{TT}, this would imply that if there exists any other multilateral index, it would deviate from I_{kj}^{TT} more than I_{kj}^{CCD}. But this may not happen for each pair but over all pairs of countries. Considering the many statistical and economic theoretic property of the binary TT index, the CCD multilateral index derived in (3.12) retains all these

properties. This is why the dissertation considered the log-changed form of I_{kj}^{CCD} in the estimation of the PPP's and their standard errors for all the (M-1) countries considered in the study. Moreover the transformed model in (3.6) will be used as building block for the stochastic model which will incorporates spatial autocorrelation in the disturbance term. This model is discussed in Chapter 5.

3.3 The Generalised CCD Multilateral Index Numbers

It has been mentioned in the previous section that CCD multilateral index number is a simple geometric mean of all the indirect comparisons between country k and j. However, the CCD index number came under criticism due to this characteristic of using the simple unweighted geometric mean averaging procedure (Prasada Rao and Selvanathan 1992a, 1994). A possibility of a weighted version of the index could be explored.

Intuition suggests that some indirect comparisons between two countries through a third country would be intrinsically more reliable than others. An example is that, if k refers to United States of America (USA) and j refers to Australia, then an indirect comparison between USA and Australia through United Kingdom (UK) would be more reliable than an indirect comparison through India. In relation with this, a differential weighting scheme was suggested in Prasada Rao and Selvanathan (1992a, 1994).

In Prasada Rao and Selvanathan (1992) paper, a basic regression model was postulated with the disturbance term exhibiting a more general form of heteroscedasticity. Rao and Selvanathan made use of the model (3.4) with the disturbance term u_{ikj} having the following properties:

$$\begin{aligned} &(i)\ \ E(u_{ikj}) = 0, \\ &(ii)\ \ V(u_{ikj}) = (\sigma_u^2 / \overline{w}_{ikj})\delta_{kj}, \quad \textit{and} \\ &(iii)\ E(u_{ikj}u_{i'k'j}) = 0, \quad \textit{for } i \neq i', k \neq k', j \neq j'. \end{aligned} \tag{3.13}$$

For the variance of the disturbance term, δ_{kj} uses the concept of economic distance between country k and j. The economic distance formula was based on the real per capita incomes associated with country j and k, and is defined as,

$$\begin{aligned}\delta_{kj} &= \left| ln\left[\frac{E_j}{I_{1j}}\right] - ln\left[\frac{E_k}{I_{1k}}\right]\right| \\ &= \left|\left(ln\,E_j - \Pi_j\right) - \left(ln\,E_k - \Pi_k\right)\right| \\ &= \left|\left(ln\,E_j - ln\,E_k\right) - \left(\Pi_j - \Pi_k\right)\right|.\end{aligned} \tag{3.14}$$

The term E_j pertains to the nominal per capita income in country j while E_j/I_{1j} converts jth per capita income into the currency unit of 1. The BLUE of the parameter vector Π, of this generalised CCD model can be estimated using a two step GLS procedure which is discussed in Prasada Rao and Selvanathan (1994, p.152). Intuition suggests that alternative specification of could be more meaningful. This requires future research.

3.4 Estimation of Purchasing Power Parities and Standard Errors

After the estimates of the parameter vector $\Pi = [\Pi_1, \Pi_2, \ldots, \Pi_M]'$, which is given in (3.10) are obtained using the GLS procedure, the purchasing power parities of the currencies of countries 1 to M-1, defined by PPP_j (j = 1, 2,..., M-1) in terms of the numeraire currency, are given by

$$\hat{PPP}_j = exp(\hat{\Pi}_j) \tag{3.15}$$

What was good in the use of stochastic approach to derive the estimates of the Π_j's is that it can also provide standard errors for the price index estimates reflecting price variability. Hence, the standard errors for the estimated PPP_j's in (3.15) may also be obtained. These standard errors could be useful in assessing the reliability of the estimated PPP for each of the M-1 countries. As the ICP of UN only published PPP's for their 1985 PPP comparisons results, the estimated standard errors of PPP's that will be derived in the dissertation would also be beneficial. The question now is how to derive these standard errors.

As the PPP_j's are obtained from the estimates of the additive stochastic model in (3.7) (which implies that the estimated parameters of (3.7) are equal to the logarithms of the PPP's), the standard errors of the purchasing power estimates that are to be obtained

would actually be the standard errors of the exponentials of the parameter estimates. Freeman (1992) proposed two procedures for the computation of these standard errors. These two procedures are outlined below.

Approximate Standard Errors for the PPP Estimate

This procedure provides an approximation to the actual standard errors based on a Taylor Series expansion. Following the result in Mood, Graybill and Boes (1974, p.181), stated as

> Let x be any random variable whose variance, defined by var(x), exists. Now consider any transformation function h(.) which defines another random variable y. If y is a function of x, that is $y = h(x)$, then an approximate formula for the variance of y is given as

$$var(y) = var(x) \cdot \left\{ \frac{\partial h(x)}{\partial x} \right\}^2 .$$

the variance of the PPP estimates in (3.15) can be *approximated* as

$$var(\hat{PPP}_j) \cong var(\hat{\Pi}_j)\left[exp(\hat{\Pi}_j)\right]^2. \tag{3.16}$$

Using the above results , the corresponding standard error of $\hat{PPP}_j$ is given by

$$se(\hat{PPP}_j) = \sqrt{var(\hat{PPP}_j)} = se(\hat{\Pi}_j)\left[exp(\hat{\Pi}_j)\right]. \tag{3.17}$$

The above estimation result is easy to apply, since the standard errors of the parameter estimates are available from the regression results of the stochastic procedure, and the above formula is easy to compute.

Exact Standard Errors for the PPP Estimates: Assuming Normality

Following the results of Prasada Rao and Selvanathan (1992b), the exact standard errors for the PPP estimates assuming normality could be obtain. Suppose we assume that the

disturbance term, u^*, from model (3.7) has a normal distribution with mean zero and variance given by $\sigma_u^2 / \overline{w}_{ikj}$. That is,

$$u^* \sim N\left(0, \frac{\sigma_u^2}{\overline{w}_{ikj}}\right). \tag{3.18}$$

The above assumption imply that the estimates, $\hat{\Pi}_j$, have a sampling distribution given by

$$\hat{\Pi}_j \sim N\left(\Pi_j, \sigma_{\hat{\Pi}_j}^2\right). \tag{3.19}$$

Considering the fact that $P\hat{P}P_j = exp(\hat{\Pi}_j)$, then $P\hat{P}P_j$ would have a log-normal distribution with parameters Π_j and $\sigma_{\hat{\Pi}_j}^2$, that is,

$$P\hat{P}P_j \sim \Lambda\left(\Pi_j, \sigma_{\hat{\Pi}_j}^2\right). \tag{3.20}$$

It could be shown that the variance of $P\hat{P}P_j$, given its log-normal distribution, is *exactly* equal to

$$var(P\hat{P}P_j) = exp(2\hat{\Pi}_j + \sigma_{\hat{\Pi}_j}^2)\left[exp(\sigma_{\hat{\Pi}_j}^2) - 1\right], \tag{3.21}$$

following Aitchison and Brown (1966). Just like the *approximate* variance estimator in (3.16), this *exact* variance estimator is easy to compute. However if we consider the expected value of $P\hat{P}P_j$, it can be proven that $P\hat{P}P_j$ is a biased estimator for PPP_j, that is

$$E(P\hat{P}P_j) = exp\left(\Pi_j + \frac{\sigma_{\hat{\Pi}_j}^2}{2}\right), \tag{3.22}$$

where the bias can be computed as

$$bias(\hat{PPP}_j) = exp(\Pi_j)\left[exp\left(\frac{\sigma^2_{\hat{\Pi}_j}}{2}\right)-1\right] . \tag{3.23}$$

Because of its biasedness, Freeman (1992) suggested that the *root mean square error* (rsme) of the estimate be considered in lieu to the standard error. By definition, the root mean square error is given by the formula

$$\begin{aligned} rsme(\hat{PPP}_j) &= \sqrt{mse(\hat{PPP}_j)} \\ &= \sqrt{var(\hat{PPP}_j)+[bias(\hat{PPP}_j)]^2} . \end{aligned} \tag{3.24}$$

Again, both the bias and rmse for each PPP estimates can be calculated easily based on the regression results of the stochastic procedure discussed in Section 3.2. The results in this section will also be used to derived the PPP estimates and their standard errors for the stochastic model that would incorporates spatial autocorrelation which is discussed in Chapter 5.

3.5 Concluding remarks

This chapter has established the general framework within which, work on the stochastic approaches to multilateral index number has extensively been undertaken. It was shown that the stochastic approach can be used to obtain a class of index numbers for multilateral spatial comparisons. The CCD indices are found to possess some useful least squares properties, most importantly, they are transitive and base invariant. Furthermore, these multilateral indices are derived using econometric models and are shown to use the Theil-Tornqvist binary indices as building blocks. The stochastic approach has also lead to the estimation of the standard errors of these multilateral indices in reference to their respective underlying econometric specification. A more generalised form of the CCD multilateral index is also discussed, however, the present study does not consider this issue as the development of a spatially autocorrelated stochastic model is the main focus of the study. Models incorporating spatial

autocorrelation are discussed in Chapter 5. Since the concept of spatial autocorrelation is new to the econometric modelling of multilateral index numbers, the present study will only be limited to the CCD multilateral index model in its additive form. The latter part of the chapter has showed also that the purchasing power parities and their standard errors can be estimated using the GLS estimates of the parameter in the CCD multilateral index model.

4. Spatial Autocorrelation

All the regression models described in Chapter 3 for the TT, CCD and GCCD indices are based on cross-sectional data on commodity prices. This suggests that the price relatives for a given commodity, for a country like India with US as base could be similar to the price ratios observed for a neighbouring country like Pakistan or Sri Lanka suggesting the presence of correlation in disturbances for these countries. Any analysis of such phenomenon requires the concept of autocorrelation across space - spatial autocorrelation.

The concept of spatial autocorrelation that will be used in conjunction with the regression model associated with the CCD multilateral index formula is outlined in this chapter. A formal definition of spatial autocorrelation and some basic notation are introduced in Section 4.1. In Section 4.2, the tools and techniques for measuring spatial autocorrelation is examined, particularly the spatial proximity variables which are considered in the development of our stochastic model in Chapter 5. In Section 4.3, we define a simple regression model which spatial dependence. The procedure used for the estimation of this simple model with spatial autocorrelated disturbance term is also included in this section. In Section 4.4, we discussed the Moran's I test statistic for the presence of spatial autocorrelation among regression residuals. Concluding remarks culminate this chapter.

4.1 The Concept of Spatial Correlation

One of the basic concerns of the geographers and statisticians in the early 60's is the analysis of spatially located data. A number of studies have been made concerning the statistical methods used for analysing spatial data. The technique of regression analysis applied to models utilising geographically or mapped data is one of those which had been often dealt with. Particular interest among geographers has been the problem of serial correlation existing on the disturbance terms of regression models in the spatial context. These phenomena had led to the concept of spatial autocorrelation which is particularly relevant to the present dissertation. 'What really is spatial autocorrelation?' is the question addressed in this section.

Suppose we consider the geographical distribution of some quality or phenomenon in the countries or states of a country, then if the presence of some quality in a state of a

country makes its presence in neighbouring states more or less likely, we say that the phenomenon exhibits a *spatial autocorrelation* (Cliff and Ord 1973). In its most general sense, spatial autocorrelation is concerned with the degree to which objects or activities at some place on earth's surface are similar to other objects or activities nearby (Goodchild 1986). This idea is reflected in the proposition of Tobler (1970) stated as: 'everything is related to everything else, but near things are more related than distant things.' This is commonly referred to as the *first law of geography*. Similarly, Griffith (1987) defined spatial autocorrelation as the relationship among values of some variable that is attributable to the manner in which the corresponding aerial units are ordered or arranged on a planar map or surface.

Goodchild (1986) interpreted spatial autocorrelation as a descriptive index, measuring aspects of the way things are distributed in space. As an index it provides a type of information about spatially distributed phenomenon which is not available in any other form of statistical analysis, and which can be vital to correct interpretation. If one were forced to summarise a spatial distribution of unequal attributes in a single statistic, one would in all likelihood choose a spatial autocorrelation index, just as one would probably choose a measure of central tendency such as mean or median to summarise a nonspatial data set. Therefore, spatial autocorrelation is important to both geographers and statisticians because it reflects upon the quantity and quality of information contained in spatial data, and ultimately then in the soundness of data interpretations.

We now formally define spatial autocorrelation following Cliff and Ord (1973, 1981). Suppose we consider a study area which is exhaustively partitioned into n non overlapping subareas (say countries). Suppose that a random variable, X, has been measured in each of the countries, and that the value of X in the typical country k is given by x_k. X could describe either (1) a single population from which repeated drawings are made to give the $\{x_k\}$; (2) n separate populations, one for each country (populations will usually assume to be identically distributed); or (3) a partition of a finite population among n countries. The choice between population models (1), (2) and (3) for generating the sample country values depends upon the problem in hand (see Cliff and Ord 1973 for examples of this). A basic property of spatially located data is that the set of values, $\{x_k\}$, are likely to be related over space. If $\{x_k\}$ displays interdependence over space then we say that the data are spatially autocorrelated. That is, when every pair of countries k and j in the study area the drawings which yield $\{x_k\}$ and $\{x_j\}$ are not all pairwise uncorrelated.

Most geographic data series may be interdependent because the data are influenced by processes that connect different places, including spatial interaction and spatial diffusion processes; or by phenomena that extend over space to occupy regions, countries and areas rather than point locations. Spatial interaction illustrates the movement of goods, people, or information over space. Events or circumstances at one place can affect the conditions at other places if these places interact. An example of which is the prices and the quantities supplied in a group of spatially located markets. They may exhibit interdependence if the markets are close enough to exchange commodities. In particular, prices at one location are likely to be dependent of prices at the other locations if they are near enough for supplies of a commodity to be moved between them. Concerning spatial diffusion, it deals on how frequent or intense the dispersion of a certain phenomenon is from a set of origins. An example of which is that innovation on computer technology may depend or distance from origin. Locations that are near to one another are likely to have similar distances from an origin, and hence would experience similar advances on computer technology.

Many geographic phenomena or activities extend over space to occupy regions rather than single point locations. The space or region that is occupied may not always be defined. Examples of which are the cultures, climates and housing markets. Rainfall statistics are likely to be autocorrelated among nearby weather stations because they experience almost the same weather events everyday. Prices of neighbouring houses are likely to be autocorrelated because they may be influenced by similar conditions of demand and supply. Interaction among places and extension of many phenomena over space mean that events and circumstances at one country are likely to be dependent of the conditions at nearby countries. This interdependence among countries brings pattern and structure to geographic data, and it is the role of spatial autocorrelation statistics to investigate hypotheses concerning the distribution of this mapped data. Some statistics that can be used to test null hypothesis that the data values are independent of values at other locations have been proposed in some studies (e.g. Moran 1948, Geary 1954, Sen 1976, Cliff and Ord 1973, 1981, Haining 1980, Upton and Fingleton 1985). One of which is the Moran's *I* statistics that was found to be the most powerful one (Cliff and Ord 1973). This is discussed in the next section.

Now we briefly turn to the question of presence of spatial autocorrelation in price data across countries. Aten (1994) explored the existence of spatial autocorrelation in the relative prices of goods and services across countries. Aten made used of the 1985 ICP benchmark data in statistical analyses about the price and income relationships across countries. Aten's study introduced a spatial component as part of the testing procedure in

detecting the presence of spatial autocorrelation in relative prices. Such spatial element is measured by, (1) the pairwise existence of common boundary; (2) the distance between the capital cities; and by, (3) the amount of trade between two countries. Aten made used of the Moran's I coefficient to measure the strength of autocorrelation. The study established the existence of spatial autocorrelation among estimated residuals in the cross-country demand analysis, thus reflecting the sensitivity of the prices to location variables. Moreover, the presence of spatial autocorrelation in the estimated residuals creates loss of efficiency in the model estimates.

With the above findings at hand, it is indeed necessary to incorporate the concept of spatial autocorrelation in any econometric modelling procedure which make use of price comparisons across countries. The principal aim of the present study is to examine spatial autocorrelation in the context of multilateral index estimation using regression models in Chapter 3. Price and quantity comparisons over space or across countries may exhibit geographic ordering (Aten 1994), that is, price levels of one country may exhibit relationships with price levels of neighbouring country. With the price levels of goods and services across countries displaying interdependence over space, spatial autocorrelation therefore exists. Hence, this would have a bearing on the efficiency of the estimates of the multilateral index formula discussed in Chapter 3. Indeed, there is really a need to incorporate the concept of spatial autocorrelation in the context of multilateral index number estimation. The next section will examine the procedures for measuring spatial autocorrelation among regression residuals. Moreover, it would consider the spatial variables relevant for the present undertakings.

4.2 Measurement of Spatial Autocorrelation

In the previous section we saw that spatial data are location specific and are therefore referenced with respect to each other so that spatial proximity can be measured. Spatial autocorrelation is really concerned with a comparison of two types of information: similarity among data attributes and similarity of location. The ways in which the former can be measured depend on the type of data present, while the calculation of spatial proximity depends on the locations.

In an attempt to devise a measure for spatial autocorrelation, two basic problems arise: (1) which function of the variable should be used; and (2) how do we allow for the degree of interaction between two locations. With the first problem on hand, one has to determine the form for x_k, and then choose on a functional form $f(x_k, x_j)$ to indicate

spatial autocorrelation. There is a variety of ways to express $f(x_k, x_j)$, but one of the most commonly used form in practice is the squared difference form, that is, $f(x_k, x_j) = (x_k - x_j)^2$. Another form which is common for joint-count statistics is the product $(x_k - \bar{x})(x_j - \bar{x})$. The latter is the one used for the Moran's I statistics which will be discussed in a later part of this Section.

Whatever the functional form for $\{x_k\}$ is finally selected, we still need to solve the second problem. To do that, we need to specify a measure of the spatial proximity W_{kj}. A variety of ways have been devised for measuring this spatial proximity variable. The existence of a linkage between objects is often a measure of spatial proximity, since nearby objects are more likely to be linked than distant ones. For area objects, a common boundary between areas is a simple, binary indicator of geographical proximity. Many of the early indices of spatial autocorrelation made use of these binary weights. Some generalisations of these spatial proximity variables are given in Cliff and Ord (1973). Some other forms can be found in Cliff, et al. (1975), Cliff and Ord (1981), and Upton and Fingleton (1985) as well as their applications. Concerning the present study, it will utilise the spatial proximity variables used by Aten (1994) as Aten's work is primarily focused on inter-country price data.

So far, the details for the measurement of spatial autocorrelation coefficient have been established already but there is a need to distil them to form a single statistic. Over the years a number of different statistics have been developed, each having the ability to provide distinctive revelations about spatial distributions. It is only recently that their close form has been clearly defined (see Hubert, et al. 1981). This general-cross product statistic is given by the equation

$$r = \sum_{k=1}^{n} \sum_{j=1}^{n} W_{kj} f(x_k, x_j), \tag{4.1}$$

where W_{kj} is a measure of the spatial proximity of locations k and j; $f(x_k, x_j)$ is a function which measures the proximity of k and j on some other dimension; and n is the number of spatial locations. The statistic r was first proposed by Knox (1964) and generalised by Mantel (1967). This r statistic bears a close relationship with the Moran's I statistic that will be used in the dissertation.

4.2.1 The Spatial Proximity Measures

Selecting a spatial weighting function is the most important step in calculating a spatial autocorrelation statistic. Odland (1988) defined a spatial weighting function as a set of rules that assign values or 'weights' to every pair of locations in a study area. The value of an autocorrelation statistic will necessarily depend on these weights as well as the data for the locations. Spatial proximity measures are often defined to represent the arrangement of areas or points relative to one another in a conventional space, but it is more general to think of it as a means of accommodating hypotheses about the relations among places. Flexibility in defining the weights makes spatial autocorrelation statistics a useful tool to investigating alternative hypotheses about the relations among places.

Generally, measures of spatial proximity are pairwise measures of physical distances between geographic regions. Usually, they are presented as square matrices (W's) of $n \times n$ dimension. Clearly n is the number of regions, and in our present study, n would refer to the number of countries. One example can be the distance between the centers of countries. Thus, w_{kj} being one element of the W matrix would represent the average distance between the centers of country k and j. Other forms of proximity structures include that of the length of common boundaries to each country; a combination of geographical distance and length of common boundaries; or a binary weighting function where $w_{kj}=1$ if countries share common boundary and zero otherwise. The use of binary weights for spatial proximity variables is the one most commonly used.

In this dissertation, three spatial proximity measures are considered in the computation of the spatial autocorrelation statistic. These proximity measures are considered also in the development of econometric models which will incorporate spatial autocorrelation in the disturbance term discussed in Chapter 5. The three measures used are: the contiguity measure, the great circle distance between capital cities, and the volume of trade between countries- measured by their exports and imports. These three measures are used by Aten (1994) in her study to established the degree of geographic relationship existing among prices of different commodities. Aten noted that what motivates her to use these three spatial proximity measures in examining the existence of spatial autocorrelation with respect to prices are the possibility of testing hypotheses that boundaries, distances or trade volumes capture differences in transport costs between countries. The succeeding paragraphs would define these three measures.

The contiguity measure

Following the definition of Hordijk (1974), any two countries are said to be *contiguous* if they have a common boundary of non-zero length or if they are spatially adjacent. Hence, a country is considered as *non-contiguous* with itself. The later claim is one of the property developed by Moran (1948). On the basis of the above definition, a simplest proximity measure could be defined as a set of binary weights that have value of one for countries that share a boundary and zero otherwise. In matrix form, a simple *nxn* contiguity matrix is given by W where each element $w_{kj}=1$, $(j,k=1,2,..,n)$, if country k and country j share a common boundary and $w_{kj}=0$ otherwise. It could be noted that $w_{kj}=0$, for $j=k$, as country is considered non-contiguous with itself. Obviously, the contiguity matrix W would then be symmetric. It is of advantageous to use this kind of spatial proximity measure when one is investigating spatial autocorrelation in irregularly spaced points or for irregular areas and locations. Furthermore, most studies which use the Moran's I statistics utilised this proximity measure (e.g. Cliff and Ord 1973, Hordijk 1974, Ord 1975, Sen and Soot 1977, Brandsma and Ketellaper 1979a, Mi on 1984). In addition, the contiguity matrix is easy to construct.

The distance measure

Another feasible alternative which will more closely reflect the quantity of inter-country interaction is to allow proximity values to depend in some way on the geographical distance between the countries. This can be both applicable to point data as well as area-based data. We will call this second measure of spatial proximity as a *distance matrix*. Measured in kilometres, it is defined as the shortest great circle distance between each country's capital city. So for a distance matrix W, $w_{kj}=0$ for $j=k$, and $w_{kj}>0$ otherwise. This will always mean that there are no off-diagonal zeroes in the W matrix. It could be noted that both the distance and contiguity matrices are symmetrical, with $w_{kj}=w_{jk}$ for all j and k. The distance matrix recognises the fact that, in the real world, interaction between countries does not usually terminate sharply beyond countries that are immediate neighbours. A similar W matrix like this is used in the study of Ripley (1977). Other kind of distance measures can be found in Cliff and Ord (1981).

The trade measure

Often a strong reason can be found for structuring the W matrix. Gatrell (1979) argues that in some applications it may be appropriate to base the measure of proximity on an

index of social or economic interaction rather than distance. The present study considers a third proximity matrix that reflects the trade flows between countries. Each element of the *trade* matrix W is calculated by using the exports or the imports between countries. That is, w_{kj} is the volume of exports from k to j or the volume of imports from j to k. Obviously, w_{kj} may not necessarily be equal to w_{jk}, hence the trade matrix will not be symmetric (see Aten 1994).

As a summary, in the contiguity matrix, a value of one indicates countries which are neighbours while a zero value indicates that there is no common boundary between them. While for the distance matrix, the larger the value the greater the distance, and hence it is the inverse of the distance that should be used as elements of the matrix. Lastly, in the case of the trade matrix, the higher the value indicates more exports or imports, or both, and hence more interaction between countries, so that the direction of this proximity matrix is the same as that of the contiguity matrix. Both the distance and trade matrices can be viewed as generalised proximity measures (Upton and Fingleton 1985, p.176). One satisfactory side effect of using generalised proximity measures is that the distribution of the test statistic will be more closely approximated by a continuous function such as the normal distribution (Cliff and Ord 1981, p.56). This latter claim is discussed much further in next section.

Normalisation of the proximity measures

The contiguity measure discussed above in the sense are sets of binary weights itself. However, for the other two proximity measures discussed previously, the elements of both the distance and trade matrices should be normalised so that the entries on the W matrix can be taken as weights. This is done by dividing each row element of the W matrix by the corresponding row totals or sum, making the entries of the W lie between zero and one, hence a new row sum of unity for the normalised proximity matrix.

As a specific example, Australia's exports from the UK are expressed as a proportion of Australia's total exports rather than absolute value. This allows the trade matrix to capture trade flows in proportion to the size of countries, that is, relative columns rather than absolute trade volumes. Similarly, for the case of the distance matrix, the distance between Sydney and London is expressed as a proportion of the total distance between Sydney and all other capital cities of the countries considered in the distance matrix. All other forms of proximity measures with the exception of the contiguity matrix should undergo the normalisation process before they can be used in the context of spatial autocorrelation analysis. One of the most important implication of this normalisation

process is for the plausibility of the Moran's *I* coefficient when computed using this proximity measures. Normalisation of some generalised proximity measures led also to the natural interpretation of the said spatial measure.

4.2.2 The Moran's *I* Statistic

Moran (1948) proposed a useful statistic to measure spatial autocorrelation. Most studies dealing with spatial processes referred to it as the *Moran's I index*. The Moran's *I* statistic is defined by

$$Moran's\ I = \frac{n \sum_{k=1}^{n} \sum_{j=1}^{n} w_{kj}(x_k - \bar{x})(x_j - \bar{x})}{S_0 \sum_{k=1}^{n} (x_k - \bar{x})^2}, \qquad (4.2)$$

where n is the number of countries; x_k is the value of the attribute of interest for country k; and, w_{kj} is the proximity variable from our proximity measure W. The term S_0 is the total of the weights in the proximity matrix W, e.i., $S_0 = \Sigma_{k=1}^{n}\Sigma_{j=1}^{n} w_{kj}$.

In Aten (1994) study, x_k is chosen to be a price relative for a certain commodity in country k. In this section we assume that the $\{x_k\}$ are raw data (independently observed values) obtained from n countries rather than residuals from a calculated regression or trend surface. Testing for spatial autocorrelation among regression residuals presents special problems and this is discussed in Section 4.4

It could be noted that equation (4.2) would be identical to r in (4.1) if we divide r by S_0 and by the sample variance of the observed x-values. In addition $f(x_k, x_j)$ is given by the distance between the x-values at country k and j. Odland (1988) noted that Moran's *I* is merely a spatial autocovariance measure standardised by two terms, namely; the variance of the data series $\Sigma_{k=1}^{n}(x_k - \bar{x})^2$, which depends on the x_k values but is invariant with their arrangement; and, $\{n/S_0\}$. The second term is a measure of connectivity for the set of countries and its value could change if the map of the countries were rearranged, but will not change with changes in the x_k .

If there is *no spatial autocorrelation*, the value of the Moran's *I* statistic in equation (4.2) approaches $-[1/(n-1)]$ (Upton and Fingleton 1985, p.170). This would imply that

x_k are independent of the values x of at neighbouring locations. When the values of I exceed -[$1/(n-1)$], it would indicate a *positive spatial autocorrelation* in which values of x_k tend to be similar to neighbouring values. However, when the computed value for I is below -[$1/(n-1)$], it would indicate a *negative spatial autocorrelation* in which neighbouring values are not independent but tend to be dissimilar. It could be notice that -[$1/(n-1)$] approaches zero as n increases, which is the expectation for an ordinary correlation coefficient to be zero when there is no correlation present.

When there is a maximum positive spatial autocorrelation, the value of the Moran's I statistic approaches one. It could be noted that the Moran's I values are not restricted to the -1 to 1 range, unlike most of the classical correlation coefficients (Moran 1948). I can exceed the value 1 for certain matrices of weights W and samples of attribute values (e.g. Cliff and Ord 1973, p.120). A more unambiguous interpretation of the Moran's I statistic is given in Goodchild (1986, p.16). The present study shall concentrate on the Moran's I statistic as it was shown by Cliff and Ord (1973, 1981) that tests based on I are consistently more powerful than those based on any other spatial autocorrelation statistics. The Moran's I statistic will be used in the measurement of spatial autocorrelation among prices of commodities in the countries considered in the present dissertation.

The sampling distribution of Moran's *I* Statistic

Cliff and Ord (1981, p.46) have shown that the mean and variance of the sampling distribution of Moran's I under randomization process are given by

$$E(I) = -1/(n-1), \tag{4.3}$$

$$Var(I) = \frac{n[(n^2-3n+3)S_1 - nS_2 + 3S_0^2] - k[n(n-1)S_1 - 2nS_2 + 6S_0^2]}{(n-1)(n-2)(n-3)\,S_0^2} - \frac{1}{(n-1)^2}, \tag{4.4}$$

where

n is the number of countries;

k is the sample kurtosis coefficient, e.i $k = m_4/m_2^2$, where m_r is the rth sample moment about the mean given by $m_r = \frac{1}{n}\Sigma_{k=1}^{n}(x_k - \bar{x})^r$;

S_0 is the total of the weights in the W matrix, i.e.

$S_0 = \Sigma_{k=1}^{n}\Sigma_{j=1}^{n} w_{kj} \quad (k \neq j);$

$S_1 = \frac{1}{2}\Sigma_{k=1}^{n}\Sigma_{j=1}^{n}(w_{kj} + w_{jk})^2 \quad (k \neq j)$:

$S_2 = \Sigma_{k=1}^{n}(w_{k\circ} + w_{\circ k})^2$; and

$w_{k\circ}$ *and* $w_{\circ k}$ are the row and column totals of the W matrix,

i.e., $w_{k\circ} = \Sigma_{j=1}^{n} w_{kj}$ and $w_{\circ k} = \Sigma_{j=1}^{n} w_{jk}$, respectively.

Following the results of Cliff and Ord (1972) and Sen (1976), it is possible to assume I to be approximately normally distributed under the null hypothesis of no spatial autocorrelation, when n is relatively large. Normality hinges on the number of countries under consideration, and on the extent and manner in which they are interconnected in the W proximity matrix. With the latter reservation, it can be justified that n=20 is sufficient enough to assume normality (Upton and Fingleton 1985). Cliff and Ord (1973, p.21) assumed that a sample size of 16 is adequate to ensure the accuracy of the normal approximation results. Moreover, Monte Carlo simulation results (Cliff and Ord 1972, p.39) established that approximations are not satisfactory when n is small,($n \leq 10$).

4.3 Spatial Autocorrelation in a Regression Model

Spatial autocorrelation is introduced in the preceding section as a characteristic that can be used to investigate interdependence among spatial data or observations. Other statistical models can also be used to investigate spatial patterns, and spatial autocorrelation statistic can play an important role if it is used in conjunction with these models. Regression models are widely used in geography, and spatial autocorrelation statistics are especially important in fitting regression models to spatial data. How to formulate spatial autocorrelation as a parameter of a statistical model will then be a major concern. In this section a general linear statistical model is considered and a spatial autocorrelation parameter is specified in its disturbance term. Two estimation procedures for the specified spatial model are also discussed in the later part of the section.

4.3.1 Spatial Disturbances Model

Suppose we consider a general linear model in matrix form defined as

$$Y = X\beta + u \tag{4.5}$$

where Y is a $(n \times 1)$ vector of observed sample values, X is a $(n \times k)$ matrix of known values of explanatory variables, β is a k-dimensional vector of unknown coefficients, and u is an unobservable $(n \times 1)$ vector of uncorrelated and identically distributed disturbance terms. Moreover, let X be a non-stochastic matrix of rank k and u is normally distributed with mean $E(u)=0$ and variance $E(uu')=\Phi=\sigma^2\Psi$, where Ψ is an $(n\times n)$ known positive definite symmetric matrix and σ^2 is an unknown scalar. Assume further that the disturbances are not heteroskedastic, that is, the diagonal elements of Ψ are identical.

Following Judge, et al. (1988, pp.320-32), the generalised least squares estimator (GLS) of β is given by

$$\hat{\beta} = \left(X'\Psi^{-1}X\right)^{-1}X'\Psi^{-1}Y, \tag{4.6}$$

where its covariance matrix is defined as

$$Cov\left(\hat{\beta}\right) = \sigma^2\left(X'\Psi^{-1}X\right)^{-1}. \tag{4.7}$$

The Gauss-Markov Theorem asserts that $\hat{\beta}$ is a best linear unbiased estimator of β (see Judge, et al. 1988, p.203). An unbiased estimator of σ^2 can be obtained using the formula

$$\hat{\sigma}^2 = \frac{(Y - X\hat{\beta})'\Psi^{-1}(Y - X\hat{\beta})}{n-k}. \tag{4.8}$$

The most popular form of autocorrelation used in econometric literature and one that has proved to be useful in many applications, is the first-order autoregressive process which leads to a specific form for Ψ and Φ. The presence of autocorrelation among

regression residuals leads to biased estimate of the residual variance σ^2 and inefficient estimates of the regression coefficients β when the method of ordinary least squares (OLS) is used. On the other hand, if there is no autocorrelation and GLS procedure is used, our estimates of β will be inefficient. It is therefore important to test for the presence of autocorrelation. Testing for the presence of spatial autocorrelation among regression residuals is discussed in the succeeding subsection.

Spatial autocorrelation in regression residuals has two closely related implications. Odland (1988, pp. 48-9) stated them as follows.

> 1. Spatial autocorrelation in the residuals indicates that the model is incorrect or, at least, incomplete. A complete and correct model would explain all of the systematic spatial organisation in the data and leave residuals that displayed no spatial organisation at all. The model needs to be corrected in order to provide a suitable explanation of how phenomena are organised in space.
>
> 2. Spatial autocorrelation in the residuals indicates that the regression model fails to fulfill an important independence condition. Consequently, it will not be a reliable basis for making statistical inferences and inferences based on the model are likely to be mistaken. The model needs to be corrected before it can be a reliable basis for testing hypotheses.

Based on the above implications, if the regression residuals are spatially autocorrelated, then the model given in (4.5) may be misspecified. An additional variable related to location or proximity variables needs to be incorporated in the model. Spatial autocorrelation in the error structures means that the error at each location depends on the error at the other locations.

When regression errors possessed geographic ordering, the first-order Markovian scheme can be specified for the error structure of model (4.5) given by

$$u_i = \sum_j^n \rho w_{ij} u_i + v_i, \qquad i = 1,2,3,\ldots,n \tag{4.9}$$

where the $\{w_{ij}\}$ are the spatial weights defined in Subsection 4.2.1, ρ is a scalar parameter and v_i is another error term. Most of the studies on spatial autocorrelation have assumed the above specification for linear statistical models (e.g. Hordijk 1974,

Aurora and Brown 1977, Cliff and Ord 1973, 1981, Miron 1984, Upton and Fingleton 1985, Griffith 1987).

In matrix form, (4.9) can be written as

$$u = \rho W u + v \tag{4.10}$$

or

$$u = (1 - \rho W)^{-1} v \tag{4.11}$$

where

u is an $n \times n$ vector of disturbance terms;
ρ is a scalar parameter;
W is an $n \times n$ matrix of proximity weights; and,
v is a column vector of disturbances.

Equation (4.11) explicitly shows the marginal direct and indirect effects of v on u (see Miron 1984, p.214). We further assume that $E(v)=0$ and $E(vv')=\sigma_v^2 I_n$, where is the variance σ_v^2 is the variance of v_t, a tth element of v, for all t's.

In general, the proximity matrix W can be defined in any fashion, depending on the researcher or upon the study in hand. It is important to stress that care must be used in the choice of W to avoid spurious correlation (Cliff and Ord 1973, p.12). That is why in the present study, three different sets of proximity weights will be utilised. A general rule must be observed for the elements of these W matrix. This rule as previously mentioned in Subsection 4.2.1, is such that W matrix should be normalised for each row summing to unity, that is, $\Sigma_j w_{ij} = 1$. This has the effect of making W asymmetrical and usefully restrains ρ within the range -1 to 1 when it is estimated. However, the magnitude of ρ would depend on the W matrix and sometimes gives a value outside this range. Moreover, Odland (1988) stressed that such normalisation is really necessary for ρ not to imply a spatial trend in each of the element in u. The parameter ρ, just like the Moran's I measures spatial association in the residuals.

Following the assumption that $v \sim N(0, \sigma_v^2 I_n)$ and assuming $(I - \rho W)$ is positive definite, the sampling properties of u in model (4.5) if it suffer from spatial autocorrelation is given by

$$
\begin{aligned}
E(u) &= 0, \\
E(uu') &= (I - \rho W)^{-1} E(vv')(I - \rho W')^{-1} \\
&= \sigma_v^2 \, [(I - \rho W')(I - \rho W)]^{-1} \\
&= \sigma_v^2 \, \Psi
\end{aligned}
\tag{4.12}
$$

where

$$\Psi^{-1} = [(I - \rho W')(I - \rho W)]. \tag{4.13}$$

Equations (4.5), (4.10) and (4.12) summarise the specification of the *spatial disturbances model.* The empirical task now is to estimate ρ, β, σ^2 and the standard errors of these estimates which is discussed in Subsection 4.3.3.

4.3.2 Testing Regression Residuals for Spatial Autocorrelation

Cliff and Ord (1981) developed a test statistic based on the Moran's *I* statistic defined in Section 4.2.2 to investigate the presence of spatial autocorrelation in the disturbance term of any linear model for spatial data. The method for testing its significance is also suggested by these authors. This is discussed below.

Consider a linear regression model for spatial data given by (4.5) with the assumption of normality and independence in its error term. Under the null hypothesis of non-existence of spatial autocorrelation, ordinary least squares (OLS) estimation would result to an unbiased estimator for β given by

$$b = (X'X)^{-1} X'Y \tag{4.14}$$

where the sample vector of residuals is estimated using

$$u = Y - bX. \tag{4.15}$$

With spatially located data, autocorrelated errors are common, and so it is very important to be able to test whether the assumption that $E(uu')=\sigma^2\Psi=\sigma^2 I$ has been satisfied. A test statistic that would measure spatial autocorrelation among regression residuals as well as one that can be used in testing the significance using distribution theory is needed.

Following Cliff and Ord (1972), the test statistic I, designed to test the presence of spatial autocorrelation among the residuals of model (4.5) is formulated as

$$I = \frac{n}{S_0} \cdot \frac{u'Wu}{u'u}. \tag{4.16}$$

Clearly, all the terms used in the above test statistics have been defined previously. Cliff and Ord evaluated the moments of I, under the assumption of normality following the *Pitmans-Koopman Theorem* (Pitmans 1937 and Koopman 1942). The derived expression for the mean of I is given by

$$E(I) = \frac{-n \cdot trace(A)}{(n-k) \cdot S_0} \tag{4.17}$$

where $A = (X'X)^{-1}X'WX$, is a k × k matrix and the variance of I is expressed as

$$Var(I) = \frac{n^2}{S_0^2(n-k)(n-k+2)}\left\{S_1 + 2trace(A^2) - trace(B) - \frac{2[trace(A)]^2}{n-k}\right\}, \tag{4.18}$$

where

$$\begin{aligned} B &= 4(X'X)^{-1}X'C^2X \\ C &= \frac{1}{2}(W + W'); \end{aligned} \tag{4.19}$$

and both B and C are $k \times k$ matrices.

In relation to the present study, Cliff and Ord provide an alternative distribution of I under the assumption of randomization (see also Haggett, et al. 1977, p.358). That is, for any distribution of X we consider the position of the observed value of I in the set of values obtained if I is evaluated for every possible spatial arrangement of the $\{x_i\}$ in the country system. There are n permutations $(n!)$ of such values. In effect, we are determining if the observed pattern of $\{x_i\}$ values, as judge by I is in any sense unusual in the set of all possible patterns that the $\{x_i\}$ could have formed. As a result, the

distribution of I under the null hypothesis of no spatial autocorrelation for the $n!$ random permutations would be identical with that of equations (4.3) and (4.4) for $E(I)$ and $Var(I)$ respectively (see Cliff and Ord 1972, for proof to this statement). The same distribution of I is applied in the study of Aten (1994).

Cliff and Ord (1981, Section 8.5) have shown that I is asymptotically normally distributed as $n \rightarrow \infty$, under the null hypothesis of no spatial autocorrelation, provided that in the proximity matrix, W, no definite set of countries dominates the study area. Sen (1976) has also proven that the large-sample size distribution of the test statistic I is asymptotically normal. Monte Carlo results in Cliff and Ord (1973) suggest that the normal approximation is indeed satisfactory for $n > 10$, in fact, in one of their example, a sample of size 16 can be assumed to ensure asymptotic normality. However, serious inferential error is really unlikely unless $n \leq 20$.

Hence, for large samples, a test of significance for the I statistic may be carried out using the (approximately) standard normal deviate given by

$$Z^* = \frac{I - E(I)}{\sqrt{Var(I)}} \sim N(0,1), \tag{4.20}$$

where the mean, $E(I)$, and variance, $Var(I)$, under the null hypothesis of no spatial autocorrelation, might be any one of those given in equations (4.3), (4.4), (4.17) and (4.18), depending upon the data or problem in hand. The null hypothesis is rejected whenever the observed value of Z^* falls in the critical region. This implies that at 5 percent level of significance, we would reject the null hypothesis if $|Z^*| \geq 1.96$ and conclude that there is a sufficient evidence for the existence of spatial autocorrelation among regression residuals.

As a summary, test procedures for spatial autocorrelation, just like the above, should always be applied to the residuals whenever a regression model is fitted to spatial data. Test results can then be very helpful in the development of an adequate model which incorporates spatial autocorrelation in the disturbance model. The above test procedures can easily be applied in the present study which would consider large number of sample observations in the modelling procedure as shown in Chapter 5.

4.3.3 Estimation of the Spatial Model Parameters

A regression model with spatially autocorrelated error is introduced earlier in Subsection 4.3.1 with the assumptions on the disturbance terms given by equations (4.12) and (4.13). An efficient estimation procedure is now designed to estimate the following parameters ρ β, σ_v^2 of the spatial disturbances model.

If ρ is known, an efficient estimator of β and σ_v^2 can be obtained using either the GLS or Maximum Likelihood estimation (ML) procedures since the OLS procedure is inefficient. The GLS estimates for β and σ_v^2 can be computed using the equations (4.6) and (4.8) respectively. However, in practice, ρ is unknown and it has to be estimated along with β and σ_v^2 .

Following Miron 1984, all the unknown parameters of the spatial disturbances model can be jointly estimated using simultaneous estimation procedure. Ord (1975) and Doreian (1980) has provided an ML method for the spatial disturbances model and this would be the initial step for the simultaneous procedure designed by Miron. The maximum likelihood approach to fitting this spatial model calls for a likelihood function equivalent to the likelihood function of a normal general linear statistical model given in Judge, et al. (1988, p.223).

As the v are multivariate normal (i.e., $v \sim N(0, \sigma_v^2 I_n)$), the joint likelihood function of v is given by

$$L = \frac{1}{(2\pi)^{n/2}\sigma_v^n} exp\left[\frac{-v'v}{2\sigma_v^2}\right]. \tag{4.21}$$

However, v is not observed. The problem now is that the errors $\{u_i\}$ are not independent, but are interrelated. The relationship between u and v is given by equation (4.11), that is, $v = (1-\rho W)u$, where $\Omega = (1-\rho W)$. Substituting this relation to equation (4.21) we obtained a likelihood function of u

$$L = |\Omega| \frac{1}{(2\pi)^{n/2}\sigma_v^n} exp\left[\frac{-(\Omega u)'(\Omega u)}{2\sigma_v^2}\right], \tag{4.22}$$

where $|\Omega|$ is the Jacobian of the transformation between v and u. Taking the logarithms of equation (4.22), we obtain the log-likelihood function

$$log\, L = -(n/2)ln(2\pi) - (n/2)ln\,\sigma_v^2 + ln|\Omega| - (1/(2\sigma_v^2))[u'\Omega'\Omega u] \qquad (4.23)$$

Finally, as the Y are observed, rather than u, the final log-likelihood function for the spatial disturbances model can be written using (4.15)

$$log\, L = -(n/2)ln(2\pi) - (n/2)ln\,\sigma_v^2 + ln|\Omega - (1/(2\sigma_v^2))[(Y - X\beta)'\Omega'\Omega(Y - X\beta)] \qquad (4.24)$$

Equation (4.24) above is a standard result for the log-likelihood function of a spatial model as demonstrated in the work of Hepple (1976), Ord (1975), Cliff and Ord (1981), and Ripley (1981). Ord (1975) simplified the maximisation of the above log-likelihood function by establishing that

$$|\Omega| = |I - \rho W| = \prod_{i=1}^{n}(1 - \rho\lambda_i), \qquad (4.25)$$

where is λ_i the ith eigenvalue of W. Hence

$$ln|\Omega| = ln|I - \rho W| = \sum_{i=1}^{n} ln(1 - \rho\lambda_i), \qquad (4.26)$$

Taking the partial derivative of $log\, L$ with respect to β and σ_v^2, gives

$$\frac{\partial(log\, L)}{\partial\beta} = -\frac{1}{2\sigma_v^2}[-2X'\Omega'\Omega Y + 2X'\Omega'\Omega X\beta] \qquad (4.27)$$

and

$$\frac{\partial(\log L)}{\partial\sigma_v^2} = -\frac{n}{2\sigma_v^2} + \frac{1}{2\sigma_v^4}\left[(Y - X\beta)'\Omega'\Omega(Y - X\beta)\right] \tag{4.28}$$

Setting both equation (4.27) and (4.28) to zero yields

$$\tilde{\beta} = [X'\Omega'\Omega X]^{-1} X'\Omega'\Omega Y; \tag{4.29}$$

and

$$\tilde{\sigma}_v^2 = \frac{1}{n}\left[(Y - X\beta)'\Omega'\Omega(Y - X\beta)\right] \tag{4.30}$$

The above expression is identical with the equations (4.6) and (4.8) respectively provided that equation (4.13) is true. Assuming ρ is known, estimates of $\tilde{\beta}$ can be readily obtained from (4.29) and with $\tilde{\beta}$ estimated, equation (4.30) provides $\tilde{\sigma}_v^2$. Both estimators depend on ρ. This parameter can be estimated using the ML procedure too since the log-likelihood function(*log L*) in (4.24) involves ρ as part of it.

Hence, an estimate of ρ can be obtained by simultaneously maximising equation (4.24) with respect to ρ as well as with respect to β and σ_v^2 . Miron (1984) suggests that a possible scheme to achieve this is to substitute the quantities (4.29), (4.30), and (4.26) for β, σ_v^2, and $ln|\Omega|$ respectively in the log-likelihood function (4.24), yielding a *concentrated* log-likelihood function, *log L**, given by

$$log\, L^* = -(n/2)[1 + ln(2\pi)] - (n/2) ln\, \tilde{\sigma}_v^2 + \sum_{i=1}^{n} ln(1 - \rho\lambda_i). \tag{4.31}$$

Maximising *log L** with respect to ρ generates the following first-order condition:

$$-\frac{n}{\tilde{\sigma}_v^2}\left[-u'\Omega u - u'\Omega'\Omega u\rho\right] - \sum_{i=1}^{n}\frac{\lambda_i}{1 - \lambda_i\rho} = 0, \tag{4.32}$$

where $u = (Y - X\beta)$.

As n gets larger, the term in the bracket on the left of the equation (4.31) comes to dominates (see Miron 1984). Hence, Miron suggested that when n is large, a ML estimator of ρ is obtained using

$$\tilde{\rho} \cong \frac{(u'\Omega u)}{(u\ \Omega'\Omega u)}. \tag{4.33}$$

The above result confirms the convergence of GLS and ML estimators at large sample sizes.

To be able to derive the previously mentioned ML estimators for β, σ_v^2 and ρ, the following step procedures is recommended:

1. Set an initial estimate of ρ and call it $\tilde{\rho}_0$
2. Using the proximity matrix W, construct the $\Omega = (1 - \tilde{\rho}_0 W)$ matrix
3. Estimate β using equation (4.29)
4. Estimate σ_v^2 using equation (4.30)
5. Derive an estimate of u using the result in step 3.
6. Numerically solve equation (4.32) for the estimate of ρ and call it $\tilde{\rho}_1$. Note that for large samples we may use equation (4.33)
7. Compare $\tilde{\rho}_0$ with $\tilde{\rho}_1$. If they have an approximately identical value, the procedure then stop, otherwise, we set $\tilde{\rho}_0 = \tilde{\rho}_1$, proceed again to step 2 and continue the process.

The iterative process is terminated when $\tilde{\rho}$ converges to a certain value. The results of which are the final estimates of the parameters for the spatial disturbances model. It would of course necessary to check the local versus global nature of the option. This culminates the simultaneous estimation procedure recommended by Miron using the method of maximum likelihood for spatial disturbances model proposed by Ord (1975) and Doreian (1980).

It is mentioned earlier that this simultaneous estimation procedure can also be done through GLS rather than ML approach. For the least squares procedure, the main objective is to minimise the expression

$$\left[(Y-X\beta)'\Phi^{-1}(Y-X\beta)\right], \tag{4.34}$$

where $\Phi^{-1}=\left[(I-\rho W)'(I-\rho W)\right]$. Maximising the expression (4.34) with respect to β and ρ yields the following estimators:

$$\hat{\beta}=\left(X'\Phi^{-1}X\right)^{-1}X'\Phi^{-1}Y \tag{4.35}$$

$$\hat{\rho}=\frac{(u'\Omega u)}{(u\ \Omega'\Omega u)} \tag{4.36}$$

The above estimator can be solved iteratively using the same seven-step procedure mentioned earlier for the ML approach. In addition, the above estimators are identical to equations (4.6) and (4.33) for large samples.

Following Ord (1975) and Doreian (1980), the asymptotic variance-covariance matrix of the parameter estimates under the ML approach, is given by

$$V(\tilde{\sigma}_v^2,\ \tilde{\rho},\ \tilde{\beta})=\tilde{\sigma}_v^2\cdot\begin{bmatrix} n/2 & \tilde{\sigma}_v^2\ tr(B) & 0' \\ \tilde{\sigma}_v^2\ tr(B) & \tilde{\sigma}_v^4\ (tr(B'B)-\alpha) & 0' \\ 0 & 0 & \tilde{\sigma}_v^2\ X'\Omega'\Omega X \end{bmatrix}^{-1}. \tag{4.37}$$

where

$$\alpha\ =\ -\sum_{i=1}^{n}\lambda_i^2\ /\ (1-\rho\lambda_i)^2,$$

$$B\ =\ W\,\Omega^{-1},$$

0 is a column vector of zeros.

In the case of the least squares estimation, the covariance matrix for $\hat{\beta}$ is given by

$$V(\hat{\beta})=\hat{\sigma}_v^2\ (X'\Phi^{-1}X)^{-1} \tag{4.38}$$

where $\hat{\sigma}_v^2$ can be estimated using equation (4.8), and Ψ^{-1} replaced by Φ^{-1}. However, Griffith (1987) disclosed that there is no closed form for the asymptotic estimator of the standard error for both the parameter estimates of $\hat{\sigma}_v^2$ and $\hat{\rho}$ in the GLS approach. Griffith also asserted that the inability to determine such closed form solutions for the latter parameter estimates reiterates the notion that difficulties lying dormant in conventional statistical analysis emerge during the analysis of geographically mapped data. These sorts of problems have encouraged some spatial analysts to do nothing more than test for the presence of spatial autocorrelation among regression residuals. Much more, the standard error matrix (4.37) for the parameter estimates derived using ML approach highly depend on the proximity matrix, particularly on its eigenvalues. Upton and Fingleton (1985, p.300) claimed that calculating this variance-covariance matrix appears to be a formidable task, particularly on large W matrices. Numerical extraction of the eigenvalues may find to be difficult and matrix inversion of large matrices is considerably a computer burden (Ord 1975). But these computational difficulties may not be of same relevance in the modern day computing.

4.3.4 Goodness-of-fit Test and Evaluation of the Spatial Parameter Estimates

Given the specification of the spatial disturbances model, it is desirable to assess whether ρ is truly non-zero and to perform usual inference procedures on β. Assessment of the desired spatial model can also be made through comparisons of the standard error estimates for $\hat{\sigma}_v^2$. Likewise, an overall measure of goodness-of-fit may be adapted for linear spatial models.

Examining equations (4.5) and (4.10), it is evident that the effect of spatial autocorrelation is that u is linked via ρ (usually positive in magnitudes) and the proximity matrix W to u for contiguous countries, so that the residual sum of squares is a downwardly biased estimate of the error sum of squares (Upton and Fingleton 1985). Ignoring this bias in the residual variance estimate, one would then come up with inflated values of the conventional t, F, and r^2 statistics. The following alternative test approaches have been suggested by Upton and Fingleton as substitutes for the said conventional test statistics.

The r^{*2} statistic

If the specified spatial disturbances model is a good fit to the geographically ordered data, then the estimate for $\hat{\sigma}_v^2$ obtained using either equation (4.8) or (4.30) will be

comparatively small when compared to the total variability of Y. Doreian (1981) proposed a useful summarising quantity expressed as

$$r^{*2} = \frac{\left[\sum_{i=1}^{n}(Y_i - \bar{Y})^2 / n\right] - \hat{\sigma}_v^2}{\left[\sum_{i=1}^{n}(Y_i - \bar{Y})^2 / n\right]} \tag{4.39}$$

The quantity r^{*2} is simply the sample proportion of the variation in Y that is explained by the spatial disturbances model. This quantity is identical to r^2 of the classical statistical approach when $\hat{\sigma}_v^2$ is estimated using OLS. A relatively high value for r^{*2} would mean that the spatial disturbances model has been quite successful in explaining the original variability in the sample. Doreian derived equation (4.39) as the square of the observed correlation between Y and $\hat{Y}$, hence, just like r^2, this could be thought as a good measure of fit for the specified spatial disturbances model.

The ratio test

It is mentioned earlier that the t-test is, strictly speaking, invalidated by the presence of spatial dependence. An equivalent test was suggested by Upton and Fingleton (1985) which relies on the asymptotic normality and independence of the estimators defined in Section 4.3.3. The procedure involves comparing the value of the *ratio* of an estimate, $\hat{\theta}$, to its estimated standard error, $s.e.(\hat{\theta})$, with the appropriate critical value from a standard normal ($N(0,1)$) distribution. The test statistic is given by

$$Estimated\ \ Ratio = \frac{\hat{\theta}}{s.e.(\hat{\theta})} \tag{4.40}$$

which is then compared to N(0,1). This would mean that if the estimated ratio lies outside the approximate 95 percent two-sided critical values of ± 1.96, it would be evident that $\hat{\theta}$ deviates significantly from its expectation (i.e., zero under the null hypothesis) at 5 percent level of significance. The fact that the spatially autocorrelated

error model incorporates a location effect into its error structure, it has a profound effect upon the apparent significance of the explanatory variables, X.

The Likelihood Ratio Test

So far, attention has been focused on testing the significance of individual parameter estimates. An alternative approach to test the significance of these sets of parameter estimates can be accomplished through the well known likelihood ratio test (LR). Likelihood ratio test can assess the significance of either an individual parameter estimate or sets of parameter estimates. This test is necessary also in determining the adequacy of any fitted model.

Basically, one compares a pair of nested models, the second of which is identical to the first apart from the j ($j \geq 1$) restrictions that have been placed on the parameters of the first model. This pair of models is usually termed as restricted and unrestricted models respectively. In this case, the proposed spatial disturbance model can be considered as the unrestricted model while the original model without the spatial variable can be regarded as the restricted one.

In practice, the LR test considers the log-likelihood function of both the restricted ($\log(L_R)$) and unrestricted ($\log(L_U)$) models. Under the null hypothesis that the restriction is true, that is, the spatial parameter ρ is not significantly different to zero, the test statistic is defined by

$$LR = -2\left[log(L_R) - log(L_U)\right] \tag{4.41}$$

The above test statistic is asymptotically distributed as χ_j^2 , j being the number of restrictions. Rejection of the null hypothesis would mean that the proposed spatial disturbances model seems appropriate or adequate for the geographically-located data.

Comparing two spatial disturbances models would mean, looking at which form of W (proximity measures) would really give the best fit for the spatial data. As LR cannot accommodate this case, a simple procedure can be done to compare the two spatial models. An intuitive way is to look at their corresponding value of the log-likelihood function. An estimated parameter for which *log L* given by equation (4.24) is large is more likely to be the true value for the parameter than for which *log L* is small (Judge, et al. 1988, p.223). Hence, a spatial disturbances model that has relatively large value of *log L* would then have the better proximity measure, *W*, to capture the effect of *v* on *u*.

This would be the criteria to be able to detect the sensitivity of the spatial model on the location variables.

4.4 Concluding remarks

This chapter has essentially focused on the conceptual framework associated with spatial autocorrelation, its measurement and then the treatment of specification, estimation and hypothesis testing - spatial autocorrelated within a standard regression framework. Several proximity measures have been defined to account for this spatial phenomena. The next chapter will apply this framework, with appropriate modifications, to the regression models that result in the TT and CCD indices. Conceptually, spatial autocorrelation plays a role only in the case of the model for the CCD index.