Software Impacts 20 (2024) 100652

Contents lists available at ScienceDirect

Software Impacts

journal homepage: www.journals.elsevier.com/software-impacts

Original software publication

Mathematica code for the topological analysis of Thom's Catastrophes in 2×2 economic games

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A R T I C L E I N F O

Keywords:
Catastrophe theory
Game theory
Economics
Bifurcations
Criticality

A B S T R A C T

René Thom's work on topological instabilities applied new methods to questions of dynamical stability that traditionally belonged to the domain of dynamical systems theorists. Topological instability focuses on universal properties of bifurcations in systems where multiple equilibria form and disappear as a function of system parameters. However, the complete mathematical description is quite abstract and the analysis benefits from graphical intuitions. Here we provide the code, in the form of a Mathematica notebook, used in our recent Games and Economic Behaviour paper (Harriset al., 2023). It illustrates our main results providing the intuition necessary to explore the bifurcations in the formal proofs.

Current code version	v1.0
Permanent link to code/repository used for this code version	https://github.com/SoftwareImpacts/SIMPAC-2024-65
Permanent link to Reproducible Capsule	
Legal Code License	GNU General Public License Version 3.0
Code versioning system used	GIT
Software code languages, tools, and services used	Mathematica version 12.0
Compilation requirements, operating environments & dependencies	Not compiled, runs as an executable notebook, requires Wolfram's Mathematica v. 12.0
If available Link to developer documentation/manual	https://github.com/M-Harre/Game-Theory-Catastrophes/blob/main/README.md
Support email for questions	michael.harre@sydney.edu.au

1. Introduction

Catastrophe Theory (CT) was first introduced by René Thom in the 1960s (see Thom's book [1]) and brought to the broader scientific community via its applications by Zeeman [2] and many others [3–5]. Amongst these applications economics featured prominently [2,6–10] but after an initial enthusiasm for the approach it fell out of favour with researchers [11].

Recently though CT has had something of a revival in applied economics via the development and application of stochastic CT [12, 13] and its subsequent applications in the study of time series data and critical phenomena [14–16]. Critical phenomena in markets and economies have been studied extensively [17–22] and it has been, in part, attributed to the herding behaviour observed in market trading [23–28]. Alongside these applied developments the theoretical foundations of stochastic decision theory, specifically economic game theory, has progressed through the study of bifurcations in the Quantal Response Equilibrium (QRE) [29–31] amongst others.

The connection between the QRE and market dynamics has been demonstrated recently in a number of papers showing that the micro-economics of individual choice, via the QRE, can give rise to the non-linear criticality observed in the time series data [32–34]. This provides a bridge from individual decisions to the collective and emergent phenomena observed in the complex dynamics of economic markets. With this in mind, developing a CT approach to the micro-economics of the QRE has ramifications for the modelling of critical dynamics observed in markets, and so the software developed in the research program behind [35] aids in this effort.

We emphasise that the software package described here can analyse the topological structure of all 2 player, 2 strategy, symmetric

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https://doi.org/10.1016/j.simpa.2024.100652
Received 3 March 2024; Received in revised form 18 April 2024; Accepted 2 May 2024

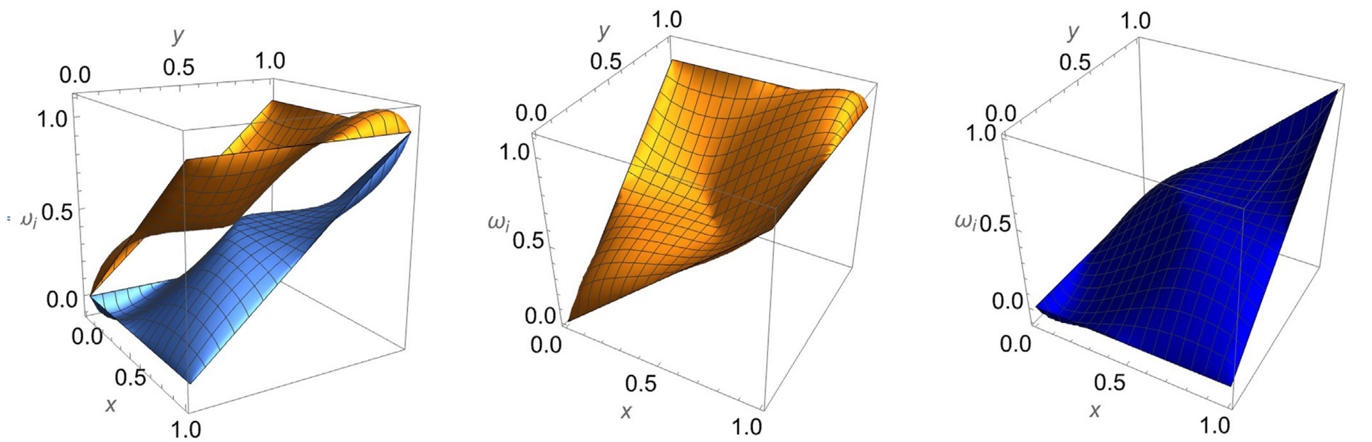

Fig. 1. Upper and lower sheets for $\omega^{+/-}$.

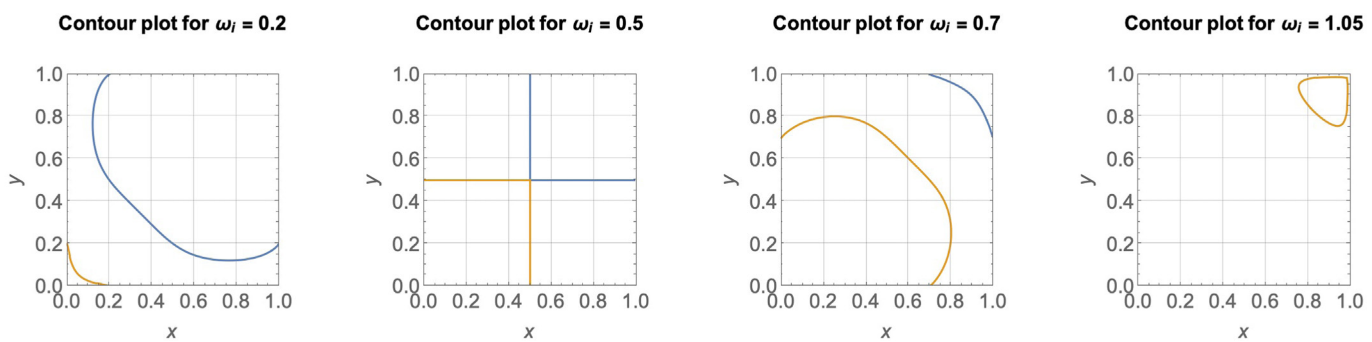

Fig. 2. Examples of the critical sets (bifurcations) in the (x, y) plane formed by choosing a specific set of values for $\omega^{+/-}$.

and asymmetric games. The examples in the code include all of the topological structures possible in this large class of games, including examples that were not included in the original article [35]. With this in mind, the audience for this software can be categorised into several different groups. One group are those economists working on the quantal response model, see for example [19,36–38]. There is also an audience in the mathematical natural sciences, such as those interested in equilibria of evolutionary systems with bifurcation structures or other non-linear phenomena, see for example [39–41]. In artificial intelligence there is also growing interest in the foundations of inter-agent interactions and their non-linearities, see for example [42–44]. The audience also includes anyone interested in the non-linear aspects of interactions between 'agents' of any variety where their interactions can be described using game theory.

2. Description and purpose

The code is a fully commented Mathematica notebook that uses a combination of symbolic and numerical methods to construct and analyse the bifurcation structures found in [35]. It is written in a tutorial style, taking advantage of the notebook format, so that the reader can reconstruct both the numerical results and the formal steps in the proofs of the original analysis in [35]. The original work carried out in developing this code appears in a number of articles [29–31,45,46], as well as the first author Harré's doctoral dissertation. The code as presented here was used in exploring and illustrating the latest paper [35] and it is run on Mathematica version 13.1.0.0 where previous versions of the code ran on earlier versions of Mathematica and there were limitations as to how well the images could be rendered as well as several complicated work arounds being needed to extract the necessary data from the code. These have since been made easier in the latest version of Mathematica (at least since version 12.0.0.0) and the code is subsequently much easier to use and understand.

The notebook runs as a standalone single file with instructions in plain text embedded in the file. The text makes reference to the location of the formulas in the original article [35] for ease of reference. However, below we also provide a quick overview to the way in which the code is intended to be used.

The first step is to solve the following equation for ω:

$$\log\left[\frac{1-x}{x}\right]\log\left[\frac{1-y}{y}\right]-\frac{x-\omega}{x(1-x)}\frac{y-\omega}{y(1-y)}=0 \tag{1}$$

For those values of ω for which solutions exist, there are two distinct sets that are called the upper ω^+ and lower ω^- sheets, see Fig. 1. Then by choosing specific values of $\omega = \{\omega_1, \omega_2\}$ corresponding to a specific (possibly asymmetric) 2×2 game it selects a horizontal plane through the left-hand plot of Fig. 1, essentially choosing the appropriate level set for the game being studied. Note that the exceptional games whose double bifurcation topology was newly discovered in [35] are related to games for which the ω_i form the "closed loop" for either $-0.1 < \omega_i < 0$ or $1.0 < \omega_i < 1.1$. For all of the examples where there are two distinct curves in the level set, these correspond to games for which there are 3 Nash equilibria, such bifurcation structures have been well known for a long time, but their formal analysis as catastrophes in the sense of Thom is the central question addressed in [35]. See Fig. 2 for examples of these level sets. These critical sets then need to be mapped from the (x, y)-plane to the (κ_1, κ_2)-plane where the numerical data points are extracted directly from the Mathematica variable that stores them. Once they are in this format the ω_i and κ_i terms are now known for each point, and so their positions can be rendered on the QRE surface. See Fig. 3.

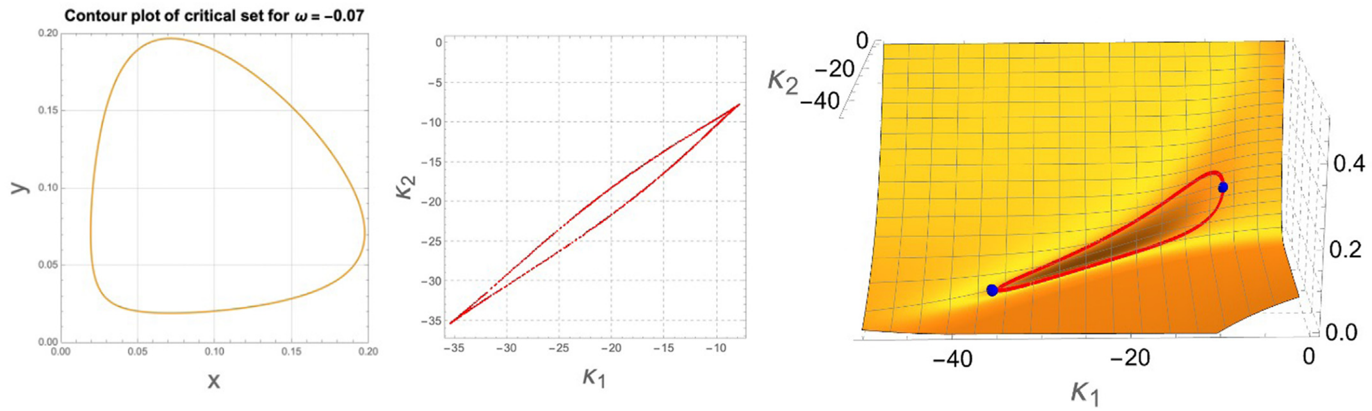

Fig. 3. An overview of the sequence of steps that are needed to take the critical sets in (x, y) for a given $\omega^{+/-}$ onto the QRE surface for the game specified by the $\omega^{+/-}$.

3. Impact overview

The significance of this code lies in the ability to directly compute the numerical values of the critical sets for the bifurcation structure of arbitrary 2×2 games, including the newly discovered exceptional cases of two triple branch points that were discovered in [35]. While one of the central roles that this code played in the development of that article was to provide intuition and numerical support for the formal results, the code is also important in that it can act as a starting point for further numerical and analytical results that extend the ideas of Catastrophe Theory into more complex cases than those considered so far. This, in part, supports the suggestion by Rosser, Jr. [11] to reconsider Catastrophe Theory in economics as an important tool that analysts should be aware of. Specifically, the nonlinear response of the agents' QRE strategies to parameter variations in micro-economic theory is much richer than has previously been assumed as the appearance of two triple branch points shown in Fig. 3 can only exist for 2×2 games in which there is a single, unique, Nash equilibria such as Prisoner's Dilemma. Consequently having observed multiple critical points in otherwise simple seeming systems deserves further analysis. This result is similar to how surprising the complex strategies were that emerged from a more detailed formal analysis of singularities in the work of Press and Dyson [47].

4. Limitations and future development

At the moment the code is only able to analyse two agent, two choice, asymmetric games in strategic form. There are theoretical reasons as to why further development of CT to an arbitrary number of players with an arbitrary number of choices in asymmetric games, as is the general case for game theory [48], is more difficult without some simplifications being made. Further development of the code to accommodate extensive form games is a highly desirable feature for future iterations, and connecting these to recent results in artificial intelligence [43] and causal game theory [42] also appears to be a very fruitful direction to explore.

CRediT authorship contribution statement

Michael S. Harré: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization. **Adam Harris:** Writing – review & editing, Validation, Methodology, Investigation, Formal analysis. **Scott McCallum:** Writing – review & editing, Validation, Methodology, Investigation, Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

This research was funded by the Australian Research Council (ARC) Discovery Project grant no. DP170102927. The ARC had no impact on the subject matter or influence on the results of this work.

DOI: 10.5281/zenodo.8395857
https://doi.org/10.5281/zenodo.10652476

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