Cubo A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 26, no. 2, pp. 239–258, August 2024
DOI: 10.56754/0719-0646.2602.239

The Levi-Civita connections of Lorentzian manifolds with prescribed optical geometries

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ABSTRACT

We explicitly derive the Christoffel symbols in terms of adapted frame fields for the Levi-Civita connection of a Lorentzian n-manifold (M, g), equipped with a prescribed optical geometry of Kähler-Sasaki type. The formulas found in this paper have several important applications, such as determining the geometric invariants of Lorentzian manifolds with prescribed optical geometries or solving curvature constraints.

RESUMEN

Derivamos explícitamente los símbolos de Christoffel en términos de los campos de marcos adaptados para la conexión de Levi-Civita de una n-variedad Lorentziana (M, g), equipada con una geometría óptica prescrita de tipo Kähler-Sasaki. Las fórmulas halladas en este artículo tienen diversas aplicaciones importantes, tales como determinar los invariantes geométricos de variedades Lorentzianas con geometrías ópticas prescritas o resolver restricciones sobre la curvatura.

Keywords and Phrases: Levi-Civita connection, optical geometry, congruence of shearfree geodesics, Sasaki manifolds.

2020 AMS Mathematics Subject Classification: 53B30, 83C05.

Published: 09 July, 2024
Accepted: 30 April, 2024
Received: 12 November, 2023

1 Introduction

An *optical geometry*, a notion introduced in the late eighties by Robinson and Trautman, is a geometrical structure that encodes the existence of an electromagnetic plane wave – or an appropriate higher dimensional generalisation [2] – propagating along a prescribed foliation by curves of a Lorentzian manifold. Let us recall the relevant definitions. A *null congruence* on a Lorentzian n-manifold (M, g), $n \geq 3$, is a foliation by curves, which are tangent to some nowhere vanishing null vector field. Given a Lorentzian n-manifold (M, g), $n \geq 3$, a null congruence is called *geodesic shearfree*, or *shearfree* for short, if there is a choice for a nowhere vanishing tangent null vector field p, whose local flow preserves both the codimension one distribution $\mathscr{W} := \mathrm{p}^{\perp_g}$ and the conformal class of the induced degenerate metric $h := g|_{\mathscr{W}\times\mathscr{W}}$ on the spaces $\mathscr{W}_x = \mathrm{p}^{\perp_g}|_x$, $x \in M$. These conditions are equivalent to requiring that the Lie derivative $\mathscr{L}_{\mathrm{p}} g$ has the form

$$\mathscr{L}_{\mathrm{p}} g = fg + \mathrm{p}^{\flat} \vee \eta \qquad \text{for some function } f \text{ and some 1-form } \eta\ . \tag{1.1}$$

If this holds, the vector field p is also geodesic, *i.e.* $\nabla_{\mathrm{p}}\mathrm{p} = \lambda\mathrm{p}$, and the curves of the congruence are geodesics (see *e.g.* [1, 2, 5, 14]). A quadruple $\mathscr{Q} := (\mathrm{p}, \mathscr{W}, [h], \{g\})$, given by

(a) a nowhere vanishing vector field p, determined up to multiplication by a nowhere zero smooth function f,

(b) a codimension one distribution $\mathscr{W}$,

(c) a conformal class $[h]$ of semi-positive metrics on $\mathscr{W}$,

(d) a non-empty set of Lorentzian metrics $\{g\}$, which are exactly all metrics g with respect to which p is a null vector field with $\mathscr{W} = \mathrm{p}^{\perp_g}$ and $[h] = [g|_{\mathscr{W}\times\mathscr{W}}]$ and both $\mathscr{W}$ and $[h]$ are preserved by the local flow of p,

is an *optical geometry* in the sense of Robinson and Trautman [2, 5, 14] [1]. The Lorentzian metrics g in the set $\{g\}$ are called *compatible with the prescribed optical geometry* $\mathscr{Q}$.

By Robinson's Theorem [8, 13], the shearfree null congruences of a real analytic four dimensional Lorentzian manifold are exactly the foliations by the lines of propagation of electromagnetic plane waves.

Many interesting examples of optical geometries $\mathscr{Q} = (\mathrm{p}, \mathscr{W}, [h], \{g\})$ are provided by connections on principal A-bundles $\pi : M \to S = M/A$ with one-dimensional structure groups $A = \mathbb{R}$ or S^1. On each bundle of this kind, one may consider an optical geometry in which p is the generator of the

[1] As a matter of fact, all four elements of $\mathscr{Q}$ can be recovered just by (i) the 1-dimensional distribution $\mathscr{K}$, which is generated by p and (ii) the set of metrics $\{g\}$, provided that they satisfy appropriate conditions. Thus, the optical geometries can be also defined as such pairs $(\mathscr{K}, \{g\})$ – see the original definition in [14].

action of the group A along the fibers, and $\mathscr{W}$ and $[h]$ are the appropriate A-invariant distribution and conformal class. In this case, the quadruple $\mathscr{M} := (\pi : M \to S, \mathrm{p}, \mathscr{W}, [h])$ is called a *regular shearfree manifold* and a metric $g \in \{g\}$ of the corresponding optical geometry $\mathscr{Q} = (\mathrm{p}, \mathscr{W}, [h], \{g\})$ is said to be a *compatible metric* of $\mathscr{M}$.

The regular shearfree manifolds are important geometric objects not only for their role in Lorentzian geometry, but also for their relations with CR geometry. Indeed, for any regular shearfree manifold $\mathscr{M} := (\pi : M \to S, \mathrm{p}, \mathscr{W}, [h])$, the base manifold $S = M/A$ is naturally equipped with the codimension one distribution $\mathscr{W}^S \subset TS$ and the positive definite conformal metric $[h^S]$ that are obtained by projecting the A-invariant distribution $\mathscr{W} := \mathrm{p}^\perp$ and conformal class $[h]$ onto $S = M/A$. If M is even dimensional and the projected distribution $\mathscr{W}^S \subset TS$ is contact then the regular shearfree manifold $\mathscr{M}$ is called *(maximally) twisting*. For any such $\mathscr{M}$, the corresponding optical geometry $\mathscr{Q} = (\mathrm{p}, \mathscr{W}, [h], \{g\})$ determines a family J^S of complex structures $J^S_x : \mathscr{W}^S_x \to \mathscr{W}^S_x$ on the projected distribution of S, that make S a *strongly pseudoconvex almost CR manifold* (see, *e.g.* [1, 2, 5, 7] and references therein).

Celebrated examples of twisting regular shearfree manifolds are given by the 4-dimensional spacetimes with Taub-NUT metrics and the 4-dimensional Kerr black holes. For such Lorentzian manifolds, the base manifold of the A-bundle $\pi : M \to S$ has an additional remarkable geometric feature: it is a principal bundle $\pi^S = S \to N$ with one dimensional structure group $A' = \mathbb{R}$ or $A' = S^1$, and the base manifold $N = S/A' = M/(A \cdot A')$ has a natural structure of a Kähler manifold. Moreover, the strongly pseudoconvex almost CR manifold $(S, \mathscr{W}^S, J^S)$ is a *regular Sasaki manifold* and the structure group A' of S preserves

(i) the CR structure $(\mathscr{W}^S, J^S)$,

(ii) a contact 1-form θ_o for $\mathscr{W}^S$, *i.e.*, $\mathscr{W}^S = \ker\theta_o$, such that $d\theta_o = \pi^{S*}\omega_o$ for some Kähler form $\omega_o = g_o(J\cdot, \cdot)$ on (N, J);

(iii) the conformal class $[h]$ on $\mathscr{W}$ contains the degenerate metric $h_o = \big((\pi^S \circ \pi)^* g_o\big)|_{\mathscr{W}}$.

The fact that the Taub-NUT and Kerr metrics have these properties is one of the reasons of the interest in twisting regular shearfree manifolds, in which the almost CR manifold $(S, \mathscr{W}^S, J^S)$ is a Sasaki manifold projecting onto a Kähler manifold. Such manifolds are called *of Kähler-Sasaki type* [2].

We recall that, according to classical results in the theory of G-structures, any local isometric invariant of a pseudo-Riemannian manifold is fully determined by the components in orthonormal bases of the Riemann tensor R and its covariant derivatives up to an appropriate order (see, for instance, [9–12, 16] and references therein). Such components are in turn given by the components of g and the Christoffel symbols of ∇^g in a frame field. This observation indicates that the explicit expressions of the Christoffel symbols in appropriate frame fields represent a fundamental tool for studying the compatible metrics of a given regular shearfree manifold of Kähler-Sasaki type and possibly finding solutions of the Einstein (or other physically relevant) equations in this class of metrics.

In this paper, we discuss in great detail the Christoffel symbols of the Levi-Civita connection ∇^g of a compatible metric g of a regular shearfree manifold $\mathscr{M} := (\pi : M \to S, \mathrm{p}, \mathscr{W}, [h])$ of Kähler-Sasaki type. More precisely, we fix a special (locally defined) frame field $(e_1, \ldots, e_n)$, which is well adapted to the optical geometry and is determined only up to a choice of a local frame field on the underlying Kähler manifold $N = M/(A \cdot A')$. Such a frame field has the following two useful properties:

(1) the last two vector fields e_{n-1}, e_n are the generators of the actions of the groups A and A', respectively, and are therefore canonically associated with the considered manifold;

(2) the vector fields e_i, $1 \leq i \leq n-2$, are tangent to the distribution $\mathscr{W}$ at all points and are $A \cdot A'$-invariant, thus projecting onto a frame field $(\widetilde{e}_1, \ldots, \widetilde{e}_{n-2})$ on N.

Note that (1) and (2) allow to minimise the number of parameters that are necessary to determine the components of a compatible metric g. Notice also that, due to the fact that $\mathscr{M}$ is twisting, a frame field satisfying (1) and (2) cannot coincide with a coordinate frame field. This forces us to avoid the use of coordinates in all subsequent computations.

After choosing an adapted frame field of this kind, we write down the general expression of a compatible metric g in terms of its dual frame field and we determine the Christoffel symbols of ∇^g in such frame and coframe fields, using just Koszul's formula and classical results on transformations of Levi-Civita connections under conformal transformations.

The expressions for the Christoffel symbols given in this paper have been originally determined during the preparation of [2] and have been successfully used to derive a coordinate-free characterisation of the generalised Taub-NUT metrics on even dimensional manifolds (see *e.g.* [3] and references therein for other characterisations of the metrics of such a kind). However, the details of the actual computations did not appear in [2] and some formulas of that paper had some minor sign errors – very few indeed and with no effect on any statement and proof. The same explicit (and amended) expressions have been later used in [6] for determining explicit expressions for the components of the Ricci tensor of compatible metrics of a shearfree manifold $\mathscr{M}$ of Kähler-Sasaki type satisfying conditions that generalise Kerr's ansatz for the 4-dimensional rotating black holes. These expressions for the Ricci tensor allowed us to translate the Einstein equations for a compatible metric into equations on its parameters in an adapted frame and to find a large class of exact solutions that naturally includes the classical Kerr black holes. We anticipate a number of further applications of the explicit expressions of these Christoffel symbols and believe that the detailed computations we present in this paper will be a helpful tool for other researchers who are interested in the developments of this field.

The paper is structured into two sections: In section 2, we define the adapted frame fields of a compatible metric, that is the frame fields in which all computations of this paper are performed; In section 3 we derive the explicit list of Christoffel symbols and provide the details of the computations.

2 The general form of a compatible metric on a shearfree manifold of Kähler-Sasaki type

2.1 Notational issues

Consider a shearfree manifold $\mathscr{M} := (\pi : M \to S,\ \mathrm{p}, \mathscr{W},\ [h])$ of Kähler-Sasaki type. We use the following notation:

(1) (N, J, g_o) is the Kähler manifold onto which S projects and $\omega_o = g_o(J\cdot, \cdot)$ is the Kähler form of N [2];

(2) A and A' are the 1-dimensional structure groups of the principal bundles $\pi : M \to S$ and $\pi^S : S \to N$, respectively;

(3) p_o and q_o^S are the fundamental vector fields of the principal bundles $\pi : M \to S$ and $\pi^S : S \to N$, corresponding to the element of the standard basis of $Lie(A) = Lie(A') = \mathbb{R}$. This means that $\Phi_s^{\mathrm{p}_o}(x) = e^s(x)$, $x \in M$, and $\Phi_s^{\mathrm{q}_o^S}(y) = e^s(y)$, $y \in S$;

(4) θ_o is the contact A'-invariant 1-form on S satisfying the conditions

$$d\theta_o = \pi^{S*}\omega_o\ , \qquad \theta_o(\mathrm{q}_o) = 1\ , \qquad \ker\theta_o|_x = \mathscr{W}_x^S\ \ ,\ x \in S\ ; \tag{2.1}$$

and ϑ_o is the pull-back $\vartheta_o = \pi^*(\theta_o)$ of θ_o on M.

It is important to note that $\mathscr{W}^S$ is an A'-invariant horizontal distribution on the principal bundle $\pi^S : S \to N$, and it is therefore a connection for this bundle. The associated connection 1-form is θ_o and its curvature 2-form is $d\theta_o = \pi^{S*}\omega_o$.

For what concerns the A-bundle $\pi : M \to S$, throughout the paper *we assume that it is trivial and equipped with the natural* flat *connection of a Cartesian product.* This apparently restrictive condition can be always locally satisfied replacing S by an open subset $\mathscr{V} \subset S$, on which the bundle

[2] Note that there is a sign difference in the definition of ω_o w.r.t. [2]. There it is defined as $\omega_o := g_o(\cdot, J\cdot)$.

is trivialisable, and identifying $\pi : M \to S$ with the trivial bundle $\pi : \pi^{-1}(\mathscr{V}) \simeq \mathscr{V} \times A \to \mathscr{V}$ equipped with the standard flat connection.

We denote by $\mathscr{H}_o$ the horizontal distribution of the flat connection of $\pi : M \to S$.

For any given vector field X on the Kähler manifold N, we denote by

- $X^{(S)}$ the unique A'-invariant horizontal vector field in $\mathscr{W}^S \subset S$ projecting onto X;
- $\widehat{X}$ the unique A-invariant horizontal vector field in $\mathscr{H}$ projecting onto $X^{(S)}$ and thus also onto X; note that, by definition of $\mathscr{W}^S$, the vector field $\widehat{X}$ takes values in $\mathscr{H}_o \cap \mathscr{W}$.

The unique A-invariant horizontal vector field in $\mathscr{H}_o$ projecting onto q_o^S is denoted by q_o.

Owing to the A- and A'-invariance of the connections of $\pi : M \to S$ and $\pi^S : S \to N$ and the properties of the connection 1-form θ_o, for any pair of vector fields X, Y on N the following Lie bracket relations hold [3]:

$$[\widehat{X}, \widehat{Y}] - \widehat{[X,Y]} = -g_o(JX, Y)\mathrm{q}_o \ , \qquad [\widehat{X}, \mathrm{p}_o] = [\widehat{X}, \mathrm{q}_o] = [\mathrm{p}_o, \mathrm{q}_o] = 0 \ . \tag{2.2}$$

2.2 The adapted frame fields

Consider a frame field $(E_1, \ldots, E_{n-2})$ on an open set $\mathscr{V} \subset N$ of the Kähler manifold and the corresponding lifted vector fields $(\widehat{E}_1, \ldots, \widehat{E}_{n-2})$ on M, taking values in the distribution $\mathscr{W}' = \mathscr{H} \cap \mathscr{W}$. The vector fields of the $(n-1)$-tuple $(\widehat{E}_1, \ldots, \widehat{E}_{n-2}, \mathrm{p}_o)$ are pointwise linearly independent and hence give linear frames for the spaces $\mathscr{W}_x \subset T_xM$, $x \in \mathscr{U} = (\pi^S \circ \pi)^{-1}(\mathscr{V})$. Since q_o projects onto q_o^S and q_o^S is transversal to $\mathscr{W}^S = \pi_*(\mathscr{W})$, the vector fields of the n-tuple

$$(\widehat{E}_1, \ldots, \widehat{E}_{n-2}, \mathrm{p}_o, \mathrm{q}_o) \tag{2.3}$$

are pointwise linearly independent and determine a frame field on $\mathscr{U}$. We call (2.3) the *adapted frame field of $\mathscr{M}$ determined by the frame field (E_i) on N.*

Note that, due to (2.2), the Lie brackets between any two vector fields of an adapted frame have the form

$$[\widehat{E}_i, \widehat{E}_j] = c_{ij}^k \widehat{E}_k - g_o(JE_i, E_j)\mathrm{q}_o \ , \qquad [\widehat{E}_i, \mathrm{p}_o] = [\widehat{E}_i, \mathrm{q}_o] = [\mathrm{p}_o, \mathrm{q}_o] = 0 \ , \tag{2.4}$$

where the c_{ij}^k are the functions such that $[E_i, E_j] = c_{ij}^k E_k$.

The dual coframe field of $(\widehat{E}_1, \ldots, \widehat{E}_{n-2}, \mathrm{p}_o, \mathrm{q}_o)$ is denoted by $(\widehat{E}^1, \ldots, \widehat{E}^{n-2}, \mathrm{p}_o^*, \mathrm{q}_o^*)$. Since the dual 1-form q_o^* satisfies $\mathrm{q}_o^*(\mathrm{q}_o) = 1$ and vanishes identically on $\mathscr{W}$ (because $\mathscr{W}$ is spanned by the

[3] The Lie bracket $[\widehat{X}, \widehat{Y}]$ differs by a sign from the one used in [2]. Since in both papers, it is assumed $d\theta_o = \omega_o$, the sign difference is a consequence of the different definitions of the Kähler form ω_o.

$\widehat{E}_i$ and p_o), it has the same kernel and takes the same value on q_o as the 1-form ϑ_o. Thus

$$\mathrm{q}_o^* = \vartheta_o \tag{2.5}$$

for any *choice of the adapted frame* $(\widehat{E}_i, \mathrm{p}_o, \mathrm{q}_o)$.

2.3 Parameterisation of the compatible metrics

Let (E_i) be a (local) frame field on N and denote by $(\widehat{E}_1, \ldots, \widehat{E}_{n-2}, \mathrm{p}_o, \mathrm{q}_o)$ the corresponding adapted frame field for $\mathscr{M}$. Since we are assuming that $\mathscr{M}$ is of Kähler-Sasaki type, the conformal class $[h]$ consists of the degenerate metrics on $\mathscr{W}$ having the form

$$h = \sigma(\pi^S \circ \pi)^*(g_o)|_{\mathscr{W}}, \qquad \sigma = \text{conformal scaling factor} . \tag{2.6}$$

By the results in [2, Section 2.5] (see also [6]), the compatible Lorentzian metrics on $\mathscr{M}$ are locally in one-to-one correspondence with the pairs (h, q) given by

- a degenerated metric h on $\mathscr{W}$ as in (2.6):
- a vector field q, which is transversal to the distribution $\mathscr{W} = \mathscr{W}' + \mathbb{R}\mathrm{p}_o$, *i.e.*, of the form

$$\mathrm{q} := a\mathrm{q}_o + b\mathrm{p}_o + c^i \widehat{E}_i \ , \qquad a \neq 0 \ . \tag{2.7}$$

More precisely, given the conformal factor σ and the vector field q, the corresponding compatible metric $g = g^{(\sigma,\mathrm{q})}$ is the unique Lorentzian metric satisfying conditions

$$\begin{gathered} g(\widehat{X}, \widehat{Y}) = \sigma g_o(X, Y) \ , \qquad g(\widehat{X}, \mathrm{p}_o) = g(\mathrm{p}_o, \mathrm{p}_o) = 0 \ , \\ g(\widehat{X}, \mathrm{q}) = 0 \ , \qquad g(\mathrm{p}_o, \mathrm{q}) = 1 \ , \qquad g(\mathrm{q}, \mathrm{q}) = 0 \ . \end{gathered} \tag{2.8}$$

From (2.7) and the first line of (2.8), the second line in (2.8) is equivalent to

$$\begin{gathered} g(\widehat{X}, \mathrm{q}_o) = -\frac{c^i\sigma}{a} g_o(X, E_i) \ , \qquad g(\mathrm{p}_o, \mathrm{q}_o) = \frac{1}{a} \ , \\ g(\mathrm{q}_o, \mathrm{q}_o) = -2\frac{b}{a^2} + \frac{1}{a^2} c^i c^j \sigma g_o(E_j, E_i) \ . \end{gathered} \tag{2.9}$$

Introducing the shorter notation

$$\alpha := \frac{2}{a\sigma} \ , \qquad \beta := \frac{2}{\sigma}\left(-2\frac{b}{a^2} + \frac{1}{a^2} c^i c^j \sigma g_o(E_j, E_i)\right) \ , \qquad \gamma^i := -2\frac{c^i}{a} \ , \tag{2.10}$$

we get that $g = g^{(\sigma,\mathrm{q})}$ is the unique Lorentzian metric satisfying the condition

$$\begin{gathered} g(\widehat{X}, \widehat{Y}) = \sigma g_o(X,Y)\ , \qquad g(\widehat{X}, \mathrm{p}_o) = g(\mathrm{p}_o, \mathrm{p}_o) = 0\ , \qquad g(\mathrm{p}_o, \mathrm{q}_o) = \frac{\sigma\alpha}{2}\ , \\ g(\mathrm{q}_o, \widehat{X}) = \frac{\sigma\gamma^i}{2} g_o(X, E_i)\ , \qquad g(\mathrm{q}_o, \mathrm{q}_o) = \frac{\sigma}{2}\beta\ . \end{gathered} \tag{2.11}$$

This means that g has the form

$$\begin{aligned} g &= \sigma g_o(E_i, E_j)\widehat{E}^i \vee \widehat{E}^j + \mathrm{q}_o^* \vee \left(\sigma\alpha \mathrm{p}_o^* + \sigma\gamma^i g_o(E_i, E_k)\widehat{E}^k + \frac{\sigma\beta}{2}\mathrm{q}_o^*\right) \\ &= \sigma\left\{(\pi^S \circ \pi)^*(g_o)\big|_{\mathscr{W}'} + \vartheta_o \vee \left(\alpha \mathrm{p}_o^* + \gamma^i g_o(E_k, E_i)\widehat{E}^k + \frac{\beta}{2}\vartheta_o\right)\right\}. \end{aligned} \tag{2.12}$$

The expression (2.12) gives a convenient parameterisation in terms of the $(n+1)$-tuple of smooth functions $(\sigma, \alpha, \beta, \gamma^i)$ for the compatible metrics of $\mathscr{M} = (\pi : M \to S, \mathrm{p}, \mathscr{W}, [h])$. We emphasise that, conversely, any metric having the form (2.12), for some $\sigma > 0$ and $\alpha \neq 0$, is a compatible metric. Indeed, it is associated with the conformal factor σ and with $\mathrm{q} = a\mathrm{q}_o + b\mathrm{p}_o + c^i\widehat{E}_i$ where a, b, c^j are solutions to (2.10) for given α, β, γ^i. They are

$$a = \frac{2}{\alpha\sigma}\ , \qquad b := -\frac{\beta}{\alpha^2\sigma} + \frac{1}{2\alpha^2\sigma}\gamma^i\gamma^j g_o(E_j, E_i)\ , \qquad c^i = -\frac{\gamma^i}{\alpha\sigma}\ .$$

3 The Christoffel symbols in an adapted frame field of the Levi-Civita connection of a compatible Lorentzian metric

3.1 The complete list of the Christoffel symbols

Let $\mathscr{M} = (\pi : M \to S, \mathrm{p}, \mathscr{W}, [h])$ be a twisting regular shearfree manifold of Kähler-Sasaki type, with S projecting onto the Kähler manifold (N, J, g_o). Let also (E_i) be a frame field on an open set $\mathscr{V} \subset N$ and $(X_A) = (\widehat{E}_1, \ldots, \widehat{E}_{n-2}, \mathrm{p}_o, \mathrm{q}_o)$ the corresponding adapted frame field on $\mathscr{U} = (\pi^S \circ \pi)^{-1}(\mathscr{V}) \subset M$. We use the notation g_{ij}, ω_{ij}, J_i^j, c_{ij}^k for the functions defined by

$$g_{ij} := g_o(E_i, E_j)\ , \qquad \omega_{ij} := g_o(JE_i, E_j)\ , \qquad JE_i = J_i^j E_j\ , \qquad [E_i, E_j] = c_{ij}^k E_k\ .$$

For what concerns the Christoffel symbols $\mathbf{\Gamma}_{AB}^C$ (*i.e.*, the functions defined by $\nabla_{X_A} X_B = \mathbf{\Gamma}_{AB}^C X_C$), we are going to use the convention that $\mathbf{\Gamma}_{ij}^m$ denotes the function which gives the component of $\nabla_{\widehat{E}_i}\widehat{E}_j$ in the direction of $\widehat{E}_m$, $\mathbf{\Gamma}_{ij}^{\mathrm{p}_o}$ is the function that gives the component of $\nabla_{\widehat{E}_i}\widehat{E}_j$ in the direction of p_o, $\mathbf{\Gamma}_{ij}^{\mathrm{q}_o}$ is the function giving the component of $\nabla_{\widehat{E}_i}\widehat{E}_j$ in the direction of q_o, and so on.

Our main result is the following:

Proposition 3.1. *Let g be a compatible metric for $\mathscr{M}$, hence of the form (2.12) for an $(n+1)$-tuple of smooth functions $(\sigma, \alpha, \beta, \gamma^i)$ on $\mathscr{U}$, with $\sigma > 0$ and $\alpha \neq 0$ at all points. The Christoffel symbols $\boldsymbol{\Gamma}_{AB}^{\ C}$ of the Levi-Civita connection of g in the frame field $(X_A) = (\widehat{E}_i, \mathrm{p}_o, \mathrm{q}_o)$ are given by*

$$\begin{aligned}\boldsymbol{\Gamma}_{ij}^{\ m} &= g^{mk} g_o(\nabla^o_{E_i} E_j, E_k) + g^{mk} S_{ij|k} + \frac{\gamma^m \omega_{ij}}{4} + \frac{1}{2\sigma}\widehat{E}_i(\sigma)\delta^m_j + \frac{1}{2\sigma}\widehat{E}_j(\sigma)\delta^m_i \\ &\quad - \frac{g_{ij}}{2\sigma}\left(g^{mk}\widehat{E}_k(\sigma) - \frac{\gamma^m}{\alpha}\mathrm{p}_o(\sigma)\right) ,\end{aligned} \tag{3.1}$$

where $S_{ij|k}$ is defined by

$$S_{ij|k} := \frac{\gamma^\ell}{4} g_o(JE_i, E_k) g_o(E_\ell, E_j) + \frac{\gamma^\ell}{4} g_o(JE_j, E_k) g_o(E_\ell, E_i) - \frac{\gamma^\ell}{4} g_o(JE_i, E_j) g_o(E_\ell, E_k) ,$$

$$\begin{aligned}\boldsymbol{\Gamma}_{ij}^{\ \mathrm{p}_o} &= \frac{1}{2\alpha}\widehat{E}_i(\gamma^k g_{jk}) + \frac{1}{2\alpha}\widehat{E}_j(\gamma^k g_{ik}) - \frac{1}{4\alpha}\gamma^m\gamma^k g_{mk}\omega_{ij} - \frac{\gamma^m}{\alpha} g_o(\nabla^o_{E_i}E_j, E_m) - \frac{\gamma^m}{\alpha} S_{ij|m} \\ &\quad - \frac{g_{ij}}{2\sigma}\left(\frac{2}{\alpha}\mathrm{q}_o(\sigma) + \frac{1}{\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\mathrm{p}_o(\sigma) - \frac{\gamma^m}{\alpha}\widehat{E}_m(\sigma)\right) ,\end{aligned} \tag{3.2}$$

$$\boldsymbol{\Gamma}_{ij}^{\ \mathrm{q}_o} = -\frac{\omega_{ij}}{2} - \frac{g_{ij}}{\alpha\sigma}\mathrm{p}_o(\sigma) , \tag{3.3}$$

$$\boldsymbol{\Gamma}_{i\mathrm{p}_o}^{\ m} = \boldsymbol{\Gamma}_{\mathrm{p}_o i}^{\ m} = \frac{\alpha g^{mk}\omega_{ik}}{4} + \frac{1}{2\sigma}\mathrm{p}_o(\sigma)\delta^m_i , \tag{3.4}$$

$$\boldsymbol{\Gamma}_{i\mathrm{p}_o}^{\ \mathrm{p}_o} = \boldsymbol{\Gamma}_{\mathrm{p}_o i}^{\ \mathrm{p}_o} = \frac{1}{2\alpha}\widehat{E}_i(\alpha) + \frac{1}{2\alpha}\mathrm{p}_o(\gamma^k)g_{ik} - \frac{\gamma^m\omega_{im}}{4} + \frac{1}{2\sigma}\widehat{E}_i(\sigma) , \tag{3.5}$$

$$\boldsymbol{\Gamma}_{i\mathrm{p}_o}^{\ \mathrm{q}_o} = \boldsymbol{\Gamma}_{\mathrm{p}_o i}^{\ \mathrm{q}_o} = 0 , \tag{3.6}$$

$$\begin{aligned}\boldsymbol{\Gamma}_{i\mathrm{q}_o}^{\ m} = \boldsymbol{\Gamma}_{\mathrm{q}_o i}^{\ m} &= \frac{g^{mk}}{4}\widehat{E}_i(\gamma^t g_{tk}) - \frac{g^{mk}}{4}\widehat{E}_k(\gamma^t g_{ti}) - \frac{\gamma^\ell}{4}c^t_{ir} g_{t\ell} g^{mr} + \frac{g^{mk}\omega_{ik}}{4}\beta - \\ &\quad - \frac{\gamma^m}{4\alpha}\widehat{E}_i(\alpha) + \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\gamma^t)g_{ti} + \frac{1}{2\sigma}\mathrm{q}_o(\sigma)\delta^m_i - \frac{\gamma^t}{4\sigma}g_{ti}\left(g^{mk}\widehat{E}_k(\sigma) - \frac{\gamma^m}{\alpha}\mathrm{p}_o(\sigma)\right) ,\end{aligned} \tag{3.7}$$

$$\begin{aligned}\boldsymbol{\Gamma}_{i\mathrm{q}_o}^{\ \mathrm{p}_o} = \boldsymbol{\Gamma}_{\mathrm{q}_o i}^{\ \mathrm{p}_o} &= \frac{1}{2\alpha}\widehat{E}_i(\beta) + \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\widehat{E}_i(\alpha) - \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{p}_o(\gamma^t)g_{it} - \frac{1}{2\alpha^2}\beta\widehat{E}_i(\alpha) + \\ &\quad + \frac{1}{2\alpha^2}\beta\mathrm{p}_o(\gamma^t)g_{it} - \frac{\gamma^m}{4\alpha}\widehat{E}_i(\gamma^t g_{tm}) + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\gamma^t g_{it}) + \frac{\gamma^m\gamma^t}{4\alpha}g_{t\ell}c^\ell_{im} + \frac{\gamma^m}{4\alpha}\omega_{im}\beta - \\ &\quad - \frac{\gamma^t}{4\sigma}g_{ti}\left(\frac{2}{\alpha}\mathrm{q}_o(\sigma) + \frac{1}{\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\mathrm{p}_o(\sigma) - \frac{\gamma^m}{\alpha}\widehat{E}_m(\sigma)\right) ,\end{aligned} \tag{3.8}$$

$$\boldsymbol{\Gamma}_{i\mathrm{q}_o}^{\ \mathrm{q}_o} = \boldsymbol{\Gamma}_{\mathrm{q}_o i}^{\ \mathrm{q}_o} = \frac{1}{2\alpha}\widehat{E}_i(\alpha) - \frac{1}{2\alpha}\mathrm{p}_o(\gamma^t)g_{it} + \frac{1}{2\sigma}\widehat{E}_i(\sigma) - \frac{\gamma^t g_{ti}}{2\alpha\sigma}\mathrm{p}_o(\sigma) , \tag{3.9}$$

$$\boldsymbol{\Gamma}_{\mathrm{p}_o\mathrm{p}_o}^{\ m} = 0 , \tag{3.10}$$

$$\boldsymbol{\Gamma}_{\mathrm{p}_o\mathrm{p}_o}^{\ \mathrm{p}_o} = \mathrm{p}_o(\log(\alpha\sigma)) , \tag{3.11}$$

$$\boldsymbol{\Gamma}_{\mathrm{p}_o\mathrm{p}_o}^{\ \mathrm{q}_o} = 0 , \tag{3.12}$$

$$\boldsymbol{\Gamma}_{\mathrm{p}_o\mathrm{q}_o}^{\ m} = \boldsymbol{\Gamma}_{\mathrm{q}_o\mathrm{p}_o}^{\ m} = \frac{1}{4}\mathrm{p}_o(\gamma^m) - \frac{g^{mk}}{4}\widehat{E}_k(\alpha) - \frac{\alpha}{4\sigma}\left(g^{mk}\widehat{E}_k(\sigma) - \frac{\gamma^m}{\alpha}\mathrm{p}_o(\sigma)\right) , \tag{3.13}$$

$$\begin{aligned}\boldsymbol{\Gamma}_{\mathrm{p}_o\mathrm{q}_o}^{\ \mathrm{p}_o} = \boldsymbol{\Gamma}_{\mathrm{q}_o\mathrm{p}_o}^{\ \mathrm{p}_o} &= \frac{1}{2\alpha}\mathrm{p}_o(\beta) - \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\gamma^i)g_{im} + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\alpha) + \frac{1}{2\sigma}\mathrm{q}_o(\sigma) - \\ &\quad - \frac{1}{2\sigma}\left(\mathrm{q}_o(\sigma) + \frac{1}{2\alpha}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\mathrm{p}_o(\sigma) - \frac{\gamma^m}{2}\widehat{E}_m(\sigma)\right) ,\end{aligned} \tag{3.14}$$

$$\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{q}_o}^{\;\;\;\mathrm{q}_o} = \mathbf{\Gamma}_{\mathrm{q}_o\mathrm{p}_o}^{\;\;\;\mathrm{q}_o} = 0\ , \tag{3.15}$$

$$\begin{aligned}\mathbf{\Gamma}_{\mathrm{q}_o\mathrm{q}_o}^{\;\;\;m} &= \frac{g^{mk}}{2}\mathrm{q}_o(\gamma^i)g_{ik} - \frac{g^{mk}}{4}\widehat{E}_k(\beta) - \frac{\gamma^m}{2\alpha}\mathrm{q}_o(\alpha) + \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\beta) - \\ &\quad - \frac{\beta}{4\sigma}\left(g^{mk}\widehat{E}_k(\sigma) - \frac{\gamma^m}{\alpha}\mathrm{p}_o(\sigma)\right)\ ,\end{aligned} \tag{3.16}$$

$$\begin{aligned}\mathbf{\Gamma}_{\mathrm{q}_o\mathrm{q}_o}^{\;\;\;\mathrm{p}_o} &= \frac{1}{2\alpha}\mathrm{q}_o(\beta) + \frac{1}{2\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{q}_o(\alpha) - \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{p}_o(\beta) - \frac{1}{\alpha^2}\beta\mathrm{q}_o(\alpha) + \\ &\quad + \frac{\beta}{2\alpha^2}\mathrm{p}_o(\beta) - \frac{\gamma^m}{2\alpha}\mathrm{q}_o(\gamma^i)g_{im} + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\beta) - \\ &\quad - \frac{\beta}{2\sigma}\left(\frac{1}{\alpha}\mathrm{q}_o(\sigma) + \frac{1}{2\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\mathrm{p}_o(\sigma) - \frac{\gamma^m}{2\alpha}\widehat{E}_m(\sigma)\right)\ ,\end{aligned} \tag{3.17}$$

$$\mathbf{\Gamma}_{\mathrm{q}_o\mathrm{q}_o}^{\;\;\;\mathrm{q}_o} = \frac{1}{\alpha}\mathrm{q}_o(\alpha) - \frac{1}{2\alpha}\mathrm{p}_o(\beta) + \frac{1}{\sigma}\mathrm{q}_o(\sigma) - \frac{\beta}{2\alpha\sigma}\mathrm{p}_o(\sigma)\ . \tag{3.18}$$

The proof will be carried out in three steps, which we provide in the next subsections. In the first step we compute all covariant derivatives $\nabla_{X_A}X_B$ determined by two vector fields of the adapted frame field $(X_A) = (\widehat{E}_i, \mathrm{p}_o, \mathrm{q}_o)$ *under the assumption* $\sigma \equiv 1$. In the second step, the determined covariant derivatives are used to compute the Christoffel symbols $\mathbf{\Gamma}_{AB}^{\;\;C}$, *still under the condition* $\sigma \equiv 1$. In the concluding third step, the Christoffel symbols $\mathbf{\Gamma}_{AB}^{\;\;C}$ are determined with no restriction on σ by using classical transformation formulas for the Levi-Civita covariant derivatives under conformal changes of the metric.

3.2 The first step

By Koszul's formula, for any triple of vector fields X_1, X_2, X_3,

$$\begin{aligned}g(\nabla_{X_1}X_2, X_3) = \frac{1}{2}\Big(&X_1(g(X_2,X_3)) + X_2(g(X_1,X_3)) - X_3(g(X_1,X_2)) - \\ &- g([X_1,X_3],X_2) - g([X_2,X_3],X_1) + g([X_1,X_2],X_3)\Big)\ .\end{aligned} \tag{3.19}$$

Using this formula, we may determine the functions $g(\nabla_{X_1}X_2, X_3)$, for a compatible metric g *with* $\sigma \equiv 1$, for with any choice of X_1, X_2, X_3 in a set of vector fields of the form

$$\left\{\ \widehat{X}, \mathrm{p}_o, \mathrm{q}_o,\ \text{where } \widehat{X} \text{ is the lift of a vector field } X \text{ on } N\ \right\}\ .$$

We get the following expressions:

$$\nabla_{\widehat{X}}\widehat{Y}: \qquad g(\nabla_{\widehat{X}}\widehat{Y}, \widehat{Z}) = g_o(\nabla^o_X Y, Z) + g(S_{XY}, Z)\ , \tag{3.20}$$

$$g(\nabla_{\widehat{X}}\widehat{Y}, \mathrm{p}_o) = -\frac{\alpha}{4}g_o(JX,Y)\ , \tag{3.21}$$

$$g(\nabla_{\widehat{X}}\widehat{Y}, \mathrm{q}_o) = \frac{1}{4}\widehat{X}(\gamma^k g_o(Y,E_k)) + \frac{1}{4}\widehat{Y}(\gamma^k g_o(X,E_k)) - \frac{1}{4}\beta g_o(JX,Y)\ , \tag{3.22}$$

where S is the tensor field of type $(0,3)$ on N, defined by

$$g(S_{XY}, Z) := \frac{\gamma^j}{4} g_o(JX, Z) g_o(E_j, Y) + \frac{\gamma^j}{4} g_o(JY, Z) g_o(E_j, X) - \frac{\gamma^j}{4} g_o(JX, Y) g_o(E_j, Z) \ ;$$

$$\nabla_{\widehat{X}} \mathrm{p}_o : \quad g(\nabla_{\widehat{X}} \mathrm{p}_o, \widehat{Z}) = \frac{\alpha}{4} g_o(JX, Z) \ , \tag{3.23}$$

$$g(\nabla_{\widehat{X}} \mathrm{p}_o, \mathrm{p}_o) = 0 \ , \tag{3.24}$$

$$g(\nabla_{\widehat{X}} \mathrm{p}_o, \mathrm{q}_o) = \frac{1}{4} \widehat{X}(\alpha) + \frac{1}{4} \mathrm{p}_o(\gamma^i) g_o(X, E_i) \ ; \tag{3.25}$$

$$\nabla_{\widehat{X}} \mathrm{q}_o : \quad g(\nabla_{\widehat{X}} \mathrm{q}_o, \widehat{Z}) = \frac{1}{4} \widehat{X}(\gamma^t g_o(E_t, Z)) - \frac{1}{4} \widehat{Z}(\gamma^i g_o(X, E_i)) - \\ - \frac{1}{4} \gamma^t g_o([X, Z], E_t) + \frac{1}{4} \beta g_o(JX, Z) \ , \tag{3.26}$$

$$g(\nabla_{\widehat{X}} \mathrm{q}_o, \mathrm{p}_o) = \frac{1}{4} \widehat{X}(\alpha) - \frac{1}{4} \mathrm{p}_o(\gamma^i) g_o(X, E_i) \ , \tag{3.27}$$

$$g(\nabla_{\widehat{X}} \mathrm{q}_o, \mathrm{q}_o) = \frac{1}{4} \widehat{X}(\beta) \ ; \tag{3.28}$$

$$\nabla_{\mathrm{p}_o} \widehat{Y} : \quad g(\nabla_{\mathrm{p}_o} \widehat{Y}, \widehat{Z}) = \frac{\alpha}{4} g_o(JY, Z) \ , \tag{3.29}$$

$$g(\nabla_{\mathrm{p}_o} \widehat{Y}, \mathrm{p}_o) = 0 \ , \tag{3.30}$$

$$g(\nabla_{\mathrm{p}_o} \widehat{Y}, \mathrm{q}_o) = \frac{1}{4} \mathrm{p}_o(\gamma^i) g_o(Y, E_i) + \frac{1}{4} \widehat{Y}(\alpha) \ ; \tag{3.31}$$

$$\nabla_{\mathrm{p}_o} \mathrm{p}_o : \quad g(\nabla_{\mathrm{p}_o} \mathrm{p}_o, \widehat{Z}) = 0 \ , \tag{3.32}$$

$$g(\nabla_{\mathrm{p}_o} \mathrm{p}_o, \mathrm{p}_o) = 0 \ , \tag{3.33}$$

$$g(\nabla_{\mathrm{p}_o} \mathrm{p}_o, \mathrm{q}_o) = \frac{1}{2} \mathrm{p}_o(\alpha) \ ; \tag{3.34}$$

$$\nabla_{\mathrm{p}_o} \mathrm{q}_o : \quad g(\nabla_{\mathrm{p}_o} \mathrm{q}_o, \widehat{Z}) = \frac{1}{4} \mathrm{p}_o(\gamma^i) g_o(E_i, Z) - \frac{1}{4} \widehat{Z}(\alpha) \ , \tag{3.35}$$

$$g(\nabla_{\mathrm{p}_o} \mathrm{q}_o, \mathrm{p}_o) = 0 \ , \tag{3.36}$$

$$g(\nabla_{\mathrm{p}_o} \mathrm{q}_o, \mathrm{q}_o) = \frac{\mathrm{p}_o(\beta)}{4} \ ; \tag{3.37}$$

$$\nabla_{\mathrm{q}_o}\widehat{Y}: \quad g(\nabla_{\mathrm{q}_o}\widehat{Y},\widehat{Z}) = \frac{1}{4}\widehat{Y}(\gamma^i g_o(E_i,Z)) - \frac{1}{4}\widehat{Z}(\gamma^t g_o(Y,E_t)) - \\ - \frac{1}{4}\gamma^t g_o([Y,Z],E_t) + \frac{1}{4}\beta g_o(JY,Z)\ , \tag{3.38}$$

$$g(\nabla_{\mathrm{q}_o}\widehat{Y},\mathrm{p}_o) = \frac{1}{4}\big(\widehat{Y}(\alpha) - \mathrm{p}_o(\gamma^i)g_o(E_i,Y)\big)\ , \tag{3.39}$$

$$g(\nabla_{\mathrm{q}_o}\widehat{Y},\mathrm{q}_o) = \frac{\widehat{Y}(\beta)}{4}\ ; \tag{3.40}$$

$$\nabla_{\mathrm{q}_o}\mathrm{p}_o: \quad g(\nabla_{\mathrm{q}_o}\mathrm{p}_o,\widehat{Z}) = \frac{1}{4}\mathrm{p}_o(\gamma^i)g_o(E_i,\widehat{Z}) - \frac{1}{4}\widehat{Z}(\alpha)\ , \tag{3.41}$$

$$g(\nabla_{\mathrm{q}_o}\mathrm{p}_o,\mathrm{p}_o) = 0\ , \tag{3.42}$$

$$g(\nabla_{\mathrm{q}_o}\mathrm{p}_o,\mathrm{q}_o) = \frac{\mathrm{p}_o(\beta)}{4}\ ; \tag{3.43}$$

$$\nabla_{\mathrm{q}_o}\mathrm{q}_o: \quad g(\nabla_{\mathrm{q}_o}\mathrm{q}_o,\widehat{Z}) = \frac{1}{2}\mathrm{q}_o(\gamma^i)g_o(E_i,Z) - \frac{1}{4}\widehat{Z}(\beta)\ , \tag{3.44}$$

$$g(\nabla_{\mathrm{q}_o}\mathrm{q}_o,\mathrm{p}_o) = \frac{1}{2}\mathrm{q}_o(\alpha) - \frac{\mathrm{p}_o(\beta)}{4}\ , \tag{3.45}$$

$$g(\nabla_{\mathrm{q}_o}\mathrm{q}_o,\mathrm{q}_o) = \frac{\mathrm{q}_o(\beta)}{4}\ . \tag{3.46}$$

From this list, we may recover the explicit expressions of the covariant derivatives of vector fields of the adapted frame field $(\widehat{E}_i,\mathrm{p}_o,\mathrm{q}_o)$ as follows. We claim that the dual coframe field $(\widehat{E}^i,\mathrm{p}_o^*,\mathrm{q}_o^*)$ is given by the following 1-forms (here, $(g^{\ell m}) := (g_{ij})^{-1} = \big(g_o(E_i,E_j)\big)^{-1}$)

$$\widehat{E}^i = g\left(g^{ik}\widehat{E}_k - \frac{\gamma^i}{\alpha}\mathrm{p}_o,\cdot\right)\ , \quad \mathrm{p}_o^* = g\left(\frac{2}{\alpha}\mathrm{q}_o + \frac{1}{\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\mathrm{p}_o - \frac{\gamma^m}{\alpha}\widehat{E}_m,\cdot\right)\ , \\ \mathrm{q}_o^* = g\left(\frac{2}{\alpha}\mathrm{p}_o,\cdot\right)\ . \tag{3.47}$$

This claim can be checked using (2.11) and observing that the right hand sides in the above equalities are 1-forms that satisfy the equalities

$$\begin{aligned}
&\widehat{E}^i(\widehat{E}_j) = g^{ik}g_{kj} = \delta^i_j\ , \quad \widehat{E}^i(\mathrm{p}_o) = 0\ , \quad \widehat{E}^i(\mathrm{q}_o) = g^{ik}\frac{\gamma^m}{2}g_{mk} - \frac{\gamma^i}{\alpha}\frac{\alpha}{2} = 0\ ,\\
&\mathrm{p}_o^*(\widehat{E}_j) = \frac{2}{\alpha}\frac{\gamma^m}{2}g_{jm} - \frac{\gamma^m}{\alpha}g_{mj} = 0\ , \quad \mathrm{p}_o^*(\mathrm{p}_o) = \frac{2}{\alpha}\frac{\alpha}{2} = 1\ ,\\
&\qquad \mathrm{p}_o^*(\mathrm{q}_o) = \frac{2}{\alpha}\frac{\beta}{2} + \frac{1}{\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\frac{\alpha}{2} - \frac{\gamma^m}{\alpha}\frac{\gamma^k}{2}g_{mk} = 0\ ,\\
&\mathrm{q}_o^*(\widehat{E}_j) = 0\ , \quad \mathrm{q}_o^*(\mathrm{p}_o) = 0\ , \quad \mathrm{q}_o^*(\mathrm{q}_o) = \frac{2}{\alpha}\frac{\alpha}{2} = 1\ .
\end{aligned}$$

Since any local vector field Z on M can be written in terms of the frame field $(\widehat{E}_i, \mathrm{p}_o, \mathrm{q}_o)$ as

$$Z = \widehat{E}^i(Z)\widehat{E}_i + \mathrm{p}_o^*(Z)\mathrm{p}_o + \mathrm{q}_o^*(Z)\mathrm{q}_o \ ,$$

from the above expressions for the 1-forms $\widehat{E}^i, \mathrm{p}_o^*$, and q_o^*, we get that for any pair of vector fields X, Y on M, the Levi-Civita covariant derivative $\nabla_X Y$ is equal to

$$\begin{aligned}\nabla_X Y = &\left(g^{mk} g(\nabla_X Y, \widehat{E}_k) - \frac{\gamma^m}{\alpha} g(\nabla_X Y, \mathrm{p}_o)\right)\widehat{E}_m + \\ &+ \left(\frac{2}{\alpha} g(\nabla_X Y, \mathrm{q}_o) + \frac{1}{\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right) g(\nabla_X Y, \mathrm{p}_o) - \frac{\gamma^m}{\alpha} g(\nabla_X Y, \widehat{E}_m)\right)\mathrm{p}_o + \\ &+ \left(\frac{2}{\alpha} g(\nabla_X Y, \mathrm{p}_o)\right)\mathrm{q}_o \ . \end{aligned} \tag{3.48}$$

Combining (3.20) – (3.46) with (3.48), we get the covariant derivatives we are looking for. We list them in (3.49) – (3.57) (here, we denote by $S_{ij|m}$ the components of the tensor field S in terms of the frame field (E_i) on N):

$$\begin{aligned}\nabla_{\widehat{E}_i}\widehat{E}_j = &\left(g^{mk} g_o(\nabla^o_{E_i} E_j, E_k) + g^{mk} S_{ij|k} + \frac{\gamma^m\omega_{ij}}{4}\right)\widehat{E}_m + \\ &+ \left(\frac{1}{2\alpha}\widehat{E}_i(\gamma^k g_{jk}) + \frac{1}{2\alpha}\widehat{E}_j(\gamma^k g_{ik}) - \right.\\ &\left. - \frac{1}{4\alpha}\gamma^m\gamma^k g_{mk}\omega_{ij} - \frac{\gamma^m}{\alpha} g_o(\nabla^o_{E_i} E_j, E_m) - \frac{\gamma^m}{\alpha} S_{ij|m}\right)\mathrm{p}_o - \frac{\omega_{ij}}{2}\mathrm{q}_o \ , \end{aligned} \tag{3.49}$$

$$\nabla_{\widehat{E}_i}\mathrm{p}_o = \frac{\alpha g^{mk}\omega_{ik}}{4}\widehat{E}_m + \left(\frac{1}{2\alpha}\widehat{E}_i(\alpha) + \frac{1}{2\alpha}\mathrm{p}_o(\gamma^k) g_{ik} - \frac{\gamma^m\omega_{im}}{4}\right)\mathrm{p}_o \ , \tag{3.50}$$

$$\begin{aligned}\nabla_{\widehat{E}_i}\mathrm{q}_o = &\left(\frac{g^{mk}}{4}\widehat{E}_i(\gamma^t g_{tk}) - \frac{g^{mk}}{4}\widehat{E}_k(\gamma^t g_{ti}) - \frac{\gamma^\ell}{4} c^t_{ir} g_{t\ell} g^{mr} + \frac{g^{mk}}{4}\beta\omega_{ik} - \right.\\ &\left. - \frac{\gamma^m}{4\alpha}\widehat{E}_i(\alpha) + \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\gamma^t) g_{it}\right)\widehat{E}_m + \\ &+ \left(\frac{1}{2\alpha}\widehat{E}_i(\beta) + \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\widehat{E}_i(\alpha) - \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{p}_o(\gamma^t) g_{it} - \frac{\beta}{2\alpha^2}\widehat{E}_i(\alpha) + \frac{\beta}{2\alpha^2}\mathrm{p}_o(\gamma^t) g_{it} - \right.\\ &\left. - \frac{\gamma^m}{4\alpha}\widehat{E}_i(\gamma^t g_{tm}) + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\gamma^t g_{it}) + \frac{\gamma^m\gamma^t}{4\alpha} g_{t\ell} c^\ell_{im} - \frac{\gamma^m}{4\alpha}\beta\omega_{im}\right)\mathrm{p}_o + \\ &+ \left(\frac{1}{2\alpha}\widehat{E}_i(\alpha) - \frac{1}{2\alpha}\mathrm{p}_o(\gamma^t) g_{it}\right)\mathrm{q}_o \ , \end{aligned} \tag{3.51}$$

$$\nabla_{\mathrm{p}_o}\widehat{E}_i = \frac{\alpha g^{mk}}{4}\omega_{ik}\widehat{E}_m + \left(\frac{1}{2\alpha}\widehat{E}_i(\alpha) + \frac{1}{2\alpha}\mathrm{p}_o(\gamma^t) g_{it} - \frac{\gamma^m}{4}\omega_{im}\right)\mathrm{p}_o \ , \tag{3.52}$$

$$\nabla_{\mathrm{p}_o}\mathrm{p}_o = \mathrm{p}_o(\log\alpha)\mathrm{p}_o \ , \tag{3.53}$$

$$\nabla_{\mathrm{p}_o}\mathrm{q}_o = \left(\frac{1}{4}\mathrm{p}_o(\gamma^m) - \frac{g^{mk}}{4}\widehat{E}_k(\alpha)\right)\widehat{E}_m + \left(\frac{1}{2\alpha}\mathrm{p}_o(\beta) - \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\gamma^i)g_{im} + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\alpha)\right)\mathrm{p}_o \ , \tag{3.54}$$

$$\begin{aligned}\nabla_{\mathrm{q}_o}\widehat{E}_i = \bigg(&\frac{g^{mk}}{4}\widehat{E}_i(\gamma^t g_{tk}) - \frac{g^{mk}}{4}\widehat{E}_k(\gamma^t g_{ti}) - \frac{\gamma^\ell}{4}c^t_{ir}g_{t\ell}g^{mr} + \frac{g^{mk}}{4}\beta\omega_{ik} - \\ &- \frac{\gamma^m}{4\alpha}\widehat{E}_i(\alpha) + \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\gamma^t)g_{ti}\bigg)\widehat{E}_m + \\ &+ \bigg(\frac{1}{2\alpha}\widehat{E}_i(\beta) + \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\widehat{E}_i(\alpha) - \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{p}_o(\gamma^t)g_{it} - \frac{1}{2\alpha^2}\beta\widehat{E}_i(\alpha) + \\ &+ \frac{1}{2\alpha^2}\beta\mathrm{p}_o(\gamma^t)g_{it} - \frac{\gamma^m}{4\alpha}\widehat{E}_i(\gamma^t g_{tm}) + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\gamma^t g_{it}) + \frac{\gamma^m\gamma^t}{4\alpha}g_{t\ell}c^\ell_{im} - \frac{\gamma^m}{4\alpha}\beta\omega_{im}\bigg)\mathrm{p}_o + \\ &+ \left(\frac{1}{2\alpha}\widehat{E}_i(\alpha) - \frac{1}{2\alpha}\mathrm{p}_o(\gamma^t)g_{it}\right)\mathrm{q}_o \ ,\end{aligned} \tag{3.55}$$

$$\begin{aligned}\nabla_{\mathrm{q}_o}\mathrm{p}_o = &\left(\frac{g^{mk}}{4}\mathrm{p}_o(\gamma^i)g_{ik} - \frac{g^{mk}}{4}\widehat{E}_k(\alpha)\right)\widehat{E}_m + \\ &+ \left(\frac{1}{2\alpha}\mathrm{p}_o(\beta) - \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\gamma^t)g_{tm} + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\alpha)\right)\mathrm{p}_o \ ,\end{aligned} \tag{3.56}$$

$$\begin{aligned}\nabla_{\mathrm{q}_o}\mathrm{q}_o = &\left(\frac{g^{mk}}{2}\mathrm{q}_o(\gamma^i)g_{ik} - \frac{g^{mk}}{4}\widehat{E}_k(\beta) - \frac{\gamma^m}{2\alpha}\mathrm{q}_o(\alpha) + \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\beta)\right)\widehat{E}_m + \\ &+ \bigg(\frac{1}{2\alpha}\mathrm{q}_o(\beta) + \frac{1}{2\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{q}_o(\alpha) - \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{p}_o(\beta) - \frac{\beta}{\alpha^2}\mathrm{q}_o(\alpha) + \frac{\beta}{2\alpha^2}\mathrm{p}_o(\beta) - \\ &- \frac{\gamma^m}{2\alpha}\mathrm{q}_o(\gamma^i)g_{im} + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\beta)\bigg)\mathrm{p}_o + \left(\frac{1}{\alpha}\mathrm{q}_o(\alpha) - \frac{\mathrm{p}_o(\beta)}{2\alpha}\right)\mathrm{q}_o \ .\end{aligned} \tag{3.57}$$

3.3 The second step

Let us now denote by $\Gamma_{AB}^{\ C}$ the Christoffel symbols of the Levi-Civita connection of a compatible metric g as in (2.12) under the assumption that the function σ is identically equal to 1. Since the $\Gamma_{AB}^{\ C}$ are the functions that appear in the expansions $\nabla_{X_A}X_B = \Gamma_{AB}^{\ C}X_C$ of the covariant derivatives (3.49) – (3.57), all such Christoffel symbols can be determined by just looking at those formulas. For convenience of the reader, we provide the complete list in the next lines

$$\Gamma_{ij}^{\ m} = g^{mk}g_o(\nabla^o_{E_i}E_j, E_k) + g^{mk}S_{ij|k} + \frac{\gamma^m\omega_{ij}}{4} \ , \tag{3.58}$$

$$\Gamma_{ij}^{\ \mathrm{p}_o} = \frac{1}{2\alpha}\widehat{E}_i(\gamma^k g_{jk}) + \frac{1}{2\alpha}\widehat{E}_j(\gamma^k g_{ik}) + - \frac{1}{4\alpha}\gamma^m\gamma^k g_{mk}\omega_{ij} - \frac{\gamma^m}{\alpha}g_o(\nabla^o_{E_i}E_j, E_m) - \frac{\gamma^m}{\alpha}S_{ij|m} \ , \tag{3.59}$$

$$\Gamma_{ij}^{\ \mathrm{q}_o} = -\frac{\omega_{ij}}{2} \ , \tag{3.60}$$

$$\Gamma_{i\mathrm{p}_o}^{\ m} = \Gamma_{\mathrm{p}_o i}^{\ m} = \frac{\alpha g^{mk}\omega_{ik}}{4} \ , \tag{3.61}$$

$$\Gamma_{i\mathrm{p}_o}^{\ \mathrm{p}_o} = \Gamma_{\mathrm{p}_o i}^{\ \mathrm{p}_o} = \frac{1}{2\alpha}\widehat{E}_i(\alpha) + \frac{1}{2\alpha}\mathrm{p}_o(\gamma^k)g_{ik} - \frac{\gamma^m\omega_{im}}{4} \ , \tag{3.62}$$

$$\Gamma_{i\mathrm{p}_o}^{\ \mathrm{q}_o} = \Gamma_{\mathrm{p}_o i}^{\ \mathrm{q}_o} = 0 \ , \tag{3.63}$$

$$\Gamma_{i\mathrm{q}_o}^{\ m} = \Gamma_{\mathrm{q}_o i}^{\ m} = \frac{g^{mk}}{4}\widehat{E}_i(\gamma^t g_{tk}) - \frac{g^{mk}}{4}\widehat{E}_k(\gamma^t g_{ti}) - \frac{\gamma^\ell}{4}c^t_{ir}g_{t\ell}g^{mr} + \frac{g^{mk}}{4}\beta\omega_{ik} - - \frac{\gamma^m}{4\alpha}\widehat{E}_i(\alpha) + \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\gamma^t)g_{it} \ , \tag{3.64}$$

$$\Gamma_{i\mathrm{q}_o}^{\ \mathrm{p}_o} = \Gamma_{\mathrm{q}_o i}^{\ \mathrm{p}_o} = \frac{1}{2\alpha}\widehat{E}_i(\beta) + \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\widehat{E}_i(\alpha) - \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{p}_o(\gamma^t)g_{it} - \frac{\beta}{2\alpha^2}\widehat{E}_i(\alpha) + + \frac{1}{2\alpha^2}\beta\mathrm{p}_o(\gamma^t)g_{it} - \frac{\gamma^m}{4\alpha}\widehat{E}_i(\gamma^t g_{tm}) + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\gamma^t g_{it}) + \frac{\gamma^m\gamma^t}{4\alpha}g_{t\ell}c^\ell_{im} - \frac{\gamma^m}{4\alpha}\beta\omega_{im} \ , \tag{3.65}$$

$$\Gamma_{i\mathrm{q}_o}^{\ \mathrm{q}_o} = \Gamma_{\mathrm{q}_o i}^{\ \mathrm{q}_o} = \frac{1}{2\alpha}\widehat{E}_i(\alpha) - \frac{1}{2\alpha}\mathrm{p}_o(\gamma^t)g_{it} \ , \tag{3.66}$$

$$\Gamma_{\mathrm{p}_o\mathrm{p}_o}^{\ \ m} = 0 \ , \tag{3.67}$$

$$\Gamma_{\mathrm{p}_o\mathrm{p}_o}^{\ \ \mathrm{p}_o} = \mathrm{p}_o(\log\alpha) \ , \tag{3.68}$$

$$\Gamma_{\mathrm{p}_o\mathrm{p}_o}^{\ \ \mathrm{q}_o} = 0 \ , \tag{3.69}$$

$$\Gamma_{\mathrm{p}_o\mathrm{q}_o}^{\ \ m} = \Gamma_{\mathrm{q}_o\mathrm{p}_o}^{\ \ m} = \frac{1}{4}\mathrm{p}_o(\gamma^m) - \frac{g^{mk}}{4}\widehat{E}_k(\alpha) \ , \tag{3.70}$$

$$\Gamma_{\mathrm{p}_o\mathrm{q}_o}^{\ \ \mathrm{p}_o} = \Gamma_{\mathrm{q}_o\mathrm{p}_o}^{\ \ \mathrm{p}_o} = \frac{1}{2\alpha}\mathrm{p}_o(\beta) - \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\gamma^i)g_{im} + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\alpha) \ , \tag{3.71}$$

$$\Gamma_{\mathrm{p}_o\mathrm{q}_o}^{\ \ \mathrm{q}_o} = \Gamma_{\mathrm{q}_o\mathrm{p}_o}^{\ \ \mathrm{q}_o} = 0 \ , \tag{3.72}$$

$$\Gamma_{\mathrm{q}_o\mathrm{q}_o}^{\ \ m} = \frac{g^{mk}}{2}\mathrm{q}_o(\gamma^i)g_{ik} - \frac{g^{mk}}{4}\widehat{E}_k(\beta) - \frac{\gamma^m}{2\alpha}\mathrm{q}_o(\alpha) + \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\beta) \ , \tag{3.73}$$

$$\Gamma_{\mathrm{q}_o\mathrm{q}_o}^{\ \ \mathrm{p}_o} = \frac{1}{2\alpha}\mathrm{q}_o(\beta) + \frac{1}{2\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{q}_o(\alpha) - \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{p}_o(\beta) - \frac{\beta}{\alpha^2}\mathrm{q}_o(\alpha) + + \frac{\beta}{2\alpha^2}\mathrm{p}_o(\beta) - \frac{\gamma^m}{2\alpha}\mathrm{q}_o(\gamma^i)g_{im} + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\beta)\ m \ , \tag{3.74}$$

$$\Gamma_{\mathrm{q}_o\mathrm{q}_o}^{\ \ \mathrm{q}_o} = \frac{1}{\alpha}\mathrm{q}_o(\alpha) - \frac{1}{2\alpha}\mathrm{p}_o(\beta) \ . \tag{3.75}$$

Note that the equalities $\Gamma_{i\mathrm{p}_o}^{\ A} = \Gamma_{\mathrm{p}_o i}^{\ A}$, $\Gamma_{i\mathrm{q}_o}^{\ A} = \Gamma_{\mathrm{q}_o i}^{\ A}$, etc. are also consequences of the fact that the torsion of the Levi-Civita connection is 0 and that the pairs of vector fields $\{\widehat{E}_i, \mathrm{p}_o\}$, $\{\widehat{E}_i\ \mathrm{q}_o\}$, etc., commute.

3.4 The third step

Assume that g is one of the metrics considered in the previous two subsections (*i.e.*, compatible with $\sigma \equiv 1$) and denote by D the Levi-Civita connection of a conformally scaled metric $g^\varphi = e^{2\varphi} g$ for some smooth φ. It is well known that, for any pair of vector fields X, Y of M (see *e.g.* [4, Th. 1.159]),

$$D_X Y = \nabla_X Y + X(\varphi) Y + Y(\varphi) X - g(X, Y) \mathrm{grad}(\varphi) \ . \tag{3.76}$$

If we expand $\mathrm{grad}\varphi$ in terms of the frame field $(\widehat{E}_i, \mathrm{p}_o, \mathrm{q}_o)$ as

$$\mathrm{grad}\varphi = (\mathrm{grad}\varphi)^{\widehat{E}_i} \widehat{E}_i + (\mathrm{grad}\varphi)^{\mathrm{p}_o} \mathrm{p}_o + (\mathrm{grad}\varphi)^{\mathrm{q}_o} \mathrm{q}_o \ , \tag{3.77}$$

we see that the Christoffel symbols $\Gamma_{AB}^{\ C}$ for a compatible metric g with $\sigma \equiv 1$, as considered in the previous subsections, and the Christoffel symbols $\mathbf{\Gamma}_{AB}^{\ C}$ for the conformally scaled metric g^φ are related to each other by

$$\mathbf{\Gamma}_{ij}^{\ m} = \Gamma_{ij}^{\ m} + \widehat{E}_i(\varphi)\delta_j^m + \widehat{E}_j(\varphi)\delta_i^m - g_{ij}(\mathrm{grad}\varphi)^{\widehat{E}_m} \ , \tag{3.78}$$

$$\mathbf{\Gamma}_{ij}^{\ \mathrm{p}_o} = \Gamma_{ij}^{\ \mathrm{p}_o} - g_{ij}(\mathrm{grad}\varphi)^{\mathrm{p}_o} \ , \tag{3.79}$$

$$\mathbf{\Gamma}_{ij}^{\ \mathrm{q}_o} = \Gamma_{ij}^{\ \mathrm{q}_o} - g_{ij}(\mathrm{grad}\varphi)^{\mathrm{q}_o} \ , \tag{3.80}$$

$$\mathbf{\Gamma}_{i\mathrm{p}_o}^{\ m} = \mathbf{\Gamma}_{\mathrm{p}_o i}^{\ \ m} = \Gamma_{i\mathrm{p}_o}^{\ m} + \mathrm{p}_o(\varphi)\delta_i^m \ , \tag{3.81}$$

$$\mathbf{\Gamma}_{i\mathrm{p}_o}^{\ \mathrm{p}_o} = \mathbf{\Gamma}_{\mathrm{p}_o i}^{\ \ \mathrm{p}_o} = \Gamma_{i\mathrm{p}_o}^{\ \mathrm{p}_o} + \widehat{E}_i(\varphi) \ , \tag{3.82}$$

$$\mathbf{\Gamma}_{i\mathrm{p}_o}^{\ \mathrm{q}_o} = \mathbf{\Gamma}_{\mathrm{p}_o i}^{\ \ \mathrm{q}_o} = \Gamma_{i\mathrm{p}_o}^{\ \mathrm{q}_o} \ , \tag{3.83}$$

$$\mathbf{\Gamma}_{i\mathrm{q}_o}^{\ m} = \mathbf{\Gamma}_{\mathrm{q}_o i}^{\ \ m} = \Gamma_{i\mathrm{q}_o}^{\ m} + \mathrm{q}_o(\varphi)\delta_i^m - \frac{\gamma^t}{2} g_{ti}(\mathrm{grad}\varphi)^{\widehat{E}_m} \ , \tag{3.84}$$

$$\mathbf{\Gamma}_{i\mathrm{q}_o}^{\ \mathrm{p}_o} = \mathbf{\Gamma}_{\mathrm{q}_o i}^{\ \ \mathrm{p}_o} = \Gamma_{i\mathrm{q}_o}^{\ \mathrm{p}_o} - \frac{\gamma^t}{2} g_{ti}(\mathrm{grad}\varphi)^{\mathrm{p}_o} \ , \tag{3.85}$$

$$\mathbf{\Gamma}_{i\mathrm{q}_o}^{\ \mathrm{q}_o} = \mathbf{\Gamma}_{\mathrm{q}_o i}^{\ \ \mathrm{q}_o} = \Gamma_{i\mathrm{q}_o}^{\ \mathrm{q}_o} + \widehat{E}_i(\varphi) - \frac{\gamma^t}{2} g_{ti}(\mathrm{grad}\varphi)^{\mathrm{q}_o} \ , \tag{3.86}$$

$$\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{p}_o}^{\ \ m} = \Gamma_{\mathrm{p}_o\mathrm{p}_o}^{\ \ m} \ , \tag{3.87}$$

$$\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{p}_o}^{\ \ \mathrm{p}_o} = \Gamma_{\mathrm{p}_o\mathrm{p}_o}^{\ \ \mathrm{p}_o} + 2\mathrm{p}_o(\varphi) \ , \tag{3.88}$$

$$\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{p}_o}^{\ \ \mathrm{q}_o} = \Gamma_{\mathrm{p}_o\mathrm{p}_o}^{\ \ \mathrm{q}_o} \ , \tag{3.89}$$

$$\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{q}_o}^{\ \ m} = \mathbf{\Gamma}_{\mathrm{q}_o\mathrm{p}_o}^{\ \ m} = \Gamma_{\mathrm{p}_o\mathrm{q}_o}^{\ \ m} - \frac{\alpha}{2}(\mathrm{grad}\varphi)^{\widehat{E}_m} \ , \tag{3.90}$$

$$\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{q}_o}^{\ \ \mathrm{p}_o} = \mathbf{\Gamma}_{\mathrm{q}_o\mathrm{p}_o}^{\ \ \mathrm{p}_o} = \Gamma_{\mathrm{p}_o\mathrm{q}_o}^{\ \ \mathrm{p}_o} + \mathrm{q}_o(\varphi) - \frac{\alpha}{2}(\mathrm{grad}\varphi)^{\mathrm{p}_o} \ , \tag{3.91}$$

$$\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{q}_o}^{\ \ \mathrm{q}_o} = \mathbf{\Gamma}_{\mathrm{q}_o\mathrm{p}_o}^{\ \ \mathrm{q}_o} = \Gamma_{\mathrm{p}_o\mathrm{q}_o}^{\ \ \mathrm{q}_o} + \mathrm{p}_o(\varphi) - \frac{\alpha}{2}(\mathrm{grad}\varphi)^{\mathrm{q}_o} \ , \tag{3.92}$$

$$\mathbf{\Gamma}_{\mathrm{q}_o\mathrm{q}_o}^{\ \ m} = \Gamma_{\mathrm{q}_o\mathrm{q}_o}^{\ \ m} - \frac{\beta}{2}(\mathrm{grad}\varphi)^{\widehat{E}_m} \ , \tag{3.93}$$

$$\mathbf{\Gamma}_{\mathrm{q}_o\mathrm{q}_o}^{\ \ \mathrm{p}_o} = \Gamma_{\mathrm{q}_o\mathrm{q}_o}^{\ \ \mathrm{p}_o} - \frac{\beta}{2}(\mathrm{grad}\varphi)^{\mathrm{p}_o} \ , \tag{3.94}$$

$$\mathbf{\Gamma}_{\mathrm{q}_o\mathrm{q}_o}^{\ \ \mathrm{q}_o} = \Gamma_{\mathrm{q}_o\mathrm{q}_o}^{\ \ \mathrm{q}_o} + 2\mathrm{q}_o(\varphi) - \frac{\beta}{2}(\mathrm{grad}\varphi)^{\mathrm{q}_o} \ . \tag{3.95}$$

We now recall that any vector field X on M decomposes into the sum

$$
\begin{aligned}
X = \widehat{E}^i(X)\widehat{E}_i + \mathrm{p}_o^*(X)\mathrm{p}_o + \mathrm{q}_o^*(X)\mathrm{q}_o = g\left(X, g^{ik}\widehat{E}_k - \frac{\gamma^i}{\alpha}\mathrm{p}_o\right)\widehat{E}_i + \\
+ g\left(X, \frac{2}{\alpha}\mathrm{q}_o + \frac{1}{\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\mathrm{p}_o - \frac{\gamma^m}{\alpha}\widehat{E}_m\right)\mathrm{p}_o + g\left(X, \frac{2}{\alpha}\mathrm{p}_o\right)\mathrm{q}_o\ .
\end{aligned}
\tag{3.96}
$$

From this, we get that the components $(\mathrm{grad}\varphi)^A$ of the gradient of φ are equal to

$$
\begin{aligned}
&(\mathrm{grad}\varphi)^{\widehat{E}_i} := g^{ik}\widehat{E}_k(\varphi) - \frac{\gamma^i}{\alpha}\mathrm{p}_o(\varphi)\ ,\\
&(\mathrm{grad}\varphi)^{\mathrm{p}_o} := \frac{2}{\alpha}\mathrm{q}_o(\varphi) + \frac{1}{\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\mathrm{p}_o(f) - \frac{\gamma^m}{\alpha}\widehat{E}_m(f)\ ,\\
&(\mathrm{grad}\varphi)^{\mathrm{q}_o} := \frac{2}{\alpha}\mathrm{p}_o(\varphi).
\end{aligned}
\tag{3.97}
$$

Inserting these expressions and (3.78) – (3.95) into (3.78) – (3.95), we get the explicit formulas for the Christoffel symbols $\mathbf{\Gamma}_{AB}^{\ \ C}$ of the scaled metric $g^\varphi = e^{2\varphi}g$. They are:

$$
\begin{aligned}
\mathbf{\Gamma}_{ij}^{\ m} = g^{mk}g_o(\nabla^o_{E_i}E_j, E_k) + g^{mk}S_{ij|k} + \frac{\gamma^m\omega_{ij}}{4} + \widehat{E}_i(\varphi)\delta_j^m + \widehat{E}_j(\varphi)\delta_i^m \\
- g_{ij}\left(g^{mk}\widehat{E}_k(\varphi) - \frac{\gamma^m}{\alpha}\mathrm{p}_o(\varphi)\right)\ ,
\end{aligned}
\tag{3.98}
$$

$$
\begin{aligned}
\mathbf{\Gamma}_{ij}^{\ \mathrm{p}_o} = \frac{1}{2\alpha}\widehat{E}_i(\gamma^k g_{jk}) + \frac{1}{2\alpha}\widehat{E}_j(\gamma^k g_{ik}) - \frac{1}{4\alpha}\gamma^m\gamma^k g_{mk}\omega_{ij} - \frac{\gamma^m}{\alpha}g_o(\nabla^o_{E_i}E_j, E_m) - \frac{\gamma^m}{\alpha}S_{ij|m} \\
- g_{ij}\left(\frac{2}{\alpha}\mathrm{q}_o(\varphi) + \frac{1}{\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\mathrm{p}_o(\varphi) - \frac{\gamma^m}{\alpha}\widehat{E}_m(\varphi)\right)\ ,
\end{aligned}
\tag{3.99}
$$

$$
\mathbf{\Gamma}_{ij}^{\ \mathrm{q}_o} = -\frac{\omega_{ij}}{2} - \frac{2g_{ij}}{\alpha}\mathrm{p}_o(\varphi)\ ,
\tag{3.100}
$$

$$
\mathbf{\Gamma}_{i\mathrm{p}_o}^{\ \ m} = \mathbf{\Gamma}_{\mathrm{p}_o i}^{\ \ m} = \frac{\alpha g^{mk}\omega_{ik}}{4} + \mathrm{p}_o(\varphi)\delta_i^m\ ,
\tag{3.101}
$$

$$
\mathbf{\Gamma}_{i\mathrm{p}_o}^{\ \ \mathrm{p}_o} = \mathbf{\Gamma}_{\mathrm{p}_o i}^{\ \ \mathrm{p}_o} = \frac{1}{2\alpha}\widehat{E}_i(\alpha) + \frac{1}{2\alpha}\mathrm{p}_o(\gamma^k)g_{ik} - \frac{\gamma^m\omega_{im}}{4} + \widehat{E}_i(\varphi)\ ,
\tag{3.102}
$$

$$
\mathbf{\Gamma}_{i\mathrm{p}_o}^{\ \ \mathrm{q}_o} = \mathbf{\Gamma}_{i\mathrm{p}_o}^{\ \ \mathrm{q}_o} = 0\ ,
\tag{3.103}
$$

$$
\begin{aligned}
\mathbf{\Gamma}_{i\mathrm{q}_o}^{\ \ m} = \mathbf{\Gamma}_{\mathrm{q}_o i}^{\ \ m} = \frac{g^{mk}}{4}\widehat{E}_i(\gamma^t g_{tk}) - \frac{g^{mk}}{4}\widehat{E}_k(\gamma^t g_{ti}) - \frac{\gamma^\ell}{4}c_{ir}^t g_{t\ell}g^{mr} + \frac{g^{mk}}{4}\beta\omega_{ik} - \\
- \frac{\gamma^m}{4\alpha}\widehat{E}_i(\alpha) + \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\gamma^t)g_{ti} + \mathrm{q}_o(\varphi)\delta_i^m - \frac{\gamma^t}{2}g_{ti}\left(g^{mk}\widehat{E}_k(\varphi) - \frac{\gamma^m}{\alpha}\mathrm{p}_o(\varphi)\right)\ ,
\end{aligned}
\tag{3.104}
$$

$$
\begin{aligned}
\mathbf{\Gamma}_{i\mathrm{q}_o}^{\ \ \mathrm{p}_o} = \mathbf{\Gamma}_{\mathrm{q}_o i}^{\ \ \mathrm{p}_o} = \frac{1}{2\alpha}\widehat{E}_i(\beta) + \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\widehat{E}_i(\alpha) - \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{p}_o(\gamma^t)g_{it} - \frac{1}{2\alpha^2}\beta\widehat{E}_i(\alpha) + \\
+ \frac{1}{2\alpha^2}\beta\mathrm{p}_o(\gamma^t)g_{it} - \frac{\gamma^m}{4\alpha}\widehat{E}_i(\gamma^t g_{tm}) + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\gamma^t g_{it}) + \frac{\gamma^m\gamma^t}{4\alpha}g_{t\ell}c_{im}^\ell - \frac{\gamma^m}{4\alpha}\beta\omega_{im} - \\
- \frac{\gamma^t}{2}g_{ti}\left(\frac{2}{\alpha}\mathrm{q}_o(\varphi) + \frac{1}{\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\mathrm{p}_o(\varphi) - \frac{\gamma^m}{\alpha}\widehat{E}_m(\varphi)\right)\ ,
\end{aligned}
\tag{3.105}
$$

$$
\mathbf{\Gamma}_{i\mathrm{q}_o}^{\ \ \mathrm{q}_o} = \mathbf{\Gamma}_{\mathrm{q}_o i}^{\ \ \mathrm{q}_o} = \frac{1}{2\alpha}\widehat{E}_i(\alpha) - \frac{1}{2\alpha}\mathrm{p}_o(\gamma^t)g_{it} + \widehat{E}_i(\varphi) - \frac{\gamma^t}{2}g_{ti}\left(\frac{2}{\alpha}\mathrm{p}_o(\varphi)\right)\ ,
\tag{3.106}
$$

$$
\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{p}_o}^{\ \ \ m} = 0\ ,
\tag{3.107}
$$

$$\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{p}_o}^{\ \ \mathrm{p}_o} = \mathrm{p}_o(\log\alpha) + 2\mathrm{p}_o(\varphi) \ , \tag{3.108}$$

$$\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{p}_o}^{\ \ \mathrm{q}_o} = 0 \ , \tag{3.109}$$

$$\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{q}_o}^{\ \ m} = \mathbf{\Gamma}_{\mathrm{q}_o\mathrm{p}_o}^{\ \ m} = \frac{1}{4}\mathrm{p}_o(\gamma^m) - \frac{g^{mk}}{4}\widehat{E}_k(\alpha) - \frac{\alpha}{2}\left(g^{mk}\widehat{E}_k(\varphi) - \frac{\gamma^m}{\alpha}\mathrm{p}_o(\varphi)\right) \ , \tag{3.110}$$

$$\begin{aligned}\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{q}_o}^{\ \ \mathrm{p}_o} = \mathbf{\Gamma}_{\mathrm{q}_o\mathrm{p}_o}^{\ \ \mathrm{p}_o} &= \frac{1}{2\alpha}\mathrm{p}_o(\beta) - \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\gamma^i)g_{im} + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\alpha) + \mathrm{q}_o(\varphi) - \\ &- \frac{\alpha}{2}\left(\frac{2}{\alpha}\mathrm{q}_o(\varphi) + \frac{1}{\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\mathrm{p}_o(\varphi) - \frac{\gamma^m}{\alpha}\widehat{E}_m(\varphi)\right) \ ,\end{aligned} \tag{3.111}$$

$$\mathbf{\Gamma}_{\mathrm{p}_o\mathrm{q}_o}^{\ \ \mathrm{q}_o} = \mathbf{\Gamma}_{\mathrm{q}_o\mathrm{p}_o}^{\ \ \mathrm{q}_o} = 0 \ , \tag{3.112}$$

$$\begin{aligned}\mathbf{\Gamma}_{\mathrm{q}_o\mathrm{q}_o}^{\ \ m} &= \frac{g^{mk}}{2}\mathrm{q}_o(\gamma^i)g_{ik} - \frac{g^{mk}}{4}\widehat{E}_k(\beta) - \frac{\gamma^m}{2\alpha}\mathrm{q}_o(\alpha) + \frac{\gamma^m}{4\alpha}\mathrm{p}_o(\beta) - \\ &- \frac{\beta}{2}\left(g^{mk}\widehat{E}_k(\varphi) - \frac{\gamma^m}{\alpha}\mathrm{p}_o(\varphi)\right) \ ,\end{aligned} \tag{3.113}$$

$$\begin{aligned}\mathbf{\Gamma}_{\mathrm{q}_o\mathrm{q}_o}^{\ \ \mathrm{p}_o} &= \frac{1}{2\alpha}\mathrm{q}_o(\beta) + \frac{1}{2\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{q}_o(\alpha) - \frac{1}{4\alpha^2}\gamma^m\gamma^k g_{mk}\mathrm{p}_o(\beta) - \frac{1}{\alpha^2}\beta\mathrm{q}_o(\alpha) + \\ &+ \frac{1}{2\alpha^2}\beta\mathrm{p}_o(\beta) - \frac{\gamma^m}{2\alpha}\mathrm{q}_o(\gamma^i)g_{im} + \frac{\gamma^m}{4\alpha}\widehat{E}_m(\beta) - \\ &- \frac{\beta}{2}\left(\frac{2}{\alpha}\mathrm{q}_o(\varphi) + \frac{1}{\alpha^2}\left(\gamma^m\gamma^k g_{mk} - 2\beta\right)\mathrm{p}_o(\varphi) - \frac{\gamma^m}{\alpha}\widehat{E}_m(\varphi)\right) \ ,\end{aligned} \tag{3.114}$$

$$\mathbf{\Gamma}_{\mathrm{q}_o\mathrm{q}_o}^{\ \ \mathrm{q}_o} = \frac{1}{\alpha}\mathrm{q}_o(\alpha) - \frac{\mathrm{p}_o(\beta)}{2\alpha} + 2\mathrm{q}_o(\varphi) - \frac{\beta}{2}\left(\frac{2}{\alpha}\mathrm{p}_o(\varphi)\right) \ . \tag{3.115}$$

In order to conclude, it is now sufficient to observe that the metric (2.12) with an arbitrary $\sigma > 0$ can be obtained from the metric considered in subsection 3.2 (*i.e.*, with $\sigma \equiv 1$) by applying the scaling factor $e^{2\varphi}$ with $\varphi := \frac{1}{2}\log\sigma$. Hence, the desired expressions for the Christoffel symbols are given by (3.98) – (3.115) with φ replaced by $\frac{1}{2}\log\sigma$ at all places. These substitutions yield (3.1) – (3.18).

Acknowledgments

D. V. Alekseevsky was supported by the Grant "Basis-foundation" (Leader) 22-7-1-34-1.

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