OPEN ACCESS
Bulletin of Mathematical Sciences
Vol. 12, No. 1 (2022) 2230001 (56 pages)

DOI: 10.1142/S1664360722300018

Propagation and reaction–diffusion models with free boundaries

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Received 16 October 2021
Accepted 24 December 2021
Published 31 January 2022

Communicated by Neil Trudinger

In this short survey, we describe some recent developments on the modeling of propagation by reaction-differential equations with free boundaries, which involve local as well as nonlocal diffusion. After the pioneering works of Fisher, Kolmogorov–Petrovski–Piskunov (KPP) and Skellam, the use of reaction–diffusion equations to model propagation and spreading speed has been widely accepted, with remarkable progresses achieved in several directions, notably on propagation in heterogeneous media, models for interacting species including epidemic spreading, and propagation in shifting environment caused by climate change, to mention but a few. Such models involving a free boundary to represent the spreading front have been studied only recently, but fast progress has been made. Here, we will concentrate on these free boundary models, starting with those where spatial dispersal is represented by local diffusion. These include the Fisher–KPP model with free boundary and related problems, where both the one space dimension and high space dimension cases will be examined; they also include some two species population models with free boundaries, where we will show how the long-time dynamics of some competition models can be fully determined. We then consider the nonlocal Fisher–KPP model with free boundary, where the diffusion operator Δu is replaced by a nonlocal one involving a kernel function. We will show how a new phenomenon, known as accelerated spreading, can happen to such a model. After that, we will look at some epidemic models with nonlocal diffusion and free boundaries, and show how the long-time dynamics can be rather fully described. Some remarks and comments are made at the end of each section, where related problems and open questions will be briefly discussed.

Keywords: Reaction–diffusion equation; free boundary; local and nonlocal diffusion; propagation speed.

Mathematics Subject Classification 2020: 35K20, 35R09, 35R36, 92D25, 92D30

1. Introduction

Propagation occurs naturally in the real world, albeit appearing in many different forms. For example, the spreading of infectious diseases, the invasion of exotic species, and the spreading of bush fires all involve a certain form of propagation. It has been observed that many common features of the propagation phenomena can be captured by reaction–diffusion models.

In a pioneer paper of 1937, Fisher [72] used a simple reaction–diffusion equation of the form

$$u_t - Du_{xx} = au(1-u), \quad t > 0, \quad x \in \mathbb{R} \tag{1.1}$$

to model the spreading of an advantageous gene in a population, where $u(t,x)$ stands for the density of the subpopulation that carries the advantageous gene at time t and spatial location x. It is assumed that individuals in the population move in space randomly following the rule of Brownian motion, which is represented by the term Du_{xx} in (1.1), with D known as the diffusion rate. The population growth is determined by the logistic growth term $au(1-u)$, with a representing the net growth rate of the population.

Fisher observed that for any constant $c \geq c_0 := 2\sqrt{aD}$, Eq. (1.1) has a special solution of the form $u(t,x) = V(ct-x)$, which he called "wave of stationary form" advancing with velocity c. Obviously, V satisfies the following ODE:

$$DV'' - cV' + aV(1-V) = 0.$$

Fisher claimed that c_0 should be the actual *spreading speed* of the advantageous gene in the environment. Such a special solution is nowadays called a *traveling wave solution* with speed c, and the associated V is called the wave profile function.

In another 1937 paper [98], Kolmogorove, Petrovski and Piskunov (KPP), independently of Fisher, studied the same gene spreading problem by a similar equation

$$u_t - Du_{xx} = f(u), \quad t > 0, \quad x \in \mathbb{R}, \tag{1.2}$$

where $f(u)$ is a C^1 function satisfying

$$f(0) = f(1) = 0 < f(u) \leq f'(0)u \quad \forall\, u \in (0,1), \quad f'(1) < 0.$$

They proved that for $c \geq c_0 := 2\sqrt{f'(0)D}$, Eq. (1.2) has a solution of the form $u(t,x) := V_c(ct-x)$ (with $V_c' > 0,\ V_c(-\infty) = 0,\ V_c(\infty) = 1$), and no such solution exists if $c < c_0$. Moreover, the solution of (1.2) with initial condition

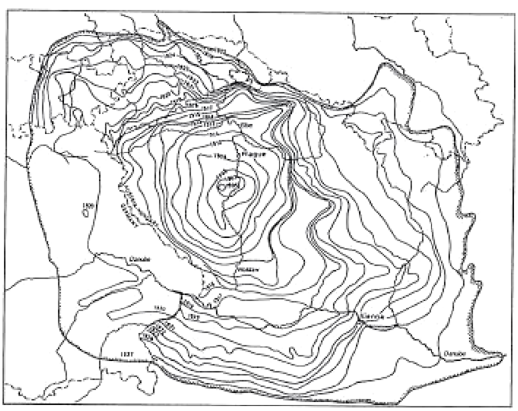
$$u(0,x) = \begin{cases} 1 & \text{for } x < 0, \\ 0 & \text{for } x \geq 0, \end{cases}$$

converges to the traveling wave solution with the minimal speed c_0 in the following sense:

$$\lim_{t\to\infty} |u(t,x) - V_{c_0}([c_0 + o(1)]t - x)| = 0 \quad \text{uniformly in } x \in \mathbb{R}.$$

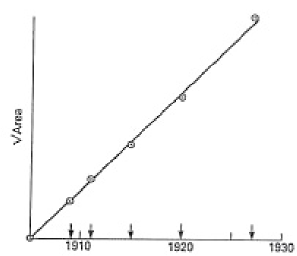
Let us note that if one takes $f(u) = au(1-u)$, then the result of Fisher is recovered, and the last conclusion is supportive to Fisher's claim that c_0 is the spreading speed of the new gene in the population.

The first real-world evidence for constant speed spreading appeared in a 1951 paper by Skellam [120], where it was demonstrated, using published data on the spreading of muskrats in Europe during 1905–1927, that the range radius of the maskrats increases linearly in time. More precisely, using a map obtained by J. Ulbrich (1930), Skellam calculated the *area of the muskrat's range* $A(t)$, took its *square root* and plotted it *against the time* t (in years), and found that the data points lay on a straight line, namely the function $t \to \sqrt{A(t)}$ is linear (see below).



Range expansion of muskrat from 1905-1927 (after Elton)

(a)



Square root of area occupied by muskrat versus time (after Skellam)

(b)

Note that the *range radius* $R(t)$ for the area $A(t)$ is determined by

$$A(t) = \pi R(t)^2, \quad \text{or} \quad R(t) = \frac{1}{\sqrt{\pi}}\sqrt{A(t)}.$$

Thus Skellam's observation says: *The range radius of the maskrats increases linearly in time.* Or worded in another way: *The spreading of muskrates has a constant speed.*

Subsequently, data on the spreading of many other species were used to show similar spreading behavior, including those for the spread of Himalayan thar in

South Island of New Zealand during 1936–1966, the spread of house finch in North America during 1956–1973, and the spread of Japanese beetle in North America during 1916–1941, to mention but a few. All these gave strong support to Fisher's claim on the existence of a constant spreading speed c_0, based on the reaction–diffusion model (1.1).

Fisher's claim on the spreading speed was convincingly proved by D. G. Aronson and H. F. Weinberger in the late 1970s. In [5], they considered the following equation in $\mathbb{R}^N$ $(N \geq 1)$:

$$u_t - D\Delta u = f(u), \tag{1.3}$$

where $\Delta u = u_{x_1x_1} + \cdots + u_{x_Nx_N}$ is the Laplacian operator. Among other things, they showed that if $f(u)$ behaves like that in the models of Fisher and KPP mentioned above, and if the initial function $u_0(x)$ is non-negative and has nonempty compact support, then the unique solution $u(t,x)$ of (1.3) with initial condition $u(0,x) = u_0(x)$ is defined for all $t > 0$ and satisfies, for any small $\epsilon > 0$,

$$\begin{cases} u(t,x) \to 1 & \text{uniformly for } x \in \{x \in \mathbb{R}^N : |x| \leq (c_0 - \epsilon)t\}, \\ u(t,x) \to 0 & \text{uniformly for } x \in \{x \in \mathbb{R}^N : |x| \geq (c_0 + \epsilon)t\}, \end{cases} \quad \text{as } t \to \infty, \tag{1.4}$$

where $c_0 := 2\sqrt{f'(0)D}$.

The behavior of u described in (1.4) indicates that, for all large time, outside of the ball of radius $[c_0 + o(1)]t$ (centered at the origin), the population density is close to 0, while inside the ball of radius $[c_0 + o(1)]t$, the population density is close to 1. Therefore, one may interpret such a behavior biologically by saying that the population spreads with asymptotic speed c_0. When $f(u) = au(1-u)$, we have $c_0 = 2\sqrt{aD}$, *as claimed by Fisher*!

The above convergence result (1.4) of Aronson and Weinberger has been improved. If the initial function u_0 is radially symmetric, then u is radially symmetric in x (i.e. $u = u(t,|x|)$) and the following holds:

$$\lim_{t\to\infty} \left| u(t,|x|) - V_{c_0}\left(c_0 t - \frac{N+2}{c_0} D \ln t + C - |x| \right) \right| = 0 \tag{1.5}$$

for some constant C, uniformly in $x \in \mathbb{R}^N$.

If u_0 is not radially symmetric, then it follows from a simple comparison argument and the above result on radial solutions that, for any small $\epsilon > 0$,

$$\begin{cases} u(t,x) \to 1 & \text{uniformly for } x \in \left\{ x \in \mathbb{R}^N : |x| \leq c_0 t - \left(\dfrac{N+2}{c_0} D + \epsilon \right) \ln t \right\}, \\ u(t,x) \to 0 & \text{uniformly for } x \in \left\{ x \in \mathbb{R}^N : |x| \geq c_0 t - \left(\dfrac{N+2}{c_0} D - \epsilon \right) \ln t \right\}. \end{cases}$$

This phenomenon is widely known as the *logarithmic shift* in the Fisher–KPP spreading.

Remarks. • When $N = 1$, the logarithmic shift term in (1.5) has coefficient $3/c_0$, which was first obtained by Bramson [14] by a *probabilistic method* for a problem concerning branching Brownian motion; it is now known as the *Bramson correction term*.

• For $N \geq 2$, (1.5) follows from Gärtner [75] (again by a probabilistic method). See [118] for further results with an analytic approach.

These classical works have inspired extensive further research in several directions, including extensive works on propagation in various heterogeneous environments, and on situations where the dispersal of the species is not governed by Brownian motion (local diffusion) but by suitable nonlocal diffusions. In this paper, we will look at a sample of the recent works on extending these classical results to equations with free boundaries, and new findings.

To see how free boundaries may arise naturally in these kind of models, let us look at a shortcoming of (1.3) as a model for propagation. As in the above-mentioned famous work of Skellam [120], the spreading behavior of a species is often measured by the expansion of its population range as time increases. Naturally, the population range at time t of a species modeled by (1.3) is given by

$$\Omega(t) := \{x \in \mathbb{R}^N : u(t,x) > 0\}.$$

However, by the strong maximum principle for parabolic equations, we have $u(t,x) > 0$ for all $x \in \mathbb{R}^N$ once $t > 0$, and therefore $\Omega(t) \equiv \mathbb{R}^N$ for all $t > 0$, although $\Omega(0)$ is bounded.

To circumvent this problem and obtain a spreading speed from (1.4), one may nominate a small positive constant δ and use

$$\Omega_\delta(t) := \{x : u(t,x) > \delta\}$$

as an approximation of the population range at time t; then (1.4) guarantees that $\Omega_\delta(t)$ is a bounded set for all time $t > 0$, and moreover, its boundary $\partial\Omega_\delta(t) = \{x : u(t,x) = \delta\}$ moves to infinity at the asymptotic speed c_0 in all radial directions, regardless of the choice of $\delta \in (0,1)$ and the initial function u_0.

Thus (1.3) can be used to successfully determine the spreading speed but it is not adequate to locate the spreading front, although the latter may provide crucial information in many applications. For example, in the spreading of an epidemic, it is important to obtain an accurate estimate of the spreading front. Unfortunately, in such a situation, models of the form (1.3) become inadequate.

To overcome this shortcoming of (1.3), Du and Lin [46] introduced a free boundary version of the Fisher equation (1.1), where the same equation for $u(t,x)$ is satisfied for x over a changing interval $(g(t), h(t))$, representing the population range at time t, together with the boundary condition $u(t,x) = 0$ for $x \in \{g(t), h(t)\}$, and free boundary conditions

$$h'(t) = -\mu u_x(t, h(t)), \quad g'(t) = -\mu u_x(t, g(t)) \quad \text{for some fixed } \mu > 0.$$

They showed that this modified model always has a unique solution and as time goes to infinity, the population $u(t,x)$ exhibits a *spreading-vanishing dichotomy*, namely it either vanishes or converges to 1; moreover, in the latter case, a finite spreading speed can be determined. So, the modified model retains the desired features of (1.1), but does not have its shortcomings. This work has motivated considerable further research, and the "spreading-vanishing dichotomy" discovered in [46] has been shown to occur in a variety of similar models; see, for example, extensions to equations with a more general nonlinear term $f(u)$ ([48, 92, 94], etc.), extensions to equations with advection ([81, 91, 134], etc.), extensions to systems of population or epidemic models ([1, 47, 61, 108, 128, 133], etc.), and development of numerical methods for treating some of these free boundary problems ([109, 110, 116], etc.).

In this paper, we will give a brief account of a selection of these results, as well as some more recent works where the diffusion operator Du_{xx} is replaced by a nonlocal diffusion operator.

For almost every model considered here, there is a corresponding version over the entire space of the spatial variable x, where no free boundary is imposed. Some of the works on such models are discussed here for comparison purposes, but most of the important works on such models are not mentioned; a proper account of the vast literature on the research of these models deserves a review of a much bigger scale, well beyond the scope of this paper.

The rest of this paper is organized as follows. In Sec. 2, we consider the Fisher–KPP model with free boundary and related problems, where both the one space dimension and high space dimension cases are discussed. In Sec. 3, we consider some two species population models, and show how the long-time dynamics of some competition models with free boundary can be determined. Section 4 considers the nonlocal Fisher–KPP model with free boundary, where the diffusion operator Du_{xx} is replaced by a nonlocal diffusion operator involving a kernel function, and a new phenomenon, known as accelerated spreading, will be examined. In Sec. 5, we look at some epidemic models with nonlocal diffusion and free boundary, and show how their long-time dynamics can be rather fully described. At the end of each section, some remarks and comments are made, where related problems and open questions are discussed.

We hope this short survey can provide the reader a glimpse of the current research on reaction–diffusion models with free boundary used to understand the propagation phenomena.

2. The Free Boundary Problem for the Fisher–KPP Model and Beyond

2.1. *The one space dimension case*

We consider the following free boundary version of (1.2), with $f(u)$ covering more general nonlinearities, to be specified in what follows:

$$\begin{cases} u_t = Du_{xx} + f(u), & g(t) < x < h(t), \quad t > 0, \\ u(t, g(t)) = u(t, h(t)) = 0, & t > 0, \\ g'(t) = -\mu u_x(t, g(t)), & t > 0, \\ h'(t) = -\mu u_x(t, h(t)), & t > 0, \\ u(0, x) = u_0(x), & -h_0 \le x \le h_0, \\ -g(0) = h(0) = h_0, \end{cases} \tag{2.1}$$

where $x = g(t)$ and $x = h(t)$ are the moving boundaries to be determined together with $u(t, x)$, μ is a given positive constant, $f : [0, \infty) \to \mathbb{R}$ is a C^1 function satisfying

$$f(0) = 0.$$

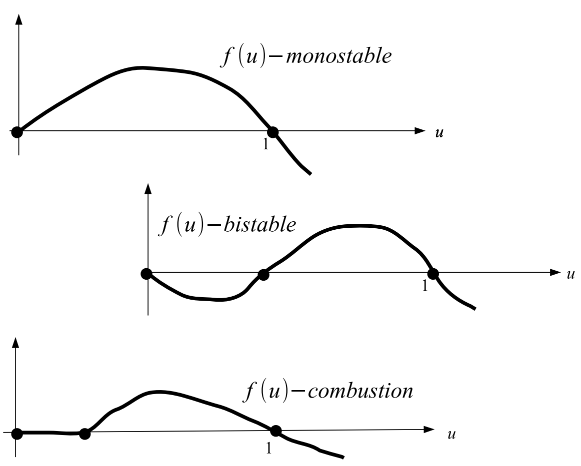
The initial function u_0 belongs to $\mathcal{X}(h_0)$ for some $h_0 > 0$, where

$$\mathcal{X}(h_0) := \big\{\phi \in C^2([-h_0, h_0]) : \phi(-h_0) = \phi(h_0) = 0, \phi'(-h_0) > 0,\ \phi'(h_0) < 0,\\ \phi(x) > 0 \text{ in } (-h_0, h_0)\big\}.$$

For any given $h_0 > 0$ and $u_0 \in \mathcal{X}(h_0)$, by a (classical) solution of (2.1) on the time-interval $[0, T]$ we mean a triple $(u(t, x), g(t), h(t))$ belonging to $C^{1,2}(G_T) \times C^1([0, T]) \times C^1([0, T])$, such that all the identities in (2.1) are satisfied pointwisely, where

$$G_T := \big\{(t, x) : t \in (0, T],\ x \in [g(t), h(t)]\big\}.$$

The nonlinear function $f(u)$ is assumed to be of the standard **monostable, bistable or combustion type**.



More precisely, f is a C^1 function and satisfies one of the following three sets of conditions:

($\mathbf{f_M}$) (Monostable): $f(0) = f(1) = 0$, $f(u) > 0$ in $(0,1)$, $f(u) < 0$ in $(1,\infty)$, $f'(0) > 0 > f'(1)$.

($\mathbf{f_B}$) (Bistable): There exists $\theta \in (0,1)$ such that $f(0) = f(\theta) = f(1) = 0$, $f(u) < 0$ in $(0,\theta)$, $f(u) > 0$ in $(\theta,1)$, $f(u) < 0$ in $(1,\infty)$, $f'(0) < 0$, $f'(1) < 0$ and $\int_0^1 f(u)du > 0$.

($\mathbf{f_C}$) (Combustion): There exists $\theta \in (0,1)$ such that $f(u) = f(1) = 0$ for $u \in [0,\theta]$, $f(u) > 0$ in $(\theta,1)$, $f(u) < 0$ in $(1,\infty)$, $f'(1) < 0$.

Problem (2.1) with $f(u)$ taking the Fisher nonlinearity $au(1-u)$ (which is a special monostable type function) was first considered by Du and Lin [46], as a model for the spreading of a new species. In the case $f(u) \equiv 0$, (2.1) reduces to the well-known *one-phase Stefan problem* [19, 73, 97], where $u(t,x)$ represents the temperature of water in the water region $(g(t),h(t))$, which is surrounded by ice. In such a case, the free boundary condition can be deduced from the law of energy conservation under phase transformation in the process of ice melting, and is known as the *Stefan condition*. However, in the biological setting, very few first principles are available to guide the modeling process. Nevertheless, if $u(t,x)$ represents the population density of a biological species in (2.1), the free boundary condition can be deduced from the assumption that k units of the species is lost per unit volume at the front [15], which gives $\mu = D/k$.

The method in [46] shows that under very general assumptions on f including ($\mathbf{f_M}$), ($\mathbf{f_B}$) and ($\mathbf{f_C}$) as special cases, (2.1) has a unique solution defined for all $t > 0$[a]. Moreover, the Hopf lemma for parabolic equations implies $g'(t) < 0 < h'(t)$ and hence

$$\begin{cases} h_\infty := \lim_{t\to+\infty} h(t) \in (h_0, +\infty], \\ g_\infty := \lim_{t\to+\infty} g(t) \in [-\infty, -h_0) \end{cases}$$

always exist.

In the following, we will focus on the long-time dynamics of (2.1), with f being one of the three types of nonlinearities ($\mathbf{f_M}$), ($\mathbf{f_B}$) and ($\mathbf{f_C}$).

2.1.1. *The monostable case*

Theorem 2.1 (Dichotomy [48]). *Assume that f is of monostable type. Then one of the following happens*:

(i) *Spreading*: $(g_\infty, h_\infty) = \mathbb{R}^1$ *and*

$$\lim_{t\to\infty} u(t,x) = 1 \quad \textit{locally uniformly in } \mathbb{R}^1.$$

[a]The smoothness requirement for the initial function, namely $u_0 \in C^2([-h_0,h_0])$, can be considerably relaxed. It is enough to require $u_0 \in C([-h_0,h_0])$; see [36].

(ii) *Vanishing*: (g_∞, h_∞) *is a finite interval with length no bigger than* $\pi/\sqrt{f'(0)}$ *and*

$$\lim_{t\to\infty} \max_{g(t)\le x\le h(t)} u(t,x) = 0.$$

Theorem 2.2 (Sharp threshold [48]). *In Theorem* 2.1, *if* $u_0 = \sigma\phi$ *with* $\phi \in \mathcal{X}(h_0)$, *then there exists* $\sigma^* = \sigma^*(h_0,\phi) \in [0,\infty]$ *such that*

(i) *Vanishing happens when* $0 < \sigma \le \sigma^*$.
(ii) *Spreading happens when* $\sigma > \sigma^*$.
(iii) σ^* *satisfies*

$$\begin{cases} \sigma^* = 0 & \text{if } 2h_0 \ge \pi/\sqrt{f'(0)}, \\ \sigma^* \in (0,\infty] & \text{if } 2h_0 < \pi/\sqrt{f'(0)}, \\ \sigma^* \in (0,\infty) & \text{if } 2h_0 < \pi/\sqrt{f'(0)} \quad \text{and if } f \text{ is globally Lipschitz.} \end{cases}$$

2.1.2. *The bistable case*

Theorem 2.3 (Trichotomy [48, 49]). *Assume that* f *is of bistable type. Then one of the following three cases must happen*:

(i) *Spreading*: $(g_\infty, h_\infty) = \mathbb{R}^1$ *and*

$$\lim_{t\to\infty} u(t,x) = 1 \quad \text{locally uniformly in } \mathbb{R}^1.$$

(ii) *Vanishing*: (g_∞, h_∞) *is a finite interval and*

$$\lim_{t\to\infty} \max_{g(t)\le x\le h(t)} u(t,x) = 0.$$

(iii) *Transition*: $(g_\infty, h_\infty) = \mathbb{R}^1$ *and there exists* $x_0 \in [-h_0, h_0]$ *such that*

$$\lim_{t\to\infty} |u(t,x) - v_\infty(x + x_0)| = 0 \quad \text{locally uniformly in } \mathbb{R}^1,$$

where v_∞ *is the unique positive solution to*

$$v'' + f(v) = 0\,(x \in \mathbb{R}^1), \quad v'(0) = 0, \quad v(-\infty) = v(+\infty) = 0.$$

Theorem 2.4 (Sharp threshold [48]). *In Theorem* 2.3, *if* $u_0 = \sigma\phi$ *for some* $\phi \in \mathcal{X}(h_0)$, *then there exists* $\sigma^* = \sigma^*(h_0,\phi) \in (0,\infty]$ *such that*

(i) *Vanishing happens when* $0 < \sigma < \sigma^*$.
(ii) *Spreading happens when* $\sigma > \sigma^*$.
(iii) *Transition happens when* $\sigma = \sigma^*$.
(iv) *There exists* $Z_B > 0$ *such that* $\sigma^* < \infty$ *if* $h_0 \ge Z_B$, *or if* $h_0 < Z_B$ *and* f *is globally Lipschitz.*

2.1.3. *The combustion case*

Theorem 2.5 (Trichotomy [48]). *Assume that f is of combustion type. Then one of the following occurs*:

(i) *Spreading*: $(g_\infty, h_\infty) = \mathbb{R}^1$ *and*

$$\lim_{t\to\infty} u(t,x) = 1 \quad \textit{locally uniformly in } \mathbb{R}^1.$$

(ii) *Vanishing*: (g_∞, h_∞) *is a finite interval and*

$$\lim_{t\to\infty} \max_{g(t)\le x\le h(t)} u(t,x) = 0.$$

(iii) *Transition*: $(g_\infty, h_\infty) = \mathbb{R}^1$ *and*

$$\lim_{t\to\infty} u(t,x) = \theta \quad \textit{locally uniformly in } \mathbb{R}^1.$$

Theorem 2.6 (Sharp threshold [48]). *In Theorem 2.7, if $u_0 = \sigma\phi$ for some $\phi \in \mathscr{X}(h_0)$, then there exists $\sigma^* = \sigma^*(h_0, \phi) \in (0, \infty]$ such that*

(i) *Vanishing happens when $0 < \sigma < \sigma^*$.*
(ii) *Spreading happens when $\sigma > \sigma^*$.*
(iii) *Transition happens when $\sigma = \sigma^*$.*
(iv) *There exists $Z_C > 0$ such that $\sigma^* < \infty$ if $h_0 \ge Z_C$, or if $h_0 < Z_C$ and f is globally Lipschitz.*

Remarks. (a) In Theorems 2.2, 2.4 and 2.6, *exactly* when $\sigma^* \ne \infty$ is still an open question. By [48], $\sigma^* = \infty$ if h_0 is small and $f(u) \sim -u^p$ for $p > p_0$ with some $p_0 > 2$.

(b) For the corresponding Cauchy problem (namely (5.11) in one space dimension), in the *bistable* and *combustion* cases, similar properties as described in Theorems 2.4 and 2.6 were conjectured by Kanel [90], and proved for *special initial functions* by Zlatǒs [146], and for *general initial functions* by Du and Matano [51].

(c) The corresponding Cauchy problem of (2.1) can be viewed as the limiting problem of (5.11) as $\mu \to \infty$; see Sec. 2.2 for a more general conclusion.

2.1.4. *Spreading profile*

When spreading happens to (2.1), for all three types of nonlinearities, the spreading speed and profile are determined by the following two theorems.

Theorem 2.7 (Semi-wave [48]). *Suppose that f is of monostable, bistable or combustion type. Then for any $\mu > 0$ there exists a unique solution pair $(c, q) = (c_0^*, q_{c_0^*})$ to*

$$\begin{cases} q'' - cq' + f(q) = 0, \quad q > 0 \quad \textit{in } (0,\infty), \\ q(0) = 0, \quad q(\infty) = 1, \quad q'(0) = c/\mu. \end{cases} \tag{2.2}$$

We call $q_{c_0^*}$ a *semi-wave with speed* c_0^*. It determines the spreading profile of (2.1), as described in the following result.

Theorem 2.8 (Spreading profile [53]). *Assume spreading happens to* (2.1) *with* f *of monostable, bistable or combustion type. Then there exist constants* C_1 *and* C_2 *such that*

$$\begin{cases} \lim_{t\to\infty} h'(t) = c_0^*, & \lim_{t\to\infty} [h(t) - c_0^* t] = C_1, \\ \lim_{t\to\infty} g'(t) = -c_0^*, & \lim_{t\to\infty} [g(t) + c_0^* t] = C_2, \end{cases}$$

$$\begin{cases} \lim_{t\to\infty} \max_{x\in[0,h(t)]} |u(t,x) - q_{c_0^*}(h(t) - x)| = 0, \\ \lim_{t\to\infty} \max_{x\in[g(t),0]} |u(t,x) - q_{c_0^*}(x - g(t))| = 0. \end{cases}$$

Here, $(c_0^*, q_{c_0^*})$ *is given by Theorem* 2.7.

2.1.5. *Transition speed*

In the transition cases of Theorems 2.4 and 2.6, it turns out that although $-g(t)$ and $h(t)$ converge to ∞ as $t \to \infty$, their rates of growth are sublinear in time, as described below.

Theorem 2.9 (Transition speed [49]). *In the transition case of Theorem* 2.4 *where* f *is bistable, there exists* $c_1 > 0$ *such that*

$$-g(t), \quad h(t) = [c_1 + o(1)]\ln t \quad \text{as } t \to \infty;$$

and in the transition case of Theorem 2.6 *where* f *is of combustion type, there exists* $c_2 > 0$ *such that*

$$-g(t), \quad h(t) = [c_2 + o(1)]\sqrt{t} \quad \text{as } t \to \infty.$$

In Theorem 2.9, $c_1 = 1/\sqrt{|f'(0)|}$ and $c = c_2 > 0$ is the unique solution to

$$2ce^{c^2} \int_0^c e^{-s^2} ds = \mu\theta.$$

The proofs of the results in this section are based on subtle constructions of upper and lower solutions in general, and in several places, including the sharp threshold results and the transition speed estimates, the so-called "zero number argument" [49] based on Angenent [4] has played a crucial role.

2.2. *The case of high space dimensions*

The corresponding free boundary problem (2.1) in space dimension $N \geq 2$ has the form

$$\begin{cases} u_t - D\Delta u = f(u) & \text{for } x \in \Omega(t), \quad t > 0, \\ u = 0 \quad \text{and} \quad u_t = \mu|\nabla_x u|^2 & \text{for } x \in \partial\Omega(t), \quad t > 0, \\ u(0,x) = u_0(x) & \text{for } x \in \Omega_0. \end{cases} \tag{2.3}$$

Here, $\Omega(t) \subset \mathbb{R}^N$ is the population range at time t, with $\Omega(0) = \Omega_0$, and we assume that Ω_0 is a bounded domain with smooth boundary, $u_0 \in C^1(\overline{\Omega}_0)$ is positive in Ω_0, and $u_0|_{\partial\Omega_0} = 0$. As for (2.1), we restrict f to be one of the three types of nonlinearities: monostable, bistable or combustion type.

We note that in (2.3), both $u(t, x)$ and $\Omega(t)$ are unknowns. The physical meaning of the free boundary condition is: Each point $x \in \partial\Omega(t)$ moves in the direction of the outer normal to $\partial\Omega(t)$ at x, with velocity $\mu|\nabla_x u(t, x)|$. In the spherically symmetric setting, where

$$\partial\Omega(t) = \{x : |x| = h(t)\} \quad \text{and} \quad u = u(t, r), \quad r = |x|,$$

this can be simplified to $h'(t) = -\mu u_r(t, h(t))$.

While the free boundary condition is meaningful when $\partial\Omega(t)$ is C^1, in general, such smoothness is not guaranteed for all $t > 0$ even if the initial data (u_0, Ω_0) are sufficiently smooth. As in the classical Stefan problem, (2.3) has to be understood in a certain weak sense. It was proved by Du and Guo [40] that (2.3) has a unique weak solution defined for all $t > 0$ (see [35] for some new development).

2.2.1. *Basic results for* (2.3)

The regularity of the free boundary of (2.3) is a difficult technical issue. By further developing techniques of Kinderlehrer and Nirenberg [97] and Cafarelli [19] used to treat the classical one phase Stefan problem, and combining them with the reflection argument as used in Matano [114], Du *et al.* [52] were able to obtain $C^{2+\alpha}$ regularity of the free boundary away from the convex hull of Ω_0, provided that f is $C^{1+\alpha}$ near $u = 0$, namely

$$f \in C^{1+\alpha}([0, \delta]) \quad \text{for some small } \delta > 0.$$

It is easy to find examples that singularity of the free boundary occurs inside the convex hull of Ω_0. The regularity and long-time dynamical behavior of the solution to (2.3) are described as follows.

Theorem 2.10 ([52]). *Let* $(u(t, x), \Omega(t))$ *be the weak solution of* (2.3). *Then the following conclusions hold*:

(1) $\Omega(t)$ *is expanding*: $\overline{\Omega}_0 \subset \Omega(t) \subset \Omega(s)$ *if* $0 < t < s$.
(2) $\partial\Omega(t)\backslash$(*convex hull of* $\overline{\Omega}_0$) *is* $C^{2+\alpha}$ *if* $f(u)$ *is* $C^{1+\alpha}$ *near* $u = 0$.
(3) Dichotomy: *Let* $\Omega_\infty := \cup_{t>0}\Omega(t)$. *Then one of the following happens*:
   (a) Ω_∞ *is a bounded set and* $\lim_{t\to\infty} \|u(t, \cdot)\|_{L^\infty(\Omega(t))} = 0$;
   (b) $\Omega_\infty = \mathbb{R}^N$ *and for all large* t, $\partial\Omega(t)$ *is a* $C^{2+\alpha}$ (*provided that* $f(u)$ *is* $C^{1+\alpha}$ *near* $u = 0$) *closed hypersurface contained in the spherical shell*

$$\left\{x \in \mathbb{R}^N : 0 \le |x| - M(t) \le \frac{\pi}{2}\mathrm{diam}(\Omega_0)\right\},$$

where $M(t)$ *is a continuous function satisfying* $\lim_{t\to\infty} M(t) = \infty$.

The following result indicates that (1.3) is the limiting problem of (2.3) as $\mu \to \infty$.

Theorem 2.11 (Limiting problem [40]). *If the solution* $(u, \Omega(t))$ *of* (2.3) *is denoted by* $(u_\mu, \Omega_\mu(t))$ *to stress its dependence on* μ, *then as* $\mu \to \infty$,

$$\Omega_\mu(t) \to \mathbb{R}^N (\forall\, t > 0), \quad u_\mu \to U \quad \textit{in } C^{1,2}_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^N),$$

where U *is the unique solution of* (1.3) *with* $U(0,x) = \begin{cases} u_0(x), & x \in \Omega_0, \\ 0, & x \notin \Omega_0. \end{cases}$

When $\Omega_\infty = \mathbb{R}^N$ in Theorem 2.10, the asymptotic profile of $(u(t,x), \Omega(t))$ as $t \to \infty$ depends on $f(u)$. If f is monostable, then spreading happens (namely $u(t,x) \to 1$ as $t \to \infty$) in such a case for Ω_∞, and if f is bistable or is of combustion type, simple sufficient conditions can be easily obtained to guarantee spreading to happen. However, sharp threshold results as in Theorems 2.4 and 2.6 are not easy to obtain in the high dimension case here, due to the existence of more complicated stationary solutions in high space dimensions (see [50] for more details). When spreading happens, we have the following result on the profile of the solution to (2.3) as $t \to \infty$.

Theorem 2.12 (Logarithmic shift [54]). *Suppose* u_0 *and* Ω_0 *are radially symmetric in* (2.3), *and thus*

$$u = u(t, |x|),$$
$$\Omega(t) = \big\{x \in \mathbb{R}^N : |x| < h(t)\big\}.$$

Then, when spreading happens and $t \to \infty$,

$$\begin{cases} u(t,|x|) - q_{c_0^*}(h(t) - |x|) \to 0 \quad \textit{uniformly in } x, \\ h(t) - [c_0^* t - (N-1)c_1^* D \ln t] \to C = C(u_0) \in \mathbb{R}, \end{cases} \tag{2.4}$$

where $(c_0^*, q_{c_0^*})$ *is given in Theorem* 2.7, *and* $c_1^* > 0$ *is given by*

$$c_1^* = \frac{1}{\zeta c_0^*}, \quad \zeta = 1 + \frac{c_0^*}{\mu^2 \int_0^\infty q'_{c_0^*}(z)^2 e^{-c_0^* z} dz}.$$

Remarks. (a) If the radial symmetry assumption on u_0 and Ω_0 in Theorem 2.12 is dropped, then by a simple comparison argument, there exist constants $C_1 \le C_2$ such that, for all large t

$$\partial\Omega(t) \subset \big\{C_1 \le |x| - \big[c_0^* t - (N-1)c_1^* D \ln t\big] \le C_2\big\}.$$

(b) Note the difference between (2.4) and the estimates in Theorem 2.8 for the case of one space dimension; now $h(t) - c_0^* t$ is no longer bounded and is of the order $-(N-1)c_1^* D \ln t$ as $t \to \infty$. Such an error term is often called a logarithmic shift: $u(t,r)$ approaches a shifted version of the semi-wave $q_{c_0^*}(c_0^* t - r)$, where $r = |x|$.

2.2.2. *Comparison of* (1.3) *and* (2.3)

Let us now compare the behavior of (1.3) and (2.3) in the case $f(u)$ is a Fisher–KPP type function, which are viewed as models for the spreading of a new species. The above results indicate that (2.3) retains the main features of the classical model (1.3), but also exhibits a number of differences, and Theorem 2.11 shows that (1.3) can be viewed as the limiting problem of (2.3) as $\mu \to \infty$. We summarize in what follows their similarities and differences.

Similarities: When spreading happens, (1.3) and (2.3) share the following asymptotic behaviors:

(i) *Shape of fronts*: In both models, the fronts can be approximated by spheres.
(ii) *Spreading speed*: The fronts go to infinity at some constant asymptotic speeds (c_0 and c_0^*, respectively).

Differences:

(i) *Location of front*:
- The front in (2.3) is located at the free boundary.
- (1.3) does not give the precise location of the front.

(ii) *Success of spreading*:
- (1.3) gives *consistent success of spreading*: Spreading succeeds whenever the non-negative initial function $u_0(x)$ is not identically zero.
- (2.3) yields a *spreading-vanishing dichotomy*: For "large" initial function u_0, spreading happens; for "small" u_0, vanishing happens.[b]

(iii) *Logarithmic shift*:
- The (approximate) front of (1.3) propagates behind the moving sphere $\{x \in \mathbb{R}^N : x = c_0 t\}$ by a distance of the order $\frac{N+2}{c_0} D \ln t$; see (1.5).
- The front of (2.3) propagates behind the moving sphere $\{x \in \mathbb{R}^N : |x| = c_0^* t\}$ by a distance of the order $(N-1)c_1^* D \ln t$ (when spreading happens).

In particular, when dimension $N = 1$, logarithmic shifting happens for (1.3) but not for (2.3).

2.3. *Further remarks and comments*

There are many related works on a number of variations of (2.1) and (2.3). We only mention and comment on a small selection of them below.

In [92, 94], Yamada and collaborators considered a new kind of $f(u)$ for (2.1), called a "positive bistable" nonlinearity, which arises in some population problems.

[b]By [50], if $u_0(x) = \sigma\phi(x)$ with $\sigma > 0$ regarded as a parameter, then there exists σ^* such that spreading happens when $\sigma > \sigma^*$ and vanishing happens when $\sigma \in (0, \sigma^*]$.

The long-time dynamics of (2.1) with such a nonlinearity becomes more complicated, exhibiting two kinds of spreading behavior, involving a "propagating terrace" consisting of a semi-wave and a normal traveling wave.

When $f(u)$ behaves like u^p with $p > 0$, solutions of (2.1) may blow up in finite time, and its dynamical behavior is very different from those discussed above; see [70, 77, 145] for further details.

Similar to Berestycki *et al.* [11, 12], where the results of Aronson and Weinberger [5] were extended to heterogeneous media, for the free boundary model (2.1) with a Fisher–KPP nonlinearity, results of [46] have been extended to several types of heterogeneous environments. For time-periodic environment, the results for (2.1) have been extended in [41], where the Fisher–KPP type nonlinearity is assumed to depend periodically on time. For space periodic environment, a similar extension has been done in [45]. For almost-periodic environment in time or in space, see [101, 102, 104, 106] by Liang, Shen and their collaborators.

In [36, 37], (2.1) with f a Fisher–KPP type nonlinearity subjected to simultaneous time-periodic and space-periodic perturbations was considered. It was proved that the model still determines a spreading speed, but not by proving the existence of an associated semi-wave as in [41] or [45]; instead, the proof relied on extending a dynamical systems approach of Weinberger [135], further developed by Liang and Zhao [105]. The existence of a semi-wave-type solution in this setting remains open.

To understand the influence of climate change on ecological invasion, [42, 62, 88, 89] considered (2.1) in a variety of shifting environments, where a Fisher–KPP type of nonlinearity was perturbed by a shifting function in the form $f = u[a(x+ct)-u]$, where c is a given constant representing the speed of the shifting environment. Depending on the behavior of the shifting function $a(x)$, several new behaviors of the long-time dynamics have been found.

When a drifting term is added to the first equation of (2.1), representing the effect of flowing water in the river for instance, interesting new behavior also arises; see [81, 91, 124] and references therein.

When time delay is incorporated into (2.1), the reaction term may become nonlocal, though the main feature of the long-time dynamics can be retained; some recent research in that direction can be found in [38, 123].

In [66], (2.1) with $\mu < 0$ was considered, which provides a model with a receding front. Other variations of the free boundary condition can be found in [7, 17, 18], where the front can both recede or invade depending on the population density or the environment.

When u_{xx} in (2.1) is replaced by the porous medium operator $(u^m)_{xx}$ $(m > 1)$ or a more general operator of a similar form, with the free boundary conditions suitably modified accordingly, semi-wave solutions have been obtained in [67, 68] by Fadai and Simpson. When the natural free boundary of the porous medium problem is used, the long-time dynamics has been determined by Audrito and Vasquez [68] and Du, Quiros and Zhou [59].

In [109, 110, 116], numerical methods have been developed for the simulation of (2.1), (2.3) and related free boundary models.

In high space dimension, if Ω_0 in (2.3) is unbounded, the dynamical behavior may change drastically. When Ω_0 is roughly an infinite cone, the dynamics of (2.3) was examined in [35], and semi-waves with curved fronts have been obtained in [39]. Much remains to be understood in this direction.

3. Some Diffusive Competition Models with Free Boundaries

In the real world, almost every species interacts with some other species, and competition is a common relationship between different species. In this section, we look at some models for the dynamics of two competing species.

When two competing species invade into unlimited space, the following Lotka–Volterra competition system has been widely used to understand their dynamical behavior:

$$\begin{cases} u_t = d_1\Delta u + u(a_1 - b_1 u - c_1 v), & x \in \mathbb{R}^N, \quad t > 0, \\ v_t = d_2\Delta v + v(a_2 - b_2 v - c_2 u), & x \in \mathbb{R}^N, \quad t > 0, \end{cases} \tag{3.1}$$

where $u(t,x)$ and $v(t,x)$ denote the population densities of the two competing species at time t and spatial location x; the positive constants d_i, a_i, b_i and c_i $(i = 1, 2)$ are the diffusion rates, intrinsic growth rates, intra-specific competition rates, and inter-specific competition rates, respectively.

For mathematical analysis, the number of parameters in (3.1) can be reduced. By using the scalings

$$\hat{u}(x,t) := \frac{b_1}{a_1} u\left(\sqrt{\frac{d_2}{a_2}}x, \frac{t}{a_2}\right), \quad \hat{v}(x,t) := \frac{b_2}{a_2} v\left(\sqrt{\frac{d_2}{a_2}}x, \frac{t}{a_2}\right),$$

$$d := \frac{d_1}{d_2}, \quad \alpha := \frac{a_1}{a_2}, \quad a := \frac{a_1 c_2}{a_2 b_1}, \quad b := \frac{a_2 c_1}{a_1 b_2},$$

and then omitting the hat signs, system (3.1) can be rewritten into the following simpler form:

$$\begin{cases} u_t = d\Delta u + \alpha u(1 - u - bv), & x \in \mathbb{R}^N, \quad t > 0, \\ v_t = \Delta v + v(1 - v - au), & x \in \mathbb{R}^N, \quad t > 0. \end{cases} \tag{3.2}$$

The system (3.2) has four constant equilibrium solutions $(u,v) = (0,0)$, $(1,0), (0,1)$ and (u^*, v^*), where $(u^*, v^*) = \left(\frac{1-b}{1-ab}, \frac{1-a}{1-ab}\right)$ is meaningful only when $(1-a)(1-b) > 0$. For the corresponding ODE problem, namely when the solutions are functions of t only, it is well known that the asymptotic behavior of the solution with initial functions $u(0), v(0) > 0$ can be classified into the following four cases:

(i) If $0 < b < 1 < a$, then

$$\lim_{t\to\infty} (u(t), v(t)) = (1, 0).$$

(ii) If $0 < a < 1 < b$, then

$$\lim_{t\to\infty}(u(t), v(t)) = (0, 1).$$

(iii) If $a, b \in (0, 1)$, then

$$\lim_{t\to\infty}(u(t), v(t)) = (u^*, v^*).$$

(iv) If $a, b > 1$, then (depending on the initial condition)

$$\lim_{t\to\infty}(u(t), v(t)) = (1, 0) \quad \text{or} \quad (0, 1) \quad \text{or} \quad (u^*, v^*).$$

Cases (i) and (ii) are known as the *weak–strong competition* cases (u strong and v weak in case (i)). Case (iii) is called the *weak competition* case and case (iv) is referred to as the *strong competition* case.

To use (3.2) to describe the spreading behavior, one typically assumes that the initial populations $u(x, 0)$ and $v(x, 0)$ are positive in a bounded region of $\mathbb{R}^N$, and then uses the diffusive system to see how the populations evolve as time t increases. In the weak–strong competition case, say in case (i), the evolution of $\big(u(x,t), v(x,t)\big)$ can often be explained by a traveling wave solution of the system with a certain speed $c > 0$. This speed c is usually interpreted as the invading speed of u (and retreating speed of v). The dynamics can be more complex though; see [79] for a very recent study of this case in space dimension $N = 1$.

As a population model for propagation, (3.2) has a similar shortcoming to (1.3), namely, although the initial population ranges may be assumed to be bounded regions in space, that is, both $\{x \in \mathbb{R}^N : u(x, 0) > 0\}$ and $\{x \in \mathbb{R}^N : v(x, 0) > 0\}$ are bounded sets, once $t > 0$, the population ranges $\{x \in \mathbb{R}^N : u(x,t) > 0\}$ and $\{x \in \mathbb{R}^N : v(x, t) > 0\}$ coincide with $\mathbb{R}^N$. Therefore (3.2) is not adequate to describe the evolution of the range boundaries of the species.

Similar to the one species Fisher–KPP model, free boundaries have been introduced to (3.2) to represent the spreading front.

3.1. *Invasion into the territory of a native competitor*

In Du and Lin [47], the following model was investigated:

$$\begin{cases} u_t - d\Delta_r u = \alpha u(1-u-bv), & t>0, \quad r \in [0, s_1(t)),\\ v_t - \Delta_r v = v(1-v-au), & t>0, \quad r \ge 0,\\ u_r(t,0) = v_r(t,0) = 0, & t > 0,\\ u(t,r) = 0, & t>0, \quad r \ge s_1(t),\\ s_1'(t) = -\mu_1 u_r(t, s_1(t)), & t>0,\\ s_1(0) = s_1^0, \quad u(0,r) = u_0(r), & r \in [0, s_1^0],\\ v(0,r) = v_0(r), & r \ge 0, \end{cases} \tag{3.3}$$

with

$$\begin{cases} u_0 \in C^2([0, s_1^0]), \quad u_0'(0) = u_0(s_1^0) = 0, \quad u_0(r) > 0 & \text{in } [0, s_1^0), \\ v_0 \in C^2([0, \infty)) \cap L^\infty(0, \infty), \quad v_0'(0) = 0, \quad v_0(r) > 0 & \text{in } [0, \infty). \end{cases}$$

Here, Δ_r denotes the radial Laplacian $\partial_{rr} + \frac{N-1}{r}\partial_r$ in $\mathbb{R}^N$ ($N \geq 1$), and (3.3) models the dynamics of a species u invading into the habitat of an existing species v in a spherically symmetric setting, with $r = |x|$, $x \in \mathbb{R}^N$. In this model, v is regarded as *already established* in the environment, while u's population range is given by $\{x : |x| < s_1(t)\}$. If $v \equiv 0$ in (3.3), then we are back to the single species problem (2.3) (with radial symmetry). Assuming radial symmetry avoids the technically difficult regularity issue, enabling us to concentrate on the long-time dynamics of the model.

The following four theorems are from [47].

Theorem 3.1 (Existence and uniqueness [47]). *Problem* (3.3) *has a unique solution* $(u(t, r), v(t, r), s_1(t))$ *and it is defined for all* $t > 0$.

Theorem 3.2 (Weak invader [47]). *If* u *is an inferior competitor, namely* $a < 1 < b$, *then the invasion of* u *always fails, in the sense that*

$$\lim_{t\to\infty} (u(t, \cdot), v(t, \cdot)) = (0, 1) \quad \text{in } L^\infty_{\text{loc}}([0, \infty)).$$

Theorem 3.3 (Strong invader [47]). *If* u *is a superior competitor, namely* $a > 1 > b$, *then the invasion of* u *is determined by a dichotomy, namely exactly one of the following two alternatives hold*:

(i) **(Invasion success)** $\lim_{t\to+\infty} s_1(t) = +\infty$ *and*

$$\lim_{t\to+\infty} \big(u(t, \cdot), v(t, \cdot)\big) = (1, 0) \quad \text{in } [L^\infty_{\text{loc}}([0, \infty))]^2.$$

(ii) **(Invasion failure)** $\lim_{t\to+\infty} s_1(t) < +\infty$ *and*

$$\lim_{t\to+\infty} \|u(t, \cdot)\|_{L^\infty([0,s_1(t)])} = 0, \quad \lim_{t\to\infty} v(t, \cdot) = 1 \quad \text{in } L^\infty_{\text{loc}}([0, \infty)).$$

Theorem 3.4 (Criteria for the dichotomy [47]). *There exists* $R^* = R^*(d, a, b) > 0$, *determined by an eigenvalue problem, such that, in Theorem* 3.3, *the invasion of* u *always succeeds when* $s_1^0 \geq R^*$. *If* $s_1^0 < R^*$, *then there exists* $\mu^* \geq 0$, *depending on* (u_0, v_0), *such that* u *invades successfully if and only if* $\mu_1 > \mu^*$.

When the invasion of u is successful, the spreading speed of u has been determined by Du *et al.* [61].

Theorem 3.5 (Spreading speed [61]). *Suppose that* $a > 1 > b$,

$$\inf_{r\geq 0} v_0(r) > 0,$$

and u invades successfully. Then there exists a unique $c_{\mu_1} > 0$ such that

$$\lim_{t\to+\infty} \frac{s_1(t)}{t} = c_{\mu_1}.$$

Moreover, c_{μ_1} is strictly increasing in μ_1 and

$$c_0 := \lim_{\mu\to+\infty} c_{\mu_1} < +\infty.$$

We call c_{μ_1} the *spreading speed* of u governed by (3.3). The key step in the proof of Theorem 3.4 is to determine c_{μ_1}, which is much more difficult to do than the determination of the spreading speed in the single species case considered in the previous section.

First, we need a result of Kan-on [93] on the following traveling wave problem:

$$\begin{cases} c\Psi' - d\Psi'' = \alpha\Psi(1-\Psi-b\Phi), \quad \Psi'>0, \quad \xi\in\mathbb{R}, \\ c\Phi' - \Phi'' = \Phi(1-\Phi-a\Psi), \quad \Phi'<0, \quad \xi\in\mathbb{R}, \\ (\Psi,\Phi)(-\infty) = (0,1), \quad (\Psi,\Phi)(\infty) = (1,0). \end{cases} \tag{3.4}$$

Proposition 3.6 (Traveling wave [93]). *Suppose that $a > 1 > b$. Then there exists a unique constant*

$$c_0 \in [2\sqrt{\alpha d(1-b)}, 2\sqrt{\alpha d}]$$

such that problem (3.4) has a solution when $c \geq c_0$ and it has no solution when $c < c_0$.

Remark. It is well known that c_0 is the spreading speed of u governed by the corresponding Cauchy problem (3.2).

The spreading speed c_{μ_1} for (3.3) is determined by the following associated *semi-wave* problem:

$$\begin{cases} c\psi' - d\psi'' = \alpha\psi(1-\psi-b\varphi), & \psi' > 0 \ (\forall\,\xi>0), \\ c\varphi' - \varphi'' = \varphi(1-\varphi-a\psi), & \varphi' < 0 \ (\forall\,\xi\in\mathbb{R}), \\ \psi \equiv 0 \ (\forall\,\xi \leq 0), & \psi(+\infty) = 1, \\ \varphi(-\infty) = 1, & \varphi(+\infty) = 0. \end{cases} \tag{3.5}$$

Theorem 3.7 (Semi-wave and c_{μ_1} [61]). *Assume that $a > 1 > b$, and c_0 is given in Proposition 3.6. Then (3.5) has a unique solution*

$$(\psi,\varphi) \in \left[C(\mathbb{R}) \cap C^2([0,\infty))\right] \times C^2(\mathbb{R})$$

for each $c \in [0, c_0)$, and it has no such solution for $c \geq c_0$. Furthermore, if we denote the unique solution by (ψ_c, φ_c) $(c \in [0, c_0))$, then the following conclusions hold:

(i) *If $0 \leq c_1 < c_2 < c_0$, then*

$$\psi'_{c_1}(0) > \psi'_{c_2}(0), \quad \psi_{c_1}(\xi) > \psi_{c_2}(\xi) \ (\forall\,\xi>0), \quad \varphi_{c_1}(\xi) < \varphi_{c_2}(\xi) \quad (\forall\,\xi\in\mathbb{R}).$$

(ii) *The operator* $c \mapsto (\psi_c, \varphi_c)$ *is continuous from* $[0, c_0)$ *to* $C^2_{\rm loc}([0,+\infty)) \times C^2_{\rm loc}(\mathbb{R})$. *Moreover,*

$$\lim_{c\to c_0} (\psi_c, \varphi_c) = (0,1) \quad \text{in } C^2_{\rm loc}([0,+\infty)) \times C^2_{\rm loc}(\mathbb{R}).$$

(iii) *For each* $\mu_1 > 0$, *there exists a unique* $c = c_{\mu_1} \in (0, c_0)$ *such that*

$$\mu_1 \psi_c'(0) = c.$$

Moreover,

$$\mu_1 \mapsto c_{\mu_1} \quad \text{is strictly increasing and } \lim_{\mu\to+\infty} c_{\mu_1} = c_0.$$

For each $c \in [0, c_0)$, the solution pair (ψ_c, φ_c) in Theorem 3.7 generates a traveling wave

$$(\tilde{u}(t,x), \tilde{v}(t,x)) := (\psi_c(ct - x), \varphi_c(ct - x)),$$

which satisfies

$$\begin{cases} \tilde{u}_t - d\tilde{u}_{xx} = \alpha\tilde{u}(1 - \tilde{u} - b\tilde{v}), & t > 0, \quad -\infty < x < ct, \\ \tilde{v}_t - \tilde{v}_{xx} = \tilde{v}(1 - \tilde{v} - a\tilde{u}), & t > 0, \quad x \in \mathbb{R}, \\ \tilde{u}(t,x) = 0, & t > 0, \quad ct \le x < +\infty. \end{cases}$$

We note that when $c = c_{\mu_1}$, one has the extra identity

$$c = -\mu_1 \tilde{u}_x(t, ct).$$

We call $(\psi_{c_{\mu_1}}, \varphi_{c_{\mu_1}})$ the *semi-wave* associated to (3.3).

3.2. *Simultaneous invasion of two competitors*

We now consider the case that two competitors invade a new territory simultaneously. Such a situation was examined in [63, 64, 82, 83, 95], and is modeled by a variation of (3.3), which has the form

$$\begin{cases} u_t - d\Delta_r u = \alpha u(1 - u - bv), & t > 0, \quad r \in [0, s_1(t)), \\ v_t - \Delta_r v = v(1 - v - au), & t > 0, \quad r \in [0, s_2(t)), \\ u_r(t,0) = v_r(t,0) = 0, & t > 0, \\ u(t,r) = 0, & t > 0, \quad r \ge s_1(t), \\ v(t,r) = 0, & t > 0, \quad r \ge s_2(t), \\ s_1'(t) = -\mu_1 u_r(t, s_1(t)), & t > 0, \\ s_2'(t) = -\mu_2 v_r(t, s_2(t)), & t > 0, \\ s_1(0) = s_1^0, \quad u(0,r) = u_0(r), & r \in [0, s_1^0], \\ s_2(0) = s_2^0, \quad v(0,r) = v_0(r), & r \in [0, s_2^0] \end{cases} \tag{3.6}$$

with $d, \alpha, a, b, \mu_1, \mu_2, s_1^0, s_2^0$ positive constants, and

$$\begin{cases} u_0 \in C^2([0, s_1^0]), \quad u_0'(0) = u_0(s_1^0) = 0, \quad u_0(r) > 0 \quad \text{in } [0, s_1^0), \\ v_0 \in C^2([0, s_2^0]), \quad v_0'(0) = v_0(s_2^0) = 0, \quad v_0(r) > 0 \quad \text{in } [0, s_2^0). \end{cases} \tag{3.7}$$

Here, $u(t,r)$ and $v(t,r)$ denote the population densities of two competing species at time t and location $r = |x|$ in $\mathbb{R}^N$, respectively; their respective population ranges at time t are $\{x : |x| < s_1(t)\}$ and $\{x : |x| < s_2(t)\}$. So, they invade into the environment through the expansion of their population ranges simultaneously.

We will focus on the following weak–strong competition case:

$$(\mathrm{H}) : a > 1 > b.$$

So, u is the *superior competitor* and v the *inferior competitor*.

When dimension $N = 1$, Guo and Wu [83] showed that under (H), the dynamics of this problem can have four possibilities:

(i) the two species vanish eventually, namely

$$\begin{cases} \lim_{t\to\infty} s_i(t) < \infty & \text{for } i = 1, 2, \\ \lim_{t\to\infty} (u(t,r), v(t,r)) = (0,0) & \text{uniformly in } r; \end{cases}$$

(ii) the species u vanishes eventually, and v spreads successfully;
(iii) the species v vanishes eventually, and u spreads successfully;
(iv) both species spread successfully.

Further results for the $N = 1$ case can be found in [112, 130] by Wang and collaborators. However, the mechanism for the situation (iv) to occur and the precise spreading profile of the two species when (iv) does occur, were not fully understood.

Let $s^*_{\mu_2}$ be the spreading speed of v in the absence of u in (3.6), as determined by Theorem 2.7 in the previous section, and let c_{μ_1} be given by Theorem 3.7, which arises from the model (3.3). It turns out that they are important for the dynamics of (3.6).

In Du and Wu [63], it was shown that at least one species must vanish when $c_{\mu_1} > s^*_{\mu_2}$. If $c_{\mu_1} < s^*_{\mu_2}$ and certain conditions on the initial functions are satisfied, the following spreading profile for the two species in the coexistence case (iv) (with $N \geq 1$) was obtained in [63], which was subsequently named *chase-and-run coexistence* in [95]:

$$u \text{ spreads at speed } c_{\mu_1}, \quad \text{and} \quad v \text{ spreads at speed } s^*_{\mu_2} > c_{\mu_1}.$$

In such a case, the inferior species v outruns the superior u, and survives in the long run; see Theorem 3.8 and Figs. 2 and 3 for a more precise description of this phenomenon.

Unfortunately, the sufficient conditions on the initial data in [64] guaranteeing the chase-and-run coexistence are rather restrictive. For example, it requires that μ_1 is sufficiently small and (u_0, v_0, s_1^0, s_2^0) satisfies

$$\begin{cases} \|u_0\|_{L^\infty} \le 1, \quad s_1^0 \ge S^* & \text{for some } S^* > 0, \\ v_0(\cdot) \ge 1 \quad \text{in } [r_0, r_0 + L] & \text{for some large } r_0 > s_1^0 \quad \text{and} \quad L > 0. \end{cases}$$

Question. Can other coexistence state happen when the initial data are varied?

For convenience, we will name the chase-and-run coexistence as case (iv)(a).

3.2.1. *Numerical simulation result*

To help to find an answer to this question, Khan *et al.* [95] numerically investigated the problem (3.6) by looking at initial functions $u_0(r; \lambda_1)$ and $v_0(r; \lambda_2)$ which vary continuously with the parameters λ_1 and λ_2, respectively. The simulations suggest that only the four outcomes (i)–(iv)(a) can be observed, implying that chase-and-run coexistence is the only possible state for the two species to live together in the long run.

In the numerical simulation, the parameters are given by $(d, a, b, \alpha, \mu_1, \mu_2) = (2, 2, 0.5, 2, 0.1, 1)$, dimension $N = 2$, and the initial functions are

$$u_0(r, \lambda_1) = \begin{cases} 1, & \text{if } r \in \left[0, \dfrac{\lambda_1 \pi}{2}\right], \\ \sin(r/\lambda_1), & \text{if } r \in \left[\dfrac{\lambda_1 \pi}{2}, \lambda_1 \pi\right], \end{cases}$$

$$v_0(r, \lambda_2) = \begin{cases} \sin \epsilon, & \text{if } r \in [0, \epsilon\lambda_2], \\ \sin(r/\lambda_2), & \text{if } r \in [\epsilon\lambda_2, \lambda_2 \pi], \end{cases}$$

where $\epsilon = \arcsin(0.1)$.

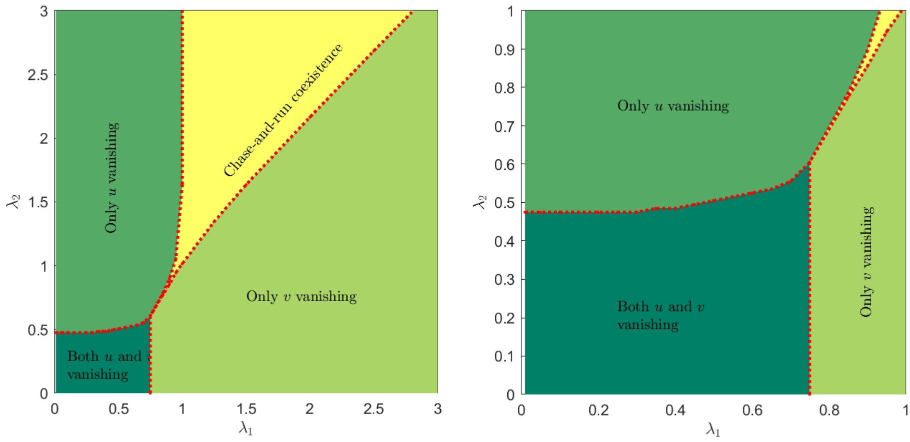
A large number of values of (λ_1, λ_2) are chosen and for each pair of these values, the solution of (3.6) is calculated until it exhibits a clear pattern for its large-time behavior. The simulation result shows that the checked region in the (λ_1, λ_2) plane can be divided into four parts, with each one yielding a particular long-time behavior of the solution, showing no new behaviors; see Fig. 1.

In Figs. 2 and 3, an example of the chase-and-run coexistence behavior captured by the numerical simulation is displayed.

3.2.2. *Further theoretical results*

In a recent work of Du and Wu [64], it is rigorously proved that there are exactly five types of long-time dynamical behaviors for (3.6) under the condition $a > 1 > b$:

apart from (i)–(iv)(a), there exists a fifth case,



(a) $(\lambda_1, \lambda_2) \in [0.01, 3.0]^2$. (b) $(\lambda_1, \lambda_2) \in [0.01, 1.0]^2$.

Fig. 1. Change of longtime dynamical behavior in the 2D radial case as (λ_1, λ_2) varies.

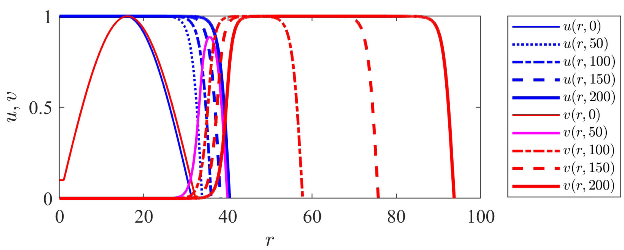


Fig. 2. (Color online) Profiles of $u(t,r)$ (blue curve) and $v(t,r)$ (red curve) with $(\lambda_1, \lambda_2) = (10, 10.28)$ at time $t = 0, 50, 100, 150, 200$, showing clear traveling wave behavior for large t.

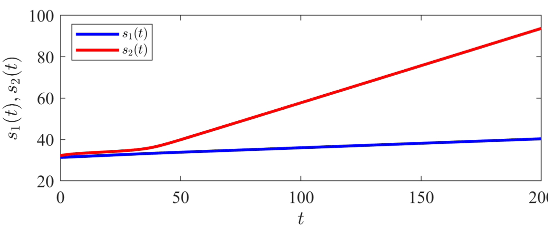


Fig. 3. (Color online) Corresponding behavior of $s_1(t)$ (blue curve) and $s_2(t)$ (red curve), showing linear growth in time for both.

which we may name it as case (iv)(b), where

$$\begin{cases} \text{both species spread successfully, and their spreading fronts} \\ \text{are kept within a finite distance to each other all the time.} \end{cases}$$

We conjecture that this new case can happen only when a parameter takes an exceptional value, and that is why it has eluded the numerical observations in [95].

In order to find all the possible coexistence cases, instead of looking at the initial functions directly, we consider the long time behavior of $s_2(t) - s_1(t)$, which clearly has three possibilities:

(S1) $\limsup_{t\to\infty} \big(s_2(t) - s_1(t)\big) = \infty$,
(S2) $\liminf_{t\to\infty} \big(s_2(t) - s_1(t)\big) = -\infty$,
(S3) $\limsup_{t\to\infty} \big|s_2(t) - s_1(t)\big| < \infty$.

We will show that under the following assumptions (which are almost necessary in order to obtain a co-existence state)

$$a > 1 > b, \quad c_{\mu_1} < s^*_{\mu_2}, \quad s_1(\infty) = \infty, \tag{3.8}$$

we have

$$\begin{cases} \text{(S1)} \Rightarrow \text{case (iv)(a) (chase-and-run coexistence)}, \\ \text{(S2)} \Rightarrow \text{case (iii) (vanishing of } v \text{ with } u \text{ spreading successfully)}. \end{cases}$$

We will also show that case (S3) can definitely occur. Note that due to $s_1(\infty) = \infty$, clearly (S3) $\Rightarrow s_2(\infty) = \infty$, and hence both species spread successfully; this is the new fifth case (iv)(b) mentioned earlier.

Theorem 3.8 (Chase-and-run coexistence [64]). *Assume that* (3.8) *holds. If* ($S1$) *happens, then the unique solution* (u, v, s_1, s_2) *of* (3.6) *satisfies*

$$\lim_{t\to\infty} \frac{s_1(t)}{t} = c_{\mu_1}, \quad \lim_{t\to\infty} \frac{s_2(t)}{t} = s^*_{\mu_2} > c_{\mu_1},$$

and for every small $\epsilon > 0$,

$$\begin{cases} \lim\limits_{t\to\infty} \sup\limits_{r\in[0,(c_{\mu_1}-\epsilon)t]} \big[|u(r,t) - 1| + |v(r,t)|\big] = 0, \\ \lim\limits_{t\to\infty} \sup\limits_{r\in[(c_{\mu_1}+\epsilon)t,(s^*_{\mu_2}-\epsilon)t]} |v(r,t) - 1| = 0. \end{cases}$$

Theorem 3.9 (Vanishing of v [64]). *Assume* (3.8) *holds. If* ($S2$) *happens, then the unique solution* (u, v, s_1, s_2) *of* (3.6) *satisfies* $s_2(\infty) < \infty$ *and*

$$\lim_{t\to\infty} [s_1(t) - (s^*_{\mu_1} t - c_N \text{logt})] = \ell, \quad \lim_{t\to\infty} s_1'(t) = s^*_{\mu_1},$$

where $\ell \in \mathbb{R}$ *depends on the initial data, and* $c_N > 0$ *depends on the space dimension* N *but not the initial data; moreover,*

$$\begin{cases} \lim\limits_{t\to\infty} \sup\limits_{r\in[0,s_1(t)]} |u(r,t) - q^*(s_1(t) - r)| = 0, \\ \lim\limits_{t\to\infty} \sup\limits_{r\in[0,s_2(t)]} |v(r,t)| = 0, \end{cases}$$

where q^* *is the unique semi-wave for* u *with* $s = s^*_{\mu_1}$.

The next result shows that (S3) (as well as (S1) and (S2)) can indeed happen when the initial functions are chosen properly.

For fixed (u_0, v_0, s_1^0, s_2^0) satisfying $v_0'(r) \le 0$, we define

$$\hat{v}^\sigma(r) := \sigma v_0\left(\frac{r}{\sigma}\right), \quad \sigma > 0.$$

Clearly $\sigma \to \hat{v}^\sigma(r)$ is continuous and monotone increasing.

Let $(u^\sigma, v^\sigma, s_1^\sigma, s_2^\sigma)$ be the unique solution of (3.6) with initial data

$$u^\sigma(r,0) = u_0(r), \quad s_1^\sigma(0) = s_1^0, \quad v^\sigma(r,0) = \hat{v}^\sigma(r), \quad s_2^\sigma(0) = \sigma s_2^0.$$

Suppose that

$$s_1^0 \ge S^* := R^* \sqrt{\frac{d}{\alpha(1-b)}}, \quad (\Rightarrow s_1^\sigma(\infty) = \infty \; \forall \, \sigma > 0), \tag{3.9}$$

where R^* is such that the first Dirichlet eigenvalue of $-\Delta$ over the ball $\{x \in \mathbb{R}^N : |x| < R^*\}$ equals 1.

We also assume

$$s_{\mu_1}^* < s_{\mu_2}^*. \quad (\Rightarrow c_{\mu_1} < s_{\mu_2}^*). \tag{3.10}$$

Note that $s_{\mu_1}^*$ and $s_{\mu_2}^*$ are independent of σ, and (3.10) holds for all small $\mu_1 > 0$ when $\mu_2 > 0$ is fixed.

We now consider the solution $(u^\sigma, v^\sigma, s_1^\sigma, s_2^\sigma)$ as σ varies, and the following result indicates that (S3) definitely occurs for some value of σ.

Theorem 3.10 (New case (iv)(b) [64]). *Suppose that $a > b > 1$, $s_1^0 \ge S^*$ and $s_{\mu_1}^* < s_{\mu_2}^*$. Then there exist $0 < \sigma_* \le \sigma^* < \infty$ such that*

(i) (S1) *happens to $(u^\sigma, v^\sigma, s_1^\sigma, s_2^\sigma)$ when $\sigma > \sigma^*$,*
(ii) (S2) *happens to $(u^\sigma, v^\sigma, s_1^\sigma, s_2^\sigma)$ when $\sigma < \sigma_*$,*
(iii) (S3) *happens to $(u^\sigma, v^\sigma, s_1^\sigma, s_2^\sigma)$ when $\sigma_* \le \sigma \le \sigma^*$.*

Remarks. (a) We believe that $\sigma_* = \sigma^*$, and so (S3) is an exceptional case, which happens only when σ takes the special value $\sigma_* = \sigma^*$, as a transition case between (S1) and (S2). This is perhaps why such a case has not been observed in the numerical simulations of [95].

(b) Moreover, we conjecture that for $\sigma = \sigma_* = \sigma^*$,

$$\lim_{t\to\infty} \frac{s_1^\sigma(t)}{t} = \lim_{t\to\infty} \frac{s_2^\sigma(t)}{t} = s^*, \quad \lim_{t\to\infty} [s_2^\sigma(t) - s_1^\sigma(t)] = \ell^*$$

for some constants $s^* \in (c_{\mu_1}, \min\{s_{\mu_1}^*, s_{\mu_2}^*\})$ and $\ell^* > 0$ independent of the initial data.

(c) Note that, for the single species Fisher–KPP model (2.3) and the competition model with a native competitor (3.6), there is no transition case.

3.3. *Further remarks and comments*

In [125], a competition model of the form (3.3) with an extra advection term was considered, which models the invasion of two competing mosquitoes. In [111], (3.3) with different competition interaction terms was used to model the spreading of Wolbachia infection in the mosquito population. In both works, some rough estimates of the spreading speed were obtained, but the precise spreading speeds were not determined.

Predator–prey is another common relationship between species, and two species predator–prey models have attracted extensive research. For similar reasons as mentioned for the competition model (3.2) above, free boundary has been introduced to predator–prey models to represent the spreading front. Generally speaking, predator–prey models are more difficult to treat than the competition counterpart, due to the lack of a certain order preserving property enjoyed by the competition models. We refer to [128, 131, 141] for some of the recent works on predator–prey models with free boundary. To determine the precise spreading speed in such models is a difficult task, which still remains open.

Cooperative relationship is also a commonly observed interaction type between species, and many epidemic models belong to this category. Models with free boundary for cholera spreading were investigated in [1, 143], and those for West Nile virus spreading were considered in [105, 133]. These models also possess an order-preserving property, which helped to have the precise spreading speeds determined [133, 143].

Some of the reaction–diffusion systems with free boundary mentioned above were considered in heterogeneous environment in the literature, but generally speaking, the understanding of the effect of the influence of heterogeneity on the dynamics is still rather poor, not yet reaching the depth achieved for the single species Fisher–KPP model mentioned in Sec. 3.3. A recent analysis for the competition model (3.6) in time-periodic or space-periodic environment based on numerical simulation can be found in [96].

Reaction–diffusion systems with free boundary in high space dimension without radial symmetry (as in (2.3)) are largely untouched so far in the literature. Such a case for (3.6) was considered by numerical simulation in [95], suggesting several properties similar to the single species case, but very little rigorous analysis is available so far.

There is another kind of free boundary problems for two interacting species, where the population ranges of the two species are separated by a free boundary, of the form

$$\begin{cases} u_t - d_1 u_{xx} = f(u), & x < s(t), \\ v_t - d_2 v_{xx} = g(v), & x > s(t), \\ s'(t) = \mu_1 u_x(t, s(t)) - \mu_2 v_x(t, s(t)). \end{cases}$$

Here, $x = s(t)$ is the free boundary. Such a problem over a finite interval $[a, b]$ with $s(t) \in (a, b)$ was first considered by Mimura *et al.* [113], subsequent works motivated by a variety of questions can be found in [23, 24, 32, 66, 80, 140] and the references therein.

4. The Nonlocal Fisher–KPP Model with Free Boundaries

In both (1.3) and (2.3), the spatial dispersal of the species is represented by the diffusion term $D\Delta u$, which means that the dispersal of the population follows the rule of Brownian motion, as in a random walk. While this is a reasonable approximation of the actual dispersal in many situations, it is increasingly recognized that such an approximation is not good enough in general [115]; for example, long-distance dispersal occurs widely in the spreading process of many species (such as spreadings caused by seeds or insects carried to new environment by modern ways of transportation), but it is not captured by such diffusion models based on Brownian motion, which is local in nature; henceforth they will be called *local diffusion models*. In the literature, several diffusion operators of nonlocal nature have been used to replace the term $D\Delta u$, and in the past 10–20 years extensive research on nonlocal versions of (1.3) and related equations has been done ([3, 8–10, 26, 30, 31, 117, 122, 135, 136, 139], etc.), and fast progress is still being made. Research on the nonlocal version of (2.3) has just started, and we will look at some recent works on this and some related problems here.

We will focus on nonlocal diffusion operators of the form

$$\mathcal{L}u := d\left[\int_{\mathbb{R}^N} J(x-y)u(t,y)dy - u(t,x)\right],$$

where $J : \mathbb{R}^N \to [0, \infty)$ is a continuous function satisfying $\int_{\mathbb{R}^N} J(x)dx = 1$. Roughly speaking, $J(x-y)$ represents the probability of an individual jumping from spatial location x to y. One may replace $J(x-y)$ by a more general function $K(x, y)$ on $\mathbb{R}^N \times \mathbb{R}^N$, but we will only consider the simpler case $J(x-y)$ here, and assume further $J(x)$ depends only on $|x|$.

For simplicity, we start by looking at some results on (1.3) and (2.3) with the local diffusion term $D\Delta u$ replaced by $\mathcal{L}u$ in space dimension 1, and with $f(u)$ the special Fisher–KPP function

$$f(u) = au(1-u), \quad a > 0.$$

The nonlocal Fisher–KPP model is given by

$$\begin{cases} u_t = d\displaystyle\int_{\mathbb{R}} J(x-y)u(t,y)dy - du(t,x) + f(u) & \text{for } x \in \mathbb{R}, \quad t > 0, \\ u(0,x) = u_0(x) \geq, \not\equiv 0 & \text{for } x \in \mathbb{R}. \end{cases} \tag{4.1}$$

The behavior of the kernel function $J(x)$ at $\pm\infty$ turns out to play a pivotal role on the propagation determined by (4.1). The kernel function $J(x)$ is called *thin-tailed* if there exists $\lambda > 0$ such that

$$\int_{\mathbb{R}} e^{\lambda x} J(x) dx < \infty.$$

Otherwise it is called *fat-tailed*. Thus any $J(x)$ with compact support is thin-tailed, and $J(x) = \xi e^{-\mu|x|}$ $(\xi, \mu > 0)$ is thin-tailed, but $J(x) = \eta(1 + |x|)^{-\mu}$ $(\eta, \mu > 0)$ is fat-tailed.

When the convolution kernel in (4.1) is *thin-tailed*, much of the basic theory for (1.1) is retained (see, for example [8–10, 26, 30, 31, 117, 122, 135, 136, 139] and the references therein). On the other hand, *accelerated spreading* happens when the kernel function is *fat-tailed*.

The following result follows from Weinberger [135].

Theorem 4.1. *Let* $u(t, x)$ *be the solution of* (4.1) *with* $f(u) = au(1 - u)$. *Then* $\lim_{t\to\infty} u(t, x) = 1$ *locally uniformly for* $x \in \mathbb{R}$. *Moreover, for any given* $\delta \in (0, 1)$, *the level set*

$$L_\delta(t) := \{x \in \mathbb{R} : u(t, x) = \delta\}$$

satisfies

$$\begin{aligned}\lim_{t\to\infty} \frac{\sup L_\delta(t)}{t} &= \lim_{t\to\infty} \frac{\inf L_\delta(t)}{-t} \\ &= \begin{cases} c_* \in (0, \infty) & \text{if } J \text{ is thin-tailed,} \\ \infty & \text{if } J \text{ is fat-tailed.} \end{cases}\end{aligned}$$

This result indicates that the spreading speed of u is finite if and only if J is thin-tailed. Moreover, in such a case, the spreading speed c_* can be similarly obtained by the associated traveling wave solutions, as in the local diffusion case. When the spreading speed is ∞, one says that *accelerated spreading* happens. Examples of fat-tailed J were given in [74] such that $\sup L_\lambda(t)$ and $-\inf L_\lambda(t)$ behave like

$$e^{\alpha t} (\alpha > 0) \quad \text{with } J(x) \sim |x|^\sigma \ (\sigma < -2),$$

or

$$t^\beta (\beta > 1) \quad \text{with } J(x) \sim e^{-|x|^{1/\beta}}.$$

Other examples of accelerated spreading can be found in [2, 13, 16, 69, 71, 121, 138], etc.

Similar to the corresponding local diffusion case, (4.1) has the shortcoming that the natural population range $\{x \in \mathbb{R} : u(t, x) > 0\}$ is the entire space $\mathbb{R}$ once $t > 0$. In order to model the precise spreading front, one naturally considers a free boundary version of (4.1).

4.1. *The nonlocal Fisher–KPP model with free boundaries in one space dimension*

In Cao *et al.* [20], the following nonlocal version of (2.3) was proposed:

$$\begin{cases} u_t = d\displaystyle\int_{g(t)}^{h(t)} J(x-y)u(t,y)dy - du(t,x) + f(u), & g(t) < x < h(t), \quad t > 0, \\ u(t,g(t)) = u(t,h(t)) = 0, & t > 0, \\ h'(t) = \mu \displaystyle\int_{g(t)}^{h(t)} \int_{h(t)}^{+\infty} J(x-y)u(t,x)dydx, & t > 0, \\ g'(t) = -\mu \displaystyle\int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x-y)u(t,x)dydx, & t > 0, \\ u(0,x) = u_0(x), \quad h(0) = -g(0) = h_0, & x \in [-h_0, h_0], \end{cases} \tag{4.2}$$

where $x = g(t)$ and $x = h(t)$ are the moving boundaries to be determined together with $u(t,x)$, which is always assumed to be identically 0 for $x \in \mathbb{R}\backslash[g(t),h(t)]$ (and so $\int_{\mathbb{R}} J(x-y)u(t,y)dy = \int_{g(t)}^{h(t)} J(x-y)u(t,y)dy$).

The initial function $u_0(x)$ satisfies $u_0 \in C([-h_0,h_0])$, and

$$u_0(-h_0) = u_0(h_0) = 0, \quad u_0(x) > 0 \quad \text{in } (-h_0,h_0),$$

so $[-h_0,h_0]$ represents the initial population range of the species.

The kernel function $J : \mathbb{R} \to \mathbb{R}$ is continuous and non-negative, and has the properties

$$(J):\ J(0) > 0, \quad \int_{\mathbb{R}} J(x)dx = 1, \quad J(x) = J(-x), \quad \sup_{\mathbb{R}} J < \infty.$$

Recall that we are taking the special Fisher–KPP nonlinearity $f(u) = au(1-u)$.

The meaning of the free boundary conditions can be understood as follows: The total population mass moved out of the range $[g(t),h(t)]$ at time t through its right boundary $x = h(t)$ per unit time is given by

$$d\int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x-y)u(t,x)dydx.$$

As we assume that $u(t,x) = 0$ for $x \notin [g(t),h(t)]$, this quantity of mass is lost in the spreading process of the species. We may call this quantity the *outward flux* at $x = h(t)$ and denote it by $J_h(t)$. Similarly, we can define the outward flux at $x = g(t)$ by

$$J_g(t) := d\int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x-y)u(t,x)dydx.$$

Then the free boundary conditions in (4.2) can be interpreted as saying that the expanding rate of the front is proportional to the outward flux (by a factor μ/d):

$$g'(t) = -\mu J_g(t),$$
$$h'(t) = \mu J_h(t).$$

For a plant species, seeds carried across the range boundary may fail to establish due to numerous reasons, such as isolation from other members of the species causing poor or no pollination, or causing overwhelming attacks from enemy species. However, some of those not very far from the range boundary may survive, which results in the expansion of the population range. The free boundary condition here assumes that this survival rate is roughly a constant for a given species. For an animal species, a similar consideration can be applied to arrive at these free boundary conditions.

Note that for most species, the living environment involves many factors, not only the resources such as food or nutrient supplies. For example, complex interactions of the concerned species with many other species in the same spatial habitat constantly occur, yet it is impossible to include all of them (even the majority of them) into a manageable model, and best treat them, or rather their combined effects, as part of the environment of the concerned species.

These free boundary conditions were proposed independently in [29], where (4.2) with $f(u) \equiv 0$ was studied, which then has very different long-time dynamical behavior from our case $f(u) = au(1-u)$.

We now describe the main results for (4.2).

Theorem 4.2 (Existence and Uniqueness [20]). *Problem* (4.2) *has a unique solution* (u, g, h) *defined for all* $t > 0$.

Theorem 4.3 (Spreading-vanishing dichotomy [20]). *Let* (u, g, h) *be the unique solution of problem* (4.2). *Then one of the following alternatives must happen*:

(i) *Spreading*: $\lim_{t\to+\infty}(g(t), h(t)) = \mathbb{R}$ *and* $\lim_{t\to+\infty} u(t, x) = 1$ *locally uniformly in* $\mathbb{R}$,
(ii) *Vanishing*: $\lim_{t\to+\infty}(g(t), h(t)) = (g_\infty, h_\infty)$ *is a finite interval and*

$$\lim_{t\to+\infty} \|u(t,\cdot)\|_{L^\infty([g(t),h(t)])} = 0.$$

Theorem 4.4 (Spreading-vanishing criteria [20]).

(α) *If* $d \le f'(0) = a$, *then spreading always happens.*
(β) *If* $d > f'(0) = a$, *then there exists a unique* $\ell^* > 0$ *such that spreading always happens if* $2h_0 \ge \ell^*$; *and for* $2h_0 \in (0, \ell^*)$, *there exists a unique* $\mu^* > 0$ *so that spreading happens exactly when* $\mu > \mu^*$.

These results are similar to that for the local diffusion model in [46], but case (α) in the *spreading-vanishing criteria* does not happen in the local diffusion case.

When spreading happens to (4.2), the spreading speed was determined in Du *et al.* [50]. In contrast to the local diffusion model (2.3), now accelerated spreading may happen. The threshold condition on the kernel function $J(x)$ governing this is

$$(J1): \int_0^\infty xJ(x)dx < +\infty.$$

Let us first note that if $J(x) := \xi(1+|x|)^\alpha$ with $\xi > 0$ and $\alpha > 2$, then (J) and $(J1)$ hold but $J(x)$ is not thin-tailed. On the other hand, it can be easily shown that for any $J(x)$ satisfying (J) and having the thin-tail property, $(J1)$ holds.

Theorem 4.5 (Spreading speed [43]). *Suppose* (J) *is satisfied, and spreading happens to the unique solution* (u, g, h) *of* (4.2). *Then*

$$\lim_{t\to\infty} \frac{h(t)}{t} = -\lim_{t\to\infty} \frac{g(t)}{t} = \begin{cases} \hat{c}_0 \in (0,\infty) & \text{if } (J1) \text{ is satisfied,} \\ \infty & \text{if } (J1) \text{ is not satisfied.} \end{cases}$$

The spreading speed $\hat{c}_0$ is determined by *semi-wave* solutions to (4.2). These are pairs (c, ϕ) determined by the following two equations:

$$\begin{cases} d\displaystyle\int_{-\infty}^0 J(x-y)\phi(y)dy - d\phi(x) + c\phi'(x) + f(\phi(x)) = 0, & x < 0, \\ \phi(-\infty) = 1, \quad \phi(0) = 0, \end{cases} \tag{4.3}$$

and

$$c = \mu \int_{-\infty}^0 \int_0^{+\infty} J(x-y)\phi(x)dydx. \tag{4.4}$$

Theorem 4.6 (Semi-wave [43]). *Suppose* (J) *holds. Then* (4.3)–(4.4) *have a solution pair* $(c,\phi) = (\hat{c}_0, \phi_0)$ *with* $\phi_0 \in C^1((-\infty, 0])$ *and* $\phi_0' \le 0$ *if and only if* (J1) *holds. Moreover, when* (J1) *holds, the solution pair is unique, and* $\hat{c}_0 > 0$, $\phi_0'(x) < 0$ *for* $x \le 0$.

By Theorems 4.1 and 4.5, the relationship between (J1) and the "thin-tail" property indicates that, accelerated spreading is less likely to happen to (4.2) than to (4.1).

Similar to the local diffusion case, problem (4.1) is the limiting problem of (4.2) when $\mu \to \infty$.

Theorem 4.7 (Limiting problem [43]). *If the solution* (u, g, h) *of* (4.2) *is denoted by* (u_μ, g_μ, h_μ) *to stress its dependence on* μ, *then as* $\mu \to \infty$,

$$-g_\mu(t), \quad h_\mu(t) \to \infty \ (\forall\, t > 0), \quad u_\mu \to U \quad \text{in } L^\infty_{\rm loc}((0,\infty)\times\mathbb{R}),$$

where U *is the unique solution of* (4.1) *with* $U(0,x) = \begin{cases} u_0(x), & x \in [-h_0, h_0], \\ 0, & x \notin [-h_0, h_0]. \end{cases}$

When spreading happens to (4.2), further estimates on the spreading rate can be obtained provided that the behavior of the kernel function $J(x)$ near infinity is suitably specified. We will write

$$\eta(t) \sim \xi(t) \quad \text{if and only if } c_1\xi(t) \le \eta(t) \le c_2\xi(t)$$

for some positive constants $c_1 \le c_2$ and all t in the concerned range. The following result follows from Du and Ni [57], where it is assumed that

$$J(x) \sim |x|^{-\alpha} \quad \text{for } |x| \gg 1, \tag{4.5}$$

and so

$$(J) \Leftrightarrow \alpha > 1, \quad \text{and} \quad (J1) \Leftrightarrow \alpha > 2.$$

Theorem 4.8 (Spreading rate [57]). *Suppose* (J) *is satisfied, and spreading happens to the unique solution* (u, g, h) *of* (4.2). *If additionally* (4.5) *holds, then for* $t \gg 1$, *we have*

$$\hat{c}_0 t + g(t), \quad \hat{c}_0 t - h(t) \sim 1 \quad \text{if } \alpha > 3,$$

$$\hat{c}_0 t + g(t), \quad \hat{c}_0 t - h(t) \sim \begin{cases} \ln t & \text{if } \alpha = 3, \\ t^{3-\alpha} & \text{if } 3 > \alpha > 2, \end{cases}$$

$$-g(t), \quad h(t) \sim \begin{cases} t\ln t & \text{if } \alpha = 2, \\ t^{\frac{1}{\alpha-1}} & \text{if } 2 > \alpha > 1. \end{cases}$$

Remarks. For the corresponding fixed boundary problems of (4.1) and (1.3), it is well known [3, 27, 28, 119] that, over any finite time interval $[0, T]$, the unique solution u of the local diffusion problem (1.3) is the limit of the unique solution of the nonlocal problem (4.1) as $\epsilon \to 0$, when the kernel function J in the nonlocal problem is replaced by

$$\tilde{J}_\epsilon(x) = \frac{C}{\epsilon^2} J_\epsilon(x) := \frac{C}{\epsilon^3} J\Big(\frac{x}{\epsilon}\Big)$$

with a suitable positive constant C, provided that J has compact support, f and the common initial function are all smooth enough. In Du and Ni [56], it was shown that (2.1) is the limiting problem of a slightly modified version of (4.2).

4.2. *The nonlocal Fisher–KPP model with free boundary in high space dimensions*

The radially symmetric version of (4.2) in $\mathbb{R}^N$ $(N \geq 2)$ is given by

$$\begin{cases} u_t = d\displaystyle\int_{B_{h(t)}} J(|x-y|)u(t,|y|)dy - du + f(u), & t>0, \quad x \in B_{h(t)}, \\ u = 0, & t>0, \quad x \in \partial B_{h(t)}, \\ h'(t) = \dfrac{\mu}{|\partial B_{h(t)}|}\displaystyle\int_{B_{h(t)}}\int_{\mathbb{R}^N\setminus B_{h(t)}} J(|x-y|)u(t,|x|)dydx, & t>0, \\ h(0) = h_0, \quad u(0,|x|) = u_0(|x|), & x \in \overline{B}_{h_0}, \end{cases} \tag{4.6}$$

where $B_{h(t)} = \{x \in \mathbb{R}^N : |x| < h(t)\}$, and $u = u(t,|x|)$ is radially symmetric. The initial function u_0 satisfies

$$\begin{cases} u_0 \quad \text{is radial and continuous in } \overline{B}_{h_0}, \\ u_0 > 0 \quad \text{in } B_{h_0}, \quad u_0 = 0 \quad \text{on } \partial B_{h_0}. \end{cases}$$

As before, for simplicity, we assume

$$f(u) = au(1-u).$$

Our basic assumptions on the kernel function $J(|x|)$ are

$$(J) \quad J \in C(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+), \quad J \geq 0, \quad J(0) > 0, \quad \int_{\mathbb{R}^N} J(|x|)dx = 1.$$

For $r := |x|$ with $x \in \mathbb{R}^N$ and $\rho > 0$, define

$$\tilde{J}(r,\rho) = \tilde{J}(|x|,\rho) := \int_{\partial B_\rho} J(|x-y|)dS_y. \tag{4.7}$$

Then (4.6) can be rewritten into the equivalent form

$$\begin{cases} u_t(t,r) = d\displaystyle\int_0^{h(t)} \tilde{J}(r,\rho)u(t,\rho)d\rho - du + f(u), & t>0, \quad r \in [0,h(t)), \\ u(t,h(t)) = 0, & t>0, \\ h'(t) = \dfrac{\mu}{h^{N-1}(t)}\displaystyle\int_0^{h(t)}\int_{h(t)}^{+\infty} \tilde{J}(r,\rho)r^{N-1}u(t,r)d\rho dr, & t>0, \\ h(0) = h_0, \quad u(0,r) = u_0(r), & r \in [0,h_0]. \end{cases} \tag{4.8}$$

(Here a universal constant is absorbed by μ).

We now describe the main results on (4.6) obtained in Du and Ni [58]. The first three theorems are almost identical to the corresponding ones for (4.2).

Theorem 4.9 (Existence and uniqueness [58]). *Suppose* (J) *is satisfied. Then problem* (4.6), *or equivalently* (4.8), *admits a unique positive solution* (u,h) *defined for all* $t>0$.

Theorem 4.10 (Spreading-vanishing dichotomy [58]). *Suppose* (J) *is satisfied. Let* (u,h) *be the solution of* (4.6). *Then one of the following alternatives must occur*:

(i) *Spreading*: $\lim_{t\to\infty} h(t)=\infty$ *and*

$$\lim_{t\to\infty} u(t,|x|)=1 \quad \textit{locally uniformly in } \mathbb{R}^N.$$

(ii) *Vanishing*: $\lim_{t\to\infty} h(t)=h_\infty<\infty$ *and*

$$\lim_{t\to\infty} u(t,|x|)=0 \quad \textit{uniformly for } x\in B_{h(t)}.$$

Theorem 4.11 (Spreading-vanishing criteria [58]). *In Theorem* 4.10,

(1) *if* $d\le f'(0)=a$, *then spreading always happens*,
(2) *if* $d>f'(0)=a$ *then there exists* $L_*>0$ *such that*
   (i) *for* $h_0\ge L_*$, *spreading always happens*,
   (ii) *for* $0<h_0<L_*$, *there is* $\mu_*>0$ *such that spreading happens if and only if* $\mu>\mu_*$.

Here, L_* is independent of u_0 and is determined by some eigenvalue problem, but μ_* depends on u_0.

Major difficulties arise when we try to determine the spreading rate. Many of these difficulties are unique for the nonlocal problem here, which do not arise in the corresponding local diffusion case. Indeed, the relationship of $J(|x|)$ and $\tilde{J}$ given by (4.7) is rather complicated, which can be more explicitly expressed by

$$\tilde{J}(r,\rho)=\omega_{N-1}2^{3-N}\frac{\rho}{r^{N-2}}\int_{|\rho-r|}^{\rho+r}\left(\left[(\rho+r)^2-\eta^2\right]\left[\eta^2-(\rho-r)^2\right]\right)^{\frac{N-3}{2}}\eta J(\eta)d\eta.$$

Therefore, it is difficult to find out how the spreading rate of (4.7) is determined by J.

To overcome this difficulty, we introduce an intermediate function as follows. For any $\xi\in\mathbb{R}$, define

$$J_*(\xi):=\int_{\mathbb{R}^{N-1}} J(|(\xi,x')|)dx', \tag{4.9}$$

where $x'=(x_2,\dots,x_N)\in\mathbb{R}^{N-1}$. Condition (J) implies

$$\begin{cases} J_*\in C(\mathbb{R})\cap L^\infty(\mathbb{R}) \quad \text{is non-negative, even,} \quad J_*(0)>0,\\ \displaystyle\int_{\mathbb{R}} J_*(\xi)d\xi=\int_{\mathbb{R}^N} J(|x|)dx=1.\end{cases}$$

Moreover,

$$J_*(\xi) = \omega_{N-1} \int_{|\xi|}^{\infty} J(r) r (r^2 - \xi^2)^{(N-3)/2} dr,$$

$$\int_0^{\infty} J_*(\xi)\xi d\xi = \frac{\omega_{N-1}}{N-1} \int_0^{\infty} J(r) r^N dr,$$

where ω_k denotes the area of the unit sphere in $\mathbb{R}^k$.

It turns out that the spreading speed of (4.6) can be easily described by making use of J_*, via a careful analysis of the relationship between J_* and $\tilde{J}$ through the above expressions. We can prove the following result.

Theorem 4.12 (Spreading speed [58]). *Assume the conditions in Theorem* 4.10 *are satisfied, and spreading happens to* (4.6). *Then*

$$\lim_{t\to\infty} \frac{h(t)}{t} = \begin{cases} \hat{c}_0 & \text{if } J_* \text{ satisfies } (J1), \\ \infty & \text{if } J_* \text{ does not satisfy } (J1), \end{cases}$$

where $\hat{c}_0$ *is given by Theorem* 4.6 *with* J *replaced by* J_*.

For kernel functions $J(x)$ with certain specific properties, we can obtain more accurate description on the behavior of $h(t)$. We now look at two rather general classes of J, one as an example satisfying (J1), the other not satisfying $(J1)$.

(a) J has compact support and hence satisfies $(J1)$.

Theorem 4.13 (Logarithmic shift [58]). *Suppose the conditions in Theorem* 4.10 *hold, and moreover the kernel function* J *has compact support. If spreading happens, then*

$$c_0 t - h(t) \sim \ln t \quad \text{for } t \gg 1.$$

In contrast, we note that in *dimension* 1, when J has compact support, it follows from [57] that

$$c_0 t - h(t) \sim 1 \quad \text{for } t \gg 1.$$

(b) $J(r) \sim r^{-\beta}$ for $r \gg 1$, with $\beta \in (N, N+1]$ and hence $(J1)$ is not satisfied.

Theorem 4.14 (Rate of accelerated spreading [58]). *Suppose the conditions in Theorem* 4.10 *are satisfied, and there exists* $\beta \in (N, N+1]$ *such that* $J(r) \sim r^{-\beta}$

for all large r. *If spreading happens, then for all large* t,

$$h(t) \sim \begin{cases} t^{\frac{1}{\beta-N}} & \textit{if } \beta \in (N, N+1), \\ t \ln t & \textit{if } \beta = N+1. \end{cases}$$

It is interesting to compare the result here with that for the case $N = 1$, where from Theorem 4.8 we have

$$h(t) \sim \begin{cases} t^{\frac{1}{\beta-1}} & \text{if } \beta \in (1, 2), \\ t \ln t & \text{if } \beta = 2. \end{cases}$$

Remark. If $J(r) \sim r^{-\beta}$ for $r \gg 1$, and $\beta > N + 1$, then (J1) holds, and it is an interesting question to determine the rate of $\hat{c}_0 t - h(t)$ for $t \gg 1$. At the moment, we only have some partial results for this case.

4.3. *Further remarks and comments*

Another well-studied nonlocal diffusion operator is the fractional Laplacian $(-\Delta)^s$ $(0 < s < 1)$. For the Cauchy problem

$$\begin{cases} u_t + (-\Delta)^s u = f(u) & \text{for } t > 0 \quad \text{and} \quad x \in \mathbb{R}^N, \\ u(0, x) = u_0(x) \geq 0 & \text{for } x \in \mathbb{R}^N, \end{cases}$$

with $f(u)$ of Fisher–KPP type, it was shown in [16] by Cabré and Roquejoffre that accelerated spreading happens with rate given by $e^{[c+o(1)]t}$ for some $c > 0$ depending on N and s. Souganidis and Tarfulea [121] further proved that such a result remains valid when f also depends periodically on x. We note that the corresponding kernel function of $(-\Delta)^s$ is given by

$$J(|x|) = |x|^{-(N+2s)},$$

which does not satisfy our basic condition (J) in Sec. 4.2 above. It would be interesting to see what happens to (4.6) if the kernel function J is allowed to behave like the kernel function of the fractional Laplacian. A related work with $f \equiv 0$ can be found in [34].

The corresponding version of (4.6) without radial symmetry is yet to be considered.

5. Two Nonlocal Epidemic Models with Free Boundaries

5.1. *A nonlocal West Nile virus model*

The West Nile virus (WNv) is the cause of an infectious disease endemic in many parts of the world. WNv spreads primarily through interacting bird and mosquito populations, with birds acting as hosts and mosquitoes as vectors.

Let $H(t,x)$ denote the density of the infective bird population (host), $V(t,x)$ denote the population density of the infective mosquitos (vector), and the population range of H and V be represented by the interval $[g(t),h(t)]$. Then the spreading of the WNv can be modeled by the following system [55]:

$$\begin{cases} H_t = d_1\mathcal{L}_1[H] + a_1(e_1 - H)V - b_1 H, & x \in (g(t),h(t)), \quad t>0,\\ V_t = d_2\mathcal{L}_2[V] + a_2(e_2 - V)H - b_2 V, & x \in (g(t),h(t)), \quad t>0,\\ H(t,x) = V(t,x) = 0, & x = g(t) \text{ or } h(t), \quad t>0,\\ g'(t) = -\mu \displaystyle\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)} J_1(x-y)H(t,x)dydx, & t>0,\\ h'(t) = \mu \displaystyle\int_{g(t)}^{h(t)}\int_{h(t)}^{\infty} J_1(x-y)H(t,x)dydx, & t>0,\\ g(0) = -h_0, \quad h(0) = h_0, & \\ H(0,x) = u_1^0(x), \quad V(0,x) = u_2^0(x), & x \in [-h_0,h_0], \end{cases} \tag{5.1}$$

where for $i \in \{1,2\}$, a_i, b_i, e_i, $d_i > 0$, and

$$\mathcal{L}_i[w] = \int_{g(t)}^{h(t)} J_i(x-y)w(t,y)dy - w(t,x).$$

The initial functions $u_i^0(x)$ $(i=1,2)$ satisfy

$$\begin{cases} u_i^0 \in C([-h_0,h_0]), \quad u_i^0(-h_0) = u_i^0(h_0) = 0,\\ 0 < u_i^0(x) \le e_i \quad \text{for } x \in (-h_0,h_0), \quad i=1,2. \end{cases} \tag{5.2}$$

The kernel functions $J_i : \mathbb{R} \to \mathbb{R}$ $(i=1,2)$ satisfy

$$(\mathbf{J}) : J_i \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \quad \text{is non-negative, symmetric,} \quad J_i(0) > 0,$$
$$\int_{\mathbb{R}} J_i(x)dx = 1, \quad i = 1,2.$$

In this model, the dispersals of the infected birds and mosquitoes are governed by the nonlocal diffusion operators $d_1\mathcal{L}_1[H]$ and $d_2\mathcal{L}_2[V]$, respectively, and it is assumed that during the epidemic period the total bird population (infective plus uninfective) is a constant e_1 and the total mosquito population is a constant e_2. It is also assumed that the range of the disease $[g(t),h(t)]$ is primarily determined by the infective birds.

The corresponding ODE model of (5.1), also known as the *Ross–Macdonold model* due to the pioneering works of Ross and Macdonald on *malaria*, whose spreading mechanism is similar to that of WNv, has been studied by many people.

The long-time behavior in this case is completely determined by the reproduction number given by

$$\mathcal{R}_0 := \sqrt{\frac{a_1 a_2 e_1 e_2}{b_1 b_2}}. \tag{5.3}$$

(Note that $\mathcal{R}_0 > 1$ if and only if $a_1 a_2 e_1 e_2 > b_1 b_2$). More precisely, if $(H(t), V(t))$ is the solution of this ODE system with $0 < H(0) \le e_1$ and $0 < V(0) \le e_2$, then as $t \to \infty$,

$$(H(t), V(t)) \to \begin{cases} (0,0) & \text{if } \mathcal{R}_0 \le 1, \\ (H^*, V^*) & \text{if } \mathcal{R}_0 > 1, \end{cases}$$

where

$$(H^*, V^*) := \left(\frac{a_1 a_2 e_1 e_2 - b_1 b_2}{a_1 a_2 e_2 + b_1 a_2}, \frac{a_1 a_2 e_1 e_2 - b_1 b_2}{a_1 a_2 e_1 + a_1 b_2} \right). \tag{5.4}$$

is the unique positive equilibrium of the ODE system.

The reaction–diffusion version of (5.1), namely

$$\begin{cases} H_t = d_1 H_{xx} + a_1(e_1 - H)V - b_1 H, & x \in \mathbb{R}, \quad t > 0, \\ V_t = d_2 V_{xx} + a_2(e_2 - V)H - b_2 V, & x \in \mathbb{R}, \quad t > 0, \end{cases} \tag{5.5}$$

was considered by Lewis *et al.* [99]. If the initial function pair $(H_0, V_0) \in C(\mathbb{R}) \times C(\mathbb{R})$ has nonempty compact supports, and satisfies $0 \le H_0 \le e_1,\ 0 \le V_0 \le e_2$, then, as $t \to \infty$,

$$(H(t,x), V(t,x)) \to \begin{cases} (0,0) & \text{if } \mathcal{R}_0 \le 1, \\ (H^*, V^*) & \text{if } \mathcal{R}_0 > 1. \end{cases}$$

Moreover, when $\mathcal{R}_0 > 1$, $(H, V) \to (H^*, V^*)$ can be characterized by a traveling wave solution with minimal speed $c^* > 0$, indicating that the virus spreads with speed c^*.

The local diffusion version of (5.1) was first considered by Lin and Zhu [108], where $(d_1 \mathcal{L}_1[H], d_2 \mathcal{L}_2[V])$ is replaced by $(d_1 U_{xx}, d_2 V_{xx})$, and the free boundary conditions are replaced by

$$g'(t) = -\mu H_x(t, h(t)), \quad h'(t) = -\mu V_x(t, h(t)).$$

It was proved in [108] that the problem has a unique solution which is defined for all $t > 0$, and when $\mathcal{R}_0 \le 1$, the virus always *vanishes eventually*, i.e.

$$\lim_{t\to\infty} [h(t) - g(t)] < \infty \quad \text{and} \quad \lim_{t\to\infty} (H(t,x), V(t,x)) = (0,0). \tag{5.6}$$

If $\mathcal{R}_0 > 1$, then a *spreading-vanishing dichotomy* holds: Either (5.6) holds, or the virus *spreads successfully*, namely,

$$\begin{cases} \lim_{t\to\infty} h(t) = -\lim_{t\to\infty} g(t) = +\infty \quad \text{and} \\ \lim_{t\to\infty} (H(t,x), V(t,x)) = (H^*, V^*). \end{cases}$$

Moreover, there is a critical length $L_* > 0$ so that either the range size $h(t) - g(t)$ reaches L_* at some finite time $t_0 \geq 0$, and then spreading happens, or $h(t) - g(t)$ stays below this critical length L_* for all time, in which case vanishing occurs. The spreading speed was determined by Wang *et al.* [133].

For the nonlocal model (5.1), the following three theorems have been proved in Du and Ni [55].

Theorem 5.1 (Existence and uniqueness [55]). *Suppose* (**J**) *holds, and the initial functions satisfy* (5.2)*. Then problem* (5.1) *admits a unique positive solution* (H, V, g, h) *defined for all* $t > 0$.

Theorem 5.2 (Spreading-vanishing dichotomy [55]). *Assume* (**J**) *holds, and the initial functions satisfy* (5.2)*. Let* (H, V, g, h) *be the solution of* (5.1)*, and denote*

$$g_\infty := \lim_{t\to\infty} g(t) \quad \text{and} \quad h_\infty := \lim_{t\to\infty} h(t).$$

Then one of the following alternatives must occur:

(i) *Spreading*: $(g_\infty, h_\infty) = \mathbb{R}$ *and*

$$\lim_{t\to\infty}(H(t,x), V(t,x)) = (H^*, V^*) \quad \text{locally uniformly in } \mathbb{R}.$$

(ii) *Vanishing*: (g_∞, h_∞) *is a finite interval and*

$$\lim_{t\to\infty}(H(t,x), V(t,x)) = (0,0) \quad \text{uniformly for } x \in [g(t), h(t)].$$

Theorem 5.3 (Spreading-vanishing criteria [55]). *Assume* (**J**) *holds, and the initial functions satisfy* (5.2)*. Let* (H, V, g, h) *be the solution of* (5.1)*, and* $\mathcal{R}_0$ *be given by* (5.9)*.*

(i) *If* $\mathcal{R}_0 \leq 1$*, then vanishing always happens.*
(ii) *If* $\mathcal{R}_0 > 1$ *and one of the following conditions holds*:

$$\text{(I)} \quad \frac{a_1 a_2 e_1 e_2}{(b_1 + d_1)(b_2 + d_2)} \geq 1,$$

$$\text{(II)} \quad \frac{a_1 a_2 e_1 e_2}{(b_1 + d_1)(b_2 + d_2)} < 1, \quad h_0 \geq L^*,$$

then spreading always happens, where L^* *is a fixed constant depending on* $(a_i, b_i, d_i, e_i, J_i)(i = 1, 2)$.

(iii) *If* $\mathcal{R}_0 > 1$ *and*

$$\text{(III)} \quad \frac{a_1 a_2 e_1 e_2}{(b_1 + d_1)(b_2 + d_2)} < 1, \quad h_0 < L^*, \quad \text{then}$$

(1) *for any given initial datum* (u_1^0, u_2^0) *satisfying* (5.2)*, there exists* $\mu^* > 0$ *such that vanishing happens for* $0 < \mu \leq \mu^*$ *and spreading happens for* $\mu > \mu^*$*,*
(2) *for fixed* $\mu > 0$ *and sufficient small initial datum* (u_1^0, u_2^0)*, vanishing happens.*

When spreading happens, the spreading rate will be determined in Sec. 5.3, by a unified approach which also covers the epidemic model in the following subsection.

5.2. *A nonlocal epidemic model for cholera*

Cholera affects an estimated 3–5 million people worldwide and causes 28,800–130,000 deaths a year [137]. It is an infection of the small intestine by some strains of the bacterium Vibrio cholerae, spread mostly by unsafe water and unsafe food that has been contaminated with human feces containing the bacteria. Most cholera cases in developed countries are a result of transmission by food, while in developing countries it is more often water. Food transmission can occur when people harvest seafood such as oysters in waters infected with sewage, as Vibrio cholerae accumulates in planktonic crustaceans and the oysters eat the zooplankton.

A nonlocal diffusion model for the spread of cholera is given by

$$\begin{cases} u_t = d_1\mathcal{L}_1[u] - au + cv, & t>0, \quad x\in(g(t),h(t)),\\ v_t = d_2\mathcal{L}_2[v] - bv + G(u), & t>0, \quad x\in(g(t),h(t)),\\ u(t,x)=v(t,x)=0, & t>0, \quad x=g(t) \quad \text{or} \quad h(t),\\ g'(t) = -\mu\displaystyle\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)} J[u,v](t,x,y)dydx, & t>0,\\ h'(t) = \mu\displaystyle\int_{g(t)}^{h(t)}\int_{h(t)}^{+\infty} J[u,v](t,x,y)dydx, & t>0,\\ g(0)=-h_0, \quad h(0)=h_0,\\ u(0,x)=u_0(x), \quad v(0,x)=v_0(x), & x\in[-h_0,h_0], \end{cases} \tag{5.7}$$

where

$$J[u,v](t,x,y) := \alpha J_1(x-y)u(t,x) + (1-\alpha)J_2(x-y)v(t,x) \quad \text{for some } \alpha\in[0,1],$$

$u(t,x)$ denotes the population density of the infective agents, $v(t,x)$ denotes the population density of the infective humans, the interval $[g(t),h(t)]$ represents the epidemic region, and $a,\ b,\ c,\ \mu,\ h_0,\ d_1>0,\ d_2\geq 0$ are given constants, $G(z) := \beta z/(1+z),\ \beta\in(0,ab/c)$, or more generally, it can be any function satisfying

$$\begin{cases} G\in C^1([0,\infty)), \quad G(0)=0, \quad G'(z)>0 \quad \text{for } z\geq 0,\\ \left[\dfrac{G(z)}{z}\right]' < 0 \quad \text{for } z>0 \quad \text{and} \quad \displaystyle\lim_{z\to+\infty}\frac{G(z)}{z} < \frac{ab}{c}. \end{cases}$$

The initial functions u_0 and v_0 are assumed to be continuous over $[-h_0,h_0]$, positive in $(-h_0,h_0)$ and 0 at $\pm h_0$. Note that the nonlocal diffusion operators $\mathcal{L}_i$ $(i=1,2)$ are defined as in the previous subsection.

The corresponding ODE model was proposed by Capasso and Paveri-Fontana [21] to describe the cholera epidemic which spread in the European Mediterranean regions in 1973. Subsequently, Capasso and Maddalena [22] considered the following diffusive model:

$$\begin{cases} u_t = d\Delta u - au + cv, & t > 0, \quad x \in \Omega, \\ v_t = -bv + G(u), & t > 0, \quad x \in \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} + \alpha u = 0, & t > 0, \quad x \in \partial\Omega, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), & x \in \overline{\Omega}, \end{cases} \tag{5.8}$$

with $\Omega \subset \mathbb{R}^N$ bounded, representing the epidemic region. Note that in this model the mobility of the infective humans is ignored and assumed to be 0.

It is shown that the number

$$\tilde{R}_0 := \frac{cG'(0)}{(a + d\lambda_1)b}$$

is a threshold value for the long-time dynamical behavior of (5.8): the epidemic will eventually tend to extinction if $0 < \tilde{R}_0 \leq 1$, and there is a globally asymptotically stable endemic state if $\tilde{R}_0 > 1$, where λ_1 is the first eigenvalue of

$$-\Delta\phi = \lambda\phi \quad \text{in } \Omega, \qquad \frac{\partial \phi}{\partial \boldsymbol{n}} + \alpha\phi = 0 \quad \text{on } \partial\Omega.$$

The local diffusion version of (5.7) with $d_2 = 0$ was first studied by Ahn *et al.* [1]. They proved a spreading-vanishing dichotomy for its long-time dynamical behavior: The unique solution (u, v, g, h) satisfies one of the following:

(i) Vanishing:

$$\lim_{t\to\infty} (g(t), h(t)) = (g_\infty, h_\infty) \quad \text{is finite and} \quad \lim_{t\to\infty} (u, v) = (0, 0).$$

(ii) Spreading:

$$\lim_{t\to\infty} (g(t), h(t)) = \mathbb{R} \quad \text{and} \quad R_0 > 1, \quad \lim_{t\to\infty} (u, v) = (K_1, K_2),$$

where

$$R_0 := \frac{cG'(0)}{ab}, \tag{5.9}$$

and (K_1, K_2) are uniquely determined by

$$\frac{G(K_1)}{K_1} = \frac{ab}{c}, \quad K_2 = \frac{G(K_1)}{b}.$$

Furthermore,

(i) if $R_0 \leq 1$, then vanishing happens;
(ii) if $R_0 \geq 1 + \frac{d}{a}\left(\frac{\pi}{2h_0}\right)^2$, then spreading happens;

(iii) if $1+\frac{d}{a}\left(\frac{\pi}{2h_0}\right)^2 > R_0 > 1$, then vanishing happens for small initial data (u_0, v_0), and spreading happens for large initial data.

When spreading happens, the spreading speed of the local diffusion model in [1] was established by Zhao *et al.* [143]. Corresponding results for the case $d_2 > 0$ were established recently by Wang and Du [132].

The following three theorems on the dynamics of (5.7) are taken from [144] (for the case $d_2 = 0$) and [25] (for the case $d_2 > 0$).

Theorem 5.4 (Global existence and uniqueness [25, 144]). *Suppose that* (J) *holds. Then problem* (5.7) *admits a unique solution* $(u(t,x), v(t,x), g(t), h(t))$ *defined for all* $t > 0$.

Theorem 5.5 (Spreading-vanishing dichotomy [25, 144]). *Let* (u, v, g, h) *be the unique solution of* (5.7). *Then one of the following must happen*:

(i) *Spreading*: $\lim_{t\to\infty}(g(t), h(t)) = \mathbb{R}$ (*and necessarily* $R_0 > 1$),

$$\lim_{t\to+\infty}(u(t,x), v(t,x)) = (K_1, K_2) \quad \textit{locally uniformly in } \mathbb{R}.$$

(ii) *Vanishing*: $\lim_{t\to\infty}(g(t), h(t)) = (g_\infty, h_\infty)$ *is a finite interval*,

$$\lim_{t\to\infty}(u(t,x), v(t,x)) = (0,0) \textit{uniformly in } x.$$

Theorem 5.6 (Spreading-vanishing criteria [25, 144]). *In Theorem* 5.5, *the dichotomy is completely determined as follows*:

(a) *If* $R_0 \le 1$, *then vanishing always occurs.*
(b) *If* $R_0 > 1$, *then spreading always occurs if one of the following holds*:

$$\text{(I)} \quad \frac{cG'(0)}{(d_1+a)(d_2+b)} \ge 1,$$

$$\text{(II)} \quad \frac{cG'(0)}{(d_1+a)(d_2+b)} < 1 \quad \textit{and} \quad h_0 \ge L^*,$$

where $L^* > 0$ *is critical length depending on* $a, b, c, d_1, d_2, J_1, J_2$ *but independent of the initial data* (u_0, v_0).
(c) *If* $R_0 > 1$ *and*

$$\text{(III)} \quad \frac{cG'(0)}{(d_1+a)(d_2+b)} < 1 \quad \textit{and} \quad h_0 < L^*, \quad \textit{then}$$

(i) *For any given admissible initial datum* (u_0, v_0), *there exists* $\mu^* > 0$ *such that vanishing happens when* $0 < \mu \le \mu^*$, *and spreading happens when* $\mu > \mu^*$.
(ii) *For fixed* $\mu > 0$ *and sufficiently small initial datum* (u_0, v_0), *vanishing occurs.*

For (5.1) and (5.7), when spreading happens, the spreading rate is determined in the following subsection, by a unified approach of Du and Ni [57].

5.3. *Spreading rate for a general nonlocal cooperative system with free boundaries*

We consider the following general system which will contain (5.1) and (5.7) as special cases:

$$\begin{cases} \partial_t u_i = d_i \mathcal{L}_i[u_i] + f_i(u_1, \ldots, u_m), & 1 \le i \le m_0, \quad t > 0, \\ & x \in (g(t), h(t)), \\ \partial_t u_i = f_i(u_1, u_2, \ldots, u_m), & m_0 < i \le m, \quad t > 0, \\ & x \in (g(t), h(t)), \\ u_i(t, g(t)) = u_i(t, h(t)) = 0, & 1 \le i \le m, \quad t > 0, \\ g'(t) = -\displaystyle\int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} \sum_{i=1}^{m_0} \mu_i J_i(x-y) u_i(t,x) dy dx, & t > 0, \\ h'(t) = \displaystyle\int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} \sum_{i=1}^{m_0} \mu_i J_i(x-y) u_i(t,x) dy dx, & t > 0, \\ u_i(0, x) = u_{i0}(x), & 1 \le i \le m, \quad x \in [-h_0, h_0], \end{cases} \tag{5.10}$$

where $1 \le m_0 \le m$, and for $i \in \{1, \ldots, m_0\}$,

$$\mathcal{L}_i[v] := \int_{g(t)}^{h(t)} J_i(x-y) v(t,y) dy - v(t,x),$$

$$d_i > 0, \quad \mu_i \ge 0 \quad \text{and} \quad \sum_{i=1}^{m_0} \mu_i > 0.$$

The initial functions satisfy, for $1 \le i \le m$,

$$u_{i0} \in \big\{ u \in C([-h_0, h_0]) : u(\pm h_0) = 0, \ u > 0 \text{ in } (-h_0, h_0) \big\}. \tag{5.11}$$

The kernel functions $J_i(x)$ $(i = 1, \ldots, m_0)$ satisfy

$$(\mathbf{J}) : J_i \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \quad \text{is non-negative, even,} \quad J_i(0) > 0,$$
$$\int_{\mathbb{R}} J_i(x) dx = 1 \quad \text{for } 1 \le i \le m_0.$$

To describe the assumptions on the function

$$F = (f_1, \ldots, f_m) \in [C^1(\mathbb{R}^m_+)]^m$$

with

$$\mathbb{R}^m_+ := \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_i \ge 0 \text{ for } i = 1, \ldots, m\},$$

we introduce some notations about vectors in $\mathbb{R}^m$.

Notations about vectors in $\mathbb{R}^m$:

(i) For $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, we simply write $(x_1, \ldots, x_m)$ as (x_i). For $x = (x_i)$, $y = (y_i) \in \mathbb{R}^m$,

$$x \succeq (\preceq)\, y \quad \text{means } x_i \geq (\leq)\, y_i \quad \text{for } 1 \leq i \leq m,$$

$$x \succ (\prec)\, y \quad \text{means } x \succeq (\preceq)\, y \quad \text{but } x \neq y,$$

$$x \succ\!\succ (\prec\!\prec)\, y \quad \text{means } x_i > (<)\, y_i \quad \text{for } 1 \leq i \leq m.$$

(ii) If $x \preceq y$, then $[x, y] := \{z \in \mathbb{R}^m : x \preceq z \preceq y\}$.

(iii) Hadamard product: For $x = (x_i)$, $y = (y_i) \in \mathbb{R}^m$,

$$x \circ y = (x_i y_i) \in \mathbb{R}^m.$$

(iv) Any $x \in \mathbb{R}^m$ is viewed as a row vector, namely a $1 \times m$ matrix, whose transpose is denoted by x^T.

Irreducible matrix and principal eigenvalue.
An $m \times m$ matrix $A = (a_{ij})$, with $m \geq 2$, is called *reducible* if the index set $\{1, \ldots, m\}$ can be split into the union of two subsets S and S' with $r \geq 1$ and $m - r \geq 1$ elements, respectively, such that $a_{ij} = 0$ for all $i \in S$ and $j \in S'$. A is called *irreducible* if it is not reducible. If D is a diagonal $m \times m$ matrix, clearly $A + D$ is irreducible if and only if A is irreducible.

If A is irreducible and all its off-diagonal elements are non-negative, then for $\sigma > 0$ large $A + \sigma \mathbf{I}_m$ is a non-negative irreducible matrix, where $\mathbf{I}_m$ denotes the $m \times m$ identity matrix. By the *Perron–Frobenius theorem*, $A + \sigma \mathbf{I}_m$ has a largest eigenvalue $\tilde{\lambda}_1 = \tilde{\lambda}_1(\sigma)$ which is the only eigenvalue that corresponds to a positive eigenvector $v_1 \succ\!\succ \mathbf{0}$: $(A + \sigma \mathbf{I}_m) v_1 = \tilde{\lambda}_1 v_1$. Hence $A v_1 = \lambda_1 v_1$ with $\lambda_1 = \tilde{\lambda}_1 - \sigma$, which is the largest eigenvalue of A and is independent of σ. We will call λ_1 the *principal eigenvalue* of A.

A 1×1 matrix is irreducible if and only if its sole element is not 0.

Assumptions on F:

($\mathbf{f_1}$) (i) $F(u) = \mathbf{0}$ has only two roots in $\mathbb{R}^m_+$: $\mathbf{0} = (0, 0, \ldots, 0)$ and $\mathbf{u}^* = (u_1^*, u_2^*, \ldots, u_m^*) \succ\!\succ \mathbf{0}$.

(ii) $\partial_j f_i(u) \geq 0$ for $i \neq j$ and $u \in [\mathbf{0}, \hat{\mathbf{u}}]$, where either $\hat{\mathbf{u}} = \infty$ meaning $[\mathbf{0}, \hat{\mathbf{u}}] = \mathbb{R}^m_+$, or $\mathbf{u}^* \prec\!\prec \hat{\mathbf{u}} \in \mathbb{R}^m$; which implies that (5.10) is a *cooperative system* in $[\mathbf{0}, \hat{\mathbf{u}}]$.

(iii) The matrix $\nabla F(\mathbf{0})$ is irreducible with principal eigenvalue positive, where $\nabla F(\mathbf{0}) = (a_{ij})_{m \times m}$ with $a_{ij} = \partial_j f_i(\mathbf{0})$.

(iv) If $m_0 < m$ then $\partial_j f_i(u) > 0$ for $1 \leq j \leq m_0 < i \leq m$ and $u \in [\mathbf{0}, \mathbf{u}^*]$.

($\mathbf{f_2}$) $F(ku) \geq kF(u)$ for any $0 \leq k \leq 1$ and $u \in [\mathbf{0}, \hat{\mathbf{u}}]$.

($\mathbf{f_3}$) The matrix $\nabla F(\mathbf{u}^*)$ is invertible, $\mathbf{u}^* \nabla F(\mathbf{u}^*) \preceq \mathbf{0}$ and for each $i \in \{1, \dots, m\}$, either

(i) $\sum_{j=1}^{m} \partial_j f_i(\mathbf{u}^*) u_j^* < 0$, or

(ii) $\sum_{j=1}^{m} \partial_j f_i(\mathbf{u}^*) u_j^* = 0$ and $f_i(u)$ is linear in $[\mathbf{u}^* - \epsilon_0 \mathbf{1}, \mathbf{u}^*]$ for some small $\epsilon_0 > 0$, where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^m$.

($\mathbf{f_4}$) The set $[\mathbf{0}, \hat{\mathbf{u}}]$ is invariant for

$$U_t = D \circ \int_{\mathbb{R}} \mathbf{J}(x-y) \circ U(t,y) dy - D \circ U + F(U), \quad t > 0, \quad x \in \mathbb{R}, \tag{5.12}$$

and the equilibrium $\mathbf{u}^*$ attracts all the nontrivial solutions in $[\mathbf{0}, \hat{\mathbf{u}}]$; namely, $U(t,x) \in [\mathbf{0}, \hat{\mathbf{u}}]$ for all $t > 0, x \in \mathbb{R}$ if $U(0,x) \in [\mathbf{0}, \hat{\mathbf{u}}]$ for all $x \in \mathbb{R}$, and $\lim_{t \to \infty} U(t, \cdot) = \mathbf{u}^*$ in $L^\infty_{\text{loc}}(\mathbb{R})$ if additionally $U(0,x) \not\equiv \mathbf{0}$.

In (5.12) we used the convention that $d_i = 0$ and $J_i \equiv 0$ for $m_0 < i \leq m$, and

$$D = (d_i), \quad \mathbf{J}(x) = (J_i(x)).$$

These assumptions imply

$$\begin{cases} \text{(5.12) is cooperative in } [\mathbf{0}, \hat{\mathbf{u}}], \text{ and monostable,} \\ \mathbf{u}^* \text{ is the unique stable equilibrium of (5.12),} \\ \text{which is the global attractor of (5.12) in } [\mathbf{0}, \hat{\mathbf{u}}] \setminus \{\mathbf{0}\}. \end{cases}$$

It can be shown that (5.10) with initial data satisfying (5.11) and $U(0,x) \in [\mathbf{0}, \hat{\mathbf{u}}]$ has a unique positive solution $(U(t,x), g(t), h(t))$ defined for all $t > 0$.

We say *spreading* happens if, as $t \to \infty$,

$$\begin{cases} (g(t), h(t)) \to \mathbb{R}, \\ U(t, \cdot) \to \mathbf{u}^* \text{ component-wise in } L^\infty_{\text{loc}}(\mathbb{R}), \end{cases}$$

and we say *vanishing* happens if

$$\begin{cases} (g(t), h(t)) \to (g_\infty, h_\infty) \text{ is a finite interval,} \\ \max_{x \in [g(t), h(t)]} |U(t,x)| \to 0. \end{cases}$$

We suppose spreading happens for (5.10) and aim to determine the spreading speed. We will need the following key condition:

$$(\mathbf{J_1})\text{: } \int_0^\infty x J_i(x) dx < \infty \quad \text{for every } i \in \{1, \dots, m_0\} \quad \text{with } \mu_i > 0.$$

We will show that if ($\mathbf{J_1}$) is satisfied, then the spreading speed is finite, otherwise it is infinite, namely *accelerated spreading* happens if ($\mathbf{J_1}$) is not satisfied.

The proof relies on the associated *semi-wave problem to* (5.10), namely (5.13) and (5.14) with unknowns $(c, \Phi(x))$:

$$\begin{cases} D \circ \displaystyle\int_{-\infty}^{0} \mathbf{J}(x-y) \circ \Phi(y)dy - D \circ \Phi + c\Phi' + F(\Phi) = 0, & -\infty < x < 0, \\ \Phi(-\infty) = \mathbf{u}^*, & \Phi(0) = \mathbf{0}, \end{cases} \tag{5.13}$$

$$c = \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{0} \int_{0}^{\infty} J_i(x-y)\phi_i(x)dydx, \tag{5.14}$$

where $D = (d_i)$, $\mathbf{J} = (J_i)$, $\Phi = (\phi_i)$.

If (c, Φ) solves (5.13), we say that Φ is a *semi-wave solution to* (5.12) with speed c. This is not to be confused with the semi-wave to (5.10), for which the extra equation (5.14) should be satisfied, yielding a semi-wave solution of (5.12) with a desired speed $c = c_0$, which determines the spreading speed of (5.10).

To fully understand the semi-wave solutions to (5.12), we also need to examine the associated traveling wave problem for (5.12). A differentiable function Ψ is called a *traveling wave solution of* (5.12) with speed c if Ψ satisfies

$$\begin{cases} D \circ \displaystyle\int_{-\infty}^{\infty} \mathbf{J}(x-y) \circ \Psi(y)dy - D \circ \Psi + c\Psi' + F(\Psi) = \mathbf{0}, \quad x \in \mathbb{R}, \\ \Psi(-\infty) = \mathbf{u}^*, \quad \Psi(\infty) = \mathbf{0}. \end{cases} \tag{5.15}$$

We are interested in semi-waves and traveling waves which are *monotone and with positive speed*. It turns out that for any fixed speed $c > 0$, either a semi-wave or traveling wave exists.

Theorem 5.7 (Semi-wave versus traveling wave [57]). *Suppose* $(\mathbf{J})$ *and* $(\mathbf{f_1}) - (\mathbf{f_4})$ *hold. Then there exists* $C_* \in (0, \infty]$ *such that* (5.13) *has a monotone solution if and only if* $0 < c < C_*$, *and* (5.15) *has a monotone solution if and only if* $c \geq C_*$.

Therefore a monotone traveling wave with some positive speed c exists if and only if $C_* < \infty$. We will show that $C_* < \infty$ if and only if the following condition is satisfied:

$$(\mathbf{J_2}): \int_0^\infty e^{\lambda x} J_i(x)dx < \infty \quad \text{for some } \lambda > 0 \quad \text{and every } i \in \{1, \ldots, m_0\}.$$

We have the following refinements of the conclusions in Theorem 5.7.

Theorem 5.8 (Semi-wave with the desired speed [57]). *Under the conditions of Theorem* 5.7, *the following hold*:

(i) *For* $0 < c < C_*$, (5.13) *has a unique monotone solution* $\Phi^c = (\phi_i^c)$, *and*

$$\lim_{c \nearrow C_*} \Phi^c(x) = \mathbf{0} \quad \textit{locally uniformly in } (-\infty, 0].$$

(ii) $C_* \neq \infty$ *if and only if* $(\mathbf{J_2})$ *holds.*

(iii) *The system* (5.13)–(5.14) *has a solution pair* (c, Φ) *with* $\Phi(x)$ *monotone if and only if* $(\mathbf{J_1})$ *holds. When* $(\mathbf{J_1})$ *holds, there exists a unique* $c_0 \in (0, C_*)$ *such that* $(c, \Phi) = (c_0, \Phi^{c_0})$ *solves* (5.13) *and* (5.14).

It is easily checked that $(\mathbf{J_2}) \Rightarrow (\mathbf{J_1})$, but $(\mathbf{J_1}) \not\Rightarrow (\mathbf{J_2})$; e.g. if $J_i(x) = \xi_i(1+|x|)^{-\eta_i}$ with $\xi_i > 0$, $\eta_i > 2$ for $1 \le i \le m_0$, then $(\mathbf{J_1})$ holds but $(\mathbf{J_2})$ does not.

Our first result on the spreading speed of (5.10) is the following theorem.

Theorem 5.9 (Spreading speed [57]). *Suppose the conditions in Theorem* 5.7 *are satisfied,* (U, g, h) *is a solution of* (5.10) *with* $U(0, x) \in [\mathbf{0}, \hat{\mathbf{u}}]$, *and spreading happens. Then the following conclusions hold for the spreading speed*:

(i) *If* $(\mathbf{J_1})$ *is satisfied, then the spreading speed is finite*:

$$-\lim_{t\to\infty} \frac{g(t)}{t} = \lim_{t\to\infty} \frac{h(t)}{t} = c_0 \quad \textit{with } c_0 \textit{ from Theorem } 5.8\text{(iii)}.$$

(ii) *If* $(\mathbf{J_1})$ *is not satisfied, then accelerated spreading happens*:

$$-\lim_{t\to\infty} \frac{g(t)}{t} = \lim_{t\to\infty} \frac{h(t)}{t} = \infty.$$

Under further conditions on F and the kernel functions, the conclusions in Theorem 5.9 can be sharpened. For $\alpha > 0$, we introduce the condition

$$(\mathbf{J}^\alpha)\text{:} \int_0^\infty x^\alpha J_i(x)dx < \infty \quad \text{for every } i \in \{1, \dots, m_0\}.$$

Theorem 5.10 (Sharp estimate of the spreading rate [57]). *In Theorem* 5.9, *suppose additionally*

(i) $(\mathbf{J}^\alpha)$ *holds for some* $\alpha \ge 2$.
(ii) F *is* C^2 *and* $\mathbf{u}^*[\nabla F(\mathbf{u}^*)]^T \lll \mathbf{0}$.

Then there exist positive constants θ, C *and* t_0 *such that, for all* $t > t_0$ *and* $x \in [g(t), h(t)]$,

$$|h(t) - c_0 t| + |g(t) + c_0 t| \le C,$$

$$\begin{cases} U(t,x) \succeq [1-\epsilon(t)]\big[\Phi^{c_0}(x - c_0 t + C) + \Phi^{c_0}(-x - c_0 t + C) - \mathbf{u}^*\big], \\ U(t,x) \preceq [1+\epsilon(t)]\min\{\Phi^{c_0}(x - c_0 t - C),\ \Phi^{c_0}(-x - c_0 t - C)\}, \end{cases}$$

where $\epsilon(t) := (t+\theta)^{-\alpha}$,

Further estimates on $g(t)$ and $h(t)$ can be obtained if we narrow down more on the class of kernel functions $\{J_i : i = 1, \dots, m_0\}$. We will write

$$\eta(t) \approx \xi(t) \quad \text{if } C_1\xi(t) \le \eta(t) \le C_2\xi(t)$$

for some constants $0 < C_1 \le C_2$ and all t is the concerned range.

Our next two theorems are about kernel functions satisfying, for some $\gamma > 0$,

$$(\hat{\mathbf{J}}^\gamma): J_i(x) \approx |x|^{-\gamma} \quad \text{for } |x| \gg 1 \quad \text{and} \quad i \in \{1, \ldots, m_0\}.$$

Note that for kernel functions satisfying $(\hat{\mathbf{J}}^\gamma)$,

$$(\mathbf{J}) \Leftrightarrow \gamma > 1, \quad (\mathbf{J_1}) \Leftrightarrow \gamma > 2.$$

The next result determines the orders of accelerated spreading when $\gamma \in (1, 2]$.

Theorem 5.11 (Rate of accelerated spreading [57]). *In Theorem* 5.9, *if additionally the kernel functions satisfy* $(\hat{\mathbf{J}}^\gamma)$ *for some* $\gamma \in (1, 2]$, *then for* $t \gg 1$,

$$\begin{aligned} -g(t), \quad h(t) &\approx t \ln t \qquad \text{if } \gamma = 2, \\ -g(t), \quad h(t) &\approx t^{1/(\gamma-1)} \quad \text{if } \gamma \in (1, 2). \end{aligned}$$

For kernel functions satisfying $(\hat{\mathbf{J}}^\gamma)$, clearly $(\mathbf{J}^\alpha)$ holds if and only if $\gamma > 1 + \alpha$. Therefore, the case $\gamma > 3$ is already covered by Theorem 5.10. The following theorem is concerned with the remaining case $\gamma \in (2, 3]$, which indicates that the result in Theorem 5.10 is sharp.

Theorem 5.12 (Order of spreading shift [57]). *In Theorem* 5.9, *suppose additionally the kernel functions satisfy*:

(i) $(\hat{\mathbf{J}}^\gamma)$ *for some* $\gamma \in (2, 3]$,
(ii) F *is* C^2 *and*

$$F(v) - v[\nabla F(v)]^T \succ\!\!\succ \mathbf{0} \quad \text{for } \mathbf{0} \prec\!\!\prec v \preceq \mathbf{u}^*. \tag{5.16}$$

Then for $t \gg 1$,

$$\begin{aligned} c_0 t + g(t), \quad c_0 t - h(t) &\approx \ln t \qquad \text{if } \gamma = 3, \\ c_0 t + g(t), \quad c_0 t - h(t) &\approx t^{3-\gamma} \quad \text{if } \gamma \in (2, 3). \end{aligned}$$

Remarks. (a) When $m = 1$, (5.16) reduces to $F(v) > F'(v)v$ for $0 < v \le u^*$, which is satisfied if, e.g. $F(v) = av - bv^p$ $(a, b > 0,\ p > 1)$.
(b) (5.1) satisfies all the conditions in Theorems 5.10–5.12 with $\hat{\mathbf{u}} = (e_1, e_2)$.
(c) (5.7) satisfies all the conditions in Theorems 5.10 and 5.11 with $\hat{\mathbf{u}} = \infty$, except (5.16) in Theorem 5.12.

The proofs of Theorems 5.10–5.12 rely on the following estimates.

Theorem 5.13 (Estimate of the semi-wave [57]). *Suppose that* F *satisfies* $(\mathbf{f_1}) - (\mathbf{f_4})$ *and the kernel functions satisfy* $(\mathbf{J})$, *and* $\Phi(x) = (\phi_i(x))$ *is a monotone solution of* (5.13) *for some* $c > 0$. *Then the following conclusions hold*:

(i) $(\mathbf{J}^\alpha)$ *holds* $\Rightarrow \sum_{i=1}^m \int_{-\infty}^{-1} \left[u_i^* - \phi_i(x)\right] |x|^{\alpha-1} dx < \infty$,

$$\text{which implies } 0 < u_i^* - \phi_i(x) \le C|x|^{-\alpha} \quad \text{for all } x < 0.$$

(ii) $(\mathbf{J}^\alpha)$ *does not hold* $\Rightarrow \sum_{i=1}^m \int_{-\infty}^{-1} \left[u_i^* - \phi_i(x)\right] |x|^{\alpha-1} dx = \infty$.
(iii) *If* $(\mathbf{J_2})$ *holds, then there exist positive constants* C *and* β *such that*

$$0 < u_i^* - \phi_i(x) \le Ce^{\beta x} \quad \text{for all } x < 0, \quad i \in \{1, \ldots, m\}.$$

5.4. *Remarks and comments*

In [60], a general two species Lotka–Volterra system with nonlocal diffusion and free boundary was considered. It is expected that a similar spreading-vanishing dichotomy would hold as in the corresponding local diffusion model. However, a technical difficulty arises in proving the vanishing case. As a result, extra assumptions on the kernel functions were made in [60] in order to obtain the desired results. Moreover, in the spreading case, the long-time dynamics was well understood only for the weak competition and weak predation cases. Part of the difficulty was caused by the lack of regularity improvement of the nonlocal diffusion operator. Some new ideas and techniques are required to overcome the difficulties here. The spreading speed for this kind of models is not considered yet in the literature, although much progress has been achieved for models discussed in Secs. 5.1–5.3.

In [103, 127], some two species models with free boundary and hybrid diffusions were considered by Wang and collaborators, where one species disperses via a nonlocal diffusion operator but the other disperses via a local diffusion operator.

There are too many questions that remain to be answered in this promising area; for example, the models in high space dimension, with more general kernels, etc. are still very poorly understood. Generally speaking, there is vast room to be explored in this fertile field of nonlocal diffusion models.

Acknowledgment

This research was supported by the Australian Research Council.

References

[1] I. Ahn, S. Beak and Z. Lin, The spreading fronts of an infective environment in a man-environment-man epidemic model, *Appl. Math. Model.* **40** (2016) 7082–7101.
[2] M. Alfaro and J. Coville, Propagation phenomena in monostable integro-differential equations: Acceleration or not? *J. Differential Equations* **263** (2017) 5727–5758.
[3] F. Andreu-Vaillo, J. M. Mazón, J. D. Rossi and J. J. Toledo-Melero, *Nonlocal Diffusion Problems*, Mathematical Surveys and Monographs (American Mathematical Society, Providence, RI, 2010).
[4] S. B. Angenent, The zero set of a solution of a parabolic equation, *J. Reine Angew. Math.* **390** (1988) 79–96.
[5] D. Aronson and H. F. Weinberger, Multidimensional nonlinear diffusion arising in population genetics, *Adv. Math.* **30** (1978) 33–76.
[6] A. Audrito and J. L. Vázquez, The Fisher–KPP problem with doubly nonlinear diffusion, *J. Differential Equations* **263**(11) (2017) 7647–7708.
[7] W. Bao, Y. Du, Z. Lin and H. Zhu, Free boundary models for mosquito range movement driven by climate warming, *J. Math. Biol.* **76** (2018) 841–875.
[8] P. Bates, P. Fife, X. Ren and X. Wang, Traveling waves in a convolution model for phase transitions, *Arch. Ration. Mech. Anal.* **138** (1997) 105–136.
[9] P. Bates and G. Zhao, Existence, uniqueness and stability of the stationary solution to a nonlocal evolution equation arising in population dispersal, *J. Math. Anal. Appl.* **332** (2007) 428–440.

[10] H. Berestycki, J. Coville and H. Vo, Persistence criteria for populations with nonlocal dispersion, *J. Math. Biol.* **72** (2016) 1693–1745.

[11] H. Berestycki and F. Hamel, Front propagation in periodic excitable media, *Comm. Pure Appl. Math.* **55** (2002) 949–1032.

[12] H. Berestycki, F. Hamel and N. Nadirashvili, The speed of propagation for KPP type problems. I. Periodic framework, *J. Eur. Math. Soc.* **7** (2005) 173–213.

[13] E. Bouin, J. Garnier, C. Henderson and F. Patout, Thin front limit of an integro-differential Fisher–KPP equation with fat-tailed kernels, *SIAM J. Math. Anal.* **50** (2018) 3365–3394.

[14] M. Bramson, Maximal displacement of branching Brownian motion, *Comm. Pure Appl. Math.* **31**(5) (1978) 531–581.

[15] G. Bunting, Y. Du and K. Krakowski, Spreading speed revisited: Analysis of a free boundary model, *Netw. Heterog. Media* **7** (2012) 583–603.

[16] X. Cabré and J.-M. Roquejoffre, The influence of fractional diffusion in Fisher–KPP equations, *Comm. Math. Phys.* **320** (2013) 679–722.

[17] J. Cai and H. Gu, Asymptotic behavior of solutions of free boundary problems for Fisher–KPP equation, *J. Dynam. Differential Equations* **33**(2) (2021) 913–940.

[18] J. Cai, B. Lou and M. Zhou, Asymptotic behavior of solutions of a reaction diffusion equation with free boundary conditions, *J. Dynam. Differential Equations* **26**(4) (2014) 1007–1028.

[19] L. Caffarelli, The regularity of free boundaries in higher dimensions, *Acta Math.* **139**(3–4) (1977) 155–184.

[20] J. F. Cao, Y. Du, F. Li and W. T. Li, The dynamics of a Fisher–KPP nonlocal diffusion model with free boundaries, *J. Funct. Anal.* **277** (2019) 2772–2814.

[21] V. Capasso and S. L. Paveri-Fontana, A mathematical model for the 1973 cholera epidemic in the European Mediterranean region, *Rev. Epidemiol. Publ.* **27** (1979) 121–132.

[22] V. Capasso and L. Maddalena, Convergence to equilibrium states for a reaction–diffusion system modeling the spatial spread of a class of bacterial and viral diseases, *J. Math. Biol.* **13** (1981/1982) 173–184.

[23] C.-H. Chang and C.-C. Chen, Travelling wave solutions of a free boundary problem for a two-species competitive model, *Comm. Pure Appl. Anal.* **12** (2013) 1065–1074.

[24] C.-H. Chang, C.-C. Chen and C.-C. Huang, Traveling wave solutions of a free boundary problem with latent heat effect, *Discrete Contin. Dyn. Syst. Ser. B* **26**(4) (2021) 1797–1809.

[25] T.-Y. Chang and Y. Du, Long-time dynamics of an epidemic model with nonlocal diffusion and free boundaries, *Electron. Res. Arch.* **30**(1) (2022) 289–313.

[26] X. Chen, Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations, *Adv. Differential Equations* **2** (1997) 125–160.

[27] C. Cortázar, M. Elgueta and J. D. Rossi, Nonlocal diffusion problems that approximate the heat equation with Dirichlet boundary conditions, *Israel J. Math.* **170** (2009) 53–60.

[28] C. Cortázar, M. Elgueta, J. D. Rossi and N. Wolanski, How to approximate the heat equation with Neumann boundary conditions by nonlocal diffusion problems, *Arch. Ration. Mech. Anal.* **187** (2008) 137–156.

[29] C. Cortázar, F. Quirós and N. Wolanski, A nonlocal diffusion problem with a sharp free boundary, *Interfaces Free Bound.* **21** (2019) 441–462.

[30] J. Coville, J. Dávila and S. Martínez, Pulsating fronts for nonlocal dispersion and KPP nonlinearity, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **30**(2) (2013) 179–223.

[31] J. Coville and L. Dupaigne, On a non-local equation arising in population dynamics, *Proc. Roy. Soc. Edinb. Sect. A* **137** (2007) 727–755.
[32] E. N. Dancer, D. Hilhorst, M. Mimura and L. A. Peletier, Spatial segregation limit of a competition-diffusion system, *European J. Appl. Math.* **10** (1999) 97–115.
[33] P. de Mottoni, Qualitative analysis for some quasilinear parabolic systems, *Inst. Math. Polish Acad. Sci. Zam.* **190** (1979) 11–79.
[34] F. del Teso, J. Endal and J. L. Vazquez, The one-phase fractional Stefan problem, *Math. Models Methods Appl. Sci.* **31** (2021) 83–131.
[35] W. Ding, Y. Du and Z. M. Guo, The Stefan problem for the Fisher–KPP equation with unbounded initial range, *Cal. Var. PDEs* **60** (2021) 69.
[36] W. Ding, Y. Du and X. Liang, Spreading in space-time periodic media governed by a monostable equation with free boundaries, Part 1: Continuous initial functions, *J. Differential Equations* **262** (2017) 4988–5021.
[37] W. Ding, Y. Du and X. Liang, Spreading in space-time periodic media governed by a monostable equation with free boundaries, Part 2: Spreading speed, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **36** (2019) 1539–1573.
[38] Y. Du, J. Fang and N. Sun, A delay induced nonlocal problem with free boundary, preprint (2021).
[39] Y. Du, C. Gui, K. Wang and M. Zhou, Semi-waves with Λ-shaped free boundary for nonlinear Stefan problems: Existence, *Proc. Amer. Math. Soc.* **149** (2021) 2091–2104.
[40] Y. Du and Z. M. Guo, The Stefan problem for the Fisher–KPP equation, *J. Differential Equations* **253** (2012) 996–1035.
[41] Y. Du, Z. M. Guo and R. Peng, A diffusive logistic model with a free boundary in time-periodic environment, *J. Funct. Anal.* **265** (2013) 2089–2142.
[42] Y. Du, Y. Hu and X. Liang, A climate shift model with free boundary: Enhanced invasion, *J. Dynam. Differential. Equations*, https://doi.org/10.1007/s10884-021-10031-3.
[43] Y. Du, F. Li and M. Zhou, Semi-wave and spreading speed of the nonlocal Fisher–KPP equation with free boundaries, *J. Math. Pure Appl.* **154** (2021) 30–66.
[44] Y. Du, W.-T. Li, W. Ni and M. Zhao, Finite or infinite spreading speed of an epidemic model with free boundary and double nonlocal effects, preprint, 2020.
[45] Y. Du and X. Liang, Pulsating semi-waves in periodic media and spreading speed determined by a free boundary model, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **32** (2015) 279–305.
[46] Y. Du and Z. Lin, Spreading-Vanishing dichotomy in the diffusive logistic model with a free boundary, *SIAM J. Math. Anal.* **42** (2010) 377–405.
[47] Y. Du and Z. Lin, The diffusive competition model with a free boundary: Invasion of a superior or inferior competitor, *Discrete Cont. Dyn. Syst. B.* **19** (2014) 3105–3132.
[48] Y. Du and B. Lou, Spreading and vanishing in nonlinear diffusion problems with free boundaries, *J. Eur. Math. Soc.* **17** (2015) 2673–2724.
[49] Y. Du, B. Lou and M. Zhou, Nonlinear diffusion problems with free boundaries: Convergence, transition speed and zero number arguments, *SIAM J. Math. Anal.* **47** (2015) 3555–3584.
[50] Y. Du, B. Lou and M. Zhou, Spreading and vanishing for nonlinear Stefan problems in high space dimensions, *J. Elliptic Parabolic Equations* **2** (2016) 297–321.
[51] Y. Du and H. Matano, Convergence and sharp thresholds for propagation in nonlinear diffusion problems, *J. Eur. Math. Soc.* **12** (2010) 279–312.

[52] Y. Du, H. Matano and K. Wang, Regularity and asymptotic behavior of nonlinear Stefan problems, *Arch. Ration. Mech. Anal.* **212** (2014) 957–1010.
[53] Y. Du, H. Matsuzawa and M. Zhou, Sharp estimate of the spreading speed determined by nonlinear free boundary problems, *SIAM J. Math. Anal.* **46** (2014) 375–396.
[54] Y. Du, H. Matsuzawa and M. Zhou, Spreading speed and profile for nonlinear Stefan problems in high space dimensions, *J. Math. Pures Appl.* **103** (2015) 741–787.
[55] Y. Du and W. Ni, Analysis of a West Nile virus model with nonlocal diffusion and free boundaries, *Nonlinearity* **33** (2020) 4407–4448.
[56] Y. Du and W. Ni, Approximation of random diffusion equation by nonlocal diffusion equation in free boundary problems of one space dimension, to appear in *Comm. Contemp. Math.*, arXiv:2003.05560.
[57] Y. Du and W. Ni, Semi-wave, traveling wave and spreading speed for monostable cooperative systems with nonlocal diffusion and free boundaries, preprint (2020), arXiv:2010.01244.
[58] Y. Du and W. Ni, The high dimensional Fisher–KPP nonlocal diffusion equation with free boundary and radial symmetry, preprint (2021), arXiv:2102.05286.
[59] Y. Du, F. Quirós and M. Zhou, Logarithmic corrections in Fisher–KPP type porous medium equations, *J. Math. Pure Appl.* **136** (2020) 415–455.
[60] Y. Du, M. Wang and M. Zhao, Two species nonlocal diffusion systems with free boundaries, *Discrete Contin. Dyn. Syst.-Ser. A* **42** (2022) 1127–1162.
[61] Y. Du, M. Wang and M. Zhou, Semi-wave and spreading speed for the diffusive competition model with a free boundary, *J. Math. Pure Appl.* **107** (2017) 253–287.
[62] Y. Du, L. Wei and L. Zhou, Spreading in a shifting environment modeled by the diffusive logistic equation with a free boundary, *J. Dynam. Differential Equations* **30** (2018) 1389–1426.
[63] Y. Du and C.-H. Wu, Spreading with two speeds and mass segregation in a diffusive competition system with free boundaries, *Cal. Var. PDEs* **57** (2018) 52.
[64] Y. Du and C.-H. Wu, Classification of the spreading behaviors of a two-species diffusion-competition system with free boundaries, *Cal. Var. PDEs* to appear.
[65] M. El-Hachem, S. W. McCue and M. J. Simpson, A sharp-front moving boundary for malignant invasion, *Physica D* **412** (2020) 132639, https://doi.org/10.1016/j.physd.2020.132639.
[66] M. El-Hachem, S. W. McCue and M. J. Simpson, Invading and receding sharp-fronted traveling waves, *Bull. Math. Biol.* **83** (2021) 35, https://doi.org/10.1007.
[67] N. T. Fadai, Semi-infinite traveling waves arising in a general reaction–diffusion Stefan model, *Nonlinearity* **34** (2021) 725–743.
[68] N. T. Fadai and M. J. Simpson, New traveling wave solutions of the Porous–Fisher model with a moving boundary, *J. Phys. A*: *Math. Theor.* **53** (2020) 095601.
[69] J. Fang and G. Faye, Monotone traveling waves for delayed neural field equations, *Math. Models Methods Appl. Sci.* **26** (2016) 1919–1954.
[70] M. Fila and P. Souplet, Existence of global solutions with slow decay and unbounded free boundary for a superlinear Stefan problem, *Interfaces Free Bound.* **3** (2001) 337–344.
[71] D. Finkelshtein and P. Tkachov, Accelerated nonlocal nonsymmetric dispersion for monostable equations on the real line, *Appl. Anal.* **98** (2019) 756–780.
[72] R. Fisher, The wave of advance of advantage genes, *Ann. Eugenics* **7** (1937) 335–369.

[73] A. Friedman, The Stefan problem in several space variables, *Trans. Amer. Math. Soc.* **132** (1968) 51–87.

[74] J. Garnier, Accelerating solutions in integro-differential equations, *SIAM J. Math. Anal.* **43** (2011) 1955–1974.

[75] J. Gärtner, Location of wave fronts for the multidimensional KPP equation and Brownian first exit densities, *Math. Nachr.* **105** (1982) 317–351.

[76] J. Gärtner and M. I. Freidlin, On the propagation of concentration waves in periodic and random media, *Sov. Math. Dokl.* **20** (1979) 1282–1286.

[77] H. Ghidouche, P. Souplet and D. Tarzia, Decay of global solutions, stability and blow-up for a reaction–diffusion problem with free boundary, *Proc. Amer. Math. Soc.* **129** (2001) 781–792.

[78] B. H. Gilding and R. Kersner, *Traveling Waves in Nonlinear Diffusion–Convection Reaction*, Progress in Nonlinear Differential Equations and their Applications, Vol. 60 (Birkhäuser Verlag, Basel, 2004).

[79] L. Girardin and K. Y. Lam, Invasion of open space by two competitors: Spreading properties of monostable two-species competition-diffusion systems, *Proc. London Math. Soc.* **119** (2019) 1279–1335.

[80] L. Girardin and G. Nadin, Traveling waves for diffusive and strongly competitive systems: Relative motility and invasion speed, *Eur. J. Appl. Math.* **26** (2015) 521–534.

[81] H. Gu, B. Lou and M. Zhou, Long time behavior of solutions of Fisher–KPP equation with advection and free boundaries, *J. Funct. Anal.* **269**(6) (2015) 1714–1768.

[82] J. Guo and C. H. Wu, On a free boundary problem for a two-species weak competition system, *J. Dynam. Differential Equations* **24** (2012) 873–895.

[83] J.-S. Guo and C.-H. Wu, Dynamics for a two-species competition-diffusion model with two free boundaries, *Nonlinearity* **28** (2015) 1–27.

[84] W. S. C. Gurney and R. M. Nisbet, The regulation of inhomogeneous populations, *J. Theoret. Biol.* **52** (1975) 441–457.

[85] M. E. Gurtin and R. C. MacCamy, On the diffusion of biological populations, *Math. Biosci.* **33** (1977) 35–49.

[86] F. Hamel and L. Roques, Fast propagation for KPP equations with slowly decaying initial conditions, *J. Differential Equations* **249** (2010) 1726–1745.

[87] D. Hilhorst, M. Mimura and R. Schätzle, Vanishing latent heat limit in a Stefan-like problem arising in biology, *Nonlinear Anal. Real World Appl.* **4** (2003) 261–285.

[88] Y. Hu, X. Hao and Y. Du, Spreading under shifting climate by a free boundary model: Invasion of deteriorated environment, *Comm. Contemp. Math.* **23** (2021) 2050077.

[89] Y. Hu, X. Hao, X. Song and Y. Du, A free boundary problem for spreading under shifting climate, *J. Differential Equations* **269** (2020) 5931–5958.

[90] Ja. I. Kanel', Stabilization of the solutions of the equations of combustion theory with finite initial functions, *Mat. Sb.* **65** (1964) 398–413.

[91] Y. Kaneko and H. Matsuzawa, Spreading speed and sharp asymptotic profiles of solutions in free boundary problems for nonlinear advection–diffusion equations, *J. Math. Anal. Appl.* **428** (2015) 43–76.

[92] Y. Kaneko, H. Matsuzawa and Y. Yamada, Asymptotic profiles of solutions and propagating terrace for a free boundary problem of nonlinear diffusion equation with positive bistable nonlinearity, *SIAM J. Math. Anal.* **52** (2020) 65–103.

[93] Y. Kan-on, Fisher wave fronts for the Lotka-Volterra competition model with diffusion, *Nonlinear Anal. TMA* **28**(1) (1997) 145–164.

[94] Y. Kawai and Y. Yamada, Multiple spreading phenomena for a free boundary problem of a reaction–diffusion equation with a certain class of bistable nonlinearity, *J. Differential Equations* **261**(1) (2016) 538–572.

[95] K. Khan, S. Liu, T. M. Schaerf and Y. Du, Invasive behavior under competition via a free boundary model: A numerical approach, *J. Math. Biol.* **83** (2021) 23.

[96] K. Khan, T. M. Schaerf and Y. Du, Effects of environmental heterogeneity on species spreading via numerical analysis of some free boundary models, to appear in *Discrete Contin. Dyn. Syst. Ser. B.*

[97] D. Kinderlehrer and L. Nirenberg, The smoothness of the free boundary in the one phase Stefan problem, *Comm. Pure Appl. Math.* **31** (1978) 257–282.

[98] A. N. Kolmogorov, I. G. Petrovski and N. S. Piskunov, A study of the diffusion equation with increase in the amount of substance, and its application to a biological problem, *Bull. Moscow Univ. Math. Mech.* **1** (1937) 1–25.

[99] M. A. Lewis, J. Renclawowicz and P. van den Driessche, Traveling waves and spread rates for a West Nile virus model, *Bull. Math. Biol.* **68** (2006) 3–23.

[100] C. Lei, H. Nie, W. Dong and Y. Du, Spreading of two competing species governed by a free boundary model in a shifting environment, *J. Math. Anal. Appl.* **462** (2018) 1254–1282.

[101] F. Li, X. Liang and W. Shen, Diffusive KPP equations with free boundaries in time almost periodic environments: I. Spreading and vanishing dichotomy, *Discrete Contin. Dyn. Syst. Ser. A* **36** (2016) 3317–3338.

[102] F. Li, X. Liang and W. Shen, Diffusive KPP equations with free boundaries in time almost periodic environments: II. Spreading speeds and semi-wave, *J. Differential Equations* **261** (2016) 2403–2445.

[103] L. Li, W. J. Sheng and M. X. Wang, Systems with nonlocal versus local diffusions and free boundaries, *J. Math. Anal. Appl.* **483** (2020) 123646.

[104] X. Liang, Semi-wave solutions of KPP–Fisher equations with free boundaries in spatially almost periodic media, *J. Math. Pures Appl.* **127** (2019) 299–308.

[105] X. Liang and X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for monotone semiflows with applications, *Comm. Pure Appl. Math.* **60** (2007) 1–40.

[106] X. Liang and T. Zhou, Transition semi-wave solutions of reaction diffusion equations with free boundaries, *J. Differential Equations* **267**(10) (2019) 5601–5630.

[107] Z. Lin, A free boundary problem for a predator-prey model, *Nonlinearity* **20** (2007) 1883–1892.

[108] Z. Lin and H. Zhu, Spatial spreading model and dynamics of West Nile virus in birds and mosquitoes with free boundary, *J. Math. Biol.* **75**(6–7) (2017) 1381–1409.

[109] S. Liu, Y. Du and X. Liu, Numerical studies of a class of reaction-diffusion equations with Stefan conditions, *Int. J. Comp. Math.* **97** (2020) 959–979.

[110] S. Liu and X. Liu, Krylov implicit integration factor method for a class of stiff reaction–diffusion systems with moving free boundaries, *Discrete Cont. Dynam. Syst. B* **25** (2020) 141–159.

[111] Y. Liu, Z. Guo, M. El Smaily and L. Wang, A Wolbachia infection model with free boundary, *J. Biol. Dyn.* **14**(1) (2020) 515–542.

[112] S. Liu, H. Huang and M. X. Wang, Asymptotic spreading of a diffusive competition model with different free boundaries, *J. Differential Equations* **266**(8) (2019) 4769–4799.

[113] M. Mimura, Y. Yamada and S. Yotsutani, A free boundary problem in ecology, *Japan J. Appl. Math.* **2** (1985) 151–186.

[114] H. Matano, Asymptotic behavior of the free boundaries arising in one phase Stefan problems in multi-dimensional spaces, *Nonlinear Partial Differential Equations in Applied Science*, eds. H. Fujita, P. D. Lax and G. Strang, North-Holland Mathematics Studies, Vol. 81 (North-Holland, Amsterdam, 1983), pp. 133–151.

[115] R. Natan, E. Klein, J. J. Robledo-Arnuncio and E. Revilla, Dispersal kernels: Review, in *Dispersal Ecology and Evolution*, eds. J. Clobert, M. Baguette, T. G. Benton and J. M. Bullock (Oxford University Press, Oxford, UK, 2012), pp. 187–210.

[116] M.-A. Piqueras, R. Company and L. Jodar, A front-fixing numerical method for a free boundary nonlinear diffusion logistic population model, *J. Comput. Appl. Math.* **309** (2017) 473–481.

[117] N. Rawal, W. Shen and A. Zhang, Spreading speeds and traveling waves of nonlocal monostable equations in time and space periodic habitats, *Discrete Contin. Dyn. Syst.* **35** (2015) 1609–1640.

[118] J.-M. Roquejoffre, L. Rossi and V. Roussier-Michon, Sharp large time behavior in N-dimensional Fisher–KPP equations, *Discrete Contin. Dyn. Syst.* **39**(12) (2019) 7265–7290.

[119] W. Shen and X. Xie, Approximations of random dispersal operators/equations by nonlocal dispersal operators/equations, *J. Differential Equations* **259**(12) (2015) 7375–7405.

[120] J. G. Skellam, Random dispersal in theoretical populations, *Biometrika* **38** (1951) 196–218.

[121] P. E. Souganidis and A. Tarfulea, Front propagation for integro-differential KPP reaction–diffusion equations in periodic media, *NoDEA Nonlinear Differential Equations Appl.* **26**(4) (2019) 29.

[122] Y.-J. Sun, W.-T. Li and Z.-C. Wang, Traveling waves for a nonlocal anisotropic dispersal equation with monostable nonlinearity, *Nonlinear Anal.* **74** (2011) 814–826.

[123] N. Sun and J. Fang, Propagation dynamics of Fisher–KPP equation with time delay and free boundaries, *Calc. Var. Partial Differential Equations* **58** (2019) 148.

[124] N. Sun, B. Lou and M. Zhou, Fisher–KPP equation with free boundaries and time-periodic advections, *Calc. Var. PDEs* **56** (2017) 61.

[125] C. Tian and S. Ruan, On an advection–reaction–diffusion competition system with double free boundaries modeling invasion and competition of Aedes albopictus and Aedes aegypti mosquitoes, *J. Differential Equations* **265** (2018) 4016–4051.

[126] J. B. Wang and X.-Q. Zhao, Uniqueness and global stability of forced waves in a shifting environment, *Proc. Amer. Math. Soc.* **147**(4) (2019) 1467–1481.

[127] J. Wang and M. Wang, Free boundary problems with nonlocal and local diffusions I: Global solution, *J. Math. Anal. Appl.* **490**(2) (2020) 123974.

[128] M. X. Wang, On some free boundary problems of the Lotka-Volterra type prey–predator model, *J. Differential Equations* **256** (2014) 3365–3394.

[129] M. X. Wang, A diffusive logistic equation with a free boundary and sign-changing coefficient in time-periodic environment, *J. Funct. Anal.* **270** (2016) 483–508.

[130] M. X. Wang and Y. Zhang, Note on a two-species competition-diffusion model with two free boundaries, *Nonlinear Anal.* **159** (2017) 458–467.

[131] M. X. Wang and J. F. Zhao, A free boundary problem for a predator-prey model with double free boundaries, *J. Dynam. Differential Equations* **29** (2017) 957–979.

[132] R. Wang and Y. Du, Long-time dynamics of a diffusive epidemic model with free boundaries, *Discrete Contin. Dyn. Syst.-Ser. B* **26** (2021) 2201–2238.

[133] Z. Wang, H. Nie and Y. Du, Spreading speed for a West Nile virus model with free boundary, *J. Math. Biol.* **79** (2019) 433–466.
[134] L. Wei, G. Zhang and M. Zhou, Long time behavior for solutions of the diffusive logistic equation with advection and free boundary, *Calc. Var. Partial Differential Equations* **55**(4) (2016) 95.
[135] H. F. Weinberger, Long-time behavior of a class of biological models, *SIAM J. Math. Anal.* **13** (1982) 353–396.
[136] H. F. Weinberger, On spreading speeds and traveling waves for growth and migration in periodic habitat, *J. Math. Biol.* **45** (2002) 511–548.
[137] WHO, Cholera vaccines: WHO position paper, *Weekly Epidemiological Record* **85**(13) (2010) 117–128, https://pubmed.ncbi.nlm.nih.gov/20349546.
[138] W. B. Xu, W.-T. Li and G. Lin, Nonlocal dispersal cooperative systems: Acceleration propagation among species, *J. Differential Equations* **268**(3) (2020) 1081–1105.
[139] H. Yagisita, Existence and nonexistence of traveling waves for a nonlocal monostable equation, *Publ. Res. Inst. Math. Sci.* **45**(4) (2009) 925–953.
[140] J. Yang and B. Lou, Traveling wave solutions of competitive models with free boundaries, *Discrete Contin. Dyn. Syst. Ser. B* **19** (2014) 817–826.
[141] W. Zhang, Z. Liu and L. Zhou, A free boundary problem of a predator–prey model with a nonlocal reaction term, *Z. Angew. Math. Phys.* **72** (2021) 77.
[142] M. Zhao, W.-T. Li and Y. Du, The effect of nonlocal reaction in an epidemic model with nonlocal diffusion and free boundaries, *Comm. Pure Appl. Anal.* **19** (2020) 4599–4620.
[143] M. Zhao, W.-T. Li and W. Ni, Spreading speed of a degenerate and cooperative epidemic model with free boundaries, *Discrete Contin. Dyn. Syst. Ser. B* **25** (2020) 981–999.
[144] M. Zhao, Y. Zhang, W.-T. Li and Y. Du, The dynamics of a degenerate epidemic model with nonlocal diffusion and free boundaries, *J. Differential Equations* **269** (2020) 3347–3386.
[145] P. Zhou and Z. G. Lin, Global existence and blowup of a nonlocal problem in space with free boundary, *J. Funct. Anal.* **262** (2012) 3409–3429.
[146] A. Zlatŏs, Sharp transition between extinction and propagation of reaction, *J. Amer. Math. Soc.* **19** (2006) 251–263.