Partial Differential Equations in Applied Mathematics 7 (2023) 100475

Contents lists available at ScienceDirect

Partial Differential Equations in Applied Mathematics

journal homepage: www.elsevier.com/locate/padiff

# Study of explicit travelling wave solutions of nonlinear evolution equations

Kamruzzaman Khan [a,b,*], M. Ali Akbar [c]

[a] Department of Mathematics, Pabna University of Science and Technology, Pabna 6600, Bangladesh
[b] School of Science and Technology, University of New England, Armidale, NSW 2351, Australia
[c] Department of Applied Mathematics, University of Rajshahi, Rajshahi 6205, Bangladesh

ARTICLE INFO

*MSC:*
35K99
35P05
35P99

*Keywords:*
Bernoulli Sub-ODE method
Zoomeron equation
mKdV–ZK equation
TRLW equation
Solitons

ABSTRACT

The Bernoulli Sub-ODE approach is used in this study to look for comprehensive travelling wave solutions to the nonlinear evolution equations (NLEEs). The analysis in the present paper shows the existence of travelling waves for the time-regularized long-wave (TRLW) equation, the modified Korteweg–de Vries –Zakharov–Kuznetsov (mKdV–ZK) equation, and the (2+1)-dimensional Zoomeron equation. The outcomes demonstrate the richness of explicit solutions of the studied models. As a result, precise solitary wave solutions to the studied problems, such as kink waves, singular kink waves, dark soliton, and periodic waves are found. The phase plane is briefly examined after the determination of the Hamiltonian function. Using Maple 13, we validated the accuracy of the obtained solutions by reintroducing them into the original equation. We will demonstrate how the amplitudes and wave profiles are impacted by free parameters. In this article, we firmly establish that the wave amplitude varies as the free parameters change. It is demonstrated that the technique is efficient and applicable to several different NLEEs in mathematical physics.

## 1. Introduction

In multiple domains of mathematics, physics, chemistry, biology, engineering, and other applications, NLEEs are observed. Exact solutions of NLEEs are crucial for understanding the qualitative characteristics of numerous occurrences and processes across a range of natural science disciplines. The mechanisms of many complicated nonlinear events, including the spatial localization of transfer processes, the existence of peaking regimes, and the multiplicity or lack of steady states under varied conditions, are visually shown and unravelled by exact solutions of nonlinear equations. The consistency and error estimates of different numerical, asymptotic, and approximate analytical techniques may be checked using even those unique precise solutions that lack a clear physical meaning. Different teams of mathematicians and physicists have successfully created numerous innovative methods to study the NLEEs, such as the $(G'/G)$ —expansion method,[1–7] the Hirota's bilinear transformation method,[8–15] the modified simple equation method,[16–18] the tanh-function method,[19] the Exp-function method,[20–22] the Jacobi elliptic function method,[23] the homotopy perturbation method,[24–26] the auxiliary equation mapping method,[27–29] the direct algebraic method[30–32] and so on.

The Korteweg–de Vries (KdV) equation describes weakly nonlinear ion-acoustic waves in a magnetized plasma.[33,34] Calogero and Degasperis studied a modification of the KdV equation and Schrödinger equation to include solitons that travel at various speeds and discovered a relationship between their speed and polarization effects.[31,33,35] This led to the emergence of two distinct types of solitons, one of which was described as an accelerated soliton that boomeranged back with the same speed in the distant past, and the other as being imprisoned and oscillating in a spatial domain while changing direction repeatedly around a fixed point.[31,33,36] These were given the names Boomeron and Trappon solitons, respectively, and the coupled Boomeron equation of the following form was designed,[31,36]

$$\begin{cases} u_t(x,t) = \vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{v}}_{\boldsymbol{x}}(x,t), \\ \vec{v}_t(x,t) = u_x(x,t)\vec{\boldsymbol{\beta}} + \vec{\boldsymbol{\alpha}} \wedge \vec{\boldsymbol{v}}(x,t) + 2\int_x^\infty dx' \vec{\boldsymbol{v}}_x(x',t) \wedge \left[\vec{\boldsymbol{v}}(x',t) \wedge \vec{\boldsymbol{\beta}}\right], \end{cases} \tag{1.1}$$

where $u(x,t)$ is a scalar field and $\vec{\boldsymbol{v}}(x,t)$ is a vector field, and $\vec{\boldsymbol{\alpha}}$ and $\vec{\boldsymbol{\beta}}$ are two vector quantities in three space dimensions.

After some manipulations, the so-called scalar Zoomeron equation can be derived from the system of Eq. (1.1).[31] Here we consider the following (2+1)-dimensional Zoomeron equation,[18]

$$\left(\frac{u_{xy}}{u}\right)_{tt} - \left(\frac{u_{xy}}{u}\right)_{xx} + 2\left(u^2\right)_{xt} = 0, \tag{1.2}$$

where $u(x,y,t)$ is the amplitude of the relative wave mode of a single scalar field. The Zoomeron equation emerges from a range of scientific fields, including laser physics, nonlinear optics, and fluid dynamics.

Another conventional model is the KdV–ZK equation. However, for more intricate plasma compositions, the soliton character might

* Corresponding author at: Department of Mathematics, Pabna University of Science and Technology, Pabna 6600, Bangladesh.
*E-mail addresses:* k.khanru@pust.ac.bd, kkamruzz@une.edu.au (K. Khan).

https://doi.org/10.1016/j.padiff.2022.100475
Received 7 November 2022; Received in revised form 9 December 2022; Accepted 11 December 2022

change from compressive to rarefactive or vice versa at critical densities and temperatures, the modified KdV–ZK equation is the alternative to be utilized. For the evolution of ion-acoustic disturbances in a magnetized plasma with two negative ion components of different temperatures, the mKdV–ZK arises.[34,37,38] In this article, our second choice is to study the modified Korteweg–de Vries–Zakharov–Kuznetsov (mKdV–ZK)equation, governing the oblique propagation of nonlinear electrostatic modes,[37,38] of the form

$$u_t + \alpha u^2 u_x + u_{xxx} + u_{xyy} + u_{xzz} = 0., \tag{1.3}$$

where $\alpha$ is a nonzero parameter.

Finally, we will study the following time-regularized long-wave (TRLW) equation, another alternative of the KdV equation,[39]

$$u_t + u_x + \alpha u u_x + u_{xtt} = 0, \tag{1.4}$$

which was proposed by Joseph and Egri[40] and Jeffrey.[41]

The objective of this paper is to find the exact solutions then the solitary wave solutions for the (2+1)-dimensional Zoomeron equation, the mKdV–ZK equation, and the TRLW equation through the Bernoulli Sub-ODE method. Our other goal is to examine the effect of the free parameters on the obtained travelling wave solutions.

The article is arranged as follows: In Section 2, the Bernoulli Sub-ODE method[42–46] is discussed. In Section 3, we apply this method to the nonlinear evolution equations pointed out above. In Section 4, results and discussion and Section 5 conclusions are given.

## 2. Description of the method

In this section, we describe the Bernoulli Sub-ODE method for finding travelling wave solutions of NLEEs. Suppose that a nonlinear partial differential equation, say in two independent variables $x$ and $t$ is given by

$$\Re(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \dots\dots\dots\dots) = 0, \tag{2.1}$$

where $u(\xi) = u(x,t)$ is an unknown function, $\Re$ is a polynomial of $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method.[42,47]

Step 1. Combining the independent variables $x$ and $t$ into one variable $\xi = x \pm \omega t$, we suppose that

$$u(\xi) = u(x,t), \qquad \xi = x \pm \omega t. \tag{2.2}$$

The travelling wave transformation Eq. (2.2) permits us to transform Eq. (2.1) to the following ODE:

$$\Re(u, u', u'', \dots\dots\dots) = 0, \tag{2.3}$$

where $\Re$ is a polynomial in $u(\xi)$ and its derivatives, while $u'(\xi) = \frac{du}{d\xi}$, $u''(\xi) = \frac{d^2u}{d\xi^2}$ and so on.

Step 2. We suppose that Eq. (2.3) has the formal solution

$$u(\xi) = \sum_{i=0}^{n} a_i G^i, \tag{2.4}$$

where $G = G(\xi)$ satisfy the equation

$$G' + \lambda G = \mu G^2, \tag{2.5}$$

in which $a_i(-n \le i \le n; n \in N)$ are constants to be determined later, and $\lambda \neq 0$.

When $\mu \neq 0$, Eq. (2.5) is the type of Bernoulli equation, we can obtain the solution as

$$G = \frac{\lambda}{\mu + \lambda E \exp(\lambda\xi)}, \tag{2.6}$$

where $E$ is an arbitrary constant.

When $\mu = 0$, Eq. (2.6) reduces to

$$G = \frac{1}{E}\exp(-\lambda\xi). \tag{2.7}$$

Setting $E = \frac{\mu}{\lambda}$ into Eq. (2.6), we get

$$G = -\frac{\lambda}{2\mu}\left(\tanh\left(\frac{\lambda}{2}\xi\right) - 1\right). \tag{2.8}$$

Setting $E = -\frac{\mu}{\lambda}$ into Eq. (2.6), we get

$$G = -\frac{\lambda}{2\mu}\left(\coth\left(\frac{\lambda}{2}\xi\right) - 1\right). \tag{2.9}$$

Step 3. The positive integer $n$ can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (2.1) or Eq. (2.3). Moreover precisely, we define the degree of $u(\xi)$ as $D(u(\xi)) = n$ which gives rise to the degree of other expressions as follows:

$$D\left(\frac{d^q u}{d\xi^q}\right) = n + q, D\left(u^p\left(\frac{d^q u}{d\xi^q}\right)^s\right) = np + s(n+q). \tag{2.10}$$

Therefore, we can find the value of $n$ in Eq. (2.4), using Eq. (2.10).

Step 4. We substitute Eq. (2.4) into Eq. (2.3) using Eq. (2.5) and then collect all terms of the same powers of $G(\xi)$ together, then set each coefficient of them to zero to yield a system of algebraic equations, solving this system we obtain the values of $a_i$ and $\omega$.

Finally, substituting $a_i$, $\omega$ and Eqs. (2.8)–(2.9) into Eq. (2.4) we obtain the exact travelling wave solutions of Eq. (2.1).

## 3. Application

### 3.1. The (2+1)-dimensional Zoomeron equation

In this sub-section, we will exert the Bernoulli Sub-ODE method to solve the (2+1)-dimensional Zoomeron equation in the form,

$$\left(\frac{u_{xy}}{u}\right)_{tt} - \left(\frac{u_{xy}}{u}\right)_{xx} + 2\left(u^2\right)_{xt} = 0, \tag{3.1}$$

where $u(x,y,t)$ is the amplitude of the relative wave mode.

The travelling wave transformation equation $u(x,y,t) = u(\xi)$, $\xi = x - cy - \omega t$ transform Eq. (3.1) to the following ordinary differential equation:

$$\omega^2\left(\frac{-cu''}{u}\right)'' - \left(\frac{-cu''}{u}\right)'' - 2\omega\left(u^2\right)'' = 0. \tag{3.2}$$

Now integrating Eq. (3.2) with respect to $\xi$ twice, by setting the first integration constant equal to zero for convenience, we obtain the following nonlinear ordinary differential equation

$$c\left(1-\omega^2\right)u'' - 2\omega u^3 + Ru = 0, \tag{3.3}$$

where $R$ is a constant of integration.

#### 3.1.1. Phase plane analysis of the (2+1)-dimensional Zoomeron equation

To proceed with phase plane analysis for the zoomeron equation, we introduce $X = u, Y = X'$. Now we may re-write Eq. (3.3) as a first-order dynamical system of the form,

$$\begin{cases} \frac{dX}{d\xi} = Y, \\ \frac{dY}{d\xi} = \frac{1}{c(1-\omega^2)}\left(2\omega X^3 - RX\right), \end{cases} \tag{3.3a}$$

which defines the well-known phase plane associated with travelling wave solutions of the (2+1)-dimensional zoomeron equation.

The ordinary differential equation Eq. (3.3) or Eq. (3.3a) comes from the Hamiltonian

$$H(X,Y) = \frac{Y^2}{2} - \frac{1}{c\left(1-\omega^2\right)}\left(\frac{\omega}{2}X^4 - \frac{R}{2}X^2\right), \tag{3.3b}$$

by using Hamilton canonical equations $X' = \frac{\partial H}{\partial Y}$ and $Y' = -\frac{\partial H}{\partial X}$.

Three equilibrium points of the dynamical system (3.3a) are $\left(\pm\sqrt{\frac{R}{2\omega}}, 0\right)$ and $(0,0)$. For values of $c = 0.25, \omega = 0.25$ and $R = 1$, the equilibrium points $(0,0), (-1.4142, 0)$ and $(1.4142, 0)$ represent a circle,

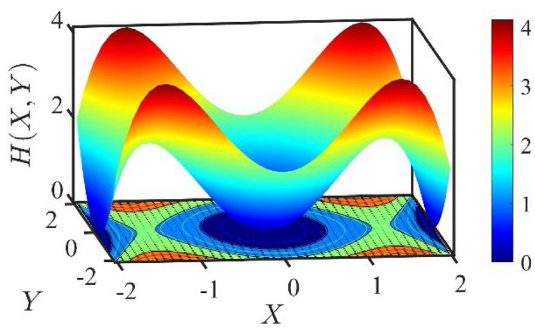


**Fig. 1a.** Hamiltonian function $H(X,Y)$ of the Zoomeron equation corresponds to Eq. (3.3b).

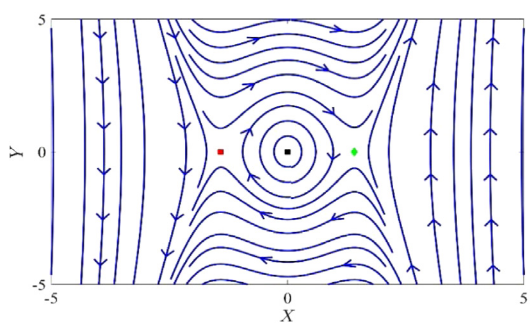


**Fig. 1b.** Phase plane visualization of the system of ODEs (3.3a) for the values of $c = 0.25, \omega = 0.25$ and $R = 1$. Three equilibrium points are $(0,0), (-1.4142, 0)$ and $(1.4142, 0)$.

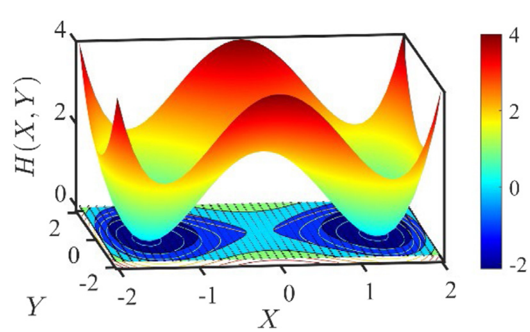


**Fig. 2a.** Hamiltonian function $H(X,Y)$ of the Zoomeron equation corresponds to Eq. (3.3b).

saddle point, and saddle point, respectively (See Fig. 1). On the other hand, for the values of $c = 0.25, \omega = -0.25$ and $R = -1$, the equilibrium points $(0,0), (-1.4142, 0)$ and $(1.4142, 0)$ represent saddle point, circle, and circle, respectively (See Fig. 2).

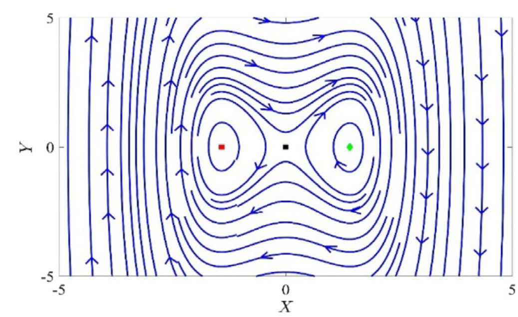


**Fig. 2b.** Phase plane visualization of the system of ODEs (3.3a) for the values of $c = 0.25, \omega = -0.25$ and $R = -1$. Three equilibrium points are $(0,0), (-1.4142, 0)$ and $(1.4142, 0)$.

### 3.1.2. Travelling wave analysis of the (2+1)-dimensional Zoomeron equation

Now, balancing the highest-order derivative term $u''$ and the nonlinear term $u^3$ from Eq. (3.3), yields $3n = n + 2$ which gives $n = 1$.

Hence for $n = 1$ Eq. (2.4) reduces to

$$u(\xi) = a_0 + a_1 G(\xi), \quad a_1 \neq 0. \tag{3.4}$$

Substitute Eq. (3.4) along with Eq. (2.5) into Eq. (3.3). As a result of this substitution, we get a polynomial of $(G(\xi))^j$. From these polynomials, we equate the coefficients of $(G(\xi))^j$ and setting them to zero, we get the following system of algebraic equations.

$$G^0 : \; -2\omega {a_0}^3 - R a_0 = 0.$$

$$G : \; -c\omega^2\lambda^2 a_1 - 6\omega {a_0}^2 a_1 - R a_1 + c a_1 \lambda^2 = 0.$$

$$G^2 : 3c\omega^2\mu\lambda a_1 - 6\omega a_0 {a_1}^2 - 3c\mu\lambda a_1 = 0.$$

$$G^3 : 2c\mu^2 a_1 - 2\omega {a_1}^3 - 2c\mu^2\omega^2 a_1 = 0.$$

Solving the above equations for $a_0, a_1$, $R$ and $\omega$, yields

$$c = -\frac{\omega a_1^2}{\mu^2(\omega^2 - 1)}, \; \lambda = \pm\frac{\mu}{a_1}\sqrt{\left(-\frac{2R}{\omega}\right)}, a_0 = I\sqrt{\frac{R}{2\omega}}.$$

Now substituting the values $R, c, \omega, a_0$ and $a_1$ into Eq. (3.4), along with Eq. (2.6), yields

$$u(\xi) = -\frac{1}{2}\frac{a_1\lambda}{\mu} + a_1\left(\frac{\lambda}{\mu + \lambda E \exp(\lambda\xi)}\right), \tag{3.5}$$

where $\xi = x - cy - \omega t$.

Now substituting the values $c, \lambda, a_0$ into Eq. (3.4), along with Eqs. (2.8) and (2.9), yields

$$u_1(x,y,t) = \pm i\sqrt{\frac{R}{2\omega}}\tanh\left(\frac{i\mu}{a_1}\sqrt{\frac{R}{2\omega}}(x - cy - \omega t)\right). \tag{3.6}$$

$$u_2(x,y,t) = \pm i\sqrt{\frac{R}{2\omega}}\coth\left(\frac{i\mu}{a_1}\sqrt{\frac{R}{2\omega}}(x - cy - \omega t)\right). \tag{3.7}$$

If $\frac{R}{2\omega} < 0$, then Eqs. (3.6) and (3.7) provide the following hyperbolic solutions:

$$u_3(x,y,t) = \pm\sqrt{\frac{R}{2\omega}}\tanh\left(\frac{\mu}{a_1}\sqrt{\frac{R}{2\omega}}(x - cy - \omega t)\right). \tag{3.6a}$$

$$u_4(x,y,t) = \pm\sqrt{\frac{R}{2\omega}}\coth\left(\frac{\mu}{a_1}\sqrt{\frac{R}{2\omega}}(x - cy - \omega t)\right). \tag{3.7a}$$

If $\frac{R}{2\omega} > 0$, then Eqs. (3.6) and (3.7) provide the following trigonometric solutions:

$$u_5(x,y,t) = \pm\sqrt{\frac{R}{2\omega}}\tan\left(\frac{\mu}{a_1}\sqrt{\frac{R}{2\omega}}(x - cy - \omega t)\right). \tag{3.6b}$$

$$u_6(x,y,t) = \pm\sqrt{\frac{R}{2\omega}}\cot\left(\frac{\mu}{a_1}\sqrt{\frac{R}{2\omega}}(x - cy - \omega t)\right). \tag{3.7b}$$

See Figs. 3 and 4.

### 3.2. The modified KdV–Zakharov–Kuznetsov equation

In this sub-section, we will exert the Bernoulli Sub-ODE method to find the exact solutions and then the solitary wave solutions to the modified KdV–Zakharov–Kuznetsov (mKdV–ZK) equation,

$$u_t + \alpha u^2 u_x + u_{xxx} + u_{xyy} + u_{xzz} = 0. \tag{3.8}$$

where $\alpha$ is a nonzero parameter.

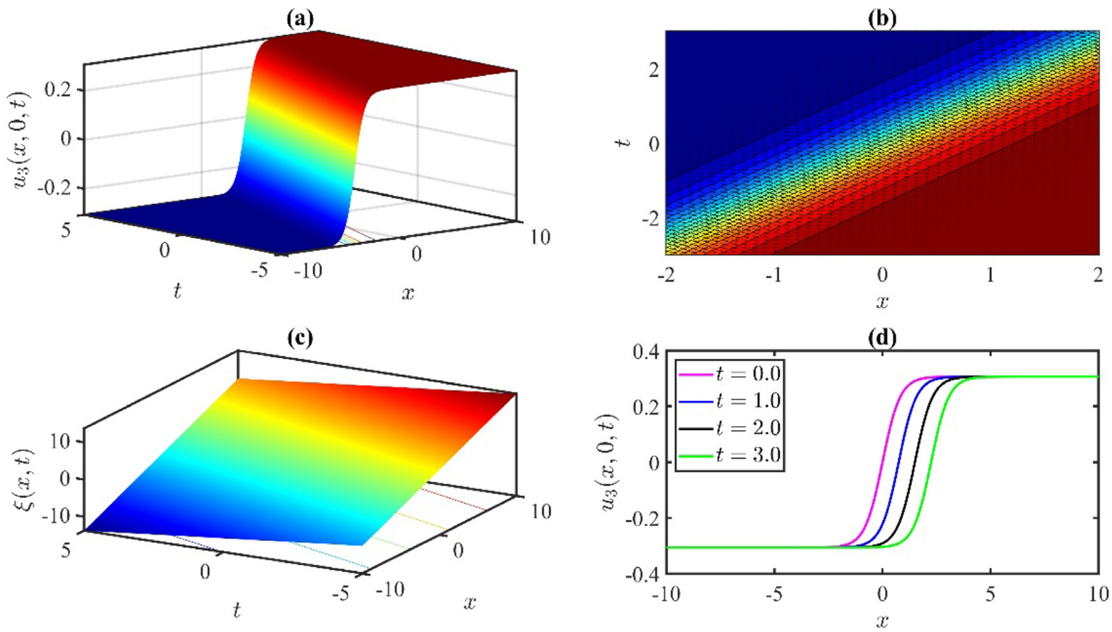


**Fig. 3.** The kink profile of the Zoomeron equation corresponds to the solution $u_3(x,0,t)$ in Eq. (3.6a) for the values of $\omega = 0.75$, $R = 0.25, a_1 = 0.5$, $\lambda = 1, \mu = 2, y = 0, c = 2$.

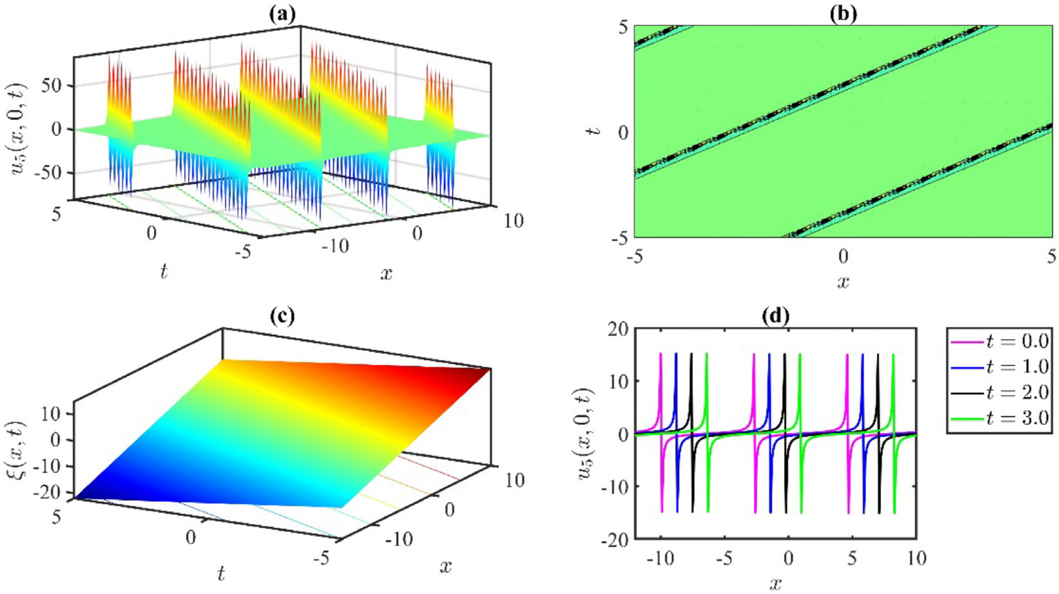


**Fig. 4.** The periodic profile of the Zoomeron equation corresponds to the solution $u_5(x,0,t)$ in Eq. (3.6b) for the values of $\omega = 1.2, R = 1.25$, $c = 1, a_1 = 1.5$, $2\mu = 2, y = 1$.

The travelling wave transformation,

$$u(x,y,z,t) = u(\xi), \qquad \xi = x + y + z - \omega t, \tag{3.9}$$

transforms Eq. (3.8) to the following ODE:

$$-\omega u' + \alpha u^2 u' + 3u''' = 0. \tag{3.10}$$

Eq. (3.10) is integrable, therefore, integrating with respect to $\xi$, setting the constant of integration to zero for convenience, we obtain

$$3u'' - \omega u + \frac{\alpha}{3}u^3 = 0. \tag{3.11}$$

### 3.2.1. Phase plane analysis of the mKdV-ZK equation

To proceed with phase plane analysis for the mKdV–ZKequation, we introduce $X = u, Y = X'$. Now we may re-write Eq. (3.11) as a first-order dynamical system of the form,

$$\begin{cases} \frac{dX}{d\xi} = Y, \\ \frac{dY}{d\xi} = \frac{\omega}{3}X - \frac{\alpha}{9}X^3, \end{cases} \tag{3.11a}$$

which defines the well-known phase plane associated with travelling wave solutions of the mKdV–ZKequation.

The ordinary differential equation Eq. (3.11) or Eq. (3.11a) comes from the Hamiltonian

$$H(X,Y) = \frac{Y^2}{2} - \frac{\omega}{6}X^2 + \frac{\alpha}{36}X^4, \tag{3.11b}$$

by using Hamilton canonical equations $X' = \frac{\partial H}{\partial Y}$ and $Y' = -\frac{\partial H}{\partial X}$.

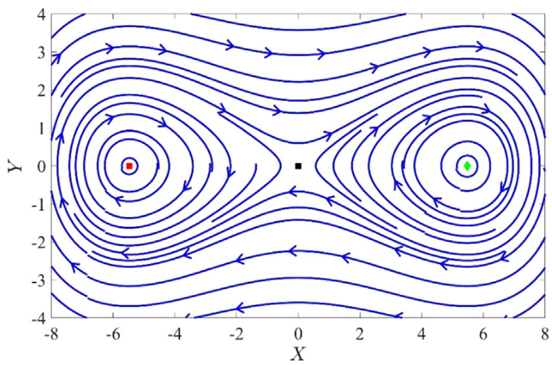


**Fig. 5.** Phase plane visualization of the system of ODEs (3.11a) for the values of $\alpha = 0.1, \omega = 1$. Three equilibrium points are $(0,0), (-5.4772, 0)$ and $(5.4772, 0)$.

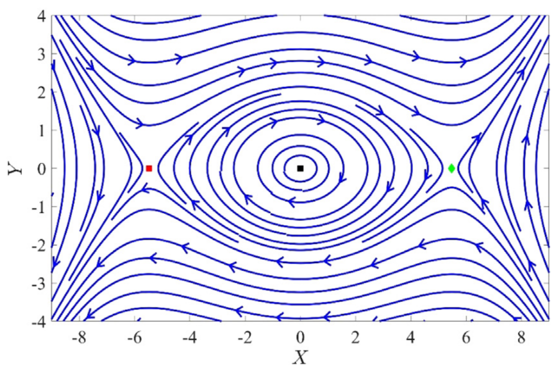


**Fig. 6.** Phase plane visualization of the system of ODEs (3.11a) for the values of $\alpha = -0.1, \omega = -1$. Three equilibrium points are $(0,0), (-5.4772, 0)$ and $(5.4772, 0)$.

Three equilibrium points of the dynamical system Eq. (3.11a) are $\left(\pm\sqrt{\frac{3\omega}{\alpha}}, 0\right)$ and $(0,0)$. For values of $\alpha = 0.1, \omega = 1$, the equilibrium points $(0,0), (-5.4772, 0)$ and $(5.4772, 0)$ represent a circle, saddle point, and circle, respectively (See Fig. 5). On the other hand, for the values of $\alpha = -0.1, \omega = -1$, the equilibrium points $(0,0), (-5.4772, 0)$ and $(5.4772, 0)$ represent saddle point, circle, and saddle point, respectively (See Fig. 6).

### 3.2.2. Solitary wave analysis of the mKdV-ZK equation

Balancing the highest-order derivative term $u''$ and the nonlinear term $u^3$ from Eq. (3.10), yields $3n = n + 2$ which gives $n = 1$.

As a result, the solution of Eq. (3.8) takes the form,

$$u(\xi) = a_0 + a_1 G(\xi), \ \ a_1 \neq 0. \tag{3.12}$$

Substitute Eq. (3.12) along with Eq. (2.5) into Eq. (3.11), and we get a polynomial in $G(\xi)$

$$\left(6\mu^2 a_1 + \frac{1}{3}\alpha {a_1}^3\right) G(\xi)^3 + \left(\alpha a_0 {a_1}^2 - 9\mu\lambda a_1\right) G(\xi)^2$$
$$+ \left(-\omega a_1 + 3a_1\lambda^2 + \alpha {a_0}^2 a_1\right) G(\xi) + \frac{1}{3}\alpha {a_0}^3 - \omega a_0 = 0.$$

From these polynomials, we equate the coefficients of $(G(\xi))^j$ and setting them to zero, we get the following system of algebraic equations.

$$G^0 : \frac{1}{3}\alpha {a_0}^3 - \omega a_0 = 0.$$
$$G : -\omega a_1 + 3a_1\lambda^2 + \alpha {a_0}^2 a_1 = 0.$$
$$G^2 : \alpha a_0 {a_1}^2 - 9\mu\lambda a_1 = 0.$$
$$G^3 : 6\mu^2 a_1 + \frac{1}{3}\alpha {a_1}^3 = 0.$$

Solving the above equations for $a_0, a_1$ and $\omega$, yields

$$\omega = -\frac{3}{2}\lambda^2, a_0 = \pm\frac{3\lambda}{\sqrt{-2\alpha}}, a_1 = \mp\frac{6\mu}{\sqrt{-2\alpha}}.$$

Now substituting the values $c, \omega, a_0$ and $a_1$ along with Eq. (2.6), yields

$$u(\xi) = \pm\frac{3\lambda}{\sqrt{-2\alpha}} \mp \frac{6\mu}{\sqrt{-2\alpha}}\left(\frac{\lambda}{\mu + \lambda E\exp(\lambda\xi)}\right), \tag{3.13}$$

where $\xi = x + y + z + \frac{3}{2}\lambda^2 t$.

Since $E$ is an arbitrary constant and $\mu, \lambda$ are free parameters, hence Setting $E = \frac{\mu}{\lambda}$ into Eq. (3.13), we obtain

$$u_1(x, y, z, t) = \pm\frac{3\lambda}{\sqrt{-2\alpha}}\tanh\left(\frac{\lambda}{2}(x + y + z + \frac{3}{2}\lambda^2 t)\right). \tag{3.14}$$

Again setting $E = -\frac{\mu}{\lambda}$ into Eq. (3.13), we obtain

$$u_2(x, y, z, t) = \pm\frac{3\lambda}{\sqrt{-2\alpha}}\coth\left(\frac{\lambda}{2}(x + y + z + \frac{3}{2}\lambda^2 t)\right). \tag{3.15}$$

See Figs. 7 and 8.

### 3.3. Time-regularized long-wave (TRLW) equation

Consider the TRLW Equation in the form:

$$u_t + u_x + \alpha u u_x + u_{xtt} = 0. \tag{3.16}$$

where $\alpha$ is a nonzero parameter.

The travelling wave transformation

$$u(x, t) = u(\xi), \qquad \xi = x - \omega t, \tag{3.17}$$

transforms the Eq. (3.16) to the following ODE:

$$(1 - \omega) u' + \alpha u u' + \omega^2 u''' = 0. \tag{3.18}$$

Eq. (3.18) is integrable, therefore, integrating with respect to $\xi$, neglecting the constant of integration, we obtain

$$\omega^2 u'' + (1 - \omega) u + \frac{\alpha}{2} u^2 = 0. \tag{3.19}$$

### 3.3.1. Phase plane analysis of the TRLW equation

To proceed with phase plane analysis for the TRLW equation, we introduce $X = u, Y = X'$. Now we may re-write Eq. (3.11) as a first-order dynamical system of the form,

$$\begin{cases} \frac{dX}{d\xi} = Y, \\ \frac{dY}{d\xi} = \frac{\omega - 1}{\omega^2} X - \frac{\alpha}{2\omega^2} X^2, \end{cases} \tag{3.19a}$$

which defines the well-known phase plane associated with travelling wave solutions of the TRLW equation.

The ordinary differential equation Eq. (3.11) or Eq. (3.11a) comes from the Hamiltonian

$$H(X, Y) = \frac{Y^2}{2} - \frac{\omega - 1}{\omega^2} X + \frac{\alpha}{2\omega^2} X^2, \tag{3.19b}$$

by using Hamilton canonical equations $X' = \frac{\partial H}{\partial Y}$ and $Y' = -\frac{\partial H}{\partial X}$.

Two equilibrium points of the dynamical system (3.19a) are $\left(\frac{2\omega(\omega - 1)}{\alpha}, 0\right)$ and $(0, 0)$.

For values of $\alpha = 0.5, \omega = 0.30$ two equilibrium points are $(0, 0)$, and $(-2.8, 0)$, respectively (See Fig. 9).

### 3.3.2. Solitary wave analysis of the TRLW equation

Balancing the linear term $u''$ and the nonlinear term $u^2$ from Eq. (3.18), yields $n = 2$.

As a result, the solution of Eq. (3.16) takes the form,

$$u(\xi) = a_0 + a_1 G(\xi) + a_2 (G(\xi))^2, \ a_2 \neq 0. \tag{3.20}$$

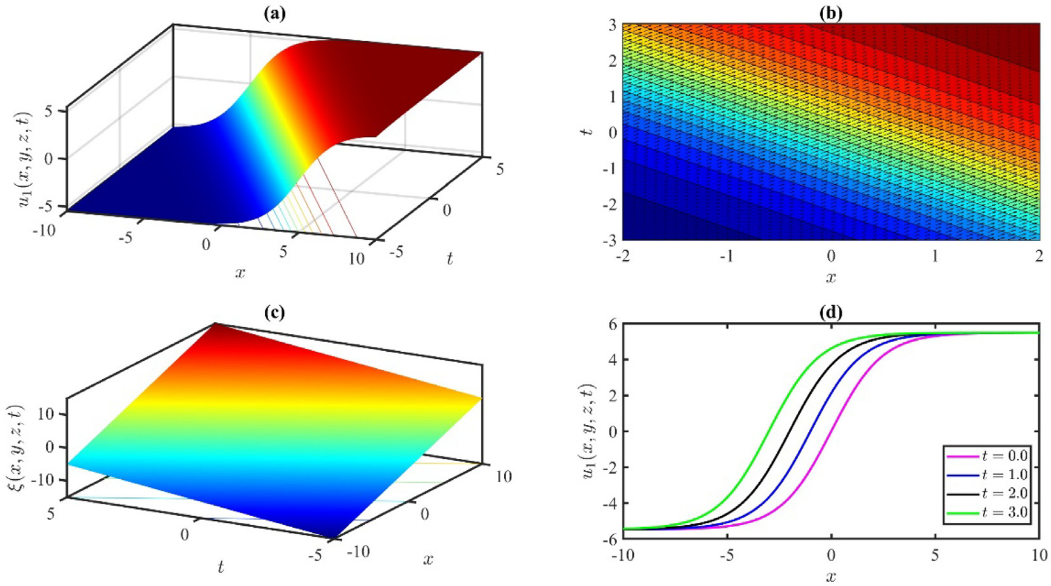


**Fig. 7.** The kink solution profile of the mKdV–ZK equation corresponds to the positive solution in Eq. (3.14) for the values of $y = z = 0, \alpha = -0.1, \lambda = \sqrt{\frac{2}{3}}$. Wave is propagating in the negative $x$-direction with constant speed $\omega = -1$.

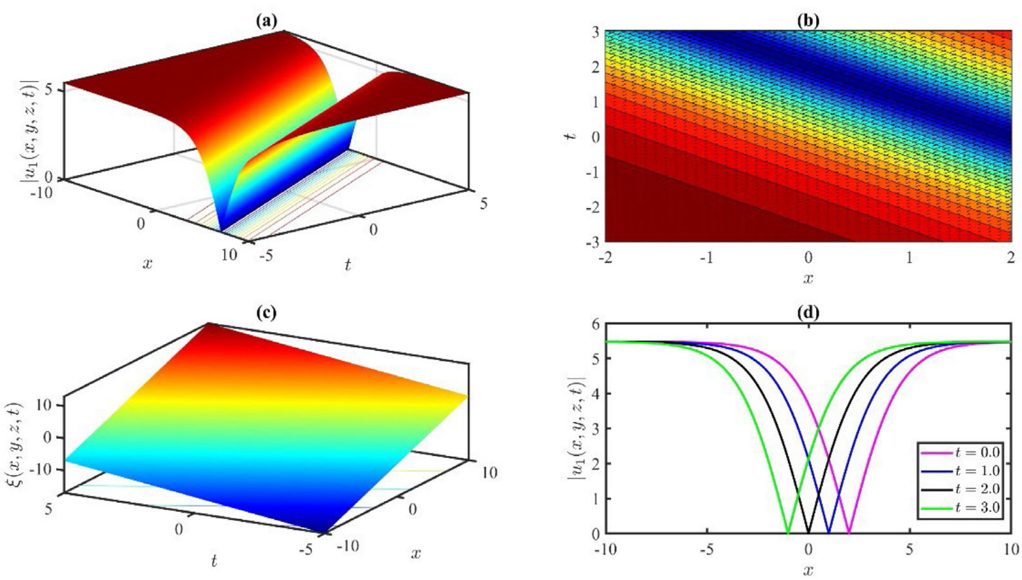


**Fig. 8.** Alphabetic-shaped soliton profile of the mKdV–ZK equation corresponds to the positive solution in Eq. (3.14) for the values of $y = 0, z = -2, \alpha = 0.1, \lambda = \sqrt{\frac{2}{3}}$. Wave is propagating in the negative $x$-direction with constant speed $\omega = -1$.

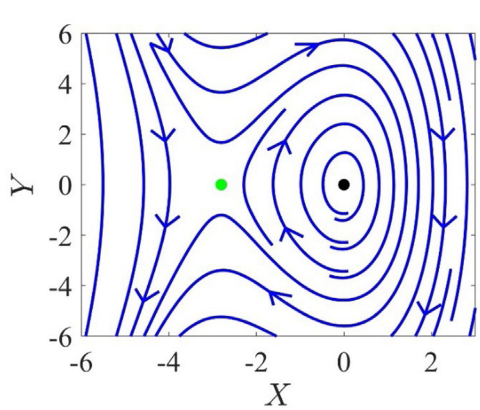


**Fig. 9.** Phase plane visualization of the system of ODEs (3.19a) for the values of $\alpha = 0.5, \omega = 0.30$. Three equilibrium points are $(0, 0)$, and $(-2.8, 0)$.

Substituting Eq. (3.20) along with Eq. (2.5) into Eq. (3.19), we get the following polynomial

$$\left(6\omega^2 a_2 \mu^2 + \frac{1}{2}\alpha {a_2}^2\right) G(\xi)^4 + \left(-10\omega^2 a_2 \mu\lambda + 2\omega^2 a_1 \mu^2 + \alpha a_1 a_2\right) G(\xi)^3$$
$$+ \left(-\omega a_2 + a_2 - 3\omega^2 a_1 \mu\lambda + \frac{1}{2}\alpha {a_1}^2 + \alpha a_0 a_2 + 4\omega^2 a_2 \lambda^2\right) G(\xi)^2$$
$$+ \left(a_1 + \alpha a_0 a_1 + \omega^2 a_1 \lambda^2 - \omega a_1\right) G(\xi) + \frac{1}{2}\alpha {a_0}^2 + a_0 - \omega a_0 = 0. \tag{3.21}$$

From Eq. (3.21), we equate the coefficients of $(G(\xi))^j$ and setting them to zero, we obtain the following system of algebraic equations.

$G^0 : \frac{1}{2}\alpha {a_0}^2 + a_0 - \omega a_0 = 0.$

$G : a_1 + \alpha a_0 a_1 + \omega^2 a_1 \lambda^2 - \omega a_1 = 0.$

$G^2 : -\omega a_2 + a_2 - 3\omega^2 a_1 \mu\lambda + \frac{1}{2}\alpha {a_1}^2 + \alpha a_0 a_2 + 4\omega^2 a_2 \lambda^2 = 0.$

$G^3 : -10\omega^2 a_2 \mu\lambda + 2\omega^2 a_1 \mu^2 + \alpha a_1 a_2 = 0.$

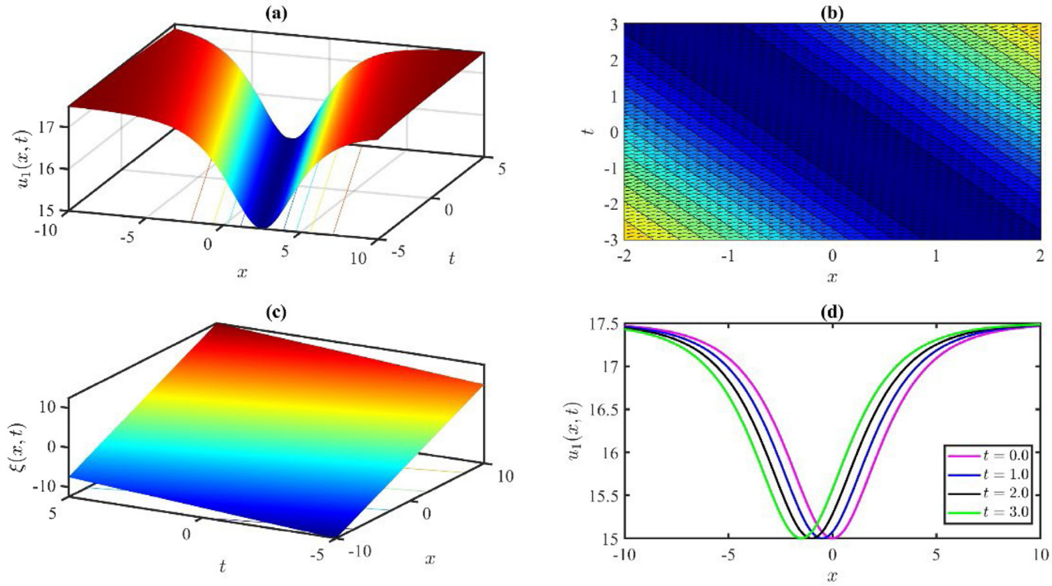


**Fig. 10.** The dark soliton profile of the TRLW equation corresponds to the solution in Eq. (3.23) for the values of $y = 0, z = -2, \omega = -0.5, \alpha = 0.1, \lambda = 1$. Wave is propagating in the negative $x$-direction with constant speed $\omega = -1$.

$G^4 : 6\omega^2 a_2 \mu^2 + \frac{1}{2}\alpha a_2{}^2 = 0.$

Solving the above equations for $a_0, a_1, a_2$ and $\omega$, yields

$$\omega = \omega, a_0 = -\left(\frac{\omega^2\lambda^2 - \omega + 1}{\alpha}\right), a_1 = \frac{12\omega^2\mu\lambda}{\alpha}, a_2 = -\frac{12\omega^2\mu^2}{\alpha}.$$

Now substituting the values $\omega, \lambda, \mu, a_0$, $a_1$ and $a_2$ into Eq. (3.20), along with Eq. (2.6), yields

$$u(\xi) = -\left(\frac{\omega^2\lambda^2 - \omega + 1}{\alpha}\right) + \frac{12\omega^2\mu\lambda}{\alpha}\left(\frac{\lambda}{\mu + \lambda E\exp(\lambda\xi)}\right) - -\frac{12\omega^2\mu^2}{\alpha}\left(\frac{\lambda}{\mu + \lambda E\exp(\lambda\xi)}\right)^2, \quad (3.22)$$

where $\xi = x - \omega t$.

According to the parallel course of action discussed in Sections 3.1 and 3.2, setting $E = \frac{\mu}{\lambda}$ into Eq. (3.22), we obtain

$$u(x,t) = -\left(\frac{1-\omega}{\alpha}\right) - \frac{\omega^2\lambda^2}{\alpha}\left(1 - 3\mathrm{sech}^2\left(\frac{\lambda}{2}(x - \omega t)\right)\right). \quad (3.23)$$

Again setting $E = -\frac{\mu}{\lambda}$ into Eq. (3.22), we obtain

$$u(x,t) = -\left(\frac{1-\omega}{\alpha}\right) - \frac{\omega^2\lambda^2}{\alpha}\left(1 + 3\operatorname{csc}h^2\left(\frac{\lambda}{2}(x - \omega t)\right)\right). \quad (3.24)$$

See Fig. 10.

## 4. Results and discussion

### 4.1. Physical explanations of the Zoomeron equation

We will now examine the impact of free parameters on the kink/shock type solitary wave profile that corresponds to the solution (3.6a) of the Zoomeron equation. We will exclude the physical explanations of the mKdV–ZK and TRLW models for convenience.

Figs. 11–14 show the diversification of Kink-type solitary wave profiles because of the influence of the free parameters. Figs. 11 and 12 show the variation of the wave profile without changing the amplitude. In Fig. 11, Kink wave profiles are depicted for a set of values of $\mu = \{0.1, 0.25, 0.5, 1, 2, 3\}$ and $\omega = 0.75$, $R = 0.25, c = 2, a_1 = 0.5$, $y = 0$. In Fig. 12, Kink wave profiles are portrayed for a set of values of $a_1 = \{0.25, 0.5, 1, 2, 3\}$ and $\omega = 0.75$, $R = 0.25, c = 2, \mu = 1, y = 0$. Snapshots are taken at time $t = 4$.

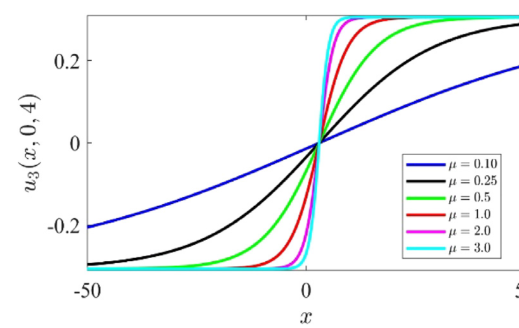


**Fig. 11.** The kink wave profile of the Zoomeron equation corresponds to Eq. (3.6a).

Figs. 13 and 14 represent the variation of the kink wave amplitude due to the parametric effect. The numerical analysis represented in Figs. 13 and 14, suggests that wave amplitude increases with the increase of the parameter $R$ and wave speed $\omega$. This completely characterizes the effect of parameters on wave amplitude. Fig. 13 illustrates the amplitude variation of the Kink wave for the values of $R = \{0.25, 0.5, 1, 2, 3\}$ and $\mu = 1$, $\omega = 1, c = 2, a_1 = 2$, $y = 0$. The numerical simulations presented in Fig. 14 have demonstrated the amplitude variation of the Kink wave for the values of wave speed $\omega = \{0.25, 0.5, 1, 2, 3\}$ and $\mu = 1$, $R = 0.25, c = 2, a_1 = 0.5$, $y = 0$. Snapshots are taken at time $t = 4$.

### 4.2. Comparisons

**Comparison with the modified simple equation**: Using the modified simple equation (MSE) approach, Khan and Akbar[18] investigated the (2+1)-dimensional Zoomeron equation and discovered four solutions (**Please see the Appendix**). On the other hand, in this article, we also have found four solutions to the (2+1)-dimensional Zoomeron equation using the Bernoulli Sub-ODE approach. The fact that all the solutions produced here using the Bernoulli Sub-ODE approach correspond with known solutions derived by Khan and Akbar[18] for specific values of the parameters is noteworthy. If we set $c = 1$, and $a_1 = \mu\sqrt{\left(\frac{\omega^2-1}{\omega}\right)}$ in our solutions Eqs. (3.6a) to (3.7b) for the Zoomeron equation, then our solutions coincide with the solutions obtained by

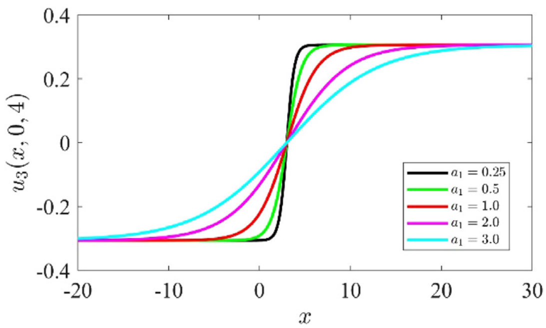


**Fig. 12.** The kink wave profile of the Zoomeron equation corresponds to Eq. (3.6a).

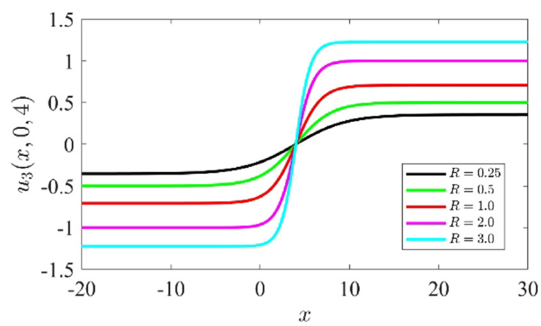


**Fig. 13.** The kink wave profile of the Zoomeron equation corresponds to Eq. (3.6a).

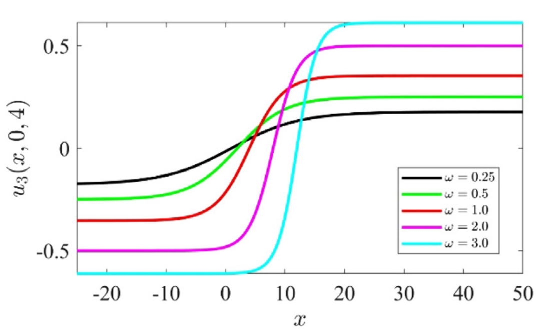


**Fig. 14.** The kink wave profile of the Zoomeron equation corresponds to Eq. (3.6a).

Khan and Akbar.[18] Analogously, we can compare the results of the other two models which are omitted for convenience.

## 5. Conclusion

The modified KdV–Zakharov–Kuznetsov (mKDV–ZK) equation, the TRLW equation, and the (2+1)-dimensional Zoomeron equation have all been solved in this study using the Bernoulli Sub-ODE approach. Consequently, accurate forms of solitary waves, including kink waves, singular kink waves, and dark soliton are discovered. The Hamiltonian function is determined, and the phase plane is analysed briefly. We ensured the correctness of the solutions with the help of Maple 13 by reintroducing them into the original equation. We thoroughly demonstrated that the wave profile varies as the parameters change. Our numerical simulation demonstrates that changing a parameter can have a variety of impacts, including a change in amplitude that affects the dynamics of solitary waves. The directness, effectiveness, and applicability of the Bernoulli Sub-ODE approach for solving several additional NLEEs in mathematical physics and engineering are demonstrated. The Bernoulli Sub-ODE approach has a larger range of applications due to its consistency.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

## Appendix

Khan and Akbar[18] found the following solution of the (2+1)-dimensional Zoomeron equation by using the MSE method:

$$u_{1,2}(x,y,t) = \pm\sqrt{\frac{R}{2\omega}}\tanh\left(\sqrt{\frac{R}{2(\omega^2-1)}}(x-cy-\omega t)\right). \tag{A.1}$$

$$u_{3,4}(x,y,t) = \pm\sqrt{\frac{R}{2\omega}}\coth\left(\sqrt{\frac{R}{2(\omega^2-1)}}(x-cy-\omega t)\right). \tag{A.2}$$

$$u_{5,6}(x,y,t) = \pm\sqrt{\frac{R}{2\omega}}\tan\left(\sqrt{\frac{R}{2(\omega^2-1)}}(x-cy-\omega t)\right). \tag{A.3}$$

$$u_{7,8}(x,y,t) = \pm\sqrt{\frac{R}{2\omega}}\cot\left(\sqrt{\frac{R}{2(\omega^2-1)}}(x-cy-\omega t)\right). \tag{A.4}$$

## References


1. Wang M, Li X, Zhang J. The $(G'/G)$-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Phys Lett A*. 2008;372(4):417–423.
2. Akbar MA, Ali NHM, Mohyud-Din ST. The alternative $(G'/G)$-expansion method with generalized Riccati equation: Application to fifth order (1+1)-dimensional Caudrey–Dodd–Gibbon equation. *Int J Phys Sci*. 2012;7(5):743–752.
3. Akbar MA, Ali NHM. The alternative $(G'/G)$-expansion method and its applications to nonlinear partial differential equations. *Int J Phys Sci*. 2011;6(35):7910–7920.
4. Kol1 GR, Tabi CB. Application of the $(G'/G)$-expansion method to nonlinear blood flow in large vessels. *Phys Scr*. 2011;83(4):045803.
5. Aslan I. Analytic solutions to nonlinear differential-difference equations by means of the extended $(G'/G)$-expansion method. *J Phys A*. 2010;43(39):395207.
6. Zhang S, Tong J, Wang W. A generalized $(G'/G)$ -expansion method for the mKdV equation with variable coeffcients. *Phys Lett A*. 2008;372(13):2254–2257.
7. Zhang J, Wei X, Lu Y. A generalized $(G'/G)$-expansion method and its applications. *Phys Lett A*. 2008;372(20):3653–3658.
8. Hirota R. Exact envelope soliton solutions of a nonlinear wave equation. *J Math Phys*. 1973;14(7):805–810.
9. Hirota R, Satsuma J. Soliton solutions of a coupled KDV equation. *Phys Lett A*. 1981;85(8–9):407–408.
10. Ma WX, Yong XL, Lü X. Soliton solutions to the B-type Kadomtsev–Petviashvili equation under general dispersion relations. *Wave Motion*. 2021;103:102719.
11. Ma WX. $N$-soliton solution and the Hirota condition of a (2+1)-dimensional combined equation. *Math Comput Simulation*. 2021;190:270–279.
12. Ma WX. $N$-soliton solution of a combined pKP-BKP equation. *J Geom Phys*. 2021;165:104191.
13. Ma WX. $N$-soliton solutions and the Hirota conditions in (2+1)-dimensions. *Opt Quantum Electron*. 2020;52(12):511.
14. Ma WX. Soliton solutions by means of Hirota bilinear forms. *Partial Differ Equ Appl Math*. 2022;5:100220.
15. Shi KZ, Ren B. Dynamics of mixed lump-soliton solutions to the (3+1)-dimensional Boiti–Leon–Manna–Pempinelli like equation. *Partial Differ Equ Appl Math*. 2022;5:100276.
16. Jawad AJM, Petkovic MD, Biswas A. Modified simple equation method for nonlinear evolution equations. *Appl Math Comput*. 2010;217(2):869–877.
17. Khan K, Akbar MA. Exact and solitary wave solutions for the Tzitzeica–Dodd–Bullough and the modified KdV–Zakharov–Kuznetsov equations using the modified simple equation method. *Ain Shams Eng J*. 2013;4(4):903–909.
18. Khan K, Akbar MA. Traveling wave solutions of the (2+1)-dimensional Zoomeron equation and the Burgers equations via the MSE method and the exp-function method. *Ain Shams Eng J*. 2014;5(1):247–256.
19. Wazwaz AM. The tanh-function method: Solitons and periodic solutions for the Dodd- bullough-mikhailov and the Tzitzeica–Dodd–Bullough equations. *Chaos Solitons Fractals*. 2005;25(1):55–63.

20. He JH, Wu XH. Exp-function method for nonlinear wave equations. *Chaos Solitons Fractals*. 2006;30(3):700–708.
21. Akbar MA, Ali NHM. Exp-function method for Duffing equation and new solutions of (2+1) dimensional dispersive long wave equations. *Prog Appl Math*. 2011;1(2):30–42.
22. Bekir A, Boz A. Exact solutions for nonlinear evolution equations using exp-function method. *Phys Lett A*. 2008;372(10):1619–1625.
23. Ali AT. New generalized Jacobi elliptic function rational expansion method. *J Comput Appl Math*. 2011;235(14):4117–4127.
24. Mohiud-Din ST. Homotopy perturbation method for solving fourth-order boundary value problems. *Math Probl Eng*. 2007;2007:98602.
25. Mohyud-Din ST, Noor MA. Homotopy perturbation method for solving partial differential equations. *Z Naturforsch A*. 2009;64a(3–4):157–170.
26. Mohyud-Din ST, Yildirim A, Sariaydin S. Numerical soliton solutions of the improved Boussinesq equation. *Int J Numer Method H*. 2011;21(7):822–827.
27. Iqbal M, Seadawy AR, Lu D. Construction of solitary wave solutions to the nonlinear modified Kortewege–de Vries dynamical equation in unmagnetized plasma via mathematical methods. *Mod Phys Lett B*. 2018;33(32):1850183.
28. Iqbal M, Seadawy AR, Lu D. Dispersive solitary wave solutions of nonlinear further modified Korteweg–de Vries dynamical equation in an unmagnetized dusty plasma. *Mod Phys Lett B*. 2018;33(37):1850217.
29. Iqbal M, Seadawy AR, Lu D. Applications of nonlinear longitudinal wave equation in a magneto-electro-elastic circular rod and new solitary wave solutions. *Mod Phys Lett B*. 2019;33(18):1950210.
30. Seadawy AR, Iqbal M. Propagation of the nonlinear damped Korteweg–de Vries equation in an unmagnetized collisional dusty plasma via analytical mathematical methods. *Math Methods Appl Sci*. 2021;44(1):737–748.
31. Calogero F, Degasperis A. *Solitons and Spectral Transform I*. Amsterdam: North-Holland; 1982.
32. Seadawy AR, Iqbal M, Lu D. Applications of propagation of long-wave with dissipation and dispersion in nonlinear media via solitary wave solutions of generalized Kadomtsev–Petviashvili modified equal width dynamical equation. *Comput Math Appl*. 2019;78(11):3620–3632.
33. Morris RM, Leach PGL. Symmetry reductions and solutions to the Zoomeron equation. *Phys Scr*. 2014;90(1):015202.
34. Sahoo S, Garai G, Saha Ray S. Lie symmetry analysis for similarity reduction and exact solutions of modified KdV–Zakharov–Kuznetsov equation. *Nonlinear Dynam*. 2017;87(3):1995–2000.
35. Calogero F, Degasperis A. Coupled nonlinear evolution equations solvable via the inverse spectral transform. *Lett Nuovo Cimento*. 1976;16(14):425–433.
36. Calogero F, Degasperi A. Baecklund transformation nonlinear superposition principle multisoliton solutions and conserved quantities for the 'boomeron' nonlinear evolution equation. *Lett Nuovo Cimento*. 1976;16(14):434–438.
37. Verheest F, Mace RL, Pillay SR, Hellberg MA. Unified derivation of Korteweg–de Vries–Zakharov–Kuznetsov equations in multispecies plasmas. *J Phys A: Math Gen*. 2002;35(3):795–806.
38. Lazarus I.J. Bharuthram R, Hellberg MA. Modified Korteweg–de Vries–Zakharov–Kuznetsov solitons in symmetric two-temperature electron–positron plasmas. *J Plasma Phys*. 2008;74(4):519–529.
39. Khan K, Akbar MA. Solving unsteady Korteweg–de Vries equation and its two alternatives. *Math Methods Appl Sci*. 2016;39(10):2752–2760.
40. Joseph RI, Egri R. Another possible model equation for long waves in nonlinear dispersive systems. *Phys Lett A*. 1977;61:429–432.
41. Jeffrey A. Nonlinear wave propagation. *Appl Math Mech*. 1978;58:T38–T56.
42. Feng QH, Liu TB. Traveling wave solution of (2+1) dimensional Boussinesq equation by Bernoulli sub-ODE method. *Adv Mater Res*. 2012;403:202–206.
43. Zheng B. Soling a nonlinear evolution equation by a proposed Bernoulli sub-ODE method. *Int Conf Image Vis Comput*. 2012;50:226–230.
44. Zheng B. Application of a generalized Bernoulli sub-ODE method for finding traveling solutions of some nonlinear equations. *WSEAS Trans Math*. 2012;7(11):618–626.
45. Zheng B. A new Bernoulli sub-ODE method for constructing traveling wave solutions for two nonlinear equations with any order. *UPB Sci Bull Ser A*. 2011;73(3):85–94.
46. Zheng B. Traveling wave solutions for some nonlinear differential equations by a generalized sub-ODE method. *Math Comp*. 2012;1(1):20–25.
47. Mirzazadeh M, Alqahtani RT, Biswas A. Optical soliton perturbation with quadratic-cubic nonlinearity by Riccati–Bernoulli sub-ODE method and Kudryashov's scheme. *Optik*. 2017;145:74–78.