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Research article

The Fisher-KPP nonlocal diffusion equation with free boundary and radial symmetry in $\mathbb{R}^3$†

Yihong Du* **and Wenjie Ni**

School of Science and Technology, University of New England, Armidale, NSW 2351, Australia

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* **Correspondence:** Email: ydu@une.edu.au.

Abstract: This paper is concerned with the radially symmetric Fisher-KPP nonlocal diffusion equation with free boundary in dimension 3. For arbitrary dimension $N \geq 2$, in [18], we have shown that its long-time dynamics is characterised by a spreading-vanishing dichotomy; moreover, we have found a threshold condition on the kernel function that governs the onset of accelerated spreading, and determined the spreading speed when it is finite. In a more recent work [19], we have obtained sharp estimates of the spreading rate when the kernel function $J(|x|)$ behaves like $|x|^{-\beta}$ as $|x| \to \infty$ in $\mathbb{R}^N$ ($N \geq 2$). In this paper, we obtain more accurate estimates for the spreading rate when $N = 3$, which employs the fact that the formulas relating the involved kernel functions in the proofs of [19] become particularly simple in dimension 3.

Keywords: nonlocal diffusion; free boundary; spreading rate

Dedicated to Professor Neil Trudinger on the occasion of his 80th birthday.

1. Introduction

We consider a radially symmetric Fisher-KPP nonlocal diffusion equation with free boundary in $\mathbb{R}^N$ ($N \geq 2$) of the form

$$\begin{cases} u_t = d\displaystyle\int_{B_{h(t)}} J(|x-y|)u(t,|y|)\mathrm{d}y - du(t,|x|) + f(u), & t>0,\ x\in B_{h(t)},\\ u(t,|x|) = 0, & t>0,\ x\in\partial B_{h(t)},\\ h'(t) = \dfrac{\mu}{|\partial B_{h(t)}|}\displaystyle\int_{B_{h(t)}}\int_{\mathbb{R}^N\setminus B_{h(t)}} J(|x-y|)u(t,|x|)\mathrm{d}y\mathrm{d}x, & t>0,\\ h(0)=h_0,\ u(0,|x|)=u_0(|x|), & x\in\overline{B}_{h_0}, \end{cases} \tag{1.1}$$

where $B_{h(t)} := \{x\in\mathbb{R}^N : |x| < h(t)\}$, with $h(t)$ an unknown function to be determined with the density function $u(t,|x|)$.

The basic assumptions on the kernel function $J(|x|)$ are

(J): $J\in C(\mathbb{R}_+)\cap L^\infty(\mathbb{R}_+)$ is nonnegative, $J(0)>0$, $\displaystyle\int_{\mathbb{R}^N} J(|x|)\mathrm{d}x = 1.$

Here and throughout the paper, $\mathbb{R}_+ = [0,\infty)$.

The nonlinear function f is of Fisher-KPP type, namely, it satisfies

$$\textbf{(f)}: \begin{cases} f \text{ is } C^1,\ f(0)=0<f'(0), \text{ there exists } u^*>0 \text{ such that}\\ f(u^*)=0>f'(u^*) \text{ and } (u^*-u)f(u)>0 \text{ for } u\in(0,\infty)\setminus\{u^*\},\\ f(u)/u \text{ is non-increasing for } u>0. \end{cases}$$

The initial function u_0 is required to satisfy

$$u_0\in C(\overline{B}_{h_0}) \text{ is radially symmetric, } u_0=0 \text{ on } \partial B_{h_0} \text{ and } u_0>0 \text{ in } B_{h_0}. \tag{1.2}$$

For $r := |x|$ with $x\in\mathbb{R}^N$ and $\rho>0$, denote

$$\tilde{J}(r,\rho) = \tilde{J}(|x|,\rho) := \int_{\partial B_\rho} J(|x-y|)\mathrm{d}S_y.$$

Then (1.1) can be rewritten into the equivalent form

$$\begin{cases} u_t(t,r) = d\displaystyle\int_0^{h(t)} \tilde{J}(r,\rho)u(t,\rho)\mathrm{d}\rho - du(t,r) + f(u), & t>0,\ r\in[0,h(t)),\\ u(t,h(t)) = 0, & t>0,\\ h'(t) = \dfrac{\mu}{h^{N-1}(t)}\displaystyle\int_0^{h(t)}\int_{h(t)}^{+\infty} \tilde{J}(r,\rho)r^{N-1}u(t,r)\mathrm{d}\rho\mathrm{d}r, & t>0,\\ h(0)=h_0,\ u(0,r)=u_0(r), & r\in[0,h_0]. \end{cases} \tag{1.3}$$

(Here a universal constant is absorbed by μ.)

Problem (1.1) may be used to model the spreading of a new or invasive species, whose population density is given by $u(t,|x|)$, and whose population range is the evolving ball $B_{h(t)}$, where the spatial dispersal of the species is assumed to obey a nonlocal diffusion law governed by the kernel function J. The one dimensional case of (1.3) was studied in [7,14,17] (see also [8] for the case $f(u)\equiv 0$).

In Du and Ni [18], we have shown that problem (1.1), or equivalently (1.3), admits a unique positive solution (u,h) defined for all $t>0$. Moreover, the long-time dynamical behaviour of (1.1) follows a spreading-vanishing dichotomy:

Proposition 1.1 (Spreading-vanishing dichotomy [18])**.** *Suppose* **(J)**, **(f)** *and* (1.2) *are satisfied. Let* (u, h) *be the solution of* (1.1)*. Then one of the following alternatives must occur* :

(i) **(Spreading)** $\lim_{t\to\infty} h(t) = \infty$ *and*

$$\lim_{t\to\infty} u(t, |x|) = u^* \text{ locally uniformly in } \mathbb{R}^N,$$

(ii) **(Vanishing)** $\lim_{t\to\infty} h(t) = h_\infty < \infty$ *and*

$$\lim_{t\to\infty} u(t, |x|) = 0 \text{ uniformly for } x \in B_{h(t)}.$$

Apart from giving the precise criteria which govern the spreading-vanishing dichotomy, the spreading speed is also determined in [18], which depends on the function J_* given by

$$J_*(l) := \int_{\mathbb{R}^{N-1}} J(|(l, x')|)dx', \ l \in \mathbb{R}, \tag{1.4}$$

where $x' = (x_2, ..., x_N) \in \mathbb{R}^{N-1}$.

It is easy to see that **(J)** implies

$$\begin{cases} J_* \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \text{ is nonnegative, even, } J_*(0) > 0, \\ \displaystyle\int_{\mathbb{R}} J_*(l)\mathrm{d}l = \int_{\mathbb{R}^N} J(|x|)\mathrm{d}x = 1. \end{cases}$$

Moreover, it was shown in [18] that

$$J_*(l) = \omega_{N-1} \int_{|l|}^{\infty} J(r)r(r^2 - l^2)^{(N-3)/2}dr, \tag{1.5}$$

where ω_k denotes the area of the unit sphere in $\mathbb{R}^k$,

$$\tilde{J}(r, \rho) = \omega_{N-1} 2^{3-N} \frac{\rho}{r^{N-2}} \int_{|\rho-r|}^{\rho+r} \Big(\big[(\rho + r)^2 - \eta^2\big] [\eta^2 - (\rho - r)^2] \Big)^{\frac{N-3}{2}} \eta J(\eta)\mathrm{d}\eta \quad \forall r, \ \rho > 0, \tag{1.6}$$

and

$$\int_0^\infty J_*(l)ldl = \frac{\omega_{N-1}}{N-1} \int_0^\infty J(r)r^N dr. \tag{1.7}$$

The threshold condition for (1.1) to have a finite spreading speed is

(J1): $\displaystyle\int_0^\infty J(r)r^N dr < +\infty.$

By [14, Theorem 1.2] and (1.7), we have the following conclusions about the associated one-dimensional semi-wave problem.

Proposition 1.2 (Semi-wave [14])**.** *Suppose* **(J)** *and* **(f)** *hold. Then the following equations*

$$\begin{cases} d \displaystyle\int_{-\infty}^{0} J_*(x - y)\phi(y)\mathrm{d}y - d\phi(x) + c\phi'(x) + f(\phi(x)) = 0, \quad x < 0, \\ \phi(-\infty) = u^*, \ \ \phi(0) = 0, \\ c = \mu \displaystyle\int_{-\infty}^{0} \int_0^\infty J_*(x - y)\phi(x)\mathrm{d}y\mathrm{d}x, \end{cases}$$

admit a solution pair (c, ϕ) *with* $c > 0$ *and* $\phi' \le 0$ *if and only if* **(J1)** *is satisfied. Moreover, when* **(J1)** *holds, the solution pair is unique, which we denote by* (c_0, ϕ_0)*, and it has the property that* $\phi_0'(x) < 0$ *for* $x \le 0$.

The result on the spreading speed in [18] is the following:

Proposition 1.3 (Spreading speed [18])**.** *Assume the conditions in Proposition 1.1 are satisfied, and spreading happens to* (1.1)*. Then*

$$\lim_{t\to\infty} \frac{h(t)}{t} = \begin{cases} c_0 & \text{if } \textbf{(J1)} \text{ is satisfied,} \\ \infty & \text{if } \textbf{(J1)} \text{ is not satisfied,} \end{cases}$$

where c_0 *is given by Proposition 1.2.*

More accurate estimates than that in Proposition 1.3 have been obtained in Du and Ni [19] for a natural class of kernel functions, namely those satisfying

$$J(|x|) \sim |x|^{-\beta} \text{ for } |x| \gg 1 \text{ in } \mathbb{R}^N. \tag{1.8}$$

It is easily seen that $\beta \in (N, N+1]$ is the exact range of β such that **(J1)** is not satisfied by such kernels while **(J)** holds. Therefore for such β accelerated spreading may happen according to Proposition 1.3. The following result on the precise rate of spreading is proved in [19].

Proposition 1.4 (Rate of accelerated spreading [19])**.** *Suppose the conditions in Proposition 1.1 hold, and the kernel function satisfies* (1.8) *with* $\beta \in (N, N+1]$*. If spreading happens, then for* $t \gg 1$,

$$\begin{cases} h(t) \sim t^{1/(\beta-N)} & \text{if } \beta \in (N, N+1), \\ h(t) \sim t \ln t & \text{if } \beta = N+1. \end{cases}$$

Recall that $\xi(s) \sim \eta(s)$ means $c_1\eta(s) \le \xi(s) \le c_2\eta(s)$ for some positive constants c_1, c_2 and all s in the specified range.

If $\beta > N+1$, then condition **(J1)** is automatically satisfied, and so by Proposition 1.3, the spreading has a finite speed c_0. The following result of [19] describes how $c_0t - h(t)$ behaves as $t \to \infty$, where a slightly more general class of kernel functions than (1.8) is allowed, namely one only requires

$$J(|x|) = O(|x|^{-\beta}) \text{ for } |x| \gg 1 \text{ in } \mathbb{R}^N. \tag{1.9}$$

Proposition 1.5 (Rate of shift [19])**.** *Suppose the conditions in Proposition 1.1 hold, and moreover* f *is* C^2 *and* J *satisfies* (1.9) *with* $\beta > N+1$*. If spreading happens, then for* $t \gg 1$,

$$\begin{cases} c_0t - h(t) \sim \ln t & \text{if } \beta > N+2, \\ |c_0t - h(t)| = O(\ln t) & \text{if } \beta = N+2, \\ |c_0t - h(t)| = O(t^{N+2-\beta}) & \text{if } \beta \in (N+1, N+2). \end{cases}$$

The purpose of this paper is to give a more accurate description of the spreading behaviour described in Proposition 1.5 when $N = 3$.

Let us note that when $N = 3$, (1.5) and (1.6) are reduced to, respectively,

$$J_*(l) = \omega_2 \int_{|l|}^{\infty} rJ(r)dr, \tag{1.10}$$

and

$$\tilde{J}(r,\rho) = \omega_2 \frac{\rho}{r} \int_{|\rho-r|}^{\rho+r} \eta J(\eta)\mathrm{d}\eta \quad \forall r,\ \rho > 0. \tag{1.11}$$

These allow considerable simplifications in the estimates of [19], and enable us to obtain more precise spreading rate when $N = 3$. The following theorem is the main result of this paper.

Theorem 1.6. *Suppose the conditions in Proposition 1.1 hold, J satisfies* (1.8)*, f is C^2 and*

$$[f(u)/u]' < 0 \text{ for } u > 0. \tag{1.12}$$

If spreading happens and $N = 3$, then for $t \gg 1$,

$$\begin{cases} c_0 t - h(t) \sim \ln t & \text{if } \beta = N+2 = 5, \\ c_0 t - h(t) \sim t^{N+2-\beta} = t^{5-\beta} & \text{if } \beta \in (N+1, N+2) = (4,5), \end{cases}$$

The above results in Propositions 1.4, 1.5 and Theorem 1.6 reveal a striking difference of the behaviour of (1.1) from the pattern exhibited in the corresponding one dimension case, when β crosses the value $N + 2$. More precisely, when $\beta > N + 2$, which guarantees finite speed spreading, Proposition 1.5 shows that logarithmic shifting occurs, while in dimension one, no such shifting happens for this kind of J according to [17]. When $\beta \in (N+1, N+2]$, where finite speed spreading still holds, Proposition 1.5 and Theorem 1.6 exhibit similar shifting behaviour to the $N = 1$ case in [17]. When $\beta \in (N, N + 1]$ (which is the exact range of β that accelerated spreading may happen with such kernel functions), Proposition 1.4 gives the exact rate of the accelerated spreading, which is again in agreement with the pattern observed in the case $N = 1$ in [17].

Let us now comment on the difficulty in treating the high dimensional radially symmetric problem (1.3) (for a general $N \geq 2$). To obtain sharp estimates for the spreading profile, the main difficulty arises from the fact that the kernel function in (1.3) is given by

$$\tilde{J}(r,\rho) = \tilde{J}(|x|,\rho) := \int_{\partial B_\rho} J(|x-y|)\mathrm{d}S_y,$$

which inherits the properties of the original kernel function $J(|x|)$ in a rather implicit way; see (1.6). On the other hand, the kernel function which determines the spreading speed of (1.3) is given by J_* in (1.5). Therefore the spreading behaviour of (1.3) involves the complicated interplays between J, $\tilde{J}$ and J_*, among other things. Note that in dimension 1, $J = J_*$ and $\tilde{J}$ is not needed.

Many of the difficulties here do not occur in the local diffusion counterpart of (1.1), which was examined in [12, 16]. It follows from [12] that the long-time dynamics of the local diffusion problem is roughly the same as that for the one dimension case considered in [15], and when spreading happens, $\lim_{t\to\infty} h(t)/t = c_*$ for some $c_* > 0$ determined by the semi-wave problem associated to the one dimensional model. So accelerated spreading never happens to the local diffusion problem. Moreover, by [16], there exists another constant $\hat{c} > 0$ independent of the dimension N such that

$$\lim_{t\to\infty} [h(t) - c_* t + (N-1)\hat{c} \ln t] = C$$

for some constant C depending on the initial function u_0.

It was shown in [13] that when $\mu \to \infty$, the limiting problem of the local diffusion version of (1.1) is the corresponding Cauchy problem

$$\begin{cases} u_t = d\Delta u + f(u) & \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{R}^N, \end{cases} \tag{1.13}$$

which, since the pioneering works of Fisher [22] and Kolmogorov, Peterovski and Piskunov [24], has been extended and used to describe the propagation phenomena arising from invasion ecology and other problems. Similarly, the argument in [14] for the one dimension case can be easily extended to show that when $\mu \to \infty$, the limiting problem of (1.1) is the nonlocal Cauchy problem

$$\begin{cases} u_t = d\left[\int_{\mathbb{R}^N} J(|x-y|)u(t,y)dy - u(t,x)\right] + f(u) & \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{R}^N. \end{cases} \tag{1.14}$$

As a nonlocal extension of (1.13), problem (1.14) and its numerous variations have been extensively studied in the last three decades (see, e.g., [1–5, 9, 10, 20, 21, 23, 25–27, 29–31] and the references therein). If f satisfies **(f)**, then the long-time behaviour of (1.14) with a compactly supported initial function u_0 is roughly the same as (1.13), namely

$$\lim_{t\to\infty} u(t,x) = u^* \text{ locally uniformly for } x \in \mathbb{R}^N, \tag{1.15}$$

where u^* is the unique positive zero of $f(u)$ given in **(f)**. A striking difference between (1.14) and (1.13) arises in the spreading speed, where accelerated spreading can happen to (1.14) when the kernel function J is fat-tailed (see, e.g., [5, 23] for space dimension 1), while (1.13) always spreads with a finite speed, determined by the minimal speed of its traveling wave solutions.

For fractional Laplacian type nonlocal diffusion operators in any dimension $N \geq 1$, it was shown in [6, 28] that the rate of accelerated spreading is given by $e^{[c+o(1)]t}$ for some $c > 0$ depending on N and the fractional Laplacian. It should be noted that our basic condition **(J)** here is not satisfied by the corresponding kernel function of the fractional Laplacian $(-\Delta)^s$, which is given by

$$J(|x|) = |x|^{-(N+2s)} \ (0 < s < 1).$$

It would be interesting to see what happens to (1.1) if the kernel function J is allowed to behave like the kernel function of the fractional Laplacian. A related work with $f \equiv 0$ can be found in [11].

Note that as a population model, (1.1) provides additional information. For example, it gives the precise spreading front of the species via the free boundaries, while (1.14) does not, since its solution $u(t,x)$ is positive for all $x \in \mathbb{R}^N$ once $t > 0$; moreover, (1.14) predicts consistent success of spreading (see (1.15)), but the long-time dynamics of (1.1) is governed by a spreading-vanishing dichotomy, which seems more natural.

The rest of the paper is organised as follows. In Section 2, for convenience of the reader, we collect several results from previous works, which will be used later in the paper. Section 3 is devoted to the proof of Theorem 1.6, by constructing subtle upper and lower solutions, based on careful estimates involving the connections of $\tilde{J}$ and J_*.

2. Some preparations

In this section, we recall some basic facts from [17, 18] for convenience of later use in the paper. Here, only $N \geq 2$ is required.

Lemma 2.1 (Maximum principle [18]). *Let $T > 0$, $d > 0$, and $g, h \in C([0,T])$ satisfy $g(0) \leq h(0)$ and $g(t) < h(t)$ for $t \in (0,T]$. Denote $D_T := \{(t,x) : t \in (0,T],\ g(t) < x < h(t)\}$ and suppose that ϕ, $\phi_t \in C(\overline{D}_T)$, $c \in L^\infty(D_T)$, and*

$$\begin{cases} \phi_t \geq d \displaystyle\int_{g(t)}^{h(t)} P(x,y)\phi(t,y)\mathrm{d}y + c(t,x)\phi, & (t,x) \in D_T, \\ \phi(t, g(t)) \geq 0, & t \in \Sigma^g_{\min}, \\ \phi(t, h(t)) \geq 0, & t \in \Sigma^h_{\max}, \\ \phi(0,x) \geq 0, & x \in [g(0), h(0)], \end{cases}$$

where

$$\begin{cases} \Sigma^g_{\min} = \{t \in (0,T] : \text{There exists } \epsilon > 0 \text{ such that } g(t) < g(s) \text{ for } s \in [t-\epsilon, t)\}, \\ \Sigma^h_{\max} = \{t \in (0,T] : \text{There exists } \epsilon > 0 \text{ such that } h(t) > h(s) \text{ for } s \in [t-\epsilon, t)\}, \end{cases}$$

and the kernel function P satisfies

$$P \in C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2),\ P \geq 0,\ P(x,x) > 0 \text{ for almost all } x \in \mathbb{R}.$$

Then $\phi \geq 0$ on $\overline{D}_T$, and if additionally $\phi(0,x) \not\equiv 0$ in $[g(0), h(0)]$, then $\phi > 0$ in D_T.

Lemma 2.2 (Comparison principle [18]). *Suppose* (**J**) *and* (**f**) *hold, and (u,h) solves* (1.3) *for $t \in [0,T]$ with some $T > 0$. For convenience we extend u by $u(t,r) = 0$ for $t \in [0,T]$ and $r > h(t)$. Let $r_*, h_* \in C([0,T])$ be nondecreasing functions satisfying $0 \leq r_*(t) < h_*(t)$, and*

$$\Omega_T := \{(t,r) : t \in (0,T], r \in (0, h_*(t))\},\ \Theta_T := \{(t,r) : t \in (0,T], r \in (r_*(t), h_*(t))\}.$$

Suppose $v \in C(\overline{\Omega}_T)$ is nonnegative with $v_t \in C(\overline{\Theta}_T)$, and

$$\hat{v}(t,r) := \begin{cases} u(t,r) \text{ for } r \in [0, r_*(t)],\ t \in [0,T], \\ v(t,r) \text{ for } r \in (r_*(t), h_*(t)],\ t \in [0,T]. \end{cases}$$

(i) *If (v, r_*, h_*) satisfy $h_*(0) \geq h(0)$,*

$$\begin{cases} v(0,r) \geq u(0,r), & r \in [0, h_*(0)], \\ v(t,r) \geq u(t,r), & t \in [0,T],\ r \in [0, r_*(t)] \end{cases}$$

and

$$\begin{cases} v_t \geq d\Big[\displaystyle\int_0^{h_*(t)} \tilde{J}(r,\rho)\hat{v}(t,\rho)\mathrm{d}\rho - v(t,r)\Big] + f(v), & t \in (0,T],\ r \in (r_*(t), h_*(t)),\\ v(t, h_*(t)) \geq 0, & t \in (0,T],\\ h_*'(t) \geq \dfrac{\mu}{h_*^{N-1}(t)} \displaystyle\int_0^{h_*(t)} \int_{h_*(t)}^{+\infty} \tilde{J}(r,\rho) r^{N-1} v(t,r)\mathrm{d}\rho\mathrm{d}r, & t \in [0,T], \end{cases}$$

then

$$h_*(t) \geq h(t),\ \ v(t,r) \geq u(t,r) \quad \text{for } t \in (0,T],\ r \in [0, h(t)].$$

(ii) *If* (v, r_*, h_*) *satisfy* $h_*(0) \leq h(0)$*,*

$$\begin{cases} v(0,r) \leq u(0,r), & r \in [0, h(0)],\\ v(t,r) \leq u(t,r), & t \in [0,T],\ r \in [0, r_*(t)] \end{cases}$$

and

$$\begin{cases} v_t \leq d\Big[\displaystyle\int_0^{h_*(t)} \tilde{J}(r,\rho)\hat{v}(t,\rho)\mathrm{d}\rho - v(t,r)\Big] + f(v), & t \in (0,T],\ r \in (r_*(t), h_*(t)),\\ v(t, h_*(t)) \leq 0, & t \in (0,T],\\ h_*'(t) \leq \dfrac{\mu}{h_*^{N-1}(t)} \displaystyle\int_0^{h_*(t)} \int_{h_*(t)}^{+\infty} \tilde{J}(r,\rho) r^{N-1} v(t,r)\mathrm{d}\rho\mathrm{d}r, & t \in [0,T], \end{cases}$$

then

$$h_*(t) \leq h(t),\ \ v(t,r) \leq u(t,r) \ \text{ for } t \in (0,T],\ r \in [0, h_*(t)].$$

Remark 2.3. *In Lemma 2.2, if* $r_*(t) \equiv 0$*, then the conclusions hold without requiring*

$$\begin{cases} v(t,r) \geq u(t,r) \textit{ for } t \in [0,T],\ r \in [0, r_*(t)] = \{0\} \textit{ in part } (i),\\ v(t,r) \leq u(t,r) \textit{ for } t \in [0,T],\ r \in [0, r_*(t)] = \{0\} \textit{ in part } (ii). \end{cases}$$

Proof. When $r_*(t) \equiv 0$, $\Sigma_{\min}^{r_*} = \emptyset$, and the conclusion follows directly from the simple proof of Lemma 2.2 in [18] when Lemma 2.1 is used for w over $t \in [t_2, t_1]$ and $r \in [r_*(t), h_\epsilon(t)]$. □

Lemma 2.4 (Behaviour of semi-waves [17])**.** *Let* $\alpha > 0$ *be a constant. Suppose that* f *satisfies* (**f**) *and the kernel function satisfies*

$$\int_0^\infty J_*(r) r^\alpha dr < +\infty \quad \text{for some } \alpha \geq 0, \tag{2.1}$$

and $\phi(x)$ *is a monotone solution of*

$$\begin{cases} d\displaystyle\int_{-\infty}^0 J_*(x-y)\phi(y)\mathrm{d}y - d\phi(x) + c\phi'(x) + f(\phi(x)) = 0, & x < 0,\\ \phi(-\infty) = u^*,\ \ \phi(0) = 0 \end{cases}$$

for some $c > 0$*. Then the following conclusions hold:*

(i) *If* (2.1) *holds for some* $\alpha > 0$, *then*

$$\int_{-\infty}^{-1} [u^* - \phi(x)]|x|^{\alpha-1} dx < \infty,$$

which implies, by the monotonicity of $\phi(x)$,

$$0 < u^* - \phi(x) \leq C|x|^{-\alpha} \text{ for some } C > 0 \text{ and all } x < 0.$$

(ii) *If* (2.1) *does not hold for some* $\alpha > 0$, *then*

$$\int_{-\infty}^{-1} [u^* - \phi(x)]|x|^{\alpha-1} dx = \infty.$$

3. Proof of Theorem 1.6

In this section, we prove Theorem 1.6. So we always assume that $N = 3$, **(J)**, **(f)** and (1.12) hold, and there exist some constants $0 < C_1 \leq C_2$ and $\beta > N + 1 = 4$ such that

$$\frac{C_1}{|x|^\beta} \leq J(x) \leq \frac{C_2}{|x|^\beta} \text{ for } |x| \geq 1 \text{ in } \mathbb{R}^3. \tag{3.1}$$

Theorem 1.6 clearly is a consequence of the following result.

Proposition 3.1. *If spreading happens to the solution* $(u(t, r), h(t))$ *of* (1.3), *then for* $t \gg 1$,

$$\begin{cases} h(t) - c_0 t \sim -t^{5-\beta} & \text{if } \beta \in (4, 5), \\ h(t) - c_0 t \sim -\ln t & \text{if } \beta \geq 5, \end{cases} \tag{3.2}$$

where $c_0 > 0$ *is given by Proposition* 1.2.

By Proposition 1.5, there is $C = C(\beta) > 0$ such that

$$\begin{cases} h(t) - c_0 t \geq -Ct^{5-\beta} & \text{if } \beta \in (4, 5), \\ h(t) - c_0 t \geq -C \ln t & \text{if } \beta \geq 5. \end{cases}$$

Hence, to prove (3.2), it suffices to obtain the desired upper bound for $h(t) - c_0 t$, which will be carried out in subsections 3.1 and 3.2, according to whether $d > f'(0)$ or $d \leq f'(0)$. The proof of the latter case is more involved, and is a modification of the proof of the former.

3.1. Upper bound of $h(t) - c_0 t$ when $d > f'(0)$

Lemma 3.2. *Let the conditions in Proposition 3.1 be satisfied. If* $d > f'(0)$, *then there is* $\widetilde{C} = \widetilde{C}(\beta) > 0$ *such that for large* t,

$$\begin{cases} h(t) - c_0 t \leq -\widetilde{C} t^{5-\beta}, & \text{if } \beta \in (4, 5), \\ h(t) - c_0 t \leq -\widetilde{C} \ln t, & \text{if } \beta \geq 5. \end{cases} \tag{3.3}$$

Proof. Let $\alpha := \min\{1, \beta - 4\} \in (0, 1]$, and (c_0, ϕ_0) be the semi-wave pair in Proposition 1.2. Define

$$\epsilon(t) := K_1(t+\theta)^{-\alpha}, \quad \delta(t) := -K_2 \int_0^t \epsilon(\tau) \mathrm{d}\tau$$

and

$$\begin{cases} \bar{h}(t) := c_0(t+\theta) + \delta(t), & t \geq 0, \\ \overline{u}(t,r) := (1+\epsilon(t))\phi_0(r - \bar{h}(t)) + \omega(t,r), & t \geq 0, \ r \leq [0, \bar{h}(t)], \end{cases}$$

where

$$\omega(t,r) := K_3 \xi(r - \overline{h}(t)) \epsilon(t),$$

with $\xi \in C^2(\mathbb{R})$ satisfying

$$0 \leq \xi(r) \leq 1, \quad \xi(r) = 1 \text{ for } |r| < \tilde{\epsilon}, \ \xi(r) = 0 \text{ for } |r| > 2\tilde{\epsilon}, \tag{3.4}$$

and the positive constants $\theta, K_1, K_2, K_3, \tilde{\epsilon}$ are to be determined.

Next we choose suitable θ, K_1, K_2, K_3 and $t_0 > 0$ such that $(\bar{u}, \bar{h})$ satisfies

$$\begin{cases} \bar{u}_t(t,r) \geq d \displaystyle\int_0^{\bar{h}(t)} \tilde{J}(r,\rho) \bar{u}(t,\rho) \mathrm{d}\rho - d\bar{u}(t,r) + f(\bar{u}(t,r)), & t > 0, \ r \in (\bar{h}(t)/2, \bar{h}(t)), \\ \bar{h}'(t) \geq \dfrac{\mu}{\bar{h}^2(t)} \displaystyle\int_0^{\bar{h}(t)} r^2 \bar{u}(t,r) \int_{\bar{h}(t)}^{+\infty} \tilde{J}(r,\rho) \mathrm{d}\rho \mathrm{d}r, & t > 0, \\ \bar{u}(t,r) \geq u(t+t_0, r), \ \bar{u}(t, \bar{h}(t)) = 0, & t > 0, \ r \in [0, \bar{h}(t)/2], \\ \bar{u}(0,r) \geq u(t_0, r), \ h(t_0) \leq \bar{h}(0), & r \in [0, h(t_0)]. \end{cases} \tag{3.5}$$

If (3.5) is proved, then we can use Lemma 2.2 to obtain

$$\begin{cases} h(t+t_0) \leq \bar{h}(t) & \text{for } t \geq 0, \\ u(t+t_0, r) \leq \overline{u}(t,r) & \text{for } t \geq 0, \ r \in [0, h(t+t_0)], \end{cases}$$

which yields (3.3).

It remains to show (3.5), which will be carried out in three steps.

Step 1. We varify the second inequality of (3.5).

From (1.10) and (1.11) we see

$$\tilde{J}(r,\rho) \leq \frac{\rho}{r} J_*(r-\rho) \ \text{ for } \ r, \rho > 0. \tag{3.6}$$

A direct computation gives

$$\begin{aligned} &\frac{\mu}{\bar{h}^2(t)} \int_0^{\bar{h}(t)} r^2 \bar{u}(t,r) \int_{\bar{h}(t)}^{+\infty} \tilde{J}(r,\rho) \mathrm{d}\rho \mathrm{d}r \\ &= \frac{\mu}{\bar{h}^2} \int_0^{\bar{h}} (1+\epsilon) r^2 [\phi_0(r-\bar{h}) + w(t,r)] \int_{\bar{h}}^{+\infty} \tilde{J}(r,\rho) \mathrm{d}\rho \mathrm{d}r \\ &\leq \frac{\mu(1+\epsilon)}{\bar{h}^2} \int_0^{\bar{h}} r[\phi_0(r-\bar{h}) + K_3\epsilon] \int_{\bar{h}}^{+\infty} \rho J_*(r-\rho) \mathrm{d}\rho \mathrm{d}r \end{aligned}$$

$$
\begin{aligned}
&= \frac{\mu(1+\epsilon)}{\bar{h}^2}\int_{-\bar{h}}^{0}(r+\bar{h})[\phi_0(r)+K_3\epsilon]\int_0^{+\infty}(\rho+\bar{h})J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r\\
&= \mu(1+\epsilon)\left[\int_{-\bar{h}}^{0}\int_0^{+\infty}[\phi_0(r)+K_3\epsilon]J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r + A\right]
\end{aligned}
$$

with

$$
A := \int_{-\bar{h}}^{0}\int_0^{+\infty}\left[\Big(1+\frac{r}{\bar{h}}\Big)\Big(1+\frac{\rho}{\bar{h}}\Big)-1\right][\phi_0(r)+K_3\epsilon]J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r.
$$

Moreover,

$$
\begin{aligned}
&\mu\int_{-\bar{h}}^{0}\int_0^{+\infty}[\phi_0(r)+K_3\epsilon]J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r\\
&= c_0 + K_3\epsilon\mu\int_{-\infty}^{0}\int_0^{+\infty}J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r - \mu\int_{-\infty}^{-\bar{h}}\int_0^{+\infty}[\phi_0(r)+K_3\epsilon]J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r\\
&= c_0 + K_3\epsilon\mu C_J - \mu\int_{-\infty}^{-\bar{h}}\int_0^{+\infty}[\phi_0(r)+K_3\epsilon]J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r,
\end{aligned}
$$

where

$$
C_J := \int_{-\infty}^{0}\int_0^{+\infty}J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r = \int_0^{\infty}\rho J_*(\rho)\mathrm{d}\rho. \tag{3.7}
$$

Here we have used change of integration order to obtain the last identity, and used (3.1) to conclude that C_J is finite.

Claim 1. There exists $C_3 > 0$ such that for all $t \geq 0$,

$$
-\int_{-\infty}^{-\bar{h}}\int_0^{+\infty}[\phi_0(r)+K_3\epsilon]J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r + A \leq -C_3\big[(t+\theta)^{-1}+(t+\theta)^{4-\beta}\big].
$$

Since $\phi(r)$ is non-increasing for $r \leq 0$, we have

$$
\begin{aligned}
&\int_{-\infty}^{-\bar{h}}\int_0^{+\infty}[\phi_0(r)+K_3\epsilon]J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r\\
&\geq [\phi_0(-\bar{h})+K_3\epsilon]\int_{-\infty}^{-\bar{h}}\int_0^{+\infty}J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r = [\phi_0(-\bar{h})+K_3\epsilon]\int_{-\infty}^{-\bar{h}}\int_{-r}^{+\infty}J_*(\rho)\mathrm{d}\rho\mathrm{d}r\\
&= [\phi_0(-\bar{h})+K_3\epsilon]\int_{\bar{h}}^{\infty}\int_{-\rho}^{-\bar{h}}J_*(\rho)\mathrm{d}r\mathrm{d}\rho = [\phi_0(-\bar{h})+K_3\epsilon]\int_{\bar{h}}^{\infty}(\rho-\bar{h})J_*(\rho)\mathrm{d}\rho
\end{aligned} \tag{3.8}
$$

and

$$
\begin{aligned}
A &= \int_{-\bar{h}}^{0}\int_0^{+\infty}\Big(\frac{r\rho}{\bar{h}^2}+\frac{r+\rho}{\bar{h}}\Big)[\phi_0(r)+K_3\epsilon]J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r\\
&\leq \int_{-\bar{h}}^{0}\int_0^{+\infty}\frac{r\rho}{\bar{h}^2}[\phi_0(r)+K_3\epsilon]J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r\\
&\quad + \int_{-\bar{h}}^{0}\int_0^{\bar{h}}\frac{r+\rho}{\bar{h}}[\phi_0(r)+K_3\epsilon]J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r
\end{aligned}
$$

$$
\begin{aligned}
&+[\phi_0(-\bar{h})+K_3\epsilon]\int_{-\bar{h}}^{0}\int_{\bar{h}}^{+\infty}\frac{r+\rho}{\bar{h}}J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r\\
&=: W_1(\bar{h})+\frac{W_2(\bar{h})}{\bar{h}}+W_3(\bar{h}).
\end{aligned}
$$

Since $\bar{h}(t)\geq \bar{h}(0)=c_0\theta$ and $\phi_0(-\infty)=u^*$, for $\theta\gg 1$ we have $\phi_0(-\bar{h}/2)\geq u_*/2$. Using this, (3.1) and $\bar{h}(t)\leq c_0(t+\theta)$, we obtain

$$
\begin{aligned}
W_1(\bar{h}) &= \int_{-\bar{h}}^{0}\int_{0}^{+\infty}\frac{r\rho}{\bar{h}^2}[\phi_0(r)+K_3\epsilon]J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r\leq \int_{-\bar{h}}^{0}\int_{0}^{+\infty}\frac{r\rho}{\bar{h}^2}\phi_0(r)J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r\\
&= \int_{-\bar{h}}^{0}\int_{-r}^{+\infty}\frac{r(\rho+r)}{\bar{h}^2}\phi_0(r)J_*(\rho)\mathrm{d}\rho\mathrm{d}r\leq \int_{-\bar{h}}^{-\bar{h}/2}\int_{2\bar{h}}^{+\infty}\frac{r(\rho+r)}{\bar{h}^2}\phi_0(r)J_*(\rho)\mathrm{d}\rho\mathrm{d}r\\
&\leq \frac{u_*}{2}\int_{-\bar{h}}^{-\bar{h}/2}\int_{2\bar{h}}^{+\infty}\frac{r\rho}{2\bar{h}^2}J_*(\rho)\mathrm{d}\rho\mathrm{d}r=-\frac{3u_*}{32}\int_{2\bar{h}}^{+\infty}\rho J_*(\rho)\mathrm{d}\rho\\
&=-\frac{3u_*}{32}\int_{2\bar{h}}^{+\infty}\rho\int_{\rho}^{\infty}\eta J(\eta)\mathrm{d}\eta\mathrm{d}\rho\leq -\frac{3u_*}{32}\int_{2\bar{h}}^{+\infty}\rho\int_{\rho}^{\infty}C_1\eta^{-\beta+1}\mathrm{d}\eta\mathrm{d}\rho\\
&=-\frac{3u_*C_1}{2^{1+\beta}(\beta-2)(\beta-4)}\bar{h}^{4-\beta}\leq -\frac{3u_*C_1c_0^{4-\beta}}{2^{1+\beta}(\beta-2)(\beta-4)}(t+\theta)^{4-\beta}.
\end{aligned}
$$

To estimate $W_2(\bar{h})$, we first prove the following claim:
Claim 2. For any constants $k,h>0$,

$$
B(h):=\int_{-h}^{0}\int_{0}^{h}(r+\rho)[\phi_0(r)+k]J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r<0,\ B'(h)\leq 0.
$$

Since J_* is even, we have

$$
\begin{aligned}
B(h) &= \int_{-h}^{0}\int_{0}^{h}(r+\rho)[\phi_0(r)+k]J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r\\
&= \int_{0}^{h}\int_{0}^{h}(\rho-r)[\phi_0(-r)+k]J_*(\rho+r)d\rho dr\\
&= \int_{0}^{h}\int_{0}^{r}(\rho-r)[\phi_0(-r)+k]J_*(\rho+r)d\rho dr+\int_{0}^{h}\int_{r}^{h}(\rho-r)[\phi_0(-r)+k]J_*(\rho+r)d\rho dr\\
&= \int_{0}^{h}\int_{\rho}^{h}(\rho-r)[\phi_0(-r)+k]J_*(\rho+r)drd\rho+\int_{0}^{h}\int_{r}^{h}(\rho-r)[\phi_0(-r)+k]J_*(\rho+r)d\rho dr\\
&= \int_{0}^{h}\int_{r}^{h}(\rho-r)[\phi_0(-r)-\phi_0(-\rho)]J_*(\rho+r)d\rho dr<0,
\end{aligned}
$$

where we have used $r\to\phi_0(-r)$ is strictly increasing and $J_*(0)>0$.

Using the first identity for $B(h)$ above, we obtain

$$
B'(h)=\int_{0}^{h}(\rho-h)[\phi_0(-h)+k]J_*(h+\rho)\mathrm{d}\rho+\int_{-h}^{0}(r+h)[\phi_0(r)+k]J_*(r-h)\mathrm{d}r
$$

$$= \int_0^h (h-\rho)[\phi_0(-\rho)-\phi_0(-h)]J_*(h+\rho)\mathrm{d}\rho \le 0.$$

Claim 2 is thus proved.

Using Claim 2, we have, due to $\bar{h}(t) \ge c_0\theta \gg 1$,

$$\begin{aligned} W_2(\bar{h}) &\le \int_{-1}^0 \int_0^1 (r+\rho)[\phi_0(r)+K_3\epsilon]J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r \\ &= \int_{-1}^0 \int_0^1 (r+\rho)\phi_0(r)J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r =: -\tilde{C}_3 < 0. \end{aligned}$$

By change of order of integration and (3.8), we have

$$\begin{aligned} W_3(\bar{h}) &= [\phi_0(-\bar{h})+K_3\epsilon]\int_{-\bar{h}}^0 \int_{\bar{h}}^{+\infty} \frac{r+\rho}{\bar{h}}J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r \\ &= [\phi_0(-\bar{h})+K_3\epsilon]\int_{-\bar{h}}^0 \int_{\bar{h}-r}^{+\infty} \frac{2r+\rho}{\bar{h}}J_*(\rho)\mathrm{d}\rho\mathrm{d}r \\ &= [\phi_0(-\bar{h})+K_3\epsilon]\left(\int_{\bar{h}}^{2\bar{h}}\int_{\bar{h}-\rho}^0 + \int_{2\bar{h}}^{\infty}\int_{-\bar{h}}^0\right)\frac{2r+\rho}{\bar{h}}J_*(\rho)\mathrm{d}r\mathrm{d}\rho \\ &= [\phi_0(-\bar{h})+K_3\epsilon]\int_{\bar{h}}^{\infty}(\rho-\bar{h})J_*(\rho)\mathrm{d}\rho \\ &\le \int_{-\infty}^{-\bar{h}}\int_0^{+\infty}[\phi_0(r)+K_3\epsilon]J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r. \end{aligned}$$

Hence, due to $\bar{h}(t) \le c_0(t+\theta)$, we have

$$\begin{aligned} &-\int_{-\infty}^{-\bar{h}}\int_0^{+\infty}[\phi_0(r)+K_3\epsilon]J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r + A \\ &\le W_1 + \frac{W_2(\bar{h})}{\bar{h}} \le -\frac{3u_*C_1c_0^{4-\beta}}{2^{1+\beta}(\beta-2)(\beta-4)}(t+\theta)^{4-\beta} - \frac{\tilde{C}_3}{c_0(t+\theta)}. \end{aligned}$$

Thus Claim 1 holds with

$$C_3 := \min\left\{\frac{\tilde{C}_3}{c_0}, \frac{3u_*C_1c_0^{4-\beta}}{2^{1+\beta}(\beta-2)(\beta-4)}\right\}.$$

We may now use Claim 1, $\alpha = \min\{1, 4-\beta\} \in (0,1]$ and $\bar{h}(t) \le c_0(t+\theta)$ to deduce

$$\begin{aligned} &\frac{\mu}{\bar{h}^2(t)}\int_0^{\bar{h}(t)} r^2\bar{u}(t,r)\int_{\bar{h}(t)}^{+\infty}\tilde{J}(r,\rho)\mathrm{d}\rho\mathrm{d}r \\ &\le (1+\epsilon)\Big(c_0 + \mu C_J K_3\epsilon - \mu C_3\Big[(t+\theta)^{-1} + (t+\theta)^{4-\beta}\Big]\Big) \\ &\le c_0 + K_1(c_0 + 2\mu C_J K_3)(t+\theta)^{-\alpha} - \mu C_3\Big[(t+\theta)^{-1} + (t+\theta)^{4-\beta}\Big] \\ &\le c_0 - K_1K_2(t+\theta)^{\alpha} = h'(t) \end{aligned}$$

provided that, apart from $\theta \gg 1$, K_1, K_2 and K_3 are small such that

$$K_1(c_0 + 2\mu C_J K_3 + K_2) \le \mu C_3. \tag{3.9}$$

Step 2. We verify the first inequality of (3.5), namely, for $t > 0$ and $r \in (\bar{h}(t)/2, \bar{h}(t))$,

$$\bar{u}_t(t,r) \ge d \int_0^{\bar{h}(t)} \tilde{J}(r,\rho)\bar{u}(t,\rho)\mathrm{d}\rho - d\bar{u}(t,x) + f(\bar{u}(t,r)). \tag{3.10}$$

We first show that for any continuous function $\phi(t,\rho)$ non-increasing in ρ,

$$\int_0^{\bar{h}(t)} \tilde{J}(r,\rho)\phi(t,\rho)\mathrm{d}\rho \le \int_0^{\bar{h}(t)} J_*(r-\rho)\phi(t,\rho)\mathrm{d}\rho \text{ for } r \in [\bar{h}(t)/2, \bar{h}(t)]. \tag{3.11}$$

From $\tilde{J}(r,\rho) \le \frac{\rho}{r} J_*(r-\rho)$, we deduce, for $r \in [\bar{h}(t)/2, \bar{h}(t)]$,

$$\begin{aligned}
\int_0^{\bar{h}} \tilde{J}(r,\rho)\phi(t,\rho)\mathrm{d}\rho &\le \int_0^{\bar{h}} \frac{\rho}{r} J_*(r-\rho)\phi(t,\rho)\mathrm{d}\rho \\
&= \int_0^{\bar{h}} J_*(r-\rho)\phi(t,\rho)\mathrm{d}\rho + \int_0^{\bar{h}} \frac{\rho - r}{r} J_*(r-\rho)\phi(t,\rho)\mathrm{d}\rho \\
&= \int_0^{\bar{h}} J_*(r-\rho)\phi(t,\rho)\mathrm{d}\rho + \frac{1}{r}\int_{-r}^{\bar{h}-r} \rho J_*(\rho)\phi(t,\rho+r)\mathrm{d}\rho.
\end{aligned}$$

Since $\phi(t,\rho)$ is nonincreasing for $\rho \le 0$, and $r \ge \bar{h}(t) - r$ for $r \in (\bar{h}(t)/2, \bar{h}(t))$, we have

$$\begin{aligned}
\int_{-r}^{\bar{h}-r} \rho J_*(\rho)\phi(t,\rho+r)\mathrm{d}\rho &\le \int_{-r}^{0} \rho J_*(\rho)\phi(t,\rho+r)\mathrm{d}\rho + \int_0^{r} \rho J_*(\rho)\phi(t,\rho+r)\mathrm{d}\rho \\
&= \int_0^{r} \rho J_*(\rho)[\phi(t,r+\rho) - \phi(t,r-\rho)]d\rho \le 0,
\end{aligned}$$

which yields (3.11).

Without loss of generality, we may assume that $\xi(r)$ is non-increasing for $r \in [0,\infty)$ and so $\bar{u}(t,r)$ is decreasing in r for $r \in [0, \bar{h}(t)]$. Therefore (3.11) holds with $\phi(t,r) = \bar{u}(t,r)$.

We are now ready to check (3.10). It is clear that

$$\bar{u}_t(t,r) = -(1+\epsilon)[c_0 + \delta'(t)]\phi_0'(r - \bar{h}(t)) + \epsilon'(t)\phi_0(r - \bar{h}(t)) + \omega_t(t,r)$$

and

$$\begin{aligned}
&-(1+\epsilon)c_0\phi_0'(r-\bar{h}(t)) \\
&= (1+\epsilon)\left[d\int_{-\infty}^{\bar{h}(t)} J_*(r-\rho)\phi_0(t,\rho)\mathrm{d}\rho - d\phi_0(r-\bar{h}(t)) + f(\phi_0(r-]barh(t)))\right] \\
&= d\int_{-\infty}^{\bar{h}(t)} J_*(r-\rho)[\bar{u}(t,\rho) - \omega]\mathrm{d}\rho - d[\bar{u}(t,r) - \omega] + (1+\epsilon)f(\phi_0(r-\bar{h}(t)))
\end{aligned}$$

$$
\begin{aligned}
&= d\int_{-\infty}^{\bar h(t)} J_*(r-\rho)\bar u(t,\rho)\mathrm{d}\rho - d\bar u(t,r)\\
&\quad + d\left[\omega(t,r) - \int_{-\infty}^{\bar h(t)} J_*(r-\rho)\omega(t,\rho)\mathrm{d}\rho\right] + (1+\epsilon)f(\phi_0(r-\bar h(t)))\\
&\geq d\int_{0}^{\bar h(t)} J_*(r-\rho)\bar u(t,\rho)\mathrm{d}\rho - d\bar u(t,r) + f(\bar u(t,r))\\
&\quad + d\left[\omega(t,r) - \int_{-\infty}^{\bar h(t)} J_*(r-\rho)\omega(t,\rho)\mathrm{d}\rho\right] + (1+\epsilon)f(\phi_0(r-\bar h(t))) - f(\bar u(t,r)).
\end{aligned}
$$

Hence by (3.11) with $\phi = \overline{u}$, for $t>0$ and $r \in [\bar h(t)/2, \bar h(t)]$,

$$
\begin{aligned}
\bar u_t(t,r) &\geq d\int_{0}^{\bar h(t)} \tilde J(r,\rho)\bar u(t,\rho)\mathrm{d}\rho - d\bar u(t,r) + f(\bar u(t,r))\\
&\quad + d\left[\omega(t,r) - \int_{-\infty}^{\bar h(t)} J_*(r-\rho)\omega(t,r)\mathrm{d}\rho\right] + (1+\epsilon)f(\phi_0(r-\bar h(t))) - f(\bar u(t,r))\\
&\quad - (1+\epsilon)\delta'(t)\phi_0'(r-\bar h(t)) + \epsilon'(t)\phi_0(r-\underline{h}(t)) + \omega'(t)\\
&=: d\int_{0}^{\bar h(t)} \tilde J(r,\rho)\bar u(t,\rho)\mathrm{d}\rho - d\bar u(t,r) + f(\bar u(t,r)) + B.
\end{aligned}
$$

In the following we show that $B \geq 0$ if $\theta \gg 1$ and K_1, K_2, K_3 are suitably chosen.

Claim 3. If $\tilde\epsilon > 0$ in (3.4) is sufficiently small and θ is sufficiently large, then

$$
\begin{aligned}
&d\left[\omega(t,r) - \int_{-\infty}^{\bar h(t)} J_*(r-\rho)\omega(t,\rho)\mathrm{d}\rho\right] + (1+\epsilon)f(\phi_0(r-\bar h(t))) - f(\overline{u}(t,r))\\
&\geq \frac{d-f'(0)}{2}\omega(t,r) > 0 \quad \text{for} \quad r\in[\bar h(t)-\tilde\epsilon, \bar h(t)].
\end{aligned} \tag{3.12}
$$

We have, for $r \in [\bar h(t)-\tilde\epsilon, \bar h(t)]$,

$$
\begin{aligned}
&d\left[\omega(t,r) - \int_{-\infty}^{\bar h(t)} J_*(r-\rho)\omega(t,\rho)\mathrm{d}\rho\right] = K_3\epsilon(t)\left[d - d\int_{-\infty}^{0} J_*(r-\bar h(t)-\rho)\xi(\rho)\mathrm{d}\rho\right]\\
&\geq K_3\epsilon(t)\left[d - d\int_{-2\tilde\epsilon}^{0} J_*(r-\bar h(t)-\rho)\mathrm{d}\rho\right] = K_3\epsilon(t)\left[d - d\int_{\bar h(t)-r-2\tilde\epsilon}^{\bar h(t)-r} J_*(\rho)\mathrm{d}\rho\right]\\
&\geq K_3\epsilon(t)\left[d - d\int_{-2\tilde\epsilon}^{\tilde\epsilon} J_*(\rho)\mathrm{d}\rho\right] \geq K_3\epsilon(t)\left[d - \frac{d-f'(0)}{4}\right] = \left[d - \frac{d-f'(0)}{4}\right]\omega(t,r),
\end{aligned}
$$

provided $\tilde\epsilon \in (0, \epsilon_1]$ for some small $\epsilon_1 > 0$ depending on $d - f'(0)$ and J_*.

Moreover, for $r \in [\bar h(t)-\tilde\epsilon, \bar h(t)]$, by (**f**) we obtain

$$
\begin{aligned}
(1+\epsilon)f(\phi_0(r-\bar h(t))) - f(\overline{u}(t,r)) &\geq f((1+\epsilon)\phi_0(r-\bar h(t))) - f(\overline{u}(t,r))\\
&= f(\overline{u}(t,r) - \omega(t,r)) - f(\overline{u}(t,r)),
\end{aligned}
$$

and

$$
0 \leq \overline{u}(t,r) \leq (1+\epsilon)\phi_0(-\tilde\epsilon) + K_3\epsilon \leq 2\phi_0(-\tilde\epsilon) + \theta^{-\alpha}
$$

if $K_1 \le 1$ and $K_3 \le 1$. So for such r, $\overline{u}(t,r)$ and $\omega(t,r)$ are small when $0 < \tilde{\epsilon} \ll 1$ and $\theta \gg 1$. Therefore

$$\begin{aligned} &f(\overline{u}(t,r) - \omega(t,r)) - f(\overline{u}(t,r)) = -\omega(t,r)[f'(\overline{u}(t,r)) + o(1)] \\ &= -\omega(t,r)[f'(0) + o(1)] \ge -\left[f'(0) + \frac{d - f'(0)}{4}\right]\omega(t,r) \end{aligned}$$

for $r \in [\bar{h}(t) - \tilde{\epsilon}, \bar{h}(t)]$, provided that $\tilde{\epsilon}$ is small and θ is large. Hence, (3.12) holds.

Let us now show $B \ge 0$ for $r \in [\bar{h}(t) - \tilde{\epsilon}, \bar{h}(t)]$. Denote

$$M := \sup_{r \le 0} |\phi'(r)|.$$

By (3.12) and $\omega(t,r) = K_3\epsilon(t)$ for $r \in [\bar{h}(t) - \tilde{\epsilon}, \bar{h}(t)]$, we have

$$\begin{aligned} B &\ge \frac{d - f'(0)}{2}\omega(t,r) - (1+\epsilon)\delta'(t)\phi_0'(r - \bar{h}(t)) + \epsilon'(t)\phi_0(r - \underline{h}(t)) + \omega_t(t,r) \\ &\ge \epsilon(t)\Big[\frac{d - f'(0)}{2}K_3 - 2K_2M - \alpha u^*(t+\theta)^{-1} - K_3\alpha(t+\theta)^{-1}\Big] \\ &\ge \epsilon(t)\Big[\frac{d - f'(0)}{2}K_3 - 2K_2M - \theta^{-1}\alpha(u^* + K_3)\Big] \\ &\ge 0 \end{aligned}$$

provided that we first fix K_2 and K_3 so that (3.9) holds and at the same time

$$\frac{d - f'(0)}{2}K_3 - 2K_2M > 0, \tag{3.13}$$

and then choose θ sufficiently large.

Next, for fixed small $\tilde{\epsilon} > 0$, we estimate B for $x \in [\bar{h}(t)/2, \bar{h}(t) - \tilde{\epsilon}]$.

Claim 4. For any $\eta \in (0, u^*)$, there exists $c_1 = c_1(\eta) > 0$ such that

$$(1+\epsilon)f(v) - f((1+\epsilon)v) \ge c_1\epsilon \ \text{ for } \ v \in [\eta, u^*] \text{ and } 0 < \epsilon \ll 1. \tag{3.14}$$

Define

$$G(v) := f(v)/v.$$

Then by (1.12), for $v \in [\eta, u^*]$ and $0 < \epsilon \ll 1$, there exists some $\tilde{v} \in [\eta, (1+\epsilon)u^*] \subset [\eta, 2u^*]$ such that

$$G(v) - G((1+\epsilon)v) = G'(\tilde{v})(-\epsilon v) \ge \tilde{c}_1\epsilon v,$$

where $\tilde{c}_1 = \min_{u \in [\eta, 2u^*]}[-G'(u)] > 0$. It follows that

$$(1+\epsilon)f(v) - f((1+\epsilon)v) = (1+\epsilon)v[G(v) - G((1+\epsilon)v)] \ge \eta^2\tilde{c}_1\epsilon$$

for $v \in [\eta, u^*]$ and $0 < \epsilon \ll 1$. This proves Claim 4.

By Claim 4, there exist positive constants C_l and C_f such that, for $v = \phi_0(x - \bar{h}(t)) \in [\phi_0(-\tilde{\epsilon}), u^*]$,

$$(1+\epsilon)f(v) - f((1+\epsilon)v + \omega)$$

$$
\begin{aligned}
&= (1+\epsilon) f(v) - f((1+\epsilon)v) + f((1+\epsilon)v) - f((1+\epsilon)v + \omega) \\
&\geq C_l \epsilon - C_f K_3 \epsilon
\end{aligned}
$$

when $\epsilon = \epsilon(t)$ is small.

We also have

$$
d\left[\omega(t,r) - \int_{-\infty}^{\bar{h}(t)} J_*(r-\rho)\omega(t,\rho)\mathrm{d}\rho\right] \geq -d\int_{-\infty}^{\bar{h}(t)} J_*(r-\rho)\omega(t,\rho)\mathrm{d}\rho \geq -dK_3\epsilon(t),
$$

and

$$
\begin{aligned}
\omega_t(t,r) &= -\xi'\bar{h}'K_3\epsilon(t) + \xi K_3\epsilon'(t) \geq -\xi_* K_3\epsilon(t) - K_3\alpha(t+\theta)^{-1}\epsilon(t) \\
&\geq -(\xi_* + \alpha\theta^{-1})K_3\epsilon(t),
\end{aligned}
$$

with $\xi_* := c_0 \max_{x\in\mathbb{R}} |\xi'(x)|$.

Using these we obtain, for $r \in [\bar{h}(t)/2, \bar{h}(t) - \tilde{\epsilon}]$,

$$
\begin{aligned}
B \geq\ & -dK_3\epsilon(t) + (1+\epsilon)f(\phi_0(r-\bar{h}(t))) - f(\bar{u}(t,r)) + 2M\delta'(t) + \epsilon'(t) + \omega_t(t,r) \\
&\geq C_l\epsilon(t) - (C_f + d)K_3\epsilon(t) - 2MK_2\epsilon(t) - \alpha(t+\theta)^{-1}\epsilon(t) - (\xi_* + \alpha\theta^{-1})K_3\epsilon(t) \\
&= \epsilon(t)\Big[C_l - K_3(C_f + d) - 2MK_2 - \alpha(t+\theta)^{-1} - (\xi_* + \alpha\theta^{-1})K_3\Big] \\
&\geq \epsilon(t)\Big[C_l - K_3(C_f + d) - 2MK_2 - \xi_* K_3 - \alpha\theta^{-1}(1+K_3)\Big] \\
&\geq 0
\end{aligned}
$$

provided that we choose K_2 and K_3 small such that

$$
C_l - K_3(C_f + C_d) - 2MK_2 - \xi_* K_3 > 0
$$

while keeping both (3.9) and (3.13) hold, and then choose $\theta > 0$ sufficiently large.

Therefore, (3.10) holds when K_2, K_3 and θ are chosen as above.

Step 3. We choose $t_0 = t_0(\theta)$ such that the last two inequalities of (3.5) hold.

Clearly, for large $\theta > 0$ depending on K_2,

$$
2c_0(t+\theta) \geq \bar{h}(t) \geq \frac{c_0}{2}(t+\theta) \ \text{ for all } \ t \geq 0. \tag{3.15}
$$

For the ODE problem

$$
v' = f(v), \quad v(0) = u^* + \epsilon_1
$$

with small $\epsilon_1 > 0$, from $f'(u^*) < 0$ we see that

$$
u^* < v(t) \leq u^* + \epsilon_1 e^{\widetilde{F}t} \text{ for all } t \geq 0,
$$

where $\widetilde{F} = \max_{u\in[u^*,u^*+\epsilon_1]} f'(u) < 0$. A simple comparison argument shows that there is $t_* > 0$ such that $u(t,r) \leq u^* + \epsilon_1$ for $t \geq t_*$ and $r \in [0, h(t)]$. Using comparison again we obtain

$$
u(t+t_*, r) \leq v(t) \leq u^* + \epsilon_1 e^{\widetilde{F}t} \ \text{ for all } t \geq 0,\ r \in [0, h(t)].
$$

We claim that there is $D_* > 0$ such that

$$J_*(\rho) \le D_*\rho^{2-\beta} \text{ for } \rho \ge 1. \tag{3.16}$$

Indeed,

$$J_*(\rho) = \omega_2 \int_{|\rho|}^{\infty} \eta J(\eta)\mathrm{d}\eta \le C_2\omega_2 \int_{\rho}^{\infty} \eta^{1-\beta}\mathrm{d}\eta \le C_2\omega_2 \frac{\rho^{2-\beta}}{\beta-2}.$$

Hence (3.16) holds, and it follows that

$$\int_0^{\infty} J_*(\eta)\eta^{\alpha_*}\mathrm{d}\eta < \infty \quad \text{for } 1 < \alpha_* < \beta - 3.$$

Now we can use Lemma 2.4 to conclude the existence of $C_\phi > 0$ such that

$$u^* - \phi_0(x) \le \frac{C_\phi}{|x|^{\alpha_*}} \text{ for } \; x \le -1. \tag{3.17}$$

In particular, for $\alpha_* \in (1, \min\{\beta-3, 2\})$ we can use (3.17) to deduce, for $t \ge 0$ and $r \in [0, \bar{h}(t)/2]$,

$$\begin{aligned}
\bar{u}(t,r) &= (1+\epsilon(t))\phi_0(r-\bar{h}(t)) \ge (1+\epsilon(t))\phi_0(-\bar{h}(t)/2) \\
&\ge (1+K_1(t+\theta)^{-1})(u^* - 2^{\alpha^*}C_\phi\bar{h}(t)^{-\alpha^*}) \ge (1+K_1(t+\theta)^{-1})[u^* - (4/c_0)^{\alpha^*}C_\phi(t+\theta)^{-\alpha^*}] \\
&= u^* + u^*K_1(t+\theta)^{-1} - (4/c_0)^{\alpha^*}C_\phi(t+\theta)^{-\alpha^*} - (4/c_0)^{\alpha^*}K_1C_\phi(t+\theta)^{-1-\alpha^*} \\
&\ge u^* + u^*K_1(t+\theta)^{-1}/2 \ge u^* + \epsilon_1 e^{\widetilde{F}(t+t_0-t_*)} \\
&\ge u(t+t_0, r)
\end{aligned}$$

provided that $\theta \gg 1$ and

$$u^*K_1(t+\theta)^{-1}/2 \ge \epsilon_1 e^{\widetilde{F}(t+t_0-t_*)} \quad \text{for all } t \ge 0. \tag{3.18}$$

We show next that this is possible if t_0 is chosen properly. Indeed, by Proposition 1.3, there is $C_1 > 0$ such that $h(t) \le 2c_0 t + C_1$ for $t \ge 0$. It follows that

$$h(t_0) \le c_0\theta/2 < \bar{h}(0) \text{ for } t_0 := \theta/4 - \frac{C_1}{2c_0} \text{ and } \theta \gg 1,$$

and (3.18) is satisfied for this choice of t_0 if

$$u^*K_1(t+\theta)^{-1}/2 \ge \epsilon_1 e^{\widetilde{F}[(t+\theta)/4 - C_1/(2c_0) - t_*]} \text{ for all } t \ge 0,$$

which is clearly valid since $\theta \gg 1$ and $\widetilde{F} < 0$.

Now all the remaining inequalities in (3.5) are satisfied and the proof of the lemma is complete. □

3.2. Upper bound of $h(t) - c_0 t$ when $d \le f'(0)$

Lemma 3.3. *In Lemma 3.2, if $f'(0) \ge d$, then* (3.3) *still holds.*

Proof. This is a modification of the proof of Lemma 3.2, where in the definition of $\bar{u}$, the term $\omega(t,r)$ is changed to $-\omega(t,r)$, and a new term $\lambda(t)$ is added; see details below.

As in the proof of Lemma 3.2, let $\alpha = \min\{\beta-4, 1\} \in (0,1]$, $\tilde{\epsilon} > 0$ be a small constant and $\xi \in C^2(\mathbb{R})$ satisfy

$$0 \le \xi(r) \le 1, \quad \xi(r) = 1 \text{ for } |r| < \tilde{\epsilon},\ \xi(r) = 0 \text{ for } |r| > 2\tilde{\epsilon}.$$

Define

$$\begin{cases} \bar{h}(t) := c_0(t+\theta) + \delta(t), & t \ge 0, \\ \bar{u}(t,r) := (1+\epsilon(t))\phi_0(r - \bar{h}(t) - \lambda(t)) - \omega(t,r), & t \ge 0,\ r \le \bar{h}(t), \end{cases}$$

where

$$\epsilon(t) := K_1(t+\theta)^{-\alpha},\ \delta(t) := -K_2 \int_0^t \epsilon(\tau)\mathrm{d}\tau,$$
$$\omega(t,r) := K_3\xi(r - \bar{h}(t))\epsilon(t),\ \lambda(t) := K_4\epsilon(t),$$

with the positive constants $\tilde{\epsilon}, K_1, K_2, K_3, K_4$ to be determined and $\theta \gg 1$.

Denote

$$C_{\tilde{\epsilon}} := \min_{r \in [-2\tilde{\epsilon}, 0]} |\phi_0'(r)| > 0.$$

Then for $r \in [\bar{h}(t) - 2\tilde{\epsilon}, \bar{h}(t)]$,

$$\bar{u}(t,r) \ge \phi_0\big(-\lambda(t)\big) - \omega(t,r) \ge C_{\tilde{\epsilon}}\lambda(t) - K_3\epsilon(t) \ge \epsilon(t)(C_{\tilde{\epsilon}}K_4 - K_3) > 0$$

if

$$K_3 = C_{\tilde{\epsilon}}K_4/2, \tag{3.19}$$

which combined with $\xi(r) = 0$ for $|r| \ge 2\tilde{\epsilon}$ implies

$$\bar{u}(t,r) \ge 0 \text{ for } t \ge 0,\ r \le \bar{h}(t). \tag{3.20}$$

Step 1. We verify that for K_1, K_2 and K_4 suitably small,

$$\bar{h}'(t) \ge \frac{\mu}{\bar{h}^2(t)} \int_0^{\bar{h}(t)} r^2\bar{u}(t,r) \int_{\bar{h}(t)}^{+\infty} \tilde{J}(r,\rho)\mathrm{d}\rho\mathrm{d}r \text{ for all } t > 0. \tag{3.21}$$

By (3.6),

$$\begin{aligned} &\frac{\mu}{\bar{h}^2(t)} \int_0^{\bar{h}(t)} r^2\bar{u}(t,r) \int_{\bar{h}(t)}^{+\infty} \tilde{J}(r,\rho)\mathrm{d}\rho\mathrm{d}r \\ \le\ &\frac{\mu(1+\epsilon)}{\bar{h}^2} \int_0^{\bar{h}} r\phi_0(r - \bar{h} - \lambda) \int_{\bar{h}}^{+\infty} \rho J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r \\ =\ &\frac{\mu(1+\epsilon)}{\bar{h}^2} \int_0^{\bar{h}} r\phi_0(t,r) \int_{\bar{h}}^{+\infty} \rho J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r \end{aligned}$$

$$
\begin{aligned}
&+\frac{\mu(1+\epsilon)}{\bar h^2}\int_0^{\bar h} r[\phi_0(r-\bar h-\lambda)-\phi_0(t,r)]\int_{\bar h}^{+\infty}\rho J_*(r-\rho)\mathrm{d}\rho\mathrm{d}r\\
&=: I+II.
\end{aligned}
$$

Let $M_1 := \sup_{x\le 0}|\phi_0'(x)|$. Then

$$
\begin{aligned}
II &\le 2\mu\frac{M_1\lambda}{\bar h^2}\int_0^{\bar h}\int_{\bar h}^{+\infty} r\rho J_*(r-\rho)d\rho dr = 2\mu\frac{M_1\lambda}{\bar h^2}\int_0^{\bar h}\int_{\bar h-r}^{+\infty} r(\rho+r)J_*(\rho)d\rho dr\\
&= 2\mu\frac{M_1\lambda}{\bar h^2}\left[\int_0^{\bar h}\int_{\bar h-\rho}^{\bar h}+\int_{\bar h}^{+\infty}\int_0^{\bar h}\right] r(\rho+r)J_*(\rho)drd\rho\\
&\le 2\mu\frac{M_1\lambda}{\bar h^2}\left[2\bar h^2\int_0^{\bar h}\rho J_*(\rho)d\rho+2\bar h^2\int_{\bar h}^{+\infty}\rho J_*(\rho)d\rho\right]\\
&= 4\mu M_1 C_J\lambda,
\end{aligned}
$$

with $C_J > 0$ given by (3.7), which is finite due to (3.16).

By the calculations in Step 1 of Lemma 3.2, we have

$$
\begin{aligned}
I &= \frac{\mu}{\bar h^2(t)}\int_0^{\bar h(t)} r^2\bar u(t,r)\int_{\bar h(t)}^{+\infty}\tilde J(r,\rho)\mathrm{d}\rho\mathrm{d}r\\
&\le c_0+K_1(c_0+2\mu C_J K_3)(t+\theta)^{-\alpha}-\mu C_3\Big[(t+\theta)^{-1}+(t+\theta)^{4-\beta}\Big].
\end{aligned}
$$

Hence, by (3.19), we have

$$
\begin{aligned}
&\frac{\mu}{\bar h^2(t)}\int_0^{\bar h(t)} r^2\bar u(t,r)\int_{\bar h(t)}^{+\infty}\tilde J(r,\rho)\mathrm{d}\rho\mathrm{d}r \le I+II\\
&\le 4\mu M_1C_JK_4\epsilon(t)+c_0+K_1(c_0+2\mu C_JK_3)(t+\theta)^{-\alpha}-\mu C_3\Big[(t+\theta)^{-1}+(t+\theta)^{4-\beta}\Big]\\
&= c_0+K_1(4\mu M_1C_JK_4+c_0+\mu C_JC_{\tilde\epsilon}K_4)(t+\theta)^{-\alpha}-\mu C_3\Big[(t+\theta)^{-1}+(t+\theta)^{4-\beta}\Big]\\
&\le c_0-K_1K_2(t+\theta)^{-\alpha}=h'(t)
\end{aligned}
$$

if K_1, K_2 and K_4 are small such that

$$
K_1(4\mu M_1C_JK_4+c_0+\mu C_JC_{\tilde\epsilon}K_4)+K_1K_2\le \mu C_3. \tag{3.22}
$$

Step 2. We show that by choosing K_2, K_4 suitably small and θ sufficiently large, for $t > 0$ and $r\in[\bar h(t)/2, \bar h(t)]$,

$$
\overline{u}_t(t,r)\ge d\int_0^{\bar h(t)}\tilde J(r,\rho)\overline{u}(t,\rho)\mathrm{d}\rho-d\,\overline{u}(t,r)+f(\overline{u}(t,r)). \tag{3.23}
$$

Firstly we notice that for $\theta\gg 1$ and all $t>0$, $\overline{u}_r(t,r)<0$. Indeed, since $\omega(t,r)=0$ for $r\notin[\bar h(t)-2\tilde\epsilon, \bar h(t)-\tilde\epsilon]$, and $\phi_0'(r)<0$ for $r\le 0$, it suffices to consider $r\in[\bar h(t)-2\tilde\epsilon, \bar h(t)-\tilde\epsilon]$. For such r,

$$
u_r(t,r)\le\phi_0'(r-\bar h(t)-K_4\epsilon(t))-K_3\xi'(r-\bar h(t))\epsilon(t)\le -C_1(\tilde\epsilon)+C_2(\tilde\epsilon)\theta^{-\alpha}<0,
$$

where

$$C_1(\tilde{\epsilon}) := \min_{s\in[-3\tilde{\epsilon},0]} |\phi_0'(s)|,\ C_2(\tilde{\epsilon}) := K_3\|\xi'\|_\infty.$$

Hence we can use (3.11) to obtain

$$\int_0^{\bar{h}(t)} \tilde{J}(r,\rho)\overline{u}(t,\rho)d\rho \le \int_0^{\bar{h}(t)} J_*(r-\rho)\overline{u}(t,\rho)d\rho \text{ for } r\in[\bar{h}(t)/2,\bar{h}(t)], t>0.$$

Using the definition of $\overline{u}$, we have

$$\begin{aligned}\overline{u}_t(t,r) &= -(1+\epsilon)(\bar{h}'+\lambda')\phi_0'(r-\bar{h}-\lambda)+\epsilon'\phi_0(r-\bar{h}-\lambda)-\omega_t\\ &= -(1+\epsilon)(c_0+\delta'+\lambda')\phi_0'(r-\bar{h}-\lambda)+\epsilon'\phi_0(r-\bar{h}-\lambda)-\omega_t\end{aligned}$$

and

$$\begin{aligned}&-(1+\epsilon)c_0\phi_0'(r-\bar{h}-\lambda)\\ &=(1+\epsilon)\left[d\int_{-\infty}^{\bar{h}+\lambda} J_*(r-\rho)\phi_0(\rho-\bar{h}-\lambda)\mathrm{d}\rho - d\phi_0(r-\bar{h}-\lambda)+f(\phi_0(r-\bar{h}-\lambda))\right]\\ &\ge(1+\epsilon)\left[d\int_0^{\bar{h}} J_*(r-\rho)\phi_0(\rho-\bar{h}-\lambda)\mathrm{d}\rho - d\phi_0(r-\bar{h}-\lambda)+f(\phi_0(r-\bar{h}-\lambda))\right]\\ &=d\int_0^{\bar{h}} J_*(r-\rho)[\overline{u}(t,\rho)+\omega]\mathrm{d}\rho - d[\overline{u}(t,r)+\omega]+(1+\epsilon)f(\phi_0(r-\bar{h}-\lambda))\\ &=\int_0^{\bar{h}} J_*(r-\rho)\overline{u}(t,\rho)\mathrm{d}\rho - d\overline{u}(t,r)\\ &\quad -d\left[\omega(t,r)-\int_0^{\bar{h}} J_*(r-\rho)\omega(t,\rho)\mathrm{d}\rho\right]+(1+\epsilon)f(\phi_0(r-\bar{h}-\lambda)).\end{aligned}$$

Hence, for $r\in[\bar{h}(t)/2,\bar{h}(t)]$ and $t>0$,

$$\overline{u}_t(t,r)\ge d\int_0^{\bar{h}(t)} \tilde{J}(r,\rho)\overline{u}(t,\rho)\mathrm{d}\rho - d\overline{u}(t,r)+f(\overline{u}(t,r))+A(t,r)$$

with

$$\begin{aligned}A(t,r) := &-d\left[\omega(t,r)-\int_0^{\bar{h}} J_*(r-\rho)\omega(t,\rho)\mathrm{d}\rho\right]+(1+\epsilon)f(\phi_0(r-\bar{h}-\lambda))-f(\overline{u}(t,r))\\ &-(1+\epsilon)(\delta'+\lambda')\phi_0'(r-\bar{h}-\lambda)+\epsilon'\phi_0(r-\underline{h}-\lambda)-\omega_t.\end{aligned}$$

To show (3.23), it remains to choose suitable K_2, K_4 and θ such that $A(t,r)\ge 0$ for $t>0$ and $r\in[\bar{h}(t)/2,\bar{h}(t)]$.

Claim: There exists $\tilde{J}_0>0$ depending on $\tilde{\epsilon}$ such that for all small $\tilde{\epsilon}_0\in(0,\tilde{\epsilon}/2)$, we have

$$\begin{aligned}&-d\left[\omega(t,r)-\int_0^{\bar{h}} J_*(r-\rho)\omega(t,\rho)\mathrm{d}\rho\right]+(1+\epsilon)f(\phi_0(r-\bar{h}-\lambda))-f(\overline{u}(t,r))\\ &\ge\ \tilde{J}_0\,\omega(t,r)\ \text{ for }\ r\in[\bar{h}(t)-\tilde{\epsilon}_0,\bar{h}(t)].\end{aligned}\tag{3.24}$$

Indeed, for $r \in [\bar{h}(t) - \tilde{\epsilon}_0, \bar{h}(t)]$,

$$
\begin{aligned}
& d\left[\omega(t,r) - \int_0^{\bar{h}(t)} J_*(r-\rho)\omega(t,\rho)\mathrm{d}\rho\right] = K_3\epsilon(t)\left[d - d\int_0^{\bar{h}(t)} J_*(r-\rho)\xi(\rho - \bar{h}(t))\mathrm{d}\rho\right] \\
\leq\ & K_3\epsilon(t)\left[d - d\int_{\bar{h}(t)-\tilde{\epsilon}}^{\bar{h}(t)} J_*(r-\rho)\mathrm{d}\rho\right] = K_3\epsilon(t)\left[d - d\int_{\bar{h}(t)-\tilde{\epsilon}-r}^{\bar{h}(t)-r} J_*(\rho)\mathrm{d}\rho\right] \\
\leq\ & d\omega(t,r)\left[1 - \int_{-\tilde{\epsilon}+\tilde{\epsilon}_0}^{0} J_*(\rho)\mathrm{d}\rho\right] \leq d\omega(t,r)\left[1 - \int_{-\tilde{\epsilon}/2}^{0} J_*(\rho)\mathrm{d}\rho\right].
\end{aligned}
$$

On the other hand, for $r \in [\bar{h}(t) - \tilde{\epsilon}_0, \bar{h}(t)]$, we have

$$
\begin{aligned}
&(1+\epsilon)f(\phi_0(x - \bar{h} - \lambda) - f(\overline{u}) \geq f((1+\epsilon)\phi_0(x - \bar{h} - \lambda)) - f(\overline{u}) \\
=\ & f(\overline{u} + \omega) - f(\overline{u}) = \omega\Big(f'(\overline{u}) + o(1)\Big) = \Big(f'(0) + o(1)\Big)\omega
\end{aligned}
$$

since both $\overline{u}(t,r)$ and $\omega(t,r)$ are close to 0 for $r \in [\bar{h}(t) - \tilde{\epsilon}_0, \bar{h}(t)]$ with $\tilde{\epsilon}_0$ small and $\theta \gg 1$.

Hence, for such r and $\tilde{\epsilon}_0$, since $f'(0) \geq d$,

$$
\begin{aligned}
& -d\left[\omega(t,r) - \int_0^{\bar{h}(t)} J_*(r-\rho)\omega(t,\rho)\mathrm{d}\rho\right] + (1+\epsilon)f(\phi_0(r - \bar{h}(t))) - f(\overline{u}(t,r)) \\
\geq\ & d\omega\left[-1 + \int_{-\tilde{\epsilon}/2}^{0} J_*(\rho)\mathrm{d}\rho\right] + f'(0)\omega + o(1)\omega \\
\geq\ & \tilde{J}_0\,\omega(t,r), \qquad \text{with} \quad \tilde{J}_0 := \frac{d}{2}\int_{-\tilde{\epsilon}/2}^{0} J_*(\rho)\mathrm{d}\rho.
\end{aligned}
$$

This proves (3.24).

Clearly for $r \in [\bar{h}(t) - \tilde{\epsilon}_0, \bar{h}(t)]$,

$$
-\omega_t(t,r) = \alpha K_3 K_1 (t+\theta)^{-\alpha-1} \geq 0.
$$

Denoting $M_1 := \sup_{x\leq 0} |\phi_0'(x)|$, we obtain, for $r \in [\bar{h}(t) - \tilde{\epsilon}_0, \bar{h}(t)]$ and small $\tilde{\epsilon}_0$,

$$
\begin{aligned}
A(t,r) &\geq \tilde{J}_0 K_3\epsilon(t) + 2(\delta'(t) + \lambda'(t))M_1 + \epsilon'(t)u^* \\
&= \tilde{J}_0 K_3\epsilon(t) + 2\epsilon(t)(-K_2 - K_4\alpha(t+\theta)^{-1})M_1 - \alpha(t+\theta)^{-1}\epsilon(t)u^* \\
&\geq \epsilon(t)\Big[\tilde{J}_0 K_3 - 2(K_2 + K_4\alpha\theta^{-1})M_1 - \alpha\theta^{-1}u^*\Big] \\
&= \epsilon(t)\Big[\tilde{J}_0 K_3 - 2K_2 M_1 - \theta^{-1}\Big(K_4\alpha M_1 + \alpha u^*\Big)\Big] \\
&\geq 0
\end{aligned}
$$

provided that K_2 is chosen small so that (3.22) holds and

$$
\tilde{J}_0 K_3 - 2K_2 M_1 > 0, \tag{3.25}
$$

and θ is chosen sufficiently large.

We next estimate $A(t,r)$ for $r \in [\bar{h}(t)/2, \bar{h}(t) - \tilde{\epsilon}_0]$. From Claim 4 in the proof of Lemma 3.2, there exist positive constants $C_l = C_l(\tilde{\epsilon}_0)$ and C_f such that, for $v = \phi_0(r - \bar{h}(t - \lambda(t))) \in [\phi_0(-\tilde{\epsilon}_0), u^*]$,

$$\begin{aligned}&(1+\epsilon)f(v) - f((1+\epsilon)v - \omega)\\ =\ &(1+\epsilon)f(v) - f((1+\epsilon)v) + f((1+\epsilon)v) - f((1+\epsilon)v - \omega)\\ \geq\ &C_l\epsilon - C_f\omega \geq C_l\epsilon - C_f K_3\epsilon\end{aligned}$$

when $\epsilon = \epsilon(t)$ is small. Hence

$$\begin{aligned}&(1+\epsilon)f(\phi_0(r - \bar{h} - \lambda)) - f(\bar{u}(t,r))\\ \geq\ &C_l\epsilon - C_f K_3\epsilon \ \text{ for } \quad r \in [\bar{h}(t)/2, \bar{h}(t) - \tilde{\epsilon}_0],\ 0 < \tilde{\epsilon}_0 \ll 1.\end{aligned}$$

Clearly,

$$-d\left[\omega(t,r) - \int_{-\infty}^{\bar{h}(t)} J(r-\rho)\omega(t,\rho)\mathrm{d}\rho\right] \geq\ -dK_3\epsilon(t),$$

and

$$\omega_t(t,r) = -K_3\xi'\bar{h}'(t)\epsilon(t) + K_3\xi\epsilon'(t) \leq \xi_* K_3\epsilon(t)$$

with $\xi_* := c_0 \max_{x\in\mathbb{R}} |\xi'(r)|$.

We thus obtain, for $r \in [\bar{h}(t)/2, \bar{h}(t) - \tilde{\epsilon}_0]$ and $0 < \tilde{\epsilon}_0 \ll 1$,

$$\begin{aligned}A(t,r) \geq\ &-K_3\epsilon(t)d + (1+\epsilon)f(\phi_0(r-\bar{h})) - f(\bar{u}) + 2M_1(\delta' + \lambda') + \epsilon' - \omega_t\\ \geq\ &C_l\epsilon(t) - K_3\epsilon(t)(d + C_f + \xi_*) + 2M_1(-K_2\epsilon(t) + K_4\epsilon'(t)) + \epsilon'(t)\\ \geq\ &\epsilon(t)\Big[C_l - K_3(d + C_f + \xi_*) - 2M_1(K_2 + K_4\alpha(t+\theta)^{-1}) - \alpha(t+\theta)^{-1}\Big]\\ \geq\ &\epsilon(t)\Big[C_l - K_3\big(d + C_f + \xi_*\big) - 2M_1K_2 - \theta^{-1}\alpha\big(2M_1K_4 + 1\big)\Big]\\ \geq\ &0\end{aligned}$$

if we choose K_2 and K_4 small so that (3.22) and (3.25) hold and at the same time, recalling (3.19),

$$C_l - K_4\big(d + C_f + \xi_*\big)C_{\tilde{\epsilon}}/2 - 2M_1K_2 > 0,$$

and then choose θ sufficiently large. Hence, (3.23) is satisfied if K_2 and K_4 are chosen small as above, and θ is sufficiently large.

Step 3. We show that (3.3) holds.

As in the proof of Lemma 3.2, we can choose sufficient large θ and t_0 such that

$$\begin{aligned}&\bar{h}(0) \geq 2h(t_0),\\ &\bar{u}(t,r) \geq u(t+t_0, r) \ \text{ for } \quad r \in [0, \bar{h}(t)/2],\ t \geq 0.\end{aligned}$$

It follows that

$$\bar{u}(0,r) \geq u(t_0,r) \text{ for } r \in [0, h(t_0)].$$

From (3.20), we have

$$\overline{u}(t, \bar{h}(t)) \geq 0 \text{ for } t \geq 0.$$

These inequalities together with (3.21) and (3.23) allow us to use the comparison principle to conclude that

$$h(t+t_0) \leq \bar{h}(t),\ u(t+t_0, r) \leq \overline{u}(t, r) \text{ for } t \geq 0,\ r \in [0, h(t+t_0)],$$

which implies (3.3). The proof of the lemma is now complete. □

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Conflict of interest

The authors declare no conflict of interest.

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