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NOTION OF FUZZY IC-BAGS

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In the present paper, the author introduces the notion of fuzzy IC-bags and does some characterizations of them. The concepts of fuzzy base sets, base equivalent fuzzy IC-Bags, cardinally equispaced fuzzy IC-Bags, and cardinally equivalent fuzzy IC-Bags have been developed. The types of peak elements, and the concerned types of peak membership grades have been discussed. It is observed that the collection of any particular type of peak elements together with their membership grades actually form fuzzy bags. A set of operations on fuzzy IC-Bags have been defined and some propositions have been proved. We note that under the conditions of uncertainty, where the counts of objects are not fixed, then the interval counts can occur with different fuzzy membership grades for each particular object in the collection, and the framework for fuzzy IC-Bags provides us with the opportunity to justify and model the organized complexity as a part of the associated intolerance embedded in the subjective patterns.

Keywords: IC-Bags; Fuzzy IC-Bags; l-peak elements; u-peak elements; fuzzy base sets; fuzzy bags.

1. Introduction

In case of modelling the situations associated with object collections involving the redundancy of objects, Yager's Bags [11] are found to be extremely useful. The notions of bags are often applied in case of relational data models with multiset semantics and they act as type constructors in case of Object Definition Language. Often the basic types such as atomic types and interface types are combined into structured types by using bags.

In case of some specific information systems, we come across situations where the individual counts of objects in the collection are not precisely defined as a fixed count, but are represented in the form of intervals of non-negative integers. In case of modeling these systems, and analyzing the decisions under these frameworks, the IC-bags as introduced by the author in [2] are found to be quite useful. It is also observed that in the event of developing the alternatives and grading the feasibility of the possible alternatives in the design phase of decision making, the associated complexities can be modelled by the application of the concept of IC-Bags [4].

In the present paper it is observed that under the conditions of uncertainty, where the counts of objects are not fixed, then the interval counts can occur with different fuzzy membership grades for each particular object in the collection. This consideration gives rise to the substantial scope for development of the structure called *Fuzzy IC-Bags* which has been introduced in this paper. The characterization involves the development of fuzzy base sets, the l-equality and u-equality of fuzzy IC-bags, the l-IC-subbag and u-IC-subbags, the cardinality of fuzzy IC-bags, as well as their l-peak and u-peak values. The concepts of base equivalent fuzzy IC-Bags, cardinally equispaced fuzzy IC-Bags, and cardinally equivalent fuzzy IC-Bags have also been developed. Some operations on fuzzy IC-bags have been defined and some propositions are proved. It is observed that fuzzy IC-bags can serve as a useful tool for modelling certain decision analysis problems where the quantitative judgment pattern relates to uncertain knowledge representation with respect to the concerned hesitation. This provides us with a considerably broader framework for formalizing the interval factor under this specific type of uncertain analysis patterns occurring in the design process with an approximate, imprecise description of the desired collection of objects.

2. Preliminaries

In this section, we represent the notion of Bags and IC-Bags as discussed in [2]. We further include some modifications of the existing notions.

A bag (or a crisp bag) B drawn from a set X is represented by a function $Count_B$ or C_B defined as

$$C_B : X \longrightarrow N$$

where N represents the set of non-negative integers.

If B be a bag drawn from a set X, then the support set of B denoted as B^* is a subset of X with the characteristic function given by

$$\phi_{B^*}(x) = min[C_B(x), 1] \quad \forall\, x \in X.$$

A bag B is called an empty bag if for all $x \in X$, $C_B(x) = 0$. The support set of an empty bag is the null set.

The cardinality of a bag B drawn from a set X is denoted by $Card(B)$ and is defined as

$$Card(B) = \sum_{x \in X} C_B(x).$$

For a bag B drawn from a set X, $\max_{x \in X} C_B(x)$ is said to be the peak value of the bag. Any $x^* \in X$ satisfying

$$C_B(x^*) = \max_{x \in X} C_B(x)$$

is called to be a peak element of the bag B.

The union of two bags B_1 and B_2 drawn from a set X is a bag denoted by $B_1 \sqcup B_2$ such that $\forall x \in X$,

$$C_{B_1 \sqcup B_2}(x) = max[C_{B_1}(x), C_{B_2}(x)].$$

The intersection of B_1 and B_2 results in a bag denoted by $B_1 \sqcap B_2$ such that $\forall x \in X$,

$$C_{B_1 \sqcap B_2}(x) = min[C_{B_1}(x), C_{B_2}(x)].$$

Let Ω be any non empty set. Then an IC bag β drawn from Ω is characterized by a pair of functions C_l^β and C_u^β such that

$$C_l^\beta : \Omega \longrightarrow N \quad and \quad C_u^\beta : \Omega \longrightarrow N$$

and $C_l^\beta(x) \leq C_u^\beta(x) \forall x \in \Omega$, where N represents the set of non-negative integers.

If $\Omega = (x_1, x_2, ..., x_n)$, then an IC bag β drawn from Ω is represented as

$$\beta = \{x_i/(C_l^\beta(x_i), C_u^\beta(x_i))\}$$

where $i = 1, 2, ...n$.

We call $C_l^\beta(x_i)$ the minimum count for the object x_i in the IC bag β and $C_u^\beta(x_i)$ the maximum count for the object x_i in the IC bag β. This simply indicates that we are not sure about the fact that how many times the object x_i occurs in the collection, but we are sure about the fact that this value can not be less than $C_l^\beta(x_i)$ and at the same time it can not be more than $C_u^\beta(x_i)$. So, the value $(C_u^\beta(x_i) - C_l^\beta(x_i))$ for any object x_i indicates the range of hesitation in the part of the decision maker. If the actual count is $C^\beta(x_i)$, then $(C_u^\beta(x_i) - C^\beta(x_i))$ is called the *upper interval of hesitation* and $(C^\beta(x_i) - C_l^\beta(x_i))$ is called the *lower interval of hesitation* for the object x_i.

If for each $x \in \Omega, C_l^\beta(x) = C_u^\beta(x)$, then β represents a crisp bag.

Let β be an IC-bag drawn from Ω. Then a subset $\sigma(\beta)$ of Ω is called the support set of the IC bag β if $\forall x \in \Omega$,

$$C_l^\beta(x) > 0 \Longrightarrow x \in \sigma(\beta),$$
$$C_l^\beta(x) = 0 \Longrightarrow x \notin \sigma(\beta).$$

Two IC bags β_1 and β_2 drawn from Ω are said to be equal if and only if $\forall x \in \Omega$ the following conditions hold:

$$C_l^{\beta_1}(x) = C_l^{\beta_2}(x),$$
$$C_u^{\beta_1}(x) = C_u^{\beta_2}(x).$$

If $C_l^{\beta_1}(x) = C_l^{\beta_2}(x)\ \forall x \in \Omega$, but $C_u^{\beta_1}(x) \neq C_u^{\beta_2}(x)\ \forall x \in \Omega$, then we call β_1 to be l-equal to β_2 (or β_2 to be l-equal to β_1).

On the other hand, if $C_u^{\beta_1}(x) = C_u^{\beta_2}(x)\ \forall x \in \Omega$, but $C_l^{\beta_1}(x) \neq C_l^{\beta_2}(x)\ \forall x \in \Omega$, then we call β_1 to be u-equal to β_2.

β_1 is called an IC sub bag of β_2 if $\forall x \in \Omega$,

$$C_l^{\beta_1}(x) \leq C_l^{\beta_2}(x),$$
$$C_u^{\beta_1}(x) \leq C_u^{\beta_2}(x).$$

But this definition does not comply with the intuitive idea of sub bag. If $C^{\beta}(x)$ denote the number of times that x appears in bag β, where $C^{\beta}(x)$ is unknown in general. All we know is that $C_l^{\beta}(x) \leq C^{\beta}(x) \leq C_u^{\beta}(x)$. Considering the notion of sub bags, a bag β_1 should be a sub bag of β_2 if for every $x \in \Omega$, $C^{\beta_1}(x) \leq C^{\beta_2}(x)$.

If we consider the example, $\beta_1 = \{x/(1,3)\}$ and $\beta_2 = \{x/(2,4)\}$, then following our definition, β_1 is a sub bag of β_2. But it could happen that $C^{\beta_1}(x) = 3$ and $C^{\beta_2}(x) = 2$, which breaks our intuition of the concept of sub bag. Perhaps it could be more appropriate to change the definition to

$$C_u^{\beta_1}(x) \leq C_l^{\beta_2}(x).$$

because this change guarantees $C^{\beta_1}(x) \leq C^{\beta_2}(x)$.
But there are the following difficulties with this proposal:

- $C^{\beta_1}(x) \leq C^{\beta_2}(x)$ does not guarantee $C_u^{\beta_1}(x) \leq C_l^{\beta_2}(x)$. So, in fact we do not have a definition but the necessary part. The sufficient part is not true in general here, although it holds true for the crisp case.
- The notion of equality as defined by us which states that two IC-Bags are equal iff

$$C_l^{\beta_1}(x) = C_l^{\beta_2}(x),$$
$$C_u^{\beta_1}(x) = C_u^{\beta_2}(x).$$

can not be defined as "β_1 is a sub bag of β_2 and β_2 is a sub bag of β_1", unless β_1 and β_2 are crisp bags. But perhaps this makes sense since, otherwise, the equality as defined by us does not guarantee that the actual number of times (unknown) that x appears in both β_1 and β_2 is the same. In other words, knowing that the upper and lower bounds are the same does not guarantee that the bounded values are the same. Since the intervals are in fact the restrictions on IC-bags, hence equal restrictions to the uncertainty about the number of times elements appear in bags does not ensure that the bags are equal. Hence from now on it is more appropriate to call this "equality" as "IC-equality" which shall necessarily mean "the equality of imposed restrictions".

β_1 is called an l-IC sub bag of β_2 if $\forall x \in \Omega$,

$$C_l^{\beta_1}(x) \leq C_l^{\beta_2}(x),$$
$$C_u^{\beta_1}(x) > C_u^{\beta_2}(x).$$

β_1 is called an u-IC sub bag of β_2 if $\forall x \in \Omega$,

$$C_l^{\beta_1}(x) > C_l^{\beta_2}(x),$$
$$C_u^{\beta_1}(x) \leq C_u^{\beta_2}(x).$$

An IC bag β drawn from Ω is called a null IC bag if $\forall x \in \Omega$,

$$C_l^{\beta}(x) = C_u^{\beta}(x) = 0.$$

The cardinality of the IC bag β drawn from $\Omega = \{x_1, x_2, ...x_n\}$ is denoted by $\#_n^{\Omega}(\beta)$ and is defined as

$$\#_n^{\Omega}(\beta) = [\sum_{i=1}^{n} C_l^{\beta}(x_i), \sum_{i=1}^{n} C_u^{\beta}(x_i)].$$

If β be an IC bag drawn from Ω, then $\max_{x\in\Omega} C_l^{\beta}(x)$ and $\max_{x\in\Omega} C_u^{\beta}(x)$ are respectively called the l-peak and the u-peak values of β. The elements ξ_1 and ξ_2 of Ω satisfying

$$C_l^{\beta}(\xi_1) = \max_{x\in\Omega} C_l^{\beta}(x)$$
$$C_u^{\beta}(\xi_2) = \max_{x\in\Omega} C_u^{\beta}(x)$$

are respectively called the l-peak and u-peak elements of β.

If for any $\zeta \in \Omega$,

$$C_l^{\beta}(\zeta) = \max_{x\in\Omega} C_l^{\beta}(x)$$
$$C_u^{\beta}(\zeta) = \max_{x\in\Omega} C_u^{\beta}(x),$$

then ζ is called the lu-peak element of β.

3. Fuzzy IC-Bags

In this section we introduce the concept of fuzzy IC-Bags and do some characterizations of them.

Definition 1. A Fuzzy IC-Bag ϕ drawn from a non-empty set $\mathcal{R}$ is characterized by the function ξ^ϕ such that

$$\xi^\phi : \mathcal{R} \longrightarrow \mathcal{U}_\mathcal{I}$$

where $\mathcal{U}_\mathcal{I}$ represents the set of all IC-Bags drawn from the continuum $\mathcal{I} = [0, 1]$.

Thus for any $\omega \in \mathcal{R}$, $\xi^\phi(\omega)$ is an IC-Bag drawn from $\mathcal{I}$. But since any IC-Bag can itself be characterized by a pair of functions over its set, hence $\xi^\phi(\omega)$ can be characterized by the pair of functions

$$C_l^{\xi^\phi(\omega)} : \mathcal{I} \longrightarrow \mathcal{N}$$
$$C_u^{\xi^\phi(\omega)} : \mathcal{I} \longrightarrow \mathcal{N}$$

where $C_l^{\xi^\phi(\omega)}(\alpha) \leq C_u^{\xi^\phi(\omega)}(\alpha) \quad \forall \quad \alpha \in \mathcal{I}$, $\mathcal{N}$ being the set of non-negative integers.

Definition 2. For any fuzzy IC-Bag ϕ drawn from $\mathcal{R}$, a fuzzy subset $\beta(\phi)$ of $\mathcal{R}$ is called a fuzzy base set of ϕ, iff $\forall\ x \in \mathcal{R}$,

$$\mu_{\beta(\phi)}(x) = \{\omega \in \mathcal{I} : C_l^{\xi^\phi(x)}(\omega) \geq C_l^{\xi^\phi(x)}(\alpha) \quad \forall \alpha \in \mathcal{I}\}$$

where $\mu_{\beta(\phi)}$ represents the fuzzy membership function of $\beta(\phi)$.

Note that different fuzzy IC-Bags drawn from $\mathcal{R}$ can have the same fuzzy set as their fuzzy base set.

Any two fuzzy IC-Bags having the same fuzzy base set are called *base-equivalent* fuzzy IC-Bags.

Proposition 1. *Associated with any fuzzy IC-Bag ϕ there can exist more than one fuzzy base set if and only if for atleast one $x \in \mathcal{R}$, $\exists$ atleast two distinct ω_i's in $\mathcal{I}$ having the same value of $C_l^{\xi^\phi(x)}(\omega_i)$ such that*

$$C_l^{\xi^\phi(x)}(\omega_i) \geq C_l^{\xi^\phi(x)}(\alpha) \quad \forall \alpha \in \mathcal{I}.$$

Proof. Let ϕ be any fuzzy IC-Bag drawn from $\mathcal{R}$ such that for atleast one $x \in R, \exists w_1, w_2 \in \mathcal{I}(w_1 \neq w_2)$ for which

$$C_l^{\xi^\phi(x)}(\omega_1) = C_l^{\xi^\phi(x)}(\omega_2) \geq C_l^{\xi^\phi(x)}(\alpha) \quad \forall \alpha \in \mathcal{I}.$$

Let $\beta_1(\phi)$ and $\beta_2(\phi)$ be such that

$$\mu_{\beta_1(\phi)}(x) = \{\omega_1 \in \mathcal{I} : C_l^{\xi^\phi(x)}(\omega_1) \geq C_l^{\xi^\phi(x)}(\alpha) \quad \forall\alpha \in \mathcal{I}\}$$
$$\mu_{\beta_2(\phi)}(x) = \{\omega_2 \in \mathcal{I} : C_l^{\xi^\phi(x)}(\omega_1) \geq C_l^{\xi^\phi(x)}(\alpha) \quad \forall\alpha \in \mathcal{I}\}$$

which indicates that $\beta_1(\phi)$ and $\beta_2(\phi)$ are two distinct fuzzy base sets of ϕ. Conversely, let us assume that $\exists$ two distinct fuzzy base sets of ϕ. Hence, we have $\forall x \in R$

$$\mu_{\beta_1(\phi)}(x) = \{\omega \in \mathcal{I} : C_l^{\xi^\phi(x)}(\omega) \geq C_l^{\xi^\phi(x)}(\alpha) \quad \forall\alpha \in \mathcal{I}\}$$
$$\mu_{\beta_2(\phi)}(x) = \{\omega \in \mathcal{I} : C_l^{\xi^\phi(x)}(\omega) \geq C_l^{\xi^\phi(x)}(\alpha) \quad \forall\alpha \in \mathcal{I}\}$$

such that atleast for one $x \in R$ we have $\mu_{\beta_1(\phi)}(x) \neq \mu_{\beta_2(\phi)}(x)$. Let us substitute ω_1 for $\mu_{\beta_1(\phi)}(x)$ and ω_2 for $\mu_{\beta_2(\phi)}(x)$. Thus we have $w_1, w_2 \in \mathcal{I}(w_1 \neq w_2)$ such that

$$C_l^{\xi^\phi(x)}(\omega_1) \geq C_l^{\xi^\phi(x)}(\alpha) \quad \forall\alpha \in \mathcal{I} \tag{1}$$
$$C_l^{\xi^\phi(x)}(\omega_2) \geq C_l^{\xi^\phi(x)}(\alpha) \quad \forall\alpha \in \mathcal{I} \tag{2}$$

If we assume that $C_l^{\xi^\phi(x)}(\omega_1) \neq C_l^{\xi^\phi(x)}(\omega_2)$, then we have the following cases:
Case-1: If $C_l^{\xi^\phi(x)}(\omega_1) < C_l^{\xi^\phi(x)}(\omega_2)$, then it is a contradiction to (1).
Case-2: If $C_l^{\xi^\phi(x)}(\omega_1) > C_l^{\xi^\phi(x)}(\omega_2)$, then it is a contradiction to (2).
Hence proved. □

Proposition 2. *Two fuzzy IC-Bags ϕ_1 and ϕ_2 drawn from $\mathcal{R}$ can have the same fuzzy subset of $\mathcal{R}$ as their fuzzy base set if and only if for each $x \in \mathcal{R}$, $\exists$ some $\omega \in \mathcal{I}$ such that*

$$C_l^{\xi^{\phi_1}(x)}(\omega) \geq C_l^{\xi^{\phi_2}(x)}(\alpha) \quad \forall\alpha \in \mathcal{I},$$
$$C_l^{\xi^{\phi_2}(x)}(\omega) \geq C_l^{\xi^{\phi_2}(x)}(\alpha) \quad \forall\alpha \in \mathcal{I}.$$

Proof. Let us assume that ϕ_1 and ϕ_2 have the same fuzzy base set β such that

$$\mu_{\beta(\phi_1)}(x) = \{\omega_i \in \mathcal{I} : C_l^{\xi^{\phi_1}(x)}(\omega_i) \geq C_l^{\xi^{\phi_1}(x)}(\alpha) \quad \forall\alpha \in \mathcal{I}\} \tag{3}$$
$$\mu_{\beta(\phi_2)}(x) = \{\omega_j \in \mathcal{I} : C_l^{\xi^{\phi_2}(x)}(\omega_j) \geq C_l^{\xi^{\phi_2}(x)}(\alpha) \quad \forall\alpha \in \mathcal{I}\} \tag{4}$$

Since the sets represented by (3) and (4) are equal, hence we put $\omega = \omega_i = \omega_j$, and for each $x \in \mathcal{R}$ we have

$$C_l^{\xi^{\phi_1}(x)}(\omega) \geq C_l^{\xi^{\phi_2}(x)}(\alpha) \quad \forall\alpha \in \mathcal{I}, \tag{5}$$
$$C_l^{\xi^{\phi_2}(x)}(\omega) \geq C_l^{\xi^{\phi_2}(x)}(\alpha) \quad \forall\alpha \in \mathcal{I}. \tag{6}$$

This proves the necessary part. Conversely, let us assume that for each $x \in \mathcal{R}$ $\exists$ some $\omega \in \mathcal{I}$ such that (5) and (6) holds. Let β_1 and β_2 be the fuzzy base sets of ϕ_1 and ϕ_2 respectively and let us assume that β_1 is never equal to β_2. Then (5) and

(6) holds. Hence $\omega_i \in \mathcal{I}$ and $\omega_j \in \mathcal{I}$ can be always replaced by $\omega \in \mathcal{I}$ which proves the sufficient part. □

Definition 3. Two fuzzy IC-Bags ϕ_1 and ϕ_2 drawn from $\mathcal{R}$ are *fuzzy IC-equal* if and only if $\forall x \in \mathcal{R}$, $\alpha \in \mathcal{I}$, the following conditions hold:

$$C_l^{\xi^{\phi_1}(x)}(\alpha) = C_l^{\xi^{\phi_2}(x)}(\alpha),$$
$$C_u^{\xi^{\phi_1}(x)}(\alpha) = C_u^{\xi^{\phi_2}(x)}(\alpha).$$

If $C_l^{\xi^{\phi_1}(x)}(\alpha) = C_l^{\xi^{\phi_2}(x)}(\alpha) \forall x \in \mathcal{R}$, $\alpha \in \mathcal{I}$; but $C_u^{\xi^{\phi_1}(x)}(\alpha) \neq C_u^{\xi^{\phi_2}(x)}(\alpha)$ $\forall x \in \mathcal{R}$, $\alpha \in \mathcal{I}$, then ϕ_1 and ϕ_2 are called *fuzzy l-equal* to each other.
If $C_u^{\xi^{\phi_1}(x)}(\alpha) = C_u^{\xi^{\phi_2}(x)}(\alpha) \quad \forall x \in \mathcal{R}$, $\alpha \in \mathcal{I}$, but $C_l^{\xi^{\phi_1}(x)}(x)(\alpha) \neq C_l^{\xi^{\phi_2}(x)}(\alpha)$ $\forall x \in \mathcal{R}$, $\alpha \in \mathcal{I}$, then ϕ_1 and ϕ_2 are called *fuzzy u-equal* to each other.

If any two fuzzy IC-Bags ϕ_1 and ϕ_2 drawn from $\mathcal{R}$ are both l-equal and u-equal to each other, then they are called equal.

Proposition 3. *If any two fuzzy IC-Bags ϕ_1 and ϕ_2 drawn from $\mathcal{R}$ are l-equal, then $\beta(\phi_1) = \beta(\phi_2)$.*

Proof. Let the two fuzzy IC-Bags ϕ_1 and ϕ_2 drawn from $\mathcal{R}$ are l-equal. From the definition of the fuzzy base set of an IC bag, we have $\forall x \in \mathcal{R}$

$$\mu_{\beta(\phi_1)}(x) = \{\omega \in \mathcal{I} : C_l^{\xi^{\phi_1}(x)}(\omega) \geq C_l^{\xi^{\phi_1}(x)}(\alpha) \quad \forall \alpha \in \mathcal{I}\} \tag{7}$$
$$\mu_{\beta(\phi_2)}(x) = \{\omega \in \mathcal{I} : C_l^{\xi^{\phi_2}(x)}(\omega) \geq C_l^{\xi^{\phi_2}(x)}(\alpha) \quad \forall \alpha \in \mathcal{I}\} \tag{8}$$

But since ϕ_1 and ϕ_2 are l-equal, hence $\forall x \in \mathcal{R}, \alpha \in \mathcal{I}$ we have

$$C_l^{\xi^{\phi_1}(x)}(\alpha) = C_l^{\xi^{\phi_2}(x)}(\alpha). \tag{9}$$

Clearly, from (7), (8), and (9) $\beta(\phi_1) = \beta(\phi_2)$. Hence proved. □

Definition 4. For any two fuzzy IC-Bags ϕ_1 and ϕ_2 drawn from $\mathcal{R}$, ϕ_1 is called a *fuzzy IC-subbag* of ϕ_2, denoted by $\phi_1 \sqsubseteq \phi_2$ iff $\forall x \in \mathcal{R}$, $\alpha \in \mathcal{I}$

$$C_u^{\xi^{\phi_1}(x)}(\alpha) \leq C_l^{\xi^{\phi_2}(x)}(\alpha),$$

ϕ_1 is called a *fuzzy l-IC-subbag* of ϕ_2 iff $\forall x \in \mathcal{R}$, $\alpha \in \mathcal{I}$

$$C_l^{\xi^{\phi_1}(x)}(\alpha) \leq C_l^{\xi^{\phi_2}(x)}(\alpha),$$
$$C_u^{\xi^{\phi_1}(x)}(\alpha) \neq C_u^{\xi^{\phi_2}(x)}(\alpha).$$

ϕ_1 is called a *fuzzy u-IC-subbag* of ϕ_2 iff $\forall x \in \mathcal{R}$, $\alpha \in \mathcal{I}$,

$$C_u^{\xi^{\phi_1}(x)}(\alpha) \leq C_u^{\xi^{\phi_2}(x)}(\alpha),$$
$$C_l^{\xi^{\phi_1}(x)}(\alpha) \neq C_l^{\xi^{\phi_2}(x)}(\alpha).$$

Proposition 4. *For any two fuzzy IC-Bags ϕ_1 and ϕ_2 drawn from $\mathcal{R}$,*

(a) if ϕ_1 is fuzzy l-equal to ϕ_2, then ϕ_1 is a l-IC-subbag of ϕ_2;

(b) if ϕ_1 is fuzzy u-equal to ϕ_2, then ϕ_1 is a u-IC-subbag of ϕ_2.

Proof. (a) Let ϕ_1 be fuzzy l-equal to ϕ_2. Hence $\forall x \in \mathcal{R}$, $\alpha \in \mathcal{I}$,

$$C_l^{\xi^{\phi_1(x)}}(\alpha) = C_l^{\xi^{\phi_2(x)}}(\alpha)$$
$$C_u^{\xi^{\phi_1(x)}}(\alpha) \neq C_u^{\xi^{\phi_2(x)}}(\alpha)$$

This proves (a).
(b) Let ϕ_1 be fuzzy u-equal to ϕ_2. Hence $\forall x \in \mathcal{R}$, $\alpha \in \mathcal{I}$,

$$C_u^{\xi^{\phi_1(x)}}(\alpha) = C_u^{\xi^{\phi_2(x)}}(\alpha)$$
$$C_l^{\xi^{\phi_1(x)}}(\alpha) \neq C_l^{\xi^{\phi_2(x)}}(\alpha)$$

This proves (b). □

Definition 5. A fuzzy IC-Bag ϕ_o drawn from $\mathcal{R}$ is called a *null fuzzy IC-Bag* iff $\forall x \in \mathcal{R}$, $\alpha \in \mathcal{I}$,

$$C_l^{\xi^{\phi_o(x)}}(\alpha) = C_u^{\xi^{\phi_o(x)}}(\alpha) = 0.$$

If ϕ_o be the null fuzzy IC-Bag drawn from $\mathcal{R}$, then $\phi_o \sqsubseteq \phi$, for any fuzzy IC-Bag ϕ drawn from $\mathcal{R}$.

Definition 6. The cardinality of the fuzzy IC-Bag ϕ drawn from $\mathcal{R}$ is denoted by $\#(\phi)$ and is defined as

$$\#(\phi) = [\sum_{x\in\mathcal{R}} \sum_{\alpha\in\mathcal{I}} \alpha * C_l^{\xi^{\phi(x)}}(\alpha), \sum_{x\in\mathcal{R}} \sum_{\alpha\in\mathcal{I}} \alpha * C_u^{\xi^{\phi(x)}}(\alpha)].$$

The value of $(\sum_{x\in\mathcal{R}} \sum_{\alpha\in\mathcal{I}} \alpha * C_u^{\xi^{\phi(x)}}(\alpha) - \sum_{x\in\mathcal{R}} \sum_{\alpha\in\mathcal{I}} \alpha * C_l^{\xi^{\phi(x)}}(\alpha))$ is called the *cardinal span* of the fuzzy IC-Bag ϕ and is denoted by $\#_s(\phi)$. Clearly for any fuzzy IC-Bag ϕ, $\#_s(\phi) \geq 0$.

If for any two fuzzy IC-Bags ϕ_1 and ϕ_2 drawn from $\mathcal{R}$, we have

$$\sum_{x\in\mathcal{R}} \sum_{\alpha\in\mathcal{I}} \alpha * C_l^{\xi^{\phi_1(x)}}(\alpha) = \sum_{x\in\mathcal{R}} \sum_{\alpha\in\mathcal{I}} \alpha * C_l^{\xi^{\phi_2(x)}}(\alpha)$$
$$\sum_{x\in\mathcal{R}} \sum_{\alpha\in\mathcal{I}} \alpha * C_u^{\xi^{\phi_1(x)}}(\alpha) = \sum_{x\in\mathcal{R}} \sum_{\alpha\in\mathcal{I}} \alpha * C_u^{\xi^{\phi_2(x)}}(\alpha)$$

then we call that ϕ_1 and ϕ_2 are *cardinally equivalent*.

If for any two fuzzy IC-Bags ϕ_1 and ϕ_2 drawn from $\mathcal{R}$, $\#_s(\phi_1) = \#_s(\phi_2)$, then they are called *cardinally equispaced*.

Proposition 5. *If any two fuzzy IC-Bags ϕ_1 and ϕ_2 drawn from $\mathcal{R}$ are cardinally equivalent, then they are cardinally equispaced.*

Proof. Let the two fuzzy IC-Bags ϕ_1 and ϕ_2 drawn from $\mathcal{R}$ be cardinally equivalent. Hence we have

$$\sum_{x\in\mathcal{R}}\sum_{\alpha\in\mathcal{I}}\alpha * C_l^{\xi^{\phi_1}(x)}(\alpha) = \sum_{x\in\mathcal{R}}\sum_{\alpha\in\mathcal{I}}\alpha * C_l^{\xi^{\phi_2}(x)}(\alpha)$$
$$\sum_{x\in\mathcal{R}}\sum_{\alpha\in\mathcal{I}}\alpha * C_u^{\xi^{\phi_1}(x)}(\alpha) = \sum_{x\in\mathcal{R}}\sum_{\alpha\in\mathcal{I}}\alpha * C_u^{\xi^{\phi_2}(x)}(\alpha)$$

This implies that the value of $\sum_{x\in\mathcal{R}}\sum_{\alpha\in\mathcal{I}}\alpha * C_u^{\xi^{\phi_1}(x)}(\alpha) - \sum_{x\in\mathcal{R}}\sum_{\alpha\in\mathcal{I}}\alpha * C_l^{\xi^{\phi_1}(x)}(\alpha)$ equals the value of $\sum_{x\in\mathcal{R}}\sum_{\alpha\in\mathcal{I}}\alpha * C_u^{\xi^{\phi_2}(x)}(\alpha) - \sum_{x\in\mathcal{R}}\sum_{\alpha\in\mathcal{I}}\alpha * C_l^{\xi^{\phi_2}(x)}(\alpha)$. Thus we have $\#_s(\phi_1) = \#_s(\phi_2)$. Hence proved. □

Definition 7. If ϕ be any fuzzy IC-Bag drawn from $\mathcal{R}$, then $\max_{x\in\mathcal{R},\alpha\in\mathcal{I}} C_l^{\xi^{\phi}(x)}(\alpha)$ and $\max_{x\in\mathcal{R},\alpha\in\mathcal{I}} C_u^{\xi^{\phi}(x)}(\alpha)$ are respectively called the *l-peak count* and *u-peak count* of ϕ.
For any fuzzy IC-Bag ϕ drawn from $\mathcal{R}$ if

$$\max_{x\in\mathcal{R},\alpha\in\mathcal{I}} C_l^{\xi^{\phi}(x)}(\alpha) = \max_{x\in\mathcal{R},\alpha\in\mathcal{I}} C_u^{\xi^{\phi}(x)}(\alpha),$$

then ϕ is called *balanced.* The elements p_i and p_j satisfying

$$C_l^{\xi^{\phi}(p_i)}(\alpha_m) = \max_{x\in\mathcal{R},\alpha\in\mathcal{I}} C_l^{\xi^{\phi}(x)}(\alpha)$$
$$C_l^{\xi^{\phi}(p_j)}(\alpha_n) = \max_{x\in\mathcal{R},\alpha\in\mathcal{I}} C_u^{\xi^{\phi}(x)}(\alpha)$$

are respectively called the *l-peak element* and *u-peak element* of ϕ. Here α_m and α_n are called the *fuzzy l-peak membership grade* of ϕ, and *fuzzy u-peak membership grade* of ϕ respectively.
If $\exists$ any $p \in \mathcal{R}, \alpha \in \mathcal{I}$ such that

$$C_l^{\xi^{\phi}(p)}(\alpha) = \max_{x\in\mathcal{R},\alpha\in\mathcal{I}} C_l^{\xi^{\phi}(x)}(\alpha)$$
$$C_u^{\xi^{\phi}(p)}(\alpha) = \max_{x\in\mathcal{R},\alpha\in\mathcal{I}} C_u^{\xi^{\phi}(x)}(\alpha)$$

then p is called the *lu-peak element* of ϕ and α is called the *fuzzy lu-peak membership grade* of ϕ.

An lu-peak element of ϕ is both an l-peak element and a u-peak element of ϕ. But an l-peak element or a u-peak element of ϕ is not necessarily an lu-peak element of ϕ.

It is obvious that any fuzzy IC-Bag ϕ drawn from $\mathcal{R}$ can have more than one l-peak element, u-peak element, or lu-peak element.

The set of all l-peak elements of ϕ is called the *l-peak set* of ϕ, the set of all u-peak elements of ϕ is called the *u-peak set* of ϕ, and the set of all lu-peak elements of ϕ is called the *lu-peak set* of ϕ.

The collection of all l-peak elements of ϕ together with their membership grades from the continuum $\mathcal{I}$ forms a fuzzy bag drawn from $\mathcal{R}$, called the *l-peak induced fuzzy bag* of ϕ.

The collection of all u-peak elements of ϕ together with their membership grades from the continuum $\mathcal{I}$ forms a fuzzy bag drawn from $\mathcal{R}$, called the *u-peak induced fuzzy bag* of ϕ.

The collection of all lu-peak elements of ϕ together with their membership grades from the continuum $\mathcal{I}$ forms a fuzzy bag drawn from $\mathcal{R}$, called the *lu-peak induced fuzzy bag* of ϕ.

Proposition 6. *For any fuzzy IC-Bag ϕ drawn from $\mathcal{R}$, $\exists$ atleast one $x \in \mathcal{R}$ for which the value of $\mu_{\beta(\phi)}(x)$ equals the fuzzy l-peak membership grade of ϕ.*

Proof. Let ϕ be any fuzzy IC-Bag drawn from $\mathcal{R}$. Then $\forall x \in \mathcal{R}$

$$\mu_{\beta(\phi)}(x) = \{\omega \in \mathcal{I} : C_l^{\xi^{\phi}(x)}(\omega) \geq C_l^{\xi^{\phi}(x)}(\alpha) \quad \forall \alpha \in \mathcal{I}\}$$

Clearly for any $x \in \mathcal{R}$, if $C_l^{\xi^{\phi}(x)}(\alpha_m) = \max\limits_{x \in \mathcal{R}, \alpha \in \mathcal{I}} C_l^{\xi^{\phi}(x)}(\alpha)$, then α_m is the l-peak membership grade of ϕ. We shall prove that $\exists$ atleast one $\overline{x} \in \mathcal{R}$ such that

$$\mu_{\beta(\phi)}(\overline{x}) = \max_{x \in \mathcal{R}, \alpha \in \mathcal{I}} C_l^{\xi^{\phi}(x)}(\alpha). \tag{10}$$

On the contrary let us suppose that there does not exist any $\overline{x} \in \mathcal{R}$ for which (10) is true. This implies if $\overline{x}$ is the fuzzy l-peak element of ϕ, then $\forall x \in \mathcal{R}, \alpha \in \mathcal{I}, C_l^{\xi^{\phi}(\overline{x})}(\alpha) < C_l^{\xi^{\phi}(x)}(\alpha)$ which is a contradiction. Hence proved. □

Definition 8. The addition of two fuzzy IC-Bags ϕ_1 and ϕ_2 drawn from $\mathcal{R}$ results in the fuzzy IC-Bag $\phi_1 \oplus \phi_2$ such that $\forall x \in \mathcal{R}, \alpha \in \mathcal{I}$

$$C_l^{\xi^{\phi_1 \oplus \phi_2}(x)}(\alpha) = C_l^{\xi^{\phi_1}(x)}(\alpha) + C_l^{\xi^{\phi_2}(x)}(\alpha)$$
$$C_u^{\xi^{\phi_1 \oplus \phi_2}(x)}(\alpha) = C_u^{\xi^{\phi_1}(x)}(\alpha) + C_u^{\xi^{\phi_2}(x)}(\alpha)$$

The removal of the fuzzy IC-Bag ϕ_2 from the fuzzy IC-Bag ϕ_1 results in the fuzzy IC-Bag $\phi_1 \ominus \phi_2$ such that $\forall x \in \mathcal{R}, \alpha \in \mathcal{I}$

$$C_l^{\xi^{\phi_1 \ominus \phi_2}(x)}(\alpha) = max(C_l^{\xi^{\phi_1}(x)}(\alpha) - C_u^{\xi^{\phi_2}(x)}(\alpha), 0)$$
$$C_u^{\xi^{\phi_1 \ominus \phi_2}(x)}(\alpha) = max(C_u^{\xi^{\phi_1}(x)}(\alpha) - C_l^{\xi^{\phi_2}(x)}(\alpha), 0)$$

Proposition 7. *For the fuzzy IC-Bags ϕ_1, ϕ_2, ϕ_3 drawn from $\mathcal{R}$, the following holds:*

(a) $\phi_1 \oplus \phi_2 = \phi_2 \oplus \phi_1$
(b) $\phi_1 \oplus (\phi_2 \oplus \phi_3) = (\phi_1 \oplus \phi_2) \oplus \phi_3$

Proof. (a) From the definition of addition of fuzzy IC-Bags, we have $\forall x \in \mathcal{R}, \alpha \in \mathcal{I}$

$$\begin{aligned} C_l^{\xi^{\phi_1 \oplus \phi_2}(x)}(\alpha) &= C_l^{\xi^{\phi_1}(x)}(\alpha) + C_l^{\xi^{\phi_2}(x)}(\alpha) \\ &= C_l^{\xi^{\phi_1}(x)}(\alpha) + C_l^{\xi^{\phi_2}(x)}(\alpha) \\ &= C_l^{\xi^{\phi_2}(x)}(\alpha) + C_l^{\xi^{\phi_1}(x)}(\alpha) \\ &= C_l^{\xi^{\phi_2 \oplus \phi_1}(x)}(\alpha) \end{aligned}$$

Similarly, we can show that $C_u^{\xi^{\phi_1 \oplus \phi_2}(x)}(\alpha) = C_u^{\xi^{\phi_2 \oplus \phi_1}(x)}(\alpha)$. This proves (a).
(b) From the definition of addition of fuzzy IC-Bags, we have $\forall x \in \mathcal{R}, \alpha \in \mathcal{I}$

$$\begin{aligned} C_l^{\xi^{(\phi_1 \oplus \phi_2) \oplus \phi_3}(x)}(\alpha) &= C_l^{\xi^{(\phi_1 \oplus \phi_2)}(x)}(\alpha) + C_l^{\xi^{\phi_3}(x)}(\alpha) \\ &= (C_l^{\xi^{\phi_1}(x)}(\alpha) + (C_l^{\xi^{\phi_2}(x)}(\alpha) + C_l^{\xi^{\phi_3}(x)}(\alpha)) \\ &= (C_l^{\xi^{\phi_1}(x)}(\alpha) + C_l^{\xi^{(\phi_2 \oplus \phi_3)}(x)}(\alpha) \\ &= C_l^{\xi^{\phi_1 \oplus (\phi_2 \oplus \phi_3)}(x)}(\alpha) \end{aligned}$$

Similarly we can show that $C_u^{\xi^{(\phi_1 \oplus \phi_2) \oplus \phi_3}(x)}(\alpha) = C_u^{\xi^{\phi_1 \oplus (\phi_2 \oplus \phi_3)}(x)}(\alpha)$. This proves (b). □

Definition 9. If ϕ_1 and ϕ_2 be the two fuzzy IC-Bags drawn from $\mathcal{R}$, then their union is the fuzzy IC-Bag $\phi_1 \sqcup \phi_2$ such that $\forall x \in \mathcal{R}, \alpha \in \mathcal{I}$

$$\begin{aligned} C_l^{\xi^{\phi_1 \sqcup \phi_2}(x)}(\alpha) &= max\{C_l^{\xi^{\phi_1}(x)}(\alpha), C_l^{\xi^{\phi_2}(x)}(\alpha)\} \\ C_u^{\xi^{\phi_1 \sqcup \phi_2}(x)}(\alpha) &= max\{C_u^{\xi^{\phi_1}(x)}(\alpha), C_u^{\xi^{\phi_2}(x)}(\alpha)\} \end{aligned}$$

By definition, we know that ϕ_1 and ϕ_2 are sub bags of $\phi_1 \sqcup \phi_2$, because we are certain that $C^{\xi^{\phi_1}(x)}(\alpha), C^{\xi^{\phi_2}(x)}(\alpha) \leq C^{\xi^{\phi_1 \sqcup \phi_2}(x)}(\alpha)$.
B The intersection of ϕ_1 and ϕ_2 is the fuzzy IC-Bag $\phi_1 \sqcap \phi_2$ such that $\forall x \in \mathcal{R}, \alpha \in \mathcal{I}$

$$\begin{aligned} C_l^{\xi^{\phi_1 \sqcap \phi_2}(x)}(\alpha) &= min\{C_l^{\xi^{\phi_1}(x)}(\alpha), C_l^{\xi^{\phi_2}(x)}(\alpha)\} \\ C_u^{\xi^{\phi_1 \sqcap \phi_2}(x)}(\alpha) &= min\{C_u^{\xi^{\phi_1}(x)}(\alpha), C_u^{\xi^{\phi_2}(x)}(\alpha)\} \end{aligned}$$

Clearly for any fuzzy IC-Bag ϕ drawn from $\mathcal{R}$, we have $\phi \sqcup \phi = \phi$, $\phi \sqcap \phi = \phi$.

Proposition 8. *For any two fuzzy IC-Bags ϕ_1 and ϕ_2 drawn from $\mathcal{R}$, the following holds:*

(a) $\phi_1 \sqcup \phi_2 = \phi_2 \sqcup \phi_1$
(b) $\phi_1 \sqcap \phi_2 = \phi_2 \sqcap \phi_1$
(c) $\phi_1 \sqcup (\phi_1 \sqcap \phi_2) = \phi_1$
(d) $\phi_1 \sqcap (\phi_1 \sqcup \phi_2) = \phi_1$

Proof. (a) For any two fuzzy IC-Bags ϕ_1 and ϕ_2 drawn from $\mathcal{R}$ we have $\forall x \in \mathcal{R}, \alpha \in \mathcal{I}$

$$C_l^{\xi^{\phi_1 \sqcup \phi_2}(x)}(\alpha) = max\{C_l^{\xi^{\phi_1}(x)}(\alpha), C_l^{\xi^{\phi_2}(x)}(\alpha)\}$$

$$= max\{C_l^{\xi^{\phi_2(x)}}(\alpha), C_l^{\xi^{\phi_1(x)}}(\alpha)\}$$
$$= C_l^{\xi^{\phi_2 \sqcup \phi_1(x)}}(\alpha)$$

Similarly we can show that $C_u^{\xi^{\phi_1 \sqcup \phi_2(x)}}(\alpha) = C_u^{\xi^{\phi_2 \sqcup \phi_1(x)}}(\alpha)$. This proves (a).

(b) For any two fuzzy IC-Bags ϕ_1 and ϕ_2 drawn from $\mathcal{R}$ we have $\forall x \in \mathcal{R}, \alpha \in \mathcal{I}$

$$C_l^{\xi^{\phi_1 \sqcap \phi_2(x)}}(\alpha) = min\{C_l^{\xi^{\phi_1(x)}}(\alpha), C_l^{\xi^{\phi_2(x)}}(\alpha)\}$$
$$= min\{C_l^{\xi^{\phi_2(x)}}(\alpha), C_l^{\xi^{\phi_1(x)}}(\alpha)\}$$
$$= C_l^{\xi^{\phi_2 \sqcap \phi_1(x)}}(\alpha)$$

Similarly we can show that $C_u^{\xi^{\phi_1 \sqcap \phi_2(x)}}(\alpha) = C_u^{\xi^{\phi_2 \sqcap \phi_1(x)}}(\alpha)$. This proves (b).

(c) For any two fuzzy IC-Bags ϕ_1 and ϕ_2 drawn from $\mathcal{R}$ we have $\forall x \in \mathcal{R}, \alpha \in \mathcal{I}$

$$C_l^{\xi^{(\phi_1 \sqcup (\phi_1 \sqcap \phi_2))(x)}}(\alpha) = max\{C_l^{\xi^{\phi_1(x)}}(\alpha), C_l^{\xi^{(\phi_1 \sqcap \phi_2)(x)}}(\alpha)\}$$
$$= max\{C_l^{\xi^{\phi_1(x)}}(\alpha), min\{C_l^{\xi^{\phi_1(x)}}(\alpha), C_l^{\xi^{\phi_2(x)}}(\alpha)\}\} \quad (11)$$

Case-I: If $C_l^{\xi^{\phi_1(x)}}(\alpha) < C_l^{\xi^{\phi_2(x)}}(\alpha)$, then from (11) we have

$$C_l^{\xi^{(\phi_1 \sqcup (\phi_1 \sqcap \phi_2))(x)}}(\alpha) = C_l^{\xi^{\phi_1(x)}}(\alpha) \quad (12)$$

Similarly we can show that

$$C_u^{\xi^{(\phi_1 \sqcup (\phi_1 \sqcap \phi_2))(x)}}(\alpha) = C_u^{\xi^{\phi_1(x)}}(\alpha) \quad (13)$$

From (12) and (13) it is clear that (c) holds for Case-I.

Case-II: If $C_l^{\xi^{\phi_1(x)}}(\alpha) = C_l^{\xi^{\phi_2(x)}}(\alpha)$, then from (11) we have

$$C_l^{\xi^{(\phi_1 \sqcup (\phi_1 \sqcap \phi_2))(x)}}(\alpha) = C_l^{\xi^{\phi_1(x)}}(\alpha) \quad (14)$$

Similarly we can show that

$$C_u^{\xi^{(\phi_1 \sqcup (\phi_1 \sqcap \phi_2))(x)}}(\alpha) = C_u^{\xi^{\phi_1(x)}}(\alpha) \quad (15)$$

From (14) and (15) it is clear that (c) holds for Case-II.

Case-III: If $C_l^{\xi^{\phi_1(x)}}(\alpha) > C_l^{\xi^{\phi_2(x)}}(\alpha)$, then from (11) we have

$$C_l^{\xi^{(\phi_1 \sqcup (\phi_1 \sqcap \phi_2))(x)}}(\alpha) = C_l^{\xi^{\phi_1(x)}}(\alpha) \quad (16)$$

Similarly we can show that

$$C_u^{\xi^{(\phi_1 \sqcup (\phi_1 \sqcap \phi_2))(x)}}(\alpha) = C_u^{\xi^{\phi_1(x)}}(\alpha) \quad (17)$$

From (16) and (17) it is clear that (c) holds for Case-III.

(d) For any two fuzzy IC-Bags ϕ_1 and ϕ_2 drawn from $\mathcal{R}$ we have $\forall x \in \mathcal{R}, \alpha \in \mathcal{I}$

$$\begin{aligned} C_l^{\xi^{(\phi_1 \sqcap (\phi_1 \sqcup \phi_2))(x)}}(\alpha) &= min\{C_l^{\xi^{\phi_1(x)}}(\alpha), C_l^{\xi^{(\phi_1 \sqcup \phi_2)(x)}}(\alpha)\} \\ &= min\{C_l^{\xi^{\phi_1(x)}}(\alpha), max\{C_l^{\xi^{\phi_1(x)}}(\alpha), C_l^{\xi^{\phi_2(x)}}(\alpha)\}\} \end{aligned} \tag{18}$$

Case-I: If $C_l^{\xi^{\phi_1(x)}}(\alpha) < C_l^{\xi^{\phi_2(x)}}(\alpha)$, then we have

$$\begin{aligned} C_l^{\xi^{(\phi_1 \sqcap (\phi_1 \sqcup \phi_2))(x)}}(\alpha) &= min\{C_l^{\xi^{\phi_1(x)}}(\alpha), C_l^{\xi^{\phi_2(x)}}(\alpha)\} \\ &= C_l^{\xi^{\phi_1(x)}}(\alpha) \end{aligned} \tag{19}$$

Similarly we can show that

$$C_u^{\xi^{(\phi_1 \sqcap (\phi_1 \sqcup \phi_2))(x)}}(\alpha) = C_u^{\xi^{\phi_1(x)}}(\alpha) \tag{20}$$

Hence (d) holds for Case-I.

Case-II: If $C_l^{\xi^{\phi_1(x)}}(\alpha) = C_l^{\xi^{\phi_2(x)}}(\alpha)$, then we have

$$\begin{aligned} C_l^{\xi^{(\phi_1 \sqcap (\phi_1 \sqcup \phi_2))(x)}}(\alpha) &= min\{C_l^{\xi^{\phi_1(x)}}(\alpha), C_l^{\xi^{\phi_1(x)}}(\alpha)\} \\ &= C_l^{\xi^{\phi_1(x)}}(\alpha) \end{aligned} \tag{21}$$

Similarly we can show that

$$C_u^{\xi^{(\phi_1 \sqcap (\phi_1 \sqcup \phi_2))(x)}}(\alpha) = C_u^{\xi^{\phi_1(x)}}(\alpha). \tag{22}$$

Hence (d) holds for Case-II.
Case-III: If $C_l^{\xi^{\phi_1(x)}}(\alpha) > C_l^{\xi^{\phi_2(x)}}(\alpha)$, then we have

$$\begin{aligned} C_l^{\xi^{(\phi_1 \sqcap (\phi_1 \sqcup \phi_2))(x)}}(\alpha) &= min\{C_l^{\xi^{\phi_1(x)}}(\alpha), C_l^{\xi^{\phi_1(x)}}(\alpha)\} \\ &= C_l^{\xi^{\phi_1(x)}}(\alpha) \end{aligned} \tag{23}$$

Similarly we can show that

$$C_u^{\xi^{(\phi_1 \sqcap (\phi_1 \sqcup \phi_2))(x)}}(\alpha) = C_u^{\xi^{\phi_1(x)}}(\alpha) \tag{24}$$

Hence (d) holds for Case-III. □

Proposition 9. *For any three fuzzy IC-Bags ϕ_1, ϕ_2 and ϕ_3 drawn from $\mathcal{R}$, the following holds:*

(a) $\phi_1 \sqcup (\phi_2 \sqcup \phi_3) = (\phi_1 \sqcup \phi_2) \sqcup \phi_3$
(b) $\phi_1 \sqcap (\phi_2 \sqcap \phi_3) = (\phi_1 \sqcap \phi_2) \sqcap \phi_3$

Proof. (a) Let ϕ_1, ϕ_2 and ϕ_3 be any three fuzzy IC-Bags drawn from $\mathcal{R}$. Then $\forall x \in \mathcal{R}, \alpha \in \mathcal{I}$ we have

$$\begin{aligned} C_l^{\xi^{\phi_1 \sqcup (\phi_2 \sqcup \phi_3)(x)}}(\alpha) &= max\{C_l^{\xi^{\phi_1(x)}}(\alpha), C_l^{\xi^{(\phi_2 \sqcup \phi_3)(x)}}(\alpha)\} \\ &= max\{C_l^{\xi^{\phi_1(x)}}(\alpha), max\{C_l^{\xi^{\phi_2(x)}}(\alpha), C_l^{\xi^{\phi_3(x)}}(\alpha)\}\} \end{aligned}$$

$$
\begin{aligned}
&= max\{C_l^{\xi^{\phi_1}(x)}(\alpha), C_l^{\xi^{\phi_2}(x)}(\alpha), C_l^{\xi^{\phi_3}(x)}(\alpha)\} \\
&= max\{max\{C_l^{\xi^{\phi_1}(x)}(\alpha), C_l^{\xi^{\phi_2}(x)}(\alpha)\}, C_l^{\xi^{\phi_3}(x)}(\alpha)\} \\
&= max\{C_l^{\xi^{(\phi_1 \sqcup \phi_2)}(x)}(\alpha), C_l^{\xi^{\phi_3}(x)}(\alpha)\} \\
&= C_l^{\xi^{(\phi_1 \sqcup \phi_2) \sqcup \phi_3}(x)}(\alpha)
\end{aligned}
$$

Similarly we can show that $\forall x \in \mathcal{R}, \alpha \in \mathcal{I}$, $C_u^{\xi^{\phi_1 \sqcup (\phi_2 \sqcup \phi_3)}(x)}(\alpha) = C_u^{\xi^{(\phi_1 \sqcup \phi_2) \sqcup \phi_3}(x)}(\alpha)$. This proves (a).

(b) Let ϕ_1, ϕ_2 and ϕ_3 be any three fuzzy IC-Bags drawn from $\mathcal{R}$. Then $\forall x \in \mathcal{R}, \alpha \in \mathcal{I}$ we have

$$
\begin{aligned}
C_l^{\xi^{\phi_1 \sqcap (\phi_2 \sqcap \phi_3)}(x)}(\alpha) &= min\{C_l^{\xi^{\phi_1}(x)}(\alpha), C_l^{\xi^{(\phi_2 \sqcap \phi_3)}(x)}(\alpha)\} \\
&= min\{C_l^{\xi^{\phi_1}(x)}(\alpha), min\{C_l^{\xi^{\phi_2}(x)}(\alpha), C_l^{\xi^{\phi_3}(x)}(\alpha)\}\} \\
&= min\{C_l^{\xi^{\phi_1}(x)}(\alpha), C_l^{\xi^{\phi_2}(x)}(\alpha), C_l^{\xi^{\phi_3}(x)}(\alpha)\} \\
&= min\{min\{C_l^{\xi^{\phi_1}(x)}(\alpha), C_l^{\xi^{\phi_2}(x)}(\alpha)\}, C_l^{\xi^{\phi_3}(x)}(\alpha)\} \\
&= min\{C_l^{\xi^{(\phi_1 \sqcap \phi_2)}(x)}(\alpha), C_l^{\xi^{\phi_3}(x)}(\alpha)\} \\
&= C_l^{\xi^{(\phi_1 \sqcap \phi_2) \sqcap \phi_3}(x)}(\alpha)
\end{aligned}
$$

Similarly we can show that $\forall x \in \mathcal{R}, \alpha \in \mathcal{I}$, $C_u^{\xi^{\phi_1 \sqcap (\phi_2 \sqcap \phi_3)}(x)}(\alpha) = C_u^{\xi^{(\phi_1 \sqcap \phi_2) \sqcup \phi_3}(x)}(\alpha)$. This proves (b). □

Proposition 10. *For any two fuzzy IC-Bags ϕ_1 and ϕ_2 drawn from $\mathcal{R}$,*

$$\phi_1 \sqcap \phi_2 = \phi_1 \Longleftrightarrow \phi_1 \sqcup \phi_2 = \phi_2.$$

Proof. Let $\phi_1 \sqcap \phi_2 = \phi_1$. Then $\forall x \in \mathcal{R}, \alpha \in \mathcal{I}$

$$
\begin{aligned}
C_l^{\xi^{\phi_1 \sqcap \phi_2}(x)}(\alpha) &= C_l^{\xi^{\phi_1}(x)}(\alpha) \\
C_u^{\xi^{\phi_1 \sqcap \phi_2}(x)}(\alpha) &= C_u^{\xi^{\phi_1}(x)}(\alpha)
\end{aligned}
$$

which implies

$$
\begin{aligned}
min\{C_l^{\xi^{\phi_1}(x)}(\alpha), C_l^{\xi^{\phi_2}(x)}(\alpha)\} &= C_l^{\xi^{\phi_1}(x)}(\alpha) \\
min\{C_u^{\xi^{\phi_1}(x)}(\alpha), C_u^{\xi^{\phi_2}(x)}(\alpha)\} &= C_u^{\xi^{\phi_1}(x)}(\alpha)
\end{aligned}
$$

Also we have $\forall x \in \mathcal{R}, \alpha \in \mathcal{I}$

$$
\begin{aligned}
C_l^{\xi^{\phi_1 \sqcup \phi_2}(x)}(\alpha) &= max\{C_l^{\xi^{\phi_1}(x)}(\alpha), C_l^{\xi^{\phi_2}(x)}(\alpha)\} \\
C_u^{\xi^{\phi_1 \sqcup \phi_2}(x)}(\alpha) &= max\{C_u^{\xi^{\phi_1}(x)}(\alpha), C_u^{\xi^{\phi_2}(x)}(\alpha)\}
\end{aligned}
$$

Clearly from the above, we have $\forall x \in \mathcal{R}, \alpha \in \mathcal{I}$

$$
\begin{aligned}
C_l^{\xi^{\phi_1 \sqcup \phi_2}(x)}(\alpha) &= C_l^{\xi^{\phi_2}(x)}(\alpha) \\
C_u^{\xi^{\phi_1 \sqcup \phi_2}(x)}(\alpha) &= C_u^{\xi^{\phi_2}(x)}(\alpha)
\end{aligned}
$$

This proves the necessary part. Conversely, let us assume that $\phi_1 \sqcup \phi_2 = \phi_2$. Then $\forall x \in \mathcal{R}, \alpha \in \mathcal{I}$

$$C_l^{\xi^{\phi_1 \sqcup \phi_2}(x)}(\alpha) = C_l^{\xi^{\phi_2}(x)}(\alpha)$$
$$C_u^{\xi^{\phi_1 \sqcup \phi_2}(x)}(\alpha) = C_u^{\xi^{\phi_2}(x)}(\alpha)$$

Thus we have $\forall x \in \mathcal{R}, \alpha \in \mathcal{I}$

$$max\{C_l^{\xi^{\phi_1}(x)}(\alpha), C_l^{\xi^{\phi_2}(x)}(\alpha)\} = C_l^{\xi^{\phi_2}(x)}(\alpha)$$
$$max\{C_u^{\xi^{\phi_1}(x)}(\alpha), C_u^{\xi^{\phi_2}(x)}(\alpha)\} = C_u^{\xi^{\phi_2}(x)}(\alpha)$$

Hence $\forall x \in \mathcal{R}, \alpha \in \mathcal{I}$

$$C_l^{\xi^{\phi_1 \sqcap \phi_2}(x)}(\alpha) = C_l^{\xi^{\phi_1}(x)}(\alpha)$$
$$C_u^{\xi^{\phi_1 \sqcap \phi_2}(x)}(\alpha) = C_u^{\xi^{\phi_1}(x)}(\alpha)$$

This proves the sufficient part. □

4. Decision Analysis and Fuzzy IC-Bags

It is often observed that the academic libraries have to take decisions regarding the ordering and re-ordering of the copies of books or any other multimedia objects that it needs from time to time depending on the ongoing types of demand patterns, backlogs, loss, and depreciation of the copies in hand, present status of reserve-room statistics etc. The hesitation factors associated with the requests received from the experts for respective numbers of copies required to be ordered for each object should be recognized and subsequently quantified for the necessary soft decision analysis needed to reach a feasible and and deterministic ordering pattern. Here we shall fundamentally analyze the patterns associated with the design phase of soft decision making, when we start developing the alternatives and it is found that in some cases, the number of possible alternatives and the number of operationally feasible alternatives can vary considerably.

Let us consider that n experts $E_1, E_2, ...E_n$ request copies of each of the m listed objects $O_1, O_2, ...O_m$. The individual requests are non-crisp and can be represented by Table 1.

$$C_l^j : O \longrightarrow N \quad and \quad C_u^j : O \longrightarrow N$$

where $C_l^j(x) \le C_u^j(x) \forall x \in O$; $O = \{O_1, O_2, ...O_m\}$,$j = 1, 2, ...n$.

For each E_j $(j = 1, 2....., n)$, and for each $x \in O$

$$H^j(x) = C_u^j(x) - C_l^j(x)$$

Table 1.

E_j	$C_l^j(x)$	$C_u^j(x)$	$\psi_j(x)$
E_1	$C_l^1(x)$	$C_u^1(x)$	$\psi_1(x)$
E_2	$C_l^2(x)$	$C_u^2(x)$	$\psi_2(x)$
-	-	-	-
-	-	-	-
-	-	-	-
E_n	$C_l^n(x)$	$C_u^n(x)$	$\psi_n(x)$

represents the psychological hesitation factor associated with the determination of the precise number of required objects which is often found to be contextual in nature. In this case, the human bias embodied in the judgment is caused due to the significance or importance of the respective object as perceived by the expert which is found to be possibilistic in nature. Hence, we associate a set ψ_j $(j = 1, 2, ...n)$ of n fuzzy membership functions which represent the contextual degree of importance for each object x in O as perceived by the experts E_j $(j = 1, 2, ...n)$. It is represented by

$$\psi_j : O \longrightarrow [0, 1]$$

Thus, each object $x \in O$ can be characterized by three attribute values $(C_l^j(x), C_u^j(x), \psi_j(x))$ by each expert E_j.

For each $x \in O$, we have the following I/P table represented by Table 2.

For each $x \in O$, $\frac{1}{j} \sum_{j=1}^{n} \psi_j(x)$ represents the possibilistic mean importance factor for the object x with respect to the concerned information system.

$$\delta_j(x) = \{\frac{1}{j} \sum_{j=1}^{n} \psi_j(x)\} - \psi_j(x)$$

represents the psychological bias factor associated with the conceptual belief measure in both positive and negative direction.

For each $x \in O$, the value of $\pi_\Delta^j(x)$ which is the product of the values of $C_\Delta^j(x)$ and $\delta_j(x)$ is called the allocation factor of x for j.

For any $x \in O$, we use the following table for representing the values of $C_\Delta^j(x)$ and $\delta_j(x)$, and $\pi_\Delta^j(x)$, $j = 1, 2, ...n$:

The final allocation count of each $x \in O$, for each $j = 1, 2, ...n$, will be determined by

Table 2.

E_j	$C^j_\Delta(x)$	$\delta_j(x)$	$\pi^j_\Delta(x)$
E_1	$C^1_\Delta(x)$	$\delta_1(x)$	$\pi^1_\Delta(x)$
E_2	$C^2_\Delta(x)$	$\delta_2(x)$	$\pi^2_\Delta(x)$
-	-	-	-
-	-	-	-
-	-	-	-
E_n	$C^n_\Delta(x)$	$\delta_n(x)$	$\pi^n_\Delta(x)$

$$\mathcal{A}_j(x) = \pi^j_\Delta(x) + C^j_\mu(x)$$

rounded upto the nearest integer value.

5. Conclusion

In this paper, we have introduced the notion of fuzzy IC-Bags and defined some operations on fuzzy IC-bags. The notions of fuzzy base sets, fuzzy l-IC-subbags, fuzzy u-IC-subbags, cardinally equivalent fuzzy IC-bags, cardinally equispaced fuzzy IC-Bags, types of peak membership grades, base-equivalent fuzzy IC-Bags, l-equality and u-equality of fuzzy IC-bags, have been all defined. Some characterizations concerning these notions have also been done. It is observed that fuzzy IC-Bags can serve as important tools for problems concerning situations where the uncertainty is associated with the interval valued counts of objects. Hence the introduction of the concept of fuzzy IC-Bags could be viewed as a significant development towards building the future theoretical basis and knowledge framework for the modelling of intelligent systems concerning these complex behaviours of interval-valued uncertainty patterns.

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