First published in the Journal of Geometric Analysis in Vol. 11, No. 3, 2001,

published by the American Mathematical Society

AUTOMORPHISMS OF NONDEGENERATE CR QUADRICS AND SIEGEL DOMAINS. EXPLICIT DESCRIPTION

VLADIMIR V. EŽOV AND GERD SCHMALZ

ABSTRACT. In this paper we give the complete explicit description of the holomorphic automorphisms of any nondegenerate CR-quadric Q of arbitrary CR-dimension and codimension and of Siegel domains of second kind with not necessarily Levi-nondegenerate Šilov-boundary.

We introduce a family of k-dimensional chains ($k = \operatorname{codim} Q$), the analogues of one-dimensional Chern-Moser chains for hyperquadrics.

We also analyse some different types of rigid quadrics.

1. Introduction

Let $z = (z^1, \ldots, z^n)$, $w = (w^1, \ldots, w^k)$ be coordinates in $\mathbb{C}^{n+k}$, $k \geq 1$, and

$$\langle z, z\rangle = \begin{pmatrix} \langle z, z\rangle^1 \\ \vdots \\ \langle z, z\rangle^k \end{pmatrix}$$

be a $\mathbb{C}^k$-valued Hermitian form on $\mathbb{C}^n$.

Consider the cone C being the interior of the convex hull of $\{\langle z, z\rangle : z \in \mathbb{C}^n\}$. Suppose C is an acute cone, i.e., C does not contain any entire line. This property takes place if and only if the form $\langle z, z\rangle$ is positive definite, i.e., in appropriate coordinates all the forms $\langle z, z\rangle^\varkappa$ are positive definite.

Let $V \supset C$ be an open acute cone in $\mathbb{R}^k$. The domain

$$\Omega_V = \{(z, w) \in \mathbb{C}^{n+k} : \operatorname{Im} w - \langle z, z\rangle \in V\}$$

is called Siegel domain of the second kind, associated with the cone V. (For simplicity we shall call them Siegel domains.)

Siegel domains were introduced by Pyatetskii-Shapiro [11] for the study of automorphic forms in several variables, homogeneous and symmetric domains. In particular, Pyatetskii-Shapiro constructed an example of a Siegel domain which is homogeneous but not symmetric. In general, a Siegel domain Ω_V is not necessarily homogeneous.

Kaup, Matsushima and Ochiai [8] proved that the infinitesimal automorphisms of Siegel domains are quadratic vector fields and that the automorphisms of Ω

Research of the first author was supported by Max-Planck-Institut Bonn.

Research of the second author was supported by Deutsche Forschungsgemeinschaft.

extend to birational maps of $\mathbb{C}^{n+k}$. An explicit description of of these infinitesimal automorphisms was given by Satake [13].

The quadric

$$Q = \{(z, w) \in \mathbb{C}^{n+k} : \operatorname{Im} w = \langle z, z\rangle\}$$

forms the Šilov boundary of Ω. Any automorphism of Ω maps Q into Q (but may have poles on Q).

Henkin and Tumanov [7] established a natural correspondence between $\operatorname{Aut}\Omega_C$ and the group of CR automorphisms of Q in the case when $V = C$. Under the assumption that the forms $\langle z, z\rangle^\varkappa$, $\varkappa = 1, \ldots, k$, are linearly independent they proved that any $\phi \in \operatorname{Aut}\Omega_C$ extends to a biholomorphic automorphism[1] of Q and, conversely, any locally defined CR automorphism of Q extends to an automorphism of the entire domain Ω and, in particular to a global automorphism of Q.

Considering the group $\operatorname{Aut} Q$ of an arbitrary Hermitian quadric Q, Belošapka [2] found a necessary and sufficient condition for $\langle z, z\rangle$ (not necessarily positive definite) which implies that $\operatorname{Aut} Q$ is a finite dimensional Lie group:

i.) The forms $\langle\cdot,\cdot\rangle^\varkappa$, $\varkappa = 1, \ldots, k$ are linearly independent. Geometrically this condition means that C has nonempty interior.

ii.) The form $\langle z, z\rangle$ does not have an annihilator, i.e., the condition $\langle a, z\rangle = 0$ for all $z \in \mathbb{C}^n$ implies that $a = 0$.

Quadrics Q which satisfy these conditions are called nondegenerate.

The nondegenerate quadrics which represent Šilov boundaries of Siegel domains should just satisfy condition *i.)* because any positive definite form $\langle z, z\rangle$ does not have an annihilator.

For nondegenerate quadrics Belošapka [1] described the infinitesimal automorphisms which are quadratic vector fields as in the case of Siegel domains (A simple proof of this was given in ([6]) by the authors). Tumanov [14] proved that their automorphisms are rational and extend to birational automorphisms of $\mathbb{C}^{n+k}$.

In this paper we obtain an explicit formula for the automorphisms of arbitrary nondegenerate quadrics, and for the automorphisms of Siegel domains of second kind.

We are grateful to V. Belošapka and A. Tumanov for inspiration and many useful discussions.

2. Infinitesimal automorphisms of CR quadrics and Siegel domains

The quadric $Q : \operatorname{Im} w = \langle z, z\rangle$ is a homogeneous manifold. The group H of Heisenberg translations $(z, w) \mapsto (z + p, w + q + 2i\langle z, p\rangle)$, $(p, q) \in Q$ acts transitively on Q. Thus, $\operatorname{Aut} Q$ splits into the semidirect product

[1] In the given non-bounded representation of Ω the birational extension of ϕ may have poles on Q, but if we choose coordinates where Ω is bounded then ϕ has no poles on Q.

$$\operatorname{Aut} Q = H \ltimes \operatorname{Aut}_0 Q,$$

where $\operatorname{Aut}_0 Q = \{\phi \in \operatorname{Aut} Q : \phi(0) = 0\}$ is the isotropy group of the origin. $\operatorname{Aut}_0 Q$ also splits:

$$\operatorname{Aut}_0 Q = L \ltimes \operatorname{Aut}_{0,\mathrm{id}} Q,$$

where $\operatorname{Aut}_{0,\mathrm{id}} Q = \{\phi \in \operatorname{Aut}_0 Q : d\phi(0)|_{T_0^{\mathbb{C}} Q} = \mathrm{id}\}$, and L is the group of linear transformations $(z, w) \mapsto (Cz, \rho w)$ $(C \in \mathrm{GL}(n, \mathbb{C}), \rho \in \mathrm{GL}(k, \mathbb{R}))$ such that $\langle Cz, Cz\rangle = \rho\langle z, z\rangle$.

Hence,

$$\operatorname{Aut} Q = H \ltimes L \ltimes \operatorname{Aut}_{0,\mathrm{id}} Q. \tag{1}$$

There is a similar splitting of the automorphism groups of Siegel domains. It is easy to verify that the Heisenberg group of the Šilov boundary Q consists of automorphisms of the Siegel domain Ω. Now, any automorphism can be represented as a composition of a Heisenberg translation and an automorphism without pole at the origin. The latter automorphisms can be uniquely decomposed as above.

As mentioned above, Satake and Belošapka gave explicit descriptions of the Lie algebras $\mathfrak{g}$ of infintesimal automorphisms of Siegel domains, resp. nondegenerate quadrics. The splitting (1) implies that $\mathfrak{g}$ can be represented as semidirect sum

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_o \oplus \mathfrak{g}_+,$$

where $\mathfrak{g}_-, \mathfrak{g}_o, \mathfrak{g}_+$ are the Lie algebras of $H, L, \operatorname{Aut}_{0,\mathrm{id}}$, respectively.

The algebra $\mathfrak{g}_-$ consists of the vector fields

$$\chi_- = \sum_{j=1}^{n} p^j \frac{\partial}{\partial z^j} + \sum_{j=1}^{k} (q^j + 2i\langle z, p\rangle^j) \frac{\partial}{\partial w^j},$$

with $p \in \mathbb{C}^n$, $q \in \mathbb{R}^k$.

The algebra $\mathfrak{g}_0$ consists of the vector fields

$$\chi_0 = \sum_{j=1}^{n} (Xz)^j \frac{\partial}{\partial z^j} + \sum_{j=1}^{k} (sw)^j \frac{\partial}{\partial w^j},$$

where $X \in \mathfrak{gl}\,(n, \mathbb{C})$, $s \in \mathfrak{gl}\,(k, \mathbb{R})$ satisfy the condition $2\operatorname{Re}\langle Xz, z\rangle = s\langle z, z\rangle$, and, in the case of a Siegel domain, s is contained in the Lie algebra $\mathfrak{g}(V)$ corresponding to the subgroup of $G(V) \subset \mathrm{GL}(k, \mathbb{R})$ of linear mapping which preserve the cone V.

For the description of $\mathfrak{g}_+$ we need the following tensors A, a, B, r:

Let $a : \mathbb{C}^k \longrightarrow \mathbb{C}^n$ be a linear operator, A be a $\mathbb{C}^n$-valued symmetric bilinear form on $\mathbb{C}^n \otimes \mathbb{C}^n$, r be an $\mathbb{R}^k$-valued symmetric bilinear form on $\mathbb{R}^k$, and B be a $\mathbb{C}^n$-valued bilinear form on $\mathbb{C}^k \otimes \mathbb{C}^n$ satisfying

$$\begin{aligned} \langle A(z,z),z\rangle &= 2i\langle z,a\langle z,z\rangle\rangle, &(2)\\ \mathrm{Re}\langle B(u,z),z\rangle &= r(u,\langle z,z\rangle), &(3)\\ \mathrm{Im}\langle B(\langle z,z\rangle,z),z\rangle &= 0 \end{aligned}$$

for all $z \in \mathbb{C}^n$ and $u \in \mathbb{R}^k$, and, in the case of a Siegel domain,

$$\begin{aligned} \mathrm{Im}\langle z, a\bar{\cdot}\rangle &\in \mathfrak{g}(V)\\ r(u,\cdot) &\in \mathfrak{g}(V)\\ \mathrm{Im}\langle B(\cdot,z),\zeta\rangle &\in \mathfrak{g}(V)\\ \mathrm{Im}\,\mathrm{tr}\,B(u,\cdot) &= 0, \end{aligned}$$

for all $z,\zeta \in \mathbb{C}^n$ and $u \in \mathbb{R}^k$.
Then

$$\chi_+ = \sum_{j=1}^{n}(aw + A(z,z) + B(w,z))^j \frac{\partial}{\partial z^j} + \sum_{j=1}^{k}(2i\langle z, a\bar{w}\rangle + r(w,w))^j \frac{\partial}{\partial w^j}.$$

The additional conditions in the case of Siegel domains are automatically satisfied when the Šilov boundary is a nondegenerate quadric and V coincides with the acute open cone C.

Any automorphism $\phi \in \mathrm{Aut}_{0,\mathrm{id}}$ is the image of some infinitesimal automorphism under the exponential map. This follows directly from the uniqueness theorem (see [6], for Siegel domains see also [8]). In fact, any $\phi \in \mathrm{Aut}_{0,\mathrm{id}}$ is determined by the parameters

$$\begin{aligned} a &:= \left.\frac{df}{dw}\right|_0\\ r &:= \left.\frac{d^2g}{dw^2}\right|_0. \end{aligned}$$

Moreover, there exist A and B such that the systems (2) and (3) are satisfied. On the other hand, the one-parametric family ϕ_t that corresponds to these parameters takes for $t = 1$ the value ϕ.

In order to calculate ϕ we have to integrate the system

$$\begin{aligned} \frac{d}{dt}f_t &= ag_t + A(f_t,f_t) + B(g_t,f_t)\\ \frac{d}{dt}g_t &= 2i\langle f_t, a\bar{g}_t\rangle + r(g_t,g_t). \end{aligned}$$

The first step will be to determine all first and second derivatives of f_t and g_t at the origin. In order to do this we derive the system of ODE from above with respect to z, resp. w and restrict it to $z = 0, w = 0$. This leads to

$$\begin{aligned} f_t &= z + taw + tA(z,z) + tB(w,z) + t^2A(z,aw) + it^2a\langle z, a\bar{w}\rangle \\ &\quad + K(w,w) + \mathrm{o}(|z|^2 + |w|^2) \\ g_t &= w + 2it\langle z, a\bar{w}\rangle + tr(w,w) + it^2\langle aw, a\bar{w}\rangle + \mathrm{o}(|z|^2 + |w|^2). \end{aligned} \tag{4}$$

(K will be determined later.)

It is still an open question whether the dimension of $\mathfrak{g}_+$ can be estimated by $2n + k$. For Siegel domains this sharp estimate was proved by Kaup, Matsushima and Ochiai [8] (see also [12]). Using Cartan's theorem they showed that the radical of $\mathfrak{g}$ intersects $\mathfrak{g}_+$ by $\{0\}$ and therefore $\mathfrak{g}_+$ is isomorphic to the dual of the factor space of $\mathfrak{g}_-$ by the radical. However, there are nondegenerate quadrics with nontrivial intersection of $\mathfrak{g}_+$ with the radical (e.g., the parabolic quadric in $\mathbb{C}^{2+2}$ being defined by $\operatorname{Im} w^1 = |z^1|^2$, $\operatorname{Im} w^2 = 2\operatorname{Re} z^1\bar{z}^2$).

3. Results

Let Q be a nondegenerate quadric, or Ω be a Siegel domain, and $\phi = (f,g) \in \mathrm{Aut}_{0,\mathrm{id}}$ be the automorphism which corresponds to the parameters (a, A, r, B). Furthermore, let $f = \sum_{l=0}^{\infty} f_l$, $g = \sum_{l=0}^{\infty} g_l$ be the expansion into homogeneous polynomials then we prove

Theorem 1. *The polynomials f_l, g_l are determined by the recursive relations*

$$(l-1)\begin{pmatrix} f_l \\ g_l \end{pmatrix} = \begin{pmatrix} \frac{\partial f_{l-1}}{\partial z} & \frac{\partial f_{l-1}}{\partial w} \\ \frac{\partial g_{l-1}}{\partial z} & \frac{\partial g_{l-1}}{\partial w} \end{pmatrix} \begin{pmatrix} A(z,z) + B(w,z) + A(aw,z) - ia\langle z, a\bar{w}\rangle \\ 2i\langle z, a\bar{w}\rangle + r(w,w) + i\langle aw, a\bar{w}\rangle \end{pmatrix}, \tag{5}$$

for $l > 1$ and the initial conditions $f_0 = 0, g_0 = 0, f_1 = z + aw, g_1 = w$.

Consider the real k-plane $\Gamma_0 = \{z = 0, \operatorname{Im} w = 0\}$ which is contained in Q. The orbit of Γ_0 under the action of $\mathrm{Aut}_0\, Q$ composes a biholomorphically invariant family of real k-manifolds on Q passing through the origin. These k-manifolds are called chains as the analogous objects on hypersurfaces. The following theorem generalizes the fact that the chains on hyperquadrics are the intersections of the hyperquadric with complex lines passing through the origin and being transversal to the complex tangent space.

Theorem 2. *Any chain $\Gamma \subset Q$ is the intersection of Q with the complex k-plane $\{z = aw\}$, with $a \in \mathcal{A}$.*

The main result of this paper is the following explicit description of the automorphisms from $\mathrm{Aut}_{0,\mathrm{id}}$.

Theorem 3. *Let* $(z^*, w^*) = \phi(z,w)$ *be from* $\mathrm{Aut}_{0,\mathrm{id}}$*, then*

$$\begin{pmatrix} z^* \\ w^* \end{pmatrix} = \left(\mathrm{id} - \begin{pmatrix} \mathfrak{P}_\mathfrak{p} & \mathfrak{P}_\mathfrak{q} \\ \mathfrak{Q}_\mathfrak{p} & \mathfrak{Q}_\mathfrak{q} \end{pmatrix} \right)^{-1} \begin{pmatrix} z - aw - A(z,z) - 2B(w,z) \\ w - i\langle z, a\bar{w}\rangle - r(w,w) \end{pmatrix},$$

where $\mathfrak{P}_\mathfrak{p}$*,* $\mathfrak{Q}_\mathfrak{p}$*,* $\mathfrak{P}_\mathfrak{q}$*,* $\mathfrak{Q}_\mathfrak{q}$ *are the following polynomial matrices:*

$$\begin{aligned}
\mathfrak{P}_\mathfrak{p} \;=\;& 2A(z,\cdot) + B(w,\cdot) + A(aw,\cdot) - ia\langle \cdot, a\bar{w}\rangle - 2A(A(z,\cdot),z) + \\
&+A(A(z,z),\cdot) + A(B(w,z),\cdot) - A(B(w,\cdot),z) - iB(\langle \cdot, a\bar{w}\rangle, z) + \\
&+iB(\langle z, a\bar{w}\rangle, \cdot) - B(w, A(z,\cdot)) + A(A(z,aw),\cdot) - A(A(z,\cdot),aw) - \\
&-A(A(\cdot,aw),z) - 2a\langle z, a\langle a\bar{w},\cdot\rangle\rangle, \\
\mathfrak{Q}_\mathfrak{p} \;=\;& i\langle \cdot, a\bar{w}\rangle + 2\langle z, a\langle a\bar{w},\cdot\rangle\rangle - \frac{i}{2}\langle B(w,\cdot), a\bar{w}\rangle + \frac{i}{2}\langle \cdot, ar(\bar{w},\bar{w})\rangle - \\
&-ir(\langle \cdot, a\bar{w}\rangle, w) + \langle aw, a\langle a\bar{w},\cdot\rangle\rangle, \\
\mathfrak{P}_\mathfrak{q} \;=\;& 2a + 2B(\cdot,z) - 2A(a\cdot,z) - 2ia\langle z, \bar{a\cdot}\rangle - 2B(w,a\cdot) - 2ia\langle aw, \bar{a\cdot}\rangle - \\
&-4iB(\langle z, \bar{a\cdot}\rangle, z) + 2A(A(z,a\cdot),z) - A(A(z,z),a\cdot) - B(w,B(\cdot,z)) + \\
&+B(\cdot,B(w,z)) - 2B(r(\cdot,w),z) + B(w,A(a\cdot,z)) + iB(\langle a\cdot, a\bar{w}\rangle, z) + \\
&+iB(\cdot, a\langle z, a\bar{w}\rangle) - iB(w, a\langle z, \bar{a\cdot}\rangle) + B(\cdot, A(aw,z)) - 2iB(\langle z, \bar{a\cdot}\rangle, aw) - \\
&-iB(\langle aw, \bar{a\cdot}\rangle, z) + 4ia\langle z, B(\bar{w}, \bar{a\cdot})\rangle - 2ia\langle z, ar(\bar{w},\bar{\cdot})\rangle - A(B(w,z),a\cdot) + \\
&+A(B(w,a\cdot),z) - A(B(\cdot,z),aw) - A(B(\cdot,aw),z) + ia\langle B(w,z), \bar{a\cdot}\rangle - \\
&-ia\langle B(\cdot,z), a\bar{w}\rangle + ia\langle z, A(a\bar{w}, \bar{a\cdot})\rangle + A(A(aw,a\cdot),z) - A(A(aw,z),a\cdot) + \\
&+A(A(a\cdot,z),aw) + ia\langle A(z,aw), \bar{a\cdot}\rangle - ia\langle A(z,a\cdot), a\bar{w}\rangle + a\langle aw, a\langle \bar{a\cdot}, z\rangle\rangle - \\
&-a\langle a\cdot, a\langle a\bar{w}, z\rangle\rangle,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{Q}_\mathfrak{q} \;=\;& 2i\langle z, \bar{a\cdot}\rangle + 2r(\cdot,w) - i\langle a\cdot, a\bar{w}\rangle + i\langle aw, \bar{a\cdot}\rangle - 2i\langle z, B(\bar{w}, \bar{a\cdot})\rangle - \\
&-2\langle z, a\langle a\bar{w}, a\cdot\rangle\rangle - 2r(r(w,\cdot),w) + r(r(w,w),\cdot) + i\langle B(w,a\cdot), a\bar{w}\rangle - \\
&-i\langle ar(w,\cdot), a\bar{w}\rangle + ir(\langle a\cdot, a\bar{w}\rangle, w) - ir(\langle aw, \bar{a\cdot}\rangle, w) + ir(\langle aw, a\bar{w}\rangle, \cdot) - \\
&-\frac{i}{2}\langle a\cdot, ar(\bar{w},\bar{w})\rangle + \frac{i}{2}\langle ar(w,w), \bar{a\cdot}\rangle - i\langle aw, ar(\bar{w},\bar{\cdot})\rangle - \langle aw, a\langle a\bar{w}, a\cdot\rangle\rangle.
\end{aligned}$$

In $\mathfrak{P}_\mathfrak{p}$ *and* $\mathfrak{Q}_\mathfrak{p}$ *the dot stands instead of a complex* n*-dimensional vector argument and in* $\mathfrak{P}_\mathfrak{q}$ *and* $\mathfrak{Q}_\mathfrak{q}$ *instead of a complex* k*-dimensional vector argument.*

4. Recursive formulas for the automorphisms

For shortness of the notations we introduce the following abbreviations: in the given fixed coordinates we will denote the vector field $\chi = \sum_{\nu=1}^n C^\nu \frac{\partial}{\partial z^\nu} + \sum_{\varkappa=1}^k D^\varkappa \frac{\partial}{\partial w^\varkappa}$ by $\chi = (C, D)$ as well. If f is an n-vector and E is an $n \times m$ matrix with columns E_μ then by $\langle f, E\rangle$ we denote the $k \times m$-matrix with columns $\langle f, E_\mu\rangle$.

We consider the canonical action of $\mathrm{Aut}_{0,\mathrm{id}}\, Q$ on the Lie algebra $\mathfrak{g}$: Let $\chi \in \mathfrak{g}$ and $\phi = (f, g) \in \mathrm{Aut}_{0,\mathrm{id}}\, Q$, then

$$\phi^*(\chi)(z, w) = (d\phi)^{-1}(\chi(f, g)).$$

Hence, if $\chi = (C, D) = \sum_{j=1}^n C^j \frac{\partial}{\partial z^j} + \sum_{m=1}^k D^k \frac{\partial}{\partial w^k}$, then $\phi^*(C, D) = (P, Q)$ equals

$$\begin{pmatrix} P(z,w) \\ Q(z,w) \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix}^{-1} \begin{pmatrix} C(f,g) \\ D(f,g) \end{pmatrix} \tag{6}$$

and is also from $\mathfrak{g}$.

Equation (6) is equivalent to

$$\begin{pmatrix} C(f,g) \\ D(f,g) \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix} \begin{pmatrix} P(z,w) \\ Q(z,w) \end{pmatrix} \tag{7}$$

Taking into account that the polynomials P and Q are of second degree and that

$$\left.\begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix}\right|_0 = \begin{pmatrix} \mathrm{id} & a \\ 0 & \mathrm{id} \end{pmatrix}$$

one can obtain the polynomials (P, Q) for given $\chi = (C, D)$ considering equation (7) up to the second order if one knows the derivatives of ϕ in 0 up to third order.

For any quadric Q and for any Siegel domain $\mathfrak{g}_0$ contains a vector field $\chi_e = (z, 2w)$. This infinitesimal automorphism corresponds to the 1-parametric subgroup

$$\begin{aligned} z^* &= e^t z \\ w^* &= e^{2t} w. \end{aligned}$$

Let now $\Phi \in \mathrm{Aut}_{0,\mathrm{id}}$ be the automorphism corresponding to (a, r). Then one can compute $\phi^*(\chi_e) = (P_e, Q_e)$ using (4):

$$\begin{aligned} P_e &= z - aw - A(z,z) - 2B(w,z) \\ Q_e &= 2w - 2i\langle z, a\bar{w}\rangle - 2r(w,w). \end{aligned}$$

Moreover, one obtains

$$K(w,w) = \frac{1}{3}B(w, aw) + \frac{2}{3}ar(w,w) + \frac{1}{3}A(aw, aw) + \frac{i}{3}a\langle aw, a\bar{w}\rangle,$$

where B is the tensor from (3) which is determined by r.

For $(C, D) = (z, 2w)$ the identity (7) takes the form

$$\begin{pmatrix} f \\ 2g \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix} \begin{pmatrix} z - aw - A(z,z) - 2B(w,z) \\ 2w - 2i\langle z, a\bar{w}\rangle - 2r(w,w) \end{pmatrix}. \tag{8}$$

Before studying this system, we will consider the action of ϕ on the vector field $\chi_i = (iz, 0)$. This infinitesimal automorphism corresponds to the 1-parametric subgroup

$$\begin{aligned} z^* &= e^{it} z \\ w^* &= w. \end{aligned}$$

One obtains $\phi^*(\chi_i) = (P_i, Q_i)$ with

$$\begin{aligned} P_i &= iz + iaw - iA(z,z) - 2iA(aw,z) - 2a\langle z, a\bar{w}\rangle \\ Q_i &= 2\langle z, a\bar{w}\rangle + 2\langle aw, a\bar{w}\rangle. \end{aligned}$$

It follows

$$\begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix} \begin{pmatrix} z + aw - A(z,z) - 2A(aw,z) + 2ia\langle z, a\bar{w}\rangle \\ -2i\langle z, a\bar{w}\rangle - 2i\langle aw, a\bar{w}\rangle \end{pmatrix}. \tag{9}$$

Combining (8) and (9) leads to

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix} \begin{pmatrix} z - A(z,z) - B(w,z) - A(aw,z) + ia\langle z, a\bar{w}\rangle \\ w - 2i\langle z, a\bar{w}\rangle - r(w,w) - i\langle aw, a\bar{w}\rangle \end{pmatrix}. \tag{10}$$

Let f_l, g_l be the homogeneous components of f and g with respect to z and w. Then

$$\begin{aligned} \frac{\partial f_l}{\partial z} z + \frac{\partial f_l}{\partial w} w &= l f_l \\ \frac{\partial g_l}{\partial z} z + \frac{\partial g_l}{\partial w} w &= l g_l. \end{aligned}$$

Isolating in (10) the component of degree l, one obtains a recursive formula which determines f_l, g_l for $l > 1$:

$$(l-1)\begin{pmatrix} f_l \\ g_l \end{pmatrix} = \begin{pmatrix} \frac{\partial f_{l-1}}{\partial z} & \frac{\partial f_{l-1}}{\partial w} \\ \frac{\partial g_{l-1}}{\partial z} & \frac{\partial g_{l-1}}{\partial w} \end{pmatrix} \begin{pmatrix} A(z,z) + B(w,z) + A(aw,z) - ia\langle z, a\bar{w}\rangle \\ 2i\langle z, a\bar{w}\rangle + r(w,w) + i\langle aw, a\bar{w}\rangle \end{pmatrix},$$

with initial conditions $f_0 = 0$, $g_0 = 0$, $f_1 = z + aw$, $g_1 = w$. Thus, we have proved Theorem 1.

5. Geometric description of k-dimensional chains

The description of the chains formulated in Theorem 2 is a direct consequence of the formula (5):

The image of Γ_0 under $\phi = (f, g)$ is $\{f(0,u), g(0,u) : u \in \mathbb{R}^k\}$. From (5) follows

$$\begin{aligned}
(l-1)f_l(0,u) &= \frac{\partial f_{l-1}(0,u)}{\partial u}(r(u,u)+i\langle au,au\rangle)\\
f_0(u) &= 0,\ f_1(u)=au\\
(l-1)g_l(0,u) &= \frac{\partial g_{l-1}(0,u)}{\partial u}(r(u,u)+i\langle au,au\rangle)\\
g_0(u) &= 0,\ g_1(u)=u.
\end{aligned}$$

For any solution $g(0,u) = \sum_{l=0}^{\infty} g_l(0,u)$, evidently, $f(0,u) = ag(0,u)$ is the uniquely determined solution for $f(0,u)$. This finishes the proof.

Any automorphism $\phi \in \mathrm{Aut}_{0,\mathrm{id}}\, Q$ with parameters (a,r) can be uniquely decomposed into $\phi_a \circ \phi_r$ corresponding to $(a,r) = (a,0)\circ(0,r)$. Then ϕ_a maps the standard chain Γ_0 onto the chain $\{z = aw\} \cap Q$, ϕ_r leaves the standard chain invariant, but changes the parameter.

6. Explicit formula for the automorphisms

We consider now the action of ϕ on the infinitesimal Heisenberg automorphisms:

$$\begin{aligned}
\chi_p &= (p, 2i\langle z,p\rangle) \text{ with } p \in \mathbb{C}^n\\
\chi_q &= (0,q) \text{ with } q \in \mathbb{R}^k.
\end{aligned}$$

Let (P_p, Q_p) and (P_q, Q_q) the images of χ_p and χ_q under ϕ^*. If p resp. q runs over the standard basis in $\mathbb{C}^n$ resp. $\mathbb{R}^k$, one can collect the resulting equations (7) into a matrix equation:

$$\begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w}\\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w}\end{pmatrix}^{-1} \begin{pmatrix} \mathrm{id} & 0\\ 2i\langle f,\mathrm{id}\rangle & \mathrm{id}\end{pmatrix} = \begin{pmatrix} \Pi_p & \Pi_q\\ \Psi_p & \Psi_q\end{pmatrix},$$

which is equivalent to

$$\begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w}\\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w}\end{pmatrix}^{-1} = \begin{pmatrix} \Pi_p & \Pi_q\\ \Psi_p & \Psi_q\end{pmatrix} \begin{pmatrix} \mathrm{id} & 0\\ -2i\langle f,\mathrm{id}\rangle & \mathrm{id}\end{pmatrix}. \tag{11}$$

Before determining the matrix blocks $\Pi_p, \Psi_p, \Pi_q, \Psi_q$ we simplify (11) and obtain an expression for the Jacobian matrix of ϕ which does not depend on f. Inserting this expression into (8) one gets an explicit formula for ϕ.

Let $\phi \in \mathrm{Aut}_{0,\mathrm{id}}$ be the automorphism corresponding to (a,A,r,B). Furthermore, set $\phi_c(z,w) = (cz, |c|^2 w)$ with $c \in \mathbb{C}^*$. Then $\phi_c^{-1} \circ \phi \circ \phi_c \in \mathrm{Aut}_{0,\mathrm{id}}\, Q$ is the automorphism corresponding to $(\bar{c}a, |c|^2 r)$. Hence, if we substitute $z, w, a, A, r, B, z^*, w^*$ by $cz, |c|^2 w, \frac{a}{\bar{c}}, \frac{A}{c}, \frac{r}{|c|^2}, \frac{B}{|c|^2}, cz^*, |c|^2 w^*$ in ϕ we obtain again ϕ. This can be reformulated using the following weights: We associate z, w, a, A, r, B with the weights $(1,0), (1,1), (0,-1), (-1,0), (-1,-1), (-1,-1)$, respectively. The vectors $p \in \mathbb{C}^n, \bar{p} \in \bar{\mathbb{C}}^n, q \in \mathbb{R}^k$ have the weights $(1,0), (0,1), (1,1)$, respectively.

The observation from above means then that f is homogeneous with weight $(1,0)$ and g is homogeneous with weight $(1,1)$. It follows

$$\begin{aligned} weight\left(\frac{\partial f}{\partial z}\right) &= (0,0) \\ weight\left(\frac{\partial f}{\partial w}\right) &= (0,-1) \\ weight\left(\frac{\partial g}{\partial z}\right) &= (0,1) \\ weight\left(\frac{\partial g}{\partial w}\right) &= (0,0). \end{aligned}$$

Set

$$H = \begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix}^{-1} = \begin{pmatrix} H_{I,I} & H_{I,II} \\ H_{II,I} & H_{II,II} \end{pmatrix},$$

where $H_{I,I}, H_{I,II}, H_{II,I}, H_{II,II}$ are blocks of dimensions $(n,n),(n,k),(k,n)$ and (k,k). Then we have

Lemma 1.

$$\begin{aligned} weight(H_{I,I}) &= (0,0) \\ weight(H_{I,II}) &= (0,-1) \\ weight(H_{II,I}) &= (0,1) \\ weight(H_{II,II}) &= (0,0). \end{aligned}$$

Proof. Set

$$J = \begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix}.$$

Then

$$H_{ij} = (-1)^{|i-j|} \frac{\det \hat{J}_{ji}}{\det J},$$

where $\hat{J}_{ji}$ is the $(n+k-1)\times(n+k-1)$-matrix which is obtained by omitting the j-th line and the i-th column in J.

It is easy to see that $weight(\det J) = (0,0)$, since $\det J$ is a sum of products containing as many factors from $\frac{\partial f}{\partial w}$ as from $\frac{\partial g}{\partial z}$. By the same reason, $weight(\det \hat{J}_{ji}) = (0,0)$ for $i,j \le n$ and $i,j > n$.

In the products of $\det \hat{J}_{ji}$ with $i \le n, j > n$ there will be one factor from $\frac{\partial f}{\partial w}$ more than factors from $\frac{\partial g}{\partial z}$. Hence, $\det \hat{J}_{ji}$ has the weight $(0,-1)$. Analogously, for $i < n, j \ge n$ $weight(\det \hat{J}_{ji})$ equals $(0,1)$. □

Now we are going to compute the weights of Π_p, Π_q, Ψ_p, Ψ_q: Let (P_p, Q_p) be the image of $(p, 2i\langle z, p\rangle)$. If p was associated with the weight $(1,0)$, then P_p would have the weight $(1,0)$ and Q_p would have the weight $(1,1)$. Passing to Π_p resp. Ψ_p we substitute p by constants of weight $(0,0)$. Consequently, the components which depend holomorphically on p get the weight $(0,0)$, resp. $(0,1)$, at the same time those components which depend antiholomorphically on p get the weight $(1,-1)$ resp. $(1,0)$.

Analogously one obtains $weight(\Pi_q) = (0,-1)$ and $weight(\Psi_q) = (0,0)$. Finally, the weight of $\langle f, \mathrm{id}\rangle$ is $(1,0)$.

From (11) follows $H_{I,I} = \Pi_p - 2i\Pi_q\langle f, \mathrm{id}\rangle$. Since $weight(\Pi_q\langle f, \mathrm{id}\rangle) = (1,-1)$ and $weight(H_{I,I}) = (0,0)$ then $H_{I,I} = (\Pi_p)_{(0,0)}$, where $(\Pi_p)_{(0,0)}$ is the $(0,0)$-component of Π_p.

In the same manner from $H_{II,I} = \Psi_p - 2i\Psi_q\langle f, \mathrm{id}\rangle$ follows $H_{II,I} = (\Psi_p)_{(0,1)}$.

Thus, the desired expression for the Jacobian matrix is

$$\begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix} = \begin{pmatrix} (\Pi_p)_{(0,0)} & \Pi_q \\ (\Psi_p)_{(0,1)} & \Psi_q \end{pmatrix}^{-1},$$

where $(\Pi_p)_{(0,0)}, (\Psi_p)_{(0,1)}$ can be obtained from (P_p, Q_p) by omitting the antiholomorphic terms with respect to p and by substituting p in the holomorphic terms by a free argument. To get (Π_q, Ψ_q) one inserts in (P_q, Q_q) q by a free holomorphic complex argument.

Now we go to compute (P_p, Q_p) and (P_q, Q_q). Therefore we need the derivatives of ϕ in 0 up to third order. They can be easily obtained by means of the recursive formula.

The recursive formula gives at once a simpler expression for f_2:

$$\begin{aligned} f_2 &= A(z,z) + B(w,z) + A(aw,z) + ia\langle z, a\bar{w}\rangle + \\ &\quad + ar(w,w) + ia\langle aw, a\bar{w}\rangle. \end{aligned}$$

Comparing with (4) leads to the following identities:

$$B(w, az) = ar(w,w) \tag{12}$$

$$A(aw, aw) = 2ia\langle aw, a\bar{w}\rangle. \tag{13}$$

Identity (13) is evidently equivalent to

$$A(aw, a\omega) = ia\langle aw, a\bar{\omega}\rangle + ia\langle a\omega, a\bar{w}\rangle.$$

Set now $f_3 = f_{zzz} + f_{zzw} + f_{zww} + f_{www}$, where the indices show the distribution of z and w variables. By means of (5) one obtaines

$$
\begin{aligned}
f_{zzz} &= A(A(z,z),z) \\
f_{zzw} &= A(B(w,z),z) + \frac{1}{2}B(w,A(z,z)) + iB(\langle z, a\bar{w}\rangle, z) + \\
&\quad + A(A(aw,z),z) + \frac{1}{2}A(A(z,z),aw) + ia\langle A(z,z), a\bar{w}\rangle
\end{aligned}
$$

$$
\begin{aligned}
f_{zww} &= \frac{1}{2}B(w,B(w,z)) + \frac{1}{2}B(r(w,w),z) + \frac{1}{2}B(w,A(aw,z)) - \\
&\quad - \frac{i}{2}B(w,a\langle z,a\bar{w}\rangle) + \frac{1}{2}A(B(w,z),aw) + 2iar(\langle z,a\bar{w}\rangle,w) + \\
&\quad + \frac{i}{2}a\langle B(w,z),a\bar{w}\rangle + \frac{i}{2}a\langle z, ar(\bar{w},\bar{w})\rangle + \frac{i}{2}B(\langle aw,a\bar{w}\rangle,z) + \\
&\quad + \frac{1}{2}A(ar(w,w),z) + \frac{1}{2}A(A(aw,z),aw) + \frac{i}{2}a\langle A(aw,z),a\bar{w}\rangle + \\
&\quad + \frac{i}{2}A(a\langle aw,a\bar{w}\rangle,z) - \frac{1}{2}a\langle z,a\langle a\bar{w},aw\rangle\rangle - \frac{1}{2}a\langle aw,a\langle a\bar{w},z\rangle\rangle.
\end{aligned}
$$

We do not need the expression for f_{www}.
For $g_3 = g_{zzz} + g_{zzw} + g_{zww} + g_{www}$ one gets

$$
\begin{aligned}
g_{zzz} &= 0 \\
g_{zzw} &= 2i\langle A(z,z),a\bar{w}\rangle, z) \\
g_{zww} &= i\langle B(w,z),a\bar{w}\rangle + 2ir(\langle z,a\bar{w}\rangle,w) + i\langle z,ar(\bar{w},\bar{w})\rangle + \\
&\quad + 2i\langle A(aw,z),a\bar{w}\rangle \\
g_{www} &= r(r(w,w),w) + ir(\langle aw,a\bar{w}\rangle,w) + \frac{i}{2}\langle aw,ar(\bar{w},\bar{w})\rangle + \\
&\quad + \frac{i}{2}\langle ar(w,w),a\bar{w}\rangle + \frac{i}{2}\langle A(aw,aw),a\bar{w}\rangle,
\end{aligned}
$$

As in the case of (P_e, Q_e) we can now determine the vector fields (P_p, Q_p) as well as (P_q, Q_q). Let $P_p = P_0^p + P_z^p + P_w^p + P_{zz}^p + P_{zw}^p$ and $Q_p = Q_0^p + Q_z^p + Q_w^p + Q_{zw}^p + Q_{ww}^p$ be the expansion into homogeneous components with respect to z and w. Then

$$
\begin{aligned}
P_0^p &= \underline{p} \\
P_z^p &= \underline{-2A(z,p)} - 2ia\langle z,p\rangle \\
P_w^p &= \underline{-B(w,p) - A(aw,p) + ia\langle p,a\bar{w}\rangle} - 2ia\langle aw,p\rangle \\
P_{zz}^p &= \underline{2A(A(z,p),z) - A(A(z,z),p)} + 2iA(z,a\langle z,p\rangle) + 2a\langle z,a\langle p,z\rangle\rangle - \\
&\quad -2iB(\langle z,p\rangle,z) \\
P_{zw}^p &= \underline{A(B(w,p),z) - A(B(w,z),p) + iB(\langle p,a\bar{w}\rangle,z) - iB(\langle z,a\bar{w}\rangle,p)} + \\
&\quad \underline{+B(w,A(z,p)) - A(A(z,aw),p) + A(A(z,p),aw)+} \\
&\quad \underline{+A(A(p,aw),z) + 2a\langle z,a\langle a\bar{w},p\rangle\rangle} - 2iB(\langle aw,p\rangle,z) + \\
&\quad +2iA(a\langle aw,p\rangle,z) + 2i\langle z,A(aw,p)\rangle - 2a\langle a\langle z,p\rangle,a\bar{w}\rangle \\
Q_0^p &= 0 \\
Q_z^p &= 2i\langle z,p\rangle \\
Q_w^p &= \underline{-2i\langle p,a\bar{w}\rangle} + 2i\langle aw,p\rangle \\
Q_{zw}^p &= \underline{-4\langle z,a\langle a\bar{w},p\rangle\rangle} - 2i\langle z,B(\bar{w},p)\rangle - 2i\langle z,A(a\bar{w},p)\rangle + 2\langle z,a\langle p,aw\rangle\rangle \\
Q_{ww}^p &= \underline{i\langle B(w,p),a\bar{w}\rangle - i\langle p,ar(\bar{w},\bar{w})\rangle + 2ir(\langle p,a\bar{w}\rangle,w)-} \\
&\quad \underline{-2\langle aw,a\langle a\bar{w},p\rangle\rangle} + 2i\langle ar(w,w),p\rangle - 4ir(\langle aw,p\rangle,w) - \\
&\quad -2\langle a\langle aw,a\bar{w}\rangle,p\rangle - 2\langle a\langle aw,p\rangle,a\bar{w}\rangle + 2\langle aw,a\langle p,aw\rangle\rangle
\end{aligned}
\tag{14}
$$

The terms which depend holomorphically on p and, therefore, contribute to the formula of the Jacobian are underlined.

The computation of (P_q, Q_q) leads to

$$
\begin{aligned}
P_0^q &= -aq \\
P_z^q &= -B(q,z) + A(aq,z) + ia\langle z,aq\rangle \\
P_w^q &= B(w,aq) + ia\langle aw,aq\rangle \\
P_{zz}^q &= 2iB(\langle z,aq\rangle,z) - A(A(z,aq),z) + \frac{1}{2}A(A(z,z),aq)
\end{aligned}
$$

$$\begin{aligned}
P^q_{zw} &= \frac{1}{2}B(w,B(q,z)) - \frac{1}{2}B(q,B(w,z)) + B(r(q,w),z) - \frac{1}{2}B(w,A(aq,z)) - \\
&\quad -\frac{i}{2}B(\langle aq,a\bar{w}\rangle,z) - \frac{i}{2}B(q,a\langle z,a\bar{w}\rangle) + \frac{i}{2}B(w,a\langle z,aq\rangle) - \\
&\quad -\frac{1}{2}B(q,A(aw,z)) + iB(\langle z,aq\rangle,aw) + \frac{i}{2}B(\langle aw,aq\rangle,z) - \\
&\quad -2ia\langle z,B(\bar{w},aq)\rangle + ia\langle z,ar(\bar{w},q)\rangle + \frac{1}{2}A(B(w,z),aq) - \\
&\quad -\frac{1}{2}A(B(w,aq),z) + \frac{1}{2}A(B(q,z),aw) + \frac{1}{2}A(B(q,aw),z) - \\
&\quad -\frac{i}{2}a\langle B(w,z),aq\rangle + \frac{i}{2}a\langle B(q,z),a\bar{w}\rangle - \frac{i}{2}a\langle z,A(a\bar{w},aq)\rangle - \\
&\quad -\frac{1}{2}A(A(aw,aq),z) + \frac{1}{2}A(A(aw,z),aq) - \frac{1}{2}A(A(aq,z),aw) - \\
&\quad -\frac{i}{2}a\langle A(z,aw),aq\rangle + \frac{i}{2}a\langle A(z,aq),a\bar{w}\rangle - \frac{1}{2}a\langle aw,a\langle aq,z\rangle\rangle + \\
&\quad +\frac{1}{2}a\langle aq,a\langle a\bar{w},z\rangle\rangle, \\
Q^q_0 &= q \\
Q^q_z &= -2i\langle z,aq\rangle \\
Q^q_w &= -2r(q,w) + i\langle aq,a\bar{w}\rangle - i\langle aw,aq\rangle \\
Q^q_{zw} &= 2i\langle z,B(\bar{w},aq)\rangle + 2\langle z,a\langle a\bar{w},aq\rangle\rangle \\
Q^q_{ww} &= 2r(r(w,q),w) - r(r(w,w),q) - i\langle B(w,aq),a\bar{w}\rangle + i\langle ar(w,q),a\bar{w}\rangle - \\
&\quad -ir(\langle aq,a\bar{w}\rangle,w) + ir(\langle aw,aq\rangle,w) - ir(\langle aw,a\bar{w}\rangle,q) + \frac{i}{2}\langle aq,ar(\bar{w},\bar{w})\rangle - \\
&\quad -\frac{i}{2}\langle ar(w,w),aq\rangle + i\langle aw,ar(\bar{w},q)\rangle + \langle aw,a\langle a\bar{w},aq\rangle\rangle.
\end{aligned}$$

Hence, all ingredients of the automorphism formula

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} (\Pi_p)_{(0,0)} & 2\Pi_q \\ \frac{1}{2}(\Psi_p)_{(0,1)} & \Psi_q \end{pmatrix}^{-1} \begin{pmatrix} P_e \\ \frac{1}{2}Q_e \end{pmatrix} \tag{15}$$

are completely described.

7. The Heisenberg sphere in $\mathbb{C}^2$

In this section we want to demonstrate the obtained formula in the simple case of the sphere in $\mathbb{C}^2$ (the Šilov boundary of the ball). Let $Q = \{(z,w) \in \mathbb{C}^2 : \operatorname{Im} w = |z|^2\}$. Then any $\phi \in \operatorname{Aut}_{0,\mathrm{id}} Q$ can be described by Poincaré's formula (see [10])

$$\begin{aligned} f &= \frac{z+aw}{1-2i\bar{a}z-(r+i|a|^2)w} \\ g &= \frac{w}{1-2i\bar{a}z-(r+i|a|^2)w}, \end{aligned} \tag{16}$$

where $a \in \mathbb{C}$ and $r \in \mathbb{R}$.

We will now obtain ϕ by means of the procedure developped above.

We have

$$\begin{aligned} (\Pi_p)_{(0,0)} &= 1-4i\bar{a}z-rw-i|a|^2w-4\bar{a}^2z^2+2i\bar{a}rzw-2\bar{a}^2azw \\ \frac{1}{2}(\Psi_p)_{(0,0)} &= -i\bar{a}w-2\bar{a}^2zw-\bar{a}^2aw^2 \\ 2(\Pi_q) &= -2a+6i|a|^2z-2rz+2arw+2ia^2\bar{a}w+4i\bar{a}rz^2+4a\bar{a}^2z^2+ \\ &\quad +2r^2zw+2|a|^4zw \\ (\Psi_q) &= 1-2i\bar{a}z-2rw+2a\bar{a}^2zw+2i\bar{a}rzw+r^2w^2+|a|^4w^2 \\ P_e &= z-aw-2i\bar{a}z^2-2rzw \\ \frac{1}{2}Q_e &= w-i\bar{a}zw-rw^2. \end{aligned}$$

Since

$$\begin{pmatrix} (\Pi_p)_{(0,0)} & 2\Pi_q \\ \frac{1}{2}(\Psi_p)_{(0,0)} & \Psi_q \end{pmatrix} = \begin{pmatrix} 1-2i\bar{a}z & -2a-2rz+2i|a|^2z \\ -i\bar{a}w & 1-rw+i|a|^2w \end{pmatrix} N,$$

with $N = 1-2i\bar{a}z-(r+i|a|^2)w$, and

$$\begin{pmatrix} P_e \\ \frac{1}{2}Q_e \end{pmatrix} = \begin{pmatrix} 1-2i\bar{a}z & -2a-2rz+2i|a|^2z \\ -i\bar{a}w & 1-rw+i|a|^2w \end{pmatrix} \begin{pmatrix} z+aw \\ w \end{pmatrix},$$

cancelling the corresponding matrices in the formula (15) we obtain the unique automorphism (16) with parameters (a, r).

The most natural way to represent the automorphisms (16) is to pass to homogeneous coordinates $(\xi : \zeta : \omega)$ in $\mathbb{P}^2$. Let $\mathbb{C}^2 \subset \mathbb{P}^2$ be the subset of all $(\xi : \zeta : \omega)$ with $\xi \neq 0$. Then $z = \frac{\zeta}{\xi}$ and $w = \frac{\omega}{\xi}$ are coordinates in $\mathbb{C}^2$. The Heisenberg sphere can be extended to

$$\operatorname{Im} \omega\bar{\xi} = |\zeta|^2$$

in $\mathbb{P}^2$.

The fractional linear automorphisms (16) can be written in the linear form

$$\begin{aligned} \xi^* &= \xi + 2i\bar{a}\zeta + (r + i|a|^2)\omega \\ \zeta^* &= \zeta + a\omega \\ \omega^* &= \omega. \end{aligned} \tag{17}$$

8. Poincaré automorphisms

A natural generalization of the automorphisms of the Heisenberg sphere are Poincaré automorphisms which were introduced in [5]. It was shown that the automorphisms from $\phi \in \mathrm{Aut}_{0,\mathrm{id}}\, Q$ with parameters (a, A, r, B) can be described by a much simpler "matrix fractional linear" formula which is similar to the Poincaré formula (16) if there exist a $\mathbb{C}^n$-valued bilinear form $\hat{A} : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n$ and a $\mathbb{C}^k$-valued Hermitian form $\hat{r} : \mathbb{C}^k \otimes \bar{\mathbb{C}}^k \to \mathbb{C}^k$ such that

$$\langle \hat{A}(z,\zeta), \xi\rangle = 2i\langle z, a\langle \xi, \zeta\rangle\rangle, \tag{18}$$

$$\langle B(w,\zeta), \xi\rangle = \hat{r}(w, \langle \xi, \zeta\rangle) \tag{19}$$

is satisfied for all $z, \zeta, \xi \in \mathbb{C}^n$, $w \in \mathbb{C}^k$. Then ϕ takes the form

$$\begin{aligned} z^* &= (\mathrm{id} - \hat{A}(z,\cdot) - B(w,\cdot) - \frac{1}{2}\hat{A}(aw,\cdot))^{-1}(z + aw), \\ w^* &= (\mathrm{id} - 2i\langle z, a\bar{\cdot}\rangle - \hat{r}(w,\bar{\cdot}) - i\langle aw, a\bar{\cdot}\rangle)^{-1} w. \end{aligned} \tag{20}$$

For proving this one needs to consider the algebra $\mathfrak{A}$ of all pairs $(D, d) \in \mathfrak{gl}\,(n, \mathbb{C}) \times \mathfrak{gl}\,(k, \mathbb{C})$ with the property $\langle Dz, z\rangle = d\langle z, z\rangle$.

It follows from (18) and (19) that

$$\begin{aligned} Da &= ad, \\ \hat{A}(Dz, \zeta) &= D\hat{A}(z, \zeta), \\ B(dw, z) &= DB(w, z), \\ \hat{r}(dw, \omega) &= d\hat{r}(w, \omega), \end{aligned} \tag{21}$$

(see Appendix).

Moreover, (18) and (19) mean that

$$(\hat{A}(z,\cdot), 2i\langle z, a\bar{\cdot}\rangle) \in \mathfrak{A} \tag{22}$$

and

$$(B(w,\cdot), \hat{r}(w,\bar{\cdot})) \in \mathfrak{A}. \tag{23}$$

Though the direct proof of the formula (20) is rather easy, we show that it can be also obtained from (15) by cancelling appropriate matrices, as in the case of the Heisenberg sphere. Using the identities (18), (19), (21), (13) and (12) one can show that

$$\begin{pmatrix} (\Pi_p)_{(0,0)} & 2\Pi_q \\ \frac{1}{2}(\Psi_p)_{(0,0)} & \Psi_q \end{pmatrix} = \begin{pmatrix} \mathrm{id} - \hat{A}(\cdot, z) & -2a - 2B(\cdot, z) + \hat{A}(a\cdot, z) \\ -i\langle \cdot, a\bar{w}\rangle & \mathrm{id} + i\langle a\cdot, a\bar{w}\rangle - \hat{r}(\cdot, \bar{w}) \end{pmatrix} \times \tag{24}$$

$$\times \begin{pmatrix} \mathrm{id} - \hat{A}(z, \cdot) - B(w, \cdot) - \frac{1}{2}\hat{A}(aw, \cdot) & 0 \\ 0 & \mathrm{id} - 2i\langle z, a\bar{\cdot}\rangle - \hat{r}(w, \bar{\cdot}) - i\langle aw, a\bar{\cdot}\rangle \end{pmatrix}$$

and

$$\begin{pmatrix} P_e \\ \frac{1}{2}Q_e \end{pmatrix} = \begin{pmatrix} \mathrm{id} - \hat{A}(\cdot, z) & -2a - 2B(\cdot, z) + \hat{A}(a\cdot, z) \\ -i\langle \cdot, a\bar{w}\rangle & \mathrm{id} + i\langle a\cdot, , a\bar{w}\rangle - \hat{r}(\cdot, \bar{w}) \end{pmatrix} \begin{pmatrix} z + aw \\ w \end{pmatrix},$$

(See the Appendix below for the proof of the first identity, the proof of the second one is simple).

There is also a linear representation of the Poincaré automorphisms (20) similar to (17). Set $L = \mathfrak{A} \oplus \mathbb{C}^{\mathfrak{n}} \oplus \mathbb{C}^{\mathfrak{k}}$ with coordinates $((\Xi, \xi), \zeta, \omega)$. The group $\mathfrak{A}^*$ of all invertible elements of $\mathfrak{A}$ acts on L by

$$(D, d) \circ ((\Xi, \xi), \zeta, \omega) = ((D\Xi, d\xi), D\zeta, d\omega).$$

We introduce homogeneous coordinates $((\Xi, \xi) : \zeta : \omega)$ in the factor space $L/\mathfrak{A}^*$.

The quadric Q can be extended by

$$\mathrm{Im}\, \bar{d}\omega = \langle \zeta, \zeta \rangle$$

to $\hat{Q} \subset L$. $\hat{Q}$ is invariant under the action of $\mathfrak{A}^*$. According to (21), the automorphisms (20) can be lifted to the linear L automorphisms

$$\begin{aligned} (\Xi^*, \xi^*) &= (\Xi, \xi) + (\hat{A}(\zeta, \cdot), 2i\langle \zeta, a\bar{\cdot}\rangle) + \\ &\quad + (B(\omega, \cdot) + \frac{1}{2}\hat{A}(a\omega, \cdot), \hat{r}(\omega, \bar{\cdot}) + i\langle a\omega, a\bar{\cdot}\rangle) \\ \zeta^* &= \zeta + a\omega \\ \omega^* &= \omega. \end{aligned}$$

The theory of Poincaré automorphisms gives a complete description of the automorphisms of nondegenerate quadrics of codimension $k = 1, 2, n^2$ and of real associative quadrics (see [3] [4]). However, Palinčak [9] found a quadric in $\mathbb{C}^6$ of codimension 3 with a 9-dimensional $\mathrm{Aut}_{0,\mathrm{id}}$ group which does not contain Poincaré automorphisms.

9. Automorphisms of different types of rigid quadrics

We introduce the following terminology: A nondegenerate quadric Q will be called s-rigid if from $(Xz, sw) \in \mathfrak{g}_o$ follows that $s = t \cdot \mathrm{id}$ with $t \in \mathbb{R}$ (in particular, any hyperquadric is s-rigid); it will be called a-rigid, resp., r-rigid if $\mathcal{A} = \{0\}$, resp., $\mathcal{R} = \{0\}$.

Proposition 1. *If a nondegenerate quadric Q is a-rigid then it is also r-rigid.*

Proof. Consider P^p_w in (14). For $a = 0$ follows that $B(\cdot, p)$ is contained in $\mathcal{A}$ for all $p \in \mathbb{C}^n$. Hence, if there was some $B \neq 0$ then would exist some $p \in \mathbb{C}^n$ such that $B(\cdot, p) \neq 0$. □

Proposition 2. *If Q is a s-rigid nondegenerate quadric then* $\mathrm{Aut}_{0,\mathrm{id}}\, Q$ *consists of fractional linear mappings. If, moreover, $k > 1$ then Q is r-rigid.*

Proof. Consider Q^p_w in (14). Then $\mathfrak{s} \cong \mathbb{R}$ implies

$$\langle p, a\bar{w}\rangle = l(p)w, \tag{25}$$

where l is a complex linear functional on $\mathbb{C}^n$. Setting in (25) $w = \langle z, \zeta\rangle$ one obtains a solution $\hat{A}$ of (18) corresponding to a: $\langle p, a\langle z, \zeta\rangle\rangle = l(p)\langle \zeta, z\rangle = \langle l(p)\zeta, z\rangle$, i.e., $\hat{A}(p, \zeta) = 2il(p)\zeta$. But then

$$\begin{aligned} z^* &= (1 - 2il(z) - il(aw))^{-1}(z + aw) \\ w^* &= (1 - 2il(z) - il(aw))^{-1}w \end{aligned}$$

is the uniquely determined automorphism corresponding to $(a, 0)$.

Now we consider Q^q_w and set there $a = 0$. It follows

$$r(q, w) = \lambda(q)w, \tag{26}$$

where λ is a real linear functional on $\mathbb{R}^k$. Setting again $w = \langle z, \zeta\rangle$, one obtains $B(u, z) = \lambda(u)z$ and $\hat{r}(w, \omega) = \lambda(w)\bar{\omega}$.

Hence,

$$\begin{aligned} z^* &= (1 - \lambda(w))^{-1}z \\ w^* &= (1 - \lambda(w))^{-1}w \end{aligned}$$

is the automorphism corresponding to r.

From the symmetry of r follows $r(u, v) = \lambda(u)v = \lambda(v)u$, i.e., if $\mathfrak{s} \cong \mathbb{R}$ and $k > 1$, then $\mathcal{R} = \{0\}$. □

Remark. The considerations of this section can be directly applied to Siegel domains.

10. Canonical parametrization of the chains

Let $\Gamma_0 = \{z = 0, \operatorname{Im} w = 0\}$ be the standard chain on Q. Then there exists a canonical family of parametrizations of Γ_0 which can be obtained from the standard parametrization $\{z = 0, w = u : u \in \mathbb{R}^k\}$ by means of a "reparametrization" automorphism corresponding to parameters $(0, r)$.

From (15) and (9) we derive a simple equation for this reparametrization map:

Proposition 3. *The automorphism ϕ_r, corresponding to $(0, r)$ has the form*

$$\begin{aligned} z^* &= (\operatorname{id} - B(w,\cdot))^{-1} z \\ w^* &= (\operatorname{id} - 2r(w,\cdot) + 2r(r(w,\cdot),w) - r(r(w,w)\cdot))^{-1}(w - r(w,w)). \end{aligned}$$

Proof. At first we set in (9) $a = 0$. It follows

$$f(z,w) = \frac{\partial f}{\partial z} z$$

Setting in

$$(\Pi_p)_{(0,0)} = \frac{\partial f}{\partial z}$$

$a = 0$ one obtains immediately $f(z,w) = (\operatorname{id} - B(w,\cdot))^{-1} z$.
The expression for g can be derived by setting $a = 0$ in (15). □

The expression for the g-component in ϕ_r can be simplified if the following condition is satisfied:

Proposition 4. *Let ϕ_r be as in Proposition 3 and $\hat{r}(\cdot)$ be a linear map $\mathbb{C}^k \to \mathfrak{gl}(k,\mathbb{C})$ with*

$$\begin{aligned} \hat{r}(w)w &= r(w,w) \\ \hat{r}(w)^2 &= \hat{r}(r(w,w)), \end{aligned} \tag{27}$$

then

$$w^* = g(z,w) = (\operatorname{id} - \hat{r}(w))^{-1} w.$$

Proof. From Proposition 3 follows that g does not depend on z. The recursive formula for g therefore takes the simple form

$$(l-1) g_l(z,w) = \frac{\partial g_{l-1}}{w} r(w,w)$$

with $g_0 = 0$ and $g_1 = w$.
One easily verifies that $g_l := \hat{r}(w)^{l-1} w$ is the solution of the recursive equations. □

It follows

Proposition 5. *Let Q be a nondegenerate quadric and $r \in \mathcal{R}$ with the property $r(w,w) = \langle aw, a\bar{w}\rangle$ (resp. $r(w,w) = -\langle aw, a\bar{w}\rangle$). Then $\hat{r}(w) = \langle aw, a\bar{\cdot}\rangle$ (resp. $\hat{r}(w) = -\langle aw, a\bar{\cdot}\rangle$) satisfies (27).*

Proof. Set $r(w,w) = \langle aw, a\bar{w}\rangle$ and $\hat{r}(w) = \langle aw, a\bar{\cdot}\rangle$. Because of (2) then

$$\hat{r}(w)^2 = \langle aw, a\langle a\bar{\cdot}, aw\rangle\rangle = \frac{1}{2i}\langle A(aw, aw), a\bar{\cdot}\rangle.$$

On the other hand it follows from (13) that

$$\hat{r}(r(w,w) = \langle a\langle a\bar{\cdot}, aw\rangle a\bar{\cdot}\rangle = \frac{1}{2i}\langle A(aw, aw), a\bar{\cdot}\rangle.$$

□

Remark. The representation $r(w,w) = \langle aw, a\bar{w}\rangle$ is not unique. Moreover, there can exist tensors $\hat{r}$ satisfying (27) which cannot be obtained in the described manner.

For automorphisms corresponding to $(a, 0)$ one can derive the following simple equation for the g-component:

Proposition 6. *Let $\Phi_a \in \mathrm{Aut}_{0,\mathrm{id}}\, Q$ with $r = 0$. Then*

$$w^* = (\mathrm{id} - 2i\langle z, a\bar{\cdot}\rangle - i\langle aw, a\bar{\cdot}\rangle)^{-1} w.$$

Proof. Set $d := 2i\langle z, a\bar{\cdot}\rangle + i\langle aw, a\bar{\cdot}\rangle$. We show by induction that $g_l = d^{l-1}w$. This implies the assertion.

For $l = 1$ we have $g_1 = w$. By inductive assumption then $g_{l-1} = d^{l-2}w$. Using the recursive formula (5) we come to

$$\begin{aligned}
(l-1)g_l &= \sum_{s=1}^{l-3} d^s(2i\langle A(z,z), a\bar{\cdot}\rangle + 2i\langle A(aw, z), a\bar{\cdot}\rangle + \\
&\quad +2\langle a\langle z, a\bar{w}\rangle, a\bar{\cdot}\rangle)d^{l-s-3}w + \\
&\quad + \sum_{s=1}^{l-3} d^s(-2\langle a\langle z, a\bar{w}\rangle, a\bar{\cdot}\rangle) - \langle a\langle aw, a\bar{w}\rangle, a\bar{\cdot}\rangle)d^{l-s-3}w + \\
&\quad +d^{l-2}(2i\langle z, a\bar{w}\rangle + i\langle aw, a\bar{w}\rangle) \\
&= \sum_{s=1}^{l-3} d^s(2i\langle A(z,z), a\bar{\cdot}\rangle + 2i\langle A(aw, z), a\bar{\cdot}\rangle - \\
&\quad -\langle a\langle aw, a\bar{w}\rangle, a\bar{\cdot}\rangle d^{l-s-3}w + \\
&\quad +d^{l-2}(2i\langle z, a\bar{w}\rangle + i\langle aw, a\bar{w}\rangle)
\end{aligned}$$

The assertion follows if we show that

$$2i\langle A(z,z), a\bar{\cdot}\rangle + 2i\langle A(aw, z), a\bar{\cdot}\rangle - \langle a\langle aw, a\bar{w}\rangle, a\bar{\cdot}\rangle) = d^2.$$

For d^2 we obtain

$$\begin{aligned} d^2 &= -4\langle z, a\langle a\cdot, z\rangle\rangle - \langle aw, a\langle \bar{a\cdot}, aw\rangle\rangle - \\ &\quad -2\langle z, a\langle a\cdot, aw\rangle\rangle - 2\langle aw, \langle \bar{a\cdot}, z\rangle\rangle. \end{aligned}$$

Because of (2) then

$$\begin{aligned} -4\langle z, a\langle a\cdot, z\rangle\rangle &= 2i\langle A(z,z), \bar{a\cdot}\rangle \\ \text{and } -2\langle z, a\langle a\cdot, aw\rangle\rangle - 2\langle aw, \langle \bar{a\cdot}, z\rangle\rangle &= 2i\langle A(aw,z), \bar{a\cdot}\rangle. \end{aligned}$$

Because of (13) and (2)

$$-\langle aw, a\langle \bar{a\cdot}, aw\rangle\rangle = \frac{1}{2}\langle A(aw,aw), \bar{a\cdot}\rangle = -\langle aw, \langle \bar{a\cdot}, aw\rangle\rangle.$$

□

Proposition 6 and Theorem 2 give a description of the chains including the canonical parameter:

Corollary 1. *The chains of the nondegenerate quadric Q have the following canonical parametrization*

$$\begin{aligned} f(u) &= a(\mathrm{id} - i\langle au, a\cdot\rangle)^{-1}u \\ g(u) &= (\mathrm{id} - i\langle au, a\cdot\rangle)^{-1}u \end{aligned}$$

with $u \in \mathbb{R}^k$.

Proof. The expression for g can be obtained by setting $z = 0$ and $w = u$ in the formula from Proposition 6. The expression for f follows then from Theorem 2. □

11. Appendix

In this Appendix we perform the rather huge verification of (24): We use (2), (3), (18), (19), its consequence (22), (23) and the following identities

$$\begin{aligned} Da &= ad, \quad \text{for } a \in \mathcal{A}, (D,d) \in \mathfrak{A} && (28) \\ \hat{A}(Dz,\zeta) &= D\hat{A}(z,\zeta) && (29) \\ \hat{r}(w, d\omega) &= \bar{d}\hat{r}(w,\omega), && (30) \\ \hat{r}(dw, \omega) &= d\hat{r}(w,\omega), && (31) \\ B(dw, z) &= DB(w,z) && (32) \end{aligned}$$

At first we verify these identities:
To show (28) it is sufficient to check

$$2i\langle z, Da\langle \xi,\zeta\rangle\rangle = 2i\langle z, ad\langle \xi,\zeta\rangle\rangle, \text{ for all } z,\zeta,\xi.$$

This follows from

$$\begin{aligned} 2i\langle z, Da\langle\xi,\zeta\rangle\rangle &= 2i\bar{d}\langle z, a\langle\xi,\zeta\rangle\rangle = \bar{d}\langle\hat{A}(z,\zeta),\xi\rangle = \\ = \langle\hat{A}(z,\zeta), D\xi\rangle &= 2i\langle z, a\langle D\xi,\zeta\rangle\rangle = 2i\langle z, ad\langle\xi,\zeta\rangle\rangle. \end{aligned}$$

Now we prove (29)

$$\begin{aligned} \langle\hat{A}(Dz,\zeta),\xi\rangle &= 2i\langle Dz, a\langle\xi,\zeta\rangle\rangle = 2id\langle z, a\langle\xi,\zeta\rangle\rangle = \\ = d\langle\hat{A}(Dz,\zeta),\xi\rangle &= \langle D\hat{A}(z,\zeta),\xi\rangle. \end{aligned}$$

(30) is a consequence of the following transformations

$$\begin{aligned} \hat{r}(w, d\langle\zeta, z\rangle &= \hat{r}(w, \langle D\zeta, z\rangle) = \langle B(w,z), D\zeta\rangle = \\ \bar{d}\langle B(w,z),\zeta\rangle &= \bar{d}\hat{r}(w, d\langle\zeta,z\rangle). \end{aligned}$$

This implies (31)

$$\hat{r}(dw,\omega) = \overline{\hat{r}(\omega, dw)} = \overline{\bar{d}\hat{r}(\omega, w)} = d\hat{r}(w,\omega),$$

and (32)

$$\begin{aligned} \langle B(dw,z),\zeta\rangle &= \hat{r}(dw, \langle\zeta,z\rangle) = d\hat{r}(w,\langle\zeta,z\rangle) = \\ d\langle B(w,z),\zeta\rangle &= \langle DB(w,z),\zeta\rangle. \end{aligned}$$

The equality of the left upper blocks in (24) is a consequence of the equalities

$$\begin{aligned}
\mathrm{id} &= \mathrm{id}, && (33)\\
-2A(z,\cdot) &= -\hat{A}(z,\cdot) - \hat{A}(\cdot,z), && (34)\\
-B(w,z) &= -B(w,z), && (35)\\
-\frac{1}{2}\hat{A}(aw,\cdot) &= -A(aw,\cdot) + ia\langle\cdot, a\bar{w}\rangle, && (36)\\
\hat{A}(\hat{A}(z,\cdot),z) &= 2A(A(z,\cdot),z) - A(A(z,z),\cdot), && (37)\\
\frac{1}{2}\hat{A}(\hat{A}(aw,\cdot),z) &= -A(A(z,aw),\cdot) + A(A(z,\cdot),aw) + && (38)\\
& \quad A(A(\cdot,aw),z) + 2a\langle z, a\langle a\bar{w},\cdot\rangle\rangle \\
\hat{A}(B(w,\cdot),z) &= A(B(w,\cdot),z) - A(B(w,z),\cdot) + iB(\langle\cdot, a\bar{w}\rangle, z) - && (39)\\
& \quad -iB(\langle z, a\bar{w}\rangle,\cdot) + B(w, A(z,\cdot)),
\end{aligned}$$

Equations (33) and (35) are tautologies, (34) follows by symmetrization of (18). To prove (36) we show that $\frac{1}{2}\hat{A}(\cdot, aw) = ia\langle\cdot, a\bar{w}\rangle$ and apply (34). The latter equality follows from the fact that $(\hat{A}(p,\cdot), 2i\langle p, a\bar{\cdot}\rangle) \in \mathfrak{A}$ and (28).

In order to obtain (37) we express the right hand side in terms of $\hat{A}$

$$\begin{aligned} & 2A(A(z,\cdot),z) - A(A(z,z),\cdot) = \\ = \; & A(\hat{A}(z,\cdot),z) + A(\hat{A}(\cdot,z),z) - A(\hat{A}(z,z),\cdot) \\ = \; & \frac{1}{2}\hat{A}(\hat{A}(z,\cdot),z) + \frac{1}{2}\hat{A}(z,\hat{A}(z,\cdot)) + \frac{1}{2}\hat{A}(\hat{A}(\cdot,z),z) + \\ & + \frac{1}{2}\hat{A}(z,\hat{A}(\cdot,z)) - \frac{1}{2}\hat{A}(\hat{A}(z,z),\cdot) - \frac{1}{2}\hat{A}(\cdot,\hat{A}(z,z)) \end{aligned}$$

Then we apply the identities

$$\begin{aligned} \frac{1}{2}\hat{A}(z,\hat{A}(z,\cdot)) &= \frac{1}{2}\hat{A}(\hat{A}(z,z),\cdot) \\ \frac{1}{2}\hat{A}(\hat{A}(\cdot,z),z) &= \frac{1}{2}\hat{A}(\cdot,\hat{A}(z,z)) \\ \frac{1}{2}\hat{A}(\hat{A}(z,\cdot),z) &= \frac{1}{2}\hat{A}(z,\hat{A}(\cdot,z)). \end{aligned}$$

Cancelling appropriate terms we get an expression that equals to the left hand side.

In (38) we use the identity

$$2a\langle z, a\langle a\bar{w},\cdot\rangle\rangle = -i\hat{A}(z, a\langle\cdot, a\bar{w}\rangle) = -\frac{1}{2}\hat{A}(\hat{A}(z,\cdot),aw).$$

The right hand side of (38) takes then the form

$$\begin{aligned} & -\frac{1}{4}\hat{A}(\hat{A}(z,aw),\cdot) - \frac{1}{4}\hat{A}(\cdot,\hat{A}(z,aw)) - \frac{1}{4}\hat{A}(\hat{A}(aw,z),\cdot) - \frac{1}{4}\hat{A}(\cdot,\hat{A}(aw,z)) + \\ & + \frac{1}{4}\hat{A}(\hat{A}(z,\cdot),aw) + \frac{1}{4}\hat{A}(aw,\hat{A}(z,\cdot)) + \frac{1}{4}\hat{A}(\hat{A}(\cdot,z),aw) + \frac{1}{4}\hat{A}(aw,\hat{A}(\cdot,z)) + \\ & + \frac{1}{4}\hat{A}(\hat{A}(\cdot,aw),z) + \frac{1}{4}\hat{A}(\hat{A}(\cdot,aw),z) + \frac{1}{4}\hat{A}(z,\hat{A}(\cdot,aw)) + \frac{1}{4}\hat{A}(z,\hat{A}(aw,\cdot)) - \\ & - \frac{1}{2}\hat{A}(\hat{A}(z,\cdot),aw) \end{aligned}$$

Using the identities

$$
\begin{aligned}
\frac{1}{4}\hat{A}(\hat{A}(z,aw),\cdot) &= \frac{1}{4}\hat{A}(z,\hat{A}(aw,\cdot)) \\
\frac{1}{4}\hat{A}(\cdot,\hat{A}(z,aw)) &= \frac{1}{4}\hat{A}(\hat{A}(\cdot,z),aw) \\
\frac{1}{4}\hat{A}(\hat{A}(aw,z),\cdot) &= \frac{1}{4}\hat{A}(aw,\hat{A}(z,\cdot)) \\
\frac{1}{4}\hat{A}(\cdot,\hat{A}(aw,z)) &= \frac{1}{4}\hat{A}(\hat{A}(\cdot,aw),z) \\
\frac{1}{4}\hat{A}(\hat{A}(z,\cdot),aw) &= \frac{1}{4}\hat{A}(\hat{A}(\cdot,aw),z) \\
\frac{1}{4}\hat{A}(aw,\hat{A}(\cdot,z)) &= \frac{1}{4}\hat{A}(z,\hat{A}(\cdot,aw))
\end{aligned}
$$

we find an expression equal to the left hand side.
It remains to prove (39). The right hand side of (39) takes the form

$$
\frac{1}{2}\hat{A}(B(w,\cdot),z)+\frac{1}{2}\hat{A}(z,B(w,\cdot))-\frac{1}{2}\hat{A}(B(w,z),\cdot)-\frac{1}{2}\hat{A}(\cdot,B(w,z))+ \\
iB(\langle\cdot,a\bar{w}\rangle,z)-iB(\langle z,a\bar{w}\rangle,\cdot)+\frac{1}{2}B(w,\hat{A}(z,\cdot))+\frac{1}{2}B(w,\hat{A}(\cdot,z))
$$

We have

$$
\begin{aligned}
\frac{1}{2}\hat{A}(z,B(w,\cdot)) &= iB(\langle z,a\bar{w}\rangle,\cdot) \\
\frac{1}{2}\hat{A}(B(w,z),\cdot) &= \frac{1}{2}B(w,\hat{A}(z,\cdot)) \\
\frac{1}{2}\hat{A}(\cdot,B(w,z)) &= iB(\langle\cdot,a\bar{w}\rangle,z) \\
\frac{1}{2}B(w,\hat{A}(\cdot,z)) &= \frac{1}{2}\hat{A}(B(w,\cdot),z).
\end{aligned}
$$

After cancelling equal terms of opposite sign we find an expression which equals to the left hand side.

$$
\begin{aligned}
\frac{1}{2}\hat{A}(B(w,\cdot),z)+\frac{1}{2}B(w,\hat{A}(\cdot,z)) &= \frac{1}{2}\hat{A}(B(w,\cdot),z)+\frac{1}{2}\hat{A}(\cdot,B(w,z) \\
&= \hat{A}(B(w,\cdot),z)
\end{aligned}
$$

The equality of the right upper blocks is a consequence of the equalities

$$
\begin{aligned}
-2a &= -2a && (40)\\
-2B(\cdot, z) &= -2B(\cdot, z) && (41)\\
4ia\langle z, \overline{a\cdot}\rangle + \hat{A}(a\cdot, z) &= 2A(a\cdot, z) + 2ia\langle z, \overline{a\cdot}\rangle && (42)\\
2a\hat{r}(w, \bar{\cdot}) &= 2B(w, a\cdot) && (43)\\
2ia\langle aw, \overline{a\cdot}\rangle &= 2ia\langle aw, \overline{a\cdot}\rangle && (44)\\
4iB(\langle z, \overline{a\cdot}\rangle, z) &= 4iB(\langle z, \overline{a\cdot}\rangle, z) && (45)\\
-2i\hat{A}(a\langle z, \overline{a\cdot}\rangle, z) &= -2A(A(z, a\cdot), z) + A(A(z, z), a\cdot) && (46)\\
2B(\hat{r}(w, \bar{\cdot}), z) &= B(w, B(\cdot, z)) - B(\cdot, B(w, z)) + && (47)\\
&\quad +2B(r(\cdot, w), z)\\
-\hat{A}(a\hat{r}(w, \bar{\cdot}), z) + & && (48)\\
+2iB(\langle aw, \overline{a\cdot}\rangle, z) &= -B(w, A(a\cdot, z)) - iB(\langle a\cdot, a\bar{w}\rangle, z) -\\
&\quad -iB(\cdot, a\langle z, a\bar{w}\rangle) + iB(w, a\langle z, \overline{a\cdot}\rangle) -\\
&\quad -B(\cdot, A(aw, z)) + 2iB(\langle z, \overline{a\cdot}\rangle, aw) +\\
&\quad +iB(\langle aw, \overline{a\cdot}\rangle, z) - 4ia\langle z, B(\bar{w}, \overline{a\cdot})\rangle +\\
&\quad +2ia\langle z, ar(\bar{w}, \bar{\cdot})\rangle + A(B(w, z), a\cdot) -\\
&\quad -A(B(w, a\cdot), z) + A(B(\cdot, z), aw) +\\
&\quad +A(B(\cdot, aw), z) - ia\langle B(w, z), \overline{a\cdot}\rangle +\\
&\quad +ia\langle B(\cdot, z), a\bar{w}\rangle\\
-i\hat{A}(a\langle aw, \overline{a\cdot}\rangle, z) &= -ia\langle z, A(a\bar{w}, \overline{a\cdot}\rangle - A(A(aw, , a\cdot), z) + && (49)\\
&\quad +A(A(aw, z), a\cdot) - A(A(a\cdot, z), aw) -\\
&\quad -ia\langle A(z, aw), \overline{a\cdot}\rangle + ia\langle A(z, a\cdot), a\bar{w}\rangle -\\
&\quad -a\langle aw, a\langle \overline{a\cdot}, z\rangle\rangle + a\langle a\cdot, a\langle a\bar{w}, z\rangle\rangle.
\end{aligned}
$$

The equalities (40), (41), (44), (45) are tautologies. Using (22) and (23), one easily proves (42) and (43).

The equality (46) is a consequence of

$$\begin{aligned}
-2A(A(z,a\cdot),z)+A(A(z,z),a\cdot) &= -\frac{1}{2}\hat{A}(\hat{A}(z,a\cdot),z)-\frac{1}{2}\hat{A}(z,\hat{A}(z,a\cdot))- \\
&\quad -\frac{1}{2}\hat{A}(\hat{A}(a\cdot,z),z)-\frac{1}{2}\hat{A}(z,\hat{A}(a\cdot,z))+ \\
&\quad +\frac{1}{2}\hat{A}(A(z,z),a\cdot) \\
&= -\hat{A}(\hat{A}(z,a\cdot),z) \\
&= -2i\hat{A}(a\langle z,\bar{a\cdot}\rangle,z).
\end{aligned}$$

(47) follows immediately after applying the identities

$$\begin{aligned}
B(\hat{r}(w,\bar{\cdot}),z) &= B(w,B(\cdot,z)) \\
B(\hat{r}(\cdot,\bar{w}),z) &= B(\cdot,B(w,z)).
\end{aligned}$$

Then we obtain

$$\begin{aligned}
2B(r(\cdot,w),z) &= B(\hat{r}(w,\bar{\cdot}),z)+B(\hat{r}(\cdot,\bar{w}),z) \\
&= B(\hat{r}(w,\bar{\cdot}),z)+B(w,B(\cdot,z)) \\
&= B(\hat{r}(w,\bar{\cdot}),z)+B(\cdot,B(w,z)) \\
&= 2B(\hat{r}(w,\bar{\cdot}),z).
\end{aligned}$$

The left hand side of (48) equals $-\hat{A}(B(w,a\cdot),z)+\hat{A}(aw,B(\cdot,z))$. The right hand side can be transformed in the following way

$$\begin{aligned}
&-\frac{1}{2}B(w,\hat{A}(a\cdot,z))-\frac{1}{2}B(w,\hat{A}(z,a\cdot))-\frac{1}{2}\hat{A}(a\cdot,B(w,z))-\frac{1}{2}B(\cdot,\hat{A}(z,aw))+ \\
&\frac{1}{2}B(w,\hat{A}(z,a\cdot))-\frac{1}{2}B(\cdot,\hat{A}(aw,z))-\frac{1}{2}B(\cdot,\hat{A}(z,aw))+\hat{A}(z,B(\cdot,aw))+ \\
&+\frac{1}{2}\hat{A}(aw,B(\cdot,z))-2\hat{A}(z,B(\cdot,aw))+\frac{1}{2}\hat{A}(z,a\hat{r}(\cdot,\bar{w}))+\frac{1}{2}\hat{A}(z,a\hat{r}(w,\bar{\cdot}))+ \\
&+\frac{1}{2}\hat{A}(B(w,z),a\cdot)+\frac{1}{2}\hat{A}(a\cdot,B(w,z))-\frac{1}{2}\hat{A}(B(w,a\cdot),z)-\frac{1}{2}\hat{A}(z,B(w,a\cdot))+ \\
&+\frac{1}{2}\hat{A}(B(\cdot,z),aw)+\frac{1}{2}\hat{A}(aw,B(\cdot,z))+\frac{1}{2}\hat{A}(B(\cdot,aw),z)+\frac{1}{2}\hat{A}(z,B(\cdot,aw))- \\
&-\frac{1}{2}\hat{A}(B(w,z),a\cdot)+\frac{1}{2}\hat{A}(B(\cdot,z),aw)
\end{aligned}$$

This equals to

$$-\frac{1}{2}\hat{A}(B(w,a\cdot),z)-\frac{1}{2}\hat{A}(B(w,z),a\cdot)-\frac{1}{2}\hat{A}(a\cdot,B(w,z))-\frac{1}{2}\hat{A}(B(\cdot,z),aw)+$$
$$\frac{1}{2}\hat{A}(B(w,z),a\cdot)-\frac{1}{2}\hat{A}(B(\cdot,aw),z)-\frac{1}{2}\hat{A}(B(\cdot,z),aw)+\hat{A}(z,B(\cdot,aw))+$$
$$+\frac{1}{2}\hat{A}(aw,B(\cdot,z))-2\hat{A}(z,B(\cdot,aw))+\frac{1}{2}\hat{A}(z,B(\cdot,aw))+\frac{1}{2}\hat{A}(z,B(w,a\cdot))+$$
$$+\frac{1}{2}\hat{A}(B(w,z),a\cdot)+\frac{1}{2}\hat{A}(a\cdot,B(w,z))-\frac{1}{2}\hat{A}(B(w,a\cdot),z)-\frac{1}{2}\hat{A}(z,B(w,a\cdot))+$$
$$+\frac{1}{2}\hat{A}(B(\cdot,z),aw)+\frac{1}{2}\hat{A}(aw,B(\cdot,z))+\frac{1}{2}\hat{A}(B(\cdot,aw),z)+\frac{1}{2}\hat{A}(z,B(\cdot,,aw))-$$
$$-\frac{1}{2}\hat{A}(B(w,z),a\cdot)+\frac{1}{2}\hat{A}(B(\cdot,z),aw)$$

Cancelling appropriate terms we just obtain the terms from the left hand side.
It remains to prove (49). We transform the right hand side:

$$-\frac{i}{2}a\langle z,\hat{A}(a\bar{w},\overline{a\cdot})\rangle-\frac{i}{2}a\langle z,\hat{A}(\overline{a\cdot},a\bar{w})\rangle-\frac{1}{4}\hat{A}(\hat{A}(aw,a\cdot),z)-\frac{1}{4}\hat{A}(z,\hat{A}(aw,a\cdot))-$$
$$-\frac{1}{4}\hat{A}(\hat{A}(a\cdot,aw),z)-\frac{1}{4}\hat{A}(z,\hat{A}(a\cdot,aw))+\frac{1}{4}\hat{A}(\hat{A}(aw,z),a\cdot)+\frac{1}{4}\hat{A}(a\cdot,\hat{A}(aw,z))+$$
$$+\frac{1}{4}\hat{A}(\hat{A}(z,aw),a\cdot)+\frac{1}{4}\hat{A}(a\cdot,\hat{A}(z,aw))-\frac{1}{4}\hat{A}(\hat{A}(a\cdot,z),aw)-\frac{1}{4}\hat{A}(aw,\hat{A}(a\cdot,z))-$$
$$-\frac{1}{4}\hat{A}(\hat{A}(z,a\cdot),aw)-\frac{1}{4}\hat{A}(aw,\hat{A}(z,a\cdot))-\frac{i}{2}a\langle\hat{A}(z,aw),\overline{a\cdot}\rangle-\frac{i}{2}a\langle\hat{A}(aw,z),\overline{a\cdot}\rangle+$$
$$+\frac{i}{2}a\langle\hat{A}(z,a\cdot),a\bar{w}\rangle+\frac{i}{2}a\langle\hat{A}(a\cdot,z),a\bar{w}\rangle+\frac{i}{2}a\langle\hat{A}(aw,z),\overline{a\cdot}\rangle-\frac{i}{2}a\langle\hat{A}(a\cdot,z),a\bar{w}\rangle.$$

The terms $\frac{i}{2}a\langle\hat{A}(aw,z),\overline{a\cdot}\rangle$ as well as $\frac{i}{2}a\langle\hat{A}(a\cdot,z),a\bar{w}\rangle$ with positive and negative sign cancel out.
Using the identities

$$\begin{aligned}
\frac{1}{4}\hat{A}(z,\hat{A}(aw,a\cdot)) &= \frac{1}{4}\hat{A}(\hat{A}(z,aw),a\cdot)\\
\frac{1}{4}\hat{A}(\hat{A}(a\cdot,aw),z) &= \frac{1}{4}\hat{A}(a\cdot,\hat{A}(aw,z))\\
\frac{1}{4}\hat{A}(\hat{A}(aw,z),a\cdot) &= \frac{1}{4}\hat{A}(aw,\hat{A}(z,a\cdot))\\
\frac{1}{4}\hat{A}(a\cdot,\hat{A}(z,aw)) &= \frac{1}{4}\hat{A}(\hat{A}(a\cdot,z),aw)
\end{aligned}$$

four more pairs cancel out.
Now, we take into account that the two equal terms

$$-\frac{1}{4}\hat{A}(\hat{A}(aw, a\cdot), z) = -\frac{1}{4}\hat{A}(aw, \hat{A}(a\cdot, z))$$

together compensate the left hand side. Thus, it remains to show that the terms

$$-\frac{i}{2}a\langle z, \hat{A}(a\bar{w}, \bar{a\cdot})\rangle - \frac{i}{2}a\langle z, \hat{A}(\bar{a\cdot}, a\bar{w})\rangle - \frac{1}{4}\hat{A}(\hat{A}(aw, a\cdot), z) - \\ -\frac{1}{4}\hat{A}(aw, \hat{A}(a\cdot, z)) - \frac{i}{2}a\langle \hat{A}(z, aw), \bar{a\cdot}\rangle + \frac{i}{2}a\langle \hat{A}(z, a\cdot), a\bar{w}\rangle$$

together do not contribute to the final expression at the right hand side.
Thus, we have to show that

$$-\frac{i}{2}a\langle z, \hat{A}(a\bar{w}, \bar{a\cdot})\rangle - \frac{i}{2}a\langle z, \hat{A}(\bar{a\cdot}, a\bar{w})\rangle - \frac{1}{2}\hat{A}(\hat{A}(z, a\cdot), aw) - \\ -\frac{i}{2}a\langle \hat{A}(z, aw), \bar{a\cdot}\rangle + \frac{i}{2}a\langle \hat{A}(z, a\cdot), a\bar{w}\rangle.$$

equals 0.
Now, the terms

$$-\frac{i}{2}a\langle z, \hat{A}(a\bar{w}, \bar{a\cdot})\rangle = -a\langle z, a\langle a\bar{w}, a\cdot\rangle\rangle = -\frac{1}{2i}\hat{A}(z, a\langle a\cdot, a\bar{w}\rangle) = \frac{1}{4}\hat{A}(z, \hat{A}(a\cdot, aw))$$

and

$$\frac{i}{2}a\langle \hat{A}(z, a\cdot), a\bar{w}\rangle = \frac{1}{4}\hat{A}(z, \hat{A}(a\cdot, aw))$$

cancel with

$$-\frac{1}{2}\hat{A}(\hat{A}(z, a\cdot), aw) = -\frac{1}{2}\hat{A}(z, \hat{A}(a\cdot, aw)),$$

and

$$-\frac{i}{2}a\langle z, \hat{A}(\bar{a\cdot}, a\bar{w})\rangle = \frac{1}{4}\hat{A}(z, \hat{A}(aw, a\cdot)) = \frac{1}{4}\hat{A}(\hat{A}(z, aw), a\cdot)$$

cancels with

$$-\frac{i}{2}a\langle \hat{A}(z, aw), \bar{a\cdot}\rangle = -\frac{1}{4}\hat{A}(\hat{A}(z, aw), a\cdot).$$

The equality of the left lower blocks is a consequence of the equalities

$$
\begin{aligned}
-i\langle\cdot, a\bar{w}\rangle &= -i\langle\cdot, a\bar{w}\rangle && (50)\\
i\langle\hat{A}(z,\cdot), a\bar{w}\rangle &= -2\langle z, a\langle a\bar{w},\cdot\rangle\rangle && (51)\\
i\langle B(w,\cdot), a\bar{w}\rangle &= \frac{i}{2}\langle B(w,\cdot), a\bar{w}\rangle - \frac{i}{2}\langle\cdot, ar(\bar{w},\bar{w})\rangle + ir(\langle\cdot, a\bar{w}\rangle, w) && (52)\\
\frac{i}{2}\langle\hat{A}(aw,\cdot), a\bar{w}\rangle &= -\langle aw, a\langle a\bar{w},\cdot\rangle\rangle. && (53)
\end{aligned}
$$

(50) is a tautology, (51) a direct consequence of (22) and (53) holds because of (18). In order to prove (52) we have to show

$$\frac{i}{2}\langle B(w,\cdot), a\bar{w}\rangle = -\frac{i}{2}\langle\cdot, a\hat{r}(\bar{w}, w)\rangle + \frac{i}{2}\hat{r}(\langle\cdot, a\bar{w}\rangle, \bar{w}) + \frac{i}{2}\hat{r}(w, \langle, a\bar{w},\cdot\rangle).$$

We have

$$
\begin{aligned}
-\frac{i}{2}\langle\cdot, a\hat{r}(\bar{w}, w)\rangle &= -\frac{i}{2}\overline{\langle a\hat{r}(\bar{w}, w),\cdot\rangle}\\
&= -\frac{i}{2}\overline{\hat{r}(\bar{w}, \langle\cdot, a\bar{w}\rangle)}\\
&= -\frac{i}{2}\hat{r}(\langle\cdot, a\bar{w}\rangle, \bar{w})
\end{aligned}
$$

which cancels with the corresponding term at the right hand side. Since

$$\frac{i}{2}\hat{r}(w, \langle, a\bar{w},\cdot\rangle) = \frac{i}{2}\langle B(w,\cdot), a\bar{w}\rangle,$$

both sides are equal.
The equality of the right lower blocks is a consequence of the equalities

$$
\begin{aligned}
\mathrm{id} &= \mathrm{id} && (54)\\
-2i\langle z, a\bar{\cdot}\rangle &= -2i\langle z, a\bar{\cdot}\rangle && (55)\\
-\hat{r}(w,\bar{\cdot}) - \hat{r}(\cdot,\bar{w}) &= -2r(\cdot, w) && (56)\\
-i\langle aw, a\bar{\cdot}\rangle + i\langle a\bar{\cdot}, a\bar{w}\rangle &= -i\langle aw, a\bar{\cdot}\rangle + i\langle a\bar{\cdot}, a\bar{w}\rangle && (57)\\
2i\hat{r}(\langle z, a\bar{\cdot}\rangle, \bar{w}) &= 2i\langle z, B(\bar{w}, a\bar{\cdot})\rangle && (58)\\
2\langle a\langle z, a\bar{\cdot}\rangle, a\bar{w}\rangle &= 2\langle z, a\langle a\bar{w}, a\cdot\rangle\rangle && (59)\\
\hat{r}(\hat{r}(w,\bar{\cdot}),\bar{w}) &= 2r(r(w,\cdot),w) - r(r(w,w),\cdot) && (60)\\
-i\langle a\hat{r}(w,\bar{\cdot}), a\bar{w}\rangle + i\hat{r}(\langle aw, a\bar{\cdot}\rangle, \bar{w}) &= -i\langle B(w, a\cdot), a\bar{w}\rangle + i\langle ar(w,\cdot), a\bar{w}\rangle - && (61)\\
&\quad -ir(\langle a\cdot, a\bar{w}\rangle, w) + ir(\langle aw, a\bar{\cdot}\rangle, w) - \\
&\quad -ir(\langle aw, a\bar{w}\rangle, \cdot) + \frac{i}{2}\langle a\cdot, ar(\bar{w},\bar{w})\rangle - \\
&\quad -\frac{i}{2}\langle ar(w,w), a\bar{\cdot}\rangle + i\langle aw, ar(\bar{w},\bar{\cdot})\rangle \\
\langle a\langle aw, a\bar{\cdot}\rangle, a\bar{w}\rangle &= \langle aw, a\langle a\bar{w}, a\bar{\cdot}\rangle\rangle. && (62)
\end{aligned}
$$

The equalities (54), (55), (57) are tautological, (56) follows from (3) and (19). (58) is a consequence of (23), (59) follows from (22). (18) and (22) imply (62). (60) can be obtained by the following transformations of the right hand side:

$$
\begin{aligned}
2r(r(w,\cdot),w) - r(r(w,w),\cdot) &= \frac{1}{2}\hat{r}(\hat{r}(w,\bar{\cdot}),\bar{w}) + \frac{1}{2}\hat{r}(w,\hat{r}(\bar{\cdot},w)) + \frac{1}{2}\hat{r}(\hat{r}(\cdot,\bar{w}),\bar{w}) + \\
&\quad + \frac{1}{2}\hat{r}(w,\hat{r}(\bar{w},\cdot)) - \frac{1}{2}\hat{r}(\hat{r}(w,\bar{w}),\bar{\cdot}) - \frac{1}{2}\hat{r}(\cdot,\hat{r}(\bar{w},w))
\end{aligned}
$$

and taking into account that

$$
\begin{aligned}
\frac{1}{2}\hat{r}(\hat{r}(w,\bar{\cdot}),\bar{w}) &= \frac{1}{2}\hat{r}(w,\hat{r}(\bar{\cdot},w))\\
\frac{1}{2}\hat{r}(\hat{r}(\cdot,\bar{w}),\bar{w}) &= \frac{1}{2}\hat{r}(\cdot,\hat{r}(\bar{w},w))\\
\frac{1}{2}\hat{r}(w,\hat{r}(\bar{w},\cdot)) &= \frac{1}{2}\hat{r}(\hat{r}(w,\bar{w}),\bar{\cdot}).
\end{aligned}
$$

It remains to prove (61). Since

$$-i\langle a\hat{r}(w,\bar{\cdot}), a\bar{w}\rangle = -i\langle B(w,a\cdot), a\bar{w}\rangle,$$

these terms cancel out immediately. In the remaining terms on the right hand side we express r by $\hat{r}$.

$$\frac{i}{2}\langle a\hat{r}(w,\bar{\cdot}), a\bar{w}\rangle + \frac{i}{2}\langle a\hat{r}(\cdot,\bar{w}), a\bar{w}\rangle - \frac{i}{2}\hat{r}(\langle a\cdot, a\bar{w}\rangle, \bar{w}) - \frac{i}{2}\hat{r}(w, \langle a\bar{w}, a\cdot\rangle) +$$
$$+\frac{i}{2}\hat{r}(\langle aw, a\bar{\cdot}\rangle, \bar{w}) + \frac{i}{2}\hat{r}(w, \langle a\bar{\cdot}, aw\rangle - \frac{i}{2}\hat{r}(\langle aw, a\bar{w}\rangle, \bar{\cdot}) - \frac{i}{2}\hat{r}(\cdot, \langle a\bar{w}, aw\rangle) +$$
$$+\frac{i}{2}\langle a\cdot, a\hat{r}(\bar{w}, w)\rangle - \frac{i}{2}\langle a\hat{r}(w,\bar{w}), a\bar{\cdot}\rangle + \frac{i}{2}\langle aw, a\hat{r}(\bar{w},\cdot)\rangle + \frac{i}{2}\langle aw, a\hat{r}(\bar{\cdot}, w)\rangle$$

Using the identities

$$\begin{aligned}
\frac{i}{2}\langle a\hat{r}(w,\bar{\cdot}), a\bar{w}\rangle &= \frac{i}{2}\hat{r}(w, \langle a\bar{w}, a\cdot\rangle)\\
\frac{i}{2}\langle a\hat{r}(\cdot,\bar{w}), a\bar{w}\rangle &= \frac{i}{2}\hat{r}(\cdot, \langle a\bar{w}, aw\rangle)\\
\frac{i}{2}\hat{r}(\langle a\cdot, a\bar{w}\rangle, \bar{w}) &= \frac{i}{2}\langle a\cdot, a\hat{r}(\bar{w}, w)\rangle\\
\frac{i}{2}\hat{r}(w, \langle a\bar{\cdot}, aw\rangle &= \frac{i}{2}\langle a\hat{r}(w,\bar{w}), a\bar{\cdot}\rangle\\
\frac{i}{2}\hat{r}(\langle aw, a\bar{w}\rangle, \bar{\cdot}) &= \frac{i}{2}\langle aw, a\hat{r}(\bar{\cdot}, w)\rangle\\
\frac{i}{2}\langle aw, a\hat{r}(\bar{w},\cdot)\rangle &= \frac{i}{2}\hat{r}(\langle aw, a\bar{\cdot}\rangle, \bar{w})
\end{aligned}$$

and cancelling out the corresponding terms in the right hand side of (61) we obtain

$$\hat{r}(\langle aw, a\bar{\cdot}\rangle, \bar{w}),$$

which coincides with the remaining term on the left hand side.

References

[1] V. K. Belošapka. Finite-dimensionality of the automorphism group of a real-analytic surface (in Russian). *Izv. Akad. Nauk SSSR Ser. Mat.*, 52(2):437–442, 1988. English transl. in Math. USSR-Izv. 32(1989).

[2] V. K. Belošapka. A uniqueness theorem for automorphisms of a nondegenerate surface in the complex space (in Russian). *Mat. Zametki*, 47(3):17–22, 1990. English transl. in Math. Notes 47(1990).

[3] V.V. Ežov and G. Schmalz. Holomorphic automorphisms of quadrics. *Math. Z.*, 216:453–470, 1994.

[4] V.V. Ežov and G. Schmalz. A matrix Poincaré formula for holomorphic automorphisms of quadrics of higher codimension. Real associative quadrics. *J. Geom. Analysis*, to appear, 1994.

[5] V.V. Ežov and G. Schmalz. Poincaré automorphisms for nondegenerate CR quadrics. *Math. Annalen*, 298:79–87, 1994.

[6] V.V. Ežov and G. Schmalz. A simple proof of Belošapka's theorem on the parametrization of the automorphism group of CR-manifolds. *Mat. Zametki (Math. Notes)*, to appear, 1996.

[7] G.M. Henkin and A.E. Tumanov. Local characterization of holomorphic automorphisms of Siegel domains. *Funkt. Analysis*, 17(4):49–61, 1983.

[8] W. Kaup, Y. Matsushima, and T. Ochiai. On the automorphisms and equivalences of generalized Siegel domains. *Amer. J. Math.*, 92(2):475–497, 1970.

[9] N. Palinčak. On quadrics of high codimension (in Russian). *Mat. zametki*, 55(5):110–115, 1994.

[10] H. Poincaré. Les fonctions analytiques de deux variables et la représentation conforme. *Rend. Circ. Math. Palermo*, pages 185–220, 1907.

[11] Pyatetskii-Shapiro. *Automorphic Functions and Geometry of Classical Domains.* Gordon and Breach, New York, 1969.

[12] O. Rothaus. Automorphisms of Siegel domains. *Amer. J. Math.*, 101(5):1167–1179, 1979.

[13] I. Satake. *Algebraic structures of symmetric domains.* Iwanami Shoten and Princeton University Press, 1980.

[14] A.E. Tumanov. Finite dimensionality of the group of CR-automorphisms of a standard CR manifold and characteristic holomorphic mappings of Siegel domains (in Russian). *USSR Izvestiya*, 32(3):655–662, 1989.

(V.V. Ežov) University of Adelaide, Department of Pure Mathematics, Adelaide 5005, South Australia

E-mail address: `ezhov@spam.maths.adelaide.edu.au`

(G. Schmalz) Mathematisches Institut der Universität Bonn, Beringstrasse 1, D-53115 Bonn

E-mail address: `schmalz@mpim-bonn.mpg.de`