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Effects of a Degeneracy in the Competition Model

Part II. Perturbation and Dynamical Behaviour

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This is the second part of our study on the competition model

$$\begin{cases} u_t(x,t) - d_1(x)\,\Delta u(x,t) = \lambda a_1(x)\,u - b(x)\,u^2 - c(x)\,uv, \\ v_t(x,t) - d_2(x)\,\Delta v(x,t) = \mu a_2(x)\,v - e(x)\,v^2 - d(x)\,uv, \end{cases}$$

where the coefficient functions are strictly positive over the underlying spatial region Ω except $b(x)$, which vanishes in a nontrivial subdomain of Ω, and is positive in the rest of Ω. In part I, we mainly discussed the existence of two kinds of steady-state solutions of this system, namely, the classical steady-states and the generalized steady-states. Here we use these solutions to determine the dynamics of the model. We do this with the help of the perturbed model where $b(x)$ is replaced by $b(x)+\varepsilon$, which itself is a classical competition model. This approach also reveals the interesting relationship between the steady-state solutions (both classical and generalized) of the above system and that of the perturbed system.

Key Words: competition model; boundary blow-up.

1. INTRODUCTION

This is the second part of our study on the competition model

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = \lambda u - b(x)\,u^2 - cuv, \\ v_t(x,t) - \Delta v(x,t) = \mu v - v^2 - duv, \end{cases} \tag{1.1}$$

where $x \in \Omega$ and $t \geqslant 0$, Ω denotes a smooth bounded domain in R^N ($N \geqslant 2$), Δ denotes the Laplacian operator on the space variable x, $b(x)$ is a nonnegative function over Ω, and λ, μ, c and d are positive constants. Moreover, we suppose that u and v satisfy homogeneous Dirichlet boundary conditions on $\partial\Omega$. Our methods and results hold when (1.1) has a more general form, and for Neumann and Robin boundary conditions as well, as indicated at the beginning of Part I (see [Du]).

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We are interested in understanding the effects of the degeneracy of $b(x)$ on the model. As in Part I, by degeneracy, we mean that $b(x) \equiv 0$ on some proper subdomain Ω_0 of Ω, and $b(x) > 0$ on $\Omega_+ = \Omega \backslash \bar{\Omega}_0$. For technical reasons, we assume further that Ω_0 has C^2 boundary $\partial \Omega_0$, is open and connected, and $\bar{\Omega}_0 \subset \Omega$. Moreover $b \in C(\bar{\Omega})$. All our notations here will follow that of Part I.

We have proved in Part I that if $\lambda < \lambda_1^{\Omega_0}(0)$, then (1.1) behaves as if $b(x)$ is a positive constant, i.e., the degeneracy has little effects on the model. However, if $\lambda > \lambda_1^{\Omega_0}(0)$, then the steady-state solution set of (1.1) is changed a great deal by this degeneracy. Moreover, in Part I, we also discussed the generalized steady-state solutions (u, v) of (1.1), where u equals ∞ on $\bar{\Omega}_0$ and is finite and positive on Ω_+, whereas v is identically zero on Ω_0 and is positive on Ω_+. These generalized steady-states are governed by the following boundary blow-up problem,

$$\begin{cases} -\Delta u = \lambda u - b(x)\, u^2 - cuv, & x \in \Omega_+, \\ -\Delta v = \mu v - v^2 - duv, & x \in \Omega_+, \\ u|_{\partial\Omega} = 0, \quad u|_{\partial\Omega_0} = \infty, \quad v|_{\partial\Omega_+} = 0. \end{cases} \tag{1.2}$$

Here in Part II, based on results obtained in Part I, we will first show that both the classical and generalized steady-states of (1.1) occur naturally as the limits, when $\varepsilon \to 0$, of the positive classical solutions of the perturbed system

$$\begin{cases} -\Delta u = \lambda u - [b(x) + \varepsilon]\, u^2 - cuv, \\ -\Delta v = \mu v - v^2 - duv, \\ u|_{\partial\Omega} = 0, \quad v|_{\partial\Omega} = 0, \end{cases} \tag{1.3}$$

where $\varepsilon > 0$ is a constant. This approach not only reveals the interesting asymptotic behaviour of the positive solution branch $S_\varepsilon = \{(\mu, u, v)\}$ of (1.3) as $\varepsilon \to 0$, but also helps to better understand the generalized steady-states of (1.1); for example, it enables us to show that, when $\lambda > \lambda_1^{\Omega_0}(0)$, the positive solution set $\{(\mu, u, v)\}$ of (1.2) contains an unbounded continuum in a suitable space. We will then discuss the dynamical behaviour of (1.1) and show that the dynamics of the model is affected greatly by the degeneracy of $b(x)$. It turns out that our perturbation approach is important for discussing the dynamical behaviour of (1.1).

In order to understand the perturbed system (1.3), we study in Section 2 the perturbed logistic equation

$$-\Delta u + \phi u = \lambda u - [b(x) + \varepsilon]\, u^2, \qquad u|_{\partial\Omega} = 0, \tag{1.4}$$

where $\phi \in C(\bar{\Omega})$. While it is easily seen that the unique positive solution u_ε of (1.4) varies continuously with $\varepsilon \in [0, \infty)$ when $\lambda \in (\lambda_1^{\Omega}(\phi), \lambda_1^{\Omega_0}(\phi))$, it is no longer the case once $\lambda \geqslant \lambda_1^{\Omega_0}(\phi)$. Indeed, we will show that in this case, u_ε blows up as $\varepsilon \to 0$ on $\bar{\Omega}_0$ while remains bounded on Ω_+. Moreover, on Ω_+, u_ε converges to the minimal positive solution $\underline{U}$ of the boundary blow-up problem

$$-\Delta u + \phi u = \lambda u - b(x)\, u^2 \text{ in } \Omega_+,\ u|_{\partial\Omega} = 0,\ u|_{\partial\Omega_0} = \infty. \tag{1.5}$$

As a by-product, we show that $\underline{U}$ varies continuously with λ for $\lambda > \lambda_1^{\Omega_0}(\phi)$.

In Section 3, we discuss how the positive solution set of (1.3) changes with ε. As is well known, (1.3) has no positive solution if $\lambda \leqslant \lambda_1^{\Omega}(0)$. Therefore, we assume $\lambda > \lambda_1^{\Omega}(0)$. By [DB], we know that there exist $\lambda_1^{\Omega}(0) < \mu_*(\varepsilon) \leqslant \mu^*(\varepsilon)$ such that (1.3) has no positive solution if $\mu \notin [\mu_*(\varepsilon), \mu^*(\varepsilon)]$, and it has at least one positive solution if $\mu \in (\mu_*(\varepsilon), \mu^*(\varepsilon))$. Moreover, the positive solutions can be chosen from a continuum of positive solutions $S_\varepsilon = \{(\mu, u, v)\}$, which connects the two semitrivial solutions $(\mu^0, 0, \theta_{\mu^0})$ and $(\mu_0^\varepsilon, u_\varepsilon, 0)$, where θ_μ denotes the unique positive solution to

$$-\Delta v = \mu v - v^2, \qquad v|_{\partial\Omega} = 0,$$

μ^0 is determined uniquely by

$$\lambda = \lambda_1^{\Omega}(c\theta_{\mu^0}),$$

u_ε is the unique positive solution of (1.4) with $\phi = 0$ and $\mu_0^\varepsilon = \lambda_1^{\Omega}(du_\varepsilon)$.

If $\lambda < \lambda_1^{\Omega_0}(0)$, then our results in this section show that the solution branch S_ε remains bounded, and as $\varepsilon \to 0$, S_ε approaches S, the branch of steady-state solutions of (1.1) given in Theorem 2.4 of Part I.

If $\lambda > \lambda_1^{\Omega_0}(0)$, however, then for any fixed $\varepsilon > 0$, S_ε is a bounded set, but by Theorem 3.1 in Part I, the branch of steady-state solutions of (1.1), S, is an unbounded set. Therefore, it is more interesting to see how S_ε and S are related when $\varepsilon \to 0$. It turns out that as $\varepsilon \to 0$, both $\mu^*(\varepsilon)$ and S_ε become unbounded. Moreover, only part of S_ε approaches the unbounded set S, while another part of S_ε converges to the generalized steady-states of (1.1). Thus, both the classical and generalized steady-state solutions of (1.1) can come from the same origin, namely, S_ε. Moreover, by using this approach, we prove that the set of positive solutions of (1.2) contains, in a suitable space, an unbounded branch $\hat{S}$ bifurcating from the semitrivial solution branch $\{(\mu, \underline{U}, 0) : \mu \in R\}$ at $\mu = \lambda_1^{\Omega_+}(d\underline{U})$ and with the μ-range of $\hat{S}$ covering $(\lambda_1^{\Omega_+}(d\underline{U}), \infty)$.

Section 4 is devoted to the study of the dynamical behaviour of (1.1). The importance of the generalized steady-state solutions of (1.1) is fully revealed here. We show that if μ is small so that (1.1) has no classical nor generalized positive steady-state solutions, then every positive solution of (1.1) has its v component converging to 0 on Ω as $t \to \infty$, while u blows up on Ω_0 as $t \to \infty$. If $\mu > \lambda_1^{\Omega_+}(d\bar{U})$, where $\bar{U}$ denotes the maximal positive solution of (1.2) with $\phi = 0$, then persistence of v is guaranteed. Moreover, in this case, for $x \in \Omega_+$, $\underline{\lim}_{t\to\infty} v(x, t) \geq \underline{v}(x)$, $\overline{\lim}_{t\to\infty} u(x, t) \leq \underline{u}(x)$, where $(\underline{u}, \underline{v})$ denotes the minimal positive solution of (1.2).

Though Theorems 3.1 and 3.6 of Part I show that a stable coexistence state of the two species is possible, results in the rest of Section 4 show that one can always find bad initial conditions such that the positive solutions of (1.1) with these bad initial conditions must have the u component blowing up in Ω_0 as $t \to \infty$. We also show that there are parameter ranges such that the global attractor of (1.1) is solely determined by the generalized steady-state solutions. Moreover, if (1.1) has a unique generalized steady-state solution in this case, then it attracts all the positive solutions.

2. PERTURBATION OF THE DEGENERATE LOGISTIC EQUATION

In order to study how the positive solutions of the perturbed system (1.3) approaches the steady-states of (1.1), we need to know how the positive solutions of the perturbed logistic equation (1.4) approach the solutions of the unperturbed equation, i.e., (1.4) with $\varepsilon = 0$. Recall that for any $\varepsilon > 0$, (1.4) has a unique positive solution when $\lambda > \lambda_1^{\Omega}(\phi)$, and (1.4) with $\varepsilon = 0$ has no positive solution if $\lambda \notin (\lambda_1^{\Omega}(\phi), \lambda_1^{\Omega_0}(\phi))$, and there is a unique positive solution if $\lambda \in (\lambda_1^{\Omega}(\phi), \lambda_1^{\Omega_0}(\phi))$. Moreover, for any real number λ, the boundary blow-up problem (1.5) has a minimal positive solution $\underline{U}$.

It can be easily seen that if $\lambda \in (\lambda_1^{\Omega}(\phi), \lambda_1^{\Omega_0}(\phi))$, then the unique positive solution of (1.4) varies continuously with ε for $\varepsilon \geq 0$. The following result describes the situation for $\lambda \geq \lambda_1^{\Omega_0}(\phi)$.

THEOREM 2.1. *Suppose that $\lambda \geq \lambda_1^{\Omega_0}(\phi)$ and $\varepsilon > 0$, and denote by u_ε the unique positive solution of* (1.4), *by $\underline{U}$ the minimal positive solution of* (1.5). *Then*

(i) $u_\varepsilon \to \infty$ *as* $\varepsilon \to 0$ *uniformly on* $\bar{\Omega}_0$;

(ii) $u_\varepsilon \to \underline{U}$ *as* $\varepsilon \to 0$ *uniformly on any compact subset of* $\bar{\Omega} \setminus \bar{\Omega}_0$.

Proof. A standard upper and lower solution argument together with the uniqueness of u_ε shows that $\varepsilon \to u_\varepsilon(x)$ is decreasing. Thus, $\lim_{\varepsilon\to 0} u_\varepsilon(x)$ is either finite or infinity. By Lemma 2.1 in [DH], we also have $u_\varepsilon \leq \underline{U}$ on Ω_+. Thus there exists some function $u_0(x)$ on Ω_+ such that

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = u_0(x) \leq \underline{U}(x), \qquad \text{for all} \quad x \in \Omega_+.$$

By elliptic regularity, one easily sees that $u_\varepsilon \to u_0$ uniformly on any compact subset of $\bar{\Omega} \setminus \bar{\Omega}_0$, and u_0 satisfies the differential equation in (1.5) together with the boundary condition on $\partial\Omega$. We will see that $u_0 = \underline{U}$.

Let

$$m_\varepsilon = \min_{x \in \bar{\Omega}_0} u_\varepsilon(x) = u_\varepsilon(x_\varepsilon), \qquad x_\varepsilon \in \bar{\Omega}_0.$$

We claim that $m_\varepsilon \to \infty$ as $\varepsilon \to 0$. Clearly this implies (i). We prove this claim by an indirect argument and divide the proof into several steps.

Step 1. If $m_\varepsilon \leqslant M$ for some constant M and all $\varepsilon > 0$, then $d(x_\varepsilon, \partial\Omega_0) \to 0$.

Since $\lambda \geqslant \lambda_1^{\Omega_0}(\phi)$, we must have $\|u_\varepsilon\|_{L^\infty(\Omega)} \to \infty$ as $\varepsilon \to 0$, for otherwise u_ε increases to a positive solution of (1.4) with $\varepsilon = 0$ as ε decreases to 0, contradicting Theorem 2.2 in Part I. Let us now pick up a sequence $\varepsilon_n \to 0$, and define $\hat{u}_n = u_n / \|u_n\|_\infty$, where $u_n = u_{\varepsilon_n}$. We easily see that

$$-\Delta\hat{u}_n + \phi\hat{u}_n = \lambda\hat{u}_n - [b(x) + \varepsilon_n] \, \|u_n\|_\infty \, \hat{u}_n^2, \qquad \hat{u}_n|_{\partial\Omega} = 0.$$

It follows that $-\Delta\hat{u}_n \leqslant (\lambda + \|\phi\|_\infty) \, \hat{u}_n$, which implies, by Lemma 2.10 in Part I, that subject to a subsequence, $\hat{u}_n$ converges weakly in $W^{1,2}$ and strongly in L^p (for all $p > 1$) to some $\hat{u} \in W_0^{1,2}(\Omega)$. Moreover, $\hat{u} \not\equiv 0$.

As u_n is bounded from above by $\underline{U}$ on Ω_+, we easily see that $\hat{u} \equiv 0$ on Ω_+. Thus, as $\partial\Omega_0$ is smooth, $\hat{u} \in W_0^{1,2}(\Omega_0)$.

Let $\|u_n\|_\infty = u(x_n)$, $x_n \in \Omega_0$. Then $-\Delta u_n(x_n) \geqslant 0$ and hence, from the equation for u_n, we obtain

$$\phi(x_n) \, u_n(x_n) \leqslant \lambda u_n(x_n) - [b(x_n) + \varepsilon_n] \, u_n(x_n)^2.$$

It follows that $\varepsilon_n \|u_n\|_\infty \leqslant \lambda + \|\phi\|_\infty$. Hence we may assume that $\varepsilon_n \, \|u_n\|_\infty \to \xi$ for some $\xi \geqslant 0$.

Now we multiply the equation for $\hat{u}_n$ by an arbitrary $\psi \in C_0^\infty(\Omega_0)$ and integrate over Ω_0, and pass to the limit $n \to \infty$, we obtain that

$$\int_{\Omega_0} [\nabla\hat{u} \cdot \nabla\psi + \phi\hat{u}\psi] \, dx = \int_{\Omega_0} (\lambda\hat{u} - \xi\hat{u}^2) \, \psi \, dx.$$

That is to say that $\hat{u}$ is a weak solution to

$$-\Delta u + \phi u = \lambda u - \xi u^2, \qquad u|_{\partial\Omega_0} = 0.$$

By the weak Harnack inequality, we know $\hat{u} > 0$ in Ω_0.

From the equation for $\hat{u}_n$, we see that $-\Delta\hat{u}_n$ is uniformly bounded on $\overline{\Omega_0}$. By standard interior L^p theory for elliptic equations (see [CW,LU]), we find that $\hat{u}_n$ is bounded in $W^{2,p}(\Omega')$ for any $p>1$ and any compact subdomain Ω' of Ω_0. By the Sobolev imbedding theorem (see [GT]), we know that subject to a subsequence, $\hat{u}_n \to \hat{u}$ in $C^1(\overline{\Omega'})$. As $\hat{u}>0$ on Ω_0, and $\|u_n\|_\infty \to \infty$, we find that $u_n(x) \to \infty$ uniformly on any compact subset of Ω_0. As $\varepsilon \to u_\varepsilon$ is monotone, $u_\varepsilon \to \infty$ uniformly on any compact subset of Ω_0 as $\varepsilon \to 0$. Thus we must have $d(x_\varepsilon, \partial\Omega_0) \to 0$ as $\varepsilon \to 0$.

Step 2. If $m_\varepsilon < M$ for some M and all $\varepsilon > 0$, then $\{\partial u_\varepsilon(x_\varepsilon)/\partial v_\varepsilon\}$ is bounded from above, where v_ε is a unit vector in R^N to be specified later.

It suffices to show that for any sequence $\varepsilon_n \to 0$, $\{\partial u_{\varepsilon_n}(x_{\varepsilon_n})/\partial v_{\varepsilon_n}\}$ has a subsequence which is bounded from above. Let us denote

$$u_n = u_{\varepsilon_n}, \qquad x_n = x_{\varepsilon_n}, \qquad \text{and} \qquad \Omega_n = \{x \in \Omega_0 : d(x, \partial\Omega_0) \geqslant d(x_n, \partial\Omega_0)\}.$$

Note that if $x_n \in \partial\Omega_0$, then $\Omega_n = \Omega_0$, and if Ω_n is different from Ω_0, then for large n, it is close to Ω_0 by Step 1. Thus for any $\Omega' \subset\subset \Omega_0$, $\Omega' \subset\subset \Omega_n$ for all large n. By a simple variant of Lemma 2.3 in [DH], we find that the problem

$$-\Delta u + \phi u = \lambda u - [b(x) + \varepsilon_n]\, u^2 \text{ in } \Omega \setminus \bar{\Omega}_n, \qquad u|_{\partial\Omega} = 0,\ u|_{\partial\Omega_n} = u_n(x_n) \tag{2.1}$$

has a unique positive solution v_n. Clearly u_n is an upper solution to this problem. Thus by Lemma 2.1 in [DH], $u_n \geqslant v_n$ in $\Omega \setminus \Omega_n$. As $u_n(x_n) = v_n(x_n)$, it follows that

$$\partial u_n(x_n)/\partial v_n \leqslant \partial v_n(x_n)/\partial v_n,$$

where v_n is the unit normal vector of $\partial\Omega_n$ at x_n pointing inward of Ω_n. Thus it suffices to show that $\partial v_n(x_n)/\partial v_n$ is bounded.

Let us now choose an open subdomain $\Omega' \subset\subset \Omega_0$ which is so close to Ω_0 such that $\lambda_1^{\Omega_0 \setminus \overline{\Omega'}}(0) > \lambda + \|\phi\|_\infty$. For example, we may choose $\Omega' = \{x \in \Omega_0 : d(x, \partial\Omega_0) < \delta\}$ with $\delta > 0$ sufficiently small. Then, $\lambda_1^{\Omega_0 \setminus \overline{\Omega'}}(\phi) > \lambda$ and we can find some λ' such that $\max\{\lambda, \lambda_1^{\Omega \setminus \overline{\Omega'}}(\phi)\} < \lambda' < \lambda_1^{\Omega_0 \setminus \overline{\Omega'}}(\phi)$. By Theorem 2.2 in Part I, the problem

$$-\Delta u + \phi u = \lambda' u - b(x)\, u^2 \text{ in } \Omega \setminus \overline{\Omega'}, \qquad u|_{\partial\Omega \cup \partial\Omega'} = 0$$

has a unique positive solution u'. We may assume that $\bar{\Omega}_n \supset \Omega'$ for all n. Then we can find a large positive constant M_1 such that $M_1 u' \geqslant M \geqslant u_n(x_n)$ on $\partial\Omega_n$ for all n. It is easily seen that $M_1 u'$ is an upper solution to (2.1) for all n. Therefore, by Lemma 2.1 of [DH], $u_n \leqslant M_1 u'$. This implies that $-\Delta v_n$ has an L^∞ bound on $\Omega \setminus \Omega_n$ which is independent of n. Since furthermore,

(a) $v_n|_{\partial\Omega_n}$ is a constant which has a bound independent of n, and

(b) for all large n, $\partial\Omega_n$ is as smooth as Ω_0 with the smoothness not depending on n, by the L^p theory of elliptic equations up to the boundary (see, e.g., [LU, pp. 190–193]), we see that, for any $p>1$, $\|v_n\|_{W^{2,p}(\Omega\setminus\Omega_n)}$ has a bound independent of n. By Sobolev imbeddings and the uniform smoothness of Ω_n, this implies that $\|v_n\|_{C^1(\bar\Omega\setminus\Omega_n)}$ has a bound independent of n. In particular $|\nabla v_n(x_n)|$ is bounded, and thus $\partial v_n(x_n)/\partial \nu_n$ is bounded, as required.

Step 3. $m_\varepsilon\to\infty$ as $\varepsilon\to 0$.

Otherwise we can find a sequence $\varepsilon_n\to 0$ such that m_{ε_n} is bounded. By Step 2, $\{\partial u_n(x_n)/\partial\nu_n\}$ is bounded from above, where ν_n is the unit normal vector of $\partial\Omega_n$ at x_n pointing inward of Ω_n. Here we follow the notations in Step 2. We show that this is impossible, and hence proving the claim. For all large n, $\partial\Omega_n$ is as smooth as $\partial\Omega_0$ and hence it satisfies a uniform interior ball condition: There exists $R>0$ such that for any large n and $x\in\partial\Omega_n$, one can find a closed ball B_x of radius R such that $B_x\in\overline{\Omega_n}$ and $B_x\cap\partial\Omega_n=\{x\}$. Let y_n denote the center of B_{x_n} and define

$$\psi(x)=e^{-\sigma|x-y_n|^2}-e^{-\sigma R^2},$$

where σ is a positive number to be specified. We may assume that $\varepsilon_n<1$ for all n. Then, for any constant c satisfying $1<c<\varepsilon_n^{-1/2}$ and $x\in B_{x_n}\setminus B^n$, where $B^n=\{x\colon |x-y_n|<R/2\}$, we have

$$\begin{aligned}
&\Delta[u_n(x_n)+c\psi]+(\lambda-\phi)[u_n(x_n)+c\psi]-\varepsilon_n[u_n(x_n)+c\psi]^2\\
&\quad=c[4\sigma^2|x-y_n|^2-2N\sigma+\lambda-\phi-2\varepsilon_n u_n(x_n)\\
&\qquad+2\varepsilon_n c e^{-\sigma R^2}-\varepsilon_n c e^{-\sigma|x-y_n|^2}]\,e^{-\sigma|x-y_n|^2}\\
&\qquad+(\lambda-\phi)[u_n(x_n)-ce^{-\sigma R^2}]-\varepsilon_n u_n(x_n)^2+2c\varepsilon_n u_n(x_n)\,e^{-\sigma R^2}-\varepsilon_n c^2 e^{-2\sigma R^2}\\
&\quad\geqslant c[\sigma^2R^2-2N\sigma-2M-1]\,e^{-\sigma R^2}-(|\lambda|+\|\phi\|_\infty)\,M-M^2-1\\
&\quad>0,
\end{aligned}$$

if σ, c and n are large enough. We fix σ at such a value.

Choose a compact set $K\subset\subset\Omega_0$ such that $K\supset\bigcup_{n=1}^\infty B^n$. By the proof of Step 1, $u_n\to\infty$ on K. Hence we can find a sequence $c_n\to\infty$ satisfying $c_n\leqslant\varepsilon_n^{-1/2}$ and

$$u_n(x)\geqslant M+c_n\psi|_{\partial B^n},\qquad \text{for all}\quad x\in\partial B^n\subset K.$$

Thus, u_n is an upper solution to the problem

$$-\Delta u+\phi u=\lambda u-\varepsilon_n u^2 \text{ in } B_{x_n}\setminus B^n,\qquad u|_{\partial B_{x_n}}=u_n(x_n),\ u|_{\partial B^n}=u_n(x_n)+c_n\psi|_{\partial B^n}.$$

By our choice of σ, for all large n, $u_n(x_n)+c_n\psi$ is a lower solution to this problem. By Lemma 2.1 in [DH], we find $u_n \geqslant u_n(x_n)+c_n\psi$ in $B_{x_n} \setminus B^n$, and it follows that

$$\partial u_n(x_n)/\partial v_n \geqslant c_n \partial\psi(x_n)/\partial v_n = c_n 2\sigma R e^{-\sigma R^2} \to \infty.$$

This contradicts the conclusion in Step 2. Thus the claim and hence conclusion (i) of the theorem is proved.

It remains to prove conclusion (ii). By (i), we see that $u_n|_{\partial\Omega_0} \to \infty$ uniformly as $n \to \infty$. It follows from Lemma 2.1 in [DH] that $w_n \leqslant u_n$, where w_n is the unique positive solution of

$$-\Delta w + \phi w = \lambda w - B w^2 \text{ in } \Omega_+, \qquad w|_{\partial\Omega} = 0, \, w|_{\partial\Omega_0} = \min_{\partial\Omega_0} u_n,$$

where the constant B is chosen such that $B \geqslant \|b\|_\infty + \varepsilon_n$ for all n. It is well-known that w_n increases to a solution w of the same problem but with $w|_{\partial\Omega_0} = \infty$. Recall that $u_n \to u_0$ as $n \to \infty$. Hence $u_0 \geqslant w$ and it follows that u_0 satisfies the boundary condition on $\partial\Omega_0$ of (1.5). Thus it is a positive solution to (1.5). It follows that $u_0 \geqslant \underline{U}$ as the latter is the minimal solution. But we have also the reversed inequality at the beginning of the proof. Thus we must have $u_0 = \underline{U}$. The proof is complete. ∎

As a simple application of Theorem 2.1, we show that the minimal positive solution $\underline{U}$ of (1.5) varies continuously with λ for $\lambda > \lambda_1^{\Omega_0}(\phi)$ in a suitable sense.

We regard $\underline{U}$ as a function in the space $C(\Omega_+ \cup \partial\Omega)$ equipped with the metric defined by

$$d(u, v) = \sum_{n=1}^{\infty} \frac{2^{-n} d_n(u, v)}{1 + d_n(u, v)},$$

with

$$d_n(u, v) = \|u - v\|_{C(\Omega_n)}, \qquad \Omega_n = \{x \in \bar{\Omega}_+ : d(x, \partial\Omega_0) \geqslant \delta/n\},$$

where δ is a small positive constant such that $\Omega_1 \neq \emptyset$. Clearly, $u_n \to u$ in this metric is equivalent to $u_n \to u$ uniformly in any compact subset of $\bar{\Omega} \setminus \bar{\Omega}_0$. Moreover, $C(\Omega_+ \cup \partial\Omega)$ is a complete metric space under this metric.

We will use the following topological result (see [W, item (9.12), p. 12]).

LEMMA 2.2. *Let A_n be a sequence of connected sets in a complete metric space X such that*

(i) $\bigcup A_n$ *is precompact; and*

(ii) $\liminf(A_n) \neq \varnothing$.

Then $\limsup(A_n)$ *is nonempty and connected.*

Here $\liminf(A_n)$ denotes the set of all $u \in X$ such that any neighbourhood of u intersects all but finitely many of A_n, and $\limsup(A_n)$ consists of all the point $u \in X$ such that any neighbourhood of u intersects infinitely many A_n. It follows easily that $\liminf(A_n) \subset \limsup(A_n)$, and both sets are closed.

To emphasize the dependence on λ, let us denote the unique positive solution of (1.4) by u_ε^λ and the minimal positive solution of (1.5) by $\underline{U}^\lambda$. Let $\varepsilon_n > 0$ be a sequence converging to zero. Now for any given $\lambda_0 > \lambda_1^{\Omega_0}(\phi)$, we fix some $\Lambda > \lambda_0$ and consider the sets

$$\Gamma_n = \{(\lambda, u_{\varepsilon_n}^\lambda|_{\Omega_+ \cup \partial\Omega}) \colon \lambda_1^{\Omega_0}(\phi) \leqslant \lambda \leqslant \Lambda\}.$$

It is well known that u_ε^λ depends continuously on λ in the $C(\bar{\Omega})$ norm. Hence Γ_n, for each n, is a continuous curve in $R \times C(\Omega_+ \cup \partial\Omega)$; in particular, it is a connected set in this space. By Theorem 2.1 (and a simple variant of its proof), we easily see that conditions (i) and (ii) of Lemma 2.2 are satisfied, and

$$\liminf(\Gamma_n) = \limsup(\Gamma_n) = \{(\lambda, \underline{U}^\lambda) : \lambda_1^{\Omega_0}(\phi) \leqslant \lambda \leqslant \Lambda\} \equiv \Gamma.$$

Thus, by Lemma 2.2, Γ is connected. Moreover, a simple argument involving upper and lower solutions shows that $\lambda \to \underline{U}^\lambda(x)$ is increasing. Hence $\underline{U}^\lambda$ must vary continuously with λ in $C(\Omega_+ \cup \partial\Omega)$ for $\lambda \in (\lambda_1^{\Omega_0}(\phi), \Lambda)$, in particular, for λ near λ_0. Thus, we have proved the following.

PROPOSITION 2.3. $\underline{U}^\lambda$ *varies continuously in* $C(\Omega_+ \cup \partial\Omega)$ *with* λ *for* $\lambda > \lambda_1^{\Omega_0}(\phi)$ *and is increasing with* λ.

Remark 2.4. (i) In fact, it can be proved that $\underline{U}^\lambda$ varies continuously in $C(\Omega_+ \cup \partial\Omega)$ with λ for all $\lambda \in (-\infty, \infty)$. To prove this, one considers the boundary blow-up problem (1.5) with $b(x)$ replaced by $b(x) + \varepsilon$ for small $\varepsilon > 0$, which has a unique positive solution U_ε^λ (see Remark 2.9 in [DH]), and it is easily seen that $U_\varepsilon^\lambda \to \underline{U}^\lambda$ in $C(\Omega_+ \cup \partial\Omega)$ as $\varepsilon \to 0$. By uniqueness, for fixed ε, U_ε^λ varies continuously with λ. Now a simple application of Lemma 2.2 as above shows that $\underline{U}^\lambda$ varies continuously with λ.

(ii) Following the approach in (i), we can also show that the maximal positive solution $\bar{U}^\lambda$ varies continuously with λ in the same sense. Indeed, let $\Omega_\varepsilon = \{x \in \Omega : d(x, \Omega_0) < \varepsilon\}$. Then since $b(x) > 0$ on $\bar{\Omega} \setminus \Omega_\varepsilon$, as was mentioned in Remark 2.9 of [DH], a simple variant of the techniques of [MV] can be used to show that

$$-\Delta u + \phi u = \lambda u - b(x)\, u^2 \text{ in } \Omega \setminus \bar{\Omega}_\varepsilon, \qquad u|_{\partial\Omega} = 0, u|_{\partial\Omega_\varepsilon} = \infty$$

has a unique positive solution U_ε^λ. It is easily seen that $U_\varepsilon^\lambda \to \bar{U}^\lambda$ as $\varepsilon \to 0$ in $C(\Omega_+ \cup \partial\Omega)$ as $\varepsilon \to 0$, and this limit is uniform in λ for λ in bounded sets. Thus, one can use Lemma 2.2 as before to deduce that $\bar{U}^\lambda$ varies continuously with λ.

3. PERTURBATION AND THE STEADY-STATE SOLUTIONS

In this section, we first use the results of Section 3.1 of Part I on classical steady-state solutions of (1.1) to obtain a better understanding of the positive solution branch S_ε of (1.3). Then this information on S_ε is used to deduce better results on the generalized steady-state solutions of (1.1); in particular, we show that the positive solution set $\{(\mu, u, v)\}$ of (1.2) contains an unbounded continuum bifurcating from $(\lambda_1^{\Omega_+}(d\underline{U}), \underline{U}, 0)$. This complements the result in Section 3.2 of Part I. We will see how S_ε evolves to give both classical and generalized steady-state solutions of (1.1) as $\varepsilon \to 0$.

Let us recall that S_ε is a continuum of positive solutions of (1.3) that connects the two semitrivial solutions $(\mu^0, 0, \theta_{\mu^0})$ and $(\mu_0^\varepsilon, u_\varepsilon, 0)$, where u_ε is the unique positive solution of (1.4) with $\phi = 0$ and $\mu_0^\varepsilon = \lambda_1^\Omega(du_\varepsilon)$. We will need the following result.

LEMMA 3.1. *Suppose $\phi_n \in C(\bar{\Omega})$ satisfies*

(i) *$\phi_n \geqslant -M$ for some constant M, $\phi_n \to \infty$ uniformly on $\bar{\Omega}_0$ as $n \to \infty$, and*

(ii) *$\phi_n \to \phi$ in $L^p(\Omega')$, for all $p > 1$ and for all $\Omega' \subset\subset \Omega_+$, where $\phi \in C(\Omega_+ \cup \partial\Omega)$.*

Then $\lambda_1^\Omega(\phi_n) \to \lambda_1^{\Omega_+}(\phi)$. Therefore, by Theorem 2.1, when $\lambda > \lambda_1^{\Omega_0}(0)$, as $\varepsilon \to 0$,

$$\mu_0^\varepsilon = \lambda_1^\Omega(du_\varepsilon) \to \lambda_1^{\Omega_+}(d\underline{U}).$$

Proof. Let B be a small ball such that $\bar{B} \subset \Omega_+$. Then

$$\lambda_1^\Omega(-M) \leqslant \lambda_1^\Omega(\phi_n) \leqslant \lambda_1^B(\phi_n) \to \lambda_1^B(\phi) < \infty.$$

Therefore, by taking a subsequence when needed, we may assume that $\mu_n = \lambda_1^\Omega(\phi_n) \to \mu^*$.

Let ψ_n be the corresponding eigenfunction of μ_n:

$$-\Delta\psi_n+\phi_n\psi_n=\mu_n\psi_n, \qquad \psi_n|_{\partial\Omega}=0, \|\psi_n\|_\infty=1, \psi_n\geqslant 0.$$

Then $-\Delta\psi_n\leqslant(\mu_n+M)\,\psi_n\leqslant M_1\psi_n$ for some large positive constant M_1. By Lemma 2.10 in Part I, subject to a subsequence, ψ_n converges weakly in $W_0^{1,2}(\Omega)$ and strongly in $L^p(\Omega)$ (for all $p>1$) to some $\psi^*\in W_0^{1,2}(\Omega)$, and $\psi^*\not\equiv 0$.

Let ξ be an arbitrary nonnegative function in $C_0^\infty(\Omega)$. Multiplying the equation for ψ_n by ξ, then integrating over Ω, we obtain

$$\int_\Omega(\phi_n+M)\,\psi_n\xi\,dx=(\mu_n+M)\int_\Omega\psi_n\xi\,dx-\int_\Omega\nabla\psi_n\cdot\nabla\xi\,dx.$$

Hence,

$$0\leqslant\min_{\bar{\Omega}_0}\,(\phi_n+M)\int_{\Omega_0}\psi_n\xi\,dx\leqslant(\mu_n+M)\int_\Omega\psi_n\xi\,dx-\int_\Omega\nabla\psi_n\cdot\nabla\xi\,dx$$

$$\to(\mu^*+M)\int_\Omega\psi^*\xi\,dx-\int_\Omega\nabla\psi^*\cdot\nabla\xi\,dx.$$

By assumption (i), $\min_{\bar{\Omega}_0}(\phi_n+M)\to\infty$. Hence

$$\int_{\Omega_0}\psi^*\xi\,dx=\lim_{n\to\infty}\int_{\Omega_0}\psi_n\xi\,dx=0.$$

This implies that $\psi^*\equiv 0$ on Ω_0. It follows that $\psi^*\in W_0^{1,2}(\Omega_+)$.

Now we choose an arbitrary function $\eta\in C_0^\infty(\Omega_+)$, and multiply the equation for ψ_n by η, then integrate over Ω. We obtain

$$\int_{\Omega_+}\nabla\psi_n\cdot\nabla\eta\,dx+\int_{\Omega_+}\phi_n\psi_n\eta\,dx=\mu_n\int_{\Omega_+}\psi_n\eta\,dx.$$

Using assumption (ii), and letting $n\to\infty$, we deduce

$$\int_{\Omega_+}\nabla\psi^*\cdot\nabla\eta\,dx+\int_{\Omega_+}\phi\psi^*\eta\,dx=\mu^*\int_{\Omega_+}\psi^*\eta\,dx.$$

That is, ψ^* is a weak solution to

$$-\Delta\psi+\phi\psi=\mu^*\psi,\ \psi|_{\partial\Omega_+}=0.$$

By Theorem 3.9 in Part I, this implies that $\mu^*=\lambda_1^{\Omega_+}(\phi)$. As μ^* is uniquely determined in this way, we find that the whole sequence μ_n converges to $\lambda_1^{\Omega_+}(\phi)$. This finishes the proof. ∎

We are now ready to analyze the behaviour of $\mu_*(\varepsilon)$ and $\mu^*(\varepsilon)$ as $\varepsilon \to 0$, where $\mu_*(\varepsilon)$ and $\mu^*(\varepsilon)$ are as in Section 1, namely, (1.3) has no positive solution when $\mu \notin [\mu_*(\varepsilon), \mu^*(\varepsilon)]$, and there is at least one positive solution if $\mu \in (\mu_*(\varepsilon), \mu^*(\varepsilon))$.

THEOREM 3.2. *The functions* $\varepsilon \to \mu^*(\varepsilon)$ *and* $\varepsilon \to \mu_*(\varepsilon)$ *are both nonincreasing. Moreover,*

(i) *if* $\lambda > \lambda_1^{\Omega_0}(0)$, *then* $\lim_{\varepsilon \to 0} \mu^*(\varepsilon) = \infty$ *and* $\lim_{\varepsilon \to 0} \mu_*(\varepsilon) = \hat{\mu} \leqslant \mu_*$, *where* μ_* *is defined in Theorem* 3.1 *of Part* I*;*

(ii) *if* $\lambda_1^{\Omega}(0) < \lambda < \lambda_1^{\Omega_0}(0)$, *then* $\lim_{\varepsilon \to 0} \mu^*(\varepsilon) = \mu^*$ *and* $\lim_{\varepsilon \to 0} \mu_*(\varepsilon) = \mu_*$, *where* μ^* *and* μ_* *are defined in Theorem* 2.4 *of Part* I.

Proof. We first show that $\varepsilon \to \mu^*(\varepsilon)$ is nonincreasing. If $\mu^*(\varepsilon) \equiv \max\{\mu^0, \mu_0^\varepsilon\}$, then, since $\mu_0^\varepsilon = \lambda_1^{\Omega}(du_\varepsilon)$ is nonincreasing with ε, there is nothing to prove. If $\mu^*(\varepsilon) > \max\{\mu^0, \mu_0^\varepsilon\}$ for some $\varepsilon = \varepsilon_0 > 0$, then by Lemma 2.5 of [DB], (1.3) (with $\varepsilon = \varepsilon_0$) has a positive solution (u_0, v_0) with $\mu = \mu^*(\varepsilon_0)$. Let $\varepsilon_1 \in (0, \varepsilon_0]$. Then

$$-\Delta u_0 \leqslant \lambda u_0 - (b(x) + \varepsilon_1) u_0^2 - c u_0 v_0.$$

We will call (u, v) a lower solution to (1.3) if

$$\begin{cases} -\Delta u \geqslant \lambda u - [b(x) + \varepsilon] u^2 - cuv, \\ -\Delta v \leqslant \mu v - v^2 - duv, \\ u|_{\partial\Omega} \geqslant 0, \qquad v|_{\partial\Omega} \leqslant 0, \end{cases}$$

and call (u, v) an upper solution to (1.3) if the inequalities above are reversed. Clearly, (u_0, v_0) is an upper solution to (1.3) with $\varepsilon = \varepsilon_1$. Since $\mu = \mu^*(\varepsilon_0) > \mu_0^{\varepsilon_0}$, if we choose ε_1 close enough to ε_0, then $\mu > \mu_0^{\varepsilon_1}$ and hence the problem

$$-\Delta v = \mu v - v^2 - d u_{\varepsilon_1} v, \qquad v|_{\partial\Omega} = 0$$

has a unique positive solution v_{ε_1}. It is easily checked that $(u_{\varepsilon_1}, v_{\varepsilon_1})$ is a lower solution to (1.3) with $\varepsilon = \varepsilon_1$. Moreover, it is easily seen that $u_{\varepsilon_1} \geqslant u_{\varepsilon_0} \geqslant u_0$ and $v_{\varepsilon_1} \leqslant v_0$. Thus, by standard upper and lower solution argument for competition models, (1.3) with $\varepsilon = \varepsilon_1$ has a positive solution (u, v) satisfying $u_0 \leqslant u \leqslant u_{\varepsilon_1}$ and $v_{\varepsilon_1} \leqslant v \leqslant v_0$. By the definition of $\mu^*(\varepsilon)$, we must have $\mu^*(\varepsilon_1) \geqslant \mu = \mu^*(\varepsilon_0)$. Thus $\varepsilon \to \mu^*(\varepsilon)$ is always nonincreasing.

The fact that $\varepsilon \to \mu_*(\varepsilon)$ is nonincreasing can be proved by a similar argument.

Next we prove that $\mu^*(\varepsilon)\to\infty$ as $\varepsilon\to 0$ provided that $\lambda>\lambda_1^{\Omega_0}(0)$. We view (1.3) as a perturbation of

$$\begin{cases} -\Delta u=\lambda u-b(x)\,u^2-cuv, \\ -\Delta v=\mu v-v^2-duv, \\ u|_{\partial\Omega}=0, \qquad v|_{\partial\Omega}=0, \end{cases} \tag{3.1}$$

and use a degree argument.

Given any $\hat{\mu}>\max\{\lambda_1^{\Omega_+}(d\underline{U}),\mu_*\}$, where μ_* is as in Theorem 3.1 of Part I. By Lemma 3.2 there, we can find a constant $M>0$ such that any positive solution (u, v) of (3.1) with $\mu\in[0,\hat{\mu}]$ satisfies $\|u\|_\infty\leqslant M$. Note also that we always have $v\leqslant\theta_\mu\leqslant\theta_{\hat{\mu}}$.

Let us choose large positive constants M_1 and M_2 such that

$$f(u,v)=(\lambda+M_1)\,u-b(x)\,u^2-cuv \qquad \text{is increasing in } u$$

and

$$g_\mu(u,v)=(\mu+M_2)\,v-v^2-duv \qquad \text{is increasing in } v$$

for all $0\leqslant u\leqslant M+1$, $0\leqslant v\leqslant\|\theta_{\hat{\mu}}\|_\infty+1$, and all $\mu\in[0,\hat{\mu}]$. Then clearly

$$A_\mu(u,v)=((-\Delta+M_1)^{-1}f(u,v),(-\Delta+M_2)^{-1}\,g_\mu(u,v))$$

maps the set

$$B=\{(u,v)\in C(\bar{\Omega})\times C(\bar{\Omega}):0\leqslant u\leqslant M+1, 0\leqslant v\leqslant\|\theta_{\hat{\mu}}\|_\infty+1\}$$

into the natural positive cone K in $C(\bar{\Omega})\times C(\bar{\Omega})$. Moreover, by our discussion above, any positive solution of (3.1) belongs to the relative interior of B with respect to K. Furthermore, it is easily seen that nonnegative solutions of (3.1) are nonnegative fixed points of A_μ and A_μ is completely continuous.

Let us now consider the fixed point index $index_K(A_\mu, B)$. When $\lambda_1^{\Omega}(0)<\mu<\mu_*$, the only nonnegative solutions of (3.1) are $(u,v)=(0,0)$ and $(u,v)=(0,\theta_\mu)$, both are linearly unstable solutions of (3.1). By Dancer's fixed point index formula [D2], for such μ,

$$index_K(A_\mu,(0,0))=index_K(A_\mu,(0,\theta_\mu))=0.$$

Therefore,

$$index_K(A_\mu,B)=index_K(A_\mu,(0,0))+index_K(A_\mu,(0,\theta_\mu))=0.$$

As A_μ has no fixed point on $\partial_K B$, the relative boundary of B with respect to K, for any $\mu \in [0, \hat{\mu}]$, by the continuity property of the fixed point index (see [A]), $index_K(A_\mu, B)$ is independent of $\mu \in [0, \hat{\mu}]$ and is thus identically zero.

Consider now $\mu \in (\mu^0, \hat{\mu}]$, where μ^0 is as in Theorem 3.1 of Part I. For such μ, the trivial solution $(0, 0)$ of (3.1) is linearly unstable and hence has fixed point index 0, but the semitrivial solution $(0, \theta_\mu)$ is linearly stable, and therefore it has fixed point index 1. It follows that we can find small neighborhoods N_0 of $(0, 0)$ and N_1 of $(0, \theta_\mu)$ such that

$$\begin{aligned} &index_K(A_{\hat{\mu}}, B\backslash(N_0 \cup N_1)) \\ &\quad = index_K(A_{\hat{\mu}}, B) - index_K(A_{\hat{\mu}}, (0, 0)) - index_K(A_{\hat{\mu}}, (0, \theta_{\hat{\mu}})) \\ &\quad = 0 - 0 - 1 = -1. \end{aligned}$$

Let $A_\mu^\varepsilon(u, v)$ denote the operator obtained by replacing $b(x)$ by $b(x) + \varepsilon$ in the definition of $A_\mu(u, v)$. For $\varepsilon > 0$ small enough, one sees that $A_{\hat{\mu}}^\varepsilon$ maps B into K (we may need to enlarge M_1 a little to ensure this), is completely continuous and varies continuously with ε. It follows from the continuity property of the fixed point index that, for all sufficiently small $\varepsilon > 0$, $index_K(A_{\hat{\mu}}^\varepsilon, B\backslash(N_0 \cup N_1))$ is well defined and equals $index_K(A_{\hat{\mu}}, B\backslash(N_0 \cup N_1)) = -1$. Thus $A_{\hat{\mu}}^\varepsilon$ has a fixed point (u, v) in $B\backslash(N_0 \cup N_1))$, i.e., (3.1) has a positive solution with $\mu = \hat{\mu}$ for all small $\varepsilon > 0$. In particular, $\mu^*(\varepsilon) \geqslant \hat{\mu}$. As $\hat{\mu}$ is arbitrary, this implies $\mu^*(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Let us now prove that $\lim_{\varepsilon \to 0} \mu_*(\varepsilon) \leqslant \mu_*$. Since $\mu_*(\varepsilon)$ is nonincreasing with ε and $\mu_*(\varepsilon) \leqslant \mu^0$, $\lim_{\varepsilon \to 0} \mu_*(\varepsilon) = \hat{\mu}$ exists. If $\hat{\mu} > \mu_*$, then, since $\hat{\mu} \leqslant \mu^0$, we must have $\mu_* < \mu^0$. By Theorem 3.1 in Part I, (3.1) with $\mu = \mu_*$ has a positive solution (u_0, v_0). It is easily checked that (u_0, v_0) is a lower solution to (1.3) with $\mu = \mu_*$ for any $\varepsilon > 0$. Moreover, since $\mu_* < \mu^0$, $\lambda = \lambda_1^\Omega(c\theta_{\mu^0}) > \lambda_1^\Omega(c\theta_{\mu_*})$, and thus, the problem

$$-\Delta u = \lambda u - (b(x) + \varepsilon)\, u^2 - c\theta_{\mu_*} u, \qquad u|_{\partial\Omega} = 0$$

has a unique positive solution u^*. Clearly (u^*, θ_{μ_*}) is an upper solution of (1.3) with $\mu = \mu_*$, and $v_0 \leqslant \theta_{\mu_*}$, $u_0 \geqslant u^*$. Thus, (1.3) with $\mu = \mu_*$ has a positive solution, and hence $\mu_*(\varepsilon) \leqslant \mu_*$. But this implies $\hat{\mu} \leqslant \mu_*$, a contradiction. Therefore, we must have $\hat{\mu} \leqslant \mu_*$, as required.

Finally we consider the case $\lambda_1^\Omega(0) < \lambda < \lambda_1^{\Omega_0}(0)$. We first show that $\lim_{\varepsilon \to 0} \mu_*(\varepsilon) = \mu_*$. The argument in the last paragraph shows that $\hat{\mu} = \lim_{\varepsilon \to 0} \mu_*(\varepsilon) \leqslant \mu_*$. We show that $\hat{\mu} \geqslant \mu_*$. Otherwise, $\hat{\mu} < \mu_*$. Thus, for any $\varepsilon > 0$ and a suitable fixed $\delta > 0$, (1.3) has a positive solution $(u_\varepsilon, v_\varepsilon)$

with $\mu = \hat{\mu} + \delta < \mu_* \leqslant \mu^0$. Choose a sequence $\varepsilon_n \to 0$ and denote $(u_n, v_n) = (u_{\varepsilon_n}, v_{\varepsilon_n})$. Since $\lambda < \lambda_1^{\Omega_0}(0)$, u_n has an L^∞ bound independent of n. It follows that $-\Delta u_n$ and $-\Delta v_n$ both have L^∞ bounds independent of n. By the L^p theory for elliptic equations, we find that $\{u_n\}$ and $\{v_n\}$ are bounded in $W^{2,p}(\Omega)$ for any $p > 1$. Thus, subject to a subsequence, $u_n \to u_0$ and $v_n \to v_0$ in the C^1 norm and (u_0, v_0) is a nonnegative solution of (3.1) with $\mu = \hat{\mu} + \delta < \mu_*$. By the definition of μ_*, (u_0, v_0) cannot be a positive solution of (3.1). If $u_0 = 0$, then $u_n \to 0$ and from the equation for u_n we obtain

$$\begin{aligned} \lambda = \lambda_1^\Omega((b+\varepsilon_n)\, u_n + cv_n) &\to \lambda_1^\Omega(cv_0) \\ &\leqslant \lambda_1^\Omega(c\theta_{\hat{\mu}+\delta}) < \lambda_1^\Omega(c\theta_{\mu^0}), \end{aligned}$$

which contradicts the definition of μ^0. If $v_0 = 0$, then $v_n \to 0$, and from the equation for v_n, we deduce

$$\hat{\mu} + \delta = \lambda_1^\Omega(v_n + du_n) \to \lambda_1^\Omega(du_0).$$

Hence $\hat{\mu} + \delta = \lambda_1^\Omega(du_0)$. But as $u_0 \neq 0$ and (u_0, v_0) solves (3.1), u_0 must be the unique positive solution of

$$-\Delta u = \lambda u - b(x)\, u^2, \qquad u|_{\partial\Omega} = 0.$$

It follows that $\lambda_1^\Omega(du_0) = \mu_0$, where μ_0 is defined in Theorem 2.4 of Part I, and $\mu_* \leqslant \mu_0$. Thus $\hat{\mu} + \delta \geqslant \mu_*$, a contradiction. This proves that $\hat{\mu} = \mu_*$.

The fact that $\lim_{\varepsilon \to 0} \mu^*(\varepsilon) = \mu^*$ is proved by an analogous argument except that now it is not evident that $\mu^*(\varepsilon)$ is bounded from above. If $\mu^*(\varepsilon)$ has no bound from above, then since it is nonincreasing with ε, for any sequence $\varepsilon_n \to 0$, $\mu^*(\varepsilon_n) \to \infty$. Let (u_n, v_n) be a positive solution of (1.3) with $\varepsilon = \varepsilon_n$ and $\mu = \mu_n = \mu^*(\varepsilon_n)$. Since $u_n \leqslant U$, the unique positive solution of (1.4) with $\phi = \varepsilon = 0$, we have $\|u_n\|_\infty \leqslant \|U\|_\infty$. From the equation for v_n, we obtain

$$-\Delta v_n \geqslant \mu_n v_n - v_n^2 - d\, \|U\|_\infty v_n,$$

and hence $v_n \geqslant \theta_{\mu_n - d\,\|U\|_\infty}$. Now we use the equation for u_n and deduce

$$\lambda \geqslant \lambda_1^\Omega(cv_n) \geqslant \lambda_1^\Omega(c\theta_{\mu_n - d\,\|U\|_\infty}) \to \infty,$$

a contradiction. Thus $\mu^*(\varepsilon)$ must be bounded from above, which implies $\hat{\mu} = \lim_{\varepsilon \to 0} \mu^*(\varepsilon)$ exists. The proof for $\hat{\mu} = \mu^*$ is analogous to that in the last paragraph and is left to the reader. ∎

Remark 3.3. Using Theorem 3.2 above and Theorem 3.6 of Part I, we can easily construct examples for the classical competition model (1.3) such that

$$\mu_*(\varepsilon) < \min\{\mu^0, \mu_0^\varepsilon\} \text{ and } \mu^*(\varepsilon) > \max\{\mu^0, \mu_0^\varepsilon\}.$$

We suppose $\lambda > \lambda_1^{\Omega_0}(0)$. If we choose c suitably, we can have $\bar{\mu} < \lambda_1^{\Omega_+}(0) < \mu^0$, where $\bar{\mu}$ is as in Theorem 3.6 of Part I, namely, $\lambda_1^{\Omega_0}(c\theta_{\bar{\mu}}) = \lambda$. By Theorem 3.6 of Part I, for all small d, μ_* is close to $\bar{\mu}$, and we can assume that d is so small that $\lambda_1^{\Omega_+}(d\underline{U})$ is close to $\lambda_1^{\Omega_+}(0)$ and hence greater than μ_*. Now for small $\varepsilon > 0$, $\mu_0^\varepsilon = \lambda_1^{\Omega}(du_\varepsilon)$ is close to $\lambda_1^{\Omega_+}(d\underline{U})$ by Lemma 3.1, and $\mu_*(\varepsilon)$ is close to some $\hat{\mu} \leqslant \mu_*$ while $\mu^*(\varepsilon)$ is very large by Theorem 3.2; therefore, for all small $\varepsilon > 0$, we have

$$\mu_*(\varepsilon) < \mu_0^\varepsilon < \mu^0 < \mu^*(\varepsilon).$$

If we choose c such that $\mu^0 < \lambda_1^{\Omega_+}(0)$, then our above argument gives an example where

$$\mu_*(\varepsilon) < \mu^0 < \mu_0^\varepsilon < \mu^*(\varepsilon).$$

Moreover, a more careful (and tedious) analysis of the construction above shows that we can choose parameters so that

$$\mu_*(\varepsilon) < \mu_0^\varepsilon = \mu^0 < \mu^*(\varepsilon).$$

In this last case, when $\mu = \mu^0$, the two semitrivial solutions $(u_\varepsilon, 0)$ and $(0, \theta_\mu)$ are both linearly neutral (i.e., the linearization has zero as the first eigenvalue), yet (1.3) has a positive solution. This contrasts to the examples of Dancer in [D1] where the two semitrivial solutions of the Lotka–Volterra competition model are linearly neutral, but there is no positive steady-state solution.

In the following, we are going to analyze how the solutions on S_ε approach the steady-state solutions of (1.1) as $\varepsilon \to 0$. To this end, we need the following useful lemma.

LEMMA 3.4. *Suppose $\lambda > \lambda_1^{\Omega_0}(0)$ and (u_n, v_n) is a positive solution of (1.3) with $\mu = \mu_n$ and $\varepsilon = \varepsilon_n > 0$. Moreover, assume that ε_n decreases to 0 as $n \to \infty$, $\{\mu_n\}$ is bounded and $\|u_n\|_\infty \to \infty$. Then, subject to a subsequence, $(\mu_n, u_n, v_n) \to (\hat{\mu}, \hat{u}, \hat{v})$, where $(\hat{u}, \hat{v}) = (\infty, 0)$ on Ω_0, and on Ω_+, $(\hat{u}, \hat{v})$ is a positive solution of (1.2) with $\mu = \hat{\mu}$ except when $\hat{\mu} = \lambda_1^{\Omega_+}(d\underline{U})$, in which case, $(\hat{u}, \hat{v}) = (\underline{U}, 0)$ is possible. Moreover, $\hat{u} \leqslant \underline{U}$. Here $v_n \to \hat{v}$ is in the norm of $L^p(\Omega)$, for all $p > 1$, and $u_n \to \hat{u}$ is in the following sense: $u_n \to \infty$ uniformly on $\bar{\Omega}_0$, and $u_n \to \hat{u}$ uniformly on any compact subset of $\Omega_+ \cup \partial\Omega$.*

Proof. We may assume that $\mu_n \leqslant \tilde{\mu} < \infty$. Since

$$-\Delta v_n = \mu_n v_n - v_n^2 - d u_n v_n \leqslant \tilde{\mu} v_n,$$

and $0 \leqslant v_n \leqslant \theta_{\tilde{\mu}}$, by Lemma 2.10 in Part I, subject to a subsequence, v_n converges weakly in $W_0^{1,2}(\Omega)$ and strongly in $L^p(\Omega)$ (for all $p > 1$) to some $\hat{v} \in W_0^{1,2}(\Omega)$. We may also assume that $\mu_n \to \hat{\mu}$.

Consider now $\hat{u}_n = u_n / \|u_n\|_\infty$. It satisfies the equation

$$-\Delta \hat{u}_n = \lambda \hat{u}_n - (b(x) + \varepsilon_n) \|u_n\|_\infty \hat{u}_n^2 - c v_n \hat{u}_n, \qquad \hat{u}_n|_{\partial\Omega} = 0. \tag{3.2}$$

Hence $-\Delta \hat{u}_n \leqslant \lambda \hat{u}_n$. By Lemma 2.10 in Part I, it follows that, subject to a subsequence, $\hat{u}_n$ converges weakly in $W_0^{1,2}(\Omega)$ and strongly in $L^p(\Omega)$ (for all $p > 1$) to some u^* and $u^* \not\equiv 0$. Moreover, since $u_n \leqslant \underline{U}$ on Ω_+, we must have $u^* \equiv 0$ on Ω_+. Thus $u^* \in W_0^{1,2}(\Omega_0)$.

If $u_n(x_n) = \|u_n\|_\infty$, $x_n \in \Omega$, we have $-\Delta u_n(x_n) \geqslant 0$ and hence

$$\lambda u_n(x_n) - (b(x_n) + \varepsilon_n)\, u_n(x_n)^2 - c u_n(x_n)\, v_n(x_n) \geqslant 0.$$

It follows that $\varepsilon_n \|u_n\|_\infty \leqslant \lambda$. Thus we may assume that $\varepsilon_n \|u_n\|_\infty \to \xi \geqslant 0$ as $n \to \infty$. We now multiply (3.2) by an arbitrary function $\psi \in C_0^\infty(\Omega_0)$, and then integrate it over Ω_0. We obtain

$$\int_{\Omega_0} \nabla \hat{u}_n \cdot \nabla \psi \, dx = \int_{\Omega_0} (\lambda \hat{u}_n - \varepsilon_n \|u_n\|_\infty \hat{u}_n^2 - c v_n \hat{u}_n)\, \psi \, dx.$$

Letting $n \to \infty$, we have

$$\int_{\Omega_0} \nabla u^* \cdot \nabla \psi \, dx = \int_{\Omega_0} (\lambda u^* - \xi (u^*)^2 - c \hat{v} u^*)\, \psi \, dx.$$

Hence, u^* is a weak solution to

$$-\Delta u = \lambda u - \xi u^2 - c \hat{v} u, \qquad u|_{\partial\Omega_0} = 0.$$

As $\lambda - \xi u^* - c\hat{v} \in L^\infty(\Omega_0)$ and u^* is nonnegative, by the weak Harnack inequality, we must have $u^* > 0$ in Ω_0. Moreover, from the right-hand side of (3.2), we find that $-\Delta \hat{u}_n$ has an $L^\infty(\Omega_0)$ bound independent of n. Thus by the interior L^p estimates, $\|\hat{u}_n\|_{W^{2,p}(\Omega')}$ has a bound independent of n for any $\Omega' \subset\subset \Omega_0$. It follows that, subject to a subsequence, $\hat{u}_n$ converges to u^* in the C^1 norm on Ω'. Thus, $u_n = \|u_n\|_\infty \hat{u}_n \to \infty$ uniformly on any $\Omega' \subset\subset \Omega_0$.

Multiplying the equation for v_n by any $\psi \in C^\infty(\Omega)$ satisfying $\psi > 0$ in Ω and $\psi|_{\partial\Omega} = 0$, and integrating over Ω, we obtain

$$\int_\Omega \nabla v_n \cdot \nabla \psi \, dx = \int_\Omega (\mu_n v_n - v_n^2)\, \psi \, dx - \int_\Omega d u_n v_n \psi \, dx.$$

It follows that

$$\int_{\Omega} d\hat{u}_n v_n \psi \, dx = (\|u_n\|_\infty)^{-1} \left[-\int_{\Omega} \nabla v_n \cdot \nabla \psi \, dx + \int_{\Omega} (\mu_n v_n - v_n^2) \, \psi \, dx \right].$$

Letting $n \to \infty$, we deduce

$$\int_{\Omega} du^* \, \hat{v} \psi \, dx = 0.$$

As $u^* = 0$ on Ω_+, it follows $\int_{\Omega_0} u^* \hat{v} \psi \, dx = 0$. But both u^* and ψ are positive in Ω_0, and $\hat{v}$ is nonnegative. So we must have $\hat{v} = 0$ on Ω_0.

Since $v_n \to \hat{v}$ in $L^p(\Omega)$ (for all $p > 1$), and $\{v_n\}$ is $L^\infty(\Omega)$ bounded, a careful check of Steps 1–3 in the proof of Theorem 2.1 shows that if we replace the function ϕ there by v_n, then we can still reach the same conclusion. Thus, $u_n \to \infty$ uniformly on $\bar{\Omega}_0$. From here, it is easily checked that, subject to a subsequence, $u_n \to \hat{u}$ uniformly on any compact subset of $\Omega_+ \cup \partial\Omega$, and $(\hat{u}, \hat{v})$ is a nonnegative solution of (1.2) with $\mu = \hat{\mu}$. To see, for example, $\hat{u}|_{\partial\Omega_0} = \infty$, we can compare $\hat{u}$ with the minimal positive solution w of (1.5) with $\phi = \|\theta_{\hat{\mu}}\|_\infty$ and with $b(x)$ replaced by some constant $B > \|b\|_\infty + \varepsilon_n$ for all n. w can be obtained as the limit of the solutions w_n of (1.5) with the above modifications and with $w_n|_{\partial\Omega_n} = \min_{\bar{\Omega}_0} u_n$. By Lemma 2.1 of [DH], we have $u_n \geqslant w_n$. Thus $\hat{u} \geqslant w$ and hence $\hat{u}|_{\partial\Omega_0} = \infty$.

If $\hat{v} = 0$, then $\hat{u}$ must be a positive solution of (1.5) and hence $\hat{u} \geqslant \underline{U}$. But each $u_n \leqslant \underline{U}$. Thus we must have $\hat{u} = \underline{U}$. But by Lemma 3.1 and the equation for v_n, we have

$$\mu_n = \lambda_1^{\Omega}(v_n + du_n) \to \lambda_1^{\Omega_+}(d\underline{U}).$$

Thus we must have $\hat{\mu} = \lambda_1^{\Omega_+}(d\underline{U})$. Hence $(\hat{u}, \hat{v})$ is a positive solution of (1.3) with $\mu = \hat{\mu}$, unless $\hat{\mu} = \lambda_1^{\Omega_+}(d\underline{U})$, in which case, it is possible that $(\hat{u}, \hat{v}) = (\underline{U}, 0)$. This finishes the proof. ∎

THEOREM 3.5. *Suppose $\lambda > \lambda_1^{\Omega_0}(0)$. Then, the closure of the positive solution set $\{(\mu, u, v)\}$ of (1.2) in the space $R \times C(\Omega_+ \cup \partial\Omega) \times L^2(\Omega_+)$ contains an unbounded connected set $\hat{S}$ such that*

(i) $(\lambda_1^{\Omega_+}(d\underline{U}), \underline{U}, 0) \in \hat{S}$;

(ii) $\hat{S} \setminus \{(\lambda_1^{\Omega_+}(d\underline{U}), \underline{U}, 0)\}$ *consists of positive solutions of* (1.2);

(iii) $\{\mu : (\mu, u, v) \in \hat{S}\} \supset (\lambda_1^{\Omega_+}(d\underline{U}), \infty)$.

Thus, we can say that (1.2) *has an unbounded positive solution branch bifurcating from the semitrivial solution branch* $\{(\mu, \underline{U}, 0) : \mu \in R\}$ *at* $\mu = \lambda_1^{\Omega_+}(d\underline{U})$.

Proof. Since $\lambda > \lambda_1^{\Omega_0}(0)$, by Lemma 3.1 and Theorem 3.2, for all small $\varepsilon > 0$, $\mu_0^\varepsilon \leqslant \lambda_1^{\Omega_+}(d\underline{U}) < \mu^*(\varepsilon)$. From Lemma 2.6 in [DB], we easily see that for each $\mu \in (\mu_0^\varepsilon, \mu^*(\varepsilon)]$, (1.3) has a minimal positive solution $(u_\mu^\varepsilon, v_\mu^\varepsilon)$ in the sense that any other positive solution (u, v) of (1.3) satisfies $u \leqslant u_\mu^\varepsilon$, and $v \geqslant v_\mu^\varepsilon$. Let us introduce, in the space $R \times C^1(\bar{\Omega}) \times C^1(\bar{\Omega})$, the set

$$\Delta_\mu^\varepsilon = (-\infty, \mu] \times [u_\mu^\varepsilon, \infty) \times (-\infty, v_\mu^\varepsilon],$$

where

$$[u_\mu^\varepsilon, \infty) = \{u \in C^1(\bar{\Omega}) : u \geqslant u_\mu^\varepsilon\}, (-\infty, v_\mu^\varepsilon] = \{v \in C^1(\bar{\Omega}) : v \leqslant v_\mu^\varepsilon\}.$$

It follows from Lemma 3.2 of [DB] and the remark following it that

$$\partial \Delta_\mu^\varepsilon \cap S_\varepsilon = \{(\mu, u_\mu^\varepsilon, v_\mu^\varepsilon)\}.$$

If we use ω to denote the interior of Δ_μ^ε, then the above identity shows that $\partial\omega \cap S_\varepsilon$ consists of a single point. By Lemma 3.1 of [DB], this implies that $S_\varepsilon \cap \omega$ is a connected set. Let us use S_ε^μ to denote this set. Then

$$S_\varepsilon^\mu \subset \Delta_\mu^\varepsilon = (-\infty, \mu] \times [u_\mu^\varepsilon, \infty) \times (-\infty, v_\mu^\varepsilon].$$

Let ε_n be a decreasing sequence of positive numbers that converges to zero. For each $\tilde{\mu} > \lambda_1^{\Omega_+}(d\underline{U})$, we can find n_0 such that $\mu^*(\varepsilon_n) > \tilde{\mu} > \mu_0^{\varepsilon_n}$ holds for all $n \geqslant n_0$. Define, for $n \geqslant n_0$,

$$A_n = S_{\varepsilon_n}^{\tilde{\mu}} \cup \{(\lambda_1^\Omega(du_{\varepsilon_n}), u_{\varepsilon_n}, 0)\},$$

and we understand that for $(\mu, u, v) \in A_n$, the functions u and v are considered as their restrictions on Ω_+.

Let (u_n^*, v_n^*) be the minimal positive solution of (1.3) with $\varepsilon = \varepsilon_n$ and $\mu = \mu^*(\varepsilon_n)$, and let $(\mu_n, u_n, v_n) \in S_{\varepsilon_n}^{\tilde{\mu}}$. Then $v_n \leqslant v_n^*$, and $u_n \geqslant u_n^*$. We first note that $\|u_n^*\|_\infty \to \infty$ as $n \to \infty$. Otherwise, we may assume $d\,\|u_n^*\|_\infty \leqslant M$ for all n. Then,

$$-\Delta v_n^* = \mu^*(\varepsilon_n)\, v_n^* - (v_n^*)^2 - d u_n^* v_n^* \geqslant (\mu^*(\varepsilon_n) - M)\, v_n^* - (v_n^*)^2.$$

It follows that $v_n^* \geqslant \theta_{\mu^*(\varepsilon_n) - M}$, and so, as $\mu^*(\varepsilon_n) \to \infty$ by Theorem 3.2,

$$\lambda_1^\Omega(cv_n^*) \geqslant \lambda_1^\Omega(c\theta_{\mu^*(\varepsilon_n)-M}) \to \infty.$$

But from the equation for u_n^*, we deduce $\lambda_1^\Omega(cv_n^*) < \lambda$. This contradiction shows that $\|u_n^*\|_\infty \to \infty$ as $n \to \infty$. It follows that $\|u_n\|_\infty \to \infty$. We now use Lemma 3.4 and see that $\bigcup A_n$ is precompact and $\limsup(A_n)$ consists of positive solutions of (1.2) satisfying $u \leqslant \underline{U}$, together with $(\lambda_1^{\Omega_+}(d\underline{U}), \underline{U}, 0)$. By Theorem 2.1 and Lemma 3.1, we have $u_{\varepsilon_n} \to \underline{U}$ in $C(\Omega_+ \cup \partial\Omega)$ and $\lambda_1^\Omega(du_{\varepsilon_n}) \to \lambda_1^{\Omega_+}(d\underline{U})$ as $n \to \infty$. Hence

$$(\lambda_1^{\Omega_+}(d\underline{U}), \underline{U}, 0) \in \liminf(A_n).$$

By Lemma 2.2, $S^{\tilde{\mu}} = \limsup(A_n)$ is a nonempty connected set consisting of positive solutions of (1.2) satisfying $u \leqslant \underline{U}$, together with $(\lambda_1^{\Omega_+}(d\underline{U}), \underline{U}, 0)$. Moreover, it follows from Lemma 3.4 that $\{\mu\colon (\mu, u, v) \in S^{\tilde{\mu}}\} \supset [\lambda_1^{\Omega_+}(d\underline{U}), \tilde{\mu}]$. Let $\hat{S} = \bigcup_{\tilde{\mu} > \lambda_1^{\Omega_+}(d\underline{U})} S^{\tilde{\mu}}$. Then clearly $\hat{S}$ is connected, consists of positive solutions of (1.2) with $u \leqslant \underline{U}$ and the point $(\lambda_1^{\Omega_+}(d\underline{U}), \underline{U}, 0)$, and $\{\mu\colon (\mu, u, v) \in \hat{S}\} \supset [\lambda_1^{\Omega_+}(d\underline{U}), \infty)$. The proof is complete. ∎

Remark 3.6. The proof of Theorem 3.5 shows that when $\lambda > \lambda_1^{\Omega_0}(0)$, for any fixed $\mu > \lambda_1^{\Omega_+}(d\underline{U})$ and all sufficiently small $\varepsilon > 0$, thanks to conclusion (i) of Theorem 3.2, (1.3) has a minimal positive solution $(u_\varepsilon, v_\varepsilon)$ and for any sequence $\varepsilon_n \to 0$, $(\mu_n, u_n, v_n) = (\mu, u_{\varepsilon_n}, v_{\varepsilon_n})$ has the property described in Lemma 3.4. This shows that (u_n, v_n) has pattern Ω_0 for large n. Note that these minimal solutions are asymptotically stable as steady-state solutions of the corresponding parabolic problem.

Theorem 3.5 and its proof also shows that when $\lambda > \lambda_1^{\Omega_0}(0)$, as $\varepsilon \to 0$, part of the positive solution branch S_ε of (1.3) evolves to an unbounded branch of generalized steady-state solutions of (1.1). Let us now see how S_ε also produces an unbounded branch of classical steady-state solutions of (1.1) when $\varepsilon \to 0$. Suppose $\lambda > \lambda_1^{\Omega_0}(0)$. By Lemma 3.2 in Part I, we know that for any fixed $\tilde{\mu} > \mu^0$, every positive solution (μ, u, v) of (3.1) with $\mu \in [\lambda_1^{\Omega}(0), \tilde{\mu}]$ satisfies

$$\|u\|_\infty \leqslant M, \ \|v\|_\infty \leqslant \|\theta_{\tilde{\mu}}\|_\infty.$$

Denote

$$B_{\tilde{\mu}} = \{(u, v) \in C(\bar{\Omega}) \times C(\bar{\Omega}) : 0 \leqslant u \leqslant M + 1, 0 \leqslant v \leqslant \|\theta_{\tilde{\mu}}\|_\infty + 1\}.$$

Then (3.1) with $\mu \in [\lambda_1^{\Omega}(0), \tilde{\mu}]$ has no nonnegative solution (u, v) lying on $\partial_K B_{\tilde{\mu}}$, the relative boundary of $B_{\tilde{\mu}}$ with respect to the natural positive cone K in $C(\bar{\Omega}) \times C(\bar{\Omega})$. A simple compactness argument shows that there exists $\varepsilon_{\tilde{\mu}} > 0$ such that, when $0 < \varepsilon \leqslant \varepsilon_{\tilde{\mu}}$, (1.3) has no nonnegative solution (μ, u, v) satisfying $\mu \in [\lambda_1^{\Omega}(0), \tilde{\mu}]$ and $(u, v) \in \partial_K B_{\tilde{\mu}}$. Since clearly $(\mu^0, 0, \theta_{\mu^0}) \in R \times B_{\tilde{\mu}}$, and $S_\varepsilon \cap (\{\lambda_1^{\Omega}(0)\} \times B_{\tilde{\mu}}]) = \varnothing$, we see that $S_\varepsilon \cap ([\lambda_1^{\Omega}(0), \tilde{\mu}] \times B_{\tilde{\mu}})$ contains a component $\hat{S}_\varepsilon^{\tilde{\mu}}$ which joins $(\mu^0, 0, \theta_{\mu^0})$ and some point $(\mu, u, v) \in S_\varepsilon$ with $\mu = \tilde{\mu}$.

Choose a decreasing sequence of numbers $\varepsilon_n \in (0, \varepsilon_{\tilde{\mu}})$ satisfying $\varepsilon_n \to 0$, and define $A_n = \hat{S}_{\varepsilon_n}^{\tilde{\mu}}$ for all large n such that $\mu_0^{\varepsilon_n} < \tilde{\mu} < \mu^*(\varepsilon_n)$. It is easily seen that $\bigcup A_n$ is precompact, $\limsup(A_n)$ consists of positive solutions of (3.1) together with $(\mu^0, 0, \theta_{\mu^0})$, and $(\mu^0, 0, \theta_{\mu^0}) \in \liminf(A_n)$. Hence by Lemma 2.2, $\hat{S}^{\tilde{\mu}} = \limsup(A_n)$ is a nonempty connected set consisting of positive solutions of (3.1) together with $(\mu^0, 0, \theta_{\mu^0})$. Define $\hat{S}' = \bigcup_{\tilde{\mu} > \mu^0} \hat{S}^{\tilde{\mu}}$. Then $\hat{S}'$ is a connected set consisting of positive solutions of (3.1) together with $(\mu^0, 0, \theta_{\mu^0})$ and $\{\mu\colon (\mu, u, v) \in \hat{S}'\} \supset [\mu^0, \infty)$. Since $(\mu^0, 0, \theta_{\mu^0})$ is a simple bifurcation point, the part of $\hat{S}'$ near $(\mu^0, 0, \theta_{\mu^0})$ is a simple curve. It follows that $S_0 = \hat{S}' \setminus \{(\mu^0, 0, \theta_{\mu^0})\}$ is connected, consists of positive solutions of (3.1), joins $(\mu^0, 0, \theta_{\mu^0})$, and satisfies $\{\mu\colon (\mu, u, v) \in S_0\} \supset (\mu^0, \infty)$.

Moreover, $S_0 \subset S$, where S is the positive solution branch of (3.1) given in Theorem 3.1 of Part I, since both connected sets coincide near $(\mu^0, 0, \theta_{\mu^0})$.

Finally, let us look briefly at the case $\lambda_1^{\Omega}(0) < \lambda < \lambda_1^{\Omega_0}(0)$. In this case, it is easily proved that for any sequence $\varepsilon_n \to 0$, $\bigcup S_{\varepsilon_n}$ is precompact and $\liminf(S_{\varepsilon_n})$ contains both $(\mu^0, 0, \theta_{\mu^0})$ and $(\mu_0, U, 0)$, where we follow the notations of Theorem 2.4 in Part I. Moreover, the connected set $\limsup(S_{\varepsilon_n})$ consists of positive solutions of (3.1) together with the two semitrivial solutions given above. As both semitrivial solutions are simple bifurcation points, it follows that $S_0 = \limsup(S_{\varepsilon_n}) \setminus \{(\mu^0, 0, \theta_{\mu^0}), (\mu_0, U, 0)\}$ is a connected set which consists of positive solutions of (3.1) and joins the two semitrivial solutions. Moreover, $S_0 \subset S$, where S is the positive solution branch given in Theorem 2.4 of Part I.

Since in both Theorems 2.4 and 3.1 of Part I, not much is known about the positive solution branch S apart from those part of it which are close to the semitrivial solutions, our above arguments do not exclude the possibility that S_0 is a proper subset of S. On the other and, by Theorem 3.2, $\{\mu: (\mu, u, v) \in S_0\} \supset \{\mu: (\mu, u, v) \in S\}$.

Finally in this section, let us look at a case where all the positive solutions of (1.3) for a certain range of μ converge, as $\varepsilon \to 0$, to generalized steady-state solutions of (1.1).

THEOREM 3.7. *Let (u_n, v_n) be a positive solution of* (1.3) *with a fixed $\mu < \underline{\mu}$ and $\varepsilon = \varepsilon_n \to 0$, where $\underline{\mu}$ is determined by $\lambda = \lambda_1^{\Omega_0}(c\theta_{\underline{\mu}})$. Then subject to a subsequence, $(u_n, v_n) \to (\hat{u}, \hat{v})$, where $(\hat{u}, \hat{v}) = (\infty, 0)$ on Ω_0, and on Ω_+, $(\hat{u}, \hat{v})$ is a positive solution of* (1.2). *Here $v_n \to \hat{v}$ is in the norm of $L^p(\Omega)$, for all $p > 1$, and $u_n \to \hat{u}$ is in the following sense: $u_n \to \infty$ uniformly on $\bar{\Omega}_0$, and $u_n \to \hat{u}$ uniformly on any compact subset of $\Omega_+ \cup \partial\Omega$.*

Proof. Since $v_n \leqslant \theta_\mu$, we have $u_n \geqslant \hat{u}_n$ where $\hat{u}_n$ denotes the unique positive solution of

$$-\Delta u = \lambda u - (b(x) + \varepsilon_n)\, u^2 - c\theta_\mu u,\ u|_{\partial\Omega} = 0.$$

The existence of $\hat{u}_n$ follows from $\mu < \underline{\mu}$ which implies $\lambda > \lambda_1^{\Omega_0}(c\theta_\mu) > \lambda_1^{\Omega}(c\theta_\mu)$. Also from $\lambda > \lambda_1^{\Omega_0}(c\theta_\mu)$, we deduce by Theorem 2.1 that $\hat{u}_n \to \infty$ uniformly on $\bar{\Omega}_0$. Therefore, $\|u_n\|_\infty \to \infty$ as $n \to \infty$. The conclusion of the theorem now follows from Lemma 3.4. This completes the proof. ∎

Remark 3.8. A sufficient condition for the existence of (u_n, v_n) as in Theorem 3.7 is that $\lambda_1^{\Omega_+}(d\underline{U}) < \underline{\mu}$, which is satisfied, from the definition of $\underline{\mu}$, if c is sufficiently small. Indeed, by Theorem 3.1, $\mu_*(\varepsilon_n) \leqslant \mu_0^{\varepsilon_n} = \lambda_1^{\Omega}(du_{\varepsilon_n}) \to \lambda_1^{\Omega_+}(d\underline{U})$ as $n \to \infty$, and by Theorem 3.2, $\mu^*(\varepsilon_n) \to \infty$ as $n \to \infty$. Thus for any μ satisfying $\lambda_1^{\Omega_+}(d\underline{U}) < \mu < \underline{\mu}$, for all large n, $\mu_*(\varepsilon_n) < \mu < \mu^*(\varepsilon_n)$, and (1.3) has a positive solution (u_n, v_n) with $\varepsilon = \varepsilon_n$.

4. DYNAMICAL BEHAVIOUR

In this section, we discuss the dynamical behaviour of (1.1). As the case $\lambda < \lambda_1^{\Omega_0}(0)$ was considered in Part I already, we only discuss here the case

$$\lambda > \lambda_1^{\Omega_0}(0). \tag{4.1}$$

We assume (4.1) throughout this section.

THEOREM 4.1 (Extinction of v). *Suppose that* $\mu < \min\{\lambda_1^{\Omega_+}(0), \mu_*\}$, *where* μ_* *is given in Theorem* 3.1 *of Part* I. *Then any positive solution* (u, v) *of* (1.1) *satisfies* $\|v(\cdot, t)\|_\infty \to 0$ *and* $\min_{x \in \bar{\Omega}_0} u(x, t) \to \infty$, *as* $t \to \infty$.

Proof. We may assume that both $u(x, 0) = u_0(x)$ and $v(x, 0) = v_0(x)$ are nonnegative and not identically zero. Let $\varepsilon > 0$ and $(U_\varepsilon, V_\varepsilon)$ be the unique solution of the problem

$$\begin{cases} u_t - \Delta u = \lambda u - (b(x) + \varepsilon)\, u^2 - cuv, & t > 0,\ x \in \Omega, \\ v_t - \Delta v = \mu v - v^2 - duv, & t > 0,\ x \in \Omega, \\ u(x, t) = v(x, t) = 0, & t > 0,\ x \in \partial\Omega, \\ u(x, 0) = u_0(x),\ v(x, 0) = v_0(x), & x \in \Omega. \end{cases} \tag{4.2}$$

It follows from the order preserving property of this system that $u \geqslant U_\varepsilon$ and $v \leqslant V_\varepsilon$ for all $t > 0$ and $x \in \Omega$. (See, e.g., [HL, Ma, Sa, S].)

If we can prove that for some $\varepsilon > 0$, $\mu < \mu_*(\varepsilon)$ holds, then by the known dynamical behaviour of (4.2) in this case (compare case (ii) in Theorem 2.6 of Part I), $V_\varepsilon(x, t) \to 0$ uniformly in x as $t \to \infty$. As $0 \leqslant v(x, t) \leqslant V_\varepsilon(x, t)$, it follows that $\|v(\cdot, t)\|_\infty \to 0$ as $t \to \infty$. By (4.1), we know that for some $\delta > 0$ small, $\lambda > \lambda_1^{\Omega_0}(0) + \delta$. Assume that $c\|v(\cdot, t)\|_\infty \leqslant \delta$ when $t \geqslant T > 0$. Then $u(x, t) \geqslant w(x, t)$ for $t > T$, where w is the unique solution of

$$w_t - \Delta w = (\lambda - \delta)\, w - b(x)\, w^2, \qquad w|_{\partial\Omega} = 0,\ w(x, T) = u(x, T).$$

By Theorem 2.3 of Part I, it follows from $\lambda - \delta > \lambda_1^{\Omega_0}(0)$ that $w(x, t) \to \infty$ uniformly for $x \in \bar{\Omega}_0$ as $t \to \infty$. Therefore, $\min_{x \in \bar{\Omega}_0} u(x, t) \to \infty$ as $t \to \infty$.

Thus it suffices to show that $\mu < \mu_*(\varepsilon)$ for all small positive ε. We argue indirectly. Suppose that there is a sequence of positive numbers ε_n such that $\varepsilon_n \to 0$ and $\mu_*(\varepsilon_n) \leqslant \mu$. Choose μ' satisfying $\mu < \mu' < \min\{\lambda_1^{\Omega_+}(0), \mu_*\}$. By Theorem 3.2, we may assume that $\mu^*(\varepsilon_n) > \mu'$ for all n. Thus, (1.3) has a positive solution (u_n, v_n) with $\varepsilon = \varepsilon_n$ and $\mu = \mu'$. If $\{\|u_n\|_\infty\}$ is bounded, then a simple compactness argument shows that, subject to a subsequence, (u_n, v_n) converges in $C^1(\bar{\Omega})$ to a nonnegative steady-state solution (u^*, v^*) of (1.1) with $\mu = \mu'$. If (u^*, v^*) is a positive solution, then we must have $\mu' \geqslant \mu_*$ by the definition of μ_*. But this contradicts the choice of μ'. Thus we necessarily have $(u^*, v^*) = (0, \theta_\mu)$ or $(u^*, v^*) = (0, 0)$. If the first alternative occurs, then we must have $\mu' = \mu^0 \geqslant \mu_*$, contradicting the choice of μ'. This implies that the second alternative must occur. But then $\lambda = \lambda_1^\Omega(bu_n + cv_n) \to \lambda_1^\Omega(0)$, contradicting (4.1). This shows that $\{\|u_n\|_\infty\}$ must be unbounded. We may assume that $\|u_n\|_\infty \to \infty$ as $n \to \infty$. Now we can use Lemma 3.4 to conclude that, subject to a subsequence, $v_n \to \hat{v}$ in $L^p(\Omega)$ (for all $p > 1$) and $u_n \to \infty$ uniformly on $\bar{\Omega}_0$ and $u_n \to U^*$ in $C(\Omega')$ for any compact subset Ω' of $\Omega_+ \cup \partial\Omega$, and U^* is a positive solution of (1.5) with $\phi = \hat{v}$. By Lemma 3.1, we obtain

$$\mu' = \lambda_1^\Omega(v_n + du_n) \to \lambda_1^{\Omega_+}(\hat{v} + dU^*) > \lambda_1^{\Omega_+}(0),$$

again contradicting the choice of μ'. This finishes the proof. ∎

THEOREM 4.2 (Persistance of v). *If $\mu > \lambda_1^{\Omega_+}(d\bar{U})$, where $\bar{U}$ denotes the maximal positive solution of* (1.5) *with $\phi = 0$, then for any compact subset Ω' of Ω_+,*

$$\varliminf_{t \to \infty} \min_{x \in \Omega'} v(x, t) > 0.$$

Proof. If we can find a subdomain Ω'' of Ω_+ such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega_+$ and $\mu > \lambda_1^{\Omega''}(d\bar{U})$, then the classical logistic problem

$$-\Delta w = \mu w - w^2 - d\bar{U}w \text{ in } \Omega'', \qquad w|_{\partial\Omega''} = 0$$

has a unique positive solution w and any positive solution $w(x, t)$ of

$$w_t - \Delta w = \mu w - w^2 - d\bar{U}w, \qquad w|_{\partial\Omega''} = 0$$

satisfies $w(x, t) \to w(x)$ uniformly on Ω'' as $t \to \infty$. But we have $v(x, t) \geqslant w(x, t)$ for $t > 0$ and $x \in \Omega''$ by the maximum principle. Therefore,

$$\varliminf_{t \to \infty} \min_{x \in \Omega'} v(x, t) \geqslant \lim_{t \to \infty} \min_{x \in \Omega'} w(x, t) = \min_{x \in \Omega'} w(x) > 0.$$

It remains to find such a subdomain Ω''. Define

$$\Omega_n = \{x \in \Omega_+ : d(x, \partial\Omega_+) > \delta/n\},$$

where $\delta > 0$ is chosen such that $\Omega_1 \neq \emptyset$. Then clearly $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \cdots \subset\subset \Omega_+$, and $\Omega' \subset\subset \Omega_n$ for all large n.

Denote $\mu_n = \lambda_1^{\Omega_n}(d\bar{U})$. Then μ_n is a decreasing sequence bounded from below by $\lambda_1^{\Omega_+}(d\bar{U})$. Therefore $\mu_n \to \hat{\mu}$ exists and $\hat{\mu} \geqslant \lambda_1^{\Omega_+}(d\bar{U})$. We show that $\hat{\mu} = \lambda_1^{\Omega_+}(d\bar{U})$. This would complete the proof for we can then choose $\Omega'' = \Omega_n$ for some large n.

Let v_n be the eigenfunction corresponding to μ_n as

$$-\Delta v_n + d\bar{U} v_n = \mu_n v_n \text{ in } \Omega_n, \qquad v_n|_{\partial\Omega_n} = 0,\ v_n > 0 \text{ in } \Omega_n,\ \|v_n\|_{L^\infty(\Omega_n)} = 1.$$

Define $\tilde{v}_n = v_n$ on Ω_n, and $\tilde{v}_n = 0$ outside Ω_n. Then $\tilde{v}_n \in W_0^{1,2}(\Omega_+)$ and

$$\int_{\Omega_+} |\nabla \tilde{v}_n|^2\, dx = \int_{\Omega_n} |\nabla \tilde{v}_n|^2\, dx \leqslant \mu_n \int_{\Omega_n} \tilde{v}_n^2\, dx \leqslant \mu_1\, |\Omega_+|.$$

It follows that, subject to a subsequence, $\tilde{v}_n$ converges weakly in $W_0^{1,2}(\Omega_+)$ and strongly in $L^2(\Omega_+)$ to some $\tilde{v} \in W_0^{1,2}(\Omega_+)$. Since $\|\tilde{v}_n\|_\infty = 1$, $\tilde{v}_n \to \tilde{v}$ in $L^p(\Omega_+)$ for all $p > 1$.

Let u_n denote the unique solution to

$$-\Delta u_n = \tilde{v}_n \text{ in } \Omega_+, \qquad u_n|_{\partial\Omega_+} = 0,$$

and let w_n be the unique solution to

$$-\Delta w_n = v_n \text{ in } \Omega_n, \qquad w_n|_{\partial\Omega_n} = 0.$$

Then it follows from the maximum principle that $u_n \geqslant w_n$ in Ω_n. If $\tilde{v} \equiv 0$, that is, $\tilde{v}_n \to 0$ in $L^p(\Omega_+)$ for any $p > 1$, then $u_n \to 0$ in $C^1(\Omega_+)$. It then follows from $0 \leqslant w_n \leqslant u_n$ in Ω_n that $\|w_n\|_{L^\infty(\Omega_n)} \to 0$ as $n \to \infty$. From $-\Delta v_n \leqslant \mu_1 v_n = -\Delta(\mu_1 w_n)$ in Ω_n and $(v_n - \mu_1 w_n)|_{\partial\Omega_n} = 0$, we deduce $0 \leqslant v_n \leqslant \mu_1 w_n$ in Ω_n, and hence, $\|v_n\|_{L^\infty(\Omega_n)} \to 0$ as $n \to \infty$. But this contradicts the fact that $\|v_n\|_{L^\infty(\Omega_n)} = 1$. Thus we have proved that $\tilde{v} \not\equiv 0$.

Let ψ be an arbitrary function in $C_0^\infty(\Omega_+)$. We multiply the equation for v_n by ψ, integrate over Ω_+, and obtain, for all large n such that $\Omega_n \supset \mathit{support}(\psi)$,

$$\int_{\Omega_+} \nabla \tilde{v}_n \cdot \nabla \psi\, dx + \int_{\Omega_+} d\bar{U} \tilde{v}_n \psi\, dx = \mu_n \int_{\Omega_+} \tilde{v}_n \psi\, dx.$$

Letting $n \to \infty$, we obtain

$$\int_{\Omega_+} \nabla \tilde{v} \cdot \nabla \psi\, dx + \int_{\Omega_+} d\bar{U} \tilde{v} \psi\, dx = \hat{\mu} \int_{\Omega_+} \tilde{v} \psi\, dx.$$

That is to say that $\tilde{v}$ is a weak solution of

$$-\Delta v + d\bar{U} v = \hat{\mu} v,\ v|_{\partial\Omega_+} = 0.$$

By Theorem 3.9 of Part I, we must have $\hat{\mu} = \lambda_1^{\Omega_+}(d\bar{U})$, as required. The proof is complete. ∎

The conclusion of Theorem 4.2 can be sharpened considerably. Indeed, we have the following better result.

THEOREM 4.3. *Suppose* $\mu > \lambda_1^{\Omega_+}(d\bar{U})$ *and let* $(\underline{u}, \underline{v})$ *denote the minimal positive solution of* (1.2) *guaranteed by Theorem* 3.15 *of Part* I. *Then for any positive solution* (u, v) *of* (1.1) *and any compact subdomain* Ω' *of* Ω_+, *we have*

$$\varliminf_{t\to\infty} v(x,t) \geqslant \underline{v}(x), \qquad \varlimsup_{t\to\infty} u(x,t) \leqslant \underline{u}(x), \text{ uniformly in } \Omega'.$$

Proof. Let u_k and v_k, $k = 1, 2, ...,$ be defined as in the proof of Theorem 3.15 in Part I. The proof there shows that u_k decreases to $\underline{u}$ and v_k increases to $\underline{v}$ as $k \to \infty$, uniformly on any compact subset of Ω_+. Thus it suffices to show that for all large k,

$$\varliminf_{t\to\infty} v(x,t) \geqslant v_k(x), \qquad \varlimsup_{t\to\infty} u(x,t) \leqslant u_k(x), \tag{4.3}$$

uniformly on any compact subset of Ω_+.

Let Ω_n be defined as in the proof of Theorem 4.2. We know from there $\mu > \lambda_1^{\Omega_n}(d\bar{U})$ for all large n. Without loss of generality, we may assume that this is true for every $n \geqslant 1$. Denote by w_n the unique positive solution of

$$-\Delta w = \mu w - w^2 - d\bar{U} w \text{ in } \Omega_n, \qquad w|_{\partial\Omega_n} = 0.$$

Then w_n increases with n by a simple variant of Lemma 2.1 in [DH]. We may regard w_n as extended to be zero outside Ω_n. Then from the equation for w_n and the fact that $w_n \leqslant \mu$, we easily deduce that w_n converges weakly in $W_0^{1,2}(\Omega_+)$ and strongly in $L^p(\Omega_+)$ (for all $p > 1$) to some $w^* \in W_0^{1,2}(\Omega_+)$ which is a positive solution to

$$-\Delta w = \mu w - w^2 - d\bar{U} w \text{ in } \Omega_+, \qquad w|_{\partial\Omega_+} = 0.$$

By Theorem 3.8 of Part I, w^* must agree with v_1 given in the proof of Theorem 3.15 of Part I.

If we denote by w_n^k the unique positive solution of

$$-\Delta w = \mu w - w^2 - d u_{k-1} w \text{ in } \Omega_n, \qquad w|_{\partial\Omega_n} = 0,$$

then a similar consideration shows that the extended w_n^k converges, as $n \to \infty$, weakly in $W_0^{1,2}(\Omega_+)$ and strongly in $L^p(\Omega_+)$ (for all $p > 1$) to v_k. Moreover, by the L^p interior estimate and the Sobolev imbedding theorem, one easily sees that $w_n \to v_1$ and $w_n^k \to v_k$ are uniform on any compact subset of Ω_+.

We are now ready to show (4.3). We use an induction argument. From the proof of Theorem 4.2 we see that, for all large n, $\varliminf_{t\to\infty} v(x,t) \geqslant w_n(x)$ uniformly on any given compact subdomain of Ω_+. Since $w_n \to v_1$, we deduce

$$\varliminf_{t\to\infty} v(x,t) \geqslant v_1(x) \tag{4.4}$$

uniformly on any compact subdomain of Ω_+. It follows that, for any $\varepsilon > 0$, we can find $T > 0$ such that

$$v(x, t) \geqslant v_1(x) - \varepsilon, \text{ for all } t \geqslant T, \text{ for all } x \in \Omega_\varepsilon = \{x \in \Omega_+ : v_1(x) > \varepsilon\}.$$

If we extend $v_1(x)$ to be zero outside Ω_+, then, since $v(x, t) \geqslant 0$ for all $t > 0$ and $x \in \Omega$, we have

$$v(x, t) \geqslant v_1(x) - \varepsilon, \qquad \text{for all} \quad t \geqslant T, \text{ for all } x \in \Omega.$$

Let $z(x, t)$ be the unique solution of the problem

$$z_t - \Delta z = \lambda z - b(x) z^2 - c(v_1 - \varepsilon) z, \qquad z|_{\partial\Omega} = 0, z(x, T) = u(x, T). \tag{4.5}$$

Then $u(x, t) \leqslant z(x, t)$ for all $t \geqslant T$ and $x \in \Omega$.

By Theorem 2.3 of Part I, according as $\lambda + c\varepsilon < \lambda_1^{\Omega_0}(cv_1)$ or $\lambda + c\varepsilon \geqslant \lambda_1^{\Omega_0}(cv_1)$, when $t \to \infty$, $z(x, t)$ converges uniformly on $\bar{\Omega}$ to the unique steady-state z_ε of (4.5), or $\overline{\lim}_{t\to\infty} z(x, t) \leqslant \bar{Z}_\varepsilon$ uniformly on any compact subset of $\Omega_+ \cup \partial\Omega$, where $\bar{Z}_\varepsilon$ denotes the maximal positive solution of

$$-\Delta z = \lambda z - b(x) z^2 - c(v_1 - \varepsilon) z \text{ in } \Omega_+, \qquad z|_{\partial\Omega} = 0, z|_{\partial\Omega_0} = \infty.$$

Since $z_\varepsilon \leqslant \bar{Z}_\varepsilon$ when both exist, we see that $\overline{\lim}_{t\to\infty} z(x, t) \leqslant \bar{Z}_\varepsilon$ always holds. It follows that $\overline{\lim}_{t\to\infty} u(x, t) \leqslant \bar{Z}_\varepsilon$. By Remark 2.4, we know that $\bar{Z}_\varepsilon(x) \to u_1(x)$ as $\varepsilon \to 0$ uniformly on any compact subset of $\Omega_+ \cup \partial\Omega$. Hence,

$$\overline{\lim_{t\to\infty}} u(x, t) \leqslant u_1(x) \tag{4.6}$$

uniformly on any compact subset of $\Omega_+ \cup \partial\Omega$. Thus we have proved that (4.3) holds for $k = 1$.

Suppose that (4.3) holds for $k = m \geqslant 1$. We want to show that it holds for $k = m + 1$. For $\varepsilon > 0$ sufficiently small such that $\mu - d\varepsilon > \lambda_1^{\Omega_+}(d\bar{U})$, by (4.3) with $k = m$, we can find $T_n > 0$ such that $u(x, t) \leqslant u_m(x) + \varepsilon$ for $t \geqslant T_n$ and $x \in \Omega_n$. For any $n \geqslant 1$, let $w(x, t)$ be the unique solution to

$$w_t - \Delta w = \mu w - w^2 - d(u_m + \varepsilon) w \text{ for } t > T_n, x \in \Omega_n,$$
$$w|_{\partial\Omega_n} = 0, w(x, T_n) = v(x, T_n).$$

Then clearly $v(x, t) \geqslant w(x, t)$ for $t \geqslant T_n$ and $x \in \Omega_n$. For all large n, $\mu - d\varepsilon > \lambda_1^{\Omega_n}(d\bar{U}) > \lambda_1^{\Omega_n}(du_m)$, and hence, as $t \to \infty$, $w(x, t)$ converges uniformly on Ω_n to the unique positive solution w_ε of

$$-\Delta w = \mu w - w^2 - d(u_m + \varepsilon) w, \qquad w|_{\partial\Omega_n} = 0.$$

Hence $\underline{\lim}_{t\to\infty}\ v(x,t) \geqslant w_\varepsilon(x)$ uniformly on Ω_n. But $w_\varepsilon(x) \to w_n^{m+1}$ as $\varepsilon \to 0$, uniformly on Ω_n. Hence $\underline{\lim}_{t\to\infty}\ v(x,t) \geqslant w_n^{m+1}(x)$ uniformly on Ω_n. As $w_n^{m+1}(x) \to v_{m+1}(x)$ uniformly on any compact subset of Ω_+ as $n \to \infty$, we deduce finally that

$$\underline{\lim}_{t\to\infty}\ v(x,t) \geqslant v_{m+1}(x)$$

uniformly on any compact subset of Ω_+.

Now we can repeat the argument for the proof of the fact that (4.4) implies (4.6) but with v_1 replaced by v_{m+1}, and deduce that

$$\overline{\lim}_{t\to\infty}\ u(x,t) \leqslant u_{m+1}(x)$$

uniformly on any compact subset of Ω_+. Hence (4.3) is true for $k=m+1$. This finishes the induction argument and hence the proof of the theorem. ▌

When $\mu > \lambda_1^{\Omega_+}(d\bar{U})$, if we define $U(x) = \underline{u}(x)$ for $x \in \Omega_+$, and $U(x) = \infty$ for $x \in \bar{\Omega}_0$, and let $V(x) = \underline{v}(x)$ for $x \in \Omega_+$, and $V(x) = 0$ for $x \in \bar{\Omega}_0$, where $(\underline{u}, \underline{v})$ is the minimal positive solution of (1.2), then it follows from Theorem 4.3 that the set of function pairs

$$A = \{(u,v)\colon u|_{\Omega_+}, v|_{\Omega_+} \in C(\Omega_+ \cup \partial\Omega), 0 \leqslant u(x) \leqslant U(x), V(x) \leqslant v(x) \leqslant \theta_\mu\}$$

is a global attractor for (1.1). The dynamical behaviour of (1.1) inside A is rather complicated. For example, if μ is also greater than μ^0, then A contains at least one classical steady-state solution, one generalized steady-state solution, and the stable semitrivial solution $(0, \theta_\mu)$.

From a biological point of view, it is important to know whether (1.1) can have a stable classical steady-state solution for the case $\lambda > \lambda_1^{\Omega_0}(0)$, where without a competitor v, u always blows up in $\bar{\Omega}_0$ as $t \to \infty$. This was answered in Theorems 3.1 and 3.6 of Part I. More precisely, Theorem 3.1 there states that when $\mu_* < \mu^0$ and $\mu \in [\mu_*, \mu^0)$, then (1.1) has at least one asymptotically stable steady-state, while Theorem 3.6 shows that $\mu_* < \mu^0$ is guaranteed to happen if d is sufficiently small. Thus a stable coexistence state of the two species is possible if d is small compared with the other parameters. This is reasonable since d measures the effects of the species u on v; so d small means that the effects of u on v is small when compared with that of v on u, measured by the constant c. Clearly, such a competitor v (with d small compared to c) is a good choice in order to avoid over growth of u.

Though a stable coexistence state of the two species is possible, our results below show that this depends very much on a suitable choice of the initial conditions apart from a good choice of the competitor v. Indeed, we will show that one can always find bad initial conditions such that the positive solution (u, v) of (1.1) has its u component blowing up as $t \to \infty$. We divide our discussions into two cases:

(a) $\mu < \lambda_1^{\Omega_+}(d\underline{U})$ and (b) $\mu \geqslant \lambda_1^{\Omega_+}(d\underline{U})$

Let us recall that we always assume (4.1) holds.
The following result covers case (a).

THEOREM 4.4. *Suppose $\mu < \lambda_1^{\Omega_+}(d\underline{U})$. Then there exists $(u_0, v_0) \in C(\bar{\Omega}) \times C(\bar{\Omega})$ with $u_0|_{\partial\Omega} = v_0|_{\partial\Omega} = 0$ and $u_0 > 0$, $v_0 > 0$ in Ω, such that if (u, v) is a solution of (1.1) with $u(x, 0) \geqslant u_0(x)$ and $0 \leqslant v(x, 0) \leqslant v_0(x)$, then, as $t \to \infty$, $u \to \infty$ uniformly on $\bar{\Omega}_0$, $v \to 0$ uniformly on $\bar{\Omega}$, and*

$$\varliminf_{t \to \infty} u(x, t) \geqslant \underline{U}(x), \qquad \varlimsup_{t \to \infty} u(x, t) \leqslant \bar{U}(x),$$

uniformly on any compact subset of $\Omega_+ \cup \partial\Omega$.

Proof. By Theorem 3.9 of Part I and Remark 2.4, we know that $\lambda_1^{\Omega_+}(d\underline{U}^\lambda)$ depends continuously on λ. Let us choose $\lambda' < \lambda$ such that $\mu < \lambda_1^{\Omega_+}(d\underline{U}^{\lambda'})$ and $\lambda' > \lambda_1^{\Omega_0}(0)$. Then we let w_ε denote the unique positive solution of the problem

$$-\Delta w = \lambda' w - (b(x) + \varepsilon)\, w^2, \qquad w|_{\partial\Omega} = 0.$$

By Theorem 2.1, as $\varepsilon \to 0$, $w_\varepsilon \to \infty$ uniformly on $\bar{\Omega}_0$, and $w_\varepsilon \to \underline{U}^{\lambda'}$ uniformly on any compact subset of $\Omega_+ \cup \partial\Omega$. By Lemma 3.1, this implies $\lambda_1^{\Omega}(dw_\varepsilon) \to \lambda_1^{\Omega_+}(d\underline{U}^{\lambda'}) > \mu$. It follows that $\lambda_1^{\Omega}(dw_\varepsilon) > \mu$ for all small $\varepsilon > 0$.
Let ψ_ε be determined by

$$-\Delta\psi_\varepsilon + dw_\varepsilon\psi_\varepsilon = \lambda_1^{\Omega}(dw_\varepsilon)\, \psi_\varepsilon, \qquad \psi_\varepsilon|_{\partial\Omega} = 0,\ \psi_\varepsilon \geqslant 0,\ \|\psi_\varepsilon\|_\infty = 1.$$

Define $v_\varepsilon = \varepsilon\psi_\varepsilon$ and let u_ε be the unique positive solution to

$$-\Delta u = \lambda u - (b(x) + \varepsilon)\, u^2 - cv_\varepsilon u, \qquad u|_{\partial\Omega} = 0.$$

Since $v_\varepsilon \leqslant \varepsilon$, when $\varepsilon \leqslant (\lambda - \lambda')/c$, we have $\lambda - cv_\varepsilon \geqslant \lambda'$ and hence $u_\varepsilon \geqslant w_\varepsilon$. Moreover, since $\lambda_1^{\Omega}(dw_\varepsilon) \to \lambda_1^{\Omega_+}(d\underline{U}^{\lambda'}) > \mu$ and $v_\varepsilon \to 0$ as $\varepsilon \to 0$, we easily see that, for all small $\varepsilon > 0$,

$$\lambda_1^{\Omega}(dw_\varepsilon)\, v_\varepsilon > \mu v_\varepsilon - v_\varepsilon^2.$$

It follows that

$$-\Delta v_\varepsilon > \mu v_\varepsilon - v_\varepsilon^2 - dw_\varepsilon v_\varepsilon \geqslant \mu v_\varepsilon - v_\varepsilon^2 - du_\varepsilon v_\varepsilon.$$

Clearly

$$-\Delta u_\varepsilon \leqslant \lambda u_\varepsilon - b(x)\, u_\varepsilon^2 - cu_\varepsilon v_\varepsilon.$$

Hence, for all small $\varepsilon > 0$, $(u_\varepsilon, v_\varepsilon)$ is an upper steady-state solution of (1.1) in the sense defined in the proof of Theorem 3.2. By the order preserving property of (1.1), any positive solution (u, v) of (1.1) with $u(x, 0) = u_\varepsilon(x)$, $v(x, 0) = v_\varepsilon(x)$ has the following property: $u(x, t)$ is increasing in t and $v(x, t)$ is decreasing in t.

Let $z(x, t)$ be the unique solution to

$$z_t - \Delta z = \lambda z - b(x)\, z^2 - c v_\varepsilon z, \qquad z|_{\partial\Omega} = 0,\, z(x, 0) = u_\varepsilon(x).$$

Since $v(x, t) \leqslant v_\varepsilon(x)$ for all $t > 0$, we deduce $u(x, t) \geqslant z(x, t)$ for all $t > 0$. But when $\varepsilon > 0$ is sufficiently small such that $\lambda - c\varepsilon > \lambda_1^{\Omega_0}(0)$ and therefore $\lambda > \lambda_1^{\Omega_0}(c v_\varepsilon)$, it follows from Theorem 2.3 of Part I that, as $t \to \infty$, $z(x, t) \to \infty$ uniformly on $\bar{\Omega}_0$. Since $u(x, t) \geqslant z(x, t)$, the same is true for $u(x, t)$. Comparing u with the unique solution of

$$w_t - \Delta w = \lambda w - b(x)\, w^2, \qquad w|_{\partial\Omega} = 0,\, w(x, 0) = u_\varepsilon(x),$$

and using Theorem 2.3 of Part I, we deduce that $\overline{\lim}_{t \to \infty} u(x, t) \leqslant \bar{U}^\lambda(x)$ uniformly on any compact subset of $\Omega_+ \cup \partial\Omega$.

Since $\mu < \lambda_1^\Omega(d w_\varepsilon) \leqslant \lambda_1^\Omega(d u_\varepsilon)$, the unique solution $w(x, t)$ of

$$w_t - \Delta w = \mu w - w^2 - d u_\varepsilon w, \qquad w|_{\partial\Omega} = 0,\, w(x, t) = v_\varepsilon(x),$$

satisfies $\lim_{t \to \infty} w(x, t) = 0$ uniformly on $\bar{\Omega}$. But it follows from $u(x, t) \geqslant u_\varepsilon(x)$ that $v(x, t) \leqslant w(x, t)$. Hence $v(x, t) \to 0$ uniformly on $\bar{\Omega}$ as $t \to \infty$.

Now for any given $\delta > 0$ satisfying $\lambda - \delta > \lambda_1^{\Omega_0}(0)$, we can find $T > 0$ such that $0 \leqslant v(x, t) \leqslant \delta$ for all $t \geqslant T$ and all $x \in \Omega$. Let $Z(x, t)$ be the unique solution to

$$Z_t - \Delta Z = \lambda Z - b(x)\, Z^2 - \delta Z, \qquad Z|_{\partial\Omega} = 0,\, Z(x, T) = v(x, T).$$

Then by Theorem 2.3 of Part I, $\underline{\lim}_{t \to \infty} Z(x, t) \geqslant \underline{U}^{\lambda - \delta}(x)$ uniformly on any compact subset of $\Omega_+ \cup \partial\Omega$. But clearly $u(x, t) \geqslant Z(x, t)$ for $t \geqslant T$. Hence $\underline{\lim}_{t \to \infty} u(x, t) \geqslant \underline{U}^{\lambda - \delta}(x)$ uniformly on any compact subset of $\Omega_+ \cup \partial\Omega$. As $\delta > 0$ can be arbitrarily small and by Remark 2.4, $\underline{U}^{\lambda - \delta} \to \underline{U}^\lambda$ as $\delta \to 0$, we obtain $\underline{\lim}_{t \to \infty} u(x, t) \geqslant \underline{U}^\lambda(x)$ uniformly on any compact subset of $\Omega_+ \cup \partial\Omega$.

It is now easily seen that we can choose $(u_0, v_0) = (u_\varepsilon, v_\varepsilon)$ for some small $\varepsilon > 0$. The proof is complete. ∎

Remark 4.5. If (3.12) of Part I holds, then $\underline{U} = \bar{U}$ and Theorem 4.4 implies that $(\tilde{U}, 0)$ is locally stable when $\mu < \lambda_1^{\Omega_+}(d\underline{U})$, where $\tilde{U}$ denotes $\underline{U}$ extended to Ω_0 with value ∞. Moreover, Theorem 4.3 implies that $(\tilde{U}, 0)$ is unstable if $\mu > \lambda_1^{\Omega_+}(d\underline{U})$.

Next we consider case (b), where $\mu \geqslant \lambda_1^{\Omega_+}(d\underline{U})$. Let u_ε be defined as in Theorem 2.1. We know from Theorem 2.1 and Lemma 3.1 that $\lambda_1^{\Omega}(du_\varepsilon)$ converges to $\lambda_1^{\Omega_+}(d\underline{U})$. Since u_ε decreases with ε, we must have $\lambda_1^{\Omega}(du_\varepsilon) < \lambda_1^{\Omega_+}(d\underline{U}) \leqslant \mu$. By Theorem 3.2, $\mu^*(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Hence $\mu_0^\varepsilon = \lambda_1^{\Omega}(du_\varepsilon) < \mu < \mu^*(\varepsilon)$ for all small $\varepsilon > 0$. As in the proof of Theorem 3.5, we know that in this case (1.3) has a minimal positive solution $(u^\varepsilon, v^\varepsilon)$. One easily shows that u^ε decreases and v^ε increases with ε. Moreover, as in the proof of Theorem 3.5, we must have $\|u^\varepsilon\|_\infty \to \infty$ as $\varepsilon \to 0$. Hence, by Lemma 3.4, as $\varepsilon \to 0$, $(u^\varepsilon, v^\varepsilon) \to (\hat{u}, \hat{v})$, where $(\hat{u}, \hat{v}) = (\infty, 0)$ on Ω_0, and on Ω_+, $(\hat{u}, \hat{v})$ is a positive solution of (1.2) except when $\mu = \lambda_1^{\Omega_+}(d\underline{U})$, in which case, $(\hat{u}, \hat{v}) = (\underline{U}, 0)$ is possible. Moreover, $\hat{u} \leqslant \underline{U}$. Here $v_n \to \hat{v}$ is in the norm of $L^p(\Omega)$, for all $p > 1$, and $u_n \to \hat{u}$ is in the following sense: $u_n \to \infty$ uniformly on $\bar{\Omega}_0$, and $u_n \to \hat{u}$ uniformly on any compact subset of $\Omega_+ \cup \partial\Omega$.

THEOREM 4.6. *Suppose that $\mu \geqslant \lambda_1^{\Omega_+}(d\underline{U})$. Then for any small $\varepsilon > 0$, the set*

$$A_\varepsilon = \{(u, v) \in C(\bar{\Omega}) \times C(\bar{\Omega}) : u^\varepsilon \leqslant u < \infty, 0 \leqslant v \leqslant v^\varepsilon\}$$

is invariant for $t > 0$ under the flow generated by (1.1). *Moreover, any solution (u, v) of* (1.1) *lying in A_ε satisfies $\underline{\lim}_{t \to \infty} u(x, t) = \infty$ uniformly on $\bar{\Omega}_0$.*

Proof. Let $(U_\varepsilon, V_\varepsilon)$ be the unique solution of (1.1) with initial condition $U_\varepsilon(x, 0) = u^\varepsilon(x)$, $V_\varepsilon(x, 0) = v^\varepsilon(x)$. It is easily seen that $(u^\varepsilon, v^\varepsilon)$ is an upper steady-state solution of (1.1). Hence, by the order preserving property of (1.1), $U_\varepsilon(x, t)$ is increasing in t and $V_\varepsilon(x, t)$ is decreasing in t. Thus $(U_\varepsilon, V_\varepsilon)$ stays in A_ε for all $t > 0$. If (u, v) is an arbitrary solution of (1.1) with $u(x, 0) \geqslant u^\varepsilon(x)$, $0 \leqslant v(x, 0) \leqslant v^\varepsilon(x)$, then $u(x, t) \geqslant U_\varepsilon(x, t)$ and $0 \leqslant v(x, t) \leqslant V_\varepsilon(x, t)$ for all $t > 0$. Therefore (u, v) remains in A_ε. By our discussion before this theorem, $v^\varepsilon \to 0$ in $L^p(\Omega_0)$, for all $p > 1$. Hence $\lambda_1^{\Omega_0}(cv^\varepsilon) \to \lambda_1^{\Omega_0}(0)$. It follows that for all small $\varepsilon > 0$, $\lambda > \lambda_1^{\Omega_0}(cv^\varepsilon)$.

Let $w(x, t)$ be the unique solution to

$$w_t - \Delta w = \lambda w - b(x)\, w^2 - cv^\varepsilon w, \qquad w|_{\partial\Omega} = 0,\ w(x, 0) = u^\varepsilon(x).$$

Then, by Theorem 2.3 of Part I , we have $w(x, t) \to \infty$ uniformly on $\bar{\Omega}_0$ as $t \to \infty$. If (u, v) is a solution of (1.1) in A_ε, then $u(x, t) \geqslant w(x, t)$ and hence $u(x, t) \to \infty$ uniformly on $\bar{\Omega}_0$ as $t \to \infty$. The proof is complete. ∎

Finally, let us look at a case that every positive solution (u, v) of (1.1) must blow up in u as $t \to \infty$, while v is persistent.

THEOREM 4.7. *Suppose $\lambda_1^{\Omega_+}(d\bar{U}) < \underline{\mu}$, where $\underline{\mu}$ is given by $\lambda = \lambda^{\Omega_0}(c\theta_{\underline{\mu}})$. Let $\mu \in (\lambda_1^{\Omega_+}(d\bar{U}), \underline{\mu})$. Then any positive solution (u, v) of* (1.1) *satisfies*

$$\lim_{t\to\infty} u(x,t)=\infty, \qquad \lim_{t\to\infty} v(x,t)=0, \ \textit{uniformly on } \bar{\Omega}_0, \tag{4.7}$$

$$\overline{\lim_{t\to\infty}}\, u(x,t)\leqslant \underline{u}(x), \qquad \underline{\lim_{t\to\infty}}\, v(x,t)\geqslant \underline{v}(x), \tag{4.8}$$

$$\underline{\lim_{t\to\infty}}\, u(x,t)\geqslant \bar{u}(x), \qquad \overline{\lim_{t\to\infty}}\, v(x,t)\leqslant \bar{v}(x), \tag{4.9}$$

uniformly on any compact subset of Ω_+, *where* $(\underline{u}, \underline{v})$ *and* $(\bar{u}, \bar{v})$ *are the minimal and maximal positive solutions of* (1.2), *respectively.*

Proof. We may assume that both $u(x,0)=u_0(x)$ and $v(x,0)=v_0(x)$ are nonnegative and not identically zero. Let $\varepsilon>0$ and $(U_\varepsilon, V_\varepsilon)$ be the unique solution of (4.2). We may assume that $\varepsilon>0$ is small enough such that $\mu>\lambda_1^\Omega(du_\varepsilon)=\mu_0^\varepsilon$. Since we also have $\mu<\underline{\mu}<\mu^0$, by the known dynamical behaviour of (4.2) (compare Theorem 2.7 of Part I),

$$\underline{\lim_{t\to\infty}}\, U_\varepsilon(x,t)\geqslant \bar{u}_\varepsilon(x), \qquad \overline{\lim_{t\to\infty}}\, V_\varepsilon(x,t)\leqslant \bar{v}_\varepsilon(x)$$

uniformly on $\bar{\Omega}$, where $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ denotes the maximal positive solution of (1.3).

Since $u(x,t)\geqslant U_\varepsilon(x,t)$ and $v(x,t)\leqslant V_\varepsilon(x,t)$, (4.7) and (4.9) now follow from Theorem 3.7 by letting $\varepsilon\to 0$ in the above inequalities. Clearly (4.8) follows from Theorem 4.3. The proof is complete. ∎

Remark 4.8. (i) A discussion on when the condition of Theorem 4.7 is satisfied can be found in Remark 3.8.

(ii) When (1.2) has a unique positive solution, then $(\underline{u}, \underline{v})=(\bar{u}, \bar{v})$, and Theorem 4.7 shows that the unique generalized steady-state solution of (1.1) attracts all the positive solutions of (1.1).

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