Investigating Structures of Knowing within Three Relational Constructs, Leading to Higher-Order Mathematical Thinking

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Abstract

This paper considers three constructs that address different, yet complementary perspectives on student mathematical performance. The three constructs are the relationship between procedural and conceptual forms of knowledge by Hiebert and Lefevre (1986), the psychology of mathematical abilities in schoolchildren as researched by Krutetskii (1976), and the developmental construct of the SOLO model as devised by Biggs and Collis (1982, 1991). It will be appreciated how these constructs derive three different viewpoints of ‘structures of knowing’ in mathematics resulting in the possibility of transitional pathways to higher-order thinking.

> *In simplest terms, higher-order thinking measures include all intellectual tasks that call for more than information retrieval.*
>
> *(Baker 1990 p.7)*

Introduction

The impetus for this research comes from disquiet amongst some educators of secondary mathematics that the subject is being reduced to information retrieval for exams.

> *One of the problems with an outcome based curriculum is that underqualified teachers teach to outcomes and tests and the view of mathematics that is conveyed to students is fragmented.*
>
> *(Thomas 2000 p.20)*

Fragmentation, or non-connectedness of conceptual thought, results from assessment tasks requiring a higher-order thinking response, being answered without higher-order thinking. This perennial problem in mathematics education arises because the student has been taught to the test or has been taught how to derive a solution without understanding.

Indeed many students are now not choosing to undertake mathematics courses ‘in which higher-level mathematics skills are taught’ (Barrington 2006 p.iv). They give the impression of becoming more and more disengaged in the learning process avoiding mathematically demanding courses in favour of elementary ones

(Barrington 2006). A lack of understanding of mathematics appears to be a relevant factor in disengaging from higher-level mathematics (Forgasz 2005).

This phenomenon may be perceived as being self-generating and is possibly best understood within the context of the following inward spiralling cycle: disengaged students are not learning concepts; they then rely on procedures for their learning, directed at passing tests; teacher instruction becomes more procedural to cope with the student's inability to grasp concepts; they become more disengaged with a compensatory lowering of standards and expectations within the learning environment. Each cycle brings with it less variation in instruction. This process is portrayed in the following diagram in which the three components increase in quantity as instruction progresses (in a downward direction):

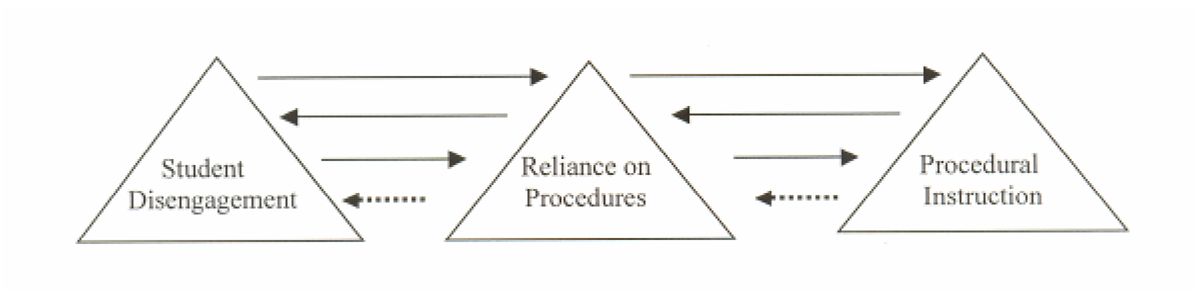

Figure 1

This cycle is alluded to by Kulm who maintained that pupils gain knowledge of 'how to do numerical computation at the expense of learning how to think and solve problems' (1990 p.71). In contrast, the advancement of technology highlights that 'the ability to learn by thinking conceptually, critically, and creatively is a fundamental competency for the workplace' (Johnson 1997 p.161).

The discrepancy between numerical computation and the requirements of critical thinking is being challenged within learning environments by the dynamics of technology exposing students to:

> *the kinds of positions they find themselves in the real world i.e. where the demands of tasks may be to some extent unpredictable, and the knowledge and skills needed are not necessarily set by some prior instruction on a topic, concept or process.*
>
> *(McCormick 1997 p.141)*

Research by Romberg et al. (1990) suggests that the scenario described above by Kulm (1990), and McCormick (1997) has a history, and any development of knowledge must take this into account. They maintain that the industrial age of the past century brought about sequential processes of reductionism, analysis, and mechanisation. This process insisted that understanding came from dismembering something to see how it worked. Information consequently took on the auspices of following predetermined rules to achieve a consensus, resulting in a procedural approach to knowledge construction. Internal cognitive mechanisms were ignored in favour of external mechanisms of synthesis.

Against such historical background, the object of learning in secondary mathematics was more the production of a solution than alternative pathways displaying higher-order thinking. Consequently the instigation of a rule dependent teaching methodology to deliver answers, without proof of conceptual understanding, within a rigid timeframe became an easy option for the instruction of disengaged students.

Reflections of this rule-based approach to knowledge construction can be found in a recent report into mathematics teaching in the United States. It describes such teaching as 'a system of … teaching in eighth grade characterised by frequent reviews of relatively unchallenging, procedurally oriented mathematics during lessons that are unnecessarily fragmented' (Hiebert et al. 2005; Givvin et al. 2005 p.116). The report also notes that 'a growing set of data indicates that classroom practice currently is tailored to support students' execution of low-level skills' (Hiebert et al. 2005 p.128). Jonassen (2002) classified this learning environment as being dominated by receiving knowledge rather than creating it.

How students combine procedural (skills) and conceptual knowledge (understanding) is of great importance in the creation of new knowledge. Krutetskii (1976) makes distinction of a similar connection that exists between students' skills and abilities. For him, this relationship is indicative of their capacity to generate new knowledge. In *The Psychology of Mathematical Abilities in Schoolchildren,* he develops the connection further by relating personal traits or abilities to achievement levels – capable, average, or incapable. In a similar vein the Structure of Observed Learning Outcome (SOLO) model, assesses student responses to reveal various achievement levels of forming mathematical relationships. Subsequently the model may provide a structure for viewing a student's progress towards the creation of new knowledge.

Procedural and conceptual knowledge, along with Krutetskii's research, and the SOLO model, embody knowledge development through relationship formation in the domain of formative assessment.

The Centre for Educational Research and Innovation (CERI) (2005) describes this type of assessment as 'frequent interactive assessments of student understanding and progress to identify learning needs and shape teaching' (p.5) and promoting 'the goals of lifelong learning, including raising levels of student achievement' (p.21). The integration of formative assessment into teaching practices has become an important tool for the creation of new knowledge. Such practices are dependent on four elements that are found collectively within the three aforementioned constructs: the creation of a benchmark for student achievement; establishing present level of student performance; developing a tool for the comparison of both performance levels; and developing intervention that bridges the gap effectively allowing teachers and students to construct learning (CERI 2005 p.45).

This paper explores the meaning of procedural and conceptual knowledge, Krutetskii's research on aligning mathematical abilities (also referred to as personal traits), with problem-solving and the use of the SOLO taxonomy as a diagnostic tool. In so doing it investigates three constructs of bridging the gap between actual and potential student performance. Each forms a framework for conceptual understanding of structures of knowing in mathematics. This conceptual interpretation informs research into how students can progress in the development of higher-order thinking skills.

Procedural and Conceptual Knowledge

This section deals with the ideas of procedural knowledge (skills) and conceptual knowledge (understanding). The dynamics between procedural and conceptual knowledge are investigated in terms of which form of knowledge develops first culminating in an interesting proposition that each develops the other. Finally, implications for learning derived from the sequence of implementation of procedural and conceptual knowledge are discussed.

Procedural knowledge may be regarded 'as the ability to execute action sequences to solve problems' (Rittle-Johnson et al. 2001 p.346). This is reflective of Krutetskii's findings that students possess a unique ability to execute actions, which in some respects, may be regarded as distinct from their mathematical ability (Krutetskii 1976). Hiebert and Lefevre also noted that sequentially performed procedures are a form of knowledge that can be seen as separate from other knowledge classifications (1986 p.6).

One traditional aspect of *procedural knowledge* is the notion of *skill acquisition,* also acknowledged by Krutetskii as being an important aspect of learning (1976). The idea began in the early 1920s through the work of Thorndike (1922), but met opposition in the early 1930s when Brownell (1935), propositioned for greater understanding with less attention to non-connected skills. This debate reappeared in the 1970s with Gagné (1977) advocating skill learning in contrast to Bruner's (1973) predominant drive for understanding. Today the connection between skill learning (the ability to execute procedures leading to a solution) and understanding is still being investigated (Rittle-Johnson et al. 2001, Hiebert et al. 2005).

Such investigations encompass the idea of *conceptual knowledge*, which can be defined as 'implicit or explicit understanding of the principles that govern a domain and of the interrelations between units of knowledge in a domain' (Rittle-Johnson et al. 2001 p.346). These authors offer an interesting view of conceptual knowledge, claiming that it 'is flexible and not tied to specific problem types and is therefore generalizable' (2001 p346). Importantly Hiebert and Lefevre maintain that conceptual knowledge "cannot be an isolated piece of information; by definition it is a part of conceptual knowledge only if the holder recognises its relationship to other pieces of information" (1986, p3).

First attempts at understanding the connection between procedural and conceptual knowledge were based around the debate concerning which develops first and therefore contributes to the development of the other. Many educational practices have been dependent on being able to address this dilemma (Hiebert & Lefevre 1986).

Some consider that conceptual knowledge, either innate or developed is used first to dictate necessary procedures in a specific domain (Geary 1994; Gelman & Williams 1998; Halford 1993). Evidence of this phenomenon has been found by several researchers (Byrnes 1992; Cowan & Renton 1996; Dixon & Moore 1996; Hiebert & Wearne 1996; Siegler & Crowley 1998; Wynn 1992). Others perceived that students learn a procedure and later, after much practice, recognise the concept (Fuson 1988; Karmiloff-Smith 1992; Siegler & Stern 1998). Supporting research for this position is also extensive (Briars & Siegler 1984; Byrnes & Wasik 1991; Frye et al. 1989, Fuson 1988; Hiebert & Wearne 1996). Rittle-Johnson et al. reconciled the two alternatives by propositioning a 'bi-directional relationship' between both forms of knowledge in which 'increases in on type … leads to gains in the other' (2001 p.347). This iterative process allows one form of knowledge to produce small gains in the other, which consequently demands additional gains in the former. Thus, according to these authors, new knowledge is being continually constructed through the interplay of both types of knowledge (Rittle-Johnson et al. 2001 p.347). Understanding how knowledge may be constructed is important in the process of discovering structure of knowing.

This proposed bi-directional relationship is alluded to by the National Research Council (2001) that promotes the idea of procedures and concepts existing without one compromising the other – both are seen as essential in the learning process. However, the debate is far from over with several researchers asking if one is being given precedence over the other as a consequence of teacher emphasis (Hiebert et al. 2005). Despite recent findings, there is still an inability to reconcile the two forms of knowledge. Hiebert and Lefevre recognise that this is reflected in two approaches, '… either to present more problems that emphasise procedures, or to present more problems that emphasise concepts' (2005 p.128). They warn that the first approach contains inherent dangers:

1. Procedures are divorced from relationships except possibly in their execution. Their capacity to create new knowledge is severely restricted because they are tied to context and structure devoid of understanding.
2. The propagation of procedural knowledge through rote learning connects structures not ideas. Consequently such knowledge may consist of pockets of information that can only lead to greater understanding provided the learner is able to connect the pockets together in a meaningful way.

Boaler (1998) reports an additional danger – procedural approaches may not transfer to life beyond the classroom:

> *There is a growing concern among mathematics educators that many students are able to learn mathematics for 11 years or more but are then completely unable to use this mathematics in situations outside the classroom context. ... Research projects have shown that in real-world mathematical situations, adults and students do not use school-learned mathematical methods or procedures. (p.41)*

The second approach addresses the concerns of Johnson (1997) that ‘the ability to learn by thinking conceptually, critically, and creatively is a fundamental competency for the workplace’ and is an important part of daily living (p.161) The danger however is that concepts cannot be fully developed if they lack supporting procedural structures (Rittle-Johnson et al. 2001) .

Whilst the importance of the connection between procedural and conceptual knowledge is recognised, the precise nature of the relationship that creates new knowledge, is still problematic.

In summary, procedural and conceptual knowledge interact to produce new knowledge and play an important role in life long learning. That is, procedural knowledge (skills, actions) may further the development of conceptual knowledge which in turn may require a more advanced procedural framework from which to develop. An understanding of the connections between both forms of knowledge allows teachers and students to implement strategies towards bridging the gap between what is achieved, and what can be achieved.

Krutetskii

Krutetskii (1976), in his book, *The Psychology of Mathematical Abilities in Schoolchildren*, argued that students possess a unique ability to execute actions that in some respects may be regarded as distinct from their mathematical ability. He noted that these actions might also be perceived as aiding the development of mathematical ability. This increased ability has the possibility of allowing further actions to take place that not only extend what is currently being achieved but also raise the potential for greater achievement.

Over a twelve-year period, principally in the 1960s, he researched the gap between students who displayed a high level of thinking and students who had yet to reach this level. This was accomplished through an investigation of gifted mathematicians and research literature resulting in a hypothesis about the structure and formation of mathematical abilities. He postulated that these abilities corresponded to certain stages of problem solving which itself could be organised into a sequential framework consisting of information gathering, processing, retention, and spatial aspects of perception. The overall correlation between abilities and these stages of problem solving is set out below:

Information gathering (selection of information for structuring a solution pathway)

The ability for formalised perception of mathematical material, for grasping the formal structure of a problem

Information processing (mental activity performed on gathered information)

The ability for logical thought in the sphere of quantitative and spatial relationships, number and letter symbols; the ability to think in mathematical symbols
The ability for rapid and broad generalisation of mathematical objects, relations and operations
The ability to curtail the process of mathematical reasoning and the system of corresponding operations; the ability to think in curtailed structures
Flexibility of mental processes in mathematical activity
Striving for clarity, simplicity, economy, and rationality of solutions
Ability for rapid and free reconstruction of the direction of a mental process, switching from a direct to a reverse train of thought (reversibility of the mental process in mathematical reasoning)

Retaining mathematical information (retention of completed solution processes)

Mathematical memory (generalised memory for mathematical relationships, type characteristics, schemes of arguments and proofs, methods of problem solving, and principles of approach)

Spatial aspects of perception

Mathematical cast of mind (pp.350-351)

Clearly, the successful completion of each stage of problem solving is now aligned with the acquisition of certain mathematical abilities or personal traits. Consequently he was able to confirm the association of varying degrees and interaction of specific abilities with levels of student responses to problem solving. Kilpatrick and Wirszup who translated Krutetskii's (1976) work evidence this in their preface:

> *Just as Piaget's notions of intellectual growth have made mathematics educators aware of differences in children's thinking at various ages, so Krutetskii's notions on the structure of mathematical abilities could make them aware of different components of ability and how they might function together. (p.xv)*

Krutetskii's (1976) research led him to formulate a set of problems consisting of 79 tests divided into 26 series. These exhibited the functionality of certain mathematical abilities as they pertained to problem solving:

> *After a number of trials, a special system of experimental problems was designed, to expose characteristics of the mental activity of pupils with various abilities in mathematics. (p.98)*

He appears to use three frames of reference to give structure to these mathematical abilities.

First, he perceived mathematical abilities as being creative within both the school, and post-school domains. Many educators of the time regarded student establishment of new knowledge as non-creative since it already existed. They believed that the only truly creative knowledge existed within the scientific domain of discovering the unknown. Krutetskii, however, maintained that school children were capable of being creative in their knowledge construction, as well as mastering the school mathematics course. For him a student could be creative if the knowledge he discovered was new to him. Thus, he stressed the importance of both learning and creativity. This concept finds similar expression in the New South Wales Mathematics Syllabus, which portrays, education existing within the domains of ‘students learn about’ and ‘students learn to’ (NSW Board of Studies, 2002). It also confirms a place, within Krutetskii’s research, of students structuring their own knowledge according to their abilities or personal traits.

Secondly, he followed on from this initial idea, contrasting abilities with skills or habits. Skills were regarded as ‘the qualities or *features of the activity of a person’* and abilities as the ‘qualities or *traits of the person carrying out an activity’* (p71). Furthermore, it was recognised that abilities and skills combined with knowledge and habits to form a bi-directional relationship:

> *On the one hand, when knowledge, skills, and habits are acquired, abilities are developed; their formation and development is impossible outside the process of mastering the appropriate knowledge, skills and habits. On the other hand, the acquisition of knowledge, skills, and habits depends, along with other conditions, on the pupil’s individual traits; abilities permit the appropriate knowledge, skills and habits to be mastered more easily and more thoroughly. Just as it would be incorrect to sever them, so it would be wrong to identify them with each other. (p.70)*

Hence, the development of personal traits can lead to development of knowledge.

Thirdly, Krutetskii (1976) separated mathematical abilities from the general abilities required for the completion of any task. For example, abilities such as a positive attitude, diligence, self-discipline, concentration, minimum knowledge etc., are perceived as relating to any general activity. He regarded “mathematical ability proper” as being represented by “definite individual psychological characteristics in the sensory and mental spheres, answering the requirements of the given activity” (p.73). In this way clarification was given to what specific abilities related to the development of mathematical thought in a particular instance.

These three perspectives are central in Krutetskii's focus on the processes within problem solving i.e. how students are able to use their abilities to establish a pathway to a solution. He was not concerned with the specific processes of problem solving, i.e. how problems dictate specific methodologies. Rather, he was more interested in the overall role of abilities in solving problems (Goldin 1977).

To achieve this goal Krutetskii derived six critical guidelines for the understanding of ability within a mathematical activity. These have important considerations for research and are outlined below:

i. Specific activities require specific abilities.

ii. Abilities only develop in specific activities.

iii. Abilities develop in optimal age periods.

iv. An activity's progress is dependent on a complex of abilities.

v. Different combinations of abilities determine high achievement.

vi. Undeveloped abilities can be compensated for by other abilities.

(pp.66-67)

These guidelines are foundational in the construction of questions designed to illicit ability levels belonging to the first three categories of problem solving (information gathering, information processing, and retention of solution methods. Problems pertaining to these categories delineate differences between capable, average, and incapable students and in so doing show an emergence of higher-order thinking skills.

It is noteworthy that Krutetskii considered the fourth category – spatial aspects of perception – as being unhelpful to this differentiation of abilities. Subsequently there is little information about this category with respect to the three types of students mentioned above. A further important consideration in reviewing the results of his research is that the characteristics of the capable students are used as benchmarks for average and incapable students. The later are perceived in terms of the degree to which they possess characteristics of the former. Hence the most extensive information provided, concerns capable students. This process does allow however, the presence common traits to be investigated across various achievement levels.

Consequently the first three categories of problem solving will now be examined to reveal a structure of knowing within capable, average, and incapable students. However, to give a more precise picture of abilities, the second category – information processing – will be examined from three perspectives: generalisation ability (ability to abstract a universal process); flexibility (the ability to switch readily from one method of operation, or train of thought, to another); and reversibility (reversing the mathematical reasoning process.

Structures of knowing for capable, average, and incapable students are described in the following analysis of student responses to Krutetskii's (1976) research questions.

Characteristics of Capable Students

Capable students *gathered information* quickly, grasped the structure of the problem immediately and are able to dismember it into connecting parts, often curtailing their reasoning process through understanding the role of each element within the structure:

> *The able pupil perceives each … complex as a composite whole. First he perceives individual elements in this complex, each element as part of the whole, and second, he perceives these elements as interrelated and forming an integral structure, as well as the role of each element in its structure. (p.228)*

Capable students recognised what was relevant or superfluous by virtue of an understanding of the problem's essential elements:

> *Capable students perceive the mathematical material of a problem analytically (they isolate different elements in its structure, assess them differently, systematise them, determine their "hierarchy') and synthetically (they combine them into complexes, they seek out mathematical relationships and functional dependencies. (p.227)*

Analysing the characteristics of *generalisation* reveals that they were not something capable students 'worked towards' but 'worked out of'. Hence, capable students were able to rapidly and broadly extend relationships without outside intervention and readily transfer information to new problem types. Their ability to do so appeared to come from their powers of initial perception, focusing on the structure of the problem rather than on its individual elements. It is not surprising then to find that very capable students actually had a need to generalise.

Further, capable students were characterised by the ability to immediately shorten the reasoning process and corresponding procedures. Recognising a set of sequenced connections that led to a solution, the capable student often made a connection by disregarding intervening steps. In some instances he was able to establish a connection to the final step immediately. The advantage for the capable student was that it took less time to complete a normal lengthy and involved mathematical process. The student was able to shorten deductions although the links themselves were not removed and could be recalled upon request.

Capable students also displayed *flexibility* of mental processes and were able to offer several pathways towards a solution – they switched readily from one operation to another without being inhibited by previous solutions. This reflected their skill at generalisation and not being tied down by concrete elements within the solution or process. They appeared to be free of the confines of a preordained method of solution and possessed significant working memory in which to perform alternate operations. In so doing they seemed to achieve the balance between discovering a solution and detaching themselves from its framework in order to seek other frameworks. Such alternatives were often sought after for being more economical and efficient, i.e. they involved less complicated steps and provided the most direct pathway to the solution.

These students were able to perform *reversibility* of mental operations freely. The resultant reverse bonds were established at the same time as direct bonds. An example of the relationship between the different types of connections appears in Krutetskii's problem series: 'A saw in a saw mill saws off a 1m piece of log every minute. How many minutes will it take to saw 16m of log? [direct bond]', and 'In 3 minutes a log is sawed into half metre pieces, with each cutting taking one minute. Find the length of the log [reverse bond]' (p.144):

> *Moreover, in approximately half the cases it turned out that a reverse problem given [to capable students] right after the direct one was solved more rapidly, more easily, than a reverse problem given independently of the direct one, as an original problem. (p.288)*

Finally, capable students possessed excellent *memory retention*. They tended to loose irrelevant and concrete data fairly quickly after establishing a solution. Often, they forgot the content of a problem, retaining the method of solution. For these students, recalling information about a problem was not a recollection of facts but a bringing to mind a possible pathway to a solution. Their minds were not cluttered with detail – only a generalised process. Subsequently it could be concluded that perhaps capable students had a larger working memory than other students.

In summary, capable students were able to solve problems quickly because they perceived the overall structure of the problem immediately. Once a problem was solved they remembered the relationships within it and guidelines for a solution. They were free of the constraints of having to form connections between essential elements within the problem and were able to apply generalised principles to the task at hand. This seems to have allowed them increased working memory with greater flexibility and reversibility of mental processes. Consequently capable students displayed a considerable degree of higher-order thinking.

Characteristics of Average Students

In contrast to capable students, average students' *information gathering* was characterised by perceiving elements within a problem as disconnected. These students required additional exercises to establish the required connections. In addition, superfluous data was an obstacle to their reasoning since it was perceived as part of a possible solution pathway.

Average students had to carry out several exercises in order to achieve an adequate level of *generalisation*. They exhibited difficulty in transferring information to other problems, possibly because the connections they had formed were only made after great effort using a high degree of working memory. Their inability to initially curtail processes leading to a solution also gives credence to the argument that average students were operating with restricted working memories. Only after doing a problem completely several times could a shortcut be made. Such shortcuts then had the effect of increasing working memory thereby improving the chances of working through other problems. Average students also became fixated on a past methodology, which impeded their thinking process in the development of a pathway to solution. Thus the *flexibility* of mental processes for the average student was restricted by previous solutions decreasing the possibility of engaging in unfamiliar problems.

The concept of prior methodology inhibiting further development of thought seems to have been responsible for the average student's lack of *reversibility* of mental processes. These students lacked confidence in establishing reverse bonds and required special exercises to create them. For average students direct bonds tended to inhibit the formation of reverse ones. They often had to undergo a time delay before undertaking a reverse process. Noteworthy was their preference to engage in reverse bonds in the absence of direct bonds. It would seem that their working memory was limited to processing only one bond at a time.

It should be noted that average students placed greater strain on their working memory by not being selective in choosing data assigned to memory retention. They tended to remember everything about the solution – the concrete, abstract, essential, non-essential, and generalised solution.

In summary, average students perceived elements within a problem as being disconnected but were able to make connections after practice. They tended not to remember relationships but all facts, both necessary and unnecessary. This process is reflected in their difficulty at generalisation, flexibility, and reversibility of thought. The retention of non-essential information also appears detrimental to the transfer of information to new problems. Generally average students displayed higher order thinking in some instances only after extensive exercises.

Characteristics of Incapable Students

Information gathering by incapable students was characterised by fragmentation. Like average students, they viewed elements within the problem as isolated and disconnected. Subsequently, they were unable to organise the elements into a hierarchy reflecting a structure for solution even after several exercises. Incapable students perceived all elements as equal. Consequently, their initial approach to a problem was to focus on features that made it different from other problems, rather than to focus on any essential elements within its structure:

> *As for the incapable pupils … connections and correlations between the elements of a problem, even with outside help, are established with great difficulty. (p.228)*

Consequently incapable students were unable to offer any *generalisation*. They were unable to engage in *flexibility* of thought, often finding the prospect of several solutions too daunting, even when the second solution method was easier than the first. Indeed 'for many of them [younger students] the idea that a problem might have several solutions (all correct) was unacceptable' (p.338). In order to succeed at a second solution, they had to completely forget the first method.

Incapable students did not display any *reversibility* of thought. They were unable to perceive connections between reverse and direct bonds, not even recognising one bond as the reverse of the other. For them problems requiring reverse bonds were perceived as similar to ones that required direct bonds. *Memory retention* was poor with essential elements often forgotten before the end of the lesson. Even a mastered solution was easily forgotten without regular review.

In summary, the inability of relatively incapable students, to establish connections between elements, to recognise redundant data, and to create a hierarchy of elements appears to interact with their ability to develop flexibility and reversibility of thought. Their inability to form connections between essential elements apparently denied them the opportunity of developing a reasoning process that would lead to generalisation. Consequently incapable students were unable to transfer knowledge onto other problems. They displayed no higher-order thinking.

Krutetskii categorised students who easily formed and remembered relationships as 'capable'. The degree to which other students were able to achieve this allowed them to be classified as 'average' or 'incapable' of relationship formation. Therefore a method of measuring the degree of student success in establishing mathematical relationships, in a particular instance, would complement his research. Such a measuring tool is provided by the SOLO model.

Structure of the Observed Learning Outcome (SOLO) Model

This section looks at the three main components of the SOLO model, namely, modes of construction, levels of understanding, and cycles of learning. These ideas are discussed in relation to Krutetskii's research. The construction of current research questions is then developed from this synthesis.

The SOLO model, created by Biggs and Collis (1982, 1991), is a developmental construct that evaluates student responses in terms of the degree to which they make connections between the elements of a problem. Hence the model may be regarded as one method of measuring the degree of student success in establishing mathematical relationships.

Biggs and Collis (1982, 1991) outline five hierarchical *modes* of knowledge construction: sensorimotor; ikonic; concrete symbolic; formal; and post formal. Students develop certain abilities within each mode to aid in the creation of knowledge. In the sensorimotor mode, they develop and use motor skills to react to their surrounding environment. Within the ikonic mode, which is developed between the ages of one and a half to six years, students are internalising these motor skills to develop the imagery necessary for the development of intuitive knowledge. From six years to sixteen years, within the concrete symbolic mode students are developing language and number systems to display thoughts. The formal mode is a time when students engage in abstract concepts resulting in the development of theoretical thought. Such theories are reviewed and challenged with resultant structural changes within the post formal mode.

Development through these modes is similar to Piaget's (1954) stage development, except that Biggs and Collis (1982, 1991) regard learning as uni-modal or multi-modal. Students who utilise a combination of modes to help them develop their present mode or transit to a higher mode are said to be engaging in multi-modal learning. For example a student may use a diagram (utilising the ikonic mode) to help solve a problem requiring understandings from the symbolic mode. Those who construct knowledge within a single mode partake in uni-modal learning.

However progression from one mode to the next and therefore the development of higher-order thinking is not assured. Biggs and Collis (1991) drew attention to at least five factors contributing towards progression to the next mode: physical maturity, achievement of the relational response in the lower mode, sufficient working memory, social support, and cognitive conflict.

Contained within each mode are three basic hierarchical *levels of understanding* that signify student ability to make connections between elements within problems – unistructural, multistructural and relational. Unistructural responses are composed of one pertinent element of the mode; multistructural responses comprise several non-connected elements usually in order; relational responses contain several connected elements constituting a composite whole (Kulm, 1990).

An important characteristic of these levels in terms of how students structure their own knowledge is that whilst unistructural and multistructural responses can be taught, the relational response can only be derived by the student. Instructional teaching of this later response would provide solution, but not progress the students to higher levels of thinking. The pupil would follow the procedure to the solution but remain at the multistructural level.

These three response levels constitute a *learning cycle*. Each mode may have one or more learning cycles with combinations of learning cycles arranged into a hierarchy of knowledge construction. This constitutes a unique structure of student knowledge displaying what he or she understands, and has yet to accomplish.

The structures of knowledge outlined within the SOLO model and those contained within Krutetskii's construct of mathematical abilities possess certain similarities in their formation. This overlap contributes to the development of an interactive relationship between them.

Biggs and Collis (1982, 1991) and Krutetskii (1976) employed the technique of observing a student outside the context of rigid testing. Biggs and Collis observed student responses in the light of problem solving, and Krutetskii observed student psychological traits also in the light of problem solving. Both, to a degree, perceived the pupil in isolation to overall factors affecting knowledge construction. Krutetskii distinguished mathematical ability from general ability while Biggs and Collis looked at student responses outside of other general influences. Importantly, the developmental construct of the SOLO model echoes Krutetskii's construct which has been described as 'developmental rather than experimental' (Bright 1977 p.55) .

In addition to these similarities there appears to be a connection between the SOLO model and student abilities in mathematics as depicted by Krutetskii (1976). It is contained within his statement of very capable students:

> *As a result there arises in the pupils a concept of the essence of hidden mathematical relationships not given directly in the problem (but following from the essence of relations given in the problem), and on this basis the plan of future operation is worked out. In these cases the examinees would say: "I am not solving the problem yet; I want to get to know it better"; "Can I try to look into the problem before solving it? (p.240)*

This passage is suggestive of the capable student unknowingly establishing a higher learning cycle, within the SOLO framework. He is obviously trying to utilise known relationships to create something new. The SOLO model postulates that learning cycles are connected in a particular way. Higher cycles of learning contain elements or relationships from preceding cycles, which in Krutetskii's (1976) terminology could be labelled as 'hidden mathematical relationships'. In this case, the capable student may be said to exhibit the personal trait of 'grasping the formal structure of a problem' associated with information processing (p.350). This student may also be regarded as displaying at the very least, a second cycle response within the SOLO model (since the student is looking to make use of a previous relationship.

Krutetskii looked at student responses as indicating personal traits or abilities while the SOLO model assessed student responses as indicating the degree to which a problem is understood. When the two constructs are compared it is apparent that Krutetskii's 'capable students' are able to respond relationally to problems. Those he describes as 'average' initially give multistructural responses while 'incapable students' mostly provide unistructural responses. Thus, with regard to student abilities, there is an apparent link between Krutetskii's research and the SOLO model.

The question then becomes, is it possible that Krutetskii's research (1976) and the SOLO model, as devised by Biggs and Collis (1982), could provide information that would develop or enhance the constructs of the other? Can a possible intervention be established to validate Krutetskii's claims that in some cases average students can progress to the point of displaying similar abilities as capable students? Does an emphasis on procedural knowledge prevent incapable students from developing greater mathematical ability? Does it also prevent a student transitioning from giving multistructural responses to relational responses within the SOLO model? Is it possible that the works of Krutetskii may provide the 'missing link' of transitional pathways within the SOLO model? Could the merging of the two constructs result in a better understanding of the structures of knowing?

The pathway to higher-order thinking appears to be multifaceted as indicated above with the distinct possibility of there being several contributory factors operating within a dynamic relationship. An attempt to elucidate these factors leads to the following research questions:

1. Is attainment of a relational response, within the SOLO model, inhibited by a lack of understanding of either essential elements or procedural processes or a combination of both, as is the case with Krutetskii's 'incapable students'?
2. Is the possibility of forming relational responses enhanced by the automatic performance multi-procedural steps or by an iterative development of conceptual and procedural knowledge?
3. Can exposure to degrees of flexibility, reversibility, and generalisation as assessed by Krutetskii, bring about the cognitive conflict that Biggs and Collis mention as a factor in transition to higher-order thinking?
4. Can the information from the above questions constitute a hierarchy of developmental activities and interventions, producing a pathway towards higher-order thinking in mathematics? Can progress along this pathway be assessed using the SOLO model?
5. Is such a pathway to higher-order thinking likely to engage more students in their learning and motivate to undertake higher-level mathematics courses?

Summarising, the SOLO model establishes a framework for assessing a student's ability to construct mathematical relationships. It signifies prior development of learning in a particular domain (since the second cycle of learning is only achieved upon completion of the first cycle), and indicates what has yet to be created. Krutetskii's research establishes a framework for investigating which student abilities are necessary for the formation of mathematical relationships within a particular type of problem. Consequently, there appears the possibility of associating these abilities with various student response levels within the SOLO model This offers the distinct likelihood of achieving a higher-level response within the model, through the development of specific abilities.

Conclusion

Clearly there are significant links among procedural and conceptual knowledge, Krutetskii's research into mathematical abilities, and the SOLO model.

The procedural and conceptual knowledge debate indicates how new knowledge can be created. Krutetskii's research outlines what ability traits need to be developed for this to happen and the SOLO model provides an assessable pathway along which development takes place. The connections between all three constructs warrant further investigation to establish an overall pathway to higher-order thinking.

Research to establish these connections utilises the constructs of some of the questions within Krutetskii's problem series. These, in conjunction with an understanding of the relationship between procedural and conceptual knowledge, underpin intervention activities by the researcher designed to monitor and progress student thinking.

Student responses to specified problems are analysed according to the SOLO model to present a coherent understanding of any gap between the expected level (as determined by the curriculum) and actual level of relationship formation. Established learning theories, such as 'constructivism' and 'transfer learning' will underscore intervention activities. This process will include interviewing techniques similar to Krutetskii's and will determine the zones of proximal development (the extent of student advancement in problem solving after intervention) – a concept developed by Vygotsky (1986).

The type and composition of intervention will be recorded, and subsequent student progress monitored to establish any improved zones of proximal development. Increases of each zone will indicate the ability to advance procedural and conceptual knowledge towards relationship formation through the development of various personal traits. Each pupil's movement within the zone will be analysed using the SOLO model, indicating extent of progression towards higher-order thinking. The aim of the research is to determine and assess how dynamic structures of knowing implicit within the domains of various mathematical activities can be developed. A possible ramification of this research will be an insight into improving student motivation to undertake mathematics courses that require higher-order thinking. Student inability to develop their own thinking processes would be reason enough not to engage in a level of mathematics that demanded higher-order thinking skills for success.

This paper began with the concept of a particular cycle of learning that had the capacity to disengage students and teachers from developing higher-order thinking skills. Lack of students possessing these skills has dire repercussions for Australia as a whole. It sets in motion a much wider cyclical event that is best described by Thomas in *Mathematical Sciences in Australia – Looking for a Future* (2000):

> *In the long term, solving the supply of mathematics teachers is intimately connected to the number of students studying advanced level mathematics in schools and strong mathematical sciences in the universities. However, both of these have shrunk – when they should have been expanding – so Australia now suffers a crisis throughout the mathematical sciences. (p.1)*

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