Existence and Exact Multiplicity for Quasilinear Elliptic Equations in Quarter-Spaces

Yihong Du[1(✉)] and Messoud Efendiev[2]

[1] School of Science and Technology, University of New England, Armidale, NSW 2351, Australia
ydu@turing.une.edu.au
[2] Institute of Computational Biology, Helmholtz Zentrum München, Ingostädter Landstrasse 1, 85764 Neuherberg, Germany
messoud.efendiyev@helmholtz-muenchen.de

Abstract. We consider positive solutions of quasilinear elliptic problems of the form $\Delta_p u + f(u) = 0$ over the quarter-space $Q = \{x \in \mathbb{R}^N : x_1 > 0, x_2 > 0\}$, with $u = 0$ on ∂Q. For a general class of nonlinearities $f \geq 0$ with finitely many positive zeros, we show that, for each $z > 0$ such that $f(z) = 0$, there is a bounded positive solution satisfying

$$\lim_{x_1\to\infty} u(x_1, x_2, ..., x_N) = V(x_2), \ \lim_{x_2\to\infty} u(x_1, x_2, ..., x_N) = V(x_1),$$

where V is the unique solution of the one-dimensional problem

$$\Delta_p V + f(V) = 0 \text{ in } [0, \infty), \ V(0) = 0, \ V(t) > 0 \text{ for } t > 0, V(\infty) = z.$$

When $p = 2$, we show further that such a solution is unique, and there are no other types of bounded positive solutions to the quarter-space problem. Thus in this case the number of bounded positive solutions to the quarter-space problem is exactly the number of positive zeros of f.

Keywords: P-laplacian equation · Positive solution · Asymptotic behavior · Quarter-space

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1 Introduction

Consider the quasilinear elliptic problem

$$\Delta_p u + f(u) = 0 \text{ in } Q, \ u = 0 \text{ on } \partial Q, \tag{1.1}$$

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where $Q = (0,\infty) \times (0,\infty) \times \mathbb{R}^{N-2}$ is a quarter space in $\mathbb{R}^N$ $(N \geq 2)$, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the usual p-Laplacian operator with $p > 1$.

For the nonlinear function f, we assume that

$$\begin{cases} f : \mathbb{R}_+ \to \mathbb{R} \text{ is continuous, nonnegative and} \\ \text{locally Lipschitz continuous except possibly at its zeros,} \end{cases} \tag{1.2}$$

$$\{z > 0 : f(z) = 0\} = \{z_1, ..., z_k\},\ k \geq 1, \tag{1.3}$$

and for $i = 1, ..., k$,

$$\liminf_{s \searrow z_i} \frac{f(s)}{(s - z_i)^{\sigma_{N,p}}} \in (0, +\infty],\ \limsup_{s \nearrow z_i} \frac{f(s)}{(z_i - s)^{p-1}} < +\infty, \tag{1.4}$$

where

$$\sigma_{N,p} = (p-1)\frac{N}{N-p} \text{ if } N > p,$$

and $\sigma_{N,p}$ stands for an arbitrary number in $[1,\infty)$ if $N \leq p$.

Moreover, we assume

$$\text{either } f(0) > 0, \text{ or } f(0) = 0 \text{ and } \liminf_{s \searrow 0} \frac{f(s)}{s^{p-1}} > 0. \tag{1.5}$$

Let us note that since f is nonnegative, we automatically have

$$\liminf_{s \searrow z_i} \frac{f(s)}{(s - z_i)^{p-1}} \geq 0.$$

This and the second inequality in (1.4) guarantee that the ODE problem (1.6) below has at most one solution. The first inequality in (1.4) ensures that any bounded nonnegative solution of $\Delta_p u + f(u) = 0$ in $\mathbb{R}^N$ must be a constant (see Theorem 2.8 of [7]). This is not needed in Proposition 1.1 below, but is required in the other results.

Since $f(s) > 0$ for $s \in (0, +\infty) \setminus \{z_1, ..., z_k\}$, we automatically have

$$\int_0^z f(s)ds < \int_0^{z_i} f(s)ds \text{ for } z \in [0, z_i),\ i = 1, ..., k.$$

Hence by Theorems 2.2 and 2.4 of [7], we have the following result.

Proposition 1.1. *Let f satisfy (1.2), (1.3), (1.4) and (1.5). Then for every z_i, $i = 1, ..., k$, the problem*

$$\Delta_p V + f(V) = 0 \text{ in } \mathbb{R}_+,\ V(0) = 0,\ V(t) > 0 \text{ for } t > 0, V(\infty) = z_i \tag{1.6}$$

has a unique solution, which we denote by V_{z_i}. Moreover, $V_{z_i}(t)$ is a strictly increasing function.

Let us note that if $p = 2$ and $0 < z_1 < z_2 < ... < z_k$, then

$$f_1(u) = \Pi_{i=1}^k |u - z_i| \text{ and } f_2(u) = |u| f_1(u) \tag{1.7}$$

satisfy all the conditions (1.2), (1.3), (1.4) and (1.5).

Our first main result in this paper is the following:

Theorem 1.2. *Let f satisfy (1.2), (1.3), (1.4) and (1.5). Then for each $z \in \{z_1, ..., z_k\}$, (1.1) has a bounded positive solution u satisfying*

$$\lim_{x_1 \to \infty} u(x_1, x_2, ..., x_N) = V_z(x_2), \ \lim_{x_2 \to \infty} u(x_1, x_2, ..., x_N) = V_z(x_1). \tag{1.8}$$

Our next result shows that when $p = 2$, (1.1) has no other types of bounded positive solutions.

Theorem 1.3. *Suppose that f is as in Theorem 1.2 and $p = 2$. Let u be any bounded positive solution of (1.1) (with $p = 2$). Then it satisfies (1.8) for some $z \in \{z_1, ..., z_k\}$.*

If we assume further that there exists $\epsilon > 0$ small such that

$$f(s) \text{ is nonincreasing in } (z_i - \epsilon, z_i) \text{ for each } i \in \{1, ..., k\}, \tag{1.9}$$

then we have the following uniqueness and exact multiplicity result.

Theorem 1.4. *Suppose that, in addition to the conditions in Theorem 1.3, f satisfies (1.9). Then for each $z \in \{z_1, ..., z_k\}$, (1.1) (with $p = 2$) has exactly one bounded positive solution satisfying (1.8). Therefore (1.1) (with $p = 2$) has exactly k bounded positive solutions.*

We note that $f_1(u)$ and $f_2(u)$ given in (1.7) also satisfy (1.9).

Problem (1.1) with the boundary condition $u|_{\partial Q} = 0$ replaced by

$$u = 0 \,\text{for}\, x_2 = 0 \,\text{and}\, u \geq, \not\equiv 0 \,\text{for}\, x_1 = 0$$

was considered in [8] (for the case $p = 2$) and [7] (for the case $p > 1$). The main concern in these papers is the asymptotic limit of the solution as $x_1 \to \infty$; the question of uniqueness and exact multiplicity of bounded positive solutions was not discussed.

When $p = 2$, the existence of a positive solution of (1.1) (with $p = 2$) satisfying (1.8) was essentially proved in [9], where the special case $k = N = 2$ was considered. Problem (1.1) with $p = N = 2$ was also considered in [4], where it was assumed that f is C^1, $f(0) = 0$ and all the positive zeros of f are nondegenerate (i.e., $f(c) = 0$ and $c > 0$ imply $f'(c) \neq 0$), which forces f to change sign, and therefore the case treated in this paper is excluded in [4].

2 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. A key step is the following result.

Lemma 2.1. *With f as in Proposition 1.1, for each $z \in \{z_1, ..., z_k\}$ and any given small $\delta > 0$, there exists $R = R_\delta > 0$ and a function $v \in W_0^{1,p}(B)$, with $B = B_R := \{x \in \mathbb{R}^N : |x| < R\}$, satisfying*

(i) $\Delta_p v + f(v) \geq 0$ *in* B, $v = 0$ *on* ∂B,
(ii) $0 < v < z$ *in* B,
(iii) $v(x_1, x_2, ..., x_N) \leq \min\{V_z(x_1 + R + 1), V_z(x_2 + R + 1)\}$ *in* B,
(iv) $\sup_B v \geq z - \delta$.

Proof. To find such a function v we follow the construction in sub-step 2.1 of the proof of Theorem 3.1 in [7]. We provide the details for convenience of the reader. Since the zeros of f are isolated, we can find $0 < M_0 < z$ such that $f(s) > 0$ in $[M_0, z)$ and $M_0 > z - \delta$. Define

$$F_1(s) = \int_s^z f(t)dt.$$

Clearly $F_1(s) > 0$ in $[0, z)$. For any small $\epsilon > 0$, we consider

$$g(s) = g_\varepsilon(s) := f(s) - \varepsilon s^\sigma \text{ in } [0, z],$$

where $\sigma = \max\{1, \sigma_{N,p}\}$ in the case $f(0) = 0$, and $\sigma = 1$ when $f(0) > 0$. There exists $M_\varepsilon \in (M_0, z)$ such that $g(M_\varepsilon) = 0$ and $g(s) > 0$ in $[M_0, M_\varepsilon)$. Set

$$G(s) = G_\varepsilon(s) := \int_s^{M_\varepsilon} g(t)dt.$$

Clearly $G(s) > 0$ in $[M_0, M_\varepsilon)$, and $M_\varepsilon \to z$ as $\varepsilon \to 0$. Since $G_\varepsilon(s) \to F_1(s)$ uniformly in $[0, z]$ as $\varepsilon \to 0$, and $F_1(s) \geq F_1(M_0) > 0$ in $[0, M_0]$, we thus find that there exists $\varepsilon_0 > 0$ sufficiently small such that for each $\varepsilon \in (0, \varepsilon_0]$,

$$\begin{cases} M_\varepsilon - \varepsilon > M_0, \\ G_\varepsilon(s) > 0 \text{ in } [0, M_\varepsilon), \\ G_\varepsilon(s) \geq G_\varepsilon(M_\varepsilon - \varepsilon) \text{ for } s \in [0, M_\varepsilon - \varepsilon), \\ G_\varepsilon(s) \text{ is decreasing in } [M_0, M_\varepsilon). \end{cases}$$

Let us also notice that due to (1.4), we always have $f(s) > g_\varepsilon(s) > 0$ for small positive s, say $s \in (0, s_0)$, and s_0 can be chosen independent of $\varepsilon \in (0, \varepsilon_0]$.

Set

$$\tilde{g}(s) = \begin{cases} g(0) & \text{for } s < 0, \\ g(s) & \text{for } s \in [0, M_\varepsilon], \\ 0 & \text{for } s > M_\varepsilon, \end{cases}$$

and

$$\tilde{G}(s) = \int_s^{M_\varepsilon} \tilde{g}(t)dt.$$

Clearly $\tilde{G}(s) \geq 0$ for all $s \in \mathbb{R}$.

We now consider the functional

$$I_r(v) = \frac{1}{p} \int_{B_r(0)} |\nabla v|^p + \int_{B_r(0)} \tilde{G}(v)$$

for all $v \in H_0^p(B_r(0))$. It is well-known that a critical point of I_r corresponds to a weak solution of

$$\Delta_p v + \tilde{g}(v) = 0 \text{ in } B_r(0),\ v|_{\partial B_r(0)} = 0.$$

Since $\tilde{g} \geq 0$ in $(-\infty, 0]$ and $\tilde{g} = 0$ for $s \geq M_\varepsilon$, by the weak maximum principle, any such solution satisfies $0 \leq v \leq M_\varepsilon$. Consequently for any such solution we have $\tilde{g}(v) = g(v)$. Moreover, by elliptic regularity for p-Laplacian equations we know that such a solution also belongs to $C^{1,\alpha}(\overline{B}_r(0))$.

It is easily seen that the functional I_r is well-defined and is coercive. Thus by standard argument we know that it has a minimizer v_r, which is a critical point of I_r and thus, as discussed above, is a nonnegative solution to

$$\Delta_p v_r + g(v_r) = 0 \text{ in } B_r(0),\ v_r|_{\partial B_r(0)} = 0.$$

Since v_r is a minimizer, by well-known rearrangement theory it must be radially symmetric and decreasing away from the center of the domain. Thus $0 \leq v_r(x) \leq v_r(0) \leq M_\varepsilon$ in $B_r(0)$.

We claim that there exists $r > 0$ such that $v_r(0) \geq M_\varepsilon - \varepsilon$. Otherwise $v_r \leq M_\varepsilon - \varepsilon$ for all $r > 0$. Hence, recalling $G(s) \geq G(M_\varepsilon - \varepsilon)$ for $s \in [0, M_\varepsilon - \varepsilon]$, we obtain

$$I_r(v_r) \geq \int_{B_r(0)} G(v_r) \geq \int_{B_r(0)} G(M_\varepsilon - \varepsilon) = \alpha_N r^N G(M_\varepsilon - \varepsilon),\ \forall r > 0,$$

where α_N stands for the volume of $B_1(0)$. On the other hand, for $r > 1$ define

$$w_r(x) = \begin{cases} M_\varepsilon & \text{for } |x| < r - 1, \\ M_\varepsilon(r - |x|) & \text{for } r - 1 \leq |x| \leq r. \end{cases}$$

This function belongs to $W_0^{1,p}(B_r(0))$, and $|\nabla w_r|^p$ and $G(w_r)$ are supported on the annulus $\{r - 1 \leq |x| \leq r\}$. Thus there exists a constant C independent of r such that

$$I_r(w_r) \leq C[r^N - (r-1)^N]\ \ \forall r > 1.$$

Since v_r is the minimizer of I_r, we have $I_r(v_r) \leq I_r(w_r)$. Thus

$$\alpha_N G(M_\varepsilon - \varepsilon) r^N \leq C[r^N - (r-1)^N]\ \ \forall r > 1.$$

Since $G(M_\varepsilon - \varepsilon) > 0$, the above inequality does not hold for large r. This contradiction shows that $v_r(0) \geq M_\varepsilon - \varepsilon > z - \delta$ for all large r, say $r \geq R = R_\varepsilon,\ \varepsilon = \varepsilon(\delta)$.

Therefore if we take $v = v_R$ then

$$\Delta_p v + g(v) = 0 \text{ in } B_R,\ v = 0 \text{ on } \partial B_R,$$

and $v(0) = \sup_{B_R} v \in (z - \delta, z)$ provided that $\epsilon > 0$ is small enough. Thus v has properties (i), (ii) and (iv). (The fact that $v > 0$ in B_R follows from the strong maximum principle.)

It remains to prove (iii). We make use of the week sweeping principle (see Proposition 2.1 in [7]). Denote $u(x) = V_z(x_1)$ and let x^* be an arbitrary point in $[R+1, +\infty) \times \mathbb{R}^{N-1}$. Let $x^n = (x_1^n, 0, ..., 0)$ with $x_1^n \to +\infty$. Clearly $u(x^n) \to z_i$. Thus we can find a point $x^0 = (x_1^0, 0, ..., 0)$ with $x_1^0 > R+1$ such that $u(x) \geq M_\varepsilon$ in $B_R(x^0)$. We now define $x^t = tx^* + (1-t)x^0$ and $u^t(x) = u(x + x^t)$. Clearly $x_1^t \geq R+1$ and thus $B_R(x^t) \subset [1, +\infty) \times \mathbb{R}^{N-1}$ for all $t \in [0, 1]$.

Since $u > 0$ on the compact set $\cup_{t\in[0,1]}\overline{B_r(x^t)}$, we can find $\delta > 0$ such that $u \geq \delta$ on this set. Let $r_1 \in (0, R)$ be chosen so that $v \in (0, \delta/2)$ on $\{|x| = r_1\}$. Denote $\mathcal{D} := B_{r_1}(0)$. Then we have, for $t \in [0, 1]$,

$$v + \delta/2 \leq u^t \text{ on } \partial\mathcal{D}$$

and

$$-\Delta_p v = g(v) \leq f(v) - \zeta, \; -\Delta_p u^t = f(u^t) \text{ in } \mathcal{D},$$

where

$$\zeta := \inf_{x\in B_{r_1}(0)} [f(v(x)) - g(v(x))] > 0. \tag{2.1}$$

Moreover, $u^0 \geq M_\varepsilon \geq v$ on $\mathcal{D}$. Thus we can apply Proposition 2.1 in [7] to conclude that $u^t \geq v$ in $\mathcal{D}$ for all $t \in [0, 1]$. In particular, $u(x^* + x) = u^1(x) \geq v(x)$ in $\mathcal{D}$. Letting $r_1 \to R$ we obtain $u(x^* + x) \geq v(x)$ in B_R. Taking $x^* = (R+1, 0, ..., 0)$ yields

$$v(x) \leq V_z(x_1 + R + 1) \text{ for } x \in B_R.$$

Similarly we can take $u(x) = V_z(x_2)$ and use the weak sweeping principle to prove that

$$v(x) \leq V_z(x_2 + R + 1) \text{ for } x \in B_R.$$

The proof of the lemma is now complete. □

Proof of Theorem 1.2. Let $\delta > 0$ be small enough such that $f(s) > 0$ in $[z-\delta, z)$. Then let $R = R_\delta$ and v be given by Lemma 2.1. Fix $x_0 \in \mathbb{R}^N$ such that the ball $B_{R+1}(x_0) := \{x \in \mathbb{R}^N : |x - x_0| < R+1\}$ is contained in Q. Then define

$$v_{x_0}(x) = \begin{cases} v(x - x_0) & \text{if } x \in B_R(x_0), \\ 0 & \text{otherwise} \end{cases}$$

Since $f(0) \geq 0$, it is clear that v_{x_0} is a subsolution of (1.1). Define

$$\underline{u} = \sup_{B_{R+1}(x_0)\subset Q} v_{x_0}.$$

Then $\underline{u}$ is again a subsolution of (1.1), and it satisfies

$$\underline{u}(x) \geq z - \delta \text{ when } x_1 \geq R+1, \; x_2 \geq R+1. \tag{2.2}$$

Define

$$\overline{u} = \min\{V_z(x_1), V_z(x_2)\}.$$

Then $\overline{u}$ is a supersolution to (1.1), and by Lemma 2.1 and the monotonicity of V_z, we have $\overline{u} \geq \underline{u}$ in Q. Therefore we can apply the standard sub- and supersolution argument to conclude that (1.1) has a positive solution u satisfying

$$\underline{u} \leq u \leq \overline{u} \text{ in } Q.$$

By equation (3.21) in the proof of Theorem 3.7 in [7], we find that

$$\lim_{h\to\infty} u(x_1 + h, x_2 + h, ..., x_N) = m$$

and m is a positive zero of f. By (2.2) and the definition of $\overline{u}$, we necessarily have $m \in [z-\delta, z]$. It then follows from the choice of δ that $m = z$. On the other hand, we have $u \leq \overline{u} < z$ in Q. Therefore we are able to apply Remark 3.4 of [7] to the proof of Theorem 3.7 in [7] to conclude that

$$\lim_{x_1\to\infty} u(x_1, x_2, ..., x_N) = V_z(x_2)$$

uniformly for $(x_2, ..., x_N) \in \mathbb{R}_+ \times \mathbb{R}^{N-2}$. We similarly have

$$\lim_{x_2\to\infty} u(x_1, x_2, ..., x_N) = V_z(x_1)$$

uniformly for $(x_1, x_3, ..., x_N) \in \mathbb{R}_+ \times \mathbb{R}^{N-2}$. Thus (1.8) holds. □

3 Proof of Theorems 1.3 and 1.4

In this section we prove Theorems 1.3 and 1.4. We note that here we only need to consider the case $p = 2$.

Proof of Theorem 1.3. Under the conditions of Theorem 1.3, it is well known (see [6] and [1]) that the monotonicity condition for half-space solutions in Theorem 3.3 of [7] is automatically satisfied. Therefore, by Remark 3.8 in [7], for any bounded positive solution of (1.1) (with $p = 2$), we can apply Theorem 3.7 of [7] to conclude that

$$\lim_{h\to\infty} u(x_1 + h, x_2, ..., x_N) = V_{z_i}(x_2) \text{ uniformly in } [A, \infty) \times \mathbb{R}_+ \times \mathbb{R}^{N-2}$$

and

$$\lim_{h\to\infty} u(x_1, x_2 + h, ..., x_N) = V_{z_j}(x_1) \text{ uniformly in } \mathbb{R}_+ \times [A, \infty) \times \mathbb{R}^{N-2}$$

for every $A \in \mathbb{R}$, where $i, j \in \{1, ..., k\}$. Using the moving plan method as in [3], one deduces that u is symmetric about the hyperplane $x_1 = x_2$, and it is strictly increasing in any direction $\zeta = (\zeta_1, \zeta_2, ..., \zeta_N)$ with $\zeta_1 > 0$ and $\zeta_2 > 0$. It follows that $i = j$ in the above limits. Thus (1.8) holds with $z = z_i$. The proof is complete. □

Proof of Theorem 1.4. It suffices to show that for each $i \in \{1, ..., k\}$, (1.1) (with $p = 2$) has exactly one bounded positive solution u satisfying (1.8) with $z = z_i$. The existence is shown in Theorem 1.3. It remains to prove the uniqueness.

From equation (3.21) in [7], and the symmetry of u with respect to the hyperplane $x_1 = x_2$, we obtain that

$$\lim_{d(x)\to\infty} u(x) = z_i,$$

where $d(x)$ denotes the distance of $x \in Q$ to ∂Q. As observed above, by the moving plan method $u(x)$ is strictly increasing in any direction $\zeta = (\zeta_1, \zeta_2, ..., \zeta_N)$ with $\zeta_1 > 0$ and $\zeta_2 > 0$. In particular, it is increasing in the direction $\zeta_0 = (1, 1, 0, ..., 0)$.

We may now use the sliding method in the direction ζ_0 as in Sect. 5 of [2] to prove the uniqueness of u. For completeness, we give the details below.

Firstly, making use of (1.5) we can show that any positive solution u satisfying (1.8) with $z = z_i$ has the property that, for every $\tau > 0$,

$$\inf_{x\in Q, d(x)\geq\tau} u(x) > 0, \quad \sup_{x\in Q, d(x)\leq\tau} u(x) < z_i. \tag{3.1}$$

Now suppose that u_1 and u_2 are two positive solutions satisfying (1.8) with $z = z_i$. For τ, $\sigma \geq 0$ we define

$$Q_\sigma := Q + \sigma\zeta_0 = \{x \in Q : x = y + \sigma\zeta_0 \text{ for some } y \in Q\}$$

and

$$u_1^\tau(x) := u_1(x + \tau\zeta_0).$$

Fix large σ so that $u_1(x), u_2(x) \geq z_i - \epsilon$ when $x \in Q_\sigma$, where $\epsilon > 0$ appears in (1.9). We may then choose $\tau > 0$ large to ensure that $u_1^\tau(x) > u_2(x)$ on $Q \setminus Q_\sigma$. It then follows from (1.9) and the maximum principle (see Lemma 2.1 in [2]) that

$$u_1^\tau(x) \geq u_2(x) \text{ for } x \in Q_\sigma.$$

We thus obtain

$$u_1^\tau(x) \geq u_2(x) \text{ for } x \in Q \text{ and all large } \tau > 0.$$

Define

$$\tau_* := \inf\{\tau > 0 : u_1^\tau(x) \geq u_2(x) \text{ for } x \in Q \setminus Q_\sigma\}.$$

We want to show that $\tau_* = 0$. If this is proved, then $u_1 \geq u_2$ in $Q \setminus Q_\sigma$ and using Lemma 2.1 of [2] as above we deduce $u_1 \geq u_2$ in Q. We may similarly show $u_2 \geq u_1$. Hence $u_1 \equiv u_2$ and the required uniqueness is established.

So to complete the proof, it suffices to show $\tau_* = 0$. Arguing indirectly we assume that $\tau_* > 0$. By the definition of τ_* we have $u_1^{\tau_*} \geq u_2$ in $Q \setminus Q_\sigma$, and there exists a sequence $\tau_n \nearrow \tau_*$ and $x_n \in Q \setminus Q_\sigma$ such that

$$u_1^{\tau_n}(x_n) < u_2(x_n) \text{ for all } n \geq 1. \tag{3.2}$$

Applying Lemma 2.1 of [2] again we obtain $u_1^{\tau_*} \geq u_2$ in Q. In view of the monotonicity of u_1 and (3.1), we have

$$u_2(x_n) > u_1^{\tau_n}(x_n) \geq u_1^{\tau_*/2}(x_n) \geq \inf_{x \in \partial Q} u_1^{\tau_*/2} > 0$$

for all large n, and hence the sequence $\{x_n\}$ is bounded away from ∂Q. Define

$$Q_n := Q - x_n = \{x : x + x_n \in Q\}$$

and for $\phi \in \{u_1, u_2\}$, $x \in Q_n$, set

$$\phi_n(x) := \phi(x + x_n).$$

It can be easily shown that by passing to a subsequence, Q_n converges to some $\tilde{Q}$ which is either a quarter space or half space in R^N, and $u_{1n} \to \tilde{u}_1$, $u_{2n} \to \tilde{u}_2$ in $C^2_{loc}(\tilde{Q})$, and for $i = 1, 2$,

$$\Delta \tilde{u}_i + f(\tilde{u}_i) = 0, \ \tilde{u}_1(\cdot + \tau_* \zeta_0) \geq \tilde{u}_2 \text{ in } \tilde{Q}.$$

By (3.2) we deduce $\tilde{u}_1(\tau_* \zeta_0) \leq \tilde{u}_2(0)$ and so necessarily $\tilde{u}_1(\tau_* \zeta_0) = \tilde{u}_2(0)$. Since $0 \in \tilde{Q}$ (due to $\{x_n\}$ being bounded away from ∂Q) the strong maximum principle infers $\tilde{u}_1(\cdot + \tau_* \zeta_0) \equiv \tilde{u}_2$ in $\tilde{Q}$. On the other hand, by (3.1) we have $\tilde{u}_1(x + \tau_* \zeta_0) - \tilde{u}_2(x) > 0$ for all $x \in \tilde{Q}$ close to $\partial \tilde{Q}$. Thus $\tau_* > 0$ leads to a contradiction. Hence we must have $\tau_* = 0$, as we wanted. □

Remark 3.1. It is unclear whether the conclusions in Theorems 1.3 and 1.4 remain valid for $p \neq 2$. The proof for the $p = 2$ case relies on the use of the strong comparison principle. A general strong comparison principle is lacking when $p \neq 2$. However, under various restrictions on p and on the nonlinear function $f(u)$, some strong comparison principles for p-Laplacian equations of the form $\Delta_p u + f(u) = 0$ are known; see [5,10,11] and the references therein.

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