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A Note on Fixed Point Theorem for Fuzzy Mappings

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This paper explores Heilpern's notions of fuzzy mapping and the fixed point theorem for fuzzy mappings. The fixed point theorem for fuzzy mappings as introduced by Heilpern has been generalized and some characterizations are done in this context.

Keywords: Fuzzy mapping, fixed point theorem for fuzzy mapping, Hausdorff distance, metric linear space.

1. Introduction

The notion of fuzzy sets [6] relates to objects with unsharp boundaries and the degrees of membership plays a crucial role in describing them. In case of fuzzy sets, we allow the truths of propositions to be expressed in terms of degrees which offers a considerably broader framework for knowledge representation while dealing with uncertainty.

In his paper [4], Heilpern introduced the concept of a fuzzy mapping which neccesarily implies a mapping from an arbitrary set to a subfamily of fuzzy sets in a metric linear space X. Heilpern further proved a fixed point theorem for such fuzzy mappings as a generalization of the fixed point theorem for point-to-set maps [2] arising from the set representation of fuzzy sets [5]. The present paper extends Heilpern's result and a generalized fixed point theorem is proved. In this context, some further characterizations are also done.

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2. Preliminaries

In this section we briefly present the preliminaries which mostly includes the work of Heilpern [4]. These concepts will be required for the sake of completeness of this paper.

Definition 1. A fuzzy set A is said to be an approximate quantity if and only if A_α is compact and convex for each $\alpha \in [0,1]$ and $\sup\limits_{x\in X} A(x) = 1$.

In case A is an approximate quantity and $A(x_0) = 1$ for some $x_0 \in X$, then A is identified with an approximation of x_0.

We consider $\mathcal{F}(X)$ to be the collection of all fuzzy sets in X and we also consider $W(X)$ to be a subcollection of all approximate quantities. Then the notion of distance between two approximate quantities is defined in [4] as below:

Definition 2. Let $A, B \in W(X), \quad \alpha \in [0,1]$. Then it is defined

$$\begin{aligned} p_\alpha(A,B) &= \inf_{x\in A_\alpha, y\in B_\alpha} d(x,y), \\ D_\alpha(A,B) &= H(A_\alpha, B_\alpha) \end{aligned} \tag{1}$$

H being the Hausdorff distance.

$$\begin{aligned} D(A,B) &= \sup_\alpha D_\alpha(A,B), \\ p(A,B) &= \sup_\alpha p_\alpha(A,B). \end{aligned} \tag{2}$$

where the function p_α is called an α-space, D_α an α-distance, and D a distance between A and B (It should be noted that P_α is a non-decreasing function of α).

Definition 3. Let us consider $A, B \in W(X)$. Then A is said to be more accurate than B, denoted by $A \subset B$, if and only if $A(x) \leq B(x)$ for each $x \in X$. The relation $\subset$ induces a partial order on the family $W(X)$.

Definition 4. Let us consider X be an arbitrary set and Y be any metric linear space. F is called a fuzzy mapping if and only if F is a mapping from the set X into $W(Y)$.

A fuzzy mapping F is a fuzzy subset on $X \times Y$ with membership function $F(x, y)$. The functional value $F(x, y)$ represents the grade of membership of y in $F(X)$.

We consider $A \in \mathcal{F}(X), B \in \mathcal{F}(Y)$. Fuzzy set $F(A)$ in $\mathcal{F}(Y)$ is defined by

$$F(A)(y) = \sup_{x \in X}(F(x,y)\hat{A}(y)), \quad y \in Y. \tag{3}$$

Fuzzy set $F^{-1}(B)$ in $\mathcal{F}(X)$ is defined by

$$F^{-1}(B)(x) = \sup_{y \in Y}(F(x,y)\hat{B}(y)), \quad x \in X. \tag{4}$$

Lemma 1. *Let us consider $x \in X$, $A \in W(X)$ and $\{x\}$ be a fuzzy set with membership function equal to the characteristic function of the set $\{x\}$. If $\{x\} \subset A$, then $\rho_\alpha(x, A) = 0$ for each $\alpha \in [0, 1]$.*

Lemma 2. *For any $x, y \in X, p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A)$.*

Lemma 3. *If $\{x_0\} \subset A$, then $p_\alpha(x_0, B) \leq D_\alpha(A, B)\ \forall\ B \in W(X)$.*

3. Fixed Point Theorem

In this section, we shall prove a fixed point theorem that generalizes Heilpern's theorem [4].

Theorem 1.

Consider (X, d) to be a complete metric linear space and consider F to be a fuzzy mapping from X into $W(X)$ satisfying the following condition:

$$D(F(x), F(y)) \leq \phi(d(x, y)) \quad \textit{for any} x, y \in X \tag{5}$$

where ϕ represents a function which is upper semicontinuous from right from R^+ into itself such that $\phi(t) < t \quad \forall\ t > 0$. Then there exists $z \in X$ such that $\{z\} \subset F(z)$.

Proof. We consider $x_0 \in X$ and $\{x_1\} \subset F(x_0)$. Then clearly $\exists\ x_2 \in X$ such that $\{x_2\} \subset F(x_1)$ and $d(x_2, x_1) \leq D_1(F(x_1), F(x_0))$. By Continuing in this way, we produce a sequence (x_n) in X such that

$$\{x_n\} \subset F(x_{n-1}) \quad \text{and} \quad d(x_n, x_{n+1}) \leq D_1(F(x_{n-1}), F(x_n)) \quad \forall n \in N. \tag{6}$$

If $d_n = d(x_n, x_{n+1})$, then for $n > 1$

$$\begin{aligned} d_n \leq D_1(F(x_{n-1}), F(x_n)) &\leq D(F(x_{n-1}), F(x_n)) \\ &\leq \phi(d_{n-1}) \\ &< d_{n-1}. \end{aligned} \tag{7}$$

which implies that the sequence (d_n) is convergent.

Let us consider $d = \lim_{n\to\infty} d_n$. If $d > 0$, then clearly from above we are able to find that $d \leq \phi(t)$ which leads to a contradiction. Hence $d = 0$.

After this, we shall prove that (x_n) is a Cauchy's sequence. On the contrary let us assume that (x_n) is not a Cauchy's sequence. Hence, for $\varepsilon > 0$, and for each positive integer K, there exists integers $n(K)$ and $m(K)$ with $K \leq m(K) \leq n(K)$ such that

$$r_K = d(x_{m(K)}, x_{n(K)}) \geq \varepsilon. \tag{8}$$

In this case, without loss of generality we assume that $n(K)$ is the smallest possible integer greater than $m(K)$ satisfying the above inequality.
Thus we have

$$\begin{aligned} \varepsilon \leq r_k &\leq d(x_{m(K)}, x_{n(K)-1}) + d(x_{n(K)-1}, x_{n(K)}) \\ &\leq \varepsilon + d_{n(K)-1} \end{aligned} \tag{9}$$

which necessarily implies $\lim_{k\to\infty} r_k = \varepsilon$.

$$\begin{aligned} \varepsilon \leq r_k &\leq d(x_{m(K)}, x_{m(K)+1}) + d(x_{m(K)+1}, x_{n(K)+1}) + d(x_{n(K)+1}, x_{n(K)}) \\ &\leq d_{m(K)} + \phi(r_k) + d_{n(K)} \\ &\to \quad \phi(\varepsilon) \quad \text{as} \quad K \to \infty \end{aligned} \tag{10}$$

which represents a contradiction. Hence, (x_n) is a Cauchy's sequence.

Let $\lim_{n\to\infty} x_n = z$.

$$\begin{aligned} p_0(z, F(z)) &\leq d(z, x_n) + p_0(x_n, F(z)) \\ &\leq d(z, x_n) + D_0(F(x_{n-1}), F(z)) \\ &\leq d(z, x_n) + \phi(d(x_{n-1}, z)) \\ &< d(z, x_n) + d(x_{n-1}, z) \\ &\to 0 \quad \text{as} \quad n \to \infty \end{aligned} \tag{11}$$

Thus from Lemma 1, it necessarily follows that $\{z\} \subset F(z)$. Hence the theorem. □

Remark 1. If $\phi(t) = qt$ where $q \in (0, 1)$, then in this case we are able to find Heilpern's result.

Theorem 2.

Consider (X, d) to be a complete metric linear space and consider F to be a fuzzy mapping from X into $W(X)$ satisfying the following condition:

$\exists\, q \in (0, 1)$ such that $D(F(x), F(y)) \leq q.d(x, y)$ where $x, y \in X$. Then there exists $z \in X$ such that $\{z\} \subset F(z)$.

The following theorem generalizes the above theorems further:

Theorem 3. *Consider (X,d) to be a complete metric linear space and consider F to be a fuzzy mapping from X into $W(X)$ satisfying the following condition for any $x, y \in X$:*

$$D(F(x), F(y)) \leq \phi\{p(x, F(x)), p(y, F(y)), p(y, F(x)), p(x, F(y)), d(x, y)\} \quad (12)$$

where $\phi : [R^+]^5 \longrightarrow R^+$ represents a continuous function non-decreasing in each co-ordinate variable such that

$$\phi(t, t, a_1 t, a_2 t, t) < t \quad \forall \ t > 0, \quad (13)$$

where $a_i \in \{0, 1, 2\}$ with $a_1 + a_2 = 2$. Then there exists $z \in X$ such that $\{z\} \subset F(z)$.

Theorem 4. *Let (X,d) be a complete metric linear space and let F be a fuzzy mapping from X into $W(X)$ satisfying the following condition:*

$$\begin{aligned} D(F(x), F(y)) \leq\ & a_1 p(x, F(x)) + a_2 p(y, F(y)) + a_3 p(x, F(y)) \\ & + a_4 p(y, F(x)) + a_5 d(x, y) \end{aligned} \quad (14)$$

for any $x, y \in X$ where a_i's are non-negative real numbers such that $\sum a_i < 1$ and $a_1 = a_2$ or $a_3 = a_4$. Then $\exists\ z \in X$ such that $\{z\} \subset F(z)$.

Remark 2. It should be noted that Theorem 3 follows from [1], and if $a_1 = a_2 = a_3 = a_4 = 0$, then Theorem 4 produces Heilpern's result.

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