Shearfree Lorentzian geometry and CR geometry

By

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Declaration

I certify that the substance of the thesis has not already been submitted for and degree and is not currently being submitted for any other degree.

I certify that to the best of my knowledge, any help received in preparing this thesis, and all sources used, have been acknowledged in this thesis.

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Abstract

We introduce a CR-invariant class of Lorentzian metrics on a circle bundle over a 3-dimensional CR-structure, which we call FRT-metrics. These metrics generalise the Fefferman metric, allowing for more control of the Ricci curvature, but are more special than the shearfree Lorentzian metrics introduced by Robinson and Trautman. Our main result is a criterion for embeddability of 3-dimensional CR-structures in terms of the Ricci curvature of the FRT-metrics in the spirit of the results by Lewandowski et al. in [37] and also Hill et al. in [25]. We also study higher dimensional versions of shearfree null congruences in conformal Lorentzian manifolds. We show that such structures induce a subconformal structure and a partially integrable almost CR structure on the leaf space and we classify the Lorentzian metrics that induce the same subconformal structure.

Contents

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Introduction

This thesis concerns the relation between geometric properties of Lorentzian spaces and CR manifolds of hypersurface type. The most famous example of Lorentzian manifold is the 4-dimensional spacetime from Einstein's relativity theory. We are interested in the curved version of that, specifically, in the properties of its geometric curvature and the existence of certain solutions of Maxwell's equations.

On the other hand, a CR manifold (M, D, J) is a manifold M of dimension $2n+1$ with a rank $2n$ contact distribution D, and complex structure J on D that satisfies certain integrability conditions. Such CR manifolds naturally occur as boundaries of domains in complex manifolds.

One of the most crucial problems is the embeddability problem which asks if an abstractly given CR manifold (M, D, J) is locally realisable as a real hypersurface in $\mathbb{C}^{n+1}$. It is well-known that 3-dimensional CR manifolds are "almost never embeddable", unless they are real analytic or possess certain symmetry.

It is a notoriously hard problem to give criteria, when a CR manifold is locally embeddable. In the 1960's, physicists discovered an intimate relationship between the embeddability problem of a 3-dimensional CR manifold and geometric properties of an associated 4-dimensional spacetime.

Subsequently, Hill et al. [25] proved the following criterion of embeddabiltiy

Theorem [25] Let M be a sufficiently smooth strictly pseudoconvex 3-dimensional CR manifold. It is locally CR embeddable as a hypersurface in $\mathbb{C}^2$ if and only if:

1. it admits a lift to a spacetime whose complexified Ricci tensor vanishes on the corresponding distribution of α-planes, and
2. it admits a non-trivial null Maxwell field aligned with the null congruence of shearfree geodesics corresponding to the CR structure on M.

The principal aim of this thesis is to clarify the nature of the above result, we succeeded in proving an embeddability criterion that uses a more special class of metrics, but does not require the a priori assumption of the existence of an aligned Maxwell field. Our result has been published in [61].

We have introduced the family of FRT metrics (in honour of Fefferman, Robinson and Trautman) specifically for our criterion. The family of FRT metrics is more general than the celebrated conformal Fefferman metric from CR geometry but more special than the family of shearfree metrics introduced by I. Robinson and A. Trautman [52, 53].

Another principal achievement of the thesis are the theorems 4.2.3 and 4.2.4, which clarify the relation between CR manifolds, subconformal manifolds and Lorentzian manifolds in higher dimensions. The results have been published in [3].

This thesis aims to survey the pioneering works on the relation between shearfree Lorentzian geometry and CR geometry. For the sake of a self-contained exposition we give an introduction to pseudo Riemannian geometry, affine connections, Cartan's structural equations, Hodge dual operator, Maxwell's equations, Beltrami equation.

We cite results and proofs from original papers whenever it is essential for the understanding of the presented material, in several instances we have added proofs of known results which did not seem to be readily available in the literature.

The thesis is organised as follows. In the preliminary chapter we collect some well-known fundamental facts about pseudo Riemannian geometry, which will be used throughout the thesis. In particular, we invoke the Nomizu operator, which simplifies some of the proofs on Lorentzian metrics with shearfree vector fields.

In addition, we explain the notation of electromagnetic fields as wedge products of elements of admissible coframe defined on the Lorentzian manifold.

Chapter 1 starts with defining contact manifolds. We then characterise the Reeb vector fields on contact manifolds as infinitesimal automorphisms of the contact distribution (Proposition 1.1.2).

Next we introduce Cartan's approach to CR structures, which enables us to define the shearfree metrics and FRT metrics in the proceeding chapters.

In Chapter 2, the embedding problem for 3-dimensional CR manifolds is discussed. In 1982, M. Kuranishi [32] proved that smooth CR manifolds of dimension $2n + 1$, with $n \geq 4$, are embeddable. After that, T. Akahori [1] showed that it is also true for $n = 3$.

However, CR manifolds of dimension 3, that is when $n = 1$, are rarely embeddable. L. Nirenberg gave the first counterexample in [44, 45]. See Chapter 2 of the present work and [5] for more details.

Based on Theorem 2.1.6 by H. Jacobowitz, we give criteria for the embeddability of 3-dimensional CR manifolds. We also compute the Fefferman metric explicitly for general 3-dimensional CR manifolds following the approach [29, 47], but without assuming that a non-constant CR function exists.

Chapter 3 is the main part of the thesis. The main result of this chapter is a criterion for local embeddability of 3-dimensional CR manifolds mentioned above.

In particular, we define the family of FRT metrics on a circle bundle and show that it is CR invariant.

In Chapter 4, we study higher dimensional versions of shearfree null congruences in conformal Lorentzian manifolds. We define the RT-spaces and show that such generalisations of shearfree null congruences induce a subconformal structure and a partially integrable almost CR structure on the leaf space.

Chapter 0

Preliminaries on pseudo-Riemannian geometry

The goal of this chapter is to provide an introduction to pseudo-Riemannian geometry and some facts in manifold theory which will be needed later and to fix our terminology. The topics discussed include Lorentzian geometry, structural equations, Nomizu operator and electromagnetic fields. In the first section we recall some fundamental theorems and definitions in pseudo-Riemannian geometry(Lorentzian geometry), which will be used throughout the thesis [24, 49, 50, 35, 19].

0.1 Pseudo-Riemannian geometry

In this section we collect some preliminaries about pseudo-Riemannian geometry, which are used throughout the thesis.

0.1.1 Pseudo-Riemannian metric

A pseudo-Riemannian metric is defined by $(0, 2)$ tensors on the tangent spaces. Hence, to define the pseudo-Riemannian metric in general, let V be a real vector space of finite dimension n. A symmetric bilinear form b on

V is an $\mathbb{R}$–bilinear mapping $b : V \times V \to \mathbb{R}$ such that $b(v, w) = b(w, v)$ for all $v, w \in V$.

Definition 0.1.1 *A symmetric bilinear form b on V is*

1. *positive (negative) definite, provided $v \neq 0$ implies $b(v, v) > 0 (< 0)$,*
2. *nondegenerate, provided $g(v, w) = 0$ for all $w \in V$ implies $v = 0$.*

A vector $u \in V$ is called a unit vector, if $g(u, u) = \pm 1$. Two vectors u, v are orthogonal if $g(u, v) = 0$. A set of n mutually orthogonal unit vectors is defined to be an *orthonormal basis* of V. Similar to the inner product space, i.e. the vector space equipped with a positive definite symmetric bilinear form, the scalar product space, that is, the vector space furnished with a symmetric nondegenerate bilinear form, possesses an orthonormal basis.

Definition 0.1.2 *The index ν of a symmetric bilinear form b on V is the largest integer that is the dimension of a vector subspace $W \subseteq V$ where $b|_{W \times W} : W \times W \to \mathbb{R}$ is negative definite.*

We now have the following lemma

Lemma 0.1.3 *[49] Let (V, g) be a scalar product. For any orthonormal basis $(e_1, \ldots, e_n)$ for V, the number of negative signs in the signature $(\epsilon_1, \ldots, \epsilon_n)$ is the index ν of V where $\epsilon_i = g(e_i, e_i)$.*

Proof Suppose for the first m vectors $e_1, \ldots, e_m$ we have that $\epsilon_i < 0$, $i = 1, \ldots m$. The inner product g is negative definite on the subspace S where

$$S = \text{linear span}\{e_1, \ldots, e_m\}.$$

Therefore, $\nu \geq m$. Let W be a maximal subspace of V on which g is negative definite with $\dim = m$. The linear map $L : W \to S$ defined by

$$L(w) = \sum_{j \leq m} g(w, e_j) e_j, \quad w \in W$$

is one-to-one, since on one hand, $g(w,w) \leq 0$ for $w \in W$. On the other hand, suppose $w \in \ker L$. For the indices $j \leq m$ it implies that $g(w,e_j) = 0$. Moreover, since $L(w) = 0$ it implies that

$$g(w,w) = \sum_{j\leq m} w^j g(w,e_j) + \sum_{j>m} w^j g(w,e_j) = \sum_{j>m} w^j g(w,e_j) = \sum_{j>m} g(w,e_j)^2 = 0,$$

which implies $w = 0$, where $w = \sum_{j=1}^{n} w^j e_j$. We note that $w^j = g(w,e_j)$ for $j > m$. ∎

Now we are in the position to define the pseudo-Riemannian metric.

Definition 0.1.4 *A pseudo-Riemannian metric g on a smooth manifold $\mathcal{M}$ is an assignment which smoothly assigns to each point $p \in \mathcal{M}$ a symmetric nondegenerate bilinear form g_p on the tangent space $T_p\mathcal{M}$, such that the index of g_p is the same for all $p \in \mathcal{M}$.*

The pair $(\mathcal{M}, g)$ is called a *pseudo-Riemannian* manifold. The pair $(\mathcal{M}, g)$ is called *Riemannian* manifold, if the index $\nu = 0$. If $\nu = 1$, the pair $(\mathcal{M}, g)$ is a *Lorentzian* manifold.

0.1.2 Affine connection

We also recall the definition of affine connection for any manifold $\mathcal{M}$. The set of all smooth sections of the tangent bundle is denoted by $\Gamma(T\mathcal{M})$.

Definition 0.1.5 *An affine connection on a manifold $\mathcal{M}$ is a rule which assigns to each $X \in \Gamma(T\mathcal{M})$ a mapping*

$$\nabla_X : \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M})$$

satisfying the following two properties:

(i) $\nabla_{fX+gY} = f\nabla_X + g\nabla_Y$,

(ii) $\nabla_X(fY) = f(\nabla_X Y) + (Xf)Y$,

for $f, g \in C^\infty(\mathcal{M})$ and $X, Y \in \Gamma(T\mathcal{M})$. The operator ∇_X is called covariant derivative with respect to X.

Before we go any further, we state the fundamental theorem in pseudo-Riemannian geometry, which plays an important role in other parts of the thesis.

Theorem 0.1.6 *[49] On a pseudo-Riemannian manifold $\mathcal{M}$ there is a unique connection ∇ such that*

$$[V, W] = \nabla_V W - \nabla_W V, \tag{0.1a}$$

$$Xg(V, W) = g(\nabla_X V, W) + g(V, \nabla_X W), \tag{0.1b}$$

for all $X, V, W \in \Gamma(T\mathcal{M})$. The connection ∇ is called the Levi-Civita connection of $\mathcal{M}$.

The so-called *Koszul* formula below is a direct consequence of (0.1a) and (0.1b) of the above theorem.

$$\begin{aligned} 2g(\nabla_V W, X) = {} & Vg(W, X) + Wg(X, V) - Xg(V, W) - g(V, [W, X]) \\ & + g(W, [X, V]) + g(X, [V, W]). \end{aligned} \tag{0.2}$$

We set

$$\begin{aligned} T(X, Y) &= \nabla_X Y - \nabla_Y X - [X, Y] \\ R(X, Y) &= \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \end{aligned}$$

for all $X, Y \in \Gamma(T\mathcal{M})$. The tensor fields T and R are called *torsion* and *curvature* tensor fields respectively. Let $p \in \mathcal{M}$ and suppose $(e_1, \ldots, e_n)$ is a basis for the vector fields in a neighborhood of p. We define the functions $\Gamma_{ij}^k, T_{ij}^k, R_{lij}^k$ by the formulas

$$\begin{aligned} \nabla_{e_i} e_j &= \sum_k \Gamma_{ij}^k e_k, \\ T(e_i, e_j) &= \sum_k T_{ij}^k e_k, \\ R(e_i, e_j) e_l &= \sum_k R_{lij}^k e_k. \end{aligned}$$

We note from (0.1a) that the Levi-Civita connection is torsion-free, that is, the torsion tensor vanishes or equivalently

$$T^i_{jk} = 0, \quad i, j, k = 1, \dots, n$$

is satisfied.

Suppose θ^i form a dual frame to e_i, i.e., $\theta^i(e_j) = \delta^i_j$. Then Γ^i_j are determined by

$$\Gamma^i_j = \Gamma^i_{kj}\theta^k, \quad i, j, k = 1, \dots, n, \tag{0.3}$$

and therefore, $\Gamma^i_{jk} = \Gamma^i_k(e_j)$. The 1-forms Γ^i_j are called *connection forms*. The structural equations which will be explained below show the relation between the 1-forms Γ^i_j and the tensor fields T and R.

0.2 The structural equations

The structural equations of Cartan play an important role in the computation of the components of the Ricci curvature in the next chapters.

Let $\mathcal{M}$ be an n-dimensional manifold with an affine connection ∇.

Theorem 0.2.1 (Cartan) *[24] For the 1-forms θ^i, Γ^i_j and tensor fields T, R the following statements are fulfilled:*

$$d\theta^i = -\Gamma^i_p \wedge \theta^p + \frac{1}{2}T^i_{jk}\theta^j \wedge \theta^k, \tag{0.4a}$$

$$d\Gamma^i_j = -\Gamma^i_p \wedge \Gamma^p_j + \frac{1}{2}R^i_{jk\ell}\theta^k \wedge \theta^\ell. \tag{0.4b}$$

The next fact relates the metric defined on the manifold to the connection forms.

Proposition 0.2.2 *Let $(\mathcal{M}, g)$ be a pseudo-Riemannian manifold of dimension n and $(e_1, \dots, e_n)$ be a local frame for the vector fields on $\mathcal{M}$. Then,*

$$dg_{ij} = g_{ik}\Gamma^k_j + g_{jk}\Gamma^k_i \tag{0.5}$$

is satisfied, where $g_{ij} = g(e_i, e_j)$ and Γ^i_j as was introduced by (0.3).

We provide a proof, analogous to the proof for Riemannian manifolds in [19].

Proof Let $Y \in \Gamma(T\mathcal{M})$ be any arbitrary vector field. We first note that

$$dg_{ij}(Y) = Y(g(e_i, e_j)).$$

On the other hand,

$$\begin{aligned}
(g_{ik}\Gamma^k_j + g_{jk}\Gamma^k_i)(Y) =& g\left(\Gamma^k_j(Y)e_k, e_i\right) + g\left(\Gamma^k_i(Y)e_k, e_j\right) \\
&= g\left(y_1\Gamma^k_j(e_1)e_k + \cdots + y_n\Gamma^k_j(e_n)e_k, e_i\right) \\
&\quad + g\left(y_1\Gamma^k_i(e_1)e_k + \cdots + y_n\Gamma^k_i(e_n)e_k, e_j\right) \\
&= g\left(y_1\Gamma^k_{1j}e_k + \cdots + y_n\Gamma^k_{nj}e_k, e_i\right) \\
&\quad + g\left(y_1\Gamma^k_{1i}e_k + \cdots + y_n\Gamma^k_{ni}e_k, e_j\right) \\
&= g\left(y_1\nabla_{e_1}e_j + \cdots + y_n\nabla_{e_n}e_j, e_i\right) \\
&\quad + g\left(y_1\nabla_{e_1}e_i + \cdots + y_n\nabla_{e_n}e_i, e_j\right) \\
&= g\left(\nabla_Y e_j, e_i\right) + g\left(\nabla_Y e_i, e_j\right) = Y(g(e_i, e_j)) \\
&= dg_{ij}(Y). \quad \blacksquare
\end{aligned}$$

We use the metric to lower and raise the indices. For example for the connection forms Γ^i_j, we have $g_{ik}\Gamma^k_j = \Gamma_{ij}$.

0.2.1 Frobenius Theorem

The Frobenius theorem plays a crucial role in the next chapters. For a detailed proof of this famous theorem, refer to, e.g. [18, 7].

Theorem 0.2.3 (Frobenius) *Let $\theta^1, \dots, \theta^r$ be 1-forms in $\mathbb{R}^n$, which are linearly independent at a point $p \in \mathbb{R}^n$. Suppose there are 1-forms ω^i_j satisfying*

$$d\theta^i = \sum_{j=1}^{r} \omega^i_j \wedge \theta^j, \quad i = 1, \dots, r.$$

Then, there are functions f^i_j, x^j, $j = 1, \dots, r$ satisfying

$$\theta^i = \sum_{j=1}^{r} f^i_j dx^j.$$

We finalise this section by recalling the following lemma, and for the convenience of the readers, the proof is provided.

Lemma 0.2.4 *[35] Let M be a smooth manifold of dimension n. Take $(e_1, \dots e_n)$ to be a local frame for M with corresponding dual coframe $(\theta^1, \dots, \theta^n)$, and, furthermore,*

$$[e_i, e_j] = c^l_{ij}\, e_l, \qquad l = 1, \cdots, n.$$

Then, the exterior derivative of each of the 1-forms θ^i is given by

$$d\theta^k = -c^k_{ij}\, \theta^i \wedge \theta^j, \qquad i < j.$$

Proof For the proof we note that $d\theta^k$ is a 2-form and can be expressed as

$$d\theta^k = b^k_{ij}\, \theta^i \wedge \theta^j, \quad k = 1, \cdots, n, \quad i < j,$$

and therefore,

$$d\theta^k(e_i, e_j) = b^k_{ij}\, \theta^i \wedge \theta^j(e_i, e_j) = \frac{1}{2} b^k_{ij}.$$

On the other hand,

$$\begin{aligned} d\theta^k(e_i, e_j) &= \frac{1}{2}\{e_i\theta^k(e_j) - e_j\theta^k(e_i) - \theta^k([e_i, e_j])\} = -\frac{1}{2}\theta^k([e_i, e_j]) \\ &= -\frac{1}{2} c^l_{ij}\, \theta^k(e_l) = -\frac{1}{2} c^k_{ij}. \end{aligned}$$

So, $b^k_{ij} = -c^k_{ij}$. ∎

0.3 Nomizu operator

The Nomizu operator is an important tool which relates the differential geometry to algebra and is defined as follows.

Definition 0.3.1 *Let $(\mathcal{M}, g)$ be a Lorentzian manifold with the Levi-Civita connection ∇. For any smooth vector field $X \in \Gamma(T\mathcal{M})$, the Nomizu operator L_X is defined as*

$$\begin{aligned} &L_X : \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M}) \\ &\qquad L_X Y := -\nabla_Y X, \qquad \forall Y \in \Gamma(T\mathcal{M}). \end{aligned}$$

The g-adjoint L_X^* is the operator defined by

$$g(L_X^* V, W) = g(V, L_X W) \quad \forall X, V, W \in \Gamma(T\mathcal{M}).$$

The symmetric and skew-symmetric parts of the Nomizu operator are denoted by L_X^s and L_X^a respectively. That is,

$$L_X^s := \frac{1}{2}(L_X + L_X^*), \qquad L_X^a := \frac{1}{2}(L_X - L_X^*).$$

The Nomizu operator has some known facts listed in the following proposition

Proposition 0.3.2 *The following statements hold*

1. *For any null vector field p, $L_p^* p = 0$.*
2. *For arbitrary vector fields X, Y, V*

   $$(\mathscr{L}_V g)(X, Y) = -2g(X, L_V^s Y), \tag{0.6}$$

   where $\mathscr{L}_V X$ is the Lie derivative of X along V.
3. *For any vector field X and the 1-form $\theta = g(X, \cdot)$*

   $$d\theta(Y, V) = -g(L_X^a Y, V), \quad Y, V \in \Gamma(TM).$$

Proof To see that $L_p^* p = 0$, where $g(p, p) = 0$, we notice that for the adjoint operator $g(L_p^* p, X) = g(p, L_p X)$ is satisfied. Therefore,

$$g(L_p^* p, X) = g(p, L_p X) = g(p, -\nabla_X p) = -\frac{1}{2} X g(p, p) = 0$$

for all vector fields X, where the last equality follows from (0.1b). Since g is non-degenerate, it follows that $L_p^* p = 0$. To prove the second property, for arbitrary vector fields X, Y, V

$$\begin{aligned}
(\mathscr{L}_V g)(X, Y) &= \mathscr{L}_V\big(g(X, Y)\big) - g(\mathscr{L}_V X, Y) - g(X, \mathscr{L}_V Y) \\
&= V g(X, Y) - g(\nabla_V X - \nabla_X V, Y) - g(X, \nabla_V Y - \nabla_Y V) \\
&= V g(X, Y) - g(\nabla_V X, Y) + g(\nabla_X V, Y) - g(X, \nabla_V Y) + g(X, \nabla_Y V) \\
&= g(-L_V X, Y) + g(X, -L_V Y) = -g(X, L_V^* Y + L_V Y) = -2g(X, L_V^s Y).
\end{aligned}$$

is satisfied. In order to show the last statement, we notice that for arbitrary vector fields Y, V it follows that

$$\begin{aligned}
d\theta(Y,V) &= \frac{1}{2}\{Yg(X,V) - Vg(X,Y) - g(X,[Y,V])\} \\
&= \frac{1}{2}\{Yg(X,V) - Vg(X,Y) - g(X,\nabla_Y V) + g(X,\nabla_V Y)\} \\
&= \frac{1}{2}\Big\{g(\nabla_Y X,V) + g(X,\nabla_Y V) - g(\nabla_V X,Y) \\
&\quad - g(X,\nabla_V Y)) - g(X,\nabla_Y V) + g(X,\nabla_V Y)\Big\} \\
&= \frac{1}{2}\{g(\nabla_Y X,V) - g(\nabla_V X,Y)\} = \frac{1}{2}\{-g(L_X Y,V) + g(L_X V,Y)\} \\
&= \frac{1}{2}\{-g(L_X Y,V) + g(L_X^* Y,V)\} \\
&= -\frac{1}{2}\{g(L_X Y - L_X^* Y,V)\} = -g(L_X^a Y,V). \qquad \blacksquare
\end{aligned}$$

0.4 Maxwell's equations

The solutions of Maxwell's equations which describe electromagnetic fields can be conveniently represented by means of differential 2-forms. In order to recall the Maxwell's equations, we first need to review some facts about the Hodge operator. The solutions of Maxwell's equations are needed in chapeter 3.

0.4.1 Hodge dual operator

Let $\mathcal{M}$ be an oriented n-dimensional manifold equipped with a metric g, (Riemannian or pseudo-Riemannian), and $\Lambda^p(T^*\mathcal{M})$ the set of all smooth differential p-forms on $\mathcal{M}$. We recall that, by *Riesz representation* theorem, we can use the metric tensor to relate the vector fields and 1-forms.

For any 1-form η there exists a unique vector field V on $\mathcal{M}$ such that $\eta = g(V,\cdot)$, which means $\eta(X) = g(V,X)$ for all vector fields X on $\mathcal{M}$.

The metric g defined on the tangent bundle of $\mathcal{M}$ induces a metric $\hat{g}$ on the cotangent bundle of $\mathcal{M}$ as follows:

$$\hat{g}(\eta,\zeta) = g(V,W)$$

where $\eta = g(V,\cdot)$ and $\zeta = g(W,\cdot)$. Similarly, the metric g induces a metric on $\Lambda^2(T^*\mathcal{M})$. In order to describe the induced metric $\Lambda^2(T^*\mathcal{M})$, we recall the universal mapping property as follows.

Universal Mapping Property [50] Let V, W be two vector spaces. Given any skew-symmetric multilinear mapping

$$A : \overbrace{V \times \cdots \times V}^{p\text{-times}} \to W$$

there exists a unique linear mapping

$$\ell : \Lambda^p(V) \to W$$

such that

$$\ell(v_1 \wedge \cdots \wedge v_p) = A(v_1, \ldots, v_p), \quad v_1, \ldots v_n \in V.$$

Now we consider the following real-valued function

$$A : \overbrace{\Lambda(T^*\mathcal{M}) \times \cdots \times \Lambda(T^*\mathcal{M})}^{4\text{-times}} \to \mathbb{R}$$

defined by

$$A(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \begin{vmatrix} \hat{g}(\alpha_1, \alpha_3) & \hat{g}(\alpha_1, \alpha_4) \\ \hat{g}(\alpha_2, \alpha_3) & \hat{g}(\alpha_2, \alpha_4) \end{vmatrix},$$

where $\alpha_i, i = 1, \ldots 4$, is a 1-form on $\mathcal{M}$. The mapping A is biskew-symmetric and mulitlinear and two applications of the universal mapping property yields a bilinear form, still denoted by g, on $\Lambda^2(T^*\mathcal{M})$ characterized by

$$g(\alpha_1 \wedge \alpha_2, \beta_1 \wedge \beta_2) = \begin{vmatrix} \hat{g}(\alpha_1, \beta_1) & \hat{g}(\alpha_1, \beta_2) \\ \hat{g}(\alpha_2, \beta_1) & \hat{g}(\alpha_2, \beta_2) \end{vmatrix}.$$

The bilinear form g is nondegenerate, because for all 1-forms X, Y,

$$g(\alpha_1 \wedge \alpha_2, X \wedge Y) = \begin{vmatrix} \hat{g}(\alpha_1, X) & \hat{g}(\alpha_1, Y) \\ \hat{g}(\alpha_2, X) & \hat{g}(\alpha_2, Y) \end{vmatrix} = 0$$

implies that the first row of the determinant is a b multiple of the second row. Precisely,

$$\hat{g}(\alpha_1, X) = b\hat{g}(\alpha_2, X), \quad \hat{g}(\alpha_1, Y) = b\hat{g}(\alpha_2, Y),$$

which implies $\alpha_1 = b\alpha_2$ since $\hat{g}$ is nondegenerate. That means $\alpha_1 \wedge \alpha_2 = 0$.

Definition 0.4.1 *Let $(\mathcal{M}, g)$ be a 4-dimensional Lorentzian manifold. The Hodge dual operator is a linear operator*

$$* : \Lambda^2(T^*\mathcal{M}) \to \Lambda^2(T^*\mathcal{M})$$

which assigns to any 2-form $\eta \in \Lambda^p(T^*\mathcal{M})$, a 2-form $*\eta \in \Lambda^2(T^*\mathcal{M})$ defined by

$$\zeta \wedge *\eta = -g(\zeta, \eta)\,\mathrm{vol}, \quad \forall \zeta \in \Lambda^2(T^*\mathcal{M}).$$

Here vol is the volume form on the manifold $\mathcal{M}$ [18].

Let (e_1, e_2, e_3, e_4) be an orthonormal frame for the set of vector fields on $\mathcal{M}$ such that $g(e_1, e_1) = -1$ and $g(e_i, e_i) = 1$ for $i = 2, 3, 4$. The set of 2-forms

$$e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_3 \wedge e_4, e_4 \wedge e_2, e_2 \wedge e_3$$

is a frame for $\Lambda^2(T^*\mathcal{M})$. The following relations are also satisfied

$$\begin{aligned} *\,(e_1 \wedge e_2) &= e_3 \wedge e_4, & *(e_3 \wedge e_4) &= -e_1 \wedge e_2, \\ *\,(e_1 \wedge e_3) &= e_4 \wedge e_2, & *(e_4 \wedge e_2) &= -e_1 \wedge e_3, \\ *\,(e_1 \wedge e_4) &= e_2 \wedge e_3, & *(e_2 \wedge e_3) &= -e_1 \wedge e_4. \end{aligned}$$

The Hodge operator has the property $*^2 = -\,\mathrm{id}$. The $*$ operator is also self-adjoint, that is,

$$g(*\zeta, \eta) = g(\zeta, *\eta)$$

for all 2-forms $\zeta, \eta \in \Lambda^2(T^*M)$. Indeed,

$$g(*\zeta, \eta)\,\mathrm{vol} = - * \zeta \wedge *\eta = - * \eta \wedge *\zeta = g(*\eta, \zeta)\,\mathrm{vol}\,.$$

0.4.2 Electromagnetic Fields

Let $(\mathcal{M}, g)$ be an oriented 4-dimensional Lorentzian manifold.

Definition 0.4.2 *An electromagnetic field is a real 2-form F such that*

$$dF = d*F = 0,$$

where $*$ is the Hodge operator associated with the metric g.

Now consider the complex 2-form $\mathcal{F} = F + \mathrm{i} * F$. It follows that $d\mathcal{F} = 0$. We call $\mathcal{F}$, the complex electromagnetic field. The complex electromagnetic field $\mathcal{F}$, has the property of being anti-self-dual, that is, $*\mathcal{F} = -\mathrm{i}\,\mathcal{F}$ [42].

Definition 0.4.3 *A differential 2-form on the Lorentzian manifold $(\mathcal{M}, g)$ is called null, if*

$$g(F, F) = g(F, *F) = 0.$$

In particular, null solutions of the Maxwell's equations are called null electromagnetic fields.

We extend the metric g defined on $\Lambda^2(T^*\mathcal{M})$ by complex linearity to a metric on $\Lambda^2(T^*\mathcal{M}) \otimes \mathbb{C}$, the complexification of $\Lambda^2(T^*\mathcal{M})$, which we also denote by g as follows

$$g(F_1 + \mathrm{i}F_2, E_1 + \mathrm{i}E_2) = g(F_1, E_1) - g(F_2, E_2) + \mathrm{i}\{g(F_1, E_2) + g(F_2, E_1)\}.$$

We now say the complex electromagnetic field $\mathcal{F}$ is null if, $g(\mathcal{F}, \mathcal{F}) = 0$.

Proposition 0.4.4 *The nonzero complex electromagnetic field $\mathcal{F}$ is null if and only if*

$$\mathcal{F} \wedge \mathcal{F} = 0.$$

Proof Assume $\mathcal{F} = F + \mathrm{i} * F$ is null, i.e. $g(\mathcal{F}, \mathcal{F}) = 0$. It follows that

$$g(F, F) = g(*F, *F), \quad \text{and} \quad g(F, *F) = 0.$$

It also follows that

$$F \wedge F = -F \wedge **F = g(F, *F)\,\mathrm{vol} = 0,$$

and furthermore,

$$*F \wedge *F = -g(*F, F)\,\mathrm{vol} = 0.$$

Moreover, on one hand,

$$F \wedge *F = *F \wedge F$$

is satisfied. On the other hand,

$$F \wedge *F = -g(F, F)\,\mathrm{vol} = -g(*F, *F)\,\mathrm{vol} = *F \wedge **F = - * F \wedge F.$$

Hence,

$$F \wedge *F = *F \wedge F = 0.$$

Thus, we now clearly observe that

$$\mathcal{F} \wedge \mathcal{F} = F \wedge F - *F \wedge *F + \mathrm{i}F \wedge *F + \mathrm{i} * F \wedge F = 0.$$

To show the converse statement we note that from $\mathcal{F} \wedge \mathcal{F} = 0$, it follows that

$$F \wedge F = *F \wedge *F, \quad \textit{and} \quad F \wedge *F = - * F \wedge F.$$

Hence,

$$g(F, F)\,\mathrm{vol} = -F \wedge *F = *F \wedge F = - * F \wedge **F = g(*F, *F)\,\mathrm{vol}$$

implies $g(F, F) = g(*F, *F)$. Furthermore,

$$g(F, *F)\,\mathrm{vol} = -F \wedge **F = F \wedge F = *F \wedge *F = -g(*F, F)\,\mathrm{vol}$$

also implies $g(*F, F) = 0$ and consequently, $g(\mathcal{F}, \mathcal{F}) = 0$. ∎

We next show that a null complex electromagnetic field $\mathcal{F}$ can always be written as $\mathcal{F} = \varphi\lambda \wedge \mu$, where φ is a complex function, λ is a smooth real 1-form and μ is a smooth complex 1-form. Before proving that, we need to state the following definition.

Definition 0.4.5 *Let $(\mathcal{M}, g)$ be a Lorentzian manifold of dimension m and*

$$p^{\perp} = \{V \in \Gamma(T\mathcal{M}) \mid g(p, V) = 0\}$$

be the codimension one distribution that is orthogonal to the null vector field p. A field of frames $(p, e_1, \dots, e_m, q)$ is called admissible if

$$p^{\perp} = \mathrm{span}(p, e_1, \dots, e_m), \; g(p, q) = 1,$$

and

$$g(p, e_i) = g(q, e_i) = 0, \; g(e_i, e_j) = \delta_{ij}, \quad \textit{for} \quad i, j = 1 \dots m.$$

We denote by

$$p^* = g(q, \cdot), e_1^* = g(e_1, \cdot), \dots, e_m^* = g(e_m, \cdot), q^* = g(p, \cdot)$$

the dual coframe to $p, e_1, \dots, e_m, q$.

We also recall that a 2-form ω is decomposable if

$$\omega = \alpha \wedge \beta,$$

where α, β are 1-forms.

Lemma 0.4.6 *Let $\mathcal{M}$ be a 4-dimensional Lorentzian manifold. Any nonzero 2-form ω is decomposable if and only if*

$$\omega \wedge \omega = 0.$$

Proof Assume that a 2-form ω is decomposable, that is there are 1-forms α, β such that $\omega = \alpha \wedge \beta$. It is now obvious that

$$\omega \wedge \omega = \alpha \wedge \beta \wedge \alpha \wedge \beta = 0.$$

The proof in the other direction is by contradiction. Assume ω is not decomposable. Therefore, there are linearly independent 1-forms $\alpha, \beta, \gamma, \eta$ such that

$$\omega = \alpha \wedge \beta + \gamma \wedge \eta,$$

which implies

$$\omega \wedge \omega = 2\alpha \wedge \beta \wedge \gamma \wedge \eta \neq 0. \quad \blacksquare$$

Lemma 0.4.7 *Let $(\mathcal{M}, g)$ be a 4-dimensional Lorentzian manifold. For the null electromagnetic field F there exists a local admissible frame (p, e, f, q) such that $F = q^* \wedge e^*$ and $*F = q^* \wedge f^*$.*

Proof Since F is null, it follows that $F \wedge F = 0$. Hence, by Lemma 0.4.6, F is decomposable, that is, there are two real 1-forms α, β such that $F = \alpha \wedge \beta$. Moreover, if follows from

$$g(F, F) = \begin{vmatrix} \hat{g}(\alpha, \alpha) & \hat{g}(\alpha, \beta) \\ \hat{g}(\beta, \alpha) & \hat{g}(\beta, \beta) \end{vmatrix} = 0$$

that there exists a function t such that

$$\hat{g}(\alpha, \alpha) + t\hat{g}(\alpha, \beta) = \hat{g}(\alpha, \alpha + t\beta) = 0,$$
$$\hat{g}(\beta, \alpha) + t\hat{g}(\beta, \beta) = \hat{g}(\beta, \alpha + t\beta) = 0.$$

Set $q^* := \alpha + t\beta$. From the construction

$$\hat{g}(q^*, q^*) = \hat{g}(q^*, \alpha) = \hat{g}(q^*, \beta) = 0$$

are satisfied. Then, there exists a null vector field p^* such that

$$\hat{g}(p^*, q^*) = 1,$$

since $\hat{g}$ is nondegenerate. One can complement p^*, q^* by orthonormal vectors e^*, f^* such that

$$(p^*, q^*, e^*, f^*)$$

forms a null coframe for the cotangent bundle, more precisely,

$$\hat{g}(p^*, e^*) = \hat{g}(p^*, f^*) = \hat{g}(q^*, e^*) = \hat{g}(q^*, f^*) = 0.$$

It now follows that

$$F = \alpha \wedge \beta = (q^* - t\beta) \wedge \beta = q^* \wedge \beta.$$

Furthermore, β is expressed as

$$\beta = xq^* + ye^* + zf^*,$$

where x, y, z are some real functions, since F is null. Therefore,

$$F = q^* \wedge \beta = q^* \wedge (ye^* + zf^*).$$

If

$$\hat{g}(ye^* + zf^*, ye^* + zf^*) = y^2 + z^2 \neq 1,$$

the coframe

$$(p^{**}, q^{**}, e^{**}, f^{**}) = (X_1, X_2, X_3, X_),$$

where

$$\begin{aligned} p^{**} &= \frac{1}{\sqrt{y^2+z^2}}\, p^*, q^{**} = \sqrt{y^2+z^2}\, q^*, \\ e^{**} &= \frac{y}{\sqrt{y^2+z^2}} e^* + \frac{z}{\sqrt{y^2+z^2}} f^*, f^{**} = \frac{z}{\sqrt{y^2+z^2}} e^* - \frac{y}{\sqrt{y^2+z^2}} f^* \end{aligned}$$

has the property that

$$F = q^{**} \wedge e^{**},$$

On the other hand, the 2-form $*F$ can be written as

$$*F = c_{ij} X^i \wedge X^j, \quad i < j.$$

The only nonzero coefficient is $c_{24} = 1$, since

$$p^{**} \wedge e^{**} \wedge *F = -g(p^{**} \wedge e^{**}, q^{**} \wedge e^{**}) \,\mathrm{vol} = -c_{24} \,\mathrm{vol},$$

and, moreover,

$$g(p^{**} \wedge e^{**}, q^{**} \wedge e^{**}) = 1.$$

Thus,

$$*F = q^{**} \wedge f^{**},$$

and the vectors (q, e, f, p) given by

$$p^{**} = g(q, \cdot), \; q^{**} = g(p, \cdot), \; e^{**} = g(e, \cdot), \; f^{**} = g(f, \cdot)$$

form an admissible frame for the tangent bundle. ∎

As a consequence of the above lemma, we have the following proposition for the null complex electromagnetic field $\mathcal{F}$.

Proposition 0.4.8 *Any null complex electromagnetic field $\mathcal{F}$ can be written as*

$$\mathcal{F} = a\lambda \wedge \mu,$$

where a is a complex function, λ a real 1-form and μ a complex 1-form.

Proof Using Lemma 0.4.7, there exists an admissible frame

$$(p^*, q^*, e^*, f^*)$$

such that the null complex electromagnetic field can be expressed as

$$\mathcal{F} = F + \mathrm{i} * F = q^* \wedge \omega,$$

where $\omega = e^* + \mathrm{i} f^*$. It also follows that

$$q^* \wedge \mathcal{F} = 0.$$

Moreover, there exists two complex 1-forms γ, μ such that $\mathcal{F} = \gamma \wedge \mu$, since $\mathcal{F}$ is null. Therefore,

$$q^* \wedge \gamma \wedge \mu = 0,$$

implies that there are two complex functions a, b such that

$$\gamma = aq^* + b\mu.$$

Wedging both sides of the above expression by μ, we get

$$\mathcal{F} = \gamma \wedge \mu = aq^* \wedge \mu,$$

where $\lambda = q^* = g(p, \cdot)$ and p is a null vector field. ∎

Corollary 0.4.9 It follows from the identity

$$\mathscr{L}_p \mathcal{F} = d(p \lrcorner \mathcal{F}) + p \lrcorner d\mathcal{F} = d(p \lrcorner \mathcal{F})$$

that $\mathscr{L}_p \mathcal{F} = 0$. Indeed,

$$p \lrcorner \mathcal{F} = ag(p,p)\omega - ag(p, \cdot) \wedge \left[(y+z)p \lrcorner e^* + (z-y)p \lrcorner f^*\right] = 0.$$

The null complex electromagnetic field also possesses two properties stated below, which will be used later.

Proposition 0.4.10 *For the null complex electromagnetic field described in Proposition* (0.4.8)*, the following statements are satisfied:*

$$\lambda \wedge \mu \wedge \mathscr{L}_p\mu = 0, \tag{0.7a}$$

$$\lambda \wedge \mathscr{L}_p\lambda = 0, \tag{0.7b}$$

where $\lambda = g(p, \cdot)$.

Proof Taking the Lie derivative along the null vector field p from both sides of the electromagnetic field $\mathcal{F} = a\lambda \wedge \mu$, we see

$$\mathscr{L}_p \mathcal{F} = p(a)\lambda \wedge \mu + a\mathscr{L}_p\lambda \wedge \mu + a\lambda \wedge \mathscr{L}_p\mu = 0, \tag{0.8}$$

which implies $\lambda \wedge \mu \wedge \mathscr{L}_p\mu = 0$. Wedging both sides of (0.8) by λ gives us

$$\lambda \wedge \mathscr{L}_p\lambda \wedge \mu = 0,$$

which is equivalent to

$$\mathscr{L}_p\lambda = x\lambda + y\mu,$$

where x, y are two complex functions.

Wedging the last identity with λ, implies that

$$\lambda \wedge \mathscr{L}_p\lambda = t\,\mathcal{F},$$

where $t = \frac{y}{a}$. If $t \neq 0$, and since the left hand side is real, it follows that

$$\bar{y}\lambda \wedge \bar{\mu} = y\lambda \wedge \mu,$$

which means that λ is a linear combination of μ and $\bar{\mu}$ contradicting the definition of the electromagnetic field. ∎

0.4.3 Isothermal coordinates

In this subsection we briefly recall the notion of isothermal coordinates, which will be used later. Let

$$g = E(x,y)dx^2 + 2F(x,y)dxdy + G(x,y)dy^2$$

satisfying

$$EG - F^2 > 0, \quad E > 0$$

be a positive definite Riemannian metric defined in a neighborhood of a surface with the local coordinates x, y. By isothermal coordinates we mean local coordinates u, v relative to which the metric takes the form

$$g = \lambda(u,v)(du^2 + dv^2), \quad \lambda(u,v) > 0.$$

A special case first observed and studied by Gauss, is a surface in three dimensions, (coordinates x, y, ϕ) given by $\phi = \phi(x,y)$. He also proved the existence of real analytic isothermal coordinates when the Riemannian metric defined on a neighborhood of $0 \in \mathbb{R}^2$ is real analytic. One can refer to, e.g. [62] for a proof. The weakest conditions under which the isothermal coordinates are known were found by A. Korn [31] in 1914 and Lichtenestein in 1916. A simpler proof was given by S. S. Chern in [11] in 1955.

0.4.4 Beltrami equation

Let (x,y) be real coordinates on a 2-dimensional surface. Also assume $u(x,y), v(x,y)$ be two smooth functions such that $w = u + \mathrm{i}v$. As customary, we denote

$$w_z = \frac{1}{2}(\frac{\partial w}{\partial x} - \mathrm{i}\frac{\partial w}{\partial y}), \quad w_{\bar{z}} = \frac{1}{2}(\frac{\partial w}{\partial x} + \mathrm{i}\frac{\partial w}{\partial y}).$$

The differential equation defined by

$$w_{\bar{z}} = \mu w_z, \quad w_z \neq 0 \tag{0.9}$$

where μ is a complex function satisfying $\mid \mu \mid < 1$, is called the *Beltrami* equation.

Now we recall a theorem in [40], which actually states the problem of finding isothermal coordinates is reduced to finding a solution of Beltrami equation.

Theorem 0.4.11 *[40] Let E, F, G be $\mathcal{C}^1$−functions such that $EG - F^2 > 0$ and $E, G > 0$. Suppose*

$$\mu = \frac{\frac{1}{2}(E-G) + \mathrm{i}F}{\frac{1}{2}(E+G) + \sqrt{EG - F^2}}.$$

If $w(z)$ is a $\mathcal{C}^1$−solution of the Beltrami equation (0.9) *near the origin, and $w(0) \neq 0$, then in a neighborhood of the origin, writing $w(z) = u + \mathrm{i}v$, the coordinates u, v are isothermal, i.e.*

$$du^2 + dv^2 = \lambda(u,v)\Big(Edx^2 + 2Fdxdy + Gdy^2\Big).$$

Chapter 1

Shearfree geometry and CR geometry

In this chapter we first collect some known facts about contact and also CR manifolds. The Sasakian manifolds which are examples of CR manifolds are introduced and we show that if the CR structure is preserved by the Reeb vector field, then a CR manifold is Sasakian.

We conclude this chapter by introducing the notion of shearfree geometry and its connection with CR geometry. We also provide some equivalent definitions of the shearfree vector fields which are useful in the next chapter.

1.1 Contact geometry

1.1.1 Contact manifolds

Definition 1.1.1 *A contact structure on a smooth manifold M of dimension $2n+1$ is a distribution D of the tangent bundle of co-dimension 1 which can be defined by a 1-form ω, $\Gamma(D) = \ker \omega$, satisfying*

$$\omega \wedge (d\omega)^n \neq 0.$$

Any such 1-form ω is called contact form. The manifold M equipped with a contact structure D is called a contact manifold.

We first note that any nonzero rescaling of a contact form is also a contact form. Indeed, for any nonzero function α, we see that

$$d(\alpha\omega) \equiv \alpha\, d\omega \quad \mod\{\omega\},$$

and

$$\alpha\omega \wedge (d\alpha\omega)^n = \alpha^{n+1}\omega \wedge (d\omega)^n \neq 0.$$

For any choice of the contact form ω, the *Reeb* vector field Z on M, is defined by the conditions

$$Z \lrcorner\, \omega = 1, \qquad Z \lrcorner\, d\omega = 0.$$

The Reeb vector field Z, is an infinitesimal automorphism of the contact structure, i.e.

$$\mathscr{L}_Z X \in \Gamma(D) \quad \forall X \in \Gamma(D).$$

Cartan's magic formula implies

$$\mathscr{L}_Z\omega = d(Z \lrcorner\, \omega) + Z \lrcorner\, d\omega = 0.$$

Furthermore, for any section X of D

$$\mathscr{L}_Z(\omega X) = \mathscr{L}_Z\,(\omega(X)) + \omega(\mathscr{L}_Z X) = \omega(\mathscr{L}_Z X)$$

is satisfied, which implies

$$\omega(\mathscr{L}_Z X) = \mathscr{L}_Z\,(\omega(X)) = 0.$$

The following proposition shows that the converse statement is also true.

Proposition 1.1.2 *A vector field Z on a contact manifold (M, D) that is transversal to D is a Reeb vector field for some contact form ω if and only if*

$$\mathscr{L}_Z\Gamma(D) \subset \Gamma(D).$$

Proof We only need to proof the "if" statement. Assume that Z is an infinitesimal automorphism of the contact structure D and transversal to D.

Let ω be the unique contact form such that $Z \lrcorner \omega = 1$. We show that Z is the Reeb vector field for ω, i.e.

$$Z \lrcorner d\omega = 0.$$

Indeed, let X be a section of D then

$$d\omega(Z, X) = \frac{1}{2}\{Z\omega(X) - X\omega(Z) - \omega([Z, X])\} = 0.$$

Together with $d\omega(Z, Z) = 0$, this proves the claim. ∎

In addition, for $\omega' = \beta\omega$ the new Reeb vector field is the following

$$Z' = \frac{1}{\beta}(Z + X_0),$$

where X_0 is a section of D such that

$$X_0 \lrcorner (\beta d\omega + d\beta \wedge \omega) = -Z \lrcorner (d\beta \wedge \omega)$$

is fulfilled. Refer also to [33].

1.1.2 Partially integrable almost CR manifold

Definition 1.1.3 *A partially integrable almost CR manifold M is a contact manifold with contact distribution D and a smooth family of endomorphisms $J_x: D_x \to D_x$ with $J_x^2 = -\mathrm{Id}$. We assume that (M, D, J) is partially integrable, i.e. the complex eigen-distribution*

$$D^{1,0} \subset D \otimes \mathbb{C}$$

of J with eigenvalue i satisfies

$$[\Gamma(D^{1,0}), \Gamma(D^{1,0})] \subseteq \Gamma(D \otimes \mathbb{C}).$$

Lemma 1.1.4 *Let (M, D, J) be an almost CR manifold. Then, M is partially integrable if and only if*

$$d\lambda(JX, JY) = d\lambda(X, Y), \quad \forall X, Y \in \Gamma(D),$$

where λ is a contact form for the contact distribution D. This is also equivalent to $d\lambda(\cdot, J\cdot)$ being symmetric.

Proof For any $X, Y \in \Gamma(D)$ the following

$$[X - \mathrm{i}JX, Y - \mathrm{i}JY] = [X, Y] - [JX, JY] - \mathrm{i}([X, JY] + [JX, Y])$$

is satisfied. Therefore, M is partially integrable if and only if

$$[X, Y] - [JX, JY] \in \Gamma(D), \quad [X, JY] + [JX, Y] \in \Gamma(D).$$

Moreover,

$$\begin{aligned} d\lambda(JX, JY) &= \frac{1}{2}\{(JX)\lambda(JY) - (JY)\lambda(JX) - \lambda([JX, JY])\} \\ &= -\frac{1}{2}\lambda([JX, JY]), \end{aligned}$$

since $JX, JY \in \Gamma(D)$. Similarly,

$$d\lambda(X, Y) = -\frac{1}{2}\lambda([X, Y]).$$

Thus, $d\lambda(JX, JY) = d\lambda(X, Y)$ is equivalent to

$$[X, Y] - [JX, JY] \in \Gamma(D).$$

We now assume that $d\lambda(JX, JY) = d\lambda(X, Y)$, is satisfied. It follows that

$$d\lambda(X, JY) = -d\lambda(J^2X, JY) = -d\lambda(JX, Y),$$

which is equivalent to

$$[X, JY] + [JX, Y] \in \Gamma(D).$$

Together with $[X, Y] - [JX, JY] \in \Gamma(D)$, implies that M is partially integrable. We also see this is equivalent to

$$d\lambda(X, JY) = d\lambda(Y, JX),$$

since

$$d\lambda(X, JY) = -d\lambda(J^2X, JY) = -d\lambda(JX, Y) = d\lambda(JY, X). \quad \blacksquare$$

1.2 CR geometry

In this section we initially introduce the notion of an almost complex structure on a manifold.

1.2.1 Almost complex structure

Definition 1.2.1 *Let M be a manifold and let V be a subbundle of $TM \otimes \mathbb{C}$. The pair (M, V) is an almost complex manifold if*

$$V \cap \overline{V} = \{0\}, \quad \textit{and} \quad V \oplus \overline{V} = TM \otimes \mathbb{C}$$

are satisfied. The subbundle V is called an almost complex structure.

Set $\dim_{\mathbb{C}} V = n$, then it follows that $\dim_{\mathbb{R}} M = 2n$.

Consider the complex manifold M with the coordinate system $(z^1, \ldots, z^n)$ at a point $p \in M$. The underlying almost complex structure is given by

$$V = \text{linear span}_{\mathbb{C}} \{\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n}\},$$

where

$$(\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n})$$

is a basis of T_pM.

Here is another definition of an almost complex manifold.

Definition 1.2.2 *Let M be a manifold and let $J_p : T_pM \to T_pM$ be a family of endomorphisms satisfying $J_p^2 = -\mathrm{Id}$ at each point $p \in M$. The pair (M, J) is called an almost complex manifold. The endomorphism J is called an almost complex structure.*

The above definitions of the almost complex manifold are equivalent. Indeed, For a given (M, V), we define the linear map J

$$J : TM \otimes \mathbb{C} \to TM \otimes \mathbb{C}$$

such that

$$J(V) = \mathrm{i}V, \quad J(\overline{V}) = -\mathrm{i}\overline{V}.$$

The restriction of the map J to TM satisfies $J^2 = -\mathrm{Id}$.

Conversely, for a given (M, J), where $J : TM \to TM$ with $J^2 = -\mathrm{Id}$, we extend J by complex linearity, still denoted by J, from $TM \otimes \mathbb{C}$ to itself and let V be the eigenspace corresponding to i.

Definition 1.2.3 *An almost complex manifold* (M, V) *is called* integrable *if*

$$[\Gamma(V), \Gamma(V)] \subset \Gamma(V),$$

that is, for all sections X, Y *of the subbundle* V*, their commutator* $[X, Y]$ *is also a section of* V*.*

Equivalently, (M, J) is integrable if the *Nijenhuis tensor* defined by

$$N^J(X, Y) = J\{[JX, Y] + [X, JY]\} - [JX, JY] + [X, Y] \tag{1.1}$$

vanishes for all vector fields X, Y on M.

Example Let M be a complex manifold on complex dimension n. Also assume that $(z^1 = x^1 + \mathrm{i}y^1, \ldots, z^n = x^n + \mathrm{i}y^n)$, is a holomorphic coordinate system at point $p \in M$. The underlying almost complex structure at each point $p \in M$ is given by

$$J(\frac{\partial}{\partial x^\alpha}) = \frac{\partial}{\partial y^\alpha}, \quad J(\frac{\partial}{\partial y^\alpha}) = -\frac{\partial}{\partial x^\alpha}, \quad \alpha = 1, \ldots n.$$

The converse statement of the example is not true in general. There are almost complex manifolds which are not complex manifolds. The following theorem shows that under some circumstances the almost complex structure is complex.

Let the manifold M be real analytic, i.e. a manifold with real analytic atlas. Fix some coordinates system from this real analytic atlas. Any almost complex structure on the manifold has a basis of vectors with real analytic complex-valued coefficients. We finalise this part with the following important theorem.

Theorem 1.2.4 (Newlander-Nirenberg) *[43] A* $\mathcal{C}^k$*,* $k > 1$*, integrable almost complex structure is complex.*

1.2.2 CR manifolds

First of all, we give two equivalent definitions of CR manifolds.

Definition 1.2.5 *Let M be a smooth real manifold of dimension $2n+1$. We also assume that a complex subbundle $V \subset TM \otimes \mathbb{C}$(complexified tangent bundle) satisfies the following properties*

(i) $\dim_{\mathbb{C}} V = n$.

(ii) V is integrable, i.e. $[\Gamma(V), \Gamma(V)] \subset \Gamma(V)$, which means the commutator of sections of V is again a section of V.

(iii) $V \cap \overline{V} = \{0\}$.

Then, (M, V) is called a CR manifold.

Definition 1.2.6 *Let M be a smooth real manifold of dimension $2n+1$. The triple (M, D, J) is called a CR manifold if*

(i) $D \subset TM$ is a real subbundle of rank $2n$.

(ii) $J : \Gamma(D) \to \Gamma(D)$ is an endomorphism such that $J^2 = -\mathrm{Id}$.

(iii) If X, Y are in $\Gamma(D)$, then so is $[JX, Y] + [X, JY]$ and $N^J(X, Y) = 0$.

We show that these two definitions are equivalent. Indeed, assume that (M, V) is given, then consider the subbundle

$$D := \operatorname{Re} V = \{X + \overline{X} : X \in \Gamma(V)\},$$

and the endomorphism J defined by

$$J(X + \overline{X}) = \mathrm{i}(X - \overline{X}).$$

The subbundle D is of codimension 1 with $J^2 = -\mathrm{Id}$. The integrability conditions are also satisfied.

On the other hand, given (M, D, J) as the CR structure, consider the complexified of D, $D \otimes \mathbb{C}$. By complex linearity we define the endomorphism on $D \otimes \mathbb{C}$, still denoted by J, and set

$$D^{1,0} = \{X \in D \otimes \mathbb{C} \mid JX = \mathrm{i}X\} = \{X - \mathrm{i}JX \mid X \in D\}.$$

We show that $D^{1,0} \cap D^{0,1} = \{0\}$. Let T be a section of $D^{1,0} \cap D^{0,1}$. Therefore, there exists a vector field X on D such that

$$T = X - \mathrm{i}JX = X + \mathrm{i}JX,$$

which implies $X = 0$, since J is an invertible endomorphism. Thus,

$$D^{1,0} \cap D^{0,1} = \{0\}.$$

Let $X - \mathrm{i}JX$ and $Y - \mathrm{i}JY$ be sections of $D^{1,0}$, where $X, Y \in \Gamma(D)$. We then have

$$\begin{aligned}
[X - \mathrm{i}JX, Y - \mathrm{i}JY] &= [X, Y] - \mathrm{i}[X, JY] - \mathrm{i}[JX, Y] - [JX, JY] \\
&= -J\{[JX, Y] + [X, JY]\} - \mathrm{i}([X, JY] + [JX, Y]) \\
&= -\mathrm{i}\big([X, JY] + [JX, Y] - \mathrm{i}J([X, JY] + [JX, Y])\big).
\end{aligned}$$

Since $[X, JY] + [JX, Y] \in \Gamma(D)$, it follows that

$$\mathrm{i}\big([X, JY] + [JX, Y] + \mathrm{i}J([X, JY] + [JX, Y]) \in \Gamma(D^{1,0}).$$

Therefore, $[X + \mathrm{i}JX, Y + \mathrm{i}JY] \in \Gamma(D^{1,0})$.

1.2.3 Some examples of CR manifolds

Below we give some examples of the CR manifolds.

Examples 1. A real hypersurface in $\mathbb{C}^{n+1}$ is a subset M of $\mathbb{C}^{n+1}$ such that for every point $p \in M$ there is a neighborhood U of p in $\mathbb{C}^{n+1}$ and a real-valued function ρ defined in U such that

$$M \cap U = \{z \in U : \rho(z, \bar{z}) = 0\}$$

with differential $d\rho \neq 0$ in U where $z = (z^1, \dots, z^{n+1})$. The function ρ is called a *defining function*.

On any real hypersurface M in $\mathbb{C}^{n+1}$, there exists a CR structure induced from $\mathbb{C}^{n+1}$. Indeed, let $(z^1 = x^1 + \mathrm{i}y^1, \dots, z^{n+1} = x^{n+1} + \mathrm{i}y^{n+1})$ be a coordinate system at a point $p \in \mathbb{C}^{n+1}$ and J be the canonical almost complex structure on $\mathbb{C}^{n+1}$, i.e.

$$J : T\mathbb{C}^{n+1} \to T\mathbb{C}^{n+1}$$

given by

$$J(\frac{\partial}{\partial x^\alpha}) = \frac{\partial}{\partial y^\alpha}, \quad J(\frac{\partial}{\partial y^\alpha}) = -\frac{\partial}{\partial x^\alpha}, \quad \alpha = 1, \dots, n+1.$$

For any $p \in M$, we set

$$D = TM \cap JTM. \tag{1.2}$$

We first note that

$$\dim D \geq 2n,$$

because

$$\begin{aligned} \dim TM + \dim JTM - \dim D &= \dim(TM + JTM) \\ &\leq \dim T\mathbb{C}^{n+1} = 2n+2. \end{aligned}$$

Assume that $\dim D = 2n+1$. Since $D \subset TM$, it follows that $D = TM$. That is, $TM \subset JTM$ and because $\dim JTM = 2n+1$, eventuality, $TM = JTM$. Without loss of any generality we may assume that $\frac{\partial}{\partial y^{n+2}} \notin TM = JTM$, which implies $\frac{\partial}{\partial x^{n+2}} \notin TM$, which contradicts the dimension of M. Hence, $\dim D = 2n$. The restriction of J to D has the property that $J^2 = -\mathrm{Id}$. We note that because $\mathbb{C}^{n+1}$ is a complex manifold, it is integrable, i.e. the Nijenhuis tensor vanishes and

$$J\{[JX, Y] + [X, JY]\} = [JX, JY] - [X, Y].$$

Moreover, if both W, JW are tangent to M, then $W \in \Gamma(D)$. Thus, $[JX, Y] + [X, JY]$ and $[JX, JY] - [X, Y]$ are tangent to M, which implies $[JX, Y] + [X, JY] \in \Gamma(D)$. Hence, (M, D, J) is a CR manifold.

2. In the next example we provide the details for the CR structure defined on the *Heisenberg group* [13]. We consider $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ with the coordinates $(z,t) = (z^1,\dots,z^n,t)$, where $z^j \in \mathbb{C}, t \in \mathbb{R}$. By its definition $\mathbb{H}_n$ is a smooth manifold of real dimension $2n+1$. Furthermore, $\mathbb{H}_n$ becomes a group with the operation defined by

$$(z,t).(w,s) = (z+w, t+s+2\mathrm{Im}\langle z,w\rangle)$$

where $\langle z,w\rangle = \delta_{jk} z^j \bar{w}^k$. The smooth manifold $\mathbb{H}_n$ with the group operation defined above is called *Heisenberg group*. We now explain the construction of a CR structure on $\mathbb{H}_n$. Let us consider the complex vector fields X_j on $\mathbb{H}_n$ defined by

$$X_j = \frac{\partial}{\partial z^j} + \mathrm{i}\bar{z}^j \frac{\partial}{\partial t}, \quad j = 1,\dots,n.$$

We then define at each point $(z,t) \in \mathbb{H}_n$, the space

$$V = \{\alpha^j X_j | \alpha^j \in \mathbb{C}\}.$$

We only need to check that the complex subbundle $\mathbb{H}_n$ is integrable, that is, $[X_j, X_k] \in \mathbb{H}_n$ for all $X_j, X_k \in \mathbb{H}_n$, $j,k = 1\dots,n$. We compute the commutator of X_j, X_k.

$$\begin{aligned}[X_j,X_k] &= [\frac{\partial}{\partial z^j} + \mathrm{i}\bar{z}^j\frac{\partial}{\partial t}, \frac{\partial}{\partial z^k} + \mathrm{i}\bar{z}^k\frac{\partial}{\partial t}] = [\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}] + \mathrm{i}[\frac{\partial}{\partial z^j}, \bar{z}^k\frac{\partial}{\partial t}] \\ &\quad + \mathrm{i}[\bar{z}^j\frac{\partial}{\partial t}, \frac{\partial}{\partial z^k}] - [\bar{z}^j\frac{\partial}{\partial t}, \bar{z}^k\frac{\partial}{\partial t}] \\ &= \mathrm{i}\bar{z}^k[\frac{\partial}{\partial z^j}, \frac{\partial}{\partial t}] + \mathrm{i}\bar{z}^j[\frac{\partial}{\partial t}, \frac{\partial}{\partial z^k}] - \bar{z}^k\bar{z}^j[\frac{\partial}{\partial t}, \frac{\partial}{\partial t}] = 0.\end{aligned}$$

Therefore, the pair $(\mathbb{H}_n, V)$ is a CR manifold.

1.2.4 The Levi form

Definition 1.2.7 *Let (M,V) be a $(2n+1)$-dimensional CR manifold. For $p \in M$, the Levi form is the bilinear form defined by*

$$\begin{aligned}&\mathcal{L}_p : V_p \times V_p \longrightarrow (T_pM \otimes \mathbb{C})/(V_p \oplus \overline{V}_p) \\ &\mathcal{L}_p(X_p, Y_p) = \frac{1}{2\mathrm{i}}\pi_p([X,\overline{Y}]_p),\end{aligned} \tag{1.3}$$

where X, Y are sections of V with $X(p) = X_p$ and $Y(p) = Y_p$. Here the map π_p is the natural projection

$$\pi_p : T_pM \otimes \mathbb{C} \longrightarrow (T_pM \otimes \mathbb{C})/(V_p \oplus \overline{V}_p).$$

The Levi form defined as above is Hermitian since,

$$\mathcal{L}_p(X_p, Y_p) = \frac{1}{2\mathrm{i}}\pi_p([X, \overline{Y}]_p) = -\frac{1}{2\mathrm{i}}\pi_p([\overline{Y}, X]_p) = \overline{\frac{1}{2\mathrm{i}}\pi_p([Y, \overline{X}]_p)} = \overline{\mathcal{L}_p(Y_p, X_p)}.$$

Alternatively, for a given CR manifold (M, D, J), the Levi form is defined as follows.

$$\mathcal{L}_p : D_p \times D_p \longrightarrow T_pM/D_p$$
$$\mathcal{L}_p(X_p, Y_p) = \pi_p([X, JY]_p),$$

where X, Y are sections of D with $X(p) = X_p$ and $Y(p) = Y_p$. The map π_p is the projection

$$\pi_p : T_pM \longrightarrow T_pM/D_p.$$

The Levi form defined in the real case is symmetric. Indeed,

$$\begin{aligned}\mathcal{L}_p(X_p, Y_p) - \mathcal{L}_p(Y_p, X_p) &= \pi_p\big([X, JY]_p - [Y, JX]_p\big) = \pi_p\big([X, JY]_p + [JX, Y]_p\big)\\ &= 0_{T_pM/D_p},\end{aligned}$$

since $[X, JY] + [JX, Y]$ is also a section of D.

Definition 1.2.8 *A CR manifold (M, V) is called*

i) nondegenerate at the point $p \in M$, if $\mathcal{L}_p(X_p, Y_p) = 0$ for all $Y \in \Gamma(D)$ with $Y(p) = Y_p$ implies $X_p = 0$,

ii) strictly pseudoconvex at the point $p \in M$, if the Levi form is positive definite ($\mathcal{L}_p > 0$) or negative definite ($\mathcal{L}_p < 0$).

1.2.5 Cartan's approach to 3-dimensional CR manifolds

Following Cartan [10] a 3-dimensional CR structure of M can also be (locally) encoded as a choice of a real 1-form λ and a complex 1-form μ such that

(i) $\lambda \wedge \mu \wedge \bar{\mu} \neq 0$

(ii) $D = \ker \lambda$

(iii) $\mu|_D \circ J = \mathrm{i}\mu|_D$ for all sections X of D.

Then any other pair (λ', μ') of 1-forms defines the same CR structure if it is related to (λ, μ) by

$$\lambda' = f\lambda, \qquad \mu' = h\mu + l\lambda \tag{1.4}$$

where $f \neq 0$ (real), $h \neq 0$ and l are complex functions.

A 3-dimensional manifold M equipped with the class of pairs of 1-forms $[(\mu, \lambda)]$ where the equivalence relation is defined by the transformations (1.4) is also called CR manifold and denoted by $(M, [(\mu, \lambda)])$.

1.2.6 Strictly pseudoconvex CR manifolds

A 3-dimensional CR structure on M can be alternatively defined by means of vector fields as follows. Let (M, D, J) be a strictly pseudoconvex CR manifold, i.e. for any (local) non-vanishing section X of D, $[X, JX] \notin D$, and $\partial = X - \mathrm{i}JX$ and $\bar{\partial} = X + \mathrm{i}JX$ are the generators of $D^{1,0}$ and $D^{0,1}$ respectively. One can complement $\partial, \bar{\partial}$ with

$$\partial_0 = \mathrm{i}[\partial, \bar{\partial}] = -2[X, JX],$$

such that $(\partial, \bar{\partial}, \partial_0)$ forms a frame for the set of all sections of the complexified tangent bundle on M. In the terminology of the CR geometry, e.g. [30] the linear differential operator $\bar{\partial}$ is called a CR operator.

We denote the corresponding dual coframe by $(\mu, \bar{\mu}, \lambda)$. Strict pseudoconvexity of M translates to

$$d\lambda \wedge \lambda \neq 0,$$

since otherwise, $d\lambda \wedge \lambda = 0$ implies

$$d\lambda = c\mu \wedge \lambda + \bar{c}\bar{\mu} \wedge \lambda$$

for some complex function c defined on M. On one hand, it is clear that

$$d\lambda(\partial, \bar{\partial}) = 0.$$

On the other hand,

$$d\lambda(\partial, \bar{\partial}) = \frac{1}{2}\big\{\partial\lambda(\bar{\partial}) - \bar{\partial}\lambda(\partial) - \lambda([\partial, \bar{\partial}])\big\} = -\frac{1}{2}\lambda([\partial, \bar{\partial}])$$

implies

$$[\partial, \bar{\partial}] = -2[X, JX] \in \Gamma(D),$$

which contradicts the strict pseudoconvexity of M. Our choice of the vector fields $\partial, \bar{\partial}, \partial_0$ allows us to normalize λ and μ in the following way.

Lemma 1.2.9 *Let (M, D, J) be a strictly pseudoconvex CR manifold. The choice of the frame $(\partial, \bar{\partial}, \partial_0 = \mathrm{i}[\partial, \bar{\partial}])$ implies*

$$d\lambda = \mathrm{i}\mu \wedge \bar{\mu} + c\mu \wedge \lambda + \bar{c}\bar{\mu} \wedge \lambda \tag{1.5a}$$

$$d\mu = \alpha\mu \wedge \lambda + \beta\bar{\mu} \wedge \lambda, \tag{1.5b}$$

where $(\mu, \bar{\mu}, \lambda)$ is the corresponding coframe and c, α, β are some complex-valued functions on M.

Proof We first note that

$$[\partial, \bar{\partial}] = c^1_{12}\partial + c^2_{12}\bar{\partial} + c^3_{12}\partial_0 = -\mathrm{i}\partial_0.$$

Then, Lemma 0.2.4 implies

$$d\lambda = -c^3_{12}\,\mu \wedge \bar{\mu} - c^3_{13}\,\mu \wedge \lambda - c^3_{23}\,\bar{\mu} \wedge \lambda = \mathrm{i}\mu \wedge \bar{\mu} - c^3_{13}\,\mu \wedge \lambda - c^3_{23}\,\bar{\mu} \wedge \lambda,$$

and since λ is a real 1-form it follows that

$$c_{13}^3 = \bar{c}_{23}^3 = -c,$$

which implies

$$d\lambda = \mathrm{i}\mu \wedge \bar{\mu} + c\mu \wedge \lambda + \bar{c}\bar{\mu} \wedge \lambda.$$

To show (1.5b) is satisfied, we note that

$$d\mu = -c_{12}^1\, \mu \wedge \bar{\mu} - c_{13}^1\, \mu \wedge \lambda - c_{23}^1\, \bar{\mu} \wedge \lambda = \alpha\mu \wedge \lambda + \beta\bar{\mu} \wedge \lambda,$$

since $c_{12}^1 = 0$. ∎

Now, we are able to compute the commutators of the vector fields stated above.

Corollary 1.2.10 *The following statements are satisfied:*

(1.6a) $$[\partial, \partial_0] = -\alpha\partial - \bar{\beta}\bar{\partial} - c\partial_0$$

(1.6b) $$[\bar{\partial}, \partial_0] = -\beta\partial - \bar{\alpha}\bar{\partial} - \bar{c}\partial_0.$$

Proof Lemma 0.2.4 and equations (1.5) give us the above statements for the commutators. ∎

From now on, without loss of any generality, we assume that the 1-forms μ, λ satisfy (1.5).

We consider the representatives (μ, λ) and (μ', λ') with the transformation (1.4) satisfying

$$d\lambda = \mathrm{i}\mu \wedge \bar{\mu} \quad \mathrm{mod}\{\lambda\}, \quad d\lambda' = \mathrm{i}\mu' \wedge \bar{\mu}' \quad \mathrm{mod}\{\lambda'\}.$$

Then there exists complex functions $f \neq 0$ and h such that

(1.7) $$\lambda' = |f|^2\lambda, \qquad \mu' = f(\mu + h\lambda).$$

This is because for $\lambda' = a\lambda$ and $\mu' = f\mu + \ell\lambda$, where $a \neq 0$ is a real function and $b \neq 0, \ell$ are complex functions, the normalization of λ',

$$d\lambda' = \mathrm{i}\frac{a}{|f|^2}\mu' \wedge \bar{\mu}' \quad \mathrm{mod}\{\lambda'\},$$

implies $a = |f|^2$, and as such μ' can be written as

$$\mu' = f(\mu + h\lambda),$$

where $h = \frac{\ell}{f}$.

Now we can fix our convention as follows: any other distinguished frame $(\partial', \bar{\partial}', \partial_0')$ and coframe $(\mu', \bar{\mu}', \lambda')$ may be expressed through the original frame and coframe by

$$\partial' = \mathrm{e}^{-\tau-\mathrm{i}\varphi}\,\partial, \qquad \partial_0' = \mathrm{e}^{-2\tau}\left(\partial_0 - h\partial - \bar{h}\bar{\partial}\right), \tag{1.8a}$$

$$\mu' = \mathrm{e}^{\tau+\mathrm{i}\varphi}(\mu + h\lambda), \qquad \lambda' = \mathrm{e}^{2\tau}\,\lambda, \tag{1.8b}$$

where τ is a real-valued function, $\varphi \in [0, 2\pi)$ and h is a complex function.

For the choice (μ', λ') the functions α', β', c' take the following forms:

Proposition 1.2.11 *For the choice of the pair* (λ', μ') *defined by* (1.8)*, the following statements are satisfied*

1. $h = -\mathrm{i}\bar{\partial}(\tau + \mathrm{i}\varphi)$
2. $\alpha' = \mathrm{e}^{2\tau}\left(\alpha - \partial_0(\tau + \mathrm{i}\varphi) + h\partial(\tau + \mathrm{i}\varphi) + \partial h + hc\right)$
3. $\beta' = \mathrm{e}^{-2\tau+2\mathrm{i}\varphi}\left(\beta + h\bar{\partial}(\tau + \mathrm{i}\varphi) + \bar{\partial}h + \bar{c}h\right)$
4. $c' = \mathrm{e}^{-\tau-\mathrm{i}\varphi}\left(c - 2\mathrm{i}\bar{h} + \partial(\tau + \mathrm{i}\varphi)\right)$.

Proof Let $f := \mathrm{e}^{\tau+\mathrm{i}\varphi}$. From (1.8) we have that

$$\begin{aligned} d\mu' &= df \wedge \mu + hdf \wedge \lambda + fd\mu + fdh \wedge \lambda + fhd\lambda \\ &= (-\bar{\partial}f + \mathrm{i}fh)\mu \wedge \bar{\mu} + (-\partial_0 f + h\partial f + f\alpha + f\partial h + cfh)\mu \wedge \lambda \\ &\quad + (h\bar{\partial}f + \beta f + f\bar{\partial}h + \bar{c}fh)\bar{\mu} \wedge \lambda. \end{aligned} \tag{1.9}$$

On the one hand,

$$\alpha'\mu' \wedge \lambda' + \beta'\bar{\mu}' \wedge \lambda' = \alpha' f|f|^2\mu \wedge \lambda + \beta'\bar{f}|f|^2\bar{\mu} \wedge \lambda. \tag{1.10}$$

Comparing (1.9) and (1.10), we get

$$\bar{\partial}\log f = \mathrm{i}h, \quad -\partial_0 f + h\partial f + f\alpha + f\partial h + cfh = \alpha' f|f|^2,$$

and also

$$\beta' \bar{f}|f|^2 = h\bar{\partial} f + \beta f + f\bar{\partial} h + \bar{c} f h.$$

Substituting $f = \mathrm{e}^{\tau + \mathrm{i}\varphi}$ into the above expressions gives α' and β'.

To prove the last property, we notice that

$$\begin{aligned} d\lambda' &= d(f\bar{f}\lambda) = \bar{f} df \wedge \lambda + f d\bar{f} \wedge \lambda + |f|^2 d\lambda \\ &= \bar{f}(\partial f)\mu \wedge \lambda + \bar{f}(\bar{\partial} f)\bar{\mu} \wedge \lambda + f(\partial \bar{f})\mu \wedge \lambda + f(\bar{\partial}\bar{f})\bar{\mu} \wedge \lambda \\ &\quad + \mathrm{i}|f|^2 \mu \wedge \bar{\mu} + c|f|^2 \mu \wedge \lambda + \bar{c}|f|^2 \bar{\mu} \wedge \lambda \\ &= \mathrm{i}|f|^2 \mu \wedge \bar{\mu} + \big(\bar{f}\partial f + f\partial \bar{f} + c|f|^2\big)\mu \wedge \lambda \\ &\quad + \big(f\bar{\partial}\bar{f} + \bar{f}\bar{\partial} f + \bar{c}|f|^2\big)\bar{\mu} \wedge \lambda \end{aligned} \tag{1.11}$$

On the other hand,

$$\begin{aligned} \mathrm{i}\mu' \wedge \bar{\mu}' + c'\mu' \wedge \lambda' + \bar{c}'\bar{\mu} \wedge \lambda' = \mathrm{i}|f|^2 \mu \wedge \bar{\mu} &+ \big(\mathrm{i}|f|^2 \bar{h} + c' f |f|^2\big)\mu \wedge \lambda \\ &+ \big(-\mathrm{i}|f|^2 h + \bar{c}' \bar{f} |f|^2\big)\bar{\mu} \wedge \lambda. \end{aligned} \tag{1.12}$$

Comparing (1.11) and (1.12), it follows that

$$\bar{f}\partial f + f\partial \bar{f} + c|f|^2 = \mathrm{i}|f|^2 \bar{h} + c' f |f|^2$$

and thus,

$$c' = \frac{1}{f}\big(c - \mathrm{i}\bar{h} + \partial \log |f|^2\big). \quad \blacksquare$$

1.3 Sasakian geometry

Sasakian manifolds are examples of CR manifolds. We first give the definition of Sasakian manifolds based on Kähler manifolds. We begin with Kähler structures which may be considered as special Riemannian structures.

1.3.1 Kähler manifolds

Definition 1.3.1 *A Kähler structure on a Riemannian manifold (M, g) is given by a closed real 2-form Ω, i.e. $d\Omega = 0$ and an endomorphism J on $\Gamma(TM)$ satisfying the following conditions:*

(i) J *is an almost complex structure of the tangent bundle, i.e.* $J^2 = -\mathrm{Id}$.

(ii) $g(X,Y) = g(JX, JY) \qquad \forall X, Y \in \Gamma(M)$.

(iii) $\Omega(X,Y) = g(JX,Y) \qquad \forall X, Y \in \Gamma(M)$.

(iv) J *is integrable, i.e. Nijenhuis tensor defined by* (1.1) *vanishes.*

Then (M, g, Ω, J) is called a Kähler manifold. We note that because of the condition (i) the manifold M is of even dimension.

Let (S, g_S, θ) be a contact Riemannian manifold of dimension $2n+1$, that is, a manifold S equipped with a Riemannian metric g_S and a contact form θ. We now consider the Riemannian metric

$$g = dr^2 + r^2 g_S$$

defined on the cone $C(S) = \mathbb{R}^+ \times S$ where r is the coordinate along $\mathbb{R}^+$. Now we have the following definition of a Sasakian manifold

1.3.2 Sasakian manifolds

Definition 1.3.2 *A contact Riemannian manifold* (S, g_S, θ) *is called Sasakian, if* $(C(S), g, \Omega, J)$ *with* $\Omega = d(r^2\theta)$ *and the almost complex structure* J *defined by*

$$\Omega(\cdot,\cdot) = g(J\cdot,\cdot)$$

is a Kähler manifold.

A Sasakian manifold S is naturally embedded into its cone as a real hypersurface. Indeed, a Sasakian manifold can be considered as

$$S = S \times \{1\} \subset C(S),$$

and the CR structure is naturally defined by

$$D = TS \cap J(TS),$$

where J is the almost complex structure on the cone.

We now recall from [2] the characterisation of the Sasakian manifolds in the class of strictly pseudoconvex CR manifolds. For the convenience of the readers we provide the proof.

Proposition 1.3.3 (Alekseevsky et al. [2]) *Let (M, D, J) be a strictly pseudoconvex CR manifold M of dimension $2n+1$ with contact form θ where $D = \ker\theta$ and Z is the corresponding Reeb vector field to θ. Assume that Z is an infinitesimal CR-automorphism, i.e.*

$$\mathscr{L}_Z J = 0.$$

Then, M is Sasakian.

Proof The proof is local. Let $\zeta_1, \dots, \zeta_n$ be the generators of $D^{1,0}$ and set $N = M \times \mathbb{R}^+$. We also assume that r is the coordinate along $\mathbb{R}^+$. We then consider the vectors $\zeta_1, \dots, \zeta_n, Z - \mathrm{i}r\partial_r$, such that

$$(\zeta_1, \dots, \zeta_n, \bar{\zeta}_1, \dots, \bar{\zeta}_n, Z - \mathrm{i}r\partial_r, Z + \mathrm{i}r\partial_r)$$

is a frame for the complexified tangent bundle of N. The endomorphism J_N defined on the tangent bundle of N given by

$$J_N|_D = J, \quad J_N(Z) = -r\partial_r, \quad J_N(r\partial_r) = Z,$$

is an almost complex structure on the tangent bundle of N and

$$(\zeta_1, \dots, \zeta_n, Z - \mathrm{i}r\partial_r)$$

are generators of $T^{1,0}N$. Now we check the integrability of $J|_N$, that is, the involutivity of $T^{1,0}N$. The endomorphism $J|_N$ is integrable on D due to the involutivity of $D^{1,0}$. We only need to check that

$$[Z - \mathrm{i}r\partial_r, \zeta_j] \in \Gamma(T^{1,0}N).$$

$$\begin{aligned} [Z - \mathrm{i}r\partial_r, \zeta_j] &= [Z - \mathrm{i}r\partial_r, X - \mathrm{i}JX] = [Z, X - \mathrm{i}JX] \\ &= [Z, X] - \mathrm{i}[Z, JX] = ([Z, X] - \mathrm{i}J[Z, X]) \in T^{1,0}N \end{aligned}$$

because $\mathcal{L}_Z J = 0$, where $\zeta_j = X - \mathrm{i}JX,\ X \in \Gamma(D)$. Moreover, notice that the metric defined on M by

$$g = d\theta(J\cdot,\cdot) + \theta^2$$

is a Riemannian metric. We are now in the position to define the metric g_N given by

$$g_N = r^2 g + dr^2$$

and the 2-form Ω by

$$\Omega = d(r^2\theta)$$

on N. The form Ω is clearly exact and hence, closed. We now show that these structures are compatible, i.e. (N, g, Ω, J) is a Kähler manifold. We need to show that $g(J_N X, Y) = \Omega(X,Y)$ or equivalently $g_N(X,Y) = \Omega(J_N X, Y)$ is satisfied, where $X, Y \in \Gamma(N)$. If $X, Y \in \ker\theta \cap \ker dr$, then

$$g_N(X,Y) = r^2 g(X,Y) = r^2 d\theta(JX,Y) = \Omega(JX,Y) = \Omega(J_N X, Y).$$

If $X = Z$ and $Y \in \ker\theta$, then

$$g_N(X,Y) = \Omega(J_N X, Y) = 0.$$

If $X = Y = Z$, then

$$g_N(Z,Z) = r^2 = \Omega(r\partial_r, Z) = \Omega(J_N Z, Z).$$

If $X = r\partial_r$ and $Y \in \ker dr$, then

$$g_N(X,Y) = \Omega(J_N X, Y) = 0.$$

If $X = Y = r\partial_r$, then

$$g_N(r\partial_r, r\partial_r) = r^2 = \Omega(-Z, r\partial_r) = \Omega(J_N r\partial_r, r\partial_r). \qquad \blacksquare$$

We also notice that the choice of the scaling function is unique: Let $JZ = h\,dr$ and $g_N = fg + dr^2$. Then the 2-form Ω is defined to be

$$\Omega = d(f\theta),$$

since

$$g_N(X,Y) = \Omega(J_N X, Y),$$

for $X, Y \in \ker\theta \cap \ker dr$ and $d\Omega = 0$. Furthermore,

$$f = \frac{1}{2} f' h$$

because

$$g_N(Z,Z) = f = f'\Omega(h\partial_r, Z) = \Omega(J_N Z, Z)$$

for $X = Y = Z$, and

$$h^2 = \frac{1}{2} f' h$$

because

$$g_N(h\partial_r, h\partial_r) = h^2 = f'\Omega(-Z, h\partial_r) = \Omega(J_N h\partial_r, h\partial_r).$$

Hence, because of

$$\frac{1}{2} f' = h,$$

and

$$f = h^2, \quad f' = 2hh',$$

it follows

$$h' = 1, \quad h = r, \quad f = r^2.$$

1.4 Shearfree congruences

In this section we first introduce the notion of shearfreeness which may be interpreted as a generalisation of the conformal Killing equation.

A vector field p is called a *conformal Killing* if the following equation is satisfied.

$$\mathscr{L}_p g = \rho g \tag{1.13}$$

where ρ is a function defined on $\mathcal{M}$. If $\rho = 0$, the vector field p is called *Killing vector*.

1.4.1 Shearfree vector fields

Definition 1.4.1 *A shearfree congruence is an even-dimensional Lorentzian manifold* $(\mathcal{M}, g)$ *equipped with a foliation into integral curves of a nowhere vanishing vector field* p*, such that*

(i) The vector field p *is null, i.e.* $g(p,p) = 0$.

(ii) $\mathscr{L}_p g = \rho g + \theta \vee \psi$*, where* $\theta = g(p,\cdot)$*,* ρ *is a real function on* $\mathcal{M}$ *and* ψ *is a 1-form. This condition means that the metric* g *changes conformally under the flow of* p *if restricted to the subspaces*

$$p^{\perp} = \{X \in \Gamma(T\mathcal{M}); g(X,p) = 0\}.$$

We call p *a shearfree vector field (with respect to* $(\mathcal{M}, g)$*) if it satisfies conditions (i) and (ii) above. The pair* (g,p)*, is called a shearfree metric.*

One can find some examples of shearfree metrics in the relevant chapters of [63]. Here, we denote the symmetrized tensor product,

$$\frac{1}{2}\left(g(p,.) \otimes \psi + \psi \otimes g(p,.)\right)$$

by $\theta \vee \psi$.

Below we show that any null conformal Killing vector field is geodesic.

Lemma 1.4.2 *Let* $(\mathcal{M}, g)$ *be a Lorentzian manifold and* p *be a null conformal Killing vector field, that is,* $\mathscr{L}_p g = \rho g$*, where* ρ *is a real function defined on* $\mathcal{M}$*. Then,* p *is geodesic, i.e.*

$$\nabla_p p = f p,$$

where f *is a real function on* $\mathcal{M}$ *and* ∇ *is the Levi-Civita connection.*

Proof For any vector field W on $p^{\perp}$, the identity (0.1b), for $X = V = p$ implies that

$$g(\nabla_p p, W) + g(p, \nabla_p W) = 0.$$

On the other hand, the Kosul formula (0.2) yields

$$g(\nabla_W p, p) = 0.$$

Moreover,

$$[p, W] = \nabla_p W - \nabla_W p$$

implies

$$g(p, [p, W]) = g(p, \nabla_p W).$$

Therefore, on one hand,

$$(\mathscr{L}_p g)(p, W) = \mathscr{L}_p\left(g(p, W)\right) - g(\mathscr{L}_p p, W) - g(p, \mathscr{L}_p W) = -g(p, \mathscr{L}_p W),$$

and on the other hand,

$$(\mathscr{L}_p g)(p, W) = \rho g(p, W) = 0,$$

which implies that

$$g(p, \mathscr{L}_p W) = 0.$$

Finally, from

$$g(\nabla_p p, W) = -g(p, \nabla_p W) = -g(p, [p, W]) = g(p, \mathscr{L}_p W) = 0$$

it follows that $\nabla_p p \in W^{\perp}$ which means there exists a function f such that $\nabla_p p = fp$. ∎

A shearfree vector field also possesses the same property. Hence, a shearfree congruence is in fact a foliation of $\mathcal{M}$ into null-geodesics, which can be interpreted as light rays.

Proposition 1.4.3 *Any shearfree vector field p of a Lorentzian manifold $(\mathcal{M}, g)$ is geodesic.*

Proof It follows from Proposition 0.3.2 that for any null vector field p, $L_p^* p = 0$ is satisfied. Substituting $V = Y = p$ into (0.6),

$$(\mathscr{L}_p g)(X, p) = -g(L_p p + L_p^* p, X) = g(\nabla_p p, X) \tag{1.14}$$

is satisfied. On the other hand, since p is shearfree, there exists a function ρ and 1-form ψ such that

$$(1.15) \qquad (\mathscr{L}_p g)(X,p) = \big(\rho g + \theta \vee \psi\big)(X,p) = \rho g(X,p) + \frac{1}{2} g(p,X)\psi(p)$$
$$= g\left([\rho + \frac{1}{2}\psi(p)]p, X\right) = g(\beta p, X),$$

where $\beta = \rho + \frac{1}{2}\psi(p)$. Comparing the right hand sides of (1.14) and (1.15) and also taking into account that g is non-degenerate, we see

$$\nabla_p p = \beta p. \qquad \blacksquare$$

Notice that shearfreeness of p depends only on the conformal class of g and is preserved under scaling of p. Such rescalings can be used to simplify the structure in the sense of the following definitions.

Definition 1.4.4 *A shearfree congruence is called diverging if the function ρ in (ii) of the definition 1.4.1 does not vanish; it is called distinguished in the opposite case, i.e. if $\rho = 0$.*
A shearfree vector field p is said to be standard if

$$\nabla_p p = 0.$$

We summarize some properties of the shearfree vector fields below.

Proposition 1.4.5 *For a shearfree vector field p, the following properties hold*

1. *Any rescaling of p is also a shearfree vector field.*
2. *The vector field p can be rescaled, so that $\nabla_p p = 0$, i.e. p is standard.*
3. *Being standard is equivalent to $p \lrcorner d\theta = 0$, which, in turn, is equivalent to*

$$\mathscr{L}_p \theta = d(p \lrcorner \theta) + p \lrcorner d\theta = 0.$$

Proof For any nonzero function t, notice that

$$\begin{aligned}\mathscr{L}_{tp}g &= t\mathscr{L}_p g + g(p,\cdot)\vee dt = t\rho g + tg(p,.)\vee\psi + g(p,\cdot)\vee dt\\ &= \hat{\rho}g + g(tp,\cdot)\vee\hat{\psi},\end{aligned}$$

where $\hat{\rho} = t\rho$ and $\hat{\psi} = \psi + d\log f$, that is, tp is a shearfree vector field. To prove that p can be made standard, notice that there exists a function β such that

$$\nabla_p p = \beta p,$$

since p is geodesic. Furthermore,

$$\nabla_{fp} fp = 0$$

is equivalent to

$$\beta = -p(\log f),$$

since

$$\nabla_{fp} fp = fp(f)p + f\nabla_{fp}p = fp(f)p + f^2\nabla_p p = \left(fp(f) + f^2\beta\right)p.$$

To prove the third property, we first notice that for any vector field X

$$\begin{aligned}d\theta(p,X) &= \frac{1}{2}\left\{p(X\lrcorner\theta) - X(p\lrcorner\theta) - [p,X]\lrcorner\theta\right\}\\ &= \frac{1}{2}\left\{pg(p,X) - g(p,[p,X])\right\} = \frac{1}{2}\{g(\nabla_p p,X) + g(p,\nabla_p X)\\ &\qquad - g(p,\nabla_p X) + g(p,\nabla_X p)\} = \frac{1}{2}\left\{g(\nabla_p p,X) + g(p,\nabla_X p)\right\}\\ &= \frac{1}{2}g(\nabla_p p,X),\end{aligned}$$

since it follows from the Koszul formula (0.2), that $g(\nabla_X p,p) = 0$. Therefore, $\nabla_p p = 0$ is equivalent to $p\lrcorner d\theta = 0$. ∎

1.4.2 Shearfree congruences in Lorentzian manifolds

Some of the material provided in this subsection are well-known, and since we are unaware of a precise reference to the literature we provide the details.

Let $\mathcal{M}$ be a 4-dimensional manifold equipped with the Lorentzian metric g of signature (3,1), and p be a nowhere vanishing null vector field over $\mathcal{M}$. We then consider the complexified tangent bundle of the manifold $\mathcal{M}$, $\mathbb{C} \otimes T\mathcal{M}$, and extend the metric g by complex linearity, denoted also by g, in the following form

$$g(X + \mathrm{i}Y, U + \mathrm{i}V) = g(X,U) - g(Y,V) + \mathrm{i}(g(X,V) + g(Y,U)),$$

for all $X, Y, U, V \in \Gamma(T\mathcal{M})$.

Then, there exists a complex frame (e_1, e_2, ℓ, p) for the complexified tangent bundle satisfying $e_1 = \bar{e}_2$ and

$$g(e_1, e_2) = g(\ell, p) = 1, \quad g(\ell,\ell) = g(\ell, e_1) = \overline{g(\ell, e_2)} = 0.$$

Indeed, because of the non-degeneracy of the metric g, there exists a null vector field ℓ such that $g(\ell, p) = 1$. One can complement ℓ, p by two real orthonormal vectors ϵ_1, ϵ_2 such that $(\epsilon_1, \epsilon_2, \ell, p)$ forms a frame for the tangent bundle. Now we set

$$e_1 = \frac{1}{\sqrt{2}}(\epsilon_1 - \mathrm{i}\epsilon_2), \quad e_2 = \frac{1}{\sqrt{2}}(\epsilon_1 + \mathrm{i}\epsilon_2).$$

Also assume that $(\theta^1, \theta^2, \theta^3, \theta^4)$ is the dual coframe to (e_1, e_2, ℓ, p) satisfying

$$\theta^1 = \bar{\theta}^2, \quad \theta^3 = g(p, \cdot). \tag{1.16}$$

Hence, the metric g takes the following form

$$g = 2(\theta^1\theta^2 + \theta^3\theta^4), \tag{1.17}$$

and the Gram matrix for both g and its inverse g^{-1} with respect to the frame (e_1, e_2, ℓ, p) and coframe (1.16) is of the following form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{1.18}$$

The 1-forms defined by (1.16) are not unique and are defined up to the subgroup of the Lorentz group preserving the null direction p. We consider the null vector field

$$p' = Ap,$$

where A is a nowhere vanishing real function defined on $\mathcal{M}$. We also consider the frame (e_1', e_2', ℓ', p') defined by the following transformations

$$\begin{pmatrix} e_1' \\ e_2' \\ \ell' \\ p' \end{pmatrix} = T \begin{pmatrix} e_1 \\ e_2 \\ \ell \\ p \end{pmatrix}, \quad T_1^4 = T_2^4 = T_3^4 = 0,\, T_4^4 = A \tag{1.19}$$

such that the Gram matrix of the metric with respect to the transformations (1.19), takes the form (1.18). After some straightforward computations, the matrix T takes the form

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi} & 0 & 0 & B \\ 0 & \mathrm{e}^{-\mathrm{i}\varphi} & 0 & \bar{B} \\ -\frac{\bar{B}}{A}\mathrm{e}^{\mathrm{i}\varphi} & -\frac{B}{A}\mathrm{e}^{-\mathrm{i}\varphi} & \frac{1}{A} & -\frac{|B|^2}{A} \\ 0 & 0 & 0 & A \end{pmatrix}, \tag{1.20}$$

where B is a complex-valued function. Therefore, the corresponding matrix for the coframe $(\theta'^1, \theta'^2, \theta'^3, \theta'^4)$ is of the form

$$\begin{pmatrix} \theta'^1 \\ \theta'^2 \\ \theta'^3 \\ \theta'^4 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\varphi} & 0 & \bar{B} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\varphi} & B & 0 \\ 0 & 0 & A & 0 \\ -\frac{B}{A}\mathrm{e}^{-\mathrm{i}\varphi} & -\frac{\bar{B}}{A}\mathrm{e}^{\mathrm{i}\varphi} & -\frac{|B|^2}{A} & \frac{1}{A} \end{pmatrix} \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \\ \theta^4 \end{pmatrix}. \tag{1.21}$$

We now consider the complex functions σ and κ defined by

$$d\theta^1 \wedge \theta^1 \wedge \theta^3 = \sigma\, \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 \tag{1.22a}$$

$$d\theta^3 \wedge \theta^1 \wedge \theta^3 = \kappa\, \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4. \tag{1.22b}$$

The next proposition relates the complex quantities σ and κ to the null vector field p. One can see that vanishing or not, of both κ and σ, is independent of the choice of the coframe. The proof can be found in [25], and for the convenience of the readers and in the sake of having the thesis self-contained, we give the proof.

Proposition 1.4.6 (Hill et al. [25]) *Vanishing or not of both functions σ and κ simultaneously, is an invariant property of the congruence.*

Proof Let $(\theta'^1, \theta'^2, \theta'^3, \theta'^4)$ be the coframe given by (1.21). By straightforward computations, it follows that

$$\begin{aligned}
d\theta'^3 \wedge \theta'^1 \wedge \theta'^3 &= -\kappa A^2 \,\mathrm{e}^{-\mathrm{i}\varphi}\, \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4, \\
d\theta'^1 \wedge \theta'^1 \wedge \theta'^3 &= A\,\mathrm{e}^{-2\mathrm{i}\varphi}\, d\theta^1 \wedge \theta^1 \wedge \theta^3 + AB\,\mathrm{e}^{-\mathrm{i}\varphi}\, d\theta^3 \wedge \theta^1 \wedge \theta^3 \\
&= (\sigma A\,\mathrm{e}^{-2\mathrm{i}\varphi} + \kappa AB\,\mathrm{e}^{-\mathrm{i}\varphi})\theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4. \qquad \blacksquare
\end{aligned}$$

The functions σ and κ are related to the frame and the metric as follows.

Proposition 1.4.7 *$(\mathcal{M}, g)$ be a Lorentzian manifold and ∇ the corresponding Levi-Civita connection. For the null vector field p, the relations*

$$\sigma = g(e_2, \nabla_{e_2} p), \quad \text{and} \quad \kappa = g(\nabla_p e_2, p)$$

are satisfied.

Proof From (1.22a), it follows that

$$d\theta^1 \wedge \theta^1 \wedge \theta^3(e_1, e_2, \ell, p) = \frac{1}{4}\sigma.$$

On the other hand,

$$\begin{aligned}
p \lrcorner (d\theta^1 \wedge \theta^1 \wedge \theta^3) &= (p \lrcorner d\theta^1) \wedge \theta^1 \wedge \theta^3 + (p \lrcorner \theta^1) d\theta^1 \wedge \theta^3 - (p \lrcorner \theta^3) d\theta^1 \wedge \theta^1 \\
&= (p \lrcorner d\theta^1) \wedge \theta^1 \wedge \theta^3,
\end{aligned}$$

because $p \lrcorner \theta^1 = p \lrcorner \theta^3 = 0$. Therefore,

$$d\theta^1 \wedge \theta^1 \wedge \theta^3(e_1, e_2, \ell, p) = -\frac{1}{2} d\theta^1(e_2, p).$$

Moreover, since

$$d\theta^1(e_2,p) = \frac{1}{2}\left(e_2(p \lrcorner \theta^1) - p(e_2 \lrcorner \theta^1) - \theta^1([e_2,p])\right) = -\frac{1}{2}g(e_2,[e_2,p])$$

is satisfied thus,

$$\sigma = g(e_2,[e_2,p]).$$

Now the property (0.1b) of the Levi-Civita connection for $X = p, Y = W = e_2$ gives

$$g(\nabla_p e_2, e_2) = 0.$$

Therefore,

$$\sigma = g(e_2, \nabla_{e_2}p) - g(e_2, \nabla_p e_2) = g(e_2, \nabla_{e_2}p).$$

A similar argument can be applied for κ. ∎

We now show that vanishing of κ is equivalent to p being geodesic.

Proposition 1.4.8 *The null vector field p is geodesic if and only if $\kappa = 0$.*

Proof First we see that for the vector fields X, Y

$$\begin{aligned} d\theta^3(X,Y) &= X\big(\theta^3(Y)\big) - Y\big(\theta^3(X)\big) - \theta^3([X,Y]) \\ &= Xg(p,Y) - Yg(p,X) - g(p, \nabla_X Y - \nabla_Y X) \\ &= g(\nabla_X p, Y) + g(p, \nabla_X Y) - g(\nabla_Y p, X) - g(p, \nabla_Y X) \\ &\quad - g(p, \nabla_X Y) + g(p, \nabla_Y X) = g(\nabla_X p, Y) - g(\nabla_Y p, X). \end{aligned}$$

Moreover, the 2-form $d\theta^3$ can be expressed as a combination of the basis, that is,

$$\begin{aligned} d\theta^3 &= c_{12}\theta^1 \wedge \theta^2 + c_{13}\theta^1 \wedge \theta^3 + c_{14}\theta^1 \wedge \theta^4 + c_{23}\theta^2 \wedge \theta^3 \\ &\quad + c_{24}\theta^2 \wedge \theta^4 + c_{34}\theta^3 \wedge \theta^4. \end{aligned}$$

Now, let $\kappa = 0$, which implies

$$d\theta^3 \wedge \theta^1 \wedge \theta^3 = 0.$$

Wedging $d\theta^3$ by $\theta^1 \wedge \theta^3$ gives us that $c_{24} = 0$. Moreover, $\bar{\kappa} = 0$ yields that

$$d\theta^3 \wedge \theta^2 \wedge \theta^3 = 0.$$

Wedging $d\theta^3$ by $\theta^2 \wedge \theta^3$ implies that $c_{14} = 0$. We see then that

$$p \lrcorner d\theta^3 = \beta\theta^3,$$

where $\beta = -\frac{1}{2}c_{34}$. Furthermore,

$$Xg(p,p) = g(\nabla_X p, p) + g(p, \nabla_X p) = 2g(p, \nabla_X p) = 0$$

implies that for any vector field X

$$d\theta^3(p, X) = g(\nabla_p p, X) = \beta\theta^3(X) = g(\beta p, X).$$

Hence, $g(\nabla_p p, X) = g(\beta p, X)$ and non-degeneracy of the metric implies

$$\nabla_p p = \beta p.$$

To see the converse statement, we assume that $\nabla_p p = \beta p$. Because of (0.1b), it follows that

$$\begin{aligned} pg(e_2, p) &= g(\nabla_p e_2, p) + g(e_2, \nabla_p p) \\ &= g(\nabla_p e_2, p) + \beta g(e_2, p) = g(\nabla_p e_2, p) = \kappa = 0. \qquad \blacksquare \end{aligned}$$

Vanishing of κ and σ simultaneously, is equivalent to shearfreeness of the vector field p. In order to clarify that we state the following lemma first.

Lemma 1.4.9 (Hill et al. [25]) *Let $\sigma = \kappa = 0$ everywhere for the null vector field p. Then the following statements are satisfied:*

(1.23a) $$(\mathscr{L}_p\theta^3) \wedge \theta^3 = 0$$

(1.23b) $$(\mathscr{L}_p\theta^1) \wedge \theta^1 \wedge \theta^3 = 0.$$

Proof The 2-form $d\theta^3$ can be written as

$$\begin{aligned} d\theta^3 = {} & c_{12}\theta^1 \wedge \theta^2 + c_{13}\theta^1 \wedge \theta^3 + c_{14}\theta^1 \wedge \theta^4 + c_{23}\theta^2 \wedge \theta^3 \\ & + c_{24}\theta^2 \wedge \theta^4 + c_{34}\theta^3 \wedge \theta^4, \end{aligned}$$

where c_{ij}'s are functions on $\mathcal{M}$. Vanishing of κ implies that $c_{14} = c_{24} = 0$, that is,

$$d\theta^3 = c_{12}\theta^1 \wedge \theta^2 + c_{13}\theta^1 \wedge \theta^3 + c_{23}\theta^2 \wedge \theta^3 + c_{34}\theta^3 \wedge \theta^4,$$

since

$$d\theta^3 \wedge \theta^3 \wedge \theta^2 = 0, \qquad d\theta^3 \wedge \theta^3 \wedge \theta^1 = 0.$$

Moreover, Cartan's formula yields

$$\mathscr{L}_p\theta^3 = p \lrcorner\, d\theta^3 + d(p \lrcorner\, \theta^3) = p \lrcorner\, d\theta^3 = \beta\theta^3,$$

where $\beta = -\frac{1}{2}c_{34}$. Wedging both sides of the last equality with θ^3 gives us (1.23a). To show the second identity we note that the 2-form $d\theta^1$ is also expressed as

$$\begin{aligned} d\theta^1 &= a_{12}\theta^1 \wedge \theta^2 + a_{13}\theta^1 \wedge \theta^3 + a_{14}\theta^1 \wedge \theta^4 + a_{23}\theta^2 \wedge \theta^3 \\ &\quad + a_{24}\theta^2 \wedge \theta^4 + a_{34}\theta^3 \wedge \theta^4, \end{aligned}$$

where a_{ij} is a function on $\mathcal{M}$. The condition $\sigma = 0$ implies

$$d\theta^1 = a_{12}\theta^1 \wedge \theta^2 + a_{13}\theta^1 \wedge \theta^3 + a_{23}\theta^2 \wedge \theta^3 + a_{34}\theta^3 \wedge \theta^4.$$

Therefore, Cartan's magic formula yields

$$\mathscr{L}_p\theta^1 = \beta\theta^3,$$

where $\beta = -\frac{1}{2}a_{34}$. Wedging the last equation with $\theta^1 \wedge \theta^3$ gives (1.23b). ∎

Theorem 1.4.10 $\sigma = \kappa = 0$ *if and only if the null vector field* p *is shearfree.*

Proof First we assume that the vector field p is shearfree, that is,

$$\mathscr{L}_p g = \rho g + g(p,\cdot) \vee \nu,$$

where ρ is a function and ν is a real 1-form. Therefore,

$$(\mathscr{L}_p g)(e_2, e_2) = \rho g(e_2, e_2) + g(p,\cdot) \vee \nu(e_2, e_2) = 0.$$

On the other hand,

$$(\mathscr{L}_p g)(e_2, e_2) = g(\nabla_{e_2} p, e_2) + g(e_2, \nabla_{e_2} p) = 2g(e_2, \nabla_{e_2} p) = 2\sigma$$

is satisfied. Comparing the above expressions, if follows that $\sigma = 0$. Moreover, from Proposition 1.4.3, the vector field p is geodesic and due to Lemma 1.4.8, $\kappa = 0$.

For the converse statement, we notice that from Lemma 1.4.9, the conditions $\sigma = \kappa = 0$ imply that

$$\mathscr{L}_p \theta^3 = a\theta^3, \qquad \mathscr{L}_p \theta^1 = b\theta^1 + c\theta^3, \tag{1.24}$$

where b, c are complex functions and a is real. We then have

$$\begin{aligned} \mathscr{L}_p g = {} & \mathscr{L}_p \theta^1 \otimes \theta^2 + \theta^1 \otimes \mathscr{L}_p \theta^2 + \mathscr{L}_p \theta^2 \otimes \theta^1 + \theta^2 \otimes \mathscr{L}_p \theta^1 \\ & + \mathscr{L}_p \theta^3 \otimes \theta^4 + \theta^3 \otimes \mathscr{L}_p \theta^4 + \mathscr{L}_p \theta^4 \otimes \theta^3 + \theta^4 \otimes \mathscr{L}_p \theta^3. \end{aligned} \tag{1.25}$$

Substituting (1.24) into (1.25), we get

$$\mathscr{L}_p g = \rho g + \theta^3 \vee \psi,$$

where $\rho = b + \bar{b}$ and $\psi = \left(c\theta^1 + \bar{c}\theta^2 - (b + \bar{b} - a)\theta^4 + \mathscr{L}_p \theta^4\right)$. ∎

1.5 A Lift of a CR manifold

Three-dimensional CR manifolds are very closely related to the shearfree congruences in 4-dimensional Lorentzian manifolds. They are well-known to physicists for constructing nontrivial solutions for the Einstein equations in the 4-dimensional Lorentzian space[55, 56, 57].

1.5.1 Shearfree metrics

Given a 3-dimensional CR manifold M, it is known [25] that, one can construct a class of CR invariant metrics on the line bundle $\mathcal{M} = M \times \mathbb{R}$ equipped with a shearfree congruence.

For a given pair of the forms (μ, λ) on the CR manifold M, satisfying (1.5) we introduce the conformal class of Lorentzian metrics defined on the line bundle $\mathcal{M} = M \times \mathbb{R}$ by

$$g = 2P^2 \left(\mu\bar{\mu} + \lambda(dr + W\mu + \overline{W}\bar{\mu} + H\lambda)\right) \tag{1.26}$$

where r is the coordinate in the direction of $\mathbb{R}$. Here, P is nowhere zero (real), H (real) and W (complex) are some functions defined on $\mathcal{M}$.

Here the symmetric tensor product $\mu\bar{\mu}$ is defined by

$$\mu\bar{\mu} = \frac{1}{2}\left(\mu \otimes \bar{\mu} + \bar{\mu} \otimes \mu\right)$$

The function r can also be another trivialising function. Precisely, let $r' = r'(x, y, z, r)$ when (x, y, z) is the coordinate system at a point $p \in M$. If $\frac{\partial r'}{\partial r} \neq 0$, then r can be replaced by r'.

We also note that the 4-dimensional Lorentzian manifold $\mathcal{M}$ projects to M by the natural projection

$$\pi : \mathcal{M} \longrightarrow M,$$

and by abuse of notation, we denote the pullback of the forms on M by the same notation. For instance, we denote by μ the pullback 1-form $\pi^*(\mu)$.

Lemma 1.5.1 *The metric defined by* (1.26)*, possesses the following properties*

1. *The vector field $p = \partial_r$ is a shearfree vector field, that is, (g, p) is a sherafree metric.*

2. *The family of shearfree metrics is CR invariant. That is, the metric g' corresponding to the alternative choice (μ', λ'), given by* (1.4)*, belongs to the class of metrics.*

Proof To show that the vector field p is shearfree, we just need to look at the Lie derivative of the shearfree metric defined by (1.26) along the vector field p.

$$\begin{aligned}\mathscr{L}_p g &= \rho g + 2P^2\big(\mathscr{L}_p\lambda\big)\big(dr + W\mu + \overline{W}\bar{\mu} + H\lambda\big)\\ &\quad + 2P^2\lambda(\mathscr{L}_p\big(dr + W\mu + \overline{W}\bar{\mu} + H\lambda\big)\\ &= \rho g + P^2\lambda\psi = \rho g + g(p,\cdot)\vee\psi,\end{aligned}$$

where $\psi = 2\mathscr{L}_p\big(dr + W\mu + \overline{W}\bar{\mu} + H\lambda\big)$. We note that

$$\mathscr{L}_p\lambda = p \lrcorner d\lambda + d(p \lrcorner \lambda) = 0,$$

and also

$$g(p,p) = 0, \quad g(p,\partial) = 0, \quad g(p,\bar{\partial}) = 0, \quad g(p,\lambda) = P^2.$$

Let (μ', λ') be another representative relating to (μ, λ) with the transformation given by (1.7). We then have

$$\begin{aligned}g' &= 2P'^2\left(\mu'\bar{\mu}' + \lambda(dr + W'\mu' + \overline{W'}\bar{\mu}' + H'\lambda')\right)\\ &= 2P^2\left(\mu\bar{\mu} + \lambda(dr + W\mu + \overline{W}\bar{\mu} + H\lambda)\right),\end{aligned}$$

where

$$P = P'|f|, \quad W = \bar{h} + W'f, \quad H = |h|^2 + W'fh + \overline{W'}\bar{f}\bar{h} + |f|^2 H'. \quad \blacksquare$$

Having chosen a representative (μ, λ) from the CR structure and a Lorentzian metric $g \in [g]$, will give the following definition.

Definition 1.5.2 *Let* $(M, [(\mu, \lambda)])$ *be a 3-dimensional CR manifold satisfying* (1.5). *The pair* (g, ∂_r), *where* $g \in [g]$ *defined by* (1.26) *on the trivial line bundle* $M \times \mathbb{R}$ *is called a lift of the CR manifold to a spacetime.*

Now the following theorem holds.

Theorem 1.5.3 (Hill et al.[25]) *Let* $(\mathcal{M}, g)$ *be a 4-dimensional Lorentzian manifold. Suppose that* $\mathcal{M}$ *is foliated by a 3-parameter family of curves which are shearfree geodesics. Then* $\mathcal{M}$ *is locally a cartesian product* $\mathcal{M} = M \times \mathbb{R}$ *with* M *being a 3-dimensional CR manifold. The CR structure* $(M, [(\mu, \lambda)])$

on M is uniquely determined by $(\mathcal{M}, g)$ and the shearfree congruence on $\mathcal{M}$. If r is a real coordinate such that $k = \partial_r$ is tangent to the congruence, then the Lorentzian metric g on $\mathcal{M}$ can be locally represented by (1.26) *with some specific functions P, W, H depending on the choice of the representatives (μ, λ) of the corresponding CR structure.*

Refer to, e.g. [25] and references therein for a detailed proof.

In chapter 4, we generalise Theorem 1.5.3 for the subconformal and almost CR manifolds.

Chapter 2

Shearfree geometry and the embedding of CR manifolds

In this chapter, we first explain the embedding problem in more details and give some examples of embeddable CR manifolds. We also explain the approach used in [25] to show that the embeddability of a 3-dimensional CR manifold is related to the existence of a solution of Maxwell's equations in addition to vanishing of some components of the Ricci curvature of the shearfree Lorentzian metric corresponding to the CR structure.

The most famous approach relating a CR manifold to a conformal Lorentzian manifold is the Fefferman metric, which will be explained in this chapter.

2.1 Embedding of 3-dimensional CR manifolds

We recall from (1.2), that any smooth real hypersurface in $\mathbb{C}^2$ has a CR structure inherited from $\mathbb{C}^2$. Now let (M^3, D, J) be any CR manifold. The (local) embeddability (or realisability) problem, asks if there exists a (local) embedding

$$\iota : M \to \mathbb{C}^2$$

such that the CR structures on ιM induced by ι and by $\mathbb{C}^2$, coincide. In other words,

$$\iota_* \partial = a(z_1, z_2)\frac{\partial}{\partial z_1} + b(z_1, z_2)\frac{\partial}{\partial z_2}$$

at each point of ιM, where (z_1, z_2) are holomorphic coordinates on $\mathbb{C}^2$ and

$$\bar{\partial} = X + \mathrm{i}JX, \quad \text{and} \quad \partial = X - \mathrm{i}JX$$

for some non-vanishing section X of D. This problem is equivalent to finding two non-constant functionally independent CR functions, that is, solutions to the complex linear PDE

$$\bar{\partial}\phi = 0, \tag{2.1}$$

where two CR functions ϕ_1 and ϕ_2 are called functionally independent provided

$$d\phi_1 \wedge d\phi_2 \neq 0.$$

Indeed, if there exist two non-constant functionally independent CR functions $\phi_1 = \phi_1(x, y, u)$ and $\phi_2 = \phi_2(x, y, u)$, where (x, y, u) are coordinates on M, then we are able to define the embedding $\iota : M \to \mathbb{C}^2$ as follows

$$\iota(x, y, u) = (\phi_1(x, y, u), \phi_2(x, y, u)) \in \mathbb{C}^2,$$

since for any vector field X in M

$$\iota_*(X) = X(\phi_1)\frac{\partial}{\partial \phi_1} + X(\bar{\phi}_1)\frac{\partial}{\partial \bar{\phi}_1} + X(\phi_2)\frac{\partial}{\partial \phi_2} + X(\bar{\phi}_2)\frac{\partial}{\partial \bar{\phi}_2}$$

is satisfied. Hence, $X = \partial$ implies

$$\iota_* \partial = \partial\phi_1 \frac{\partial}{\partial \phi_1} + \partial\phi_2 \frac{\partial}{\partial \phi_2},$$

since $\bar{\partial}\phi_1 = \bar{\partial}\phi_2 = 0$. For the converse statement assume that $\bar{\partial}$ is the CR operator on M and that (z_1, z_2) are the holomorphic coordinates on $\mathbb{C}^2$. At any point, $\bar{\partial}$ is a linear combination of $\frac{\partial}{\partial \bar{z}_1}$ and $\frac{\partial}{\partial \bar{z}_2}$. The restriction of (z_1, z_2) to M implies that

$$\bar{\partial}\phi_1 = \bar{\partial}\phi_2 = 0$$

with $d\phi_1 \wedge d\phi_2 \neq 0$ where $\phi_1 = z_1$ and $\phi_2 = z_2$.

It is also worthwhile to mention that any holomorphic function of a CR function is also a CR function. Indeed, let $\phi : M \to \mathbb{C}$ be a CR function, i.e. $\bar{\partial}\phi = 0$. Take (x, y, u) to be a coordinate system at a point $p \in M$, and $f : \mathbb{C} \to \mathbb{C}$ a holomorphic function, i.e $\frac{\partial f}{\partial \bar{w}} = 0$, where $w \in \mathbb{C}$. Therefore, the CR operator $\bar{\partial}$ can be expressed as

$$\bar{\partial} = \gamma_1 \frac{\partial}{\partial z^1} + \gamma_2 \frac{\partial}{\partial \bar{z}^1} + \gamma_3 \frac{\partial}{\partial z^2} + \gamma_3 \frac{\partial}{\partial \bar{z}^2},$$

where γ_i is a complex function and $z^1 = x + \mathrm{i}y$ and $u = \mathrm{Rel}\, z^2$, where (z^1, z^2) is the coordinate system of $\mathbb{C}^2$. Then the chain rule implies

$$\frac{\partial}{\partial \bar{z}^j}(f \circ \phi) = \frac{\partial f}{\partial w}\frac{\partial \phi}{\partial \bar{z}^j} + \frac{\partial f}{\partial \bar{w}}\frac{\partial \bar{\phi}}{\partial \bar{z}^j} = \frac{\partial f}{\partial w}\frac{\partial \phi}{\partial \bar{z}^j},$$

and

$$\frac{\partial}{\partial z^j}(f \circ \phi) = \frac{\partial f}{\partial w}\frac{\partial \phi}{\partial z^j} + \frac{\partial f}{\partial \bar{w}}\frac{\partial \bar{\phi}}{\partial z^j} = \frac{\partial f}{\partial w}\frac{\partial \phi}{\partial z^j},$$

where $j = 1, 2$. Therefore,

$$\bar{\partial}(f \circ \phi) = \frac{\partial f}{\partial w}\left(\gamma_1 \frac{\partial \phi}{\partial z^1} + \gamma_2 \frac{\partial \phi}{\partial \bar{z}^1} + \gamma_3 \frac{\partial \phi}{\partial z^2} + \gamma_3 \frac{\partial \phi}{\partial \bar{z}^2}\right) = \frac{\partial f}{\partial w}\bar{\partial}\phi = 0.$$

Not all 3-dimensional CR manifolds are embeddable. The first counterexamples appeared in the context of linear partial differential equations by H. Lewy in [39, 38]. L. Nirenberg proved in the works [44, 45] that the PDE (2.1) has no solution but constant functions where the operator ∂ is defined by

$$\partial = a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial u} = \partial_1 + \mathrm{i}\partial_2$$

for complex functions $a_j = a_j(x, y, u)$, where (x, y, u) is a coordinate system at a point in a neighborhood of $\mathbb{R}^3$, with ∂_1, ∂_2 and $[\partial_1, \partial_2]$ linearly independent [44, 45].

There are also some CR structures, which admit a non-constant CR function ϕ and every other CR function is a holomorphic function of ϕ [59]. Refer to [27, 51, 15, 14] for more examples of non-embeddable CR manifolds.

We now consider a 3-dimensional CR manifold $(M, [(\mu, \lambda)])$ satisfying (1.5) and also assume a CR function ζ exists. Let $\zeta = x + \mathrm{i}y$ and $\mu = d\zeta$. One can choose a real coordinate u such that (x, y, u) is a coordinate system at a point $p \in M$. The operator ∂ can be written as a linear combination of $\partial_\zeta, \partial_{\bar{\zeta}}, \partial_u$. By rescaling ∂, we get

$$\partial = \partial_\zeta - L\partial_u, \quad \partial_0 = \mathrm{i}(\bar{\partial}L - \partial\bar{L})\partial_u, \tag{2.2}$$

where $L(z, \bar{z}, u)$ is a complex-valued function subject to

$$\bar{\partial}L - \partial\bar{L} \neq 0.$$

The 1-form λ takes the form

$$\lambda = \frac{du + Ldz + \bar{L}d\bar{z}}{\mathrm{i}(\bar{\partial}L - \partial\bar{L})}. \tag{2.3}$$

In addition, the complex function c defined by (1.5a) is of the following form

$$c = -\partial\log(\bar{\partial}L - \partial\bar{L}) - \partial_u L.$$

Refer also to [23].

The following lemma plays a crucial role in finding a CR function.

Lemma 2.1.1 *[25] Let φ be a smooth complex-valued 1-form defined locally in $\mathbb{R}^n$, $n \geq 3$, such that $\varphi \wedge \bar{\varphi} \neq 0$. Then,*

$$d\varphi \wedge \varphi \equiv 0$$

if and only if there exists a smooth complex function ζ and a smooth non-vanishing complex function h such that

$$\varphi = hd\zeta, \qquad d\zeta \wedge d\bar{\zeta} \neq 0.$$

The proof provided here is slightly different from the one given in [25].

Proof Let φ be a complex 1-form defined in a neighborhood $U \in \mathbb{R}^n$ such that $\varphi \wedge \bar{\varphi} \neq 0$ and $d\varphi \wedge \varphi = 0$. We define the real 1-forms $\varphi_1 = \mathrm{Re}\,\varphi$ and $\varphi_2 = \mathrm{Im}\,\varphi$. The condition $\varphi \wedge \bar{\varphi} \neq 0$ implies $\varphi_1 \wedge \varphi_2 \neq 0$, since

$$\varphi \wedge \bar{\varphi} = (\varphi_1 + \mathrm{i}\varphi_2) \wedge (\varphi_1 - \mathrm{i}\varphi_2) = -2\mathrm{i}\varphi_1 \wedge \varphi_2.$$

On the other hand, from $d\varphi \wedge \varphi = 0$, the following identity

$$d\varphi_1 \wedge \varphi_1 - d\varphi_2 \wedge \varphi_2 + \mathrm{i}(d\varphi_2 \wedge \varphi_1 + d\varphi_1 \wedge \varphi_2) = 0,$$

is satisfied. We now consider the real and imaginary parts of the above expression.

$$d\varphi_1 \wedge \varphi_1 - d\varphi_2 \wedge \varphi_2 = 0,$$
$$d\varphi_2 \wedge \varphi_1 + d\varphi_1 \wedge \varphi_2 = 0.$$

For dimensions $n \geq 4$, wedging the above equations with φ_2 gives

$$d\varphi_1 \wedge \varphi_1 \wedge \varphi_2 = 0, \qquad d\varphi_2 \wedge \varphi_1 \wedge \varphi_2 = 0.$$

In dimension $n = 3$, the above expressions are automatically satisfied. We can now apply the Frobenius theorem for the real 1-forms φ_1, φ_2. Therefore, there exists a coordinate system $(x, y, u^\ell), \ell = 3, \dots, n$ in U such that

$$\varphi_1 = f_1^1 dx + f_2^1 dy, \qquad \varphi_2 = f_1^2 dx + f_2^2 dy,$$

where f_ℓ^k are some real functions such that $f_1^1 f_2^2 - f_2^1 f_1^2 \neq 0$. Thus the 1-form φ can be expressed as

$$\varphi = c_1 dx + c_2 dy,$$

where $c_1 = f_1^1 + \mathrm{i} f_1^2$ and $c_2 = f_2^1 + \mathrm{i} f_2^2$, satisfying

$$c_1 \bar{c}_2 - \bar{c}_1 c_2 = -2\mathrm{i}(f_1^1 f_2^2 - f_2^1 f_1^2) \neq 0.$$

We also see that neither c_1 nor c_2 can be zero. Using the complex coordinate $z = x + \mathrm{i}y$ and taking into account that

$$dz = dx + \mathrm{i}dy, \qquad d\bar{z} = dx - \mathrm{i}dy$$

the complex 1-form φ takes the following form

$$\varphi = \frac{c_1 - \mathrm{i}c_2}{2} dz + \frac{c_1 + \mathrm{i}c_2}{2} d\bar{z}.$$

We first consider the cases that either $c_1 = \mathrm{i}c_2$ or $c_1 = -\mathrm{i}c_2$. In any of these cases, the 1-form φ can be written as $\varphi = h d\zeta$ where $h \neq 0$. Suppose neither

$c_1 = \mathrm{i}c_2$ nor $c_1 = -\mathrm{i}c_2$ are satisfied. Therefore, If $c_1 - \mathrm{i}c_2 \equiv 0$, then $\varphi = hd\zeta$ where $h = 2\mathrm{i}c_2$ and $\zeta = \bar{z}$. for the functions $h(z,\bar{z})$ and $\zeta(z,\bar{z})$, we set

$$\varphi = \frac{c_1 - \mathrm{i}c_2}{2}\Big[dz + \frac{c_1 + \mathrm{i}c_2}{c_1 - \mathrm{i}c_2}d\bar{z}\Big] = h\zeta_z\big(dz + \frac{\zeta_{\bar{z}}}{\zeta_z}d\bar{z}\big),\ \zeta_z \neq 0.$$

We now consider the equation

$$\zeta_{\bar{z}} = \mu\zeta_z, \quad \zeta_z \neq 0, \qquad \mu = \frac{c_1 + \mathrm{i}c_2}{c_1 - \mathrm{i}c_2}. \tag{2.4}$$

The rest of the proof splits into two cases:

1. If $f_1^1 f_2^2 - f_2^1 f_1^2 > 0$, then
$$\mid \mu \mid < 1.$$
In fact, it follows that
$$\Big|\frac{c_1 + \mathrm{i}c_2}{c_1 - \mathrm{i}c_2}\Big|^2 = \frac{\mid c_1 \mid^2 + \mid c_2 \mid^2 - \mathrm{i}(c_1\bar{c}_2 - \bar{c}_1 c_2)}{\mid c_1 \mid^2 + \mid c_2 \mid^2 + \mathrm{i}(c_1\bar{c}_2 - \bar{c}_1 c_2)} = \frac{\mid c_1 \mid^2 + \mid c_2 \mid^2 - 2(f_1^1 f_2^2 - f_2^1 f_1^2)}{\mid c_1 \mid^2 + \mid c_2 \mid^2 + 2(f_1^1 f_2^2 - f_2^1 f_1^2)} < 1.$$
The Beltrami equation given by (2.4) has a smooth solution ζ since the function μ is smooth. Then it follows that
$$\varphi = hd\zeta, \quad h = \frac{c_1 - \mathrm{i}c_2}{2\zeta_z}.$$

2. If $f_1^1 f_2^2 - f_2^1 f_1^2 < 0$, we then consider the conjugate of φ. We have that
$$\bar{\varphi} = \frac{\bar{c}_1 + \mathrm{i}\bar{c}_2}{2}\Big[\frac{\bar{c}_1 - \mathrm{i}\bar{c}_2}{\bar{c}_1 + \mathrm{i}\bar{c}_2}dz + d\bar{z}\Big] = \bar{h}\bar{\zeta}_{\bar{z}}\big(\frac{\bar{\zeta}_z}{\bar{\zeta}_{\bar{z}}}dz + d\bar{z}\big).$$
A straightforward computation shows that
$$\frac{1}{\mid \bar{\mu} \mid} < 1.$$
Therefore, the Beltrami equation
$$\bar{\zeta}_{\bar{z}} = \frac{1}{\bar{\mu}}\bar{\zeta}_z$$
has a solution. Thus,
$$\varphi = hd\zeta, \quad h = \frac{c_1 - \mathrm{i}c_2}{2\zeta_z},$$
is satisfied. ∎

2.1.1 The Embedding problem

We now recall some well-known results regarding the embaddability of strictly pseudoconvex CR manifolds. The first one is a global result by Boutet de Monvel, who proved that

Theorem [Boutet de Monvel, [6]] Let M be a $(2n+1)$-dimensional compact strictly pseudoconvex CR manifold with $n \geq 2$. Then M is embeddable in $\mathbb{C}^{n+1}$.

For the case $n = 1$, the counterexamples were provided by H. Grauert [22], H. Rossi [60] and D. Burns [9]. See also [41] for more details.

In 1982, M. Kuranishi proved that CR manifolds of dimension $2n + 1$, with $n \geq 4$, are embeddable. After that, T. Akahori showed that it is also true for $n = 3$. Thus, the following theorem holds.

Theorem [Kuranishi [32] and Akahori, [1]] Any strictly pseudoconvex $(2n + 1)$-dimensional CR manifold with $n \geq 3$, is embeddable in $\mathbb{C}^{n+1}$.

2.1.2 Some examples of embeddable CR manifolds

The next theorem guarantees that any real analytic CR manifold of dimension $2n + 1$ is locally embeddable. For the convenience of the readers we provide the following proof [29].

Theorem 2.1.2 *[29] Any real analytic CR manifold is locally embeddable.*

Proof Let (M, V) be a CR manifold of dimension $2n + 1$. Also assume $(x_1, \ldots, x_{2n+1})$ is the coordinate system at a point $x \in U \subset M$. The vectors

$$L_j = \sum_{k=1}^{2n+1} \alpha_{jk}(x_1, \ldots, x_{2n+1}) \frac{\partial}{\partial x_k}, \quad j = 1, \ldots, n,$$

form a basis for V with each component α_{jk} a real analytic function at $x \in U$. Because of the integrability condition we have

$$[L_j, L_\ell] = \sum_{r=1}^{n} \beta_{jkr}(x_1, \ldots, x_{2n+1}) L_r, \quad j, \ell = 1, \ldots, n,$$

where each component β_{jkr} is a real analytic complex-valued function. Since $\dim_{\mathbb{C}} V = n$ it implies that there is an element of the basis which does not belong to the set of generators of V and without loss of generality we may assume $\frac{\partial}{\partial x_{2n+1}} \notin V$. We now complexify the coordinate x_{2n+1} and extend the function α_{jk} by real analyticity and set

$$M_j = \sum_{k=1}^{2n+1} \alpha_{jk}(x_1, \dots, x_{2n}, x_{2n+1} + \mathrm{i}t)\frac{\partial}{\partial x_k}, \quad j = 1, \dots, n,$$

and also define the operator

$$M_{n+1} = \frac{\partial}{\partial_{2n+1}} + \mathrm{i}\frac{\partial}{\partial t}$$

with the real coordinate t. We now consider the subbundle of the complexified tangent bundle defined by

$$V_0 = \text{linear span}_{\mathbb{C}}\{M_1, \dots, M_{n+1}\}.$$

We show that the almost complex structure V_0 is integrable. For the indices $j, \ell \neq n+1$, the commutators $[M_j, M_\ell]$ are sections of V_0 because of the integrability condition on V. It just remains to check that the commutator of M_{n+1} and M_j for $j = 1, \dots, n$ is also a section of V_0. To do that, we notice

$$[M_j, \frac{\partial}{\partial t}] = \sum_{r=k}^{2n+1} \gamma_{jk}(x_1, \dots, x_{2n}, x_{2n+1} + \mathrm{i}t)\frac{\partial}{\partial x_k},$$

which is a section of V_0. Thus, the subbundle V_0 is a real analytic integrable almost complex structure and by Newlander-Nirenberg theorem 1.2.4, is complex. Now, M is given as the hypersurface $\{t = 0\}$ in this complex structure. ∎

It is interesting that the theorem above is also true globally [4].

Other interesting class of examples of embeddable CR manifolds are Sasakian manifolds. Indeed, any Sasakian manifold S is embedded into its cone

$$C(S) = \mathbb{R}^+ \times S,$$

and thus, is locally embedded into $\mathbb{C}^2$.

There are lots of surveys around embeddability of 3-dimensional CR manifolds. For instance, we conclude this part by recalling two remarkable theorems by H. Jacobowitz in [28], which we use later.

The first theorem is the following

Theorem 2.1.3 (Jacobowitz) *[28] The following statements are equivalent*

1. *A CR manifold (M^{2n+1}, D, J) is embeddable in a neighborhood of the point $p \in M$.*
2. *There exists a complex vector field Y with $\mathscr{L}_Y \Gamma(D^{0,1}) \subset \Gamma(D^{0,1})$ and $Y \notin \Gamma(D^{1,0} \oplus D^{0,1})$.*

Here the Lie bracket of a complex vector field $X = X_1 + \mathrm{i}X_2$ along the complex vector field $Y = Y_1 + \mathrm{i}Y_2$, means

$$[Y, X] = [Y_1 + \mathrm{i}Y_2, X_1 + \mathrm{i}X_2] = [Y_1, X_1] - [Y_2, X_2] + \mathrm{i}([Y_1, X_2] + [Y_2, X_1]).$$

In our setting (1.5), the complex vector field Y is expressed as

$$Y = \gamma_1 \partial + \gamma_2 \bar{\partial} + \gamma_3 \partial_0,$$

where γ_1, γ_2 and $\gamma_3 \neq 0$ are complex functions satisfying the following PDE's

$$\begin{aligned} \mathrm{i}\gamma_1 - \bar{c}\gamma_3 + \bar{\partial}(\gamma_3) &= 0, \\ \bar{\partial}(\gamma_1) - \gamma_3 \beta &= 0. \end{aligned}$$

Another theorem by H. Jacobowitz [28] provides a criterion for embeddability of CR manifolds in terms of the canonical bundle of the CR structure. We first introduce the notion of the canonical bundle of a CR manifold.

2.1.3 The canonical bundle of a CR manifold

Definition 2.1.4 *Let (M, D, J) be a CR manifold of dimension $2n+1$ and $D^{\mathbb{C}} = D^{1,0} \oplus D^{0,1}$ the eigenspace decomposition of J. The canonical bundle is*

$$\mathcal{K} = \left\{ \Omega \in \Lambda^{n+1}(M) \otimes \mathbb{C} : \forall L \in D^{0,1}, L \lrcorner \Omega = 0 \right\},$$

where $\Lambda^{n+1}(M)$ *is the space of smooth* $(n+1)$*-forms on* M*.*

The canonical bundle $\mathcal{K}$ is a complex line bundle over M. Indeed, let

$$D^{1,0} = \text{linear span}_{\mathbb{C}}\{L_1, \dots, L_n\},$$

and $(\mu_1, \dots, \mu_n)$ be the dual coframe of $(L_1, \dots, L_n)$. One can complement $(\mu_1, \dots, \mu_n, \bar{\mu}_1, \dots, \bar{\mu}_n)$ with a real 1-form λ such that

$$(\mu_1, \dots, \mu_n, \bar{\mu}_1, \dots, \bar{\mu}_n, \lambda)$$

is a coframe of the space of 1-forms on the complexified tangent bundle over M. Then any $(n+1)$-form Ω can be written as a linear combination of $(n+1)$-forms generated by $(\mu_1, \dots, \mu_n, \bar{\mu}_1, \dots, \bar{\mu}_n, \lambda)$. The conditions

$$\bar{L}_i \lrcorner \Omega = 0, \quad i = 1, \dots, n,$$

simply imply that Ω is a multiple of

$$\mu_1 \wedge \dots \wedge \mu_n \wedge \lambda,$$

which means $\mathcal{K}$ is a complex bundle of rank one.

In dimension 3, in our setting, the canonical bundle corresponding to the CR manifold $(M, [(\mu, \lambda)])$ is the bundle spanned by the complex 2-form $\mu \wedge \lambda$.

Among the non-vanishing sections of the canonical bundle, the d-closed ones play an important role in the embedding of CR manifolds. The following proposition relates the existence of a non-vanishing d-closed section of the canonical bundle to a $\bar{\partial}$-problem.

Proposition 2.1.5 *A CR manifold* $(M, [(\mu, \lambda)])$ *admits locally a nowhere zero d-closed section of the canonical bundle if and only if the* $\bar{\partial}$*-problem*

$$\bar{\partial} \log \psi = -\bar{c} \tag{2.5}$$

has a solution. Here c is the structure function from (1.5a).

Proof Taking into account (1.5a),

$$d(\psi\,\mu\wedge\lambda) = \bar\partial\psi\,\bar\mu\wedge\mu\wedge\lambda - \bar c\psi\mu\wedge\bar\mu\wedge\lambda = (\bar\partial\psi + \bar c\psi)\,\bar\mu\wedge\mu\wedge\lambda$$

vanishes if and only if (2.5) is satisfied with non-vanishing ψ. ∎

We also recall the notion of strongly independent functions. The complex-valued functions $\phi_1,\dots,\phi_n$ are strongly independent at a point if

$$d\phi_1\wedge\cdots\wedge d\phi_n\wedge d\bar\phi_1\wedge\cdots\wedge d\bar\phi_n \neq 0$$

is satisfied. In dimension 3, the above condition simply reduces to the existence of a CR function ϕ with $d\phi\wedge d\bar\phi\neq 0$. Moreover, the condition $d\phi\wedge d\bar\phi\neq 0$ is equivalent to $\partial\phi\neq 0$, since

$$d\phi\wedge d\bar\phi = |\partial\phi|^2\mu\wedge\bar\mu + \partial\phi\partial_0\bar\phi\mu\wedge\lambda - \bar\partial\bar\phi\partial_0\phi\bar\mu\wedge\lambda,$$

which implies $\partial\phi\neq 0$. The converse is also true.

Theorem 2.1.6 (Jacobowitz) *[28] Let (M,D,J) be a CR manifold of dimension $2n+1$. Suppose that near some point $p\in M$, the CR manifold has n strongly independent CR functions. If the canonical bundle associated with the CR manifold has a non-vanishing d-closed section, then the CR manifold is embeddable near p.*

The converse of the above theorem is also true. Let $\phi_1,\dots,\phi_{n+1}$ be $n+1$ functionally independent CR functions, i.e

$$d\phi_1\wedge\cdots\wedge d\phi_{n+1}\neq 0,\quad \bar L_j(\phi_k)=0,\quad j,=1,\dots,n,\quad k=1,\dots,n+1,$$

where

$$D^{1,0} = \text{linear span}_{\mathbb{C}}\{L_1,\dots,L_n\}.$$

The non-vanishing $(n+1)$-form Ω defined by

$$\Omega = d\phi_1\wedge\cdots\wedge d\phi_{n+1}$$

is a d-closed section of the canonical bundle.

As a consequence of Theorem 2.1.6 we have the following theorem.

Theorem 2.1.7 *Let $(M, [(\mu, \lambda)])$ be a CR manifold satisfying* (1.5)*. The CR manifold M is embeddable if and only if there exists a representative (μ, λ) with $\beta = 0$ and α nowhere zero.*

Proof Assume for the representative (μ, λ) of 1-forms, $\beta = 0$ and $\alpha \not\equiv 0$. From (1.5b) it follows that

$$d\mu = \alpha\mu \wedge \lambda.$$

The right hand side of the above expression is a non-zero section of the canonical bundle, since $\alpha \neq 0$ and the left hand side guarantees that it is d-closed. Moreover, wedging both sides with μ implies that

$$d\mu \wedge \mu = 0, \quad \textit{with} \quad \mu \wedge \bar{\mu} \not\equiv 0.$$

Therefore, Lemma 2.1.1 guarantees that there exists complex functions γ, ζ such that $\mu = \gamma d\zeta$ satisfying $d\zeta \wedge d\bar{\zeta} \neq 0$. The CR function ζ and the nonzero d-closed section of the canonical bundle fulfil the conditions of Theorem 2.1.6 and hence, the CR manifold M is embeddable.

For the converse statement we first note that for the representative (μ', λ'), because of Proposition 1.2.11, the condition $\beta' = 0$ is equivalent to saying that $-\mathrm{i}\bar{h}$ is a solution of the PDE

$$\partial u + u(c - u) = \mathrm{i}\bar{\beta},$$

and also non-vanishing of α' is equivalent to

$$\alpha \neq \partial_0 \log f + h\partial(\log f) + \partial h + hc,$$

where β, α are corresponding functions to the choice (μ, λ). Now let the CR manifold be embeddable, i.e. there are CR functions $z = x + \mathrm{i}y$ and ζ such that

$$dz \wedge d\zeta \neq 0.$$

One can choose the real coordinate u in such a way that (x, y, u) forms a coordinate system on M. We define the forms (μ', λ') as follows

$$\lambda' = |\zeta|^2\lambda, \quad \mu' = \zeta\mu,$$

where $\mu = dz$ and λ given by (2.3). It is now obvious that the functions $\beta = \alpha = 0$. By definition the function $h = 0$, introduced in (1.7), is a solution of

$$\partial u + u(u - c) = 0.$$

Therefore, for the new representative we have $\beta' = 0$. It just remains to show that

$$\alpha' = -\frac{1}{|f|^2}\partial_0\zeta \neq 0$$

which is automatically satisfied from

$$dz \wedge d\zeta = (\partial z dz + \partial_0 z\lambda) \wedge (\partial\zeta dz + \partial_0\zeta\lambda) = \partial_0\zeta dz \wedge \lambda,$$

because $\partial z = \partial_z z = 1$ and $\partial_0 z = A\partial_u z = 0$, where A is defined by (2.2). ∎

We note that the case $\alpha = 0, \beta = 0$ gives one CR function and the CR manifold M, may or may not be embeddable.

The following lemma which can be also found in [25], Lemma 3.23, is a consequence of Theorem 2.1.6. Here we give a shorter proof.

Lemma 2.1.8 *Let M be a CR manifold satisfying* (1.5) *with $\mu = d\zeta$ and $d\zeta \wedge d\bar{\zeta} \neq 0$. If in addition, the CR manifold admits a solution to the equation*

$$\partial_0\partial\eta = 0, \quad \textit{with} \quad \partial_0\eta \neq 0.$$

Then, M is embeddable.

Proof We first notice from (1.6a) that

$$[\partial, \partial_0]\eta = \partial\partial_0\eta - \partial_0\partial\eta = \partial\partial_0\eta = -c\partial_0\eta,$$

which is also equivalent to

$$\bar{\partial}\log\psi = -\bar{c},$$

where $\psi = \partial_0\eta$. Therefore, the non-zero 2-form

$$\psi\mu \wedge \lambda$$

is a d-closed section of the canonical bundle and hence, the CR manifold is embeddable. ∎

The next lemma which can also be found in [70] is another consequence of Theorem 2.1.6. The proof in [70] is completed by finding two functionally independent CR functions directly while showing that the existence of one CR function also admits a d-closed section of the canonical bundle.

We first note from (1.5) that taking exterior derivative of $d\lambda$ implies

$$\partial\bar{c} - \bar{\partial}c = \mathrm{i}(\alpha + \bar{\alpha}). \tag{2.6}$$

In the case where a CR function ζ exists, one can choose the 1-form μ to be $\mu = d\zeta$, and, therefore, $\alpha = 0$. Hence,

$$\bar{\partial}c = \partial\bar{c}.$$

Lemma 2.1.9 *Suppose a given CR manifold M admits a CR function ζ. Also assume the complex function c defined by* (1.5a) *does not depend on u, where (x,y,u) is the coordinate system at a point $p \in M$ and $\mu = d\zeta$ with $\zeta = x + \mathrm{i}y$. Then the CR manifold is locally embeddable.*

Proof We consider the system of complex PDEs

$$\begin{cases} \partial_\zeta \varphi = -c \\ \partial_{\bar{\zeta}} \bar{\varphi} = -\bar{c} \end{cases}$$

and we claim that the system has at least one real-valued solution. Since $\partial_u c = 0$, it follows from $\partial\bar{c} = \bar{\partial}c$ that

$$\partial_\zeta \bar{c} = \partial_{\bar{\zeta}} c.$$

Substituting $c = a + \mathrm{i}b$ and $\partial_\zeta = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$ into the above equation gives us

$$b_x = -a_y, \tag{2.7}$$

where $a = a(x,y)$ and $b = b(x,y)$. On the other hand, for a real function φ, the system is equivalent to

$$\begin{cases} \varphi_x = -2a \\ \varphi_y = 2b. \end{cases} \tag{2.8}$$

Therefore, the condition (2.7) implies the existence of a real-valued function φ which does not depend on u. In addition, we take into account that

$$\partial\varphi = \partial_\zeta\varphi - L\partial_u\varphi = \partial_\zeta\varphi = -c,$$

admits a non-vanishing d-closed section of the canonical bundle and thus, by Theorem 2.1.6, the CR manifold is embeddable. ∎

2.2 Embedding problem and shearfree metric

Lewandowski et al. in [37] and also Hill et al. in [25] prove a series of embeddability results in terms of shearfree congruences of Lorentzian manifolds. In order to present the main result of [25] we first need to recall the notion of distribution of α-planes and a null Maxwell field aligned with the null congruence of shearfree geodesics.

2.2.1 Lorentzian geometry and α-planes

Let $(\mathcal{M}, g)$ be a 4-dimensional Lorentzian manifold equipped with a foliation into integral curves of a non-vanishing null vector field p. We have the following canonical objects

(i) the 1-form $\theta = g(p, \cdot)$

(ii) the distribution $p^\perp = \{X \in \Gamma(T\mathcal{M}) \colon g(X,p) = 0\}$

(iii) the distribution of *screen spaces* $S \coloneqq p^\perp/p$.

Proposition 2.2.1 *On each screen space S_x there are two canonical almost complex structures J_x and $-J_x$.*

Proof Because the metric g is nondegenerate, at each point $X \in \mathcal{M}$, there exists a null vector ℓ and orthonormal vectors e, f so that (p, ℓ, e, f) forms an admissible frame for the tangent space at the point x, i.e.

$$g(p,\ell) = g(e,e) = g(f,f) = 1, \quad g(p,e) = g(\ell,e) = g(p,f) = g(p,e) = 0.$$

On the screen space S_x the bilinear form

$$h : S_x \times S_x \longrightarrow \mathbb{R}$$

defined by

$$h([e], \cdot) = g(e, \cdot), \qquad h([f], \cdot) = g(f, \cdot)$$

is a positive definite metric where $[e], [f]$ are generators of S_x. We notice that

$$h([e], [e]) = h([f], [f]) = 1.$$

On S_x the endomorphisms J_1 and J_2 defined by

$$J_1([e]) = [f], \qquad J_1([f]) = -[e],$$

and

$$J_2([e]) = -[f], \qquad J_2([f]) = [e]$$

are almost complex structures. It is now clear that $J_1 = -J_2$. ∎

Choose one of the two almost complex structures on S. Now we consider the complexification of the screen space, $\mathbb{C} \otimes S$. We also denote by J, the complex linear extension of J on $\mathbb{C} \otimes S$, that is,

$$J(X + \mathrm{i}Y) = JX + \mathrm{i}JY, \quad X, Y \in \Gamma(S).$$

Therefore, $\mathbb{C} \otimes S$ splits into its eigenspaces $S^{1,0} \oplus S^{0,1}$ where

$$S^{1,0} = \{X \in \Gamma(\mathbb{C} \otimes S) \colon JX = \mathrm{i}X\}, \quad \textit{and} \quad S^{0,1} = \{X \in \Gamma(\mathbb{C} \otimes S) \colon JX = -\mathrm{i}X\}.$$

Let

$$\pi \colon \mathbb{C} \otimes p^{\perp} \longrightarrow \mathbb{C} \otimes S$$

be the canonical projection map. The subspaces $\mathcal{P}^{1,0}$ and $\mathcal{P}^{0,1}$ of $\mathbb{C} \otimes p^{\perp}$ defined by

$$\mathcal{P}^{1,0} = \pi^{-1} S^{1,0}, \quad \textit{and} \quad \mathcal{P}^{0,1} = \pi^{-1} S^{0,1} \tag{2.9}$$

are called α-planes and β-planes, respectively. Notice that changing the orientation used in the definition of J results in interchanging the α-planes and the β-planes. Clearly,

$$\mathcal{P}^{1,0} \cap \mathcal{P}^{0,1} = \text{linear span}_{\mathbb{C}}\{p\}, \quad \textit{and} \quad \mathcal{P}^{1,0} + \mathcal{P}^{0,1} = \mathbb{C} \otimes p^{\perp}.$$

Definition 2.2.2 *We say that the complexified Ricci tensor of g vanishes on the α-planes $\mathcal{P}^{1,0}$, if $\mathrm{Ric}\,|_{\mathcal{P}^{1,0}} = 0$, i.e.*

$$\mathrm{Ric}(X_1, X_2) = 0 \quad \textit{for all} \quad X_1, X_2 \in \Gamma(\mathcal{P}^{1,0}).$$

Notice that vanishing of the complexified Ricci tensor on α-planes is equivalent to its vanishing on β-planes. Hence the definition above does not depend on the choice of J.

Definition 2.2.3 *Let $\mathcal{M}$ be a 4-dimensional manifold equipped with a Lorentzian metric g and a non-vanishing null vector field p. A complex frame (e_1, e_2, ℓ, p) is called adapted to (g, p), if e_1 is a section of α-planes, $e_2 = \bar{e}_1$ (and, hence, is a section of β-planes), and*

$$g(e_1, e_2) = 1, \quad g(\ell, \ell) = 0, \quad g(\ell, p) = 1, \quad g(\ell, e_1) = \overline{g(\ell, e_2)} = 0.$$

Proposition 2.2.4 *A 4-dimensional Lorentzian manifold $(\mathcal{M}, g)$ with a non-vanishing null vector field p, possesses (locally) a complex adapted frame.*

Proof Let ℓ be a null vector field such that $g(\ell, p) = 1$. Choose a unit vector field $\varepsilon_1 \in p^{\perp} \cap \ell^{\perp}$. Choose $\varepsilon_2 \in p^{\perp}$ such that $\pi\varepsilon_2 = J\pi(\varepsilon_1)$ and $g(\varepsilon_2, \ell) = 0$. Now, set

$$e_1 = \frac{1}{\sqrt{2}}(\varepsilon_1 - \mathrm{i}\varepsilon_2), \quad e_2 = \frac{1}{\sqrt{2}}(\varepsilon_1 + \mathrm{i}\varepsilon_2). \quad \blacksquare$$

It follows that the α-planes are spanned by (e_1, p) and the β-planes are spanned by (e_2, p).

Now, vanishing of the Ricci curvature on α-planes is equivalent to

(i) $\mathrm{R}_{11} = \mathrm{Ric}(e_1, e_1) = \overline{\mathrm{Ric}(e_2, e_2)} = 0$

(ii) $\mathrm{R}_{14} = \mathrm{Ric}(e_1, p) = \overline{\mathrm{Ric}(e_2, p)} = 0$

(iii) $\mathrm{R}_{44} = \mathrm{Ric}(p, p) = 0$.

Let the vector field p be shearfree for the metric g and $(\theta^1, \theta^2, \theta^3, \theta^4)$ be the dual coframe to (e_1, e_2, ℓ, p) achieved in Proposition 2.2.4. Then the Lorentzian metric g takes the form

$$g = 2(\theta^1\theta^2 + \theta^3\theta^4).$$

Below we cite a version of the celebrated Goldberg-Sachs theorem [21, 25, 20], which is a useful tool for computing certain components of the Weyl tensor of the Lorentzian metric g:

$$\mathrm{C}_{ijkl} = \mathrm{R}_{ijkl} + \frac{1}{6}\mathrm{R}\left(g_{ik}g_{lj} - g_{il}g_{kj}\right) + \frac{1}{2}\left(g_{il}\,\mathrm{R}_{kj} - g_{ik}\,\mathrm{R}_{lj} + g_{jk}\,\mathrm{R}_{li} - g_{jl}\,\mathrm{R}_{ki}\right),$$

where R_{ijkl} is the Riemann curvature, R_{kj} is the Ricci curvature and R is the scalar curvature. The following quantities are called Weyl scalars:

$$\Psi_0 = \mathrm{C}(k, e_1, k, e_1) = \mathrm{C}_{4141}, \qquad \Psi_1 = \mathrm{C}(k, \ell, k, e_1) = \mathrm{C}_{4341}\,.$$

2.2.2 Goldberg-Sachs Theorem

We now quote below a version of the Goldberg-Sachs theorem proved in [21] in terms of vanishing of certain components of the Ricci curvature.

Theorem 2.2.5 (Gover et al. [21, 25]) *Let $(\mathcal{M}, g)$ be a 4-dimensional Lorentzian manifold and p be a shearfree vector field. Also assume that the complexified Ricci curvature of g vanishes on the α-planes, i.e. $R_{11} = R_{14} = R_{44} = 0$ with respect to an adapted coframe $(\theta^1, \theta^2, \theta^3, \theta^4)$. Then*

$$\Psi_0 = \Psi_1 = 0.$$

As the first application of the Goldberg-Sachs theorem, we are now able to compute the function W defined in the shearfree metric (1.26).

We introduce the coframe $(\theta^1, \theta^2, \theta^3, \theta^4)$ with

$$\theta^1 = P\mu, \qquad \theta^2 = P\bar{\mu}, \tag{2.10a}$$

$$\theta^3 = P\lambda, \qquad \theta^4 = P\big(dr + W\mu + \overline{W}\bar{\mu} + H\lambda\big). \tag{2.10b}$$

Then, the metric (1.26) becomes

$$g = 2(\theta^1\theta^2 + \theta^3\theta^4). \tag{2.11}$$

The dual frame (e_1, e_2, e_3, e_4) to $(\theta^1, \theta^2, \theta^3, \theta^4)$ takes the form

$$e_1 = \frac{1}{P}(\partial - W\partial_r), \qquad e_2 = \frac{1}{P}(\bar{\partial} - \overline{W}\partial_r), \tag{2.12a}$$

$$e_3 = \frac{1}{P}(\partial_0 - H\partial_r), \qquad e_4 = \frac{1}{P}\partial_r. \tag{2.12b}$$

Therefore, the commutators of the frame fields (2.12) evaluate to

$$\begin{aligned}
[e_1, e_2] &= \left(\frac{\bar{\partial} P}{P^2} - \overline{W}\frac{P_r}{P^2}\right)e_1 + \left(-\frac{\partial P}{P^2} + W\frac{P_r}{P^2}\right)e_2 - \frac{\mathrm{i}}{P}e_3 + \left(-\frac{\mathrm{i}H}{P} + W_2 - \overline{W}_1\right)e_4,\\
[e_1, e_3] &= \left(\frac{\partial_0 P}{P^2} - H\frac{P_r}{P^2} - \frac{\alpha}{P}\right)e_1 - \frac{\bar{\beta}}{P}e_2 + \left(-\frac{\partial P}{P^2} + W\frac{P_r}{P^2} - \frac{c}{P}\right)e_3\\
&\quad + \left(-\frac{cH}{P} + W_3 - H_1 - \frac{\alpha W}{P} - \frac{\bar{\beta}\overline{W}}{P}\right)e_4,\\
[e_1, e_4] &= \frac{P_r}{P^2}e_1 + \left(-\frac{\partial P}{P^2} + W\frac{P_r}{P^2} + \frac{W_r}{P}\right)e_4,\\
[e_2, e_3] &= -\frac{\beta}{P}e_1 + \left(\frac{\partial_0 P}{P^2} - H\frac{P_r}{P^2} - \frac{\bar{\alpha}}{P}\right)e_2 + \left(-\frac{\bar{\partial} P}{P^2} + \overline{W}\frac{P_r}{P^2} - \frac{\bar{c}}{P}\right)e_3\\
&\quad + \left(-\frac{\bar{c}H}{P} + \overline{W}_3 - H_2 - \frac{\bar{\alpha}\overline{W}}{P} - \frac{\beta W}{P}\right)e_4,\\
[e_2, e_4] &= \frac{P_r}{P^2}e_2 + \left(-\frac{\bar{\partial} P}{P^2} + \overline{W}\frac{P_r}{P^2} + \frac{\overline{W}_r}{P}\right)e_4,\\
[e_3, e_4] &= \frac{P_r}{P^2}e_3 + \left(-\frac{\partial_0 P}{P^2} + H\frac{P_r}{P^2} + \frac{H_r}{P}\right)e_4,
\end{aligned}$$

where the subscripts 1, 2, 3 in the above expressions denote derivation with respect to the corresponding frame field (2.12). For example, H_1 means

$$\frac{1}{P}(\partial H - WH_r).$$

Now by using these commutator relations and Cartan's structure equation

$$d\theta^i + \Gamma^i_k \wedge \theta^k = 0,$$

for the metric (2.11) we find the connection forms listed below:

$$\Gamma^1_4 = (\frac{\mathrm{i}}{2P} + c^1_{14})\theta^1 + \frac{1}{2}(c^3_{23} + c^4_{24})\theta^3, \tag{2.13a}$$

$$\Gamma^1_1 = -c^2_{12}\theta^1 - c^1_{12}\theta^2 + \frac{1}{2}(c^2_{23} - c^1_{13} - c^4_{12})\theta^3 + \frac{\mathrm{i}}{2P}\theta^4 \tag{2.13b}$$

$$\Gamma^4_4 = c^4_{34}\theta^3 + c^3_{34}\theta^4 - \frac{1}{2}c^3_{23}\theta^2 - \frac{1}{2}c^3_{13}\theta^1 + \frac{1}{2}c^4_{14}\theta^1 + \frac{1}{2}c^4_{24}\theta^2 \tag{2.13c}$$

$$\Gamma^3_1 = (\frac{\mathrm{i}}{2P} - c^2_{24})\theta^2 - \frac{1}{2}(c^3_{13} + c^4_{14})\theta^3 \tag{2.13d}$$

$$\Gamma^4_1 = -c^2_{13}\theta^1 - \frac{1}{2}(c^4_{12} + c^2_{23} + c^1_{13})\theta^2 - c^4_{13}\theta^3 - \frac{1}{2}(c^4_{14} + c^3_{13})\theta^4 \tag{2.13e}$$

$$\Gamma^1_3 = \frac{1}{2}(-c^4_{12} + c^1_{13} + c^2_{23})\theta^1 + c^1_{23}\theta^2 + c^4_{23}\theta^3 + \frac{1}{2}(c^4_{24} + c^3_{23})\theta^4 \tag{2.13f}$$

where c^k_{mn} are the structure constants defined by

$$[e_m, e_n] = c^k_{mn} e_k.$$

We also notice that (0.5) implies $\Gamma^1_2 = \Gamma_{11} = 0$ and $\Gamma^3_4 = \Gamma_{33} = 0$.

Remark 2.2.6 *Note that, because of the choice of the coframe* (2.10a)*, complex conjugation of the connection forms interchanges the indices* 1 *and* 2 *and keeps the indices* 3*,* 4 *unchanged, for example,* $\overline{\Gamma^1_4} = \Gamma^2_4$.

In the following lemma we are able to compute the complex function W introduced in the shearfree metric.

Lemma 2.2.7 *Assume that the complexified Ricci tensor of the shearfree metric* (1.26)*, vanishes on the α-planes, i.e*

$$\mathrm{R}_{11} = \mathrm{R}_{14} = \mathrm{R}_{44} = 0,$$

with respect to the frame field (2.12)*. Then, the complex-valued function W takes the following form*

$$W = \mathrm{i}x\,\mathrm{e}^{-\mathrm{i}r} + y, \tag{2.14}$$

where x, y are complex, r-independent functions.

Proof Since the conditions $\mathrm{R}_{11} = \mathrm{R}_{14} = \mathrm{R}_{44} = 0$ are satisfied and the null vector field ∂_r is a shearfree vector field, the Goldberg-Sachs theorem 2.2.5 implies that

$$\mathrm{C}_{1414} = \mathrm{C}_{1434} = 0.$$

Moreover, since the Weyl tensor is conformally invariant we may assume, for simplicity, the conformal factor $P = 1$. We also note that

$$\mathrm{C}_{1434} = \mathrm{R}^2_{434} + \frac{1}{2}\,\mathrm{R}_{14}\,.$$

To compute the component R^2_{434} of the Riemann curvature, we look at the coefficient of the 2-form $\theta^3 \wedge \theta^4$ of the Γ^2_4 as follows

$$d\,\Gamma^2_4 + \Gamma^2_k \wedge \Gamma^k_4 = \mathrm{R}^2_{4k\ell}\,\theta^k \wedge \theta^\ell, \quad k < \ell,$$

which simplifies to

$$d\,\Gamma^2_4 + \Gamma^2_2 \wedge \Gamma^2_4 + \Gamma^2_4 \wedge \Gamma^4_4 = \mathrm{R}^2_{4k\ell}\,\theta^k \wedge \theta^\ell, \quad k < \ell.$$

Substituting $\Gamma^2_4, \Gamma^2_2, \Gamma^4_4$ into the above equation and taking the exterior derivative we see that

$$\mathrm{R}^2_{434} = -\frac{1}{2}W_{rr} - \frac{\mathrm{i}c}{4} + \frac{\mathrm{i}}{4}W_r.$$

Moreover,

$$\mathrm{R}_{14} = \mathrm{R}^1_{114} + \mathrm{R}^2_{124} + \mathrm{R}^3_{134}\,.$$

To compute R^1_{114} we look at the coefficient of the 2-from $\theta^1 \wedge \theta^4$ of the Cartan's structure equation for Γ^1_1

$$d\Gamma^1_1 + \Gamma^1_3 \wedge \Gamma^3_1 + \Gamma^1_4 \wedge \Gamma^4_1 = \mathrm{R}^1_{1k\ell}\,\theta^k \wedge \theta^\ell, \quad k < \ell.$$

After taking the exterior derivative, we get

$$\mathrm{R}^1_{114} = -\frac{3}{4}\mathrm{i}c^4_{14} - \frac{\mathrm{i}}{4}c^3_{13}.$$

The component $\mathrm{R}^2_{124} = 0$, since the coefficient of the 2-form $\theta^2 \wedge \theta^4$ of the Cartan's structure equation for Γ^2_1 is 0.

In order to compute the component R^3_{134} we look at the coefficient of the 2-form $\theta^3 \wedge \theta^4$ of the structure equation

$$d\Gamma^3_1 + \Gamma^3_1 \wedge \Gamma^1_1 + \Gamma^3_3 \wedge \Gamma^3_1 = \mathrm{R}^3_{1k\ell}\,\theta^k \wedge \theta^\ell, \quad k < \ell,$$

which reads

$$\mathrm{R}^3_{134} = \frac{1}{2}(c^3_{13})_4 + \frac{1}{2}(c^4_{14})_4 - \frac{\mathrm{i}}{4}c^3_{13} - \frac{\mathrm{i}}{4}c^4_{14}.$$

Thus, after substituting c^i_{jk}'s to R_{14} and simplifying we get

$$\mathrm{R}_{14} = +\frac{\mathrm{i}c}{2} + \frac{W_{rr}}{2} - \mathrm{i}W_r.$$

Therefore, the condition $\mathrm{C}_{1434} = 0$ is equivalent to the function W satisfying the following equation

$$W_r - \mathrm{i}W_{rr} = 0.$$

Thus, the general solution of the second order differential equation with constant coefficients is

$$W = \mathrm{i}x\,\mathrm{e}^{-\mathrm{i}r} + y$$

where $x_r = y_r = 0$. ∎

2.2.3 Maxwell field aligned with a congruence

In order to present the main theorem in [25] we also need to introduce the notion of what is called by physicists "Maxwell field aligned with the congruence".

Definition 2.2.8 *Let* $(M, [(\mu, \lambda)])$ *be a strictly pseudoconvex 3-dimensional CR manifold and* (μ', λ') *be a pair from the class* $[(\mu, \lambda)]$ *with the transformations* (1.7)*. The 2-form*

$$\mathcal{F} = \pi^*(\lambda' \wedge \mu'),$$

is called a null Maxwell field aligned with the congruence if it is closed, i.e.

$$d\mathcal{F} = 0.$$

Here $\pi : M \times \mathbb{R} \longrightarrow M$, is the natural projection.

Remark 2.2.9 *We note that the 2-form $\lambda' \wedge \mu'$ is a complex multiple of $\lambda \wedge \mu$ due to the transformations* (1.7).

The main result about embedding of strictly pseudoconvex 3-dimensional manifolds among other results in [25], is the following

Theorem 2.2.10 (Hill, Lewandowski, Nurowski) *[25] Let M be a sufficiently smooth strictly pseudoconvex 3-dimensional CR manifold. It is locally CR embeddable as a hypersurface in $\mathbb{C}^2$ if and only if:*

1. *it admits a lift to a spacetime whose complexified Ricci tensor vanishes on the corresponding distribution of α-planes, and*
2. *it admits a non-trivial null Maxwell field aligned with the null congruence of shearfree geodesics corresponding to the CR structure on M.*

The proof of the theorem above consists of two parts. The first condition actually gives a CR function ϕ such that $d\phi \wedge d\bar{\phi} \neq 0$. The procedure of finding a CR function, which we also use in the next chapter, has been known and used by physicists since 1969 in the context of finding a solution of Maxwell equations [12, 58].

A second CR function which is functionally independent from the first one, arises from imposing the second condition of the theorem above on the Lorentzian manifold.

In the next chapter we develop another approach based on a different family of metrics which allows us to prove that vanishing of the complexified Ricci tensor on the distribution of α-planes implies the embeddability of the underlying CR manifold without the additional assumption on the existence of the aligned Maxwell field.

In fact, we will show that vanishing of the complexified Ricci tensor on the distribution of α-planes not only gives a CR function but also implies the existence of a non-vanishing d-closed section of the canonical bundle which together with the CR function, by Theorem 2.1.6, implies that the CR manifold is embeddable.

We also note that the Definition 2.1.4 shows that having a non-trivial null Maxwell field aligned with the null congruence of shearfree geodesics is equivalent to having a d-closed section of the canonical bundle of the CR structure. Indeed, if

$$d(\psi\mu \wedge \lambda) = 0,$$

that is, $\psi\mu \wedge \lambda$ is a nonzero d-closed section of the canonical bundle, then

$$\mathcal{F} = \pi^*(\psi\mu \wedge \lambda)$$

is a Maxwell field aligned with the congruence and vise versa.

In order to define a class of metrics which we call the FRT metrics we first need to recall the construction of the Fefferman metric for 3-dimensional CR manifolds following [36] and [47].

2.3 Fefferman metric

The most famous approach to relate a given CR manifold to a Lorentzian space is the Fefferman metric which was introduced by C. Fefferman in [17] for real hypersurfaces in $\mathbb{C}^n$ on a circle bundle over M. Moreover, the Fefferman construction was generalised for any CR structure in [8].

The construction in [8] is based on the canonical Cartan connection associated with the CR structure while in the constructions in [34] and also in [16], one does not need to use a connection associated with the CR structure.

We also compute the Fefferman metric explicitly for general 3-dimensional CR manifolds following the approach [29, 47], but without assuming that a non-constant CR function exists. In order to do that, we present the following important theorem.

Theorem 2.3.1 (Cartan) *[10, 29] Every strictly pseudoconvex CR manifold* $(M, [(\mu, \lambda)])$ *satisfying* (1.5) *uniquely defines an eight dimensional principal bundle* P *over* M*, the following 1-forms on* P

$$
\begin{aligned}
\Omega_0 &= \mathrm{e}^{2\tau}\,\lambda,\\
\Omega_1 &= \mathrm{e}^{\tau+\mathrm{i}\varphi}(\mu + h\lambda),\\
\Omega_2 &= d\tau + \mathrm{i}d\varphi + A\mu + B\bar{\mu} + C\lambda,\\
\Omega_3 &= \mathrm{e}^{-\tau+\mathrm{i}\varphi}(dh + D\mu + E\bar{\mu} + F\lambda),\\
\Omega_4 &= \mathrm{e}^{-2\tau}\Big(d\rho + \frac{\mathrm{i}}{2}(hd\bar{h} - \bar{h}dh) + H\mu + \overline{H}\bar{\mu} + G\lambda\Big),
\end{aligned}
$$

where Ω_4 *is a real 1-form and* $\pi : P \to M$ *is the natural projection and*

$$
\Omega_0 = \mathrm{e}^{2\tau}\,\pi^*\lambda, \quad \Omega_1 = \mathrm{e}^{\tau+\mathrm{i}\varphi}(\pi^*\mu + h\pi^*\lambda).
$$

Moreover, ρ *is an arbitrary real function, the functions* $A, \dots, H$*, which are given by* (2.16)*,* (2.15)*,* (2.20)*,*(2.21)*,* (2.17)*,* (2.22)*,* (2.24) *and* (2.23) *respectively. The forms satisfy the following equations:*

$$
\begin{aligned}
d\Omega_0 &= \mathrm{i}\Omega_1 \wedge \overline{\Omega}_1 + (\Omega_2 + \overline{\Omega}_2) \wedge \Omega_0,\\
d\Omega_1 &= \Omega_2 \wedge \Omega_1 + \Omega_3 \wedge \Omega_0,\\
d\Omega_2 &= 2\mathrm{i}\Omega_1 \wedge \overline{\Omega}_3 + \mathrm{i}\overline{\Omega}_1 \wedge \Omega_3 + \Omega_4 \wedge \Omega_0,\\
d\Omega_3 &= \Omega_4 \wedge \Omega_1 + \Omega_3 \wedge \overline{\Omega}_2 + \mathcal{R}\overline{\Omega}_1 \wedge \Omega_0,\\
d\Omega_4 &= \mathrm{i}\Omega_3 \wedge \overline{\Omega}_3 + \Omega_4 \wedge (\Omega_2 + \overline{\Omega}_2) + \mathcal{S}\Omega_1 \wedge \Omega_0 + \overline{\mathcal{S}}\,\overline{\Omega}_1 \wedge \Omega_0
\end{aligned}
$$

where $\mathcal{R}$ *given by* (2.25) *and* $\mathcal{S}$ *has the property that if* $\mathcal{R} = 0$*, then* $\mathcal{S} = 0$*.*

Proof To obtain the functions $A, \dots, H$ we follow the computations in [29] according to our setting (1.5). We note that the computations in [29] are based on the assumption $d\mu = 0$ but, here we consider the general case, where the CR manifold is not necessarily embeddable in $\mathbb{C}^2$.

We first look at the structure equation

$$\begin{aligned} d\Omega_0 - \mathrm{i}\Omega_1 \wedge \overline{\Omega}_1 - (\Omega_2 + \overline{\Omega}_2) \wedge \Omega_0 = {} & \mathrm{e}^{2\tau}(2d\tau \wedge \lambda + d\lambda) - \mathrm{i}\,\mathrm{e}^{2\tau}(\mu \wedge \bar{\mu} + \bar{h}\mu \wedge \lambda + h\bar{\mu} \wedge \lambda) \\ & - (2d\tau + (A + \bar{B})\mu + (B + \bar{A})\bar{\mu} + (C + \bar{C})\lambda) \wedge \mathrm{e}^{2\tau}\,\lambda \\ = {} & 2\,\mathrm{e}^{2\tau}\,d\tau \wedge \lambda + \mathrm{i}\,\mathrm{e}^{2\tau}\,\mu \wedge \bar{\mu} + \mathrm{e}^{2\tau}\,c\mu \wedge \theta + \mathrm{e}^{2\tau}\,\bar{c}\bar{\mu} \wedge \lambda \\ & - \mathrm{i}\,\mathrm{e}^{2\tau}(\mu \wedge \bar{\mu} + \bar{h}\mu \wedge \lambda + h\bar{\mu} \wedge \lambda) \\ & - \mathrm{e}^{2\tau}(2d\tau + (A + \bar{B})\mu + (B + \bar{A})\bar{\mu} + (C + \bar{C})\lambda) \wedge \lambda \\ = {} & \mathrm{e}^{2\tau}(c - \mathrm{i}\bar{h} - A - \bar{B})\mu \wedge \lambda + \mathrm{e}^{2\tau}(\bar{c} + \mathrm{i}h - \bar{A} - B)\bar{\mu} \wedge \lambda. \end{aligned}$$

It follows

$$A = -\bar{B} + c - \mathrm{i}\bar{h}.$$

Furthermore,

$$\begin{aligned} d\Omega_1 - \Omega_2 \wedge \Omega_1 - \Omega_3 \wedge \Omega_0 = {} & \mathrm{e}^{\tau+\mathrm{i}\varphi}(d\tau + \mathrm{i}d\varphi) \wedge (\mu + h\lambda) + \mathrm{e}^{\tau+\mathrm{i}\varphi}(\alpha\mu \wedge \lambda + \beta\bar{\mu} \wedge \lambda) \\ & + \mathrm{e}^{\tau+\mathrm{i}\varphi}(dh \wedge \lambda + \mathrm{i}h\mu \wedge \bar{\mu} + hc\mu \wedge \lambda + h\bar{c}\bar{\mu} \wedge \lambda) \\ & - \mathrm{e}^{\tau+\mathrm{i}\varphi}[d\tau + \mathrm{i}d\varphi + A\mu + B\bar{\mu} + C\lambda] \wedge [\mu + h\lambda] \\ & - \mathrm{e}^{\tau+\mathrm{i}\varphi}(dh + D\mu + E\bar{\mu} + F\lambda) \wedge \lambda \\ = {} & \mathrm{e}^{\tau+\mathrm{i}\varphi}(\mathrm{i}h + B)\mu \wedge \bar{\mu} + \mathrm{e}^{\tau+\mathrm{i}\varphi}(\alpha + hc + C - hA - D)\mu \wedge \lambda \\ & + \mathrm{e}^{\tau+\mathrm{i}\varphi}(\beta + h\bar{c} - hB - E)\bar{\mu} \wedge \lambda. \end{aligned}$$

It follows

$$B = -\mathrm{i}h \tag{2.15}$$

and hence,

$$A = c - 2\mathrm{i}\bar{h} \tag{2.16}$$

and

$$C - D = hA - hc - \alpha = -2\mathrm{i}|h|^2 - \alpha$$

and therefore,

$$E = \beta + h\bar{c} - hB = \beta + h\bar{c} + \mathrm{i}h^2. \tag{2.17}$$

Moreover, the structure equation for Ω_2 is

$$d\Omega_2 + \mathrm{i}\Omega_3 \wedge \overline{\Omega}_1 + 2\mathrm{i}\overline{\Omega}_3 \wedge \Omega_1 - \Omega_4 \wedge \Omega_0 = 0,$$

where Ω_4 and Ω_0 are real. This yields

$$d(\Omega_2 - \overline{\Omega}_2) + 3\mathrm{i}\Omega_3 \wedge \overline{\Omega}_1 + 3\mathrm{i}\overline{\Omega}_3 \wedge \Omega_1 = 0.$$

We then have

$$\begin{aligned} d(\Omega_2 - \overline{\Omega}_2) = (dA - d\bar{B}) \wedge \mu + (dB - d\bar{A}) \wedge \bar{\mu} + (dC - d\bar{C}) \wedge \lambda \\ + (A - \bar{B})(\alpha\mu \wedge \lambda + \beta\bar{\mu} \wedge \lambda) + (B - \bar{A})(\bar{\beta}\mu \wedge \lambda \\ + \bar{\alpha}\bar{\mu} \wedge \lambda) + (C - \bar{C})(\mathrm{i}\mu \wedge \bar{\mu} + c\mu \wedge \lambda + \bar{c}\bar{\mu} \wedge \lambda). \end{aligned}$$

After expressing the exterior derivatives with respect to $\mu, \bar{\mu}, \lambda$ we get

$$\begin{aligned} d(\Omega_2 - \overline{\Omega}_2) = {} & (A_\mu\mu + A_{\bar{\mu}}\bar{\mu} + A_\lambda\lambda - \bar{B}_\mu\mu - \bar{B}_{\bar{\mu}}\bar{\mu} - \bar{B}_\lambda\lambda) \wedge \mu \\ & + (B_\mu\mu + B_{\bar{\mu}}\bar{\mu} + B_\lambda\lambda - \bar{A}_\mu\mu - \bar{A}_{\bar{\mu}}\bar{\mu} - \bar{A}_\lambda\lambda) \wedge \bar{\mu} \\ & + (C_\mu\mu + C_{\bar{\mu}}\bar{\mu} + C_\lambda\lambda - \bar{C}_\mu\mu - \bar{C}_{\bar{\mu}}\bar{\mu} - \bar{C}_\lambda\lambda) \wedge \lambda \\ & + (A - \bar{B})(\alpha\mu \wedge \lambda + \beta\bar{\mu} \wedge \lambda) + (B - \bar{A})(\bar{\beta}\mu \wedge \lambda \\ & + \bar{\alpha}\bar{\mu} \wedge \lambda) + (C - \bar{C})(\mathrm{i}\mu \wedge \bar{\mu} + c\mu \wedge \lambda + \bar{c}\bar{\mu} \wedge \lambda), \end{aligned}$$

and also

$$\begin{aligned} 3\mathrm{i}\Omega_3 \wedge \overline{\Omega}_1 + 3\mathrm{i}\overline{\Omega}_3 \wedge \Omega_1 &= 3\mathrm{i}(dh + D\mu + E\bar{\mu} + F\lambda) \wedge (\bar{\mu} + \bar{h}\lambda) \\ &\quad + 3\mathrm{i}(d\bar{h} + \bar{D}\bar{\mu} + \bar{E}\mu + \bar{F}\lambda) \wedge (\mu + h\lambda) \\ &= 3\mathrm{i}(h_\mu\mu + h_{\bar{\mu}}\bar{\mu} + h_\lambda\lambda + D\mu + E\bar{\mu} + F\lambda) \wedge (\bar{\mu} + \bar{h}\lambda) \\ &\quad + 3\mathrm{i}(\bar{h}_\mu\mu + \bar{h}_{\bar{\mu}}\bar{\mu} + \bar{h}_\lambda\lambda + \bar{D}\bar{\mu} + \bar{E}\mu + \bar{F}\lambda) \wedge (\mu + h\lambda). \end{aligned}$$

We will examine the $\mu \wedge \bar{\mu}$ and $\lambda \wedge \mu$ components of

$$d(\Omega_2 - \overline{\Omega}_2) + 3\mathrm{i}\Omega_3 \wedge \overline{\Omega}_1 + 3\mathrm{i}\overline{\Omega}_3 \wedge \Omega_1 = 0.$$

We then get

$$-A_{\bar{\mu}} + \bar{B}_{\bar{\mu}} + B_\mu - \bar{A}_\mu + \mathrm{i}(C - \bar{C}) + 3\mathrm{i}h_\mu + 3\mathrm{i}D - 3\mathrm{i}\bar{h}_{\bar{\mu}} - 3\mathrm{i}\bar{D} = 0, \tag{2.18}$$

and also

$$A_\lambda - \bar{B}_\lambda - C_\mu + \bar{C}_\mu - \alpha(A - \bar{B}) - \bar{\beta}(B - \bar{A}) - c(C - \bar{C}) \\ - 3\mathrm{i}\bar{h}h_\mu - \bar{h}D - 3\mathrm{i}h\bar{h}_\mu + 3\mathrm{i}\bar{h}_\lambda - 3\mathrm{i}h\bar{E} + 3\mathrm{i}\bar{F} = 0. \tag{2.19}$$

Substituting the functions A, B, E into (2.18) gives

$$-c_{\bar{\mu}} - \bar{c}_\mu + \mathrm{i}(C - \bar{C}) + 3\mathrm{i}D - 3\mathrm{i}\bar{D} = 0.$$

Using $C - D = -2\mathrm{i}|h|^2 - \alpha$, after simplifications we get

$$4\mathrm{i}(C - \bar{C}) = c_{\bar{\mu}} + \bar{c}_\mu + 12|h|^2 - 3\mathrm{i}(\alpha - \bar{\alpha}).$$

and consequently,

$$C = \rho - \mathrm{i}\frac{c_{\bar{\mu}} + \bar{c}_\mu}{8} - \frac{3}{2}\mathrm{i}|h|^2 - \frac{3}{4}\alpha, \tag{2.20}$$

where ρ is an arbitrary real function. Hence, the function D is of the form

$$D = C + 2\mathrm{i}|h|^2 + \alpha = \rho - \mathrm{i}\frac{c_{\bar{\mu}} + \bar{c}_\mu}{8} + \frac{1}{2}\mathrm{i}|h|^2 + \frac{1}{4}\alpha. \tag{2.21}$$

By straightforward computations from (2.19) we have

$$\begin{aligned} \bar{F} = \frac{1}{3\mathrm{i}}\Big(& - c_\lambda + 2\mathrm{i}\bar{h}_\lambda + \mathrm{i}\bar{h}_\lambda + \rho_\mu - \mathrm{i}\frac{c_{\bar{\mu}\mu} + \bar{c}_{\mu\mu}}{8} - \mathrm{i}\frac{3}{2}(h\bar{h})_\mu - \frac{3}{4}\alpha_\mu \\ & - \rho_\mu - \mathrm{i}\frac{c_{\bar{\mu}\mu} + \bar{c}_{\mu\mu}}{8} - \mathrm{i}\frac{3}{2}(h\bar{h})_\mu + \frac{3}{4}\alpha_\mu + \alpha(c - 3\mathrm{i}\bar{h}) \\ & + \bar{\beta}(-3\mathrm{i}h - \bar{c}) - \mathrm{i}c\frac{c_{\bar{\mu}} + \bar{c}_\mu}{4} - 3c\mathrm{i}|h|^2 - \frac{3}{4}c(\alpha + \bar{\alpha}) \\ & + 3\mathrm{i}\bar{h}\rho + 3\bar{h}\frac{c_{\bar{\mu}} + \bar{c}_\mu}{8} - \frac{3}{2}\mathrm{i}\bar{h}|h|^2 + \frac{3}{4}\mathrm{i}\bar{h}\alpha + 3\mathrm{i}(\bar{h}h)_\mu - 3\mathrm{i}\bar{h}_\lambda \\ & + 3\mathrm{i}h\bar{\beta} + 3\mathrm{i}|h|^2c + 3\bar{h}|h|^2\Big). \end{aligned}$$

After simplification we get the following expression for F

$$\begin{aligned} F = \frac{\mathrm{i}}{3}\Big(& - \bar{c}_\lambda + \mathrm{i}\frac{\bar{c}_{\mu\bar{\mu}} + c_{\bar{\mu}\bar{\mu}}}{4} - \frac{3}{4}\bar{\alpha}_{\bar{\mu}} + \frac{3}{4}\alpha_{\bar{\mu}} + \bar{\alpha}\bar{c} + 3\mathrm{i}\bar{\alpha}h - c\beta + \mathrm{i}\bar{c}\frac{c_{\bar{\mu}} + \bar{c}_\mu}{4} \\ & - \frac{3}{4}\bar{c}(\alpha + \bar{\alpha}) - 3\mathrm{i}h\rho + 3h\frac{c_{\bar{\mu}} + \bar{c}_\mu}{8} + \frac{3}{2}h|h|^2 - \frac{3}{4}\mathrm{i}h\bar{\alpha}\Big), \end{aligned} \tag{2.22}$$

which is consistent with [29] when $\alpha = \beta = 0$ and $c_{\bar{\mu}} = \bar{c}_\mu$. Notice that in our case $\bar{c}_\mu - c_{\bar{\mu}} = \mathrm{i}(\alpha + \bar{\alpha})$, i.e. it does not vanish in general.

To determine the function H appearing in Ω_4 we first notice that

$$d(\Omega_2 + \overline{\Omega}_2) = \mathrm{i}\Omega_1\overline{\Omega}_3 - \mathrm{i}\overline{\Omega}_1\Omega_3 + 2\Omega_4\Omega_0.$$

Therefore,

$$\begin{aligned}
d(\Omega_2 + \overline{\Omega}_2) = {} & dc \wedge \mu - \mathrm{i}d\bar{h} \wedge \mu + (c - \mathrm{i}\bar{h})(\alpha\mu \wedge \lambda + \beta\bar{\mu} \wedge \lambda) + d\bar{c}\bar{\mu} + \mathrm{i}dh \wedge \bar{\mu} \\
& + (\bar{c} + \mathrm{i}h)(\bar{\alpha}\bar{\mu} \wedge \lambda + \bar{\beta}\mu \wedge \lambda) + 2d\rho \wedge \lambda - \frac{3}{4}d\alpha \wedge \lambda - \frac{3}{4}d\bar{\alpha} \wedge \lambda \\
& + \mathrm{i}(2\rho - \frac{3}{4}\alpha - \frac{3}{4}\bar{\alpha})\mu \wedge \bar{\mu} + c(2\rho - \frac{3}{4}\alpha - \frac{3}{4}\bar{\alpha})\mu \wedge \lambda \\
& + \bar{c}(2\rho - \frac{3}{4}\alpha - \frac{3}{4}\bar{\alpha})\bar{\mu} \wedge \lambda.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathrm{i}\Omega_1 \wedge \overline{\Omega}_3 - \mathrm{i}\overline{\Omega}_1 \wedge \Omega_3 + 2\Omega_4 \wedge \Omega_0 = {} & \mathrm{i}\mu \wedge d\bar{h} + \mathrm{i}\bar{D}\mu \wedge \bar{\mu} + \mathrm{i}\bar{F}\mu \wedge \lambda + \mathrm{i}h\bar{D}\lambda \wedge \bar{\mu} \\
& + \mathrm{i}h\bar{E}\lambda \wedge \mu - \mathrm{i}\bar{\mu} \wedge dh - \mathrm{i}D\bar{\mu} \wedge \mu - \mathrm{i}F\bar{\mu} \wedge \lambda \\
& - \mathrm{i}\bar{h}D\lambda \wedge \mu - \mathrm{i}\bar{h}E\lambda \wedge \bar{\mu} + 2d\rho \wedge \lambda + 2H\mu \wedge \lambda \\
& + 2\overline{H}\bar{\mu} \wedge \lambda.
\end{aligned}$$

Looking at the coefficient of the term $\mu\wedge\lambda$ of the $d(\Omega_2+\overline{\Omega}_2)$ gives the following expression for H

$$\begin{aligned}
H = \frac{1}{2}\Big(& -c_\lambda + \alpha(c - \mathrm{i}\bar{h}) + \bar{\beta}(\bar{c} + \mathrm{i}h) - \frac{3}{4}\alpha_\mu - \frac{3}{4}\bar{\alpha}_\mu \\
& + c(2\rho - \frac{3}{4}\alpha - \frac{3}{4}\bar{\alpha}) - \mathrm{i}\bar{F} + \mathrm{i}h\bar{E} - \mathrm{i}\bar{h}D\Big).
\end{aligned} \tag{2.23}$$

It only remains to determine G. To do that we look at the coefficient of the term $\lambda \wedge \mu$ of $d\Omega_3$. Indeed, on one hand,

$$\begin{aligned}
d\Omega_3 &= d\Big(\mathrm{e}^{-\tau+\mathrm{i}\varphi}(dh + D\mu + E\bar{\mu} + F\lambda)\Big) \\
&= \mathrm{e}^{-\tau+\mathrm{i}\varphi}(-d\tau + \mathrm{i}d\varphi) \wedge (dh + D\mu + E\bar{\mu} + F\lambda) \\
&\quad + \mathrm{e}^{-\tau+\mathrm{i}\varphi}\Big(dD \wedge \mu + D\alpha\mu \wedge \lambda + D\beta\bar{\mu} \wedge \lambda + dE \wedge \bar{\mu} \\
&\qquad + E\bar{\alpha}\bar{\mu} \wedge \lambda + E\bar{\beta}\mu \wedge \lambda + dF \wedge \lambda + \mathrm{i}F\mu \wedge \bar{\mu} \\
&\qquad + cF\mu \wedge \lambda + \bar{c}F\bar{\mu} \wedge \lambda\Big).
\end{aligned}$$

On the other hand,

$$\begin{aligned} d\Omega_3 &= \Omega_4 \wedge \Omega_1 + \Omega_3 \wedge \overline{\Omega}_2 + \mathcal{R}\overline{\Omega}_1 \wedge \Omega_0 \\ &= \mathrm{e}^{-\tau+\mathrm{i}\varphi}\left(d\rho + \frac{\mathrm{i}}{2}(hd\bar{h} - \bar{h}dh) + H\mu + \overline{H}\bar{\mu} + G\lambda\right) \wedge (\mu + h\lambda) \\ &\quad + \mathrm{e}^{-\tau+\mathrm{i}\varphi}(-d\tau + \mathrm{i}d\varphi) \wedge (dh + D\mu + E\bar{\mu} + F\lambda) \\ &\quad + \mathrm{e}^{-\tau+\mathrm{i}\varphi}(dh + D\mu + E\bar{\mu} + F\lambda) \wedge (\bar{A}\bar{\mu} + \bar{B}\mu + \bar{C}\lambda) \\ &\quad + \mathcal{R}\,\mathrm{e}^{\tau-\mathrm{i}\varphi}(\bar{\mu} + \bar{h}\lambda)\,\mathrm{e}^{2\tau}\,\lambda. \end{aligned}$$

Comparing the two above expressions for $d\Omega_3$ and considering the coefficient of $\lambda \wedge \mu$ determines G, as follows

$$\begin{aligned} G = &-\frac{\mathrm{i}}{2}h\bar{h}_\lambda + \frac{\mathrm{i}}{2}h^2\bar{h}_\mu + \frac{\mathrm{i}}{2}\bar{h}h_\lambda - \frac{\mathrm{i}}{2}|h|^2 h_\mu + Hh + h_\mu\bar{C} - h_\lambda\bar{B} + D\bar{C} \\ &- F\bar{B} + D_\lambda - D\alpha - E\bar{\beta} - F_\mu - CF. \end{aligned} \tag{2.24}$$

To obtain $\mathcal{R}$ we look at the coefficient of the 2-form $\bar{\mu} \wedge \lambda$ of $d\Omega_3$. We then have

$$\begin{aligned} \mathcal{R} = \mathrm{e}^{-4\tau+2\mathrm{i}\varphi}\Big(&D\beta - E_\lambda + E\alpha + F_{\bar{\mu}} + \bar{c}F - \frac{\mathrm{i}}{2}h^2\bar{h}_{\bar{\mu}} + \frac{\mathrm{i}}{2}|h|^2 h_{\bar{\mu}} \\ &- \overline{H}h - \bar{C}h_{\bar{\mu}} + \bar{A}h_\lambda - E\bar{C} + F\bar{A}\Big). \qquad \blacksquare \end{aligned} \tag{2.25}$$

A straightforward computation shows

$$\begin{aligned} \Omega_2 - \overline{\Omega}_2 = &2\mathrm{i}d\varphi + (c - 3\mathrm{i}\bar{h})\mu - (\bar{c} + 3\mathrm{i}h)\bar{\mu} \\ &+ \left(\frac{c_{\bar{\mu}} + \bar{c}_\mu}{4\mathrm{i}} - 3\mathrm{i}|h|^2 - \frac{3(\alpha - \bar{\alpha})}{4}\right)\lambda. \end{aligned}$$

Now we are in the position to define the Fefferman metric as follows

Definition 2.3.2 *The bilinear symmetric tensor*

$$\begin{aligned} g_F &= \Omega_1\overline{\Omega}_1 + \frac{1}{3\mathrm{i}}\Omega_0(\Omega_2 - \overline{\Omega}_2) \\ &= \mu\bar{\mu} + \lambda\left(\frac{2}{3}d\varphi - \frac{\mathrm{i}}{3}c\mu + \frac{\mathrm{i}}{3}\bar{c}\bar{\mu} - \left(\frac{\partial\bar{c} + \bar{\partial}c}{12} - \frac{\mathrm{i}(\alpha - \bar{\alpha})}{4}\right)\lambda\right) \end{aligned} \tag{2.26}$$

defined on the circle bundle $\mathcal{M} = D^{1,0}/\mathbb{R}^+$ *where* (μ, λ) *is a distinguished coframe and* $\mathcal{M} \ni m|_p = \mathrm{e}^{-\tau-\mathrm{i}\varphi}\,\partial|_p \mapsto (p, \varphi)$ *is a local trivialisation and* $\varphi \in [0, 2\pi)$ *is called Fefferman metric.*

We again note that we kept the notations $\mu, \bar{\mu}, \lambda$ for their pull-backs under the circle bundle projection.

It is well-known that the Fefferman metric is CR invariant which precisely means that a change of the distinguished coframe $(\mu, \bar{\mu}, \lambda)$ causes only a conformal change of g_F by the factor $\mathrm{e}^{2\tau}$.

Lemma 2.3.3 *For the choice of* (μ', λ') *defined by* (1.8), *where* $f := \mathrm{e}^{\tau + \mathrm{i}\varphi}$, *the following statement for new c is satisfied*

$$\begin{aligned}\frac{\partial' \bar{c}' + \bar{\partial}' c'}{12} \lambda' = \frac{1}{12} \Big[& - \bar{c} \partial \log \bar{f} - 4 |\bar{\partial} \log f|^2 - (\partial \log \bar{f})(\bar{\partial} \log \bar{f}) + \bar{\partial} c \\ & + 2 \bar{\partial} \partial \log f - 2\mathrm{i} \partial_0 \log f + \bar{\partial} \partial \log \bar{f} - \mathrm{i} \partial_0 \log \bar{f} - c \bar{\partial} \log f \\ & - (\bar{\partial} \log f)(\partial \log f) + \partial \bar{c} + 2 \bar{\partial} \partial \log \bar{f} + \bar{\partial} \partial \log f \Big] \lambda,\end{aligned}$$

Proof The transformations (1.7) and Proposition 1.2.11 imply that

$$\begin{aligned}\frac{\partial' \bar{c}' + \bar{\partial}' c'}{12} \lambda' &= \frac{1}{12} \left[\frac{1}{f} \partial \left(\frac{1}{\bar{f}} \left(\bar{c} + 2\mathrm{i} h + \bar{\partial} \log \bar{f} \right) \right) + \frac{1}{\bar{f}} \bar{\partial} \left(\frac{1}{f} \left(c - 2\mathrm{i} \bar{h} + \partial \log f \right) \right) \right] \lambda' \\ &= \frac{1}{12} \Big[- \bar{c} \partial \log \bar{f} - 2\mathrm{i} h \partial \log \bar{f} - (\partial \log \bar{f})(\bar{\partial} \log \bar{f}) + \bar{\partial} c + 2\mathrm{i} \partial h + \partial \bar{\partial} \log \bar{f} \\ & \quad - c \bar{\partial} \log f + 2\mathrm{i} \bar{h} \bar{\partial} \log f - (\bar{\partial} \log f)(\partial \log f) + \partial \bar{c} - 2\mathrm{i} \bar{\partial} \bar{h} + \bar{\partial} \partial \log f \Big] \lambda \\ &= \frac{1}{12} \Big[- \bar{c} \partial \log \bar{f} - 2 |\bar{\partial} \log f|^2 - (\partial \log \bar{f})(\bar{\partial} \log \bar{f}) + \bar{\partial} c + 2 \partial \bar{\partial} \log f \\ & \quad + \partial \bar{\partial} \log \bar{f} - c \bar{\partial} \log f - 2 |\bar{\partial} \log f|^2 - (\bar{\partial} \log f)(\partial \log f) + \partial \bar{c} + 2 \bar{\partial} \partial \log \bar{f} \\ & \quad + \bar{\partial} \partial \log f \Big] \lambda \\ &= \frac{1}{12} \Big[- \bar{c} \partial \log \bar{f} - 4 |\bar{\partial} \log f|^2 - (\partial \log \bar{f})(\bar{\partial} \log \bar{f}) + \bar{\partial} c + 2 \bar{\partial} \partial \log f \\ & \quad - 2\mathrm{i} \partial_0 \log f + \bar{\partial} \partial \log \bar{f} - \mathrm{i} \partial_0 \log \bar{f} - c \bar{\partial} \log f - (\bar{\partial} \log f)(\partial \log f) \\ & \quad + \partial \bar{c} + 2 \bar{\partial} \partial \log \bar{f} + \bar{\partial} \partial \log f \Big] \lambda.\end{aligned}$$

Moreover,

Lemma 2.3.4 *The following statement holds.*

$$\frac{\mathrm{i}(\alpha'-\bar{\alpha}')}{4}\lambda' = \frac{\mathrm{i}}{4}\Big(\alpha - 2\partial_0\log f - \mathrm{i}(\bar\partial\log f)(\partial\log f) - \mathrm{i}\bar\partial\partial\log f - \mathrm{i}c\bar\partial\log f - \bar\alpha + \partial_0\log\bar f - \mathrm{i}(\partial\log\bar f)(\bar\partial\log\bar f) - \mathrm{i}\bar\partial\partial\log\bar f - \mathrm{i}\bar c\partial\log\bar f\Big)\lambda.$$

Proof The transformations (1.7) and Proposition 1.2.11 imply that

$$\begin{aligned}\frac{\mathrm{i}(\alpha'-\bar{\alpha}')}{4}\lambda' &= \frac{\mathrm{i}}{4}\left(\alpha - \partial_0\log f + h\partial\log f + \partial h + hc - \bar\alpha + \partial_0\log\bar f - \bar h\bar\partial\log\bar f - \bar\partial\bar h - \bar h\bar c\right)\lambda\\ &= \frac{\mathrm{i}}{4}\Big(\alpha - \partial_0\log f - \mathrm{i}(\bar\partial\log f)(\partial\log f) - \mathrm{i}\partial\bar\partial\log f - \mathrm{i}c\bar\partial\log f - \bar\alpha + \partial_0\log\bar f\\ &\qquad - \mathrm{i}(\partial\log\bar f)(\bar\partial\log\bar f) - \mathrm{i}\bar\partial\partial\log\bar f - \mathrm{i}\bar c\partial\log\bar f\Big)\lambda\\ &= \frac{\mathrm{i}}{4}\Big(\alpha - 2\partial_0\log f - \mathrm{i}(\bar\partial\log f)(\partial\log f) - \mathrm{i}\bar\partial\partial\log f - \mathrm{i}c\bar\partial\log f - \bar\alpha + \partial_0\log\bar f\\ &\qquad - \mathrm{i}(\partial\log\bar f)(\bar\partial\log\bar f) - \mathrm{i}\bar\partial\partial\log\bar f - \mathrm{i}\bar c\partial\log\bar f\Big)\lambda. \qquad \blacksquare\end{aligned}$$

Proposition 2.3.5 *The Fefferman metric is CR invariant.*

Proof Let g'_F be the Fefferman metric corresponding to the choice (μ', λ'), that is,

$$g'_F = \mu'\bar\mu' + \lambda'\left(\frac{2}{3}d\rho' - \frac{\mathrm{i}}{3}c'\mu' + \frac{\mathrm{i}}{3}\bar c'\bar\mu' - \left(\frac{\partial'\bar c' + \bar\partial' c'}{12} - \frac{\mathrm{i}(\alpha'-\bar\alpha')}{4}\right)\lambda'\right).$$

Using (1.8) and Lemmata 2.3.3 and 2.3.4 we get

$$\begin{aligned}g'_F &= |f|^2(\mu\bar\mu + \bar h\mu\lambda + h\lambda\mu + |h|^2\lambda\lambda)\\ &+ |f|^2\lambda\Bigg(\frac{2}{3}d\rho' - \frac{\mathrm{i}}{3}c\mu - \frac{\mathrm{i}}{3}ch\lambda - \frac{2}{3}\bar h\mu - \frac{2}{3}|h|^2\lambda - \frac{\mathrm{i}}{3}(\partial\log f)\mu - \frac{\mathrm{i}}{3}h(\partial\log f)\lambda\\ &\qquad + \frac{\mathrm{i}}{3}\bar c\bar\mu + \frac{\mathrm{i}}{3}\bar c\bar h\lambda - \frac{2}{3}h\bar\mu - \frac{2}{3}|h|^2\lambda + \frac{\mathrm{i}}{3}(\bar\partial\log\bar f)\bar\mu + \frac{\mathrm{i}}{3}\bar h(\bar\partial\log\bar f)\lambda\Bigg)\\ &+ \Bigg[\frac{1}{12}\bar c\partial\log\bar f + \frac{1}{3}|\bar\partial\log f|^2 + \frac{1}{12}(\partial\log\bar f)(\bar\partial\log\bar f)\\ &\quad - \frac{1}{12}\bar\partial c - \frac{1}{6}\bar\partial\partial\log f + \frac{\mathrm{i}}{6}\partial_0\log f - \frac{1}{12}\bar\partial\partial\log\bar f + \frac{\mathrm{i}}{12}\partial_0\log\bar f + \frac{1}{12}c\bar\partial\log f\\ &\quad + \frac{1}{12}(\bar\partial\log f)(\partial\log f) - \frac{1}{12}\partial\bar c - \frac{1}{6}\bar\partial\partial\log\bar f - \frac{1}{12}\bar\partial\partial\log f + \frac{\mathrm{i}}{4}\alpha - \frac{\mathrm{i}}{2}\partial_0\log f\\ &\quad + \frac{1}{4}(\bar\partial\log f)(\partial\log f) + \frac{1}{4}\bar\partial\partial\log f + \frac{1}{4}c\bar\partial\log f - \frac{\mathrm{i}}{4}\bar\alpha + \frac{\mathrm{i}}{4}\partial_0\log\bar f\\ &\quad + \frac{1}{4}(\partial\log\bar f)(\bar\partial\log\bar f) + \frac{1}{4}\bar\partial\partial\log\bar f + \frac{1}{4}\bar c\partial\log\bar f\Bigg]\lambda\Bigg).\end{aligned}$$

We notice that $|h|^2 = |\bar{\partial}\log f|^2$ and then by collecting the like terms we achieve

$$g'_F = |f|^2 g_F = \mathrm{e}^{2\tau}\, g_F,$$

where

$$g_F = \mu\bar{\mu} + \lambda\left(\frac{2}{3}d\rho - \frac{\mathrm{i}}{3}c\mu + \frac{\mathrm{i}}{3}\bar{c}\bar{\mu} - \left(\frac{\partial\bar{c} + \bar{\partial}c}{12} - \frac{\mathrm{i}(\alpha - \bar{\alpha})}{4}\right)\lambda\right)$$

and, moreover,

$$\rho = \varphi + \rho'. \qquad \blacksquare$$

The Fefferman metric possesses some interesting properties. The Fefferman metric in never globally Einstein. For computational reasons we will give the proof in the next chapter (see [69, 34]).

Chapter 3

FRT metrics and the embedding of CR manifolds

In this chapter which is inspired by [37] and [25], we introduce a CR invariant class of Lorentzian metrics on a circle bundle over a 3-dimensional CR manifold, which we call FRT metrics where FRT stands for Fefferman Robinson Trautman.

These metrics generalise the Fefferman metric but allow for more control of the Ricci curvature. Our main result is a criterion for the local embeddability of 3-dimensional CR manifolds in terms of the Ricci curvature of the FRT metrics.

This chapter is a joint work which has been already published online. See the reference [61] for more details.

3.1 Fefferman Robinson Trautman metrics

We recall from the Definition 2.26, for a given CR manifold (M, D, J) the Fefferman metric is defined on the circle bundle $\mathcal{M} = D^{1,0}/\mathbb{R}^{+}$. Denote by $\tilde{\mathcal{M}}$ the natural lift of $\mathcal{M}$ to a line bundle. It will be convenient in the computations below to rescale the coordinate ρ on $\tilde{\mathcal{M}}$ to $r = \frac{2\rho}{3}$. Then, the change of the coframe (λ, μ) induces the change

$$r' = r - \frac{2}{3}\varphi \tag{3.1}$$

of the trivialisation of $\tilde{\mathcal{M}}$ where $\varphi \in [0, 2\pi)$.

Denote the quotient bundle of the (rescaled) line bundle $\tilde{\mathcal{M}}$ mod 2π by $\mathcal{M}^{\frac{3}{2}}$. Since the Fefferman metric is invariant with respect to the principle $\mathbb{R}$-bundle action it projects to any S^1-bundle with arbitrary period. In particular, it is well-defined on $\mathcal{M}^{\frac{3}{2}}$.

Definition 3.1.1 *Let (M, D, J) be a CR structure and $\mathcal{M}^{\frac{3}{2}}$ as above. For any choice of a distinguished coframe (μ, λ) and the induced trivialisation of $\mathcal{M}^{\frac{3}{2}}$ we define the family of FRT metrics on $\mathcal{M}^{\frac{3}{2}}$ by*

$$g = 2P^2 \left[\mu\bar{\mu} + \lambda\left(dr + W\mu + \overline{W}\bar{\mu} + H\lambda\right)\right] \tag{3.2}$$

where

$$W = \mathrm{i}x\, \mathrm{e}^{-\mathrm{i}r} - \frac{\mathrm{i}}{3}c.$$

Here $P \neq 0, H$ are real-valued functions on $\mathcal{M}^{\frac{3}{2}}$ and x is a complex-valued function on M.

We note that the complex function W defined as above is consistent with (2.2.7), and the vector field ∂_r is a shearfree vector field.

The family of FRT metrics has the following important property

Theorem 3.1.2 *The family of FRT metrics is CR invariant.*

Proof Under the frame change (1.8) and the induced change of the trivialisation (3.1) the FRT metric changes as follows. Let g'

$$g' = 2P'^2\left[\mu'\bar{\mu}' + \lambda'\left(dr' + (\mathrm{i}x'\,\mathrm{e}^{-\mathrm{i}r'} - \frac{\mathrm{i}}{3}c')\mu' + (-\mathrm{i}\bar{x}'\,\mathrm{e}^{\mathrm{i}r'} + \frac{\mathrm{i}}{3}\bar{c}')\bar{\mu}' + H'\lambda'\right)\right]$$

be the representative of FRT metrics corresponding to the choice (μ', λ').

Therefore, substituting the transformations (1.8) into g' implies that

$$
\begin{aligned}
g' &= 2|f|^2 P'^2\Bigg[\mu\bar\mu + \bar h\mu\lambda + h\bar\mu\lambda + |h|^2\lambda^2 \\
&\quad + \lambda\Bigg(dr - \frac{2}{3}d\varphi + f\left(\mathrm{i}x'\,\mathrm{e}^{\mathrm{i}\frac{2}{3}\varphi}\,\mathrm{e}^{-\mathrm{i}r} - \frac{\mathrm{i}}{3f}(c - 2\mathrm{i}\bar h + \partial\log f)\right)\mu \\
&\quad + \bar f\left(-\mathrm{i}\bar x'\,\mathrm{e}^{-\mathrm{i}\frac{2}{3}\varphi}\,\mathrm{e}^{\mathrm{i}r} + \frac{\mathrm{i}}{3\bar f}(\bar c + 2\mathrm{i}h + \bar\partial\log\bar f)\right)\bar\mu \\
&\quad + \left(fh(\mathrm{i}x'\,\mathrm{e}^{\mathrm{i}\frac{2}{3}\varphi}\,\mathrm{e}^{-\mathrm{i}r} - \frac{\mathrm{i}}{3}c') + \bar f\bar h(-\mathrm{i}\bar x'\,\mathrm{e}^{-\mathrm{i}\frac{2}{3}\varphi}\,\mathrm{e}^{\mathrm{i}r} + \frac{\mathrm{i}}{3}\bar c') + |f|^2H'\right)\lambda\Bigg)\Bigg] \\
&= 2P^2\Bigg[\mu\bar\mu + \lambda\Bigg(dr + (\mathrm{i}x\,\mathrm{e}^{-\mathrm{i}r} - \frac{\mathrm{i}}{3}c)\mu + (\frac{1}{3}\bar h - \frac{\mathrm{i}}{3}\partial\log f - \frac{2}{3}\partial\varphi)\mu \\
&\quad + (-\mathrm{i}\bar x\,\mathrm{e}^{\mathrm{i}r} + \frac{\mathrm{i}}{3}\bar c)\bar\mu + (\frac{1}{3}h + \bar\partial\log\bar f - \frac{2}{3}\bar\partial\varphi)\bar\mu + H\lambda\Bigg)\Bigg] \\
&= 2P^2\Bigg[\mu\bar\mu + \lambda\Bigg(dr + (\mathrm{i}x\,\mathrm{e}^{-\mathrm{i}r} - \frac{\mathrm{i}}{3}c)\mu + (-\mathrm{i}\bar x\,\mathrm{e}^{\mathrm{i}r} + \frac{\mathrm{i}}{3}\bar c)\bar\mu + H\lambda\Bigg)\Bigg],
\end{aligned}
$$

where

$$
f = \mathrm{e}^{\tau+\mathrm{i}\varphi}, \quad P = \mathrm{e}^{\tau}\,P', \quad x = \mathrm{e}^{\tau+\mathrm{i}\frac{5}{3}\varphi}\,x', \tag{3.3a}
$$

$$
\begin{aligned}
H &= \mathrm{e}^{2\tau}\,H' + |h|^2 + \mathrm{e}^{\tau+\mathrm{i}\varphi}\,h(\mathrm{i}x'\,\mathrm{e}^{-\mathrm{i}r'} - \frac{\mathrm{i}}{3}c') \\
&\quad + \mathrm{e}^{\tau-\mathrm{i}\varphi}\,\bar h(-\mathrm{i}\bar x'\,\mathrm{e}^{\mathrm{i}r'} + \frac{\mathrm{i}}{3}\bar c') - \frac{2}{3}\partial_0\varphi. \qquad \blacksquare
\end{aligned} \tag{3.3b}
$$

In the proposition below we compute the Ricci components of the FRT metric.

Proposition 3.1.3 *Let g be an FRT metric* (3.2) *on $\mathcal{M}^{\frac{3}{2}}$ associated with a CR-manifold M that admits a non-constant CR function ζ. Let $(\mu = d\zeta, \lambda)$ be a coframe for M and R_{ik} the components of the Ricci curvature with respect to an adapted frame. Then*

(i) $\mathrm{R}_{44} = 0$ *is equivalent to*

$$
P = \frac{a}{\cos(\frac{r+s}{2})}, \tag{3.4}
$$

where a, s are arbitrary r-independent real functions.

(ii) $\mathrm{R}_{24} = \overline{\mathrm{R}}_{14} = 0$ *is equivalent to*

$$\partial \log a^2 + \mathrm{i}\partial s - 2x\,\mathrm{e}^{\mathrm{i}s} = -\frac{2c}{3}. \tag{3.5}$$

(iii) $\mathrm{R}_{22} = 0$ *if and only if the equation*

$$\partial t + t(c - t) = 0 \tag{3.6}$$

is satisfied, where

$$t = c + \partial \log a^2 - x\,\mathrm{e}^{\mathrm{i}s}\,. \tag{3.7}$$

For an alternate coframe (μ', λ') *the function* t *changes to*

$$t' = \mathrm{e}^{-\tau - \mathrm{i}\varphi}(t - \mathrm{i}\bar{h}). \tag{3.8}$$

Proof To verify that the condition $\mathrm{R}_{44} = 0$ is equivalent to the function P having the form (3.4), we first notice that $\mathrm{R}_{44} = 2\,\mathrm{R}^1_{414}$. We now consider the Cartan's structure equation for the 1-form Γ^1_4

$$d\,\Gamma^1_4 + \Gamma^1_k \wedge \Gamma^k_4 = \mathrm{R}^1_{4k\ell}\,\theta^k \wedge \theta^\ell, \quad k < \ell,$$

which simplifies to

$$d\,\Gamma^1_4 + \Gamma^1_1 \wedge \Gamma^1_4 + \Gamma^1_4 \wedge \Gamma^4_4 = \mathrm{R}^1_{4k\ell}\,\theta^k \wedge \theta^\ell, \quad k < \ell \tag{3.9}$$

since $\Gamma^1_2 = \Gamma^3_4 = 0$. Substituting the 1-forms $\Gamma^1_4, \Gamma^1_1, \Gamma^4_4$ given by (2.13a),(2.13b),(2.13c) into (3.9), we have

$$\begin{aligned} d\,\Gamma^1_4 = {} & d(\frac{\mathrm{i}}{2P} + c^1_{14}) \wedge \theta^1 - (\frac{\mathrm{i}}{2P} + c^1_{14})c^1_{ij}\theta^i \wedge \theta^j \\ & + \frac{1}{2}d(c^3_{23} + c^4_{24}) \wedge \theta^3 - \frac{1}{2}(c^3_{23} + c^4_{24})c^3_{ij}\theta^i \wedge \theta^j, \quad i < j. \end{aligned}$$

The coefficient of the 2-form $\theta^1 \wedge \theta^4$ in $d\,\Gamma^1_4$ and $\Gamma^1_1 \wedge \Gamma^1_4 + \Gamma^1_4 \wedge \Gamma^4_4$ is

$$\frac{\mathrm{i}}{2}\frac{P_r}{P^3} - \frac{\mathrm{i}}{2}\frac{P_r}{P^3} - \frac{P_{rr}}{P^3} + \frac{2P_r^2}{P^4} - \frac{P_r^2}{P^4}$$

and

$$\frac{1}{4P^2} - \frac{\mathrm{i}}{2}\frac{P_r}{P^3} + \frac{\mathrm{i}}{2}\frac{P_r}{P^3} + \frac{P_r^2}{P^4}$$

respectively. Therefore, $\mathrm{R}^1_{414} = 0$ implies

$$-4PP_{rr} + 8P_r^2 + P^2 = 0. \tag{3.10}$$

We solve the ODE (3.10) using the substitution

$$P = \mathrm{e}^{\int u dr}.$$

It follows that

$$P_r = u\,\mathrm{e}^{\int u dr}, \quad P_{rr} = \left(u' + u^2\right)\mathrm{e}^{\int u dr}$$

so that substituting P_r and P_{rr} into the ODE (3.10), it takes the form

$$-4\left(u' + u^2\right)\mathrm{e}^{2\int u dr} + 8u^2\,\mathrm{e}^{2\int u dr} + \mathrm{e}^{2\int u dr} = \left(-4u' + 4u^2 + 1\right)\mathrm{e}^{2\int u dr} = 0.$$

It also follows that

$$u' = \frac{du}{dr} = \frac{1 + 4u^2}{4},$$

which implies

$$\int \frac{4}{1 + 4u^2} du = 2\arctan 2u = r + s$$

or equivalently,

$$u = \frac{1}{2}\tan\frac{r + s}{2}.$$

By integrating both sides of the last equality, we see that

$$\int u dr = \frac{1}{2}\int \tan\frac{r + s}{2} dr = -\ln\cos\frac{r + s}{2} + C$$

is satisfied and consequently,

$$P = \mathrm{e}^{\int u dr} = \frac{a}{\cos\frac{r+s}{2}},$$

where $a = \mathrm{e}^C$.

To show that $\mathrm{R}_{14} = 0$ is equivalent to (3.5), we first note that

$$\mathrm{R}_{14} = \mathrm{R}^1_{114} + \mathrm{R}^2_{124} + \mathrm{R}^3_{134}.$$

To compute R^1_{114} we look at the coefficient of the 2-form $\theta^1 \wedge \theta^4$ of the Cartan's structure equation for Γ^1_1

$$d\Gamma^1_1 + \Gamma^1_3 \wedge \Gamma^3_1 + \Gamma^1_4 \wedge \Gamma^4_1 = \mathrm{R}^1_{1k\ell}\, \theta^k \wedge \theta^\ell, \quad k < \ell.$$

After taking the exterior derivative, we get

$$\mathrm{R}^1_{114} = (c^2_{12})_4 + c^2_{12} c^1_{14} + \frac{\mathrm{i}}{2}(\frac{1}{p})_1 - \frac{3\mathrm{i}}{4P} c^4_{14} - \frac{\mathrm{i}}{4P} c^3_{13} - \frac{1}{2} c^1_{14} c^4_{14} - \frac{1}{2} c^1_{14} c^3_{13}.$$

The component $\mathrm{R}^2_{124} = 0$, since the coefficient of the 2-form $\theta^2 \wedge \theta^4$ of the Cartan's structure equation for Γ^2_1 is 0. Indeed, the left hand side of the following equation vanishes

$$d\,\Gamma^2_1 + \Gamma^2_1 \wedge \Gamma^1_1 + \Gamma^2_2 \wedge \Gamma^2_1 + \Gamma^2_3 \wedge \Gamma^3_1 + \Gamma^2_4 \wedge \Gamma^4_1 = \mathrm{R}^2_{1k\ell}\, \theta^k \wedge \theta^\ell, \quad k < \ell,$$

due to $\Gamma^2_1 = 0$ and also

$$\Gamma^2_4 = g_{12} \Gamma^2_4 = g_{1k} \Gamma^k_4 = \Gamma_{14} = -\Gamma_{41} = -g_{4k} \Gamma^k_1 = -g_{43} \Gamma^3_1 = -\Gamma^3_1$$

and $\Gamma^4_1 = -\Gamma^2_3$. In order to compute the component R^3_{134} we study the coefficient of the 2-form $\theta^3 \wedge \theta^4$ of the structure equation

$$d\Gamma^3_1 + \Gamma^3_1 \wedge \Gamma^1_1 + \Gamma^3_3 \wedge \Gamma^3_1 = \mathrm{R}^3_{1k\ell}\, \theta^k \wedge \theta^\ell, \quad k < \ell,$$

which reads

$$\mathrm{R}^3_{134} = \frac{1}{2}(c^3_{13})_4 + \frac{1}{2}(c^4_{14})_4 - \frac{\mathrm{i}}{4P} c^3_{13} - \frac{\mathrm{i}}{4P} c^4_{14}.$$

Thus,

$$\begin{aligned} \mathrm{R}_{14} = {} & (c^2_{12})_4 + c^2_{12} c^1_{14} + \frac{\mathrm{i}}{2}(\frac{1}{p})_1 - \frac{\mathrm{i}}{P} c^4_{14} - \frac{\mathrm{i}}{2P} c^3_{13} \\ & - \frac{1}{2} c^1_{14} c^4_{14} - \frac{1}{2} c^1_{14} c^3_{13} + \frac{1}{2}(c^3_{13})_4 + \frac{1}{2}(c^4_{14})_4. \end{aligned}$$

After substituting c^i_{jk} to R_{14} and simplifying, we get

$$\begin{aligned} \mathrm{R}_{14} = \frac{1}{P}\Big(& 2\partial_r \partial(\frac{1}{P}) - W_r \partial_r(\frac{1}{P}) - 2W \partial_{rr}(\frac{1}{P}) + \frac{\mathrm{i}c}{2P} - c\partial_r(\frac{1}{P}) + \frac{W_{rr}}{2P} \\ & - \mathrm{i}\partial(\frac{1}{P}) + \mathrm{i}W \partial_r(\frac{1}{P}) - \mathrm{i}\frac{W_r}{P}\Big). \end{aligned}$$

Taking the derivatives with respect to the variable r and using the following identities

$$\cos\frac{r+s}{2} = \frac{\mathrm{e}^{\mathrm{i}\frac{r+s}{2}} + \mathrm{e}^{-\mathrm{i}\frac{r+s}{2}}}{2}, \quad \sin\frac{r+s}{2} = \frac{\mathrm{e}^{\mathrm{i}\frac{r+s}{2}} - \mathrm{e}^{-\mathrm{i}\frac{r+s}{2}}}{2\mathrm{i}} \tag{3.11}$$

we see that , after lengthy but straightforward computations and simplifications, vanishing of R_{14} is equivalent to (3.5).

To show that $\mathrm{R}_{11} = 0$ is equivalent to (3.6), we initially note that

$$\mathrm{R}_{11} = 2\,\mathrm{R}_{1413} = 2\,\mathrm{R}^2_{413}\,.$$

In order to compute the Ricci component R_{22}, we examine the coefficient of the 2-form $\theta^2 \wedge \theta^3$ of the following structure equation

$$d\,\Gamma^2_4 + \Gamma^2_2 \wedge \Gamma^2_4 + \Gamma^2_4 \wedge \Gamma^4_4 = \mathrm{R}^2_{4k\ell}\,\theta^k \wedge \theta^\ell, \quad k < \ell,$$

since $\Gamma^2_1 = \Gamma^3_4 = 0$. After taking the exterior derivative of Γ^2_4, the coefficient of the $\theta^1 \wedge \theta^3$ in $d\,\Gamma^2_4$ is as follows

$$\frac{\mathrm{i}}{2P}c^2_{13} - c^2_{24}c^2_{13} + \frac{1}{2}(c^3_{13})_1 - \frac{1}{2}(c^3_{13})^2 + \frac{1}{2}(c^4_{14})_1 - \frac{1}{2}c^4_{14}c^3_{13}.$$

Also the coefficient of the $\theta^1 \wedge \theta^3$ in $\Gamma^2_2 \wedge \Gamma^2_4 + \Gamma^2_4 \wedge \Gamma^4_4$ is in the following form

$$-\frac{1}{2}c^2_{21}c^3_{13} - \frac{1}{2}c^2_{21}c^4_{14} + \frac{1}{4}(c^3_{13})^2 - \frac{1}{4}(c^4_{14})^2.$$

Therefore,

$$\begin{aligned}\mathrm{R}_{11} = {} & \frac{\mathrm{i}}{P}c^2_{13} - 2c^2_{24}c^2_{13} + (c^3_{13})_1 - \frac{1}{2}(c^3_{13})^2 + (c^4_{14})_1 - c^4_{14}c^3_{13} \\ & + c^2_{12}c^3_{13} + c^2_{12}c^4_{14} - \frac{1}{2}(c^4_{14})^2.\end{aligned}$$

Substituting c^i_{jk} in R_{11}, it gives

$$\begin{aligned}\mathrm{R}_{11} = \frac{1}{P}\Bigg(& -\mathrm{i}\frac{\bar\beta}{P} - 2\bar\beta\partial_r(\frac{1}{P}) + 2\partial\partial(\frac{1}{P}) - 4W\partial_r\partial(\frac{1}{P}) - 2(\partial W)\partial_r(\frac{1}{P}) \\ & + 2WW_r\partial_r(\frac{1}{P}) + 2W^2\partial_r\partial_r(\frac{1}{P}) - \frac{\partial c}{P} - \frac{c^2}{2P} + \frac{\partial W_r}{P} - \frac{WW_{rr}}{P} \\ & + \frac{cW_r}{P} - \frac{(W_r)^2}{2P}\Bigg).\end{aligned}$$

By taking the derivatives and using (3.11), we see vanishing of R_{11} implies that

$$-\mathrm{i}\bar{\beta} + \partial t + t(c-t) = 0 \tag{3.12}$$

is satisfied where t is given by (3.8). Since a CR function exists one can choose $\mu = d\zeta$, where ζ is a CR function. Then it follows that $\beta = 0$ and consequently,

$$\partial t + t(c-t) = 0.$$

To verify (3.8), we first notice that

$$P' = \mathrm{e}^{-\tau} P = \frac{a\,\mathrm{e}^{-\tau}}{\cos(\frac{r+s}{2})} = \frac{a'}{\cos(\frac{r'+s'}{2})}$$

for all r and $r' = r - \frac{2}{3}\varphi$. It follows $a' = \mathrm{e}^{-\tau} a$ and $s' = \frac{2}{3}\varphi + s$. Therefore,

$$\begin{aligned}
t' &= c' + \partial' \log a'^2 - x'\,\mathrm{e}^{\mathrm{i}s'} \\
&= \mathrm{e}^{-\tau-\mathrm{i}\varphi}\left(c - 2\mathrm{i}\bar{h} + \partial(\tau + \mathrm{i}\varphi)\right) + \mathrm{e}^{-\tau-\mathrm{i}\varphi}(\partial \log a^2 - 2\partial\tau) \\
&\quad - \mathrm{e}^{-\tau-\frac{5\mathrm{i}}{3}\varphi}\, x\,\mathrm{e}^{\mathrm{i}s+\frac{2\mathrm{i}}{3}\varphi} \\
&= \mathrm{e}^{-\tau-\mathrm{i}\varphi}\left(c - 2\mathrm{i}\bar{h} + \partial(\tau+\mathrm{i}\varphi) + \partial\log a^2 - \partial(\tau+\mathrm{i}\varphi) + \mathrm{i}\bar{h} - x\,\mathrm{e}^{\mathrm{i}s}\right) \\
&= \mathrm{e}^{-\tau-\mathrm{i}\varphi}\left(t - \mathrm{i}\bar{h}\right). \quad \blacksquare
\end{aligned}$$

Now, we are in the position to show that there is no representative in the family of Fefferman metric, which is Einstein.

Proposition 3.1.4 *Let $(M, [(\mu,\lambda)])$ be a strictly pseudoconvex CR manifold satisfying* (1.5). *For the choice of pairs (μ,λ), the Fefferman metric defined by*

$$g_F = P^2\left(\mu\bar{\mu} + \lambda\left(\frac{2}{3}d\rho - \frac{\mathrm{i}}{3}c\mu + \frac{\mathrm{i}}{3}\bar{c}\bar{\mu} - \left(\frac{\partial\bar{c} + \bar{\partial}c}{12} - \frac{\mathrm{i}(\alpha-\bar{\alpha})}{4}\right)\lambda\right)\right)$$

is never globally Einstein.

Proof Set

$$\theta^1 = P\mu, \quad \theta^2 = P\bar{\mu}, \quad \theta^3 = P\lambda,$$
$$\theta^4 = \left(dr - \frac{\mathrm{i}}{3}c\mu + \frac{\mathrm{i}}{3}\bar{c}\bar{\mu} - \left(\frac{\partial\bar{c} + \bar{\partial}c}{12} - \frac{\mathrm{i}(\alpha-\bar{\alpha})}{4}\right)\lambda\right),$$

where $r = \frac{2}{3}\rho$. From (3.4), it follows that $\mathrm{R}_{44} = 0$ implies that the conformal factor P, takes the following form

$$P = \frac{a}{\cos(\frac{r+s}{2})},$$

where a and s are real-valued functions and and $s_r = 0$. We notice that the function P has global singularities. ∎

3.2 FRT metrics and embedding of the CR manifold

We are now ready to prove the main theorem of this chapter.

Theorem 3.2.1 *A strictly pseudoconvex 3-dimensional CR manifold* (M, D, J) *is (locally) embeddable if and only if there exists an associated circle bundle* $\mathcal{M}^{\frac{3}{2}}$ *with an FRT metric* g *whose complexified Ricci tensor vanishes on the distribution of* α*-planes.*

Proof Let $(M, [(\mu, \lambda)])$ be a CR manifold with the representative (μ, λ) and let g be an FRT metric defined by (3.2) on $\mathcal{M}^{\frac{3}{2}}$ for which $R_{22} = R_{24} = R_{44} = 0$, where the Ricci components are computed with respect to the frame field (e_1, e_2, e_3, e_4) defined by (2.12). We then consider the connection 1-form

$$\Gamma_{24} = \Gamma^1_4 = \delta\theta^1 + \gamma\theta^3 \tag{3.13}$$

from (2.13a) where

$$\delta = \frac{\mathrm{i}}{2P} + \frac{P_r}{P^2}, \qquad \gamma = -\frac{\bar{\partial}P}{P^2} + \frac{\overline{W}P_r}{P^2} - \frac{\bar{c}}{2P} + \frac{\overline{W}_r}{P}.$$

Clearly, $\delta \neq 0$ and therefore, the form $\Gamma_{24} \neq 0$. Moreover,

$$\Gamma_{24} \wedge \overline{\Gamma}_{24} \neq 0,$$

since

$$\Gamma_{24} \wedge \overline{\Gamma}_{24} = |\sigma|^2\, \theta^1 \wedge \theta^2 \quad \mathrm{mod}\{\theta^3\}.$$

On the other hand, the conditions of the Goldberg-Sachs theorem 2.2.5 with respect to the shearfree vector field ∂_r are satisfied and therefore,

$$\Psi_0 = C_{4141} = R_{1414} = 0, \qquad \Psi_1 = C_{4341} = \frac{1}{2}(R_{4341} + R_{1421}) = 0.$$

It follows

$$C_{4242} = \overline{C_{4141}} = 0, \quad C_{4342} = \overline{C_{4341}} = 0,$$

and furthermore, using the symmetries of the Riemann curvature

$$R_{ijk\ell} = R_{k\ell ij}, \qquad R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k}$$

it yields

$$R_{2424} = 0, \qquad R_{2434} + R_{2412} = 0. \tag{3.14}$$

Since

$$R_{44} = 2R_{2414}, \quad R_{22} = 2R_{2423}, \quad R_{24} = R_{2412} - R_{2434},$$

where $R_{ij} = R^k_{ikj}$ and $R_{ijk\ell} = g_{im}R^m_{jk\ell}$, this shows that the conditions

$$R_{44} = R_{22} = R_{24} = 0$$

are equivalent to

$$R_{2414} = R_{2423} = R_{2412} - R_{2434} = 0. \tag{3.15}$$

Combining (3.14) and (3.15) yields

$$R_{2412} = R_{2424} = R_{2414} = R_{2423} = R_{2434} = 0. \tag{3.16}$$

Therefore, Cartan's structure equation (3.9) for the connection 1-form $\Gamma_{24} = \Gamma^1_4$ becomes

$$d\Gamma_{24} - (\Gamma_{12} + \Gamma_{34}) \wedge \Gamma_{24} = R_{24k\ell}\theta^k \wedge \theta^\ell = R_{2413}\theta^1 \wedge \theta^3.$$

Wedging the equation above with Γ_{24} and taking into account that Γ_{24} is a linear combination of θ^1 and θ^3, given by (3.13), we conclude that

$$d\Gamma_{24} \wedge \Gamma_{24} = 0.$$

Now, we can apply Lemma 2.1.1 for the 1-form Γ_{24} and deduce that locally there exists complex functions $h \neq 0$ and ζ such that

$$\Gamma_{24} = h\, d\zeta \quad \text{with} \quad d\zeta \wedge d\bar{\zeta} \neq 0.$$

Wedging the equation

$$h\, d\zeta = \Gamma_{24} = P(\sigma\mu + \rho\lambda)$$

by $\lambda \wedge \mu$ shows that

$$d\zeta \wedge \lambda \wedge \mu = 0.$$

Restricting the function ζ to the CR manifold M, considered as a section $\{r = 0\}$ of $\mathcal{M}^{\frac{3}{2}}$, gives a CR function there.

Now we may assume that $\mu = d\zeta$. Since vanishing of the Ricci tensor on the α-planes does not depend on the choice of an adapted frame, the conditions $R_{44} = R_{24} = R_{22} = 0$ are still satisfied.

We consider the two cases $t \equiv 0$ and $t \not\equiv 0$, where t is defined by (3.7). In the first case it follows, from the equation (3.5), that

$$\frac{4}{3}c = -\partial \log a^2 + \mathrm{i}\partial s,$$

and hence,

$$\partial \log(a^{\frac{3}{2}}\, \mathrm{e}^{-\frac{3}{4}\mathrm{i}s}) = -c.$$

Therefore, equation (2.5) has a solution $\psi = a^{\frac{3}{2}}\, \mathrm{e}^{\frac{3}{4}\mathrm{i}s}$ and, by Proposition 2.1.5, the canonical bundle has a nowhere vanishing d-closed section. Now, by Theorem 2.1.6, there exists a second CR function that is functionally independent from ζ and therefore the CR manifold is embeddable.

In the second case, if t is not identically 0, we replace the complex coframe 1-form μ by another exact form μ' as follows. Consider

$$\omega = \mu + \mathrm{i}\bar{t}\lambda, \tag{3.17}$$

and since $R_{22} = 0$ we have

$$d\omega \wedge \omega = \mathrm{i}(\bar{\partial}\bar{t} + \bar{t}(\bar{c} - \bar{t}))\mu \wedge \bar{\mu} \wedge \lambda = 0.$$

Also $\omega \wedge \bar{\omega} \neq 0$ holds because

$$\omega \wedge \bar{\omega} = \mu \wedge \bar{\mu} - \mathrm{i} t \mu \wedge \lambda - \mathrm{i} \bar{t} \bar{\mu} \wedge \lambda.$$

Thus, the 1-form ω satisfies the conditions of Lemma 2.1.1. Consequently, there exists complex-valued functions $b \neq 0$ and η such that

$$\omega = \mu + \mathrm{i} \bar{t} \lambda = b \, d\eta. \tag{3.18}$$

Clearly,

$$d\eta \wedge d\bar{\eta} = \frac{1}{|b|^2} \omega \wedge \bar{\omega} \neq 0.$$

It follows from the definition of η and ω that $d\eta$ is a linear combination of μ and λ, and hence,

$$d\eta \wedge \lambda \wedge \mu = 0,$$

meaning that , η is a CR function. Now we switch to the coframe $(\mu' = d\eta, \lambda')$ for which, because of (3.8), $t' \equiv 0$ everywhere. This reduces the second case to the first case and proves the embeddability of M.

For the proof of the converse statement we assume that the CR structure M with adapted coframe $(\mu = d\zeta, \lambda)$ is embeddable. Then, the canonical bundle contains a nonzero d-closed section, i.e. there exists a nonzero complex function ψ such that

$$\partial \log \bar{\psi} = -c.$$

We define real functions a and s, and a complex function x as follows

$$\log a^2 = \frac{4}{3} \mathrm{Re}(\log \bar{\psi}), \qquad s = -\frac{4}{3} \mathrm{Im}(\log \bar{\psi}), \qquad x = \mathrm{e}^{-\mathrm{i}s}(c + \partial \log a^2).$$

The metric defined by

$$g = 2P^2 \big[\mu \bar{\mu} + \lambda (dr + W \mu + \overline{W} \bar{\mu} + H \lambda) \big],$$

where

$$P = \frac{a}{\cos(\frac{r+s}{2})}, \qquad W = \mathrm{i} x \, \mathrm{e}^{-\mathrm{i}r} - \frac{\mathrm{i}}{3} c,$$

and H is any real function defined on $\mathcal{M}^{\frac{3}{2}}$, is an FRT metric for (M, μ, λ) and, due to Proposition 3.1.3, $R_{44} = R_{24} = R_{22} = 0$ is satisfied. ∎

Chapter 4

Subconformal geometry and shearfree geometry

In this chapter we study higher dimensional versions of shearfree null congruences in conformal Lorentzian manifolds. We show that such structures induce a subconformal structure and a partially integrable almost CR structure on the leaf space. Furthermore, we classify the Lorentzian metrics that induce the same subconformal structure [3].

The results of this chapter is a joint work, which has been already published in [3].

4.1 Subconformal geometry

We start with the definition of the subconformal manifold and its relation with the partially integrable almost CR manifolds.

Definition 4.1.1 *A subconformal manifold is a contact manifold M with contact distribution D, which is endowed with a conformal class of Riemannian metrics $[g_D]$.*

For $\dim M = 3$, subconformal manifolds are essentially the same as CR manifolds. More precisely, the conformal metric on the contact distribution induces two mutually conjugate complex structures that rotate vectors by an angle $\frac{\pi}{2}$. Vice versa, the conformal structure can be recovered from either of these complex structures by making multiplication by complex numbers conformal mappings on the distribution.

In higher dimensions the relation between subconformal and CR manifolds is less obvious.

Theorem 4.1.2 *Let $(M, D, [g_D])$ be an orientable subconformal manifold. Then, M inherits two mutually conjugated partially integrable almost CR structures J and $-J$.*

Proof There exists a global contact 1-form λ such that $D = \ker\lambda$, since M is orientable. Let $A = g^{-1}d\lambda|_D$, i.e. $d\lambda|_D = g(A\cdot,\cdot)|_D$. The endomorphism A is then nondegenerate and skew-symmetric, hence A^2 is symmetric and negative definite. Define

$$J = \sqrt{-A^{-2}}A.$$

It follows that J depends smoothly on the coordinates of M. A different choice of the contact form λ affects only the sign of J. We now show that J, and hence $-J$, define the partially integrable almost CR structures.

For any point $x \in M$ denote the eigenvalues of A_x^2 by $-\alpha_j^2$ (with $\alpha_j > 0$) and the corresponding eigenspaces by D_j. Then D_j is invariant for A_x and the restrictions $J_x|_{D_j}$ are equal to $\frac{1}{\alpha_j}A_x|_{D_j}$. It follows

$$J_x^2|_{D_j} = \frac{1}{\alpha_j}A_x|_{D_j}\frac{1}{\alpha_j}A_x|_{D_j} = \frac{1}{\alpha_j^2}A_x^2|_{D_j} = -\mathrm{id}.$$

The partial integrability condition can also be checked pointwise. For any X, Y in D_j,

$$\begin{aligned} d\lambda(J_xX, J_xY) &= \frac{1}{\alpha_j^2}d\lambda(A_xX, A_xY) = \frac{1}{\alpha_j^2}g(A_x^2X, A_xY) = \frac{-\alpha_j^2}{\alpha_j^2}g(X, A_xY) \\ &= -g(A_xY, X) = -d\lambda(Y, X) = d\lambda(X, Y). \end{aligned}$$

For X, Y from different eigenspaces $d\lambda(X, Y) = 0$. This proves partial integrability. ∎

The theorem above indicates that CR structures in higher dimensions are weaker structures than subconformal ones. Indeed, in order to reconstruct a subconformal structure from a CR structure (M, D, J) we need to prescribe a $d\lambda$-orthogonal decomposition of

$$D = \oplus D_j$$

and positive numbers α_j as above. Then let $A|_{D_j} = \alpha_j J|_{H_j}$ and $g = d\lambda \circ A^{-1}$. The extremal choices of the decomposition of D are, on the one hand, the trivial decomposition $D = D$ and, on the other hand, the decomposition into complex one-dimensional D_j. The former choice is equivalent to the CR structure while the latter one induces a much more rigid geometric structure.

4.2 Shearfree congruences and their orbit spaces

We start with a global conformal version of shearfree congruences.

Definition 4.2.1 *Let $(\mathcal{M}, [g])$ be a $(2n{+}2)$-dimensional conformal Lorentzian manifold with a shearfree vector field p and assume that the flow of p generates a free action of $G = \mathbb{R}$ or $G = S^1$ so that the orbit space by $M = \mathcal{M}/G$ is a manifold and the canonical projection $\pi : \mathcal{M} \to M$ is a principal G-bundle. We call the $(\mathcal{M}, [g], p, M)$ a Robinson-Trautman space (RT-space) of type G. We also say that the RT-space is twisting if $(d\theta)^n \wedge \theta \neq 0$, where $\theta = g(p, \cdot)$.*

Notice that the notion of twist is invariant under scaling of p and g and hence it is well-defined. Since the notion of the shearfreeness of p is invariant with respect to rescaling of p, we can replace p in the definition of the twisting RT-space by its equivalenc class $[p]$.

We will show that the orbit space M of a twisting RT-space carries a canonical subconformal structure and hence, a CR structure.

Definition 4.2.2 *An RT-space $(\mathcal{M}, [g], [p])$ and a subconformal structure $(D, [g_D])$ with contact distribution D and subconformal metric $[g_D]$ on the* *orbit space M are called compatible, if for any contact form λ on M with the Reeb vector field Z*

(i) $\ker \pi^*\lambda = p^\perp = \{X \in T\mathcal{M}: g(X,p) = 0\}$*, and*

(ii) $\pi^* g_D^\lambda|_{p^\perp}$ *is conformally equivalent to* $g|_{p\perp}$*. Here* g_D^λ *is the extension of* g_D *to the degenerate metric on* M *with* $Z = \ker g_D^\lambda$*. That is,*

$$g = P^2(\pi^* g_D^\lambda + g(p,\cdot) \vee \psi)$$

for some positive function P^2 *and some 1-form* ψ*.*

Theorem 4.2.3 *Let* $(\mathcal{M}, [g], [p])$ *be a twisting RT-space. Then, there exists a unique compatible subconformal structure on the orbit space* M*.*

Proof Let $U \in T_Q M$. Then we call $u \in T_q\mathcal{M}$ a lift of U if $\pi(q) = Q$ and $\pi_* u = U$. A compatible contact distribution $D_Q \subset T_Q M$ must satisfy the condition $\theta(u) = g(p,u) = 0$ for any lift u, of any $U \in D_Q$. This proves the uniqueness of the contact structure. We show that this condition does not depend on the choice of the lift. Let u_0 and u_1 be two lifts at q_0 and q_1, respectively, connected by a path $u(t)$, where t is the time parameter of the flow of the vector field p. Then, with respect to some local trivialisation,

$$u(t) = U + \alpha(t)p,$$

and

$$\begin{aligned}\frac{d}{dt}\theta(u(t)) &= \mathscr{L}_p g(u(t),p) = \rho g(u(t),p) + \theta(u(t))\psi(p)\\ &= (\rho + \psi(p))\theta(u(t))\end{aligned}$$

it follows that

$$\theta(u(t)) = Ce^{\int \rho+\psi(p)dt},$$

and therefore, either equals zero for all t or nowhere.

We show that D is a contact distribution. Let λ be a form that annihilates D. Then $\pi^*\lambda = \alpha\theta$, where α is a non-vanishing function. Since

$$\pi^* d\lambda^n \wedge \lambda = \alpha^{n+1} d\theta^n \wedge \theta \neq 0$$

it follows $d\lambda^n \wedge \lambda \neq 0$. In particular, twisting RT-spaces must be even-dimensional.

The conformal metric g_D on D_Q is uniquely determined by

$$g_D(U, V) = g(u, v)$$

for $U, V \in D_Q$ and any lifts $u, v \in p^\perp$ at the same base point q. We show that this definition does not depend on the choice of the lifts. Let

$$u(t) = U + \alpha(t)p, \quad \textit{and} \quad v(t) = V + \beta(t)p$$

be two paths connecting two pairs of lifts (u_0, v_0) and u_1, v_1 with respect to some trivialisation. Then,

$$\frac{d}{dt} g(u(t), v(t)) = \mathscr{L}_p g(u(t), v(t)) = \rho g(u(t), v(t)),$$

where ρ depends on t but not on $u(t)$ and $v(t)$. It follows that $g(u(t), v(t))$ scales along the path by a multiplier that does not depend on the path. Hence $g_D(U, V)$ is well-defined as a conformal metric. ∎

The theorem below describes the RT-structures that are compatible with $(M, D, [g_M])$.

Theorem 4.2.4 *Let $\pi\colon \mathcal{M} \to M$ be a line bundle over a subconformal manifold $(M, D, [g_D])$ and p any non-vanishing vertical vector field. Then, the triple $(\mathcal{M}, [g], [p])$ is a twisting RT-structure compatible with $(M, D, [g_D])$ if and only if*

$$g = P^2(\pi^* g_D^\lambda + \pi^* \lambda \vee \psi), \tag{4.1}$$

where λ is a contact form on M, P^2 is a positive function on $\mathcal{M}$ and ψ is a 1-form on $\mathcal{M}$.

Proof Assume g has the form (4.1). Then,

(i) g is Lorentzian, and $g|_{p^\perp}$ is conformally equivalent $\pi^* g_D^\lambda|_{p^\perp}$.

(ii) p is null, and

(iii) The following is satisfied

$$\begin{aligned}\mathscr{L}_p g &= 2P\frac{\partial P}{\partial t}\left(g_D + \pi^*\lambda \vee \psi\right) + P^2(\pi^*\lambda \vee \mathscr{L}_p\psi)\\ &= 2P\frac{\partial P}{\partial v} g + \pi^*\lambda \vee \tilde{\psi}.\end{aligned}$$

That is, the vector field p is shearfree for g.

Therefore, $(\mathcal{M}, [g], [p])$ is a RT-space compatible with $(M, D, [g_M])$.

It remains to show that any conformal Lorentzian metric that satisfies (i)-(iii) has the form (4.1). Since g_D is compatible with g there exists a positive function P^2 on M such that

$$g|_{p^\perp} = P^2\pi^* g_D^\lambda|_{p^\perp}.$$

Consider the symmetric 2-form

$$T = g - P^2\pi^* g_D^\lambda$$

for some choice of the contact form λ on M. Then, $T(u, v) = 0$ for any $u, v \in T_q M$ such that $g(v, p) = 0$. Let z be a lift of the Reeb vector field Z. We can choose z such that $g(z, z) = 0$.

Consider the 1-forms

$$\theta = g(p, \cdot) = \gamma\pi^*\lambda, \qquad \psi' = g(z, \cdot).$$

We have $\theta(z) = g(z, p) = \gamma\omega(Z) = \gamma$.

If $u = u' + \alpha z$ is the decomposition of a vector field u on M such that $u' \in p^\perp$, then

$$\theta(u) = \alpha g(p, z) = \alpha\gamma\pi^*(Z) = \alpha\gamma$$

hence,

$$\alpha = \frac{1}{\gamma}\theta(u) = \pi^*\lambda(u).$$

For two vector fields u, v on $\mathcal{M}$ with $u = u' + \alpha z$, $v = v' + \beta z$, where $u', v' \in p^\perp$ we have

$$T(u,v) = \alpha g(z,v) + \beta g(u,z) = \frac{1}{\gamma}(\theta(u)\psi'(v) + \theta(v)\psi'(u))$$
$$= \pi^*\lambda \vee \psi'(u,v).$$

It follows

$$g = P^2\pi^* g_D^\lambda + T = P^2(\pi^* g_D^\lambda + \pi^*\lambda \vee \psi),$$

where $\psi = \frac{1}{P^2}\psi'$. ∎

4.3 Applications of shearfree congruences

In this section we survey some applications of shearfree congruences in dimension 4.

The correspondence between 4-dimensional shearfree congruences and 3-dimensional CR manifolds has been known by physicists and has been exploited in both directions (see, e.g. [56, 25] and references therein).

Consider the Lorentzian metrics

$$g = P^2\mu\bar{\mu} + \lambda(dr + W\mu + \overline{W}\bar{\mu} + H\lambda),$$

where

$$\mu = dz,$$
$$\lambda = du - 2\,\mathrm{Im}\,\frac{((a+b)|z|^2 + b)dz}{z(1+|z|^2)^2},$$
$$P^2 = \frac{r^2}{(1+|z|^2)^2} + \frac{(b-a) + (b+a)|z|^2}{(1+|z|^2)^4},$$
$$W = \frac{2\,\mathrm{i}\,az}{(1+|z|^2)^2},$$
$$H = \frac{2(mr+b^2)(1+|z|^2)^2 - 2ab(1-|z|^4)}{r^2(1+|z|^2)^2 + (b-a+(b+a)|z|^2)^2} - 1.$$

Here $z = x + \mathrm{i}y, u, r$ are coordinates in $\mathbb{R}^4$ and a, b, m are real parameters. The metric g is singular for $z = 0$ if $b \neq 0$, and for $r = 0$ and $|z|^2 = \frac{a-b}{a+b}$ if $|b| \leq |a|$.

The corresponding RT-space $(\mathcal{M}, [g], [\partial_r])$ is twisting, unless $a = b = 0$, the metric g is the Kerr rotating black hole with mass m and the angular momentum parameter a; if $a = b = 0$ the metric g describes the Schwarzschild black hole with mass m. For $m = a = 0$ this is the Taub-NUT vacuum metric. The orbit spaces M can be described with $\mathbb{C} \times \mathbb{R}$ with the coordinates (z, u).

If $b \neq 0$ we have to delete the singular line $z = 0$. The induced subconformal structures are $(M, [\lambda], [\mu\bar{\mu}])$ and the CR structures are defined by $(M, [(\lambda, \mu)])$. Notice that the parameter m only appears in the function D and does not affect the family of CR manifolds.

All resulting CR manifolds can be embedded into $\mathbb{C}^2$ with the coordinates (z, w) as

$$v = \operatorname{Im} w = \frac{-2a}{1 + |z|^2} + 2b \log \frac{|z|^2}{1 + |z|^2}.$$

This is the trivial Levi-flat CR manifold $v = 0$ for the Schwarzschild solution, a spherical CR manifold (with singularity at 0) in the Taub-NUT case and a non-spherical Sasakian manifold for the Kerr solution.

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