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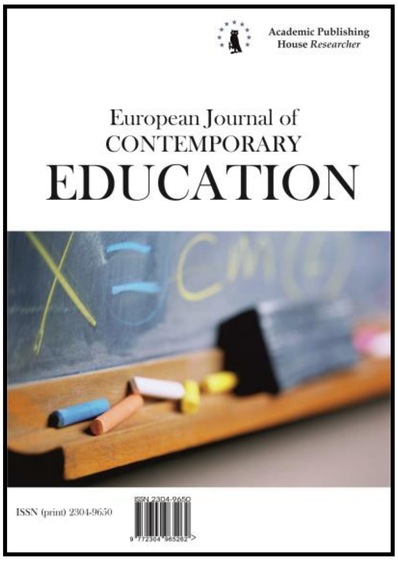

Visualisation of Selected Mathematics Concepts with Computers – the Case of Torricelli's Method and Statistics

Ján Gunčaga [a, *], Wacek Zawadowski [b], Theodosia Prodromou [c]

[a] Faculty of Education, Comenius University, Bratislava, Slovak Republic
[b] University of Warsaw, Poland
[c] School of Education, University of New England, Armidale, Australia

Abstract

Visual imagery has been an effective tool to communicate ideas connected with basic mathematics concepts since the dawn of mankind. The development of educational visualisation technology allows these ideas to be demonstrated with the help of some educational software. In this paper, we specifically consider the use of GeoGebra, a free, open-source educational application developed by an international consortium of mathematics and statistics educators, but other educational software could also be used for the same visualisation tasks.

In this study, we present Torricelli's method for measuring the area under arc of cycloid as an example of using GeoGebra to visualise he area of planar figures. This kind of introduction is suitable for secondary schools and for training pre-service teachers.

We will also show how GeoGebra can be used to develop students' understanding of representing data (i.e. the topic from statistics education). While students explore the visualisation of data, GeoGebra allows them to create and explore representations while building the understanding that is required for analysing data and drawing figural conclusions from graphical representations.

Keywords: measuring, Cavalieri's method of indivisibles, Evangelista Torricelli, the area under arc of cycloid, visualisation in statistics education.

1. Introduction

The theory of mathematics education developed by Hejný (see Hejný et al., 2006) identifies stages of gaining knowledge. Hejný described each of these stages of cognitive processes in mathematics. He defined the following stages: motivation, isolated models, generic model, abstract knowledge and crystallisation. An isolated model is a model used for explaining a concept.

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[*] Corresponding author

E-mail addresses: jguncaga@gmail.com (J. Gunčaga)

For example, one car or one pen is an isolated model for the number one. A generic model can be one finger (mostly used by children). Isolated and generic models play important roles in this theory. In order to explain a mathematical concept, it is useful to use some explanations used in the history of mathematics.

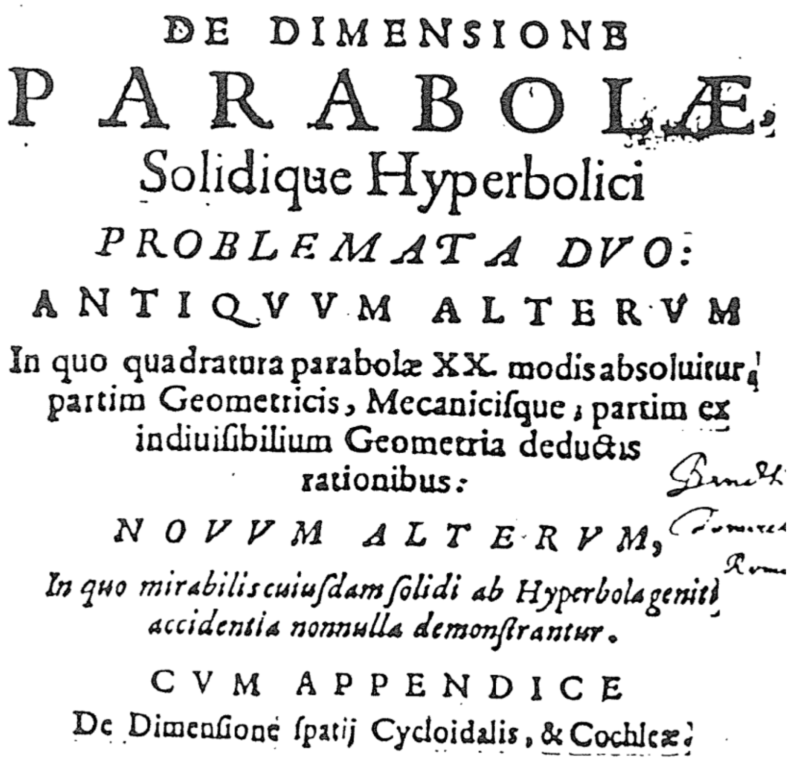
In the following section, we present an example of a visual and geometrical representation of the measurement of the area under the arc of cycloid. This representation was developed by Evangelista Torricelli (1608–1647) using the geometrical application of Cavalieri's method of indivisibles. We present some modern possibilities of geometrical visual representations prepared in GeoGebra (see also Koreňová, 2016). These presentations have dynamic components in some cases.

Torricelli lived in the beginning of the 17th century, when there was no established formal logic or style of mathematical argumentation, to say nothing of formal proof. For this reason, Torricelli used multiple kinds of argumentation to be certain about his final conclusions. Modern students can also develop better understandings of concepts when they are exposed to multiple explanations.

2. Discussion

Genesis of Torricelli's Appendix on Measuring the Cycloid

Torricelli's measurement of the area under the arc of cycloid is appended to the end of his treaty entitled *On measuring the parabola* (see Figure 1).



DE DIMENSIONE
PARABOLÆ,
Solidique Hyperbolici
PROBLEMATA DVO:
ANTIQVVM ALTERVM
In quo quadratura parabolæ XX. modis abſoluitur,
partim Geometricis, Mecaniciſque; partim ex
indiuiſibilium Geometria deductis
rationibus:
NOVVM ALTERVM,
In quo mirabilis cuiuſdam ſolidi ab Hyperbola geniti
accidentia nonnulla demonſtrantur.
CVM APPENDICE
De Dimenſione ſpatij Cycloidalis, & Cochleæ.

Fig. 1. Front page of Torricelli's treaty about measuring a parabola

The problem of the cycloid was well known at the time. In Italy, the first to consider the cycloid was Galileo Galilei (1564–1642), followed by his disciples Bonaventura Francesco Cavalieri (1598–1647), Evangelista Torricelli (1608-1647) and Vincenzo Viviani (1622–1703). In France the cycloid was the focus of the work of Marin Mersenne (1588–1648), Gilles Personne de Roberval (1602–1675), Pierre de Fermat (1607–1665), Blaise Pascal (1623–1662) and Rene Descartes (1596–1650). In England, Sir Christopher Wren (1632–1723) found that the length of the cycloid is eight times longer than the radius of the rolling circle.

Galilei had tried to estimate the area under the arc of cycloid. He assumed that this area was equal to "three times the area of the rolling circle". Not being able to prove it, he hung physical shapes on a balance to compare their weight. Due to problems with that method, he concluded that the area under the cycloid might be less than his original belief (that it was three times the area of the generating circle). Torricelli later proved that Galilei was correct by using the work of his colleague Cavalieri (see also Fulier, Tkačik, 2015).

Torricelli used expressions like "a rectangle which is equal to two circles" to prove that if we assume that two regions in a plane are included between two parallel lines in that plane, then when these two lines intersect, both figures in the line segments of equal length have equal areas (see Howard, 1991). He compares the area of a complicated planar figure with the area of a simple planar figure.

Torricelli's text on the area under an arc of cycloid shows the emergence of a new language which gave mathematics new power in the 17th century. In the original text, Torricelli used abbreviated language – for example "*AB* and *CD* are the same", meaning "the segments *AB* and *CD* have the same length"; and "The shapes *AC* and *KM* are the same", meaning "The shapes *ABCD* and *KLMN* have the same area." He used less precise argumentation because many arguments are made in the form of figures.

We present in the next parts the original Latin text in the form of a close paraphrase of the original text from the Appendix (see Appendix 1 for Torricelli's original Latin text).

We use argumentation that is more readable than in the original. Figures prepared in GeoGebra provide visualisation of the arguments, but other software could have been used (see Vančová, Šulovská, 2016).

Presentation of the Supplement (Appendix) on Measuring the Cycloid

Let us suppose that on a certain fixed line *AB*, there is a circle *AC* touching the line *AB* at the point *A*. Let us assume that point *A* is fixed on the circumference of the circle *AC*. Now let us imagine the circle is moving on the fixed line towards point *B* and at the same time revolving so that some point of the line *AB* is always touching the circle, until the fixed point returns to touch the line at the point *B*.

It is certain that point *A*, which is on the circumference of the moving circle, describes a line which at first rises from the line *AB*, culminates around *D*, and then bowing, descends towards point B. A line such as *ADB* is called a cycloid and the line *AB* was called the base of the cycloid, and the circle *AC* its generator.

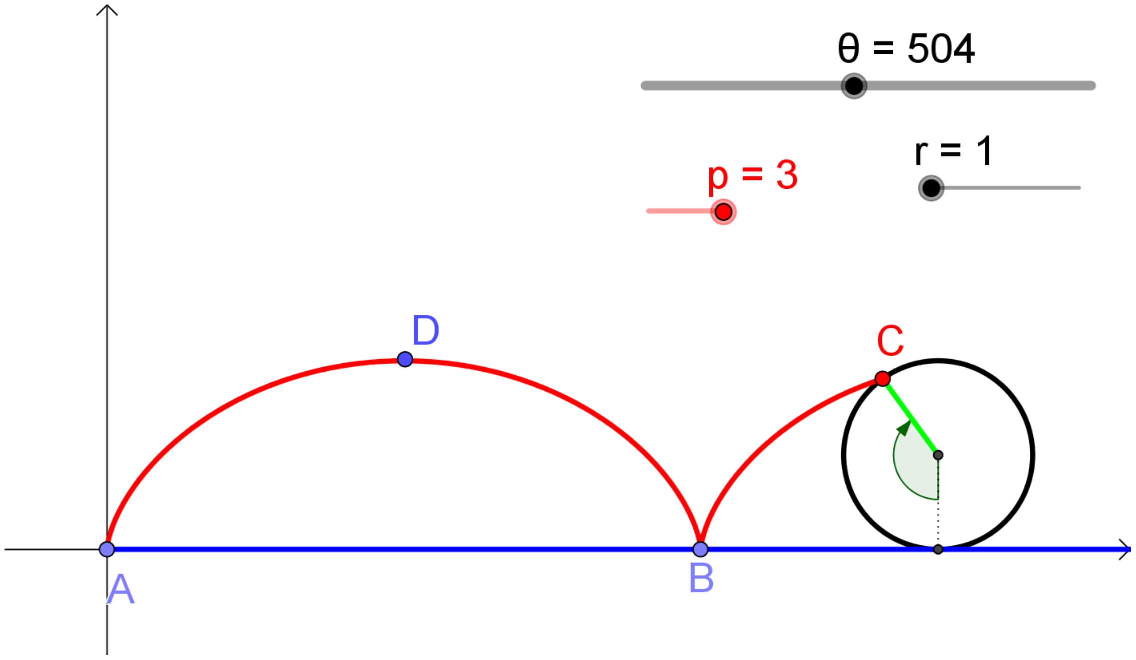


Fig. 2. Visualisation of the definition of cycloid (compare with Figure 16 in Appendix 1)

The character and property of a cycloid is such that the length of its base *AB* is equal to the circumference of the generating circle *AC*. A question arises about the ratio of the area under the arc of the cycloid *ADB* to the area of the generating circle *AC*. We shall show that it is triple. (In GeoGebra, we can move the slider θ (see Figure 2), and the circle moves with the point *C*.)

Torricelli included three proofs/arguments, each entirely different from the other. Torricelli argues as follows:

"The first and the third proceed according the new method of indivisibles. The second is by false assumption, according to the ancient customs, so that advocates of both should be satisfied. We would remind you that almost all principles according to which something is proved by the method of indivisibles could be reduced to the indirect proof, which was customary for the Ancients: this was done by us in the first and in the third of following theorems as well as in many other cases. In order not to abuse the readers' patience, most of them will be omitted, and we shall show only three."

Theorem I

The entire area of the shape between the line of a cycloid and the straight line of its base is three times the area of the generating circle or one-and-half times the area of the triangle that has the same base and height.

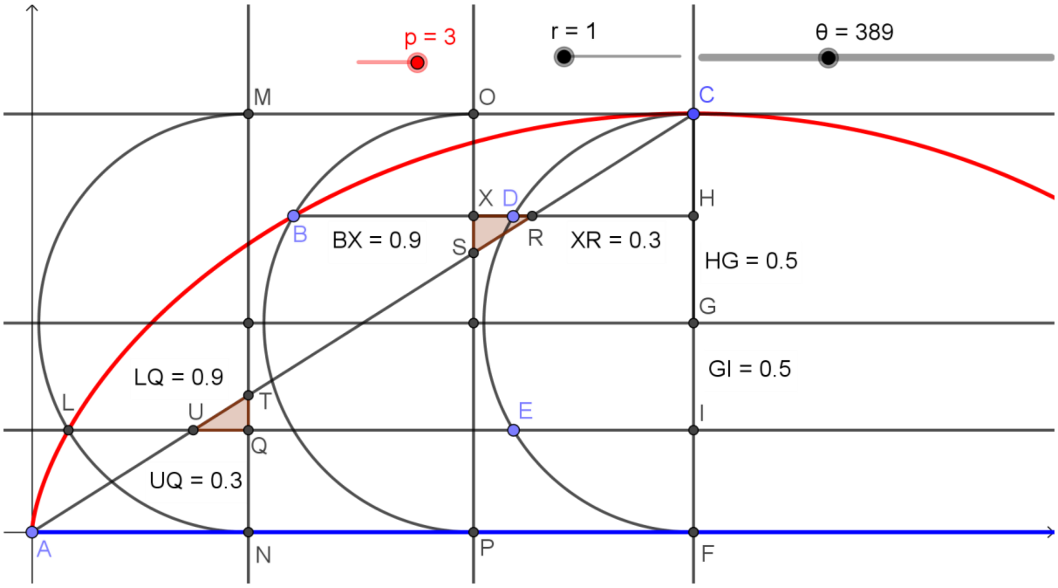


Fig. 3. Visual presentation of the picture in Figure 17 in Appendix 1

Let *ABC* be a cycloidal line traced by point *C* of the circle *CDEF* when it is rolling on a fixed base *AF* (we consider half of the circle and half of the cycloid only to avoid complicating the drawing, see Figure 3). Figure 3 presents the picture from Figure 17 in Appendix 1. It is possible to move with point B and to show that the triangles *SXR* and *UTQ* are the same.

We say that the area under half of the arc of the cycloid *ABCF* is equal to three times the area of the semicircle *CDEF*, or one-and-half times bigger than the area of the triangle *ACF*. Let us take two points *H* and *I* on the diameter *CF* at the same distance from the middle point *G*. Extending from these points are lines *HB*, *IL* and *CM*, which are parallel to the same line *FA*. *HB* passes through point *B* of the semicircle *OBP*, and *IL* passes through point *L* of the semicircles *MLN*. Both of these semicircles are equal to semicircle CDF and touch the base *FA* at points *P* and *N*. It is evident that segments *HD*, *IE*, *XB* and *QL* are equal and that using Proposition 14 of Book III (of Euclid's Elements) results in the arcs *OB*, *LN* being equal as well. The segments *GH* and *GI* are the same; hence, segments *CH* and *IF* are equal.

The whole circumference *MLN* before the cycloid (on the left) is equal to the segment *AF*. Furthermore, the arc *LN* is equal to the segment *AN* for the same reason, and because the length of the arc *LN* is the same than the length of the segment *AN*, the remaining arc *LM* will be equal to the remaining segment *NF*. For the same reason, the arc *BP* is equal to the segment *AP*, and the arc *BO* is equal to segment *PF*.

In addition, the segment *AN* is equal to the arc *LN*, to arc *BO* or to the segment *PF*. Triangles *ANT* and *COS* are the same, so the segment *AT* is equal to *SC*. Moreover, because the segment *CR* is then equal to *AU*, the remaining segments *UT*, *SR* are equal as well. Therefore the equiangular triangles *UTQ*, *RSX* have equal corresponding sides *UQ*, *RX*. It is therefore evident that the length of two segments *LU*, *BR* taken together are equal to the sum of the two segments *LQ*, *BX*, and for the same reason, they are equal to the length of the sum of the segments *EI*, *DH* – something that will always be true. When two points *H* and *I* are equally remote from the middle point *G*. Therefore, all segments of the geometric figure *ALBCA* are equal to all the segments of the semicircle *CDEF*.

However, the triangle *ACF* is twice the semicircle *CDEF* because triangle *ACF* is reciprocal to the triangle described by Archimedes in *On measuring of the circle*, when the side *AF* is equal to the semicircle and when *FC* is the diameter. Therefore, triangle *ACF* is equal to the whole circle whose diameter is *CF*.

In summary, the area under one half arc of cycloid is one-and-half times the area of the triangle *ACF* and therefore three times the area of the semicircle *CDEF*. Thus, the area under the arc of cycloid will be three times the area of the circle whose diameter is *CF* (i.e. the generating circle).

Lemma I

We suppose that on the opposite sides of an arbitrary rectangle AEFD we draw two semicircles EIF and AGD. The figure contained between their outlines and the remaining sides is equal to the initial rectangle AEFD (see Figure 4).

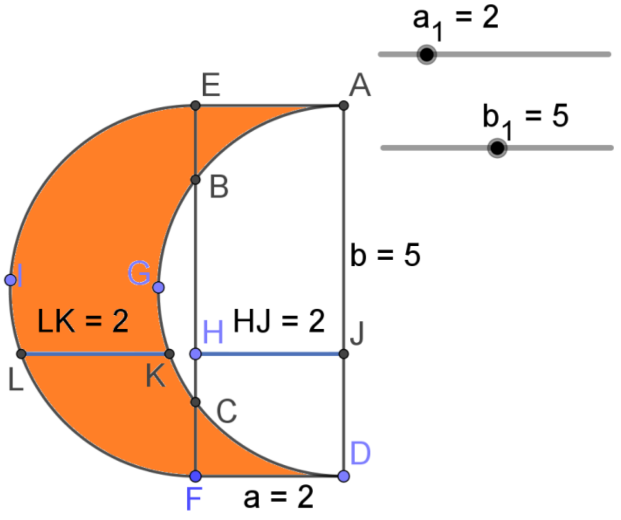


Fig. 4. Visual presentation of the picture on the Figure 18 in Appendix 1

Figure 4 presents a visualisation created by a GeoGebra applet, in which slider a_1 can change the length of the segment *AE* and slider b_1 can change the length of the segment *AD*. If we move point *H*, the segments *LK* and *HJ* remain the same. The shape *ABCDFLE*, which is marked in the Figure 4 with the colour, is called the *arc shape*.

The proof of Lemma 1 is as follows: Since the semicircles *AGD*, *EIF* are equal, after subtracting their common part *BGC* and adding the two three-sided figures *EBA* and *CFD*, the proposed thesis is clear (a geometrical application of Cavalieri's method of indivisibles).

In case there is no common part, the proof is easier. By subtraction, the arc shape, which is cut through some line parallel to segment *FD*, can be shown to be equal to the rectangle of the same height and built on the same base.

Lemma II.

Let the cycloidal line ABC be drawn from point C of a semicircle CDE, which rolls on the fixed line AE. The rectangle AFCE is completed so that a semicircle AGF rises next to AF. We say that the cycloid ABC cuts the arc shape AGFCDE in halves (see Figure 5).

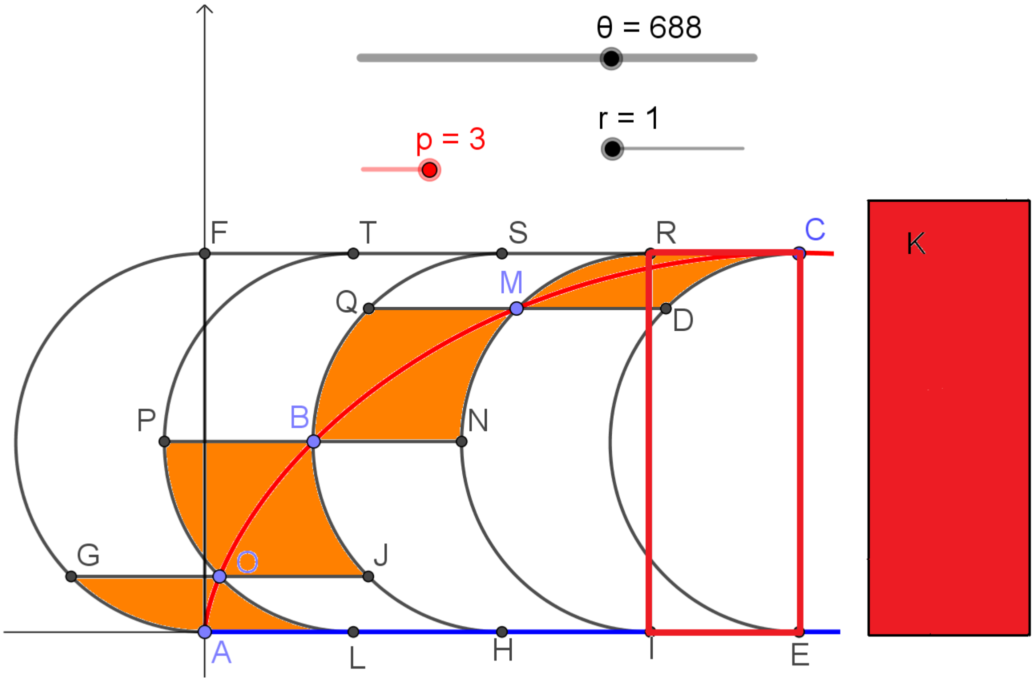


Fig. 5. Visual presentation of picture for Lemma II from Figure 19 in Appendix 1

This proof will be absurdum proof, then one of the three-sided figures *FGABC*, *ABCDE* would certainly be greater than half of the area of the arc shape *AGFCDE*. If the area of one of the arc shapes namely *ABCDE* is greater than half of the arc shape *AGFCDE*. Let the excess part, by which the three-sided figure is greater than half of the area of the arc shape, be equal to the area of a certain shape *K*. This approach is geometrical application of the "ε-δ technique". The area of a certain shape *K* is a geometrical representation of the number ε.

Let *AE* be cut into halves by a point *H*, and then *HE* by point *I*. And let it continue in cutting of *AE* (points *L*, *I*, ...)until some rectangle *IECR* is smaller than the area of the shape *K*. The whole *AE* is then divided into parts that are equal to the segment *IE*. Let semicircles be drawn through points *L*, *H*, *I* – equal to the semicircle *CDE*, touching the base *AE* at points *L*, *H*, *I* and cutting the cycloid at points *O*, *B*, *M*, through which straight lines GO, PB, QM are drawn parallel to the base *AE*.

Therefore, the areas of the arc shapes *OLHJ*, *GALO* are equal; the areas of the arc shapes *BHIN* and *PLHB* are equal; and the areas of the arc shapes *MIED*, *QHIM* are equal. Therefore, the area of the whole figure consisting of arc shapes *OLHJ*, *BHIN, MIED*, which are contained in the three-sided arc figure *ABCDE*, is equal to the area of the figure just circumscribed on the same three-sided figure, excluding the arc shape *IMRCDE* (which consists of the arc shapes *GALO*, *PLHB*, *QHIM*). And if the arc shape *IMRCDE* is added to this circumscribed figure, then its area becomes greater than the area of the one inscribed by the mentioned arc shape or by rectangle *RIEC*, which is of course less than the shape *K*. Therefore, the area of the figure contained in the three-sided arc figure *ABCDE* is greater by that amount (the area of the rectangle *RIEC*) than half of the area of the arc shape *AGFCDE*, and thus it is greater than a three-sided arc figure *FGABC*. However, it is equal to another figure composed of arc shapes in the three-sided arc figure *FGABC*. And this figure would be bigger than the figure *FGABC*, a part greater than its whole, which is impossible.

It is clear that the areas of the inscribed figures (arc shapes) are equal. Specifically, the arc *OL* is equal to the segments *LA* or *IE* or to the arc *RM* (above the cycloid). Therefore, the area of the arc shape *OLHJ* is equal to the area of the arc shape *QMRS* – and so on with each of them (pairs of the arc shapes *PBST*, *BHIN* and *GOTF*, *MIED*). If we suppose, in fact, that the area of the three-sided arc figure *FGADC* is greater than half of the area of the arc shape *AGFCDE*, the construction of figures and the proof are entirely the same. Thus, the conclusion is that the cycloidal line *ABC* divides the arc shape *AGFCDE* into two shapes with the same area.

Theorem II

The area under the arc of cycloid is three times bigger than the area of the generating circle.

Let a cycloid *ABC* be traced from point *C* of the circle *CFD*. We say that the area under half of the arc of cycloid (the shape *ABCD*) is three times bigger than the area of the semicircle *CFD*. In a rectangle *ADCE*, the side *AE* is completed by a semicircle *AGE* (see Fig. 6), and the segment *AC* is drawn.

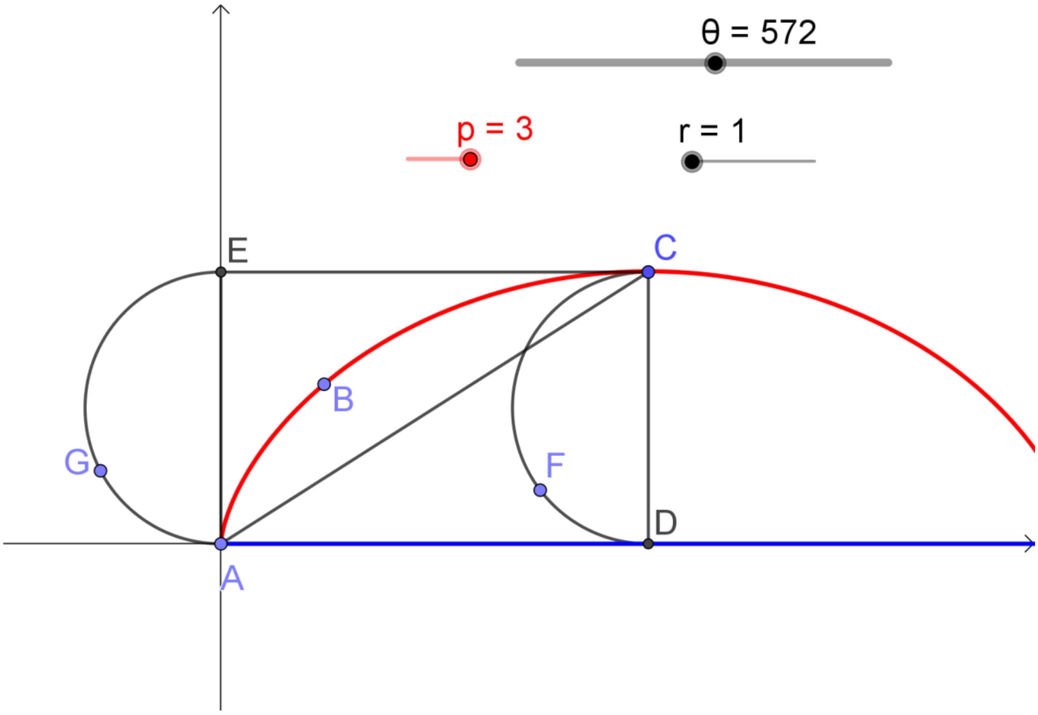


Fig. 6. Visual presentation of picture from Figure 20 in Appendix 1

The area of the triangle *ADC* is two times the area of the semicircle *CFD*, because the base *AD* is equal to the circumference *CFD* (this follows from the construction of the cycloid, and the height is equal to the diameter). Therefore, the area of the rectangle *ADCE* is four times the area of the semicircle *CFD*. Thus, the area of the arc shape *AGECFD* is four times the semicircle; the three-sided arc figure *ABCFD* (from the preceding lemma) is two times the semicircle; and the area of the shape under half of the arc of the cycloid *ABCD* is three times the area of the semicircle *CFD*. For this reason, the area of the shape under the whole arc of the cycloid is three times the area of the circle that generates the cycloid.

Theorem III.

The entire area of the shape under the arc of cycloid is three times bigger than the area of the circle that generates the cycloid.

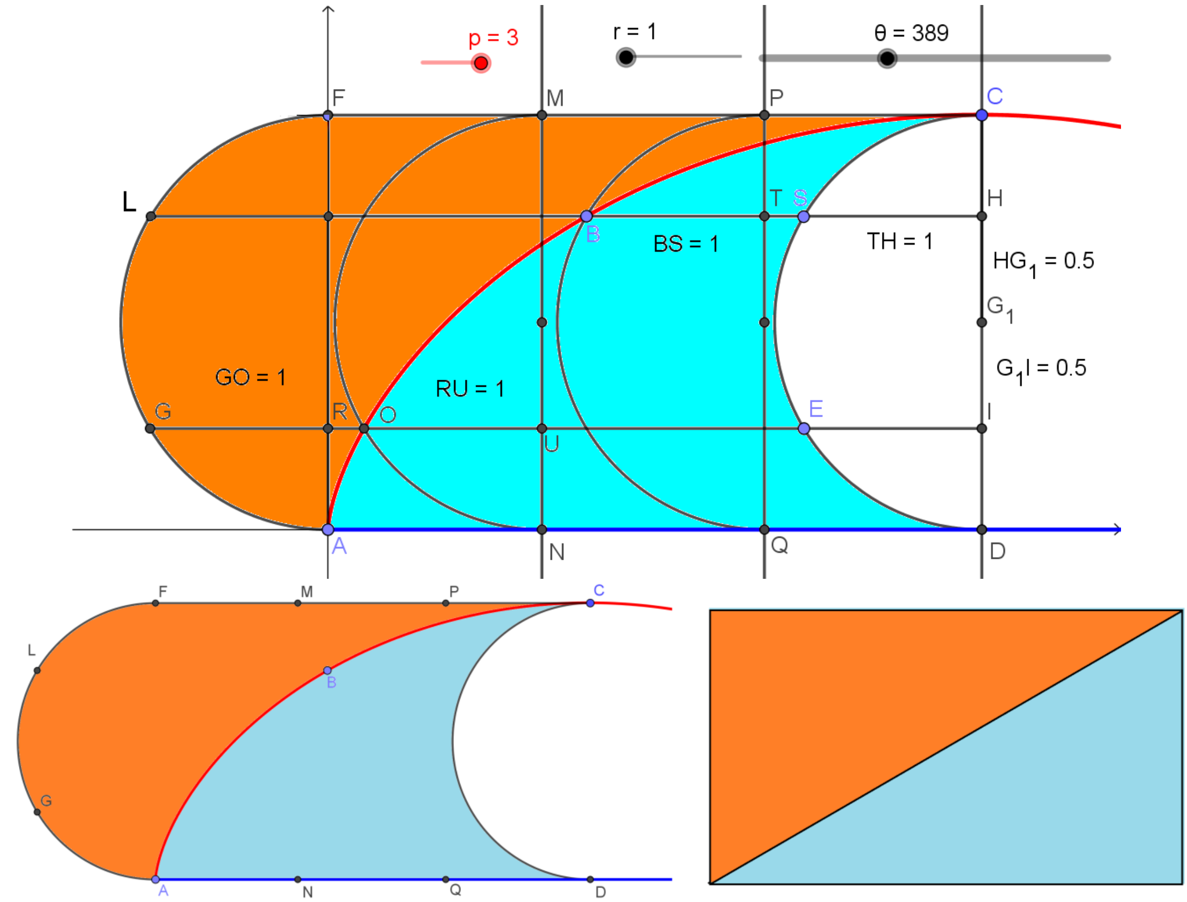


Fig. 7. Visual presentation of Figure 21 in Appendix 1 with analogy between arc shape and rectangle

Let the cycloidal line *ABC* (see Figure 7) be drawn from the point *C* of the semicircle *CED*. We say that the area of the arc figure *ABCD* is three times bigger than the area of the semicircle *CED*. Let us draw the rectangle *AFCD* and fix two points *H* and *I* on the diameter *CD* of the semicircle *CED* at the same distance from the middle G_1 of *CD*. Then, let lines *HL*, *IG* be drawn parallel to *AD*, cutting the cycloid at points *B* and *O*. Finally, let us draw through point *B* and through point *O* two semicircles *PBQ* as *MON* as done previously (with the same diameter as diameter *CD*).

Now the segment *GO* is equal to the segment *RU* (since segments *GR*, *OU* are equal and since *RO* is a common part), equal to the segment *AN* as well as to the length of arc *ON*, arc *PB*, segment *PC*, segment *TH* and segment *BS*.

Similarly, as it was shown that the segment *GO* is equal to the segment *BS*, we also show that all the segments together of the three-sided arc figure *FGABC* and each of them separately are equal to all segments of the three-sided arc figure *ABCED*. Therefore, the three-sided arc figures *FGABC*, *ABCED* are equal. Hence, as in the previous theorem (Theorem II), the area of the shape under half of the arc of the cycloid *ABCD* is three times bigger than the area of the semicircle *CED*, and the area of the shape under the arc of the cycloid is three times bigger than the area of the circle that generates the cycloid (see Figure 7).

The result is also that cycloid arc *ABC* cuts the arc shape *FGADC* into two arc shapes with the same area. Analogically, a diagonal cut of some rectangle also results in two triangles with the same area.

The following figure is a visual presentation of Cavalieri's method of indivisibles in this theorem (see Figure 8).

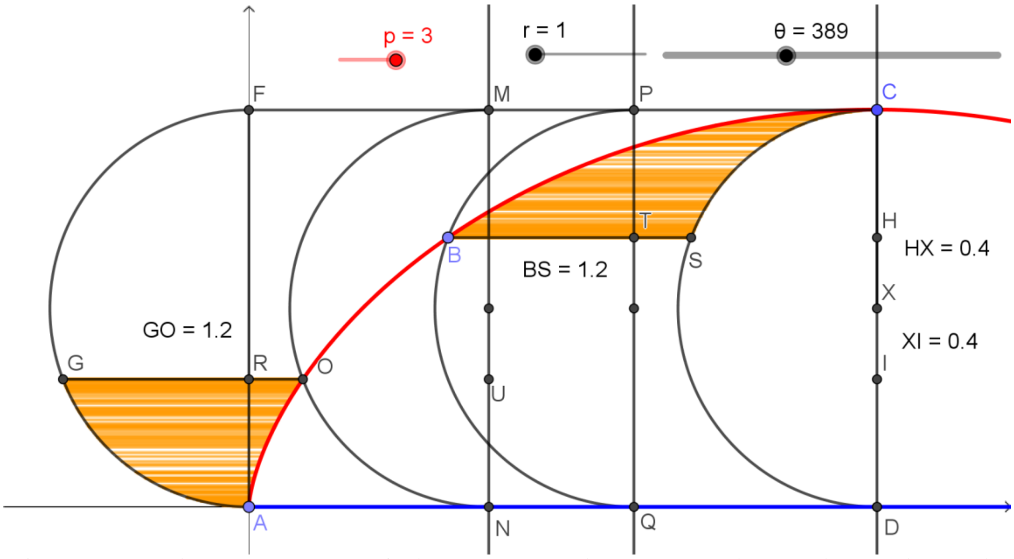


Fig. 8. Visual presentation of Theorem III (the orange planar shapes have the same area)

If we move with point *B* (see Figure 8), we obtain orange planar figures (the GeoGebra function "Trace on" used for the segments *GO*, *BS*). The segments *GO* and *BS* are always the same, and according Cavalieri's method of indivisibles, these shapes have the same area.

Remarks on the Torricelli Approach of Using Cavalieri's Method of Indivisibles

Gilles Personne de Roberval (1602–1675) also studied the cycloid and introduced the term "socia". If we have half of the arc of cycloid *AGB* (see Figure 9) in the rectangle *ADBE*, we can make a picture of this arc in the central symmetry with the centre *S*. The point *S* is a centre of the rectangle *ADBE*. We obtain an arc *AHB*. We can move with the segment *GH*, which is parallel to segment *AD*, and the points *G, H* are points of these central symmetry cycloid arcs. The centre *S* of the segment *GH* (see Figure 9) describes a part of the curve that is practically sinusoidal. Points *I*, *J*, *G*, *H* are on the same line, and the lengths of the segments *IJ* and *GH* are the same.

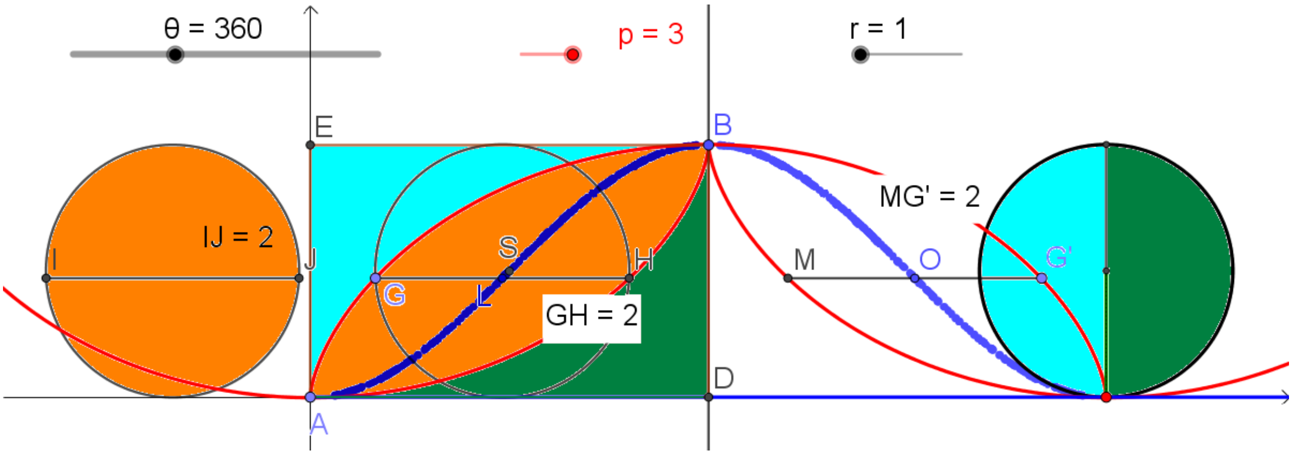


Fig. 9. Visual presentation of Roberval's "socia" (blue colour) via GeoGebra
(the plane figures with the same colour have the same area)

The area between these two cycloid curves has a "spindle" shape. It is an interesting property that points of both cycloid curves are on the same rolling circle (the orange circle in Figure 9). The segments *IJ* and *GH* are the same, and according to Cavalieri's method of indivisibles, that "spindle" shape has the same area as the rolling circle. If the area of the shape under half of the arc of a cycloid is equal to one and a half of the area of the rolling circle, then the area of the shape under the second (down) cycloid curve is equal to one half of the area of the rolling circle. This is visualised by GeoGebra in Figure 9.

Visualisation in Statistics Education

We also can use GeoGebra as a tool to help students appropriately visualise data in order to analyse and interpret that data because visualisation is critical for teaching and learning data. As Prodromou (2014) discusses, GeoGebra can be implemented into the curriculum and learning process of introductory statistics to engage college students (and secondary students) in cycles of statistical investigations, including (a) managing data, (b) developing students' understanding of specific statistical concepts, (c) conducting data analysis and inference and (d) exploring probability models.

GeoGebra is used in two distinct ways when teaching introductory statistics (Prodromou, 2014):

(1) Applets created with GeoGebra are implemented into teaching practices to demonstrate specific concepts.

(2) Students use GeoGebra as a software tool to perform data analysis and inference and to develop probability models.

GeoGebra applets can be used during teaching practices to visually represent particular fundamental concepts that are commonly difficult to conceptualise. Furthermore, most of the applets make it possible to interact with parameters and variables by altering sliders, using dynamic representations as tools for analysis, formulating personal models, calculating probabilities, communicating dynamic changes of data visualisations and storing and processing real data.

For example, when students begin to learn how the normal distribution approximates binomial probabilities, we use the following GeoGebra applet (see Figure 10) to visualise statistical distributions when the parameters and variables are altered using sliders.

More specifically, this applet allows students to manipulate *n*, which indicates the random sample of a number of people who participated in a research study, and *p*, which indicates the probability of an event occurring.

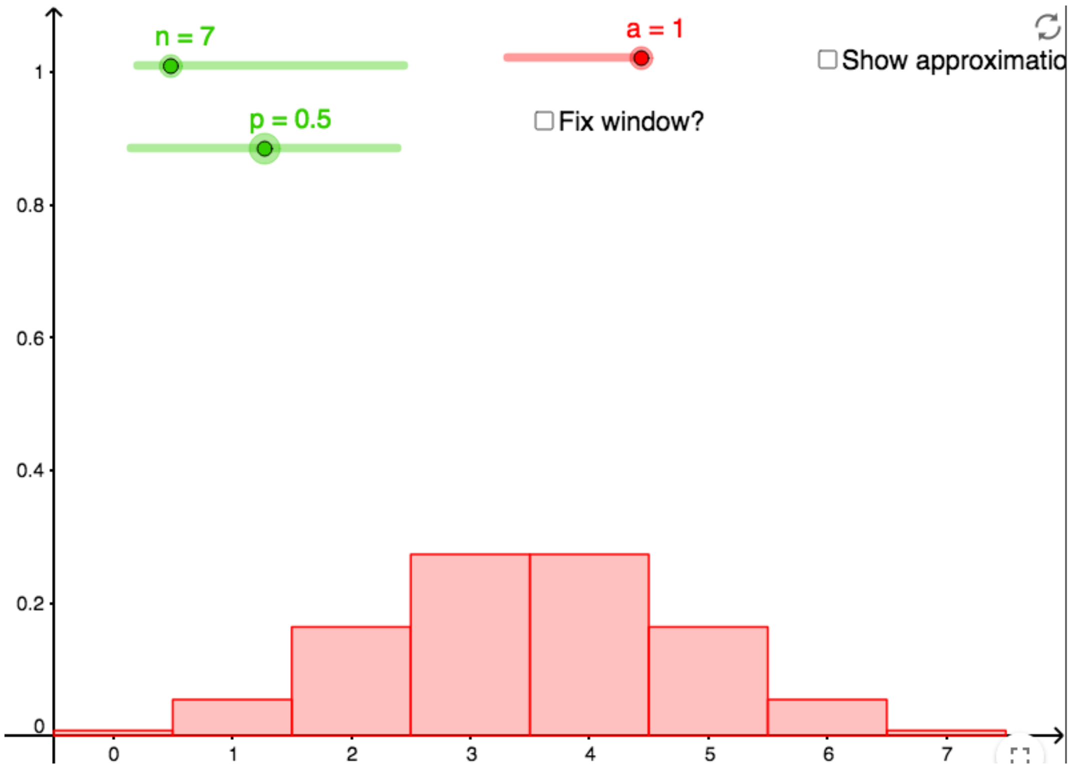


Fig. 10. Applet of binomial approximation

In particular, the mathematics shown by the applet in Figure 10 are as follows:

The central limit theorem is the tool that enables us to use the normal distribution to approximate binomial probabilities:

- let X_i = 1, if a person agrees that a particular event is occurring with probability p,
- let X_i = 0, if a person does not agree that a particular event is occurring with probability 1-p.

Let X_i is a Bernoulli random variable with mean

$$\mu = E(X) = (0)(1-p) + (1)(p) = p$$

and variance

$$\sigma^2 = Var(X) = E[(X-p)]^2 = (0-p)^2(1-p) + (1-p)^2(p) = p(1-p).$$

We conducted a research study with a random sample on n people, and let
$Y = X_1 + X_2 + \ldots + X_n$.
Y is a binomial (n, p) random variable, y = 0, 1, 2, …, n, with mean

$$\mu = np$$

and variance

$$\sigma^2 = np(1-p)$$

In a teaching context, a teacher using GeoGebra might ask students to play with the green sliders first and explain what they noticed. After doing so, students may articulate that when n decreases, the number of columns decreases as well and that each column becomes wider (see Figure 11). Moreover, when n increases, the number of columns increases, and the columns move to the right (see Figure 12).

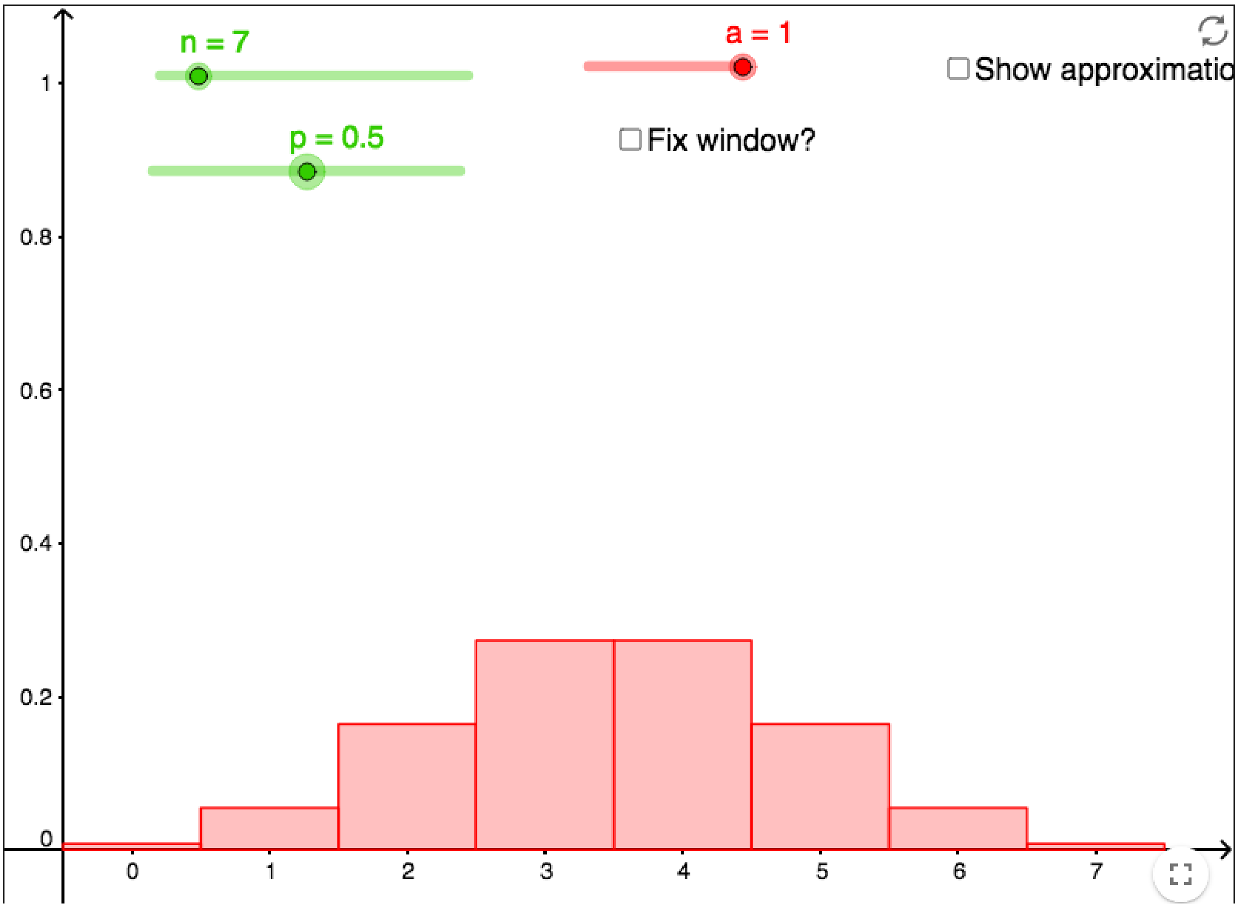


Fig. 11. When n decreases, the number of columns decreases, and the width of each column increases

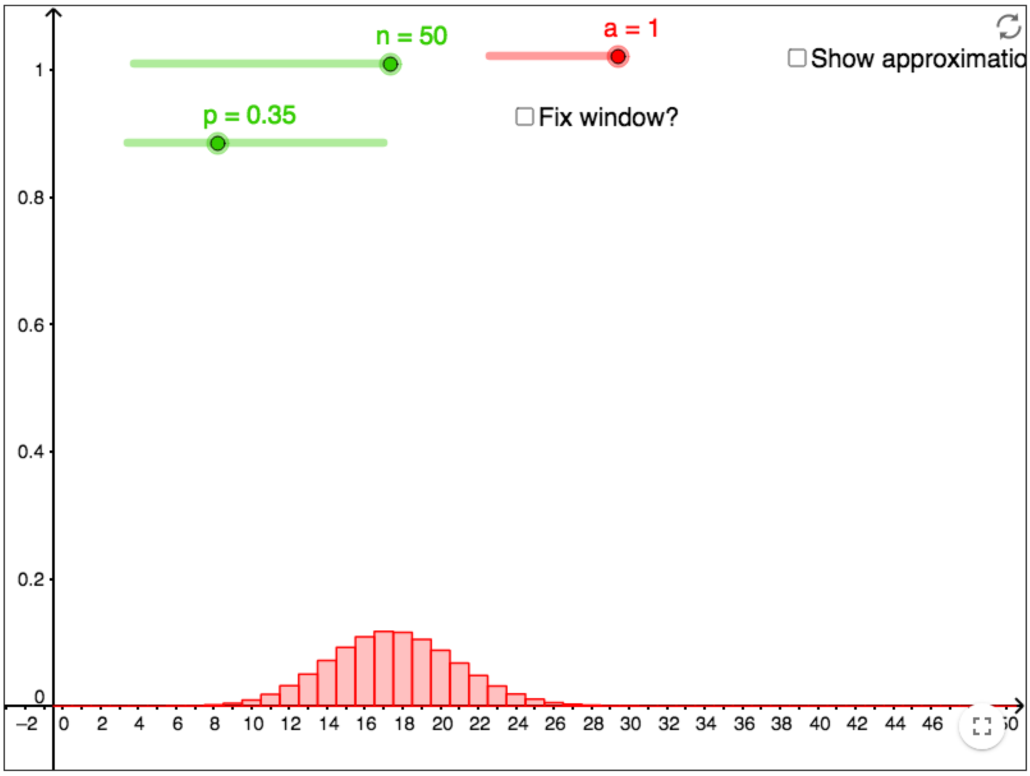


Fig. 12. When *n* increases, the number of columns increases

Students also may notice that when *p* decreases, the distribution of data moves to the left in the visualisation (see Figure 13) and that when *p* increases, the distribution of data moves to the right (see Figure 14).

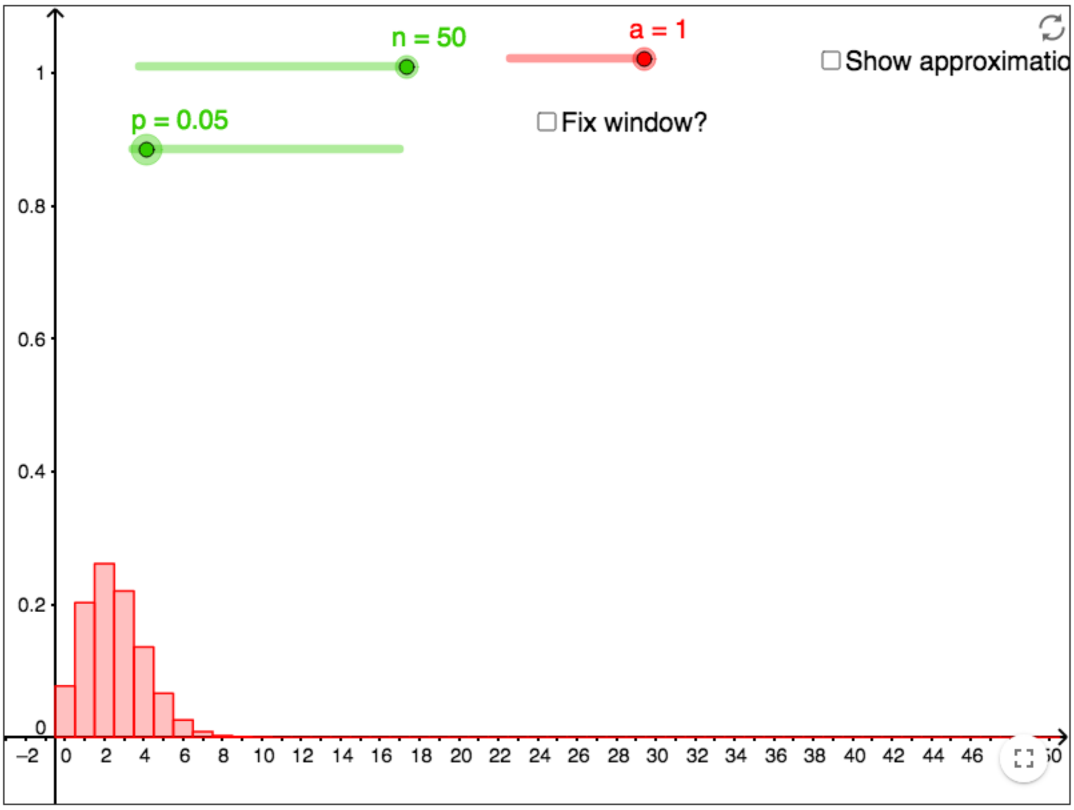


Fig. 13. When *p* decreases, the distribution of data moves to the left

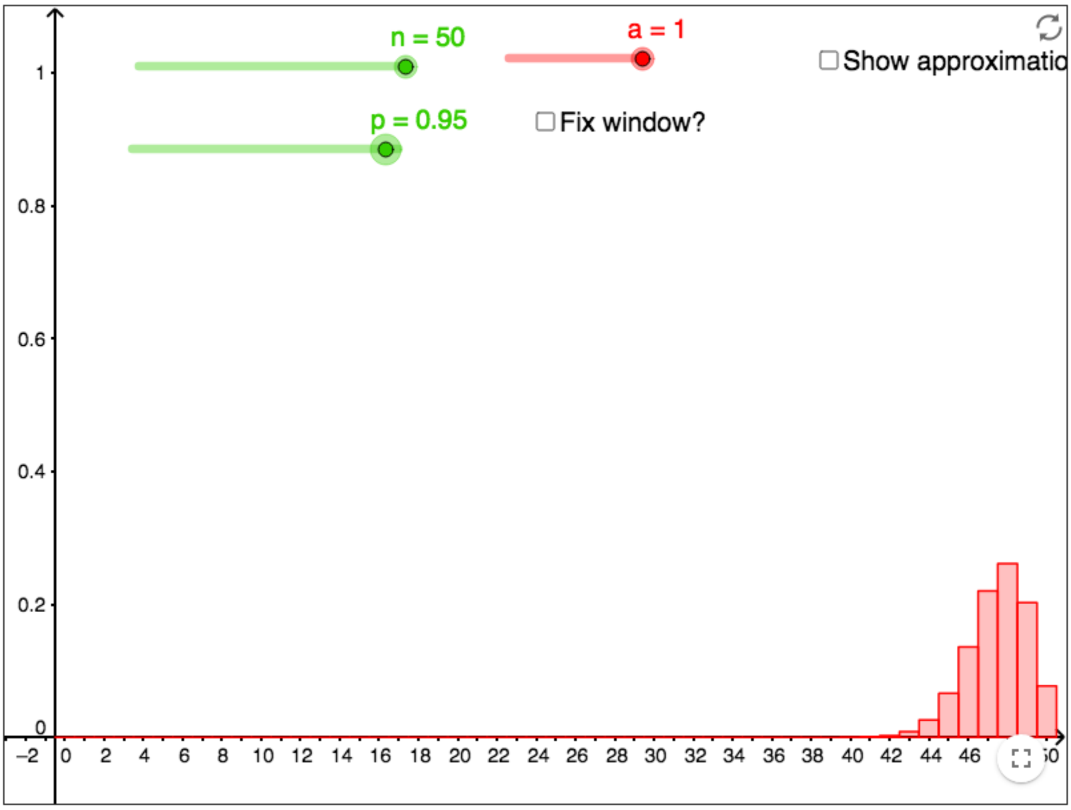


Fig. 14. When *p* increases, the distribution of data moves to the right

A teacher might ask students to assume that n = 10 and p = ½ (so that Y is binomial (10, ½) in order to calculate the probability that five people approve of a particular event occurring.

Students can adjust the sliders of the applet so that n would indicate 10 and p would indicate 0.5 The applet provides a visualisation of the probability that five people approve of a particular event occurring (Figure 15) – when x is equal to 5, the other coordinate on the continuous distribution is equal to 0.2460, representing a probability of 24.6 %.

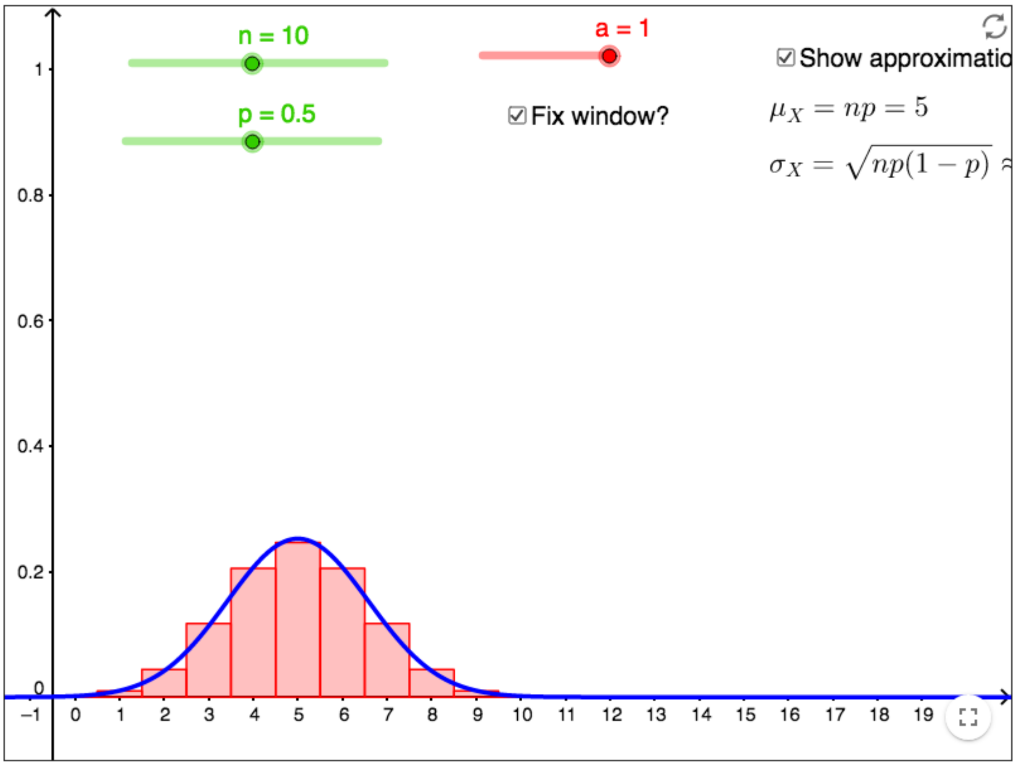


Fig. 15. When $n =$ 10 and p = ½

In particular, when we look at the graph of the binomial distribution with the vertical column corresponding to $Y = 5$, we make an adjustment that is called a “continuity correction” by using the continuous distribution (i.e. the normal distribution[*]) to approximate the discrete distribution. Specifically, the column that includes $Y = 5$ also includes any Y greater than 4.5 and less than 5.5, as follows:

$$
\begin{aligned}
P(Y = 5) &= P(4.5 < Y < 5.5) = P(4.5 < Z < 5.5)\\
&= P\left(\frac{4.5-5.5}{\sqrt{2.5}} < Z < \frac{5.5-5}{\sqrt{2.5}}\right) =\\
&= P(-0.32 < Z < 0.32) =\\
&= P(Z < 0.32) - P(Z < -0.32) =\\
&= P(Z < 0.32) - P(Z > 0.32) =\\
&= 0.6255 - 0.3745 = 0.251
\end{aligned}
$$

The visualisation of the probability that five people approve of a particular event occurring can be also determined by calculating the exact probability using the binomial table with n = 10 and p = ½. Doing so, we get

$$(Y{=}5){=}(Y{\leq}5){-}P(Y{\leq}4){=}0.6230{-}0.3770{=}0.2460.$$

Hence, there is a 24.6 % chance that five randomly selected people approve of a particular event occurring.

Visualising the above example makes it accessible to younger students, helping them understand, interpret and use the data to calculate probabilities. Moreover, the use of applets caters to the needs of diverse learners and could help younger students construct the meaning of the co-ordination of the two epistemological perspectives on distribution (Prodromou, 2012a;

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[*] Y is defined as a sum of independent random variables. When n *is large,* the Central Limit Theorem can be used to calculate probabilities for Y. Specifically, the Central Limit Theorem establishes that when independent random variables are added, their sum tends towards a normal distribution although the original variables themselves are not normally distributed: $Z{=}\frac{Y-np}{\sqrt{np(1-p)}}{\rightarrow}N(0{,}1)$.

Prodromou, Pratt, 2006) while connecting concepts of experimental probability and theoretical probability (Prodromou, 2012b).

3. Conclusion

Visualisation has many applications in the educational process, and this article presents practical examples from historical and modern mathematical contexts. Torricelli's approach to the area under a cycloid arc with software GeoGebra brings possibility to present mathematics concepts from historical materials developed by mathematicians in the past for future mathematics teachers (see Zahorec et al., 2018).

According to the theory developed by David Tall (see Tall, 2006 and Tall, Mejia-Ramos, 2009), two kinds of students exist in the classroom: one group with fast, gestalt thinking (i.e. thinking with figural characters, students see an object as a whole) and a second group that uses "step-by-step," successive thinking. Presentation of the area under the cycloid arc by Torricelli and visualisation through software makes it possible to present this topic in an appropriate way for both groups of students and to allow collaboration between them (see also the examples in Bayerl, Žilková, 2016).

Torricelli's approach has educational application in that it promotes an understanding of the area of shapes which are bordered not by a line segment but by the arc of a curve (see Moru, 2007).

Torricelli's original text uses abbreviated language, and it is difficult to translate and make a close paraphrase of some of the original text.

Many students have problems understanding, for example, the "ε-δ technique" in a purely formal way. Such students may benefit from an approach like the geometrical "ε-δ technique" presented by Archimedes and Torricelli (see Lemma II).

According to Prodromou and Lavicza (2015), GeoGebra allows for the presentation of many mathematical concepts in instrumental, relational and formal modes, with the support of visualisation and simulation.

Archimedes' approach to the area under the arc of parabolas was not only an inspiration for Torricelli but also for Slovak-Australian mathematician Igor Kluvanek, who developed his own integration theory on the exhaustion method from Eudoxos (see Nillsen, 2011).

The examples provided in this article show that the possibilities of using visualisations to display selected mathematics concepts are extensive and that such visualisations can motivate teachers to embrace the necessary technology and improve the experience of mathematics and statistics, both for themselves and their students.

The importance of technology like GeoGebra, which enables students to build their own representations and explore different aspects of those representations, must be emphasised.

Pratt, Davies, Connor (2011) discuss some general impediments to the use of technology for teaching statistics:

1. teachers not prioritising technological tools,
2. the curriculum not supporting the use of technology,
3. assessment not encouraging the use of technological tools,
4. teachers' unwillingness to attend professional development programs or up-skill on the latest technology developments, and
5. the use of technology reinforcing other skills (e.g. computation) rather than the development of concepts.

Digital technology is being introduced into many school curricula, and "visualisation has blossomed into a multidisciplinary research area, and a wide range of visualisation tools have been developed at an accelerated pace" (Prodromou, Dunne, 2017a: 1). In such an environment, it is hoped that the barriers noted by Pratt, Davies and Connor (2011) can be overcome.

In particular, research on data visualisation and statistical literacy (Prodromou, Dunne, 2017b) has discussed the role of visualisation and the need for teachers "to marshal many facets of visualisation, from elicitation of pattern to salient pictorial representation of a particular specified context" (Prodromou, Dunne, 2017b: 3). They found that visualisation assists with the basic production of contextual meaning and interpretation compared to other familiar cognitive strategies, including the following: describing and comparing observed conditions or states in a context; describing and assessing relationships amongst categories; counts and measures (often with time factors ignored); describing and comparing current changes or processes in a context (over a period, sometimes with equal inter-observation intervals); and describing and assessing associations amongst changes in observed variables (over some implicit or specified time intervals).

Prodromou and Dunne suggested (see (Prodromou and Dunne, 2017a) that fluency with visualisation is central to statistics. We would expect the same to be true in mathematics, but unfortunately, no research about the process of understanding through visualisations of mathematical concepts has been done.

This paper's demonstration of the role of GeoGebra in presenting Torricelli's proofs suggests ways in which current technologies and visualisation can be integrated into learning. Future research should experiment with GeoGebra visualisations as an aid to teaching integration (i.e. calculating the area under a curve).

7. Acknowledgements

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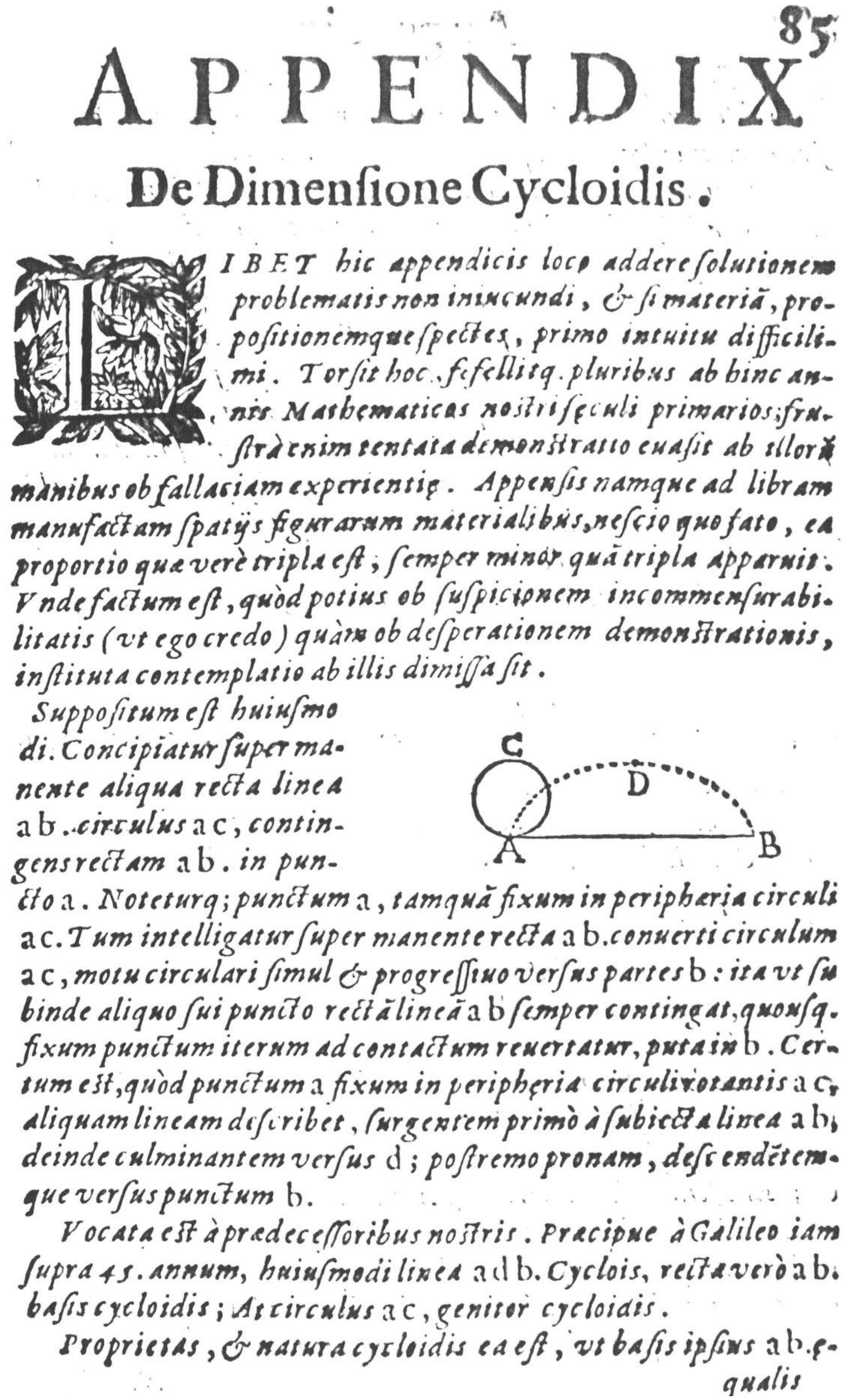
Appendix 1. Original Latin text of Torricelli's Appendix



85

APPENDIX

De Dimenſione Cycloidis.

LIBET hic appendicis loco addere ſolutionem problematis non iniucundi, & ſi materiã, propoſitionemque ſpectes, primo intuitu difficilimi. Torſit hoc, fefellitq. pluribus ab hinc annis Mathematicos noſtri ſeculi primarios; fruſtrà enim tentata demonſtratio euaſit ab illorũ manibus ob fallaciam experientiæ. Appenſis namque ad libram manufactam ſpatijs figurarum materialibus, neſcio quo fato, ea proportio quæ verè tripla eſt, ſemper minor quã tripla apparuit. Vnde factum eſt, quòd potius ob ſuſpicionem incommenſurabilitatis (vt ego credo) quàm ob deſperationem demonſtrationis, inſtituta contemplatio ab illis dimiſſa ſit.

Suppoſitum eſt huiuſmodi. Concipiatur ſuper manente aliqua recta linea ab. circulus ac, contingens rectam ab. in puncto a. Noteturq; punctum a, tamquã fixum in peripheria circuli ac. Tum intelligatur ſuper manente recta ab. conuerti circulum ac, motu circulari ſimul & progreſſiuo verſus partes b: ita vt ſubinde aliquo ſui puncto rectã lineã ab ſemper contingat, quouſq. fixum punctum iterum ad contactum reuertatur, puta in b. Certum eſt, quòd punctum a fixum in peripheria circuli rotantis ac, aliquam lineam deſcribet, ſurgentem primò à ſubiecta linea ab, deinde culminantem verſus d; poſtremo pronam, deſcendẽtemque verſus punctum b.

Vocata eſt à prædeceſſoribus noſtris. Præcipue à Galileo iam ſupra 45. annum, huiuſmodi linea adb. Cyclois, recta verò ab. baſis cycloidis; At circulus ac, genitor cycloidis.

Proprietas, & natura cycloidis ea eſt, vt baſis ipſius ab. æ-

qualis

Fig. 16. Page 85 of Torricelli's manuscript

86 Appendix

quales ſit peripheria circuli genitoris ac. Quod quidem non adeo obſcurum eſt. Nam tota peripheria ac ſe ipſam in conuerſione commenſurauit ſuper manente recta ab.

Quæritur nunc quam proportionem habeat ſpatium cycloidale adb ad circulum ſuum genitorem ac? Oſtendemuſque, Deo dante, triplum eſſe. Demonſtrationes tres erunt, inter ſe penitus diuerſae. Prima, & tertia per nouam Indiuiſibilium Geometriam nobis amiciſſimam procedent: ſecunda verò per duplicem poſitionem, more veterum recepto; vt vtriſque fautoribus ſatiſfiat. Cęterum, hoc moneo; principia ferè omnia, quibus aliquid per Indiuiſibilium Geometriam demonſtratur, ad ſolitam antiquorum demonſtrationem indirectam reduci poſſe: quod à nobis factum eſt, vt in multis alijs, ita etiam in primo, & tertio ſequentium Theorematum; ſed ne lectoris patientia nimium adhuc abuteremur plura omittenda cenſuimus, treſq; tantum demonſtrationes exibemus

THEOREMA I.

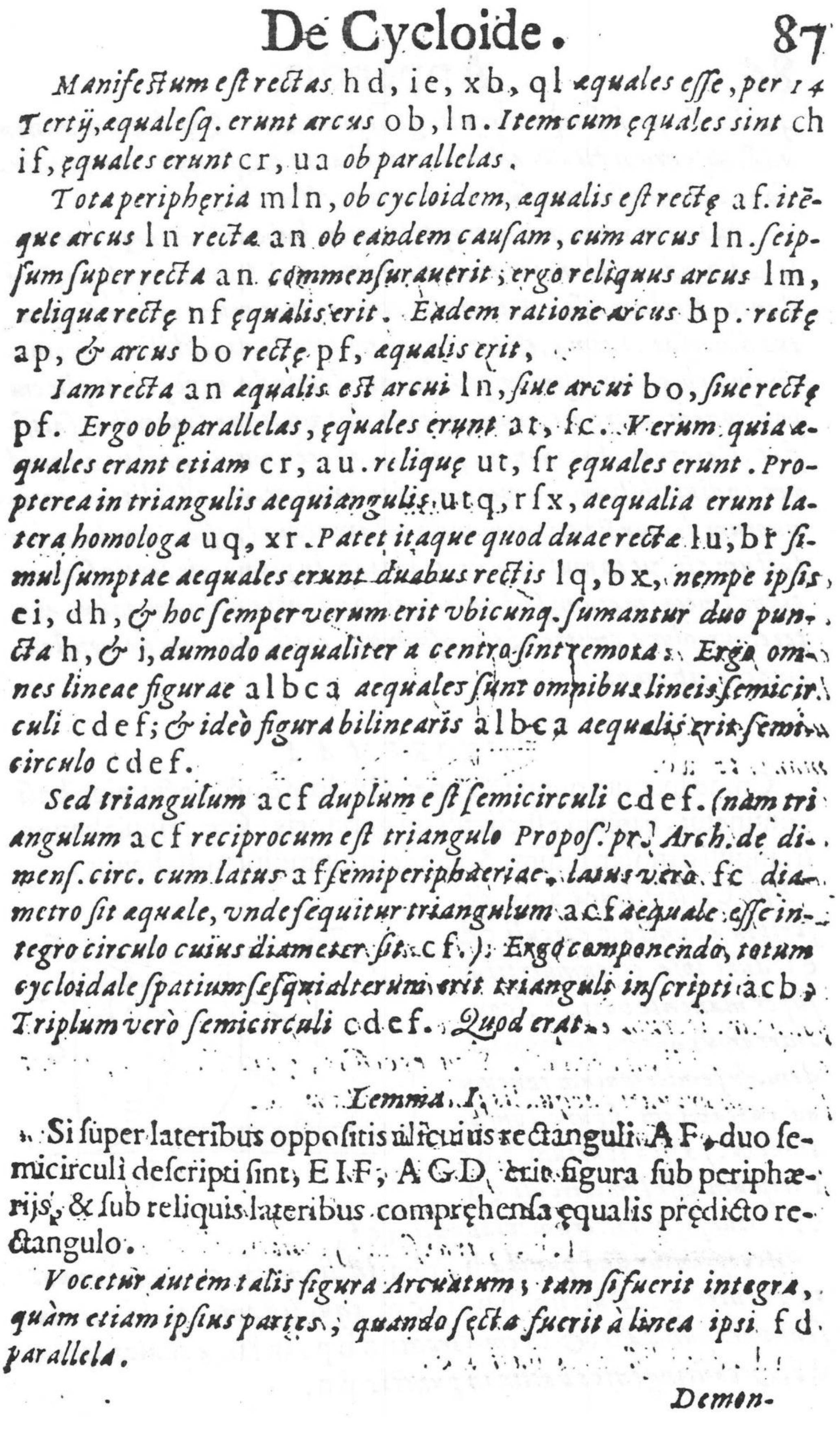
Omne ſpatium quod ſub linea Cycloide, & recta eius baſi continetur, triplum eſt circuli ſui genitoris; ſiue ſeſquialterum trianguli eandem baſim, & eandem altitudinem habentis.

Eſto Cyclois linea abc deſcripta à puncto c circuli cd ef dum ipſe circumuertitur ſuper manente baſi af. (conſideramus autem ſemicycloidem, & ſemicirculum tantum ad euitandam figuræ confuſionem.) Dico ſpatium abcf triplum eſſe ſemicirculi cdef; ſiue ſeſquialterum trianguli acf,

Accipiantur duo puncta h, & i in diametro cf. æquè remota à centro g. Ductiſq. hb, il cm æquidiſtanter ipſi fa, tranſeant per puncta b, & l ſemicirculi obp, mln, æquales ipſi cdf, & contingentes baſim in punctis pn.

Naniſt-

Fig. 17. Page 86 of Torricelli’s manuscript



De Cycloide. 87

Manifeſtum eſt rectas hd, ie, xb, ql *æquales eſſe, per 14 Tertij, æqualeſq. erunt arcus* ob, ln. *Item cum ęquales ſint* ch if, *ęquales erunt* cr, ua *ob parallelas.*

Tota peripheria mln, *ob cycloidem, æqualis eſt rectę* af. *itēque arcus* ln *rectæ* an *ob eandem cauſam, cum arcus* ln *ſeipſum ſuper recta* an *commenſurauerit; ergo reliquus arcus* lm, *reliquæ rectę* nf *ęqualis erit. Eadem ratione arcus* bp *rectę* ap, *& arcus* bo *rectę* pf, *æqualis erit,*

Iam recta an *æqualis eſt arcui* ln, *ſiue arcui* bo, *ſiue rectę* pf. *Ergo ob parallelas, ęquales erunt* at, fc. *Verum quia æquales erant etiam* cr, au *reliquę* ut, ſr *ęquales erunt. Propterea in triangulis aequiangulis* utq, rſx, *aequalia erunt latera homologa* uq, xr. *Patet itaque quod duae rectæ* lu, bt *ſimul ſumptae aequales erunt duabus rectis* lq, bx, *nempe ipſis* ei, dh, *& hoc ſemper verum erit vbicunq. ſumantur duo puncta* h, *&* i, *dumodo aequaliter a centro ſint remota. Ergo omnes lineae figurae* albca *aequales ſunt omnibus lineis ſemicirculi* cdef; *& ideò figura bilinearis* albca *aequalis erit ſemicirculo* cdef.

Sed triangulum acf *duplum eſt ſemicirculi* cdef. *(nam triangulum* acf *reciprocum eſt triangulo Propoſ. pr. Arch. de dimenſ. circ. cum latus* af *ſemiperiphaeriae, latus verò* fc *diametro ſit æquale, unde ſequitur triangulum* acf *aequale eſſe integro circulo cuius diameter ſit* cf.) *Ergo componendo, totum cycloidale ſpatium ſeſquialterum erit trianguli inſcripti* acb; *Triplum verò ſemicirculi* cdef. *Quod erat.*

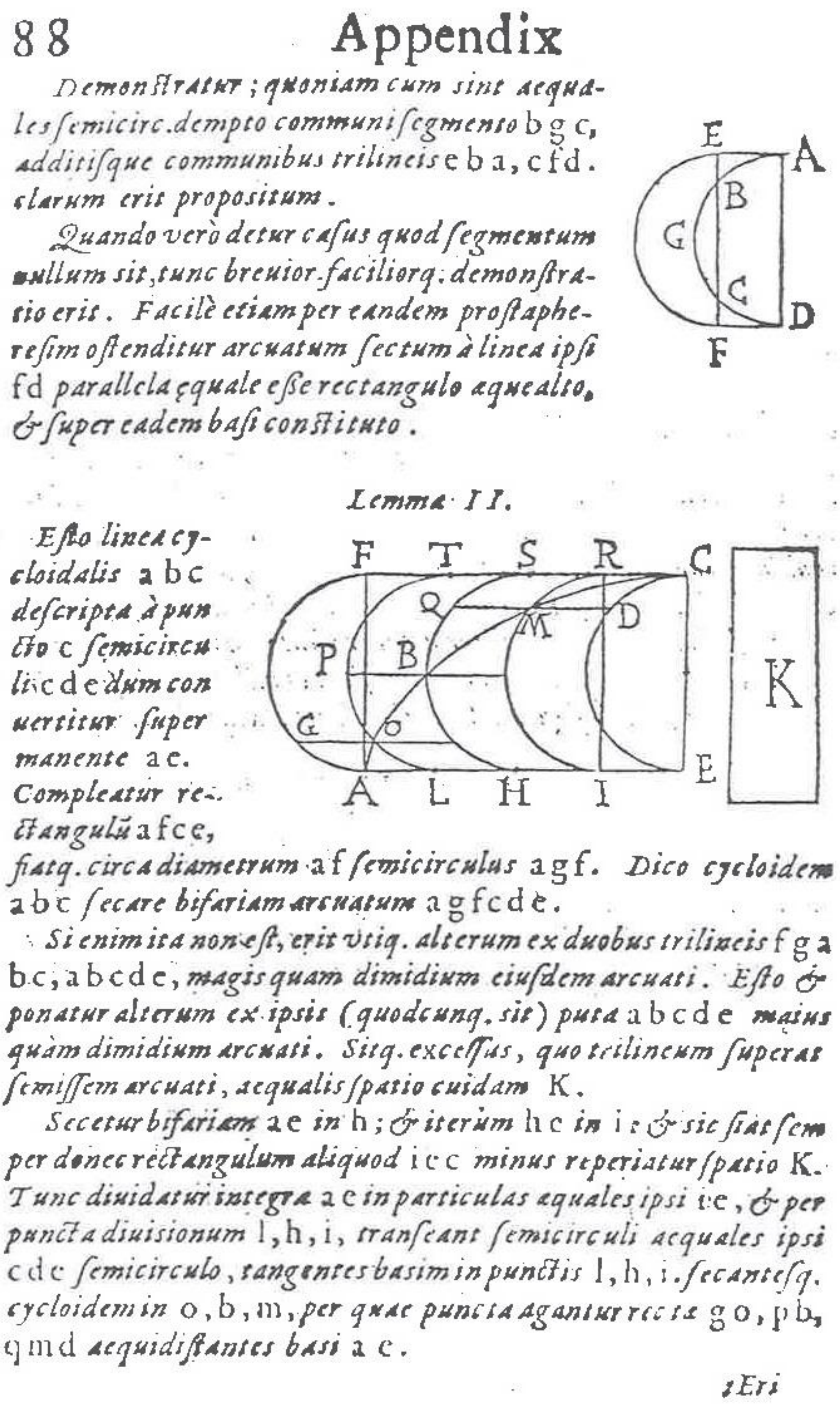
Lemma I.

Si ſuper lateribus oppoſitis alicuius rectanguli AF, duo ſemicirculi deſcripti ſint, EIF, AGD erit figura ſub periphærijs, & ſub reliquis lateribus comprehenſa ęqualis prędicto rectangulo.

Vocetur autem talis figura Arcuatum; tam ſi fuerit integra, quàm etiam ipſius partes; quando ſecta fuerit à linea ipſi fd *parallela.*

Demon-

Fig. 18. Page 87 of Torricelli's manuscript



88 Appendix

Demonſtratur; quoniam cum ſint aequales ſemicirc. dempto communi ſegmento b g c, *additiſque communibus trilineis* e b a, c f d. *clarum erit propoſitum.*

Quando verò detur caſus quod ſegmentum nullum ſit, tunc breuior faciliorq; demonſtratio erit. Facilè etiam per eandem proſtapheresim oſtenditur arcuatum ſectum à linea ipſi f d *parallela çquale eſſe rectangulo aquealto, & ſuper eadem baſi conſtituto.*

Lemma II.

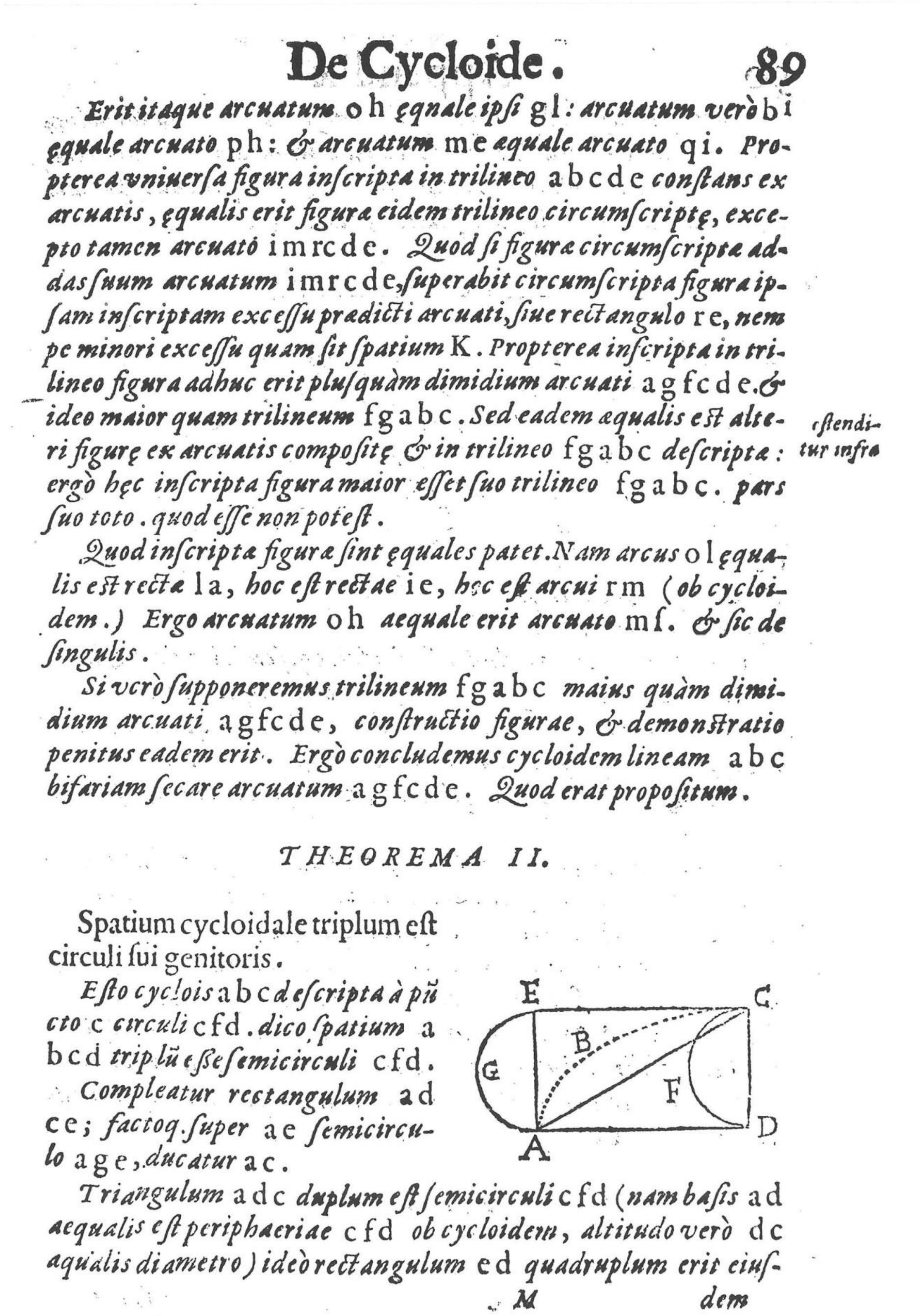
Eſto linea cycloidalis a b c *deſcripta à puncto* c *ſemicirculi* c d e *dum conuertitur ſuper manente* a e. *Compleatur rectangulũ* a f c e, *fiatq. circa diametrum* a f *ſemicirculus* a g f. *Dico cycloidem* a b c *ſecare bifariam arcuatum* a g f c d e.

Si enim ita non eſt, erit vtiq. alterum ex duobus trilineis f g a b c, a b c d e, *magis quam dimidium eiuſdem arcuati. Eſto & ponatur alterum ex ipsis (quodcunq. ſit) puta* a b c d e *maius quàm dimidium arcuati. Sitq. exceſſus, quo trilineum ſuperat ſemiſſem arcuati, aequalis ſpatio cuidam* K.

Seccetur bifariam a e *in* h; *& iterùm* h e *in* i; *& ſic fiat ſemper donec rectangulum aliquod* i e c *minus reperiatur ſpatio* K. *Tunc diuidatur integra* a e *in particulas aquales ipſi* i e, *& per puncta diuiſionum* l, h, i, *tranſeant ſemicirculi aequales ipſi* c d e *ſemicirculo, tangentes baſim in punctis* l, h, i. *ſecanteſq. cycloidem in* o, b, m, *per quae puncta agantur recta* g o, p b, q m d *aequidiſtantes baſi* a e.

Eri

Fig. 19. Page 88 of Torricelli’s manuscript



De Cycloide. 89

Erit itaque arcuatum oh ęquale ipſi gl: arcuatum verò bi ęquale arcuato ph: & arcuatum me aquale arcuato qi. Propterea vniuerſa figura inſcripta in trilineo abcde conſtans ex arcuatis, ęqualis erit figura eidem trilineo circumſcriptę, excepto tamen arcuatò imrcde. Quod ſi figurę circumſcriptę addas ſuum arcuatum imrcde, ſuperabit circumſcripta figura ipſam inſcriptam exceſſu prędicti arcuati, ſiue rectangulo re, nempe minori exceſſu quam ſit ſpatium K. Propterea inſcripta in trilineo figura adhuc erit pluſquàm dimidium arcuati agfcde. & ideo maior quam trilineum fgabc. Sed eadem ęqualis eſt alteri figurę ex arcuatis compoſitę & in trilineo fgabc deſcriptę: ergò hęc inſcripta figura maior eſſet ſuo trilineo fgabc. pars ſuo toto. quod eſſe non poteſt.

ſtenditur infra

Quod inſcriptę figurę ſint ęquales patet. Nam arcus ol ęqualis eſt rectę la, hoc eſt rectae ie, hęc eſt arcui rm (ob cycloidem.) Ergo arcuatum oh aequale erit arcuato mſ. & ſic de ſingulis.

Si verò ſupponeremus trilineum fgabc maius quàm dimidium arcuati agfcde, conſtructio figurae, & demonſtratio penitus eadem erit. Ergò concludemus cycloidem lineam abc bifariam ſecare arcuatum agfcde. Quod erat propoſitum.

THEOREMA II.

Spatium cycloidale triplum eſt circuli ſui genitoris.

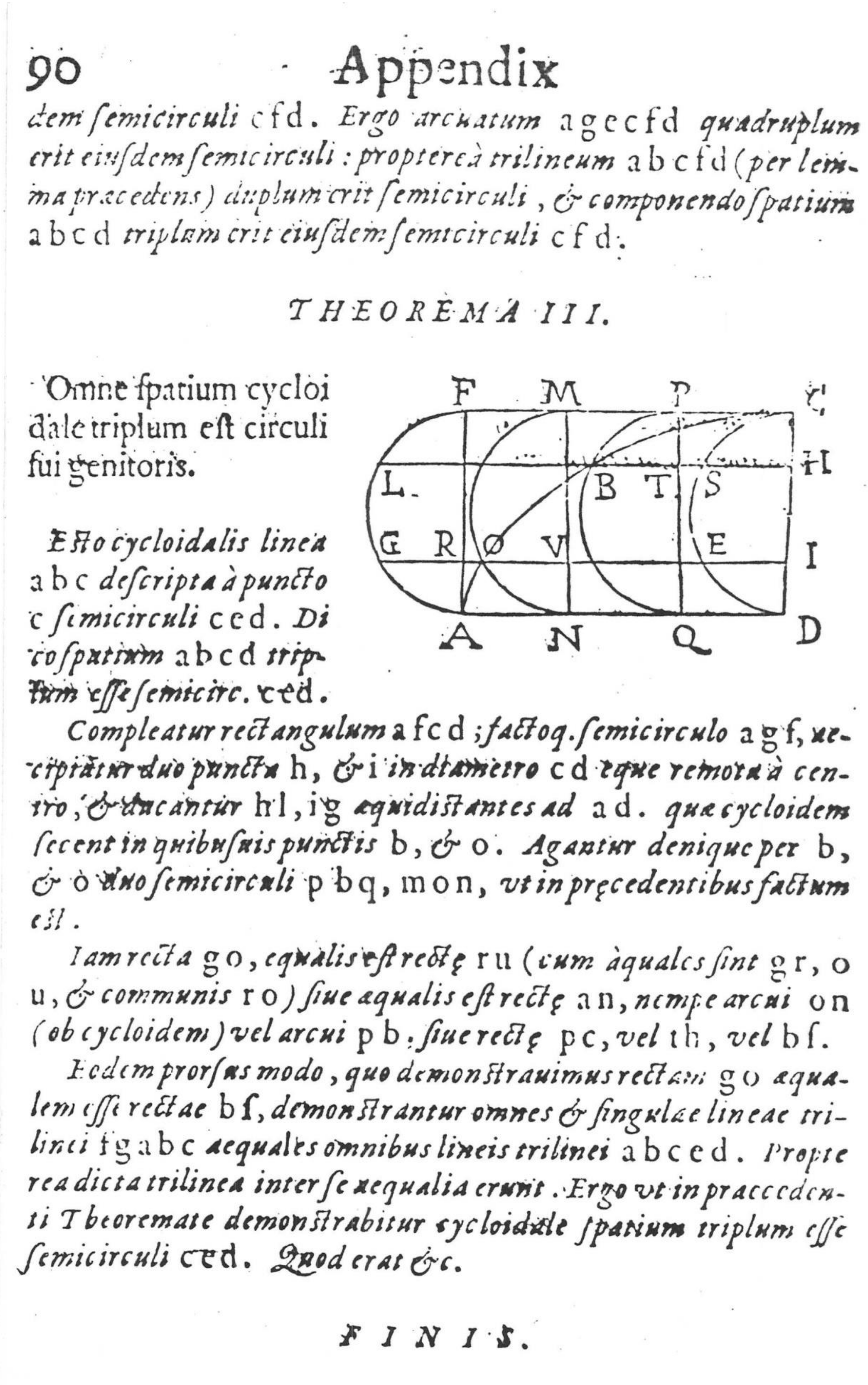
Eſto cyclois abcd deſcripta à pũcto c circuli cfd. dico ſpatium abcd triplũ eſſe ſemicirculi cfd.

Compleatur rectangulum adce; factoq. ſuper ae ſemicirculo age, ducatur ac.

Triangulum adc duplum eſt ſemicirculi cfd (nam baſis ad aequalis eſt periphaeriae cfd ob cycloidem, altitudo verò dc aqualis diametro) ideò rectangulum ed quadruplum erit eiuſ-

M dem

Fig. 20. Page 89 of Torricelli's manuscript



90 Appendix

dem ſemicirculi cfd. *Ergo arcuatum* agecfd *quadruplum erit eiuſdem ſemicirculi : propterea trilineum* abcfd (*per lemma præcedens*) *duplum erit ſemicirculi , & componendo ſpatium* abcd *triplum erit eiuſdem ſemicirculi* cfd.

THEOREMA III.

Omne ſpatium cycloidale triplum eſt circuli ſui genitoris.

Esto cycloidalis linea abc *deſcripta à puncto* c *ſemicirculi* ced. *Dico ſpatium* abcd *triplum eſſe ſemicirc.* ced.

Compleatur rectangulum afcd ; *factoq. ſemicirculo* agf, *accipiantur duo puncta* h, *&* i *in diametro* cd *æque remota à centro, & ducantur* hl, ig *æquidiſtantes ad* ad. *quæ cycloidem ſecent in quibuſuis punctis* b, *&* o. *Agantur deniquc per* b, *&* o *duo ſemicirculi* pbq, mon, *vt in præcedentibus factum eſt*.

Iam recta go, *equalis eſt rectæ* ru (*cum æquales ſint* gr, ou, *& communis* ro) *ſiue æqualis eſt rectæ* an, *nempe arcui* on (*ob cycloidem*) *vel arcui* pb. *ſiue rectæ* pc, *vel* th, *vel* bf.

Eodem prorſus modo , quo demonſtrauimus rectam go *æqualem eſſe rectae* bf, *demonſtrantur omnes & ſingulae lineae trilinei* fgabc *aequales omnibus lineis trilinei* abced. *Propterea dicta trilinea inter ſe aequalia erunt . Ergo vt in præcedenti Theoremate demonſtrabitur cycloidale ſpatium triplum eſſe ſemicirculi* ced. *Quod erat &c.*

FINIS.

Fig. 21. Page 90 of Torricelli's manuscript