AN INVESTIGATION INTO STUDENTS' UNDERSTANDINGS OF LINEAR RELATIONSHIPS WHEN USING DYNAMIC MATHEMATICAL SOFTWARE AS AN EXPLORATION TOOL

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Abstract

Numerous studies have provided substantial evidence of the benefits of technology when learning mathematical concepts. Despite this evidence, various factors inhibit successful implementation and navigation of technology. As a result, when students require technology to further develop their understanding of concepts, they may fall short because they lack the consolidated skills enabling them to competently advance. This study investigated students' understandings of Linear Relationships concepts using the dynamic software GeoGebra. The purpose was to identify developmental hurdles and issues that could be addressed by teachers in their teaching sequence. The design involved a tightly focused investigation of the understanding of Linear Relationships concepts through analysing the students' responses during the unit. Empirical evidence is provided to explain the difficulties faced by students when using technology to support their understanding of Linear Relationships. This evidence has theoretical and practical implications for the instruction of the Linear Relationships unit and the incorporation of GeoGebra as a pedagogical tool.

The theoretical base for this study was the van Hiele Teaching Phases and the SOLO model. The five Teaching Phases that form part of the van Hiele Theory presented a framework for teaching and learning in which to sequence Linear Relationships activities that facilitate students' cognitive development for the transition between van Hiele levels. While the Teaching Phases have been widely studied, particularly with respect to Geometry, this study extends research by utilising activities for the teaching sequence aligned with the Teaching Phases, and based on the Linear Relationships unit for the Australian Curriculum Stage 5.3 (approximately 14–16 years old) incorporating GeoGebra as a technological tool to support understanding. To provide a deeper insight with which to view students' understandings of Linear Relationships concepts, the SOLO model was used as an analytical tool.

Qualitative data was collected during the investigation. This comprised three Google Form Tests: a Pre-test, an End of Topic test, and a Delayed Post-test, to provide the main set of data, along with video and audio footage, photographs and workbook samples. The tests were completed by 26 students from a Year 9 Stage 5.3 class (approximately 14–16 years old), all from one secondary school.

A central finding of this study is the identification of the developmental pathway for Linear Relationships concepts. This pathway characterises student growth in understanding and recognised two cycles of responses within the concrete symbolic SOLO mode. Whilst first cycle responses indicated possible levels of support from the ikonic mode, students were mainly operating within the concrete symbolic mode. This study also identified student difficulties associated with dynamic technology, such as GeoGebra, when used to support the understanding of Linear Relationships concepts.

The research highlights the difficulties students encounter when attempting to utilise technology to develop conceptual understanding with mathematical topics such as Linear Relationships. Hence, the characterisation of a developmental pathway for conceptual understanding of Linear Relationships provides a valuable tool for teaching. In addition, this study highlights the use of the SOLO model as an interpretive tool for research in Mathematics education and the van Hiele Teaching Phases as a pedagogical tool for developing comprehensive learning experiences.

Keywords

GeoGebra, Linear Relationships, SOLO model, van Hiele Teaching Phases, technology.

Certificate

I certify that the substance of this thesis has not already been submitted for any degree and is not currently being submitted for any other degree or qualification.

I certify that any help received in preparing this thesis and all sources used have been acknowledged in this thesis.

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Signature

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Nearing completion, this thesis became more than a research project, it became a personal challenge that extended me beyond my comfort zone, and became an accomplishment that I am extremely proud of.

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Abbreviations

ACARA	Australian Curriculum, Assessment Reporting Authority
ACT	Australian Capital Territory
AFL	Assessment *for* Learning
ATAR	Australian Tertiary Admissions Rank
BDC	Board Development Courses
BEC	Board Endorsed Courses
BYOD	Bring Your Own Device
CAS	Computer Algebra System
CEC	Content Endorsed Courses
CK	Content Knowledge
CSO	Catholic Schools Office
DER	Digital Education Revolution program
DGS	Dynamic Geometry Software
GC	Graphics Calculators
ICT	Information and Communications Technology
Idk	I don't know
MSP	Mathematics and Science Partnership
MSPE	Master, Servant, Partner and Extension of Self
M-TPACK	Mathematics–Technological Pedagogical And Content Knowledge
NCTM	National Council of Teachers of Mathematics
NSW	New South Wales
NSF	National Science Foundation
NT	Northern Territory
OECD	Organisation for Economic Co-operation and Development
PCK	Pedagogical Content Knowledge
PK	Pedagogical Knowledge
QLD	Queensland
SA	South Australia
SAMR	Substitution, Augmentation, Modification and Redefinition
SCS	Screen Capture Software
SOLO	Structure of the Observed Learning Outcome
TCA	Thematic Content Analysis
TPK	Technological Pedagogical Knowledge
TK	Technological Knowledge
TPACK	Technological Pedagogical And Content Knowledge
VIC	Victoria
WA	Western Australia

CHAPTER 1: INTRODUCTION

> Mathematical ideas have evolved across all cultures over thousands of years and are constantly developing. Digital technologies facilitate this expansion of ideas, providing access to new tools for continuing mathematical exploration and invention.
>
> (NSW Board of Studies, 2012d)

The origin of this study lies in the author's desire to explore how technology could be better utilised within mathematics classrooms to assist students' understanding of mathematical concepts. As highlighted in the above the quote, from the rationale of the Mathematics K-10 NSW Syllabus for the Australian Curriculum (2012d), use of digital technology when engaging with mathematics is an important and relevant issue for teachers of today.

Unfortunately, use of technology within mathematics classrooms across Australia is inconsistent. Research has shown that despite increasing technological developments demonstrating technological tools and learning environments provide alternative techniques and strategies to increase mathematical understanding, secondary mathematics teachers have been relatively slow to incorporate technology into their classrooms (Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010; Norton, McRobbie, & Cooper, 2000). There remains a need to explore frameworks that assist mathematics teachers' implementation of technology, so that both students and teachers can appreciate that technology should not simply be used because it is available but because it is mathematically relevant (Garofalo, Drier, Harper, Timmerman, & Shockey, 2000). The van Hiele Theory (van Hiele, 1986) and SOLO model (Biggs & Collis, 1982) are two frameworks that can assist with this need.

The van Hiele Theory (van Hiele, 1986) is the theoretical framework that provided a base for this study. It hypothesises five hierarchical levels of thinking, along with five Teaching Phases that describe progression through the levels. Of particular interest to this study are the van Hiele Teaching Phases that assist with sequencing of activities and lessons for the study. Despite numerous studies exploring the van Hiele Theory with respect to Geometry, there are very few that have focused specifically on Linear Relationships. Understanding Linear Relationships has been identified as a problematic topic since it requires students to extend their algebraic thinking to consider contextual meaning (Bardini & Stacey, 2006; Brown, 2007).

The SOLO model (Biggs & Collis, 1982) is the pedagogical framework that provides an analytical tool to qualify student's responses. It offers a categorisation structure of five modes with five levels within each mode. Of interest to this study are the five levels that can be used to describe the sophistication of responses within a mode.

Three research themes were employed to guide the present study investigating students' understandings of Linear Relationships incorporating dynamic mathematical software. Each of the themes investigates Linear Relationships with technology: the first within the context of the frameworks integral to this study; the second synthesises the findings; and the third theme provides a perspective in the form of a case study.

The following discussion provides an outline of this thesis, which comprises eight chapters. This chapter, Chapter 1, orientates the reader to the purpose of this study and the background of the research.

In Chapter 2, current literature relating to technology within mathematics classrooms is reviewed. The literature is presented under three main themes. First, a broad overview of the issues currently facing mathematics teachers when incorporating technology is presented. The next theme considers frameworks that assist with incorporating technology into pedagogy. The third theme examines research relating to Linear Relationships and the dynamic software used for the study, namely, GeoGebra. The chapter concludes by linking the three themes together in an overview.

Chapter 3 provides an in-depth discussion of the two main frameworks underpinning this study, namely, the van Hiele Theory and the SOLO model. Within the van Hiele Theory, particular focus is on the van Hiele Teaching Phases as a pedagogical framework. The SOLO model is explained as a framework that can be used as an analytical tool to qualify students' responses. Chapter 3 concludes by linking together the frameworks, stating the research themes relevant to the study and the subsequent questions that address the themes.

The design and methodological approach used for conducting the study and data analysis are presented in Chapter 4. It includes issues relating to the context, evaluation of the design and data analysis plan, and the ethical considerations of the study.

Chapters 5 and 6 report the results of the study. Chapter 5 presents the responses to core content questions from the Linear Relationships unit for each of the three Google form tests. Each question is detailed and responses are categorised in accordance with thematic coding and the SOLO model. Chapter 6 presents the responses to the problem-solving component of the tests. Similar to the previous results chapter, the responses are considered using thematic coding and the SOLO model. Both chapters address two of the research themes of the study.

In Chapter 7, an in-depth case study on a student pairs educational journey throughout the Linear Relationships unit is presented. Students' responses to activities, aligned with van Hiele Teaching Phases, are examined using the SOLO model as an analytical lens. The case study provides an insight into the validity of the van Hiele Teaching Phases as a framework, while highlighting the potential of dynamic geometry software to facilitate student understanding of Linear Relationships concepts.

Finally, Chapter 8 identifies the limitations of the study then summarises and synthesises the main findings of the study as they relate to the three research themes explored. In light of the findings contained in this summary, the implications to the van Hiele Teaching Phases, SOLO model and teaching are considered. Areas of potential further research resulting from this study conclude the chapter.

CHAPTER 2: REVIEW OF LITERATURE

> It is evident that information technology is here to stay and we as mathematics educators need to come to terms with its use.
>
> (Tall, 2009, p. 9)

2.1. Introduction

Advancements in technology have founded a major revolution in the teaching and learning of mathematics. Since the scientific calculator in the early 1970s, forms of technology have evolved to become significant teaching tools for mathematics for the 21st century. Advocated as an essential element for mathematics classes for more than two decades, technology has the potential to provide "ways of doing and experiencing mathematics models that we simply did not dream of thirty years ago" (Mariotti, 2002, p. 4). Teachers continue to remain a crucial component of the classroom environment because they are the necessary link that negotiates relationships between technology, pedagogy, content and students (Drijvers, 2015; NCTM, 2000). Unfortunately, in Australia, technology has yet to reach its full potential within mathematical educational domains. Obstacles continue to hinder its progress, influencing the extent to which it is implemented and incorporated within the mathematics classroom. The apprehensiveness towards using technology as a teaching tool held by many in the mathematics education field provided the initial stimulus for this study.

This chapter presents the background literature on the use of technology within the mathematical classroom setting. For clarity, the research reviewed has been organised into three different themes. The first theme discusses research concerned with the issues currently facing mathematics teachers incorporating technology. The second theme reviews research focused on teaching with technology and considers the Master, Servant, Partner and Extension of Self (MSPE) framework (Geiger, 2009), the Substitution, Augmentation, Modification and Redefinition (SAMR) model (Puentedura, 2010), and the Technological Pedagogical And Content Knowledge (TPACK) framework (Koehler & Mishra, 2009). The third theme reviews research most relevant to the essence of this study, Linear Relationships and associated technology with specific attention to GeoGebra. The last part of this chapter is the Conclusion, which links the three themes and provides an overview of major ideas guiding this study.

2.2. The Technological Issues Facing Mathematics Teachers

This section considers the present circumstances and issues surrounding the implementation of technology within mathematics classrooms. These issues are characterised by current practice, curriculum documents and what influence they hold, and teachers' roles and beliefs and how they impact the integration of technology into their teaching.

The development of student's mathematical understanding is enhanced and strengthened through the use of technology as a teaching tool (Cheung & Slavin, 2013; Hopper, 2009; Li & Ma, 2010; NCTM, 2000; Wenglinsky, 1998). From humble beginnings, such as the abacus and slide rule, to the scientific calculator, which dominated mathematical classrooms for many decades, mathematical technology has evolved rapidly in recent times. While Computer Algebra Systems (CAS), Graphics Calculators (GC) and sophisticated computer software continue to be used, it is online resources and applications that are progressively becoming more accessible. As the capabilities of mobile technology increases, many students have some form of mathematical technological aid at their fingertips, particularly as many schools adopt a Bring Your Own Device (BYOD) approach.

Tedious hand-written computations are lessened through the use of technology, which performs procedural tasks quickly and efficiently. The use of technology or any Information and Communications Technology (ICT) supports the understanding of mathematical concepts "to enrich students' learning process" (Kilicman, Hassan, & Husain, 2010, p. 613). Hence, technology provides a challenge for mathematical educators to review their pedagogy in light of opportunities afforded by a rapidly changing technological environment (Drijvers et al., 2016; Saha, Ayub, & Tarmizi, 2010), "the existence, versatility, and power of technology make it possible and necessary to re-examine what mathematics students should learn, as well as how they can best learn it" (NCTM, 2000, p. 25). Thus, teachers must consider alternative activities that promote mathematical understanding within the technological environment available to them. By reducing the energy spent on procedural computations, technology affords students the time to explore and investigate problems, enabling them to focus on "decision making, reflection, reasoning, and problem solving" (Saha, et al., 2010, p. 687) rather than merely a solution.

Currently, technology use within mathematics classrooms varies among teachers. Despite the increase in technological developments and the changing expectations of parents and society, governments and curriculum documentation, secondary mathematics teachers (teaching students approximately 12–18 years old) have been slow to incorporate technology into their classrooms (Drijvers, et al., 2010; Kissane, 2000; Norton, et al., 2000); “nowhere is the reluctance to change from paper and pencil techniques more evident than in the case of mathematics” (Crawford, 1995, p. 113). This concern has been recognised worldwide for decades, with Balacheff and Kaput (1996) echoing that “computer use remains a relatively small part of classroom practice” (p. 470). Mariotti (2002) stated that “computers’ entry into school has been slow and their integration in school practice seems to be even slower” (p. 4), and, more recently, Drijvers and Weigand (2010) conclude that “nowadays, despite the use of digital technologies in the public and business world, despite the tremendous number of research and practical classroom papers, the use of technologies in mathematics education and their impact on curricula is still limited” (p. 666).

In 2008, the Australian Labour Government federally funded the Digital Education Revolution program (DER), an intervention to boost the integration of ICT in secondary schools. For New South Wales (NSW), emphasis was placed on the roll out of hardware into schools. While the DER initiative had merit, the support necessary to implement the hardware, such as teacher professional development, was lacking (Forgasz, 2006); the mistaken assumption being that the supply of hardware and software would result in an increase in teachers’ use of digital technology, and, in turn, would inspire them to create innovative methods of teaching (Drijvers, et al., 2016; Goos & Bennison, 2004, 2008; Lynch, 2006); with governments assessing the success and implementation of technology in terms of computer-to-student ratios and neglect the complexity of technology usage (Lynch, 2006). In reality, the consequence of the influx of hardware drained teachers of precious time as they grappled with the demands of not only understanding how to use the hardware but also where and how to include it in their teaching practices.

More technology or time does not always equate to successful learning and teaching with technology, “as with any teaching tool, it can be used well or poorly” (NCTM, 2000, p. 25). The question of how the technology is being used is important. Technology used as an add-on to lessons and for facilitating drill and practice does not develop higher-order mathematical skills. The relationship between technology usage and learning for students is not a case of cause and effect because many elements influence the learning that occurs from technology usage (Lynch, 2006). A recent study by the Organisation for Economic Co-operation and Development (OECD, 2015) concluded that “despite considerable investments in computers, Internet connections and software for educational use, there is little solid evidence that greater computer use among students leads to better scores in mathematics and reading" (OECD, 2015, p. 145). It is important, therefore, that the technology is used for the development of understanding of mathematical concepts, not merely used to fulfil an outcome.

In 2014, Australia implemented a new national curriculum, which was developed by the Australian Curriculum, Assessment Reporting Authority (ACARA). Previously, each of the eight Australian states and territories: NSW, Queensland (QLD), Victoria (VIC), Australian Capital Territory (ACT), Western Australia (WA), South Australia (SA), Northern Territory (NT) and Tasmania (TAS), had individual curriculum documents with varying content and standards. The Australian National Curriculum was developed to provide a united curriculum with expectations and standards so that every student in Australia would have access to the same content and would be judged to the same consistent standards despite their location. It recognises that the ways in which young people learn is constantly changing in the 21st century, affected by numerous external factors, both locally and globally. As a result, the Australian Curriculum promotes educational goals that "support all young Australians to become successful learners, confident and creative individuals, and active and informed citizens" (Barr et al., 2008, p. 8).

The Australian Curriculum does not specify how the content is taught, thus enabling teachers and schools the flexibility to address the diverse needs of their students through creating individual programs that offer the best opportunities for their students to learn and achieve. The curriculum is presented as a continuum of learning that progresses from Foundation (approximately 5–6 years old) to Year 10 (approximately 15–16 years old), with the curriculum of senior secondary years of schooling (approximately 16–18 years old) providing students with increased opportunities to specialise. The senior secondary program also endorses pathways choices for learning, giving students the opportunity to train with employers while completing a regular school program.

The Australian Curriculum identifies the capability to use ICT as one of the seven general capabilities that students must learn and possess. These general capabilities "encompass the knowledge, skills, behaviours and dispositions that, together with curriculum content in each learning area and the cross-curriculum priorities, will assist students to live and work successfully in the twenty-first century" (ACARA, 2010). The Australian Curriculum acknowledges that in this digital age successful learners must be highly skilled in the use of ICT across all learning areas. Naming ICT as a capability, reflects the importance with which technology is regarded for learning while at school and beyond.

For Mathematics, ICT capability involves the ability to adapt methods and strategies for mathematical problem solving as technologies such as spreadsheets, Dynamic Geometry Software (DGS) and CAS evolve. The expectation is that "students develop ICT capability when they investigate, create and communicate mathematical ideas and concepts using fast, automated, interactive and multimodal technologies" (ACARA, 2010). Dynamic software tools, such as DGS, allow the user to create and then manipulate constructions instantly within a digital environment, as opposed to traditional methods with static tools, such as a ruler and pen. DGS enables students to visualise and develop mental constructs for analytical thinking, these include transformations of geometrical shapes, moving points in relationships, or manipulating features of graphs, all of which occur instantaneously compared to using static tools which may require many drafts of diagrams to achieve the same result. Through the use of dynamic software, students are empowered and motivated to explore and investigate open-ended problems. In the same way that the DGS has revolutionised geometry explorations, CAS has transformed the computation of mathematical expressions, manipulating algebraic expressions and performing operations such as expand and simplify and factorising.

The last technological change to have such an impact on curriculum was the introduction of scientific calculators that replaced the need for slide rules and log tables in the 1970s. While curriculum documents are fixed for a certain time frame, they must evolve, they are "not written on tablets of stone" (Kissane, 2002, p. 193). Curriculum documents must adapt to accommodate changes in technology. As technology promises new and interesting opportunities for students to experience mathematics, the curriculum must value teaching and assessments involving technology (Drijvers, et al., 2016; Kissane, 2000).

Nevertheless, as curriculum documents evolve, new challenges arise concerning the choice of content required in mathematics programs. As stated by the National Council of Teachers of Mathematics (NCTM, 2000), "the relative importance of particular mathematics topics is likely to change over time in response to changing perceptions of their utility and to new demands and possibilities" (p. 16). Although, with a syllabus already overflowing, it is unreasonable to expect technological activities to be additional; rather, they require to be integrated with what is already there, otherwise attempts to cover everything becomes futile, resulting in limited mathematical understanding, because learning suffers from covering too much too quickly (Crossley, 2006). An important task is determining what is necessary within the syllabus, what may be rendered less necessary, and how to ensure a suitable balance is attained (Crossley, 2006; Herget, Heugl, Kutzler, & Lehmann, 2000; Kissane, 2000, 2007; NCTM, 2000; Schwartz, 1999). It is essential that the balance between traditional concepts and technological explorations for lesson and assessment content be sustained, "using technology with every activity and for every instructional purpose is just as futile as using direct instruction for every topic and lesson" (Guerrero, 2010, p. 136). As boundaries of mathematics transform, one of the most significant tasks relevant to mathematics education today is the modification of curriculum documents and pedagogy (Fey, 1989).

Another problem confronting mathematics curriculum developers is the rate with which technology advances and new tools emerge. Deciding which technologies should be valued and identifying the respective skills required becomes an important decision. While some skills may need to be developed earlier in order to accommodate the requirements of certain technologies, other skills must be reinforced to enable students to obtain a deeper conceptual understanding of mathematics through using the technology (Kissane, 2007). There appears to be a lack of literature available that discusses specific mathematical skills necessary for technology. One paper presented the results of a two-day debate held by the authors discussing the "manual calculation skills" they consider to be "indispensable" (Herget, et al., 2000, p. 9) with respect to CAS calculators and computer software; the authors aimed to be controversial in order to create an "impulse for a broad discussion" (Herget, et al., 2000, p. 18), hopefully bringing light to issues worthy of further investigation.

While curriculum documents provide the official position on what content and capabilities, such as ICT, should be explored by students, teachers are those responsible for implementing and delivering the curriculum. Thus, the key component that determines the fate of technology integration is the teacher (Ertmer, Addison, Lane, Ross, & Woods, 1999; Geiger, Forgasz, Tan, Calder, & Hill, 2012; Handal & Herrington, 2003; NCTM, 2000). The teacher remains the strategic link between content and technology, becoming a conveyance of change or a major obstacle inhibiting reform (Handal & Herrington, 2003; Prawat, 1992).

In order for teachers to maximise their usage of technology in classrooms, their respective beliefs and attitudes towards the new challenges technology presents must be addressed (Crawford, 1995; Forgasz, 2006; Geiger, et al., 2012; Kaput, 1992; Thomas & Chinnappan, 2008). These include their fundamental instructional beliefs about mathematics teaching, philosophies of mathematics teaching, their own attitude towards technology, and the attitude with which fellow teachers approach and utilise technology (Cavanagh, 2005; Handal & Herrington, 2003). Many research studies endorse the utilisation of technological tools in mathematics classrooms, but teachers' beliefs can be either an encouraging or obstructive influence depending on the particular beliefs held, because teachers inevitably control the delivery of content within their classrooms. Allan (2006) identified that teachers who are uncertain of the benefits that technology provides for their teaching and student understanding are less likely to integrate technology into their lessons; successful integration does not rely on the influence of academics, research findings or curriculum documents that prescribe the benefits of technology.

It is important, therefore, that teachers address the beliefs and attitudes which create barriers against technological implementation. However, the process of inciting such a change can be extremely challenging and problematic, requiring extended periods of time and nurturing of the process (Allan, 2006; Forgasz, 2006), including appropriate professional development and continual support. Training opportunities promote the realisation that technological tools are capable of enhancing learning and enriching mathematical understanding. As teachers begin to essentially "play" with the technology they have available, their overarching perception of technology improves and their confidence increases (Serow & Callingham, 2011). Subsequently, teachers can begin the process of integrating technology into what they teach and modifying their teaching practices to incorporate technology appropriately (Pierce & Ball, 2009). The benefits and successes of technology hinges on the skills that teachers have to integrate, navigate and use the tools (Drijvers, et al., 2016; Ertmer, 2005). Through supportive networks, teachers are able to share ideas, thoughts and problems, thus resolving issues and developing new concepts to enable learning through technology. Ertmer (2005) suggested that the process whereby teachers accumulate enough knowledge and confidence to successfully implement technology in a constructivist way could take around five or six years.

Time has been identified as an important area to be addressed (Drijvers & Weigand, 2010). Goos and Bennison (2008) identified that teachers required "more time to develop resources, plan lessons and curriculum units, and explore and evaluate the technology, preferably in collaboration with colleagues” (p. 118). Time was recognised as being a determining factor towards the positive and creative implementation of technology, “the pressures of external examinations, lesson preparation, motivating pupils, dealing with unruly pupil behaviour, marking and other responsibilities within the school all exact a toll on the proportion of the thinking and working time teachers have to adapt to new technologies” (Kissane, 2000, p. 69). Allocation of more time to teachers is a contentious issue that brings with it more challenges. One notable challenge is the issue of funding and determining who is responsible to fund the extra time allocation – but this becomes another issue beyond the scope of the current study.

In summary, there are a number of issues that influence the implementation of technology in today’s classrooms: curriculum documents evolve, stating the importance of technology and mandating its use, but teachers remain a crucial component towards determining the success or failure of technology utilisation within their classrooms; supporting teachers professionally, through the likes of training programs, along with the provision of extra time to explore and prepare lessons that integrate technology, and through fostering networks to promote discussion surrounding technological implementation, will assist with the hurdles many teachers face; it is essential that concerns regarding technology are addressed, to ensure new and challenging innovations are embraced and integrated within the classroom, thus enabling students to have the best opportunities for learning.

2.3. Theoretical Models of Technology and Learning

This section considers the connections between technology and pedagogy and presents three models that can be used to assist teachers with implementing technology. It discusses the changing role of the teacher and other pedagogical concerns currently identified when introducing technology into mathematics classrooms. Following from this, three frameworks are described that can support the teacher with how to use technology in the classroom: the Master, Servant, Partner and Extension of Self (MSPE) framework; the Substitution, Augmentation, Modification and Redefinition (SAMR) model; and the Technological Pedagogical And Content Knowledge (TPACK) framework. The section concludes by identifying the technological results that obstruct positive learning opportunities.

In the 1990s, some people suggested that computers and technology would eventually provide teaching, rendering the role of a teacher redundant (Dry & Lawler, 1998). Decades on, the reverse has become apparent. As previously mentioned, the teacher's role remains crucial for fostering the mathematical development of ideas and concepts, with the teacher being a valuable resource for students (Balacheff & Kaput, 1996; Drijvers & Weigand, 2010; Kissane, 2000). Teachers use various forms of technology as tools that assist with the instruction of the subject matter; "technology does not replace the mathematics teacher. When students are using technological tools, they often spend time working in ways that appear somewhat independent of the teacher, but this impression is misleading" (NCTM, 2000, p. 26).

The role of the teacher, however, transforms with the introduction of technology. The teacher shifts from being a director of learning to being a facilitator, fostering learning rather than dictating it (Cavanagh, 2005; Fey, 1989; Monaghan, 2004). Drijvers et al. (2010) explain this well using a musical metaphor when they compare the change of a teacher's role to the comparison between a conductor of a symphony orchestra and a jazz band. The symphony orchestra, representative of the traditional mathematics classroom, performs music strictly followed by accomplished musicians under careful direction of the conductor. The conductor of the jazz band, however, assists the accomplished musicians to learn the music while encouraging individual interpretation and improvisation, similar to facilitating exploration in a technological classroom.

Learning with technology encourages a more student-orientated rather than concept-orientated approach (Allan, 2006). A significant part of the learning experience involves motivating students to explore mathematical concepts for themselves, this supports cognitive development as well as consolidating necessary procedural skills (Kissane, 2007). Through appropriate activities, selected by the teacher, learning can be enhanced as the student initiates inquiry and critical thinking, while the teacher monitors and supports the learning process (Doerr & Zangor, 2000; Forster, 2004). The teacher-student relationship alters as both "learn to listen carefully to and assess the qualities of one another's arguments" (Cuban quoted in Monaghan, 2004).

For mathematics, technology provides tools that promote students' motivation and engagement (Bate, Day, & Macnish, 2013; Raines & Clark, 2011). Traditional concepts may become more enjoyable and relevant (Burrill, 2005; Hopper, 2009; Kissane, 2008), in particular “students who are easily distracted may focus more intently on computer tasks, and those who have organisational difficulties may benefit from the constraints imposed by a computer environment” (NCTM, 2000, p. 25). Teaching strategies that are designed to build motivation, particularly in mathematics, have been found to relate directly to enhancing students’ achievement (Bobis, Anderson, Martin, & Way, 2011; Stipek et al., 1998). Such tools must be encouraged to foster mathematical learning and promote the further study of mathematics, since “almost all students stop studying mathematics as soon as they are no longer obliged to opt for it” (Kissane, 2000, p. 62). Students are often unaware of the importance mathematics holds to their lives after school. The National Council of Teachers of Mathematics (NCTM, 2000) claims mathematics is necessary in everyday life and its importance continues to grow despite the increased use of technology. “Those who understand and can do mathematics will have significantly enhanced opportunities and options for shaping their futures. Mathematical competence opens doors to productive futures” (NCTM, 2000, p. 5).

It is essential that the underlying mathematical ideas be retained when technology is incorporated into the teaching and learning sequence. Simply inserting technology into a program for the sake of satisfying curriculum outcomes or making mathematics enjoyable does not foster the development of conceptual understanding. It cannot be “a finished product that can be inserted into an education setting to create a particular effect” (Lynch, 2006, p. 32). The fundamental question for teachers to consider becomes whether, and to what extent, technology can be used to influence conceptualization of the subject matter (Guerrero, 2010; Kaput, 1992; Thomas & Chinnappan, 2008, p. 173). Hence, the structure and design of activities associated with the technology is as important as the technology itself, although the “process of designing learning activities is not a simple one” (Healy, Jahn, & Frant, 2010, p. 397). Activities must be designed to facilitate technology without compromising the core mathematics (Ertmer, 2005; Haapasalo, 2007). Concerns have been raised regarding the design of worksheets, which can often become a detailed set of instructions for the technology, “give too little direction and students become bogged down with some aspect of the program; but give too much detailed instruction and students ‘cannot see the wood for the trees’: they become too preoccupied with following the requirements of running the program, at the expense of focusing on the mathematical ideas” (Little, 2009, p. 53).

The MSPE framework developed by Geiger (2009) identifies four categories of students' usage of technology. The first category, *Technology as Master*, indicates that students rely heavily on technology. They blindly accept its solutions, signifying their inability to effectively use the technology. The second category, *Technology as Servant*, indicates that students control the technology for their own benefit. It involves using technology to check answers and perform operations without any creativity, simply because the technology is more efficient than traditional methods. The third category, *Technology as Partner*, indicates that students work together with the technology to solve problems. Here technology provides scaffolding support to explore and investigate traditional problems. The visualisation of features and properties of graphs using DGS is an example of *Technology as Partner*. The last category *Technology as Extension of Self* indicates students have competent technological skills, and tasks are transformed such that they would not be possible without the technology. This category would extend to explorations and investigations that require conceptual thinking by the student.

The SAMR model provides another useful framework that enables teachers to gauge how technology is utilised in activities. Created by Dr. Ruben R. Puentedura's in the early 1990s, the model assumes technology integration of any classroom activity to be on a continuum moving from substitution through to redefinition (Puentedura, 2010). Technology challenges teachers to "identify pedagogical approaches" (Forgasz, 2006, p. 465) that accommodate all students. Through using the SAMR model, teachers can categorise activities and make adjustments to enhance the use of technology. Figure 2.1 illustrates the levels of the SAMR model.

The first most basic level, substitution, indicates that the technology has not functionally changed the activity; rather, it has simply replaced a manual technique. An example of this would be printing a worksheet for use.

The second level, augmentation, indicates some functional improvement such as taking an online quiz. Although a direct substitute of a traditional manual quiz, it is enhanced by providing immediate feedback to teacher and student. Activities that restrict the use of technology to drilling repetitive skills are indicative of augmentation. These types of closed activities limit the possibilities of using technology for conceptual mathematics understanding, along with inhibiting any deep mathematical learning available to students

(Ertmer, 2005; Monaghan, 2004; Norton, et al., 2000; Reed, Drijvers, & Kirschner, 2010; Serow & Callingham, 2011).

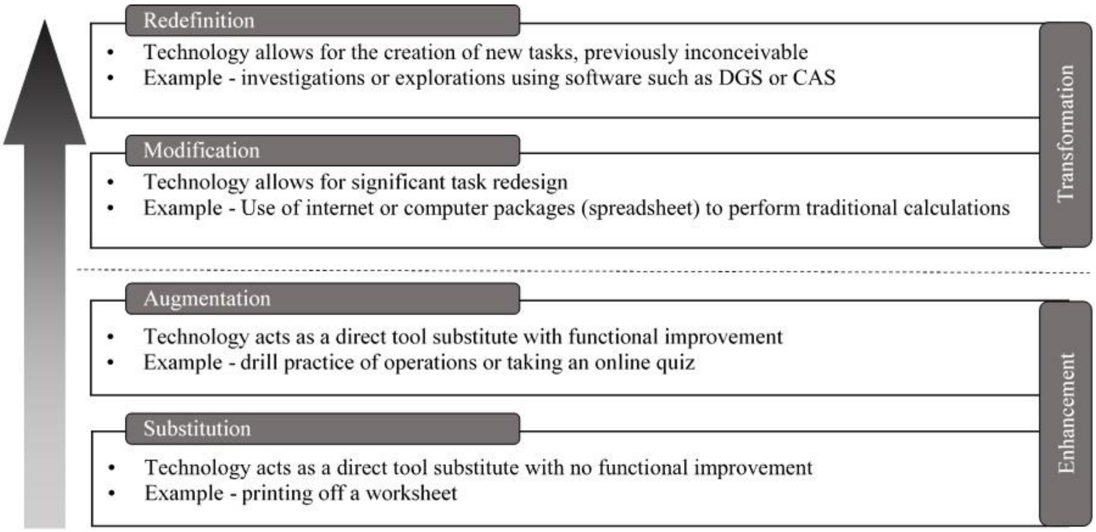

Figure 2.1: SAMR Model

The third level, modification, is the first step between enhancing the traditional pedagogy of the classroom and transforming the classroom. It involves a significant re-design for common classroom tasks through the use of computer technology. This includes the use of the Internet and computer packages, such as spreadsheets, to perform data calculations, requiring different approaches to designing traditional activities.

The final level, redefinition, indicates that such activities would not be possible without technology. Redefinition would include investigations and explorations that cannot be done without computer packages such as DGS or CAS.

The success of SAMR model relies upon two factors: the knowledge of the teacher implementing the task and the availability of technology (Jude, Kajura, & Birevu, 2014). Both the MSPE framework and SAMR model analyse how technology is applied through categorising the types of activities implemented. The following model, the TPACK framework, encompasses all aspects of technology integration.

The TPACK framework assists teachers to understand how technology, pedagogy and mathematical content relate to one another. Developed by Koehler & Mishra (2009) as an extension of the Pedagogical Content Knowledge (PCK) work by Shulman (1986), it describes “how teachers’ understanding of educational technologies and PCK interact with one another to produce effective teaching with technology” (Koehler & Mishra, 2009, p. 62).

It identifies the different types of learning that occur when using technology as a pedagogical tool because it recognises that "learning subject matter with technology is different from learning to teach that subject matter with technology" (Niess, 2005, p. 509). Others have identified types of learning relevant to specific tools. For example, Sheryn (2005) noted that learning with graphics calculators requires "learning mathematics, learning how to use a graphics calculator and learning mathematics and learning how to use a graphics calculator simultaneously" (p. 107). The TPACK framework, however, provides a generalised structure applicable to content matter of any subject and any technology. Specific to the teaching of Mathematics, the acronym M-TPACK is commonly used (Drijvers, et al., 2016).

Koehler and Mishra define TPACK as:

> [The] basis of effective teaching with technology, requiring an understanding of the representation of concepts using technologies; pedagogical techniques that use technologies in constructive ways to teach content; knowledge of what makes concepts difficult or easy to learn and how technology can help redress some of the problems that students face; knowledge of students' prior knowledge and theories of epistemology; and knowledge of how technologies can be used to build on existing knowledge to develop new epistemologies or strengthen old ones. (p. 66)

The TPACK framework consists of seven components, as shown in Figure 2.2. The three main areas represent Technological Knowledge (TK), Content Knowledge (CK) and Pedagogical Knowledge (PK). TK refers to the knowledge a teacher has regarding the technologies available, this knowledge fluctuates because new technologies constantly emerge. CK refers to the "actual subject-matter that is to be taught or learnt, and includes knowledge of concepts, theories and ideas, evidence and proof" (Serow, Callingham, & Muir, 2014). PK refers to knowledge of processes and practices pertinent to teaching and learning (Koehler & Mishra, 2009). Despite their size, the intersections of the TPACK framework are of equal importance and signify the interactions between and among the main bodies of knowledge.

PCK is the teacher's knowledge of pedagogical approaches for specific content. It represents the core business of teaching content, knowledge of curriculum, assessment and pedagogy. Technological Content Knowledge (TCK) is teachers' combined knowledge of the relationship between content and technologies that may be relevant to that content. It is knowing which technologies to use with certain content and "understanding of the manner in which technology and content influence and constrain one another" (Koehler & Mishra, 2009, p. 65). TPK is the understanding of how technology and pedagogy interact with one another. It is being able to adapt technologies with the purpose of improving the students' knowledge and understanding, including "knowing the pedagogical affordances and constraints of a range of technological tools" (Koehler & Mishra, 2009, p. 65). Using the TPACK framework assists teachers to understand and develop the "knowledge teachers must possess and access" (Guerrero, 2010, p. 132) when integrating technology.

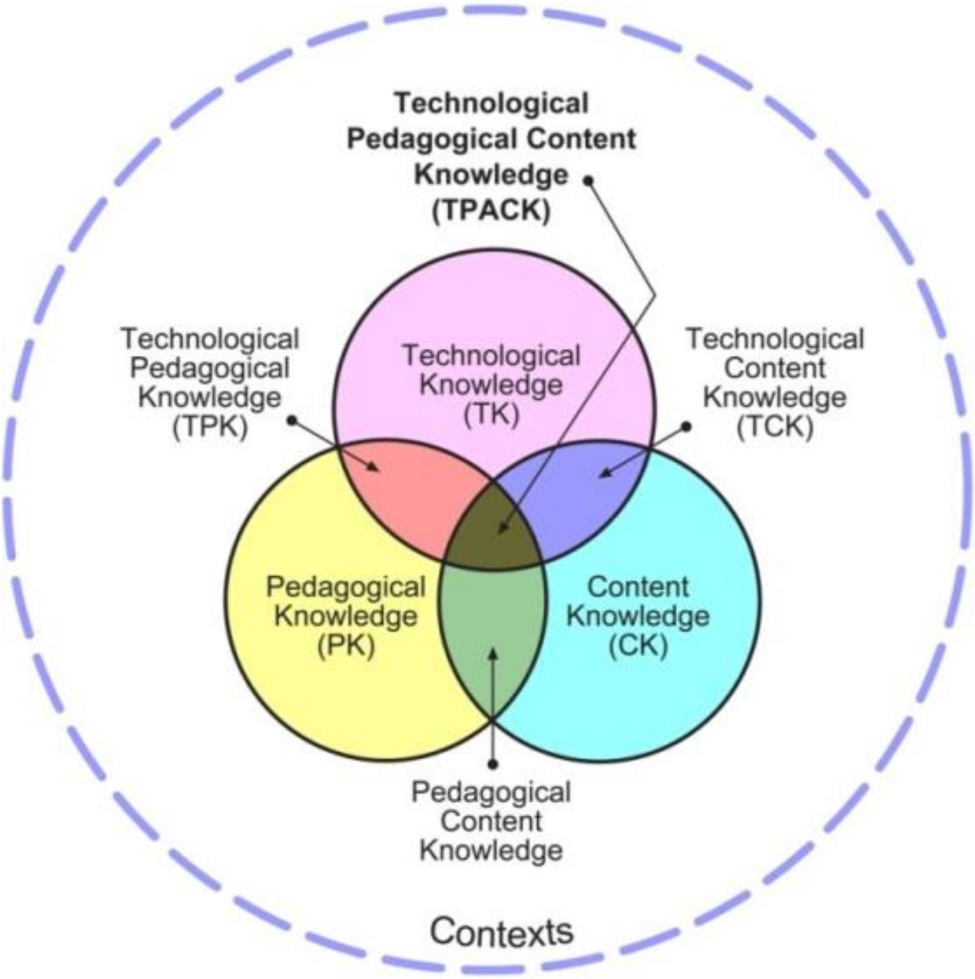

Figure 2.2: TPACK framework
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Despite attempts to re-design activities consistent with the TPACK framework and SAMR model, technological issues still arise which impede positive learning opportunities. Technology is renowned for throwing inconsistencies, limitations and unexpected results that require technological or mathematical understanding to explain. Examples include: graphing calculators or software that present partial views; and unequally scaled axes or zooming features that require interpretation and discussion to discern the required information be interpreted correctly. Traditionally, teachers have attempted to direct students away from making common mistakes in the early stages by choosing problem solution pathways that avoid the pitfalls or by demonstrating strategies that solve the pitfalls, with each step carefully articulated by the teacher (Cavanagh, 2005). This is no longer the case; as technological tools and their capabilities rapidly change it is almost impossible to know and solve every inadequacy they produce (Sheryn, 2006).

Recent studies suggest that teachers should permit students to experience the inconsistencies, limitations and unexpected results. This enables them to not only critically refine their mathematical skills but also become independent learners and improve their understanding of the technology (Cavanagh, 2005; Crossley, 2006). Through confronting unpredictable situations, students develop thinking skills and strategies that resolve the inconsistencies and assist them to make sense of the information (Cavanagh, 2005; Guin & Trouche, 1998). They make informed decisions about not only how to use the technology and software but also when and if to use it; judgemental choices that Crossley (2006) recommends “cannot start too early” (p. 177) – though, Cavanagh (2005) advises teachers to be cautious not to overwhelm students by confronting them with too many inconsistencies and unexpected results because it may become a barrier towards future learning. Thus, as previously mentioned, tasks must be carefully designed or selected to ensure an appropriate balance is obtained such that students learn to determine the best strategies for solving or investigating the problem.

In summary, technology provides many opportunities for teachers to demonstrate content in new and interesting ways. Shifting the teachers’ role, from being a dictator using explicit teaching techniques, to becoming a facilitator who guides students towards the specific learning outcome, creates a more positive environment for learning with technology. Three frameworks: the MSPE framework, the SAMR model and the TPACK framework, support teachers by providing guidelines to assist with the development of activities and lesson plans. These frameworks enable the teacher to monitor the delivery of content when using technology such that credibility of mathematical content is retained.

To conclude, this section discussed the technological issues that create barriers towards mathematical learning and considered current research providing assistance for dealing with some of the complications.

2.4. Linear Relationships and Technology

This section considers the topic of Linear Relationships and how technology has influenced the learning of its content. In particular, the discussion targets the software GeoGebra, a form of dynamic mathematical software, which is used during the teaching sequence for this study.

Linear Relationships is one of the first topics in which secondary students are exposed to algebra with some contextual meaning. Therefore, a thorough understanding of the topic Linear Relationships is important for providing a solid foundation for further algebraic studies and algebraic thinking (Beatty & Bruce, 2012; Pierce, Stacey, & Bardini, 2010; Wells, 2016). The term, *Linear Relationships*, is often used interchangeably with *Linear Functions* but the former defines a set of points (x and y coordinates), which form a relationship resulting in a straight-line graph whereas a function refers to a special kind of relationship such that every x value has a unique y value. Thus, a *Linear Function* is a function that results in a straight line. The misconception occurs because, with the exception of one particular family of lines – the vertical line family – all Linear Relationships are actually Linear Functions. The vertical line family, for example, $x = 2$, does not satisfy the condition of having a unique y value since they have many y values for their chosen x values. In the Australian National Curriculum, Linear Relationships is a unit introduced in junior secondary (approximately 12–16 years old) prior to the Linear Functions unit, which is delivered in senior years (approximately 16–18 years old). Since Linear Relationships forms a subset of the Linear Functions unit, their studies also contain meaning for Linear Relationships and will be viewed as such.

The topic of Linear Relationships and Functions is considered challenging for most junior secondary students (Brown, 2007). Numerous studies have documented the difficulties faced by students with Linear Relationships (Bardini & Stacey, 2006; Beatty & Bruce, 2012; Brasell & Rowe, 1993; Ellis, 2007; Moschkovich, 1996, 1998). The difficulties present when students attempt to simultaneously understand the algebra and its manipulation and the necessary contextual connections that give the variables meaning (Brasell & Rowe, 1993). The ability to distinguish the symbolic, numeric and graphical representations proves problematic for some students (Beatty & Bruce, 2012). A study by Acevedo Nistal, Van Dooren and Verschaffel (2013) found a common issue for them was that students seemed overwhelmed with the decision of which representation to choose to solve a problem if not directed by the teacher. Pierce, et al., (2010) found that interpretations within a real-world context also proved challenging for students, with the effect of one parameter on the graphic representation providing uncertainty for student understanding (Moschkovich, 1996). Bardini and Stacey (2006) categorised the challenges confronted by students with the equation of a straight line, $y = mx + c$, into four different dimensions, as shown in Table 2.1.

Table 2.1: The four dimensions of *m* and *c*.

	Symbolic Dimension	Graphical Dimension	Numerical Dimension	Context of Question
m	Represents the coefficient of x in the gradient form of equation of a straight line $y = mx + c$	Represents the gradient/slope of the graph of $y = mx + c$	Represents the ratio $\Delta y/\Delta x$ also defined as rise over run	Represents the real world perspective of m (e.g. price per kg)
c	Represents the constant term of the equation of a straight line $y = mx + c$	Represents the y-intercept of the graph of $y = mx + c$	Represents the value of y when $x = 0$	Represents the real world perspective of c (e.g. initial value)

Technological tools that are dynamic and interactive in nature provide renewed inspiration, for topics such as Linear Relationships. They stimulate student's engagement, thus promoting understanding. Within the unit of Linear Functions, students are frequently required to draw graphs that assist with the understanding of concepts. The ability of technologies to reliably produce a graph accurately and quickly fosters learning through a visual representation of information, providing "powerful visual information/feedback for students to use while answering the questions" (Özgün-Koca, 2008, p. 23). Technology enables students to switch their focus from plotting graphs, to exploring the context which the graph represents, such as describing features, and recognising patterns and relationships (Cavanagh, 2005; Fey, 1989). Teachers also shift their focus "from demonstration of 'how to' produce a graph to explanations and questions of 'what the graph is saying' about an algebraic expression or situation it represents" (Fey, 1989, p. 250). One study using graphics calculators reported that despite the technology requiring an initial expenditure of time and effort, both students and teachers found it beneficial and worthwhile for developing the understanding of Linear Functions (Bardini, Pierce, & Stacey, 2004). Graphing technology, such as GeoGebra, assists by linking the symbolic, graphical and numerical dimensions through "giving visual images of symbolic information" (Fey, 1989, p. 249). The graphing technology enriches the understanding of algebraic forms and a developing realisation of the associated algebra emerges.

While technology has the potential to make the drawing of graphs a trivial exercise, Berry and Graham (2005) suggest that the ability to draw a rough sketch has become even more important with the increased use of and dependence upon technology. They noted that the skills and techniques associated with pen and paper graphing had suffered in the decade prior to their study because sketching tasks were not prominently featured in mathematics examinations. The devaluing of sketching skills has resulted in students relying heavily on technological tools to generate tables and graphs then blindly accepting the result produced (Cavanagh, 2005; Garofalo, et al., 2000). In order for students to capably confirm the accuracy of results produced by graphing technology, it is essential that the manual skills and techniques are nurtured and preserved. These assist in providing the foundations towards further investigations or conclusions. The MSPE framework identifies this situation in its opening category, in which technology is considered the master, indicating that the technology dominates because the student depends on it to provide the solution without contestation (Geiger, 2009).

As highlighted previously, it is important for students to understand that technology can deliver unexpected results that may cause issues and problems for students, particularly when interpreting results. Two issues that skew findings from graphing technology are: scaled axes, as shown in Figure 2.3 (Kemp, Kissane, & Bradley, 1996, p. 5), that may cause students to misinterpret what they visualise; and partial views, as shown in Figure 2.4 (Mitchelmore & Cavanagh, 2000, p. 258), that prevent important features of a graph to be displayed in the view window (Cavanagh, 2005; Gibbs, 2006). While it is important for students to be exposed to these problems, they must also understand how to interpret and adjust their results so that the contextual meaning of the initial problem is not lost. For this to occur, students must have sound knowledge of the underlying mathematics supporting the technology's contradictions. The assumption that students know how to use a particular graphing technology simply because they have the resource available is naive (Goos & Bennison, 2004). As technologies evolve, the inconsistencies change because more powerful calculations are possible, but the mathematical skills which enable students to deal with the inconsistencies remain firm.

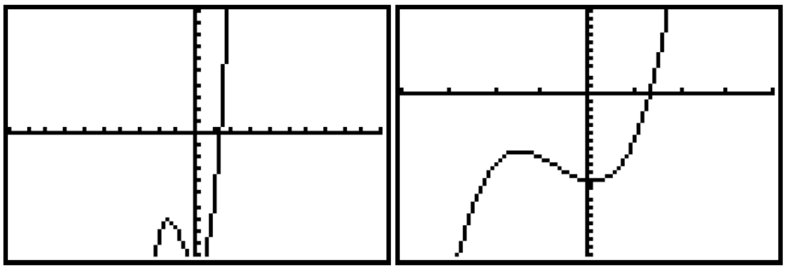


Figure 2.3: Graphs of $y = 2x^3 + 4x^2 - x - 11$ demonstrating issues with scaled axes
(Source: Kemp, Kissane, & Bradley, 1996)

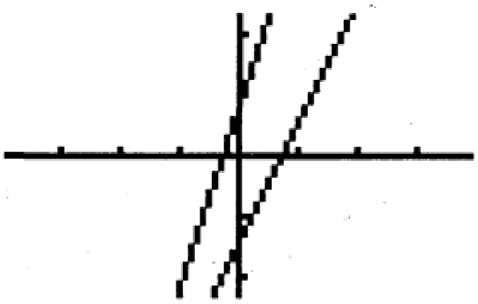


Figure 2.4: View from a graphics calculator when trying to find the intersection of the graphs $y = 2x - 1.5$ and $y = 3x + 0.8$
(Source: Mitchelmore & Cavanagh, 2000)

One mathematics program growing rapidly in popularity is the free, open source, multi-platform software, GeoGebra (https://www.geogebra.org/). GeoGebra, the brainchild of Markus Hohenwarter, is a program that was initially designed to link geometry with algebra (as defined by the name **Geo**metry and al**Gebra)** but has become a worldwide project that has expanded to include tables, spreadsheets, calculus and statistics. Geometry and algebra are widely considered to be the two most fundamental and formal pillars of mathematics and, through linking these two, Hohenwarter has provided an exciting innovation that has become a revolution for mathematics education (Atiyah, 2001; Edwards & Jones, 2006).

Geometry has been defined as the study of operations and manipulations involving space (Atiyah, 2001); a predominantly visual concept that involves spatial perception and also allows our intuition to form conjectures and hypotheses from diagrams that feed our visual senses. Algebra is considered a discipline that deals with manipulations of time, since it revolves around a sequence of operations performed in a specific order (Atiyah, 2001). Algebra uses abstract variables to represent numbers, which can be manipulated following precise rules. Previously, technologies focussed separately on one a single environment, either

- Algebra – such as CAS, which focusses on the "manipulation of symbolic expressions" (Hohenwarter & Jones, 2007, p. 126). Popular forms of CAS software included Derive (http://www.chartwellyorke.com/derive.html) and Maple (http://www.maplesoft.com/products/maple/); or
- Geometry – such as DGS, which focusses on the "relationships of lines, points, circles" (Hohenwarter & Jones, 2007, p. 126). Popular forms of DGS include Geometers Sketchpad (http://www.dynamicgeometry.com/).

Hohenwarter became interested in the idea of a single computer package that had the visual capabilities of CAS and the dynamic changeability of DGS, after attending a mathematics education lecture at the University of Salzburg, Austria, in 1997, which demonstrated the TI-92 calculator. This lecture revealed that the TI-92 had capabilities of both CAS and DGS as separate entities; however, Hohenwarter envisaged potential in a tool that combined geometry and algebra into a single easy to use package. Although others are known to have previously suggested such an idea, Hohenwarter accomplished the development of a completely new mathematical tool for secondary school education as part of his Master's thesis in 2002. After publishing GeoGebra onto the Internet in 2002, Hohenwarter was unexpectedly surprised at the number of teachers who contacted him with positive feedback and shared his enthusiasm for the tool and its benefits for mathematics classrooms (Hohenwarter & Lavicza, 2007). After winning several educational software awards, and supported by a scholarship, Hohenwarter continued to further develop GeoGebra as part of his PhD project. In 2006, GeoGebra and Hohenwarter were embraced by the Florida Atlantic University in the US, where its development continues through funded projects such as the National Science Foundation's (NSF) Mathematics and Science Partnership (MSP) initiative. The MSP partnership liaises with local schools to enable Hohenwarter and his team to continue developing GeoGebra according to the feedback from collaborating mathematics teachers.

The strength of GeoGebra exists in the fact that it offers the features of both DGS and CAS in a bidirectional capacity. As shown in Figure 2.5, it has two windows, the graphics window which enables working with points, vectors, segments, polygons, lines, and conic sections, and the algebra window which enables equations and coordinates to be entered directly into the input bar at the bottom of the window. Its bidirectional combination of geometry and algebra means that typing an equation into the algebra window will result in the graph of the equation being shown in the graphic window. Likewise, by dragging or manipulating the graph in the graphic window, the equation changes accordingly in the algebra window (Hohenwarter & Fuchs, 2005). Hence, GeoGebra more closely aligns the connections between visualisation capabilities of CAS and dynamic changeability of DGS. In recent developments, GeoGebra has expanded to include spreadsheets, tables, calculus and statistics.

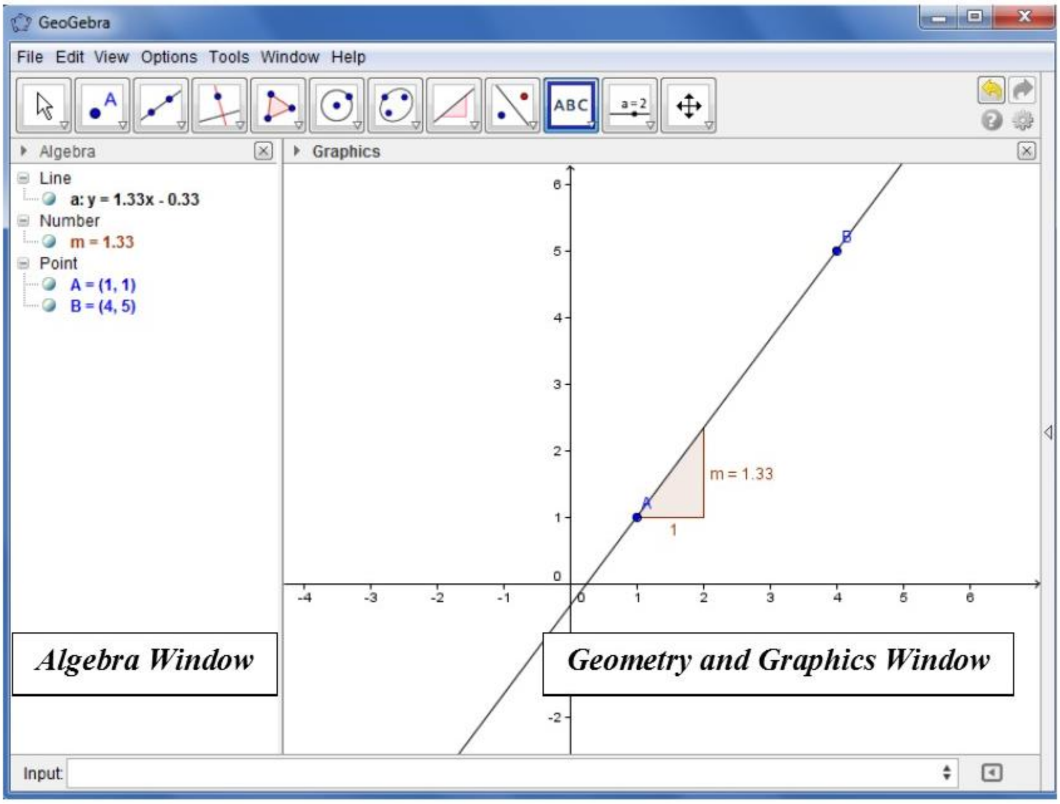


Figure 2.5: GeoGebra view window – algebra and geometry and graphics window

GeoGebra continues to be widely researched and acknowledged as a supportive tool for mathematical learning and understanding. It enables visualisation of concepts, which facilitates visual reasoning, a process widely recognised as strengthening students' mathematical problem-solving skills (Arcavi, 2003; Hohenwarter, Hohenwarter, Kreis, & Lavicza, 2008; Kllogjeri & Shyti, 2010). With the benefit of connecting symbolic and graphic representations, it promotes students involvement in investigations and explorations (Hohenwarter, et al., 2008). This enables students to focus on the conceptual understanding linking geometry and algebra rather than the procedural knowledge. Deep questioning is promoted, facilitating investigations and exploration of concepts that previously were considered too difficult to attempt. Studies have demonstrated that GeoGebra supports student learning and understanding in many mathematical topics, such as Fractions (Thambi & Eu, 2013), Trigonometry (Zengin, Furkan, & Kutluca, 2012), Coordinate Geometry (Saha, et al., 2010) and Functions (Gómez-Chacón & Prieto, 2011; Hohenwarter, 2006).

One of GeoGebra's strengths is facilitating mathematical learning to a wide range of student levels, ranging from primary to university (Hohenwarter, et al., 2008; Kllogjeri & Shyti, 2010). Another remarkable feature of GeoGebra is the multitude of free teaching materials and online fora, both of which provide an excellent free resource for teachers (Hohenwarter, et al., 2008; Zulnaidi & Zakaria, 2012). GeoGebra is known to be used educationally in over 190 countries and translated into more than 50 languages (Hohenwarter, 2013).

In summary, graphing software supports the fostering of mathematical understanding for topics such as Linear Relationships. A challenging topic for secondary students, Linear Relationships requires an understanding of symbolic, numeric and graphic representations. Through its bidirectional capabilities, the software GeoGebra facilitates the combination of geometry and algebra, and has been expanded to include spreadsheets, tables, calculus and statistics. A world renowned mathematical success, GeoGebra increases the possibility for more conceptual questioning and promotes a deeper understanding through engaging students with its dynamically changing environment. By guiding students to embrace the benefits of graphing technology, such as GeoGebra, while exposing technology's limitations and unexpected results, teachers can further the understanding of topics such as Linear Relationships, permitting mathematical investigations and explorations to flourish.

2.5. Conclusion

This chapter has discussed and presented research studies to provide a background on the relevant topics informing the current research project. Reviewing studies has highlighted that the implementation of technological tools in Australian mathematics classrooms needs addressing. In particular, practical strategies and programs to assist teachers in overcoming the obstacles that obstruct them from incorporating technology into their pedagogy would be beneficial.

A gap exists in literature to explain the specific strategies used by students when learning with technology, and how teachers can determine that student learning improves through teaching with technology. While frameworks and models exist, such as TPACK and SAMR, that assist teachers when implementing technology, and MSPE that describe students' usage of technology, an extensive investigation into how students' mathematical knowledge improves through learning with technology would also be valuable. The combination of these frameworks with appropriate theoretical and pedagogical frameworks would provide the foundation for a comprehensive teaching sequence of a topic. Subsequent analysis of work completed by students during this sequence, using an appropriate tool, would provide data enabling an achievable research project to evolve.

The next chapter describes in detail the two theoretical and pedagogical frameworks which assist the current study with these aims. The first, the van Hiele Teaching Phases provides a basis for lesson sequence structure and the Structure of the Observed Learning Outcome (SOLO) model provides an analysis tool to evaluate student responses.

CHAPTER 3: THEORETICAL AND PEDAGOGICAL FRAMEWORK

> SOLO also provides the means for the van Hiele model to move to a new phase of development, namely, the exploration and explanation of individuality in geometry education. The challenge for researchers and teachers is to investigate and identify individual paths of development: to seek out variability.
>
> (Pegg & Davey, 1998, p. 133)

3.1. Introduction

The above quote by Pegg and Davey identifies the van Hiele Theory and the Structure of the Observed Learning Outcome (SOLO) model are frameworks that provide potential benefits for both researchers and teachers of mathematics. Although the quote specifically identifies these frameworks with Geometry, they have been used in other areas of Mathematics (Li & Goos, 2013; Lian & Yew, 2012a; Pijls, Dekker, & Van Hout-Wolters, 2007) and will also be used in this study, as will be detailed in this chapter.

The van Hiele Theory and the SOLO model are well documented as effective frameworks for mathematics education (Burger & Shaughnessy, 1986; Pegg & Davey, 1998; Pegg & Panizzon, 2008; Pegg & Tall, 2001; Serow, 2007, 2008). The van Hiele Theory, in particular, has been widely accepted and used as a successful structure to describe the stages of learning for Geometry since the early 1980's (Braconne & Dionne, 1987; Burger & Shaughnessy, 1986; Fuys, Geddes, & Tischler, 1988; Serow, 2002). It has since been adapted for other mathematical topics such as Fractions (Streefland, 1991), Probability (Pijls, et al., 2007), Functions (Isoda, 1996), Pythagoras' Theorem (Flores, 1993) and Linear Relationships. Understanding the van Hiele Theory and its Teaching Phases not only assists teachers to recognise the associations students make when learning content but also enables teachers to identify why certain strategies work or fail, and how to improve their lessons in order to increase student knowledge and understanding.

The SOLO model has also been extensively used for educational research – for teaching: Science (Martin, 2011; Minogue & Jones, 2009; Panizzon, Callingham, Wright, & Pegg, 2007; Panizzon & Pegg, 2008), Design and Technology (Leung, 2000), English (McNeill & Hook, 2012), Geography (Munowenyu, 2007) and Mathematics (Campbell, Watson, & Collis, 1992; Lake, 1999; Li & Goos, 2013; Lian & Yew, 2012b; Pegg, 1992a; Pegg &

Davey, 1989; Reading, 2002; Serow, 2007). Through qualifying the student's responses, the SOLO model enables teachers to gauge learning in order to effectively understand whether students have achieved deep understanding. Together, the SOLO model and the van Hiele Theory and its Teaching Phases provide the necessary frameworks to support this study.

For clarity, the chapter is divided into three main sections. The first section discusses the van Hiele Theory and Teaching Phases as a theoretical framework, with the Teaching Phases providing a pedagogical framework for sequencing student activities and lessons. The second section discusses the SOLO model – also known as the SOLO taxonomy. The SOLO model provides the pedagogical tool and framework to assist in qualifying student responses. The final section is the conclusion, which links the frameworks and identifies the research questions for this study.

3.2. The van Hiele Theory

The van Hiele Theory concentrates on stages of learning rather than developmental stages and places great importance on the role of language to assist with the development of student understanding. The foundation of the van Hiele Theory is a hierarchy of five levels of thinking that describe growth in student understanding. The five Teaching Phases assist the teacher in designing activities that support students to move through the levels.

This section is divided into four subsections: the first subsection provides an outline of the history of the theory; the second and third sections consider the levels of thinking and the Teaching Phases; and the final subsection provides an overview of van Hiele Theory.

3.2.1. Outline

The van Hiele Theory emerged from the companion doctoral work of Dutch husband and wife team, Pierre van Hiele and Dina van Hiele-Geldof, at the University of Utrecht in 1957. Pierre's work focussed on identifying why students had difficulties learning Geometry. This work lead to the development of his theory on the levels of thinking students pass through to attain understanding when learning concepts and ideas in geometry. Dina's work focussed on the sequential Teaching Phases, otherwise known as instructional experiences, and the teacher's role in these experiences that assist a student to pass through these levels. Her work resulted in the establishment of a phase-based approach that can be used for lesson planning.

The van Hieles believed that it is the quality and method of instruction that assists students to attain higher levels of learning not age or maturation, as proposed previously by Piaget. The van Hiele's view is supported by Pegg and Davey (1998), "it is the nature and quality of the experience in the teaching/learning program that influences a genuine advancement from a lower to a higher level" (p. 111). The work of the van Hieles provided mathematical educators with a tool that not only identified students' current level of understanding in Geometry but also provided a teaching sequence to help students move from one level to the next.

Unfortunately, Dina died soon after completing her dissertation and it was left to Pierre to continue with improving and advancing the theory. Primarily, his work focussed on Geometry. However, more recent studies have demonstrated that the levels and Teaching Phases may be applied to other topics in Mathematics, as noted in the introduction, and as will be shown in this research study, Linear Relationships.

While the two fundamental aspects of the van Hiele Theory are the levels of thinking and the five Teaching Phases, in his book, *Structure and Insight*, van Hiele (1986) introduces his theory through only one simple idea: structure. He provides no formal definition of the structure construct, only specifying two distinct types; leaving the reader to develop their own interpretation from the examples provided. This empowers the reader to develop their own understanding of structure, thus having ownership of their definition. This idea of having ownership of a definition or of one's learning, is something van Hiele advocates throughout his theory.

The two types of structure van Hiele identifies are: "feeble" and "rigid" (van Hiele, 1986, p. 19). Rigid structures are those most commonly associated with Mathematics because they follow a specific rule or pattern that can be readily extended. As the word, feeble, implies, this structure type is weaker and may appear to have no rule, so that attempts to extend the structure are met with hesitation. One example is a sequence, such as 2, 3, 5, 8, that is neither arithmetic nor geometric but can be extended in different ways depending on how the structure is identified. If recognised as the Fibonacci sequence, extending it would involve adding the preceding two numbers to obtain the next term. However, it could also be extended by adding the difference between preceding numbers to obtain the next term, such that it becomes 2, 3, 5, 8, 12, 17.

The ability to extend a structure is an important element that assists towards defining the structure. Van Hiele maintains that it is important to permit and encourage students to define their own structures, as it assists in their understanding of the structure (van Hiele, 1986); every teacher's aim should be to provide students with the appropriate tools in order to visualise the structure within a problem. The issue that presents, naturally, is that the structure envisaged by the students may differ remarkably from that expected by the teacher. This uncovers another important element of the theory: that of the role of the teacher in the learning process. The teacher's role is to assist, support, monitor and guide the students, providing them with the tools that enable them to explore and investigate with the intention that they can see the structure for themselves. This empowers the students with ownership of the structure. The concept of structure provides the foundation for the van Hiele Theory on the levels of thinking.

3.2.2. Levels of Thinking

According to the van Hiele Theory there are five hierarchical levels and, more recently identified, two transitional levels of thinking that provide a window for viewing students' development and understanding of concepts. The levels are numbered from one through to five, as illustrated in Figure 3.1, where a brief description has been provided for each level. Although the levels have often been related to specific content, in particular to Geometry, "the levels are situated not in the subject matter, but in the thinking of man" (van Hiele, 1986, p. 41), and can be thought of as descriptors for stages of cognitive development. However, Pegg and Davey (1998) suggest that the van Hiele Theory may be considered a pedagogical framework since many (or most) of the issues affecting students' learning originate from the quality of teaching methods, rather than cognitive processes. Thus, the van Hiele Theory may be used to improve teaching practice by providing not only a lens for teachers to view their student's cognitive development but also to provide a structure with which to monitor the design of their teaching activities.

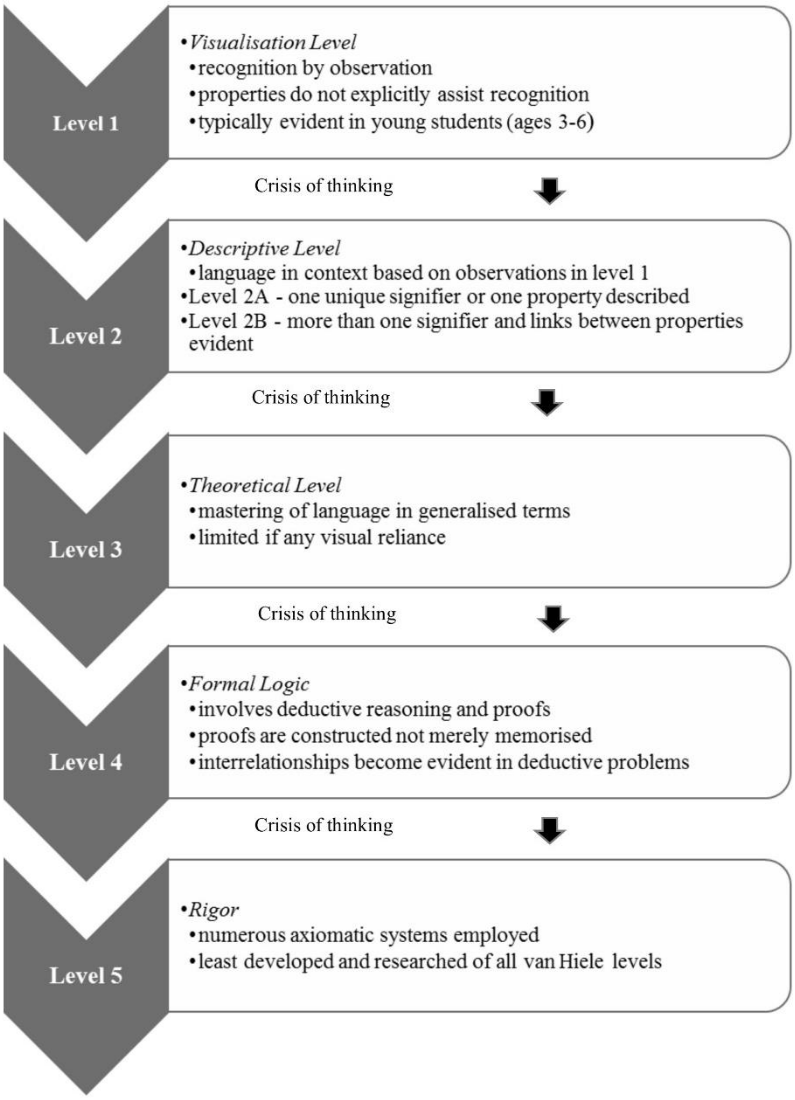


Figure 3.1: Van Hiele levels of thinking

The first level, known as the "visual level" (van Hiele, 1986, p. 53), indicates thinking such that a structure is observed without any attempt to understand it. The structure may be feeble or rigid, however, intricate details are not articulated and other structures are compared purely from a visual perspective; for example the recognition of a particular shape or figure in young students. The structure is identified by its appearance, with properties playing no explicit role in the recognition of the figure. This level is easily recognised in students between the ages of 3 years to 6 years, as they tend to rely on their senses, using visual stimulus to identify shapes, words and numbers.

Within level two, known as the "descriptive level" (van Hiele, 1986, p. 53), mathematical connections are made using language in context. In this level, difficulties often present as a result of the interpretation of the structure observed in level one, with the context of the language used being based on what was visualised in the prior level. As mentioned previously, the sequence 2, 3, 5, 8, can be extended in two ways, with each description differing as a result of how the structure was originally perceived. For Geometry, Pegg (1995) identified two sub levels within level two, labelled as level 2A and level 2B (Pegg, 1995, 1997; Serow, 2002). The difference between these sub levels relates to the number of properties used by the student when describing the figure. With Level 2A, classifications would be based on one visual property or unique signifier to describe a grouping. For example, students could identify equilateral triangles as simply having three sides equal. With Level 2B, classifications would be based on more than one visual property or unique signifier to describe a grouping, often linking the visual cues to support such grouping. For example, students relate squares and rectangles based on their both having four right-angles and opposite sides parallel but recognising that they differ because rectangles having opposites sides equal (Serow et al., 2014).

The next level, level three, known as the "theoretical level" (van Hiele, 1986, p. 53), is where the reliance on visual cues diminishes and more abstract language develops. It is at this level that logical implications and interrelationships from previous levels become evident, leading to the creation of meaningful definitions using generalised terminology. Students are able to provide informal arguments to justify reasoning, however confusion exists between the role of axiom and proof. Secondary students are commonly associated with the thinking of levels two and three.

Levels higher than level three become more difficult to define because of increasing complexity of abstraction (van Hiele, 1986). Level four, known as the level of "formal logic" (van Hiele, 1986, p. 53), involves "comparing, transposing and operation with relations" (van Hiele, 1986, p. 44). Students reason formally within a mathematical context and are able to not only use proofs, such as congruence, but also construct them (Pegg & Davey, 1998). The place of deduction is understood and students thinking at this level are also able to distinguish between necessary and sufficient conditions for developing proofs as well as determining the difference between a statement and its converse (Crowley, 1987).

Van Hiele states that level five, known as rigor or the "nature of logic laws" (van Hiele, 1986, p. 53), is overvalued since it is a highly theoretical abstraction. At this level a student can work within different axiomatic systems.

3.2.2.1 Crisis of Thinking

According to van Hiele, the transition between levels is not considered a natural or simple process, with different students making the transition at different times depending on their abilities and understanding (van Hiele, 1986). Transitioning to a higher level, is a sequential progression that can only happen if thinking on the previous level has occurred. Van Hiele identified that for a successful transition between levels, a “crisis of thinking” (van Hiele, 1986, p. 43) must occur. This involves the student challenging themselves in order to further their thinking onto the next level. Serow (2002) states that a crisis of thinking involves “the reorganisation of mental structures, which were necessary for one level, to take on a different form” (p. 15).

The transition between levels that is necessary for the development of students thinking is influenced through an effective teaching/learning program. Van Hiele listed one of the teachers’ obligations as providing the student with “appropriate subject matter to a thinking crisis at the right moment” (van Hiele, 1986, p. 44). Although a difficult hurdle to overcome, it is important that the crisis of thinking be performed by the students with the teacher illustrating how the “necessary crises of thinking can be initiated and how the pupil can be involved not to avoid it, but on the contrary to surmount it” (van Hiele, 1986, p. 44), and students must not “be forced to think at a higher level” (Pegg, 1992b, p. 22). The five Teaching Phases, that form part of the van Hiele Theory, assist the teacher to structure and prepare the teaching/learning program, ensuring that the transformation required to reach the next level is achieved.

3.2.3. Teaching Phases

Dina van Hiele-Geldof’s work suggested that the progression through levels was largely based on instruction rather than age or biological maturation; the instructional activities and experiences employed being instrumental in a successful progression between levels. The five Teaching Phases represent a framework for teaching and learning with which activities can be sequenced to facilitate students’ cognitive development for the transition between levels. The phases and the van Hiele Theory acknowledge the importance of teachers and their role in guiding students’ learning process; the opportunity for discussion, monitored by the teacher, to continue with assisting in the student learning process and the gradual advancement of more technical language. Transition between levels unassisted is difficult for students, as van Hiele (1986) acknowledged: “help from other people is necessary for so many learning processes” (p. 181), the teacher being a key component towards successful transitioning.

The Teaching Phases are summarised in Figure 3.2, and the following discussion provides further detail concerning each phase and suggestions of possible activities for the phase.

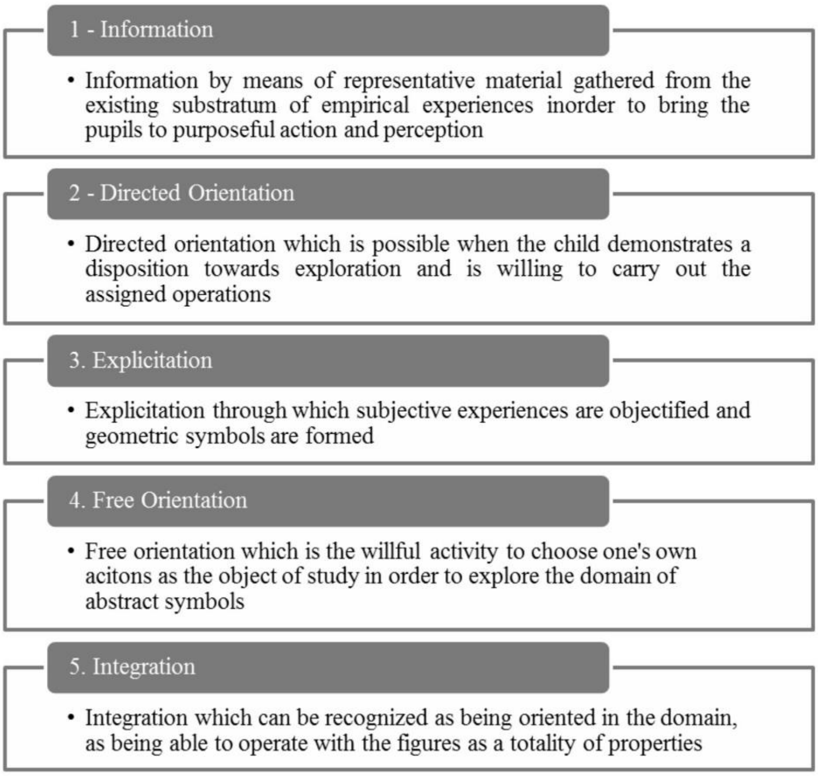


Figure 3.2: Description of van Hiele Teaching Phases
(van Hiele in Fuys et al., 1984, p. 223)

The purpose of the first phase is that of giving of “Information” (van Hiele, 1986, p. 96) to the students. This usually involves discussion with students, introducing terms and vocabulary within the context of the topic. Activities within this phase enable students to become familiar with the working domain through discussion and exploration. Discussions between the teacher and students stress the prior knowledge and content required for the main component of the lesson, either through direct questioning or brainstorming.

During the second phase, “Directed Orientation” (van Hiele, 1986, p. 97), the totality of the structure is visualised and students commence investigating the connections through a series of teacher guided tasks. The terms exploration and direction have also been used throughout the lesson plans to identify this phase. The teacher is crucial in directing activities, gradually ensuring students recognise and establish the correct relationships. It is in this phase that students initiate a process of thinking, carefully monitored by the teacher. Working through a series of teacher guided activities, students have the opportunity to express their views with the aim of identifying the concept being studied. Activities for this phase often take a practical approach, if possible, to assist students' understanding through a process of observation.

The third phase of the learning process, "Explicitation" (van Hiele, 1986, p. 97), involves the students openly discussing what they have explored in the previous second phase, "one could call this objectification of the subjective experience" (Fuys, Geddes, & Tischler, 1984, p. 219). Again, the teacher is instrumental in aiding students through this phase, monitoring conversations to ensure the desired meanings are expressed and guaranteeing the connections established in the second phase are sustained. In this phase, the teacher introduces the necessary technical terms, as students attempt to express their structure within the problem they are exploring (Fuys et al., 1984). Full class participation is advantageous for this phase because open discussion promotes the exchange of ideas, with care taken to explicitly develop language.

Time is an important factor when assisting the development of language and understanding. The ability of the students to grasp language that correctly expresses their ideas or explains the structure within the problem they are exploring may require extended periods of time to nurture and acquire. This presents problems that are beyond the scope of this study but are worthy of mention.

During the fourth phase, "Free Orientation" (van Hiele, 1986, p. 97), students complete activities where they are required to find their own way in the network of relations. They are familiar with the subject content, recognising the symbols that represent the relationships and are ready to explore further, recognising cues to assist them in combining all the information they know in order to solve the problem. The teacher's role diminishes slightly, as "the teacher appeals to the inventive ability of his pupils" (Fuys et al., 1984, p. 221), although "there is not yet a real problem setting" (Fuys et al., 1984, p. 221), as students follow an order of instructions to attain the intended result. The problems chosen for phase four are "not simply 'hard' questions they are questions in which multi-path solutions are possible" (Pegg, 1995, p. 99). Students' language continues to develop in complexity because they are able to distinguish the relationships and connections from the three phases they have already progressed through.

The final phase in the learning process, “Integration” (van Hiele, 1986, p. 177), involves students reflecting on their accomplishments as they build an overview of their investigations. The purpose of the instruction is now clear to the students and it is during this reflection that they summarise any patterns or rules they have discovered to aid them in further study. The teacher continues to assist in ensuring nothing new is presented to the student, only summaries of what they know.

It is interesting to note that summarisation and memorisation of rules and patterns investigated occurs in the last phase and not in the first phase. Hence, traditional methods of presenting rules and demonstrating application is not considered to be conducive of developing mathematical understanding. Students cannot attain understanding of a structure if they have not acquired the language in context, in order to be able to explore and investigate its properties. This offers some justification as to why concepts are revised year after year as students have not achieved an in-depth understanding initially to provide a foundation for long-term understanding. Students must have ownership of the structure as a basis for understanding and this is achieved by developing their own methods of remembering the structure (van Hiele, 1986).

The Teaching Phases assist in explaining the progression from one level to the next. While the phases are numbered, they are not always passed through sequentially from phase one to phase five. Spiralling of phases may occur, in particular, throughout a teaching sequence prior to phase four, where students are able to find their own way. Often students may spiral through phases two and three while developing and consolidating their understanding of a concept. However, skipping of phases does not occur and students will pass through each phase at some stage throughout the teaching process.

In summary, the Teaching Phases encourage teachers to examine their methods when assisting students through the levels of thinking. By adjusting examples, providing detailed information, permitting time for exploration and encouraging students to discuss their investigations to promote the development of correct language, students will progress to the next level of thinking.

3.2.3.1 Level Reduction

Level reduction involves the transformation of higher-level structures to lower-level structures. The concept of level reduction can have a positive or negative influence on the learning and understanding of mathematics depending on the person initiating the reduction.

Level reduction initiated by the teacher, rather than the learner, defines the negative aspect of level reduction. For various reasons, teachers create shortcuts and tricks or commit students to memorize procedures and methods that assist students to proceed to the next level and thus escape the challenge of a "crisis of thinking" (van Hiele, 1986, p. 43). While students may be able to reach the desired solution, they will, however, have little conceptual understanding, leaving them unequipped should the orientation or presentation of the problem be unfamiliar. Van Hiele (1986) identifies this as an important concern; namely, how to encourage students such that they do not avoid a crisis of thinking but overcome it.

Level reduction predominantly occurs between the second and third levels. This is due to the change in language used within the "descriptive level" to the "theoretical level" (van Hiele, 1986, p. 53). Working within the visual and descriptive levels of level one and two respectively, students can openly discuss the structure using simple, familiar terms. Advancing to the theoretical level, students are required to use more abstract language, and it is at this point that the teacher instils confidence through the introduction of a little trick. The student feels they have understood the problem and continues using the trick without any knowledge as to why, other than it produces a correct solution. Angles associated with parallel lines provides a typical example of level reduction, where teachers explicitly demonstrate to students the letters Z, C and F as a shortcut to remembering alternate, co-interior and corresponding angles respectively (van Hiele, 1986). The students have no ownership of the content being addressed and hence are set up for failure every time a related problem contains a variation of the processes applied.

Level reduction takes on a positive meaning when initiated by the students themselves. It supports their understanding of the structure since the students have developed the simplification themselves. This usually occurs through discussion of the structure in level three, where the students visualise the network of relations using appropriate language, then through, in finding their own rules, they develop methods of simplification, thus performing their own form of level reduction. Van Hiele recognised that, when students take ownership of their own ideas, despite having developed some simplification in order to understand the structure, they will have more confident recall later on (van Hiele, 1986). Discussion becomes a crucial component of level reduction and, in fact, all levels, because it enables ideas to not only be shared but also to be open for analysis by teacher intervention. For example, through exploring, investigating and discussing the angles associated with parallel lines, students can develop a method for distinguishing the various angles themselves. Through observations, recognition of the patterns the angles form as letters of the alphabet are established, that can be used to identify the angles. Although, encouraging students to embrace exploration and investigation often requires more class time than traditional explicit instruction since not all students progress through the levels of thinking simultaneously (Pegg, 1992b; van Hiele, 1986).

3.2.3.2 Insight

The main purpose of instruction, according to the van Hieles, was for the development of insight. Insight existing "when a person acts in a new situation adequately and with intention" (Van Hiele 1957 quoted in van Hiele, 1986, p. 24), such as when a student applies an appropriate response to a new and unfamiliar problem to those previously encountered. It is an invaluable ability that students should be encouraged to develop.

Specifically identifying if and when a student has developed or applied insight is difficult. It is important for the teacher to determine whether processes used by students are a memorisation of procedures from applying someone else's insight or a true indication of insight by the student themselves (van Hiele, 1986). The assessment of insight is an ongoing process that best occurs during lessons, through formative assessment methods such as teacher observation and questioning, now known as Assessment *for* Learning (AFL), rather than a summative assessment task, now known as Assessment *of* Learning. This enables the students to be able to experience new situations without the pressure of time constraints and stress of examinations, and the teacher can witness what processes are used to solve the problem.

3.2.2.1. Development of Language

Language plays an important role in teaching and is crucial in all the van Hiele levels of thinking and in moving through the levels (Pegg, 1995). Through discussion, students are able to articulate thoughts, enabling teachers to monitor the context of the language. Each level of thinking contains language specific to that level, with "its own linguistic symbols and its own network of relationships connecting those symbols" (Usiskin, 1982, p. 5). While certain terms may be used across different levels, the context is dependent on the level of thinking. Through the use of appropriate language, structures can be explained and extended, and vocabulary is built on through the progression of levels, “the heart of the idea of levels of thought lies in the statement that in each scientific discipline, it is possible to think and to reason at different levels, and that this reasoning calls for different languages” (van Hiele, 1959, p. 65).

3.2.4. Overview

The van Hiele Theory aims to improve teaching through the organisation of instructional activities based on a hierarchical series of levels that describe students thinking. The theory suggests that addressing a student’s levels of thinking during the teaching sequence enables a student to have ownership of the content, leading him or her to the development of insight, which, for the van Hieles, was the main reason of instruction (Pegg & Davey, 1998).

3.3. The SOLO model

This section considers the SOLO model as a useful pedagogical structure for this study. The SOLO model of Biggs and Collis (1982), provides a qualitative approach for assessing student understanding of content through analysing the nature of their responses. The foundation of the SOLO model is a hierarchy of modes and levels that categorise the complexity of the learning response. Theoretically, the SOLO model can be applied to any learning context, and has been documented with Science (Martin, 2011; Minogue & Jones, 2009), Design and Technology (Leung, 2000), English (McNeill & Hook, 2012) and Geography (Munowenyu, 2007). More recently, it has been associated with promoting and assisting the AFL strategy (Panizzon, et al., 2007). For Mathematics, in particular, the model has been applied to Numeracy in Biological Science (Lake, 1999), Patterns and Relationships (Lian & Yew, 2012b), Correlation Graphing (Li & Goos, 2013), Statistics (Pegg, 1992a; Reading, 2002), Volume and Measurement (Campbell, et al., 1992), Algebra (Lian & Yew, 2012a; Pegg, 1992a) and Geometry (Pegg, 1992a; Pegg & Davey, 1989; Serow, 2007). The SOLO model continues to be a useful tool that assists and enlightens curriculum and teaching decisions.

This section is divided into four subsections, which provide a brief description of the model. The first subsection provides an outline of the model and its history. The second and third sections consider the five modes of functioning and the levels of thinking within each mode, with the final subsection providing an overview of the SOLO model.

3.3.1. Outline

The SOLO model is a categorisation structure that assesses the quality of students' responses. Developed by Australian duo, John Biggs and Kevin Collis (1982), it identifies the sequence of learning through evaluating the responses provided by students for tasks rather than their level of thinking or stages of development. It offers a systematic, hierarchical method that gauges how students' understanding develops both in complexity and level of abstraction when mastering tasks, through observing the quality of their responses. The SOLO model is predominantly concerned "with specifying how well (qualitative) rather than how much (quantitative) has been learned" (Panizzon & Pegg, 2008), and provides a language with which to describe this quality of knowledge (Pegg, 1992a).

The SOLO model evolved from Biggs and Collis's dissatisfaction with some of the ideas of Piaget (Biggs & Collis, 1982) and his stage theory of developmental learning. Their aim was to provide a structure that rivalled his work, improving on important issues they felt were unresolved. Piaget's theory, based on discrete stages of biological cognitive development, was governed by specific student performances, where the progression through each stage was based on maturity rather than understanding. Therefore, once a student advanced to the next stage, returning to a former stage was not possible or attainable. Piagetian theory did not accommodate for varying changes in students understanding, and presumed a student would continually progress forwards and upwards. In contrast, Biggs and Collis recognised that, in reality, students may demonstrate thinking and understanding characteristic to a number of varying stages depending on the task, and performance was not always consistent. While Piaget identified this as an issue, to the point of even naming it "decalage" (Biggs & Collis, 1982, p. 20), it was not resolved within the framework of his theory.

Biggs and Collis found that by changing their frame of reference from the developmental stage to learning quality of the student, the issue of varying stages was resolved. Simply "shifting the label from the *student* to his *response* to a particular task" (Biggs & Collis, 1982, p. 21) removed the classification associated with Piagetian theory based on age and described their performance based on the response given at that particular time. This acknowledged influences affecting performance, such as motivation and prior learning experiences, that can impact student function at a particular time but not necessarily be an overall determinant of developmental stages.

The SOLO model resulted from an analysis of student responses from a wide range of subject and topic areas, which noted that their cognitive development or level of thinking was not always linked to their respective maturation. The value of the SOLO model rests with its ability to "identify in broad terms the stage at which the student is currently operating" (Hattie, Biggs, & Purdie, 1996) during the course of their learning. In particular, offering a language that describes the quality of students' responses in terms of "structural characteristics" (Panizzon & Pegg, 2008) that categorises and provides a consistent structure with which to compare students' responses at various stages of conceptual understanding. "Understanding and applying the SOLO model was seen as both a catalyst for action and a framework to guide teacher's thinking" (Sriraman & English, 2009, p. 185).

It is the structure for analysing the student responses that makes the SOLO model appeal to this study. Through recording the nature and quality of the responses to identify how they change over time, the SOLO model provides a structure for analysing the student's understanding of the stimulus items. Hence, it is envisaged that a detailed description of the development of students' descriptions of Linear Relationships is possible.

The SOLO model is context dependent and bases itself on two forms of hierarchy to detail the quality of the response: the mode of functioning, which distinguishes the degree of abstraction of understanding; and the level of response, which distinguishes the structure and complexity of the response (Biggs & Collis, 1982). Within some modes, multi-modal functioning has been identified, with at least two cycles of levels occurring. All of these are addressed below.

3.3.2. Modes

The SOLO model categorises learning into one of five modes of cognitive functioning, or a combination of modes. These modes relate to the level of abstraction demonstrated by the student or responses given by the student with respect to a particular task. The modes form a hierarchy that become sequentially available to all individuals from birth. The five modes of functioning are:

Sensori-motor
This response involves actions coordinated in a physical sense. It is associated with the motor responses given to a sensory stimulus and may be described as “tacit” (Biggs & Collis, 1989, p. 156) knowledge. Examples include performance in sport and learning in infancy.

Ikonic
This response involves internalisation of actions by forming mental images or icons, and developing language relating to the images. It represents the kind of knowledge that may be perceived or felt; knowing a solution before being able to clarify it symbolically (Collis, Biggs, & Rowe, 1991). It is associated with a pre-symbolic mode of information processing and may be described as “intuitive” (Biggs & Collis, 1989, p. 156) knowledge. Examples include children explaining reality and images using words and adults’ thoughts in aesthetic criticism.

Concrete symbolic
This response involves a significant shift in abstraction from direct imagery of reality to a higher-order symbolisation of reality. The application of a system of symbols, such as written language and number problems, are applied to concrete real world experiences. Such responses indicate logic and order between the symbol system itself and the world it represents. Primary and secondary schooling predominantly operates within the concrete symbolic mode. According to Biggs and Collis (1989), the mastery of symbolisation, such as words and numbers, and the ability to adequately apply them to real life problems is the major task of primary and secondary schooling. Responses in this mode may be described as “declarative” (Biggs & Collis, 1989, p. 156) knowledge.

Formal
This response involves examination of abstract concepts because relating to the real world referent is no longer required. It is associated with questioning rather than blind acceptance and leads to the formation of hypotheses and generalisations concerning how things are. It may be described as “theoretical” (Biggs & Collis, 1989, p. 156) knowledge. Thinking at the formal mode has been considered, by some, as an essential element for studying at a university level (Biggs & Collis, 1989).

Post formal
This response involves the maximum level of abstraction where one questions and challenges conventional theory and practice of the formal mode. It is associated with research and professional practice where further exploration of discipline establishes new

theory and/or practice. The existence of this mode is often disputed, but is expected to appear at postgraduate level.

The modes, in large part, align to the developmental stages of Piagetian Theory (Biggs & Collis, 1982), although there are fundamental differences. Most obviously, there are five SOLO modes while Piaget describes four cognitive development stages: sensori-motor (birth to two years); intuitive/pre-operational (two to six years); concrete operational (seven to fifteen years); and, formal operational (sixteen plus years). Piaget's cognitive development stages contain different core assumptions to the SOLO modes. In particular, the SOLO modes do not successively replace each other in a logical structure, like a developmental pathway continuum, as do the stages of Piaget's model, but add successively and coexist with the predecessor mode (Biggs & Collis, 1989). That is, the "modes accrue from birth to maturity" (Collis, et al., 1991, p. 61). Each earlier mode provides support for later acquired modes, and, depending on the task, students can demonstrate understanding of a previous mode. Piaget labelled this as "decalage" (Biggs & Collis, 1982, p. 20), but considered it as aberrant and too rare to warrant it being resolved in his theory. In contrast, Biggs and Collis, considered decalage to be extremely common in an educational context (Biggs & Collis, 1982). They acknowledged that a "student can be 'early formal' in mathematics and 'early concrete' in history, or even formal in mathematics one day and concrete the next. Such observations cannot indicate shifts in cognitive development, but rather shifts in more proximal constructs such as learning, performance or motivation" (Collis, et al., 1991, p. 60). Each mode contains distinct characteristics that result in unique individualities. Unlike Piaget's theory with discrete logical structure, the ages associated with the modes are only a broad indication of when to expect the emergence of that mode of thinking and may differ depending on the individual child and task. An outline of the five modes of thinking appears in Figure 3.3.

In summary, the five modes of the SOLO model: sensori-motor, ikonic, concrete symbolic, formal and post-formal, represent the level of abstraction provided by the response of an individual to a particular task. Despite appearing similar to Piaget's developmental stages, there are major differences. In particular, newly acquired SOLO modes are not presumed to be static and do not replace previous modes, instead they coexist and provide support for the development of the new mode. The SOLO model "makes learning directly measurable, which makes it a valuable evaluative and instructional tool" (Wells, 2015, p. 37). An extension of the original 1982 model, known as multi-modal functioning, is discussed below.

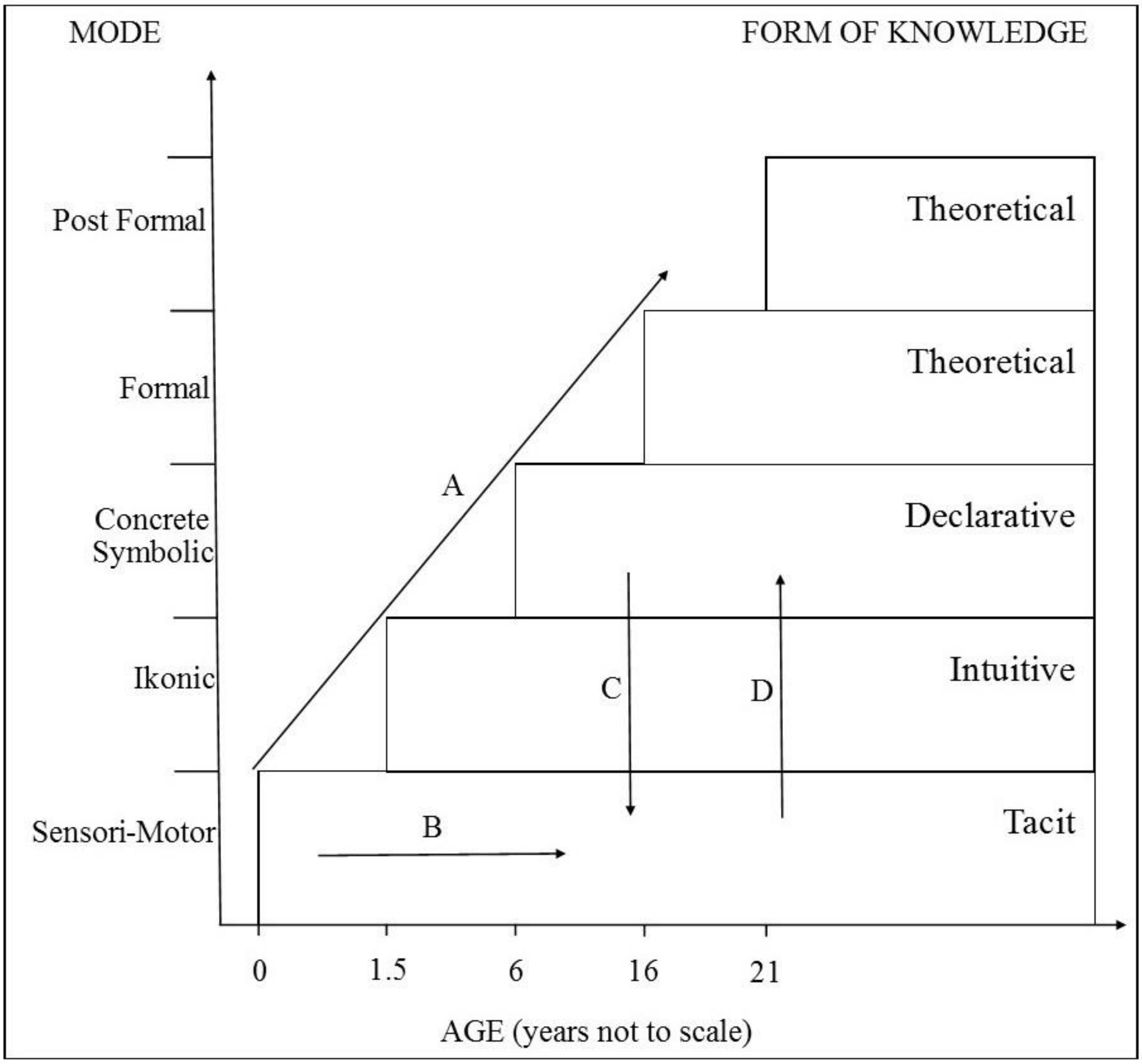


Figure 3.3: Five modes of thinking
(adapted from Collis, et al., 1991)

Multi-modal functioning

The SOLO model acknowledges that individuals operate at different modes depending on the individual and problem type. Often, the preferred mode of functioning changes and the ability to form a cycle of modes rather than a distinct progression through the modes occurs. This ability is known as multi-modal functioning. Collis and Biggs (1991) identified four different paths of development within the SOLO model, as demonstrated in Figure 3.3. The first of these paths is indicated by the diagonal line, A, and represents the "course of optimal development" (Collis, et al., 1991); this represents the development theorised by stage theorists such as Piaget. Learning is characterised by the sequential progression from one mode to the next, with each mode subsuming the previous one. The horizontal line B represents the simplest path of learning, uni-modal learning, all of which occurs within only one mode. The vertical lines, C and D, represent the "top down facilitation of lower level learning" (Collis, et al., 1991, p. 70) and "bottom up facilitation of high level learning" (Biggs & Collis, 1989, p. 71), respectively. Top down learning involves the application of a higher-order mode to develop the learning in an earlier mode, such as adults learning a sensori-motor act through the explanation at a higher level. For example, an adult learning to swim may consider the aspects of water resistance in order to improve performance. In this case, the formal mode is used to support the learning of the sensori-mode activity. In contrast, bottom up learning involves the application of a lower order mode to support the development of learning of a higher mode, as is used in progressive education where experimentation and discovery methods are used for learning. For example, using different coloured beads to represent elements when teaching molecular structure in chemistry demonstrates concrete symbolic objects being used to support the learning of an abstract concept in formal mode.

In summary, each individual operates at a different mode or cycle of modes to solve problems. Although operating in a preferred mode, the individual is able to access one or more earlier acquired modes to support and assist the learning within the context of the problem to be solved. In this way, the former modes are not subsumed, instead they remain available in the form of multi-modal functioning.

3.3.3. Levels

Within each mode, the quality or sophistication of the response and how it is handled is coded into a series of five different levels, as defined by the SOLO model. Each level is nested or subsumed by its successor. A broad explanation of the levels appears in Figure 3.4. The first and last levels, prestructural and extended abstract, are often considered as not existing within the mode in question. Prestructural responses represent learning that is low for the current mode, thus often considered to belong to the previous mode. Extended abstract, as the name implies, suggests that the level of abstraction goes beyond the current mode into the next mode, thus becoming the first level or unistructural level of the next mode.

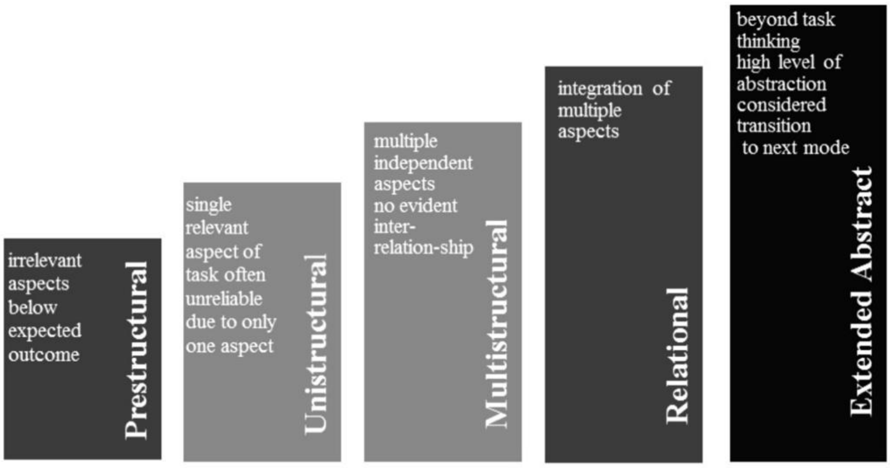


Figure 3.4: SOLO levels

Further descriptions of each level appear below illustrating the possible responses within the concrete symbolic mode that represent school age learning. The diagrams by Atherton (2013), approved by John Biggs, illustrate the concept of a house in terms of the SOLO levels and have been included to further clarify the definitions.

Prestructural

This response is based on irrelevant aspects and it is below what would be expected. It could be the result of the student being frequently distracted or not engaged with the task in the mode involved (Pegg & Davey, 1998). As demonstrated in Figure 3.5 information has no organisation as each section of information is disconnected from other parts as a whole it generally doesn't make any sense.

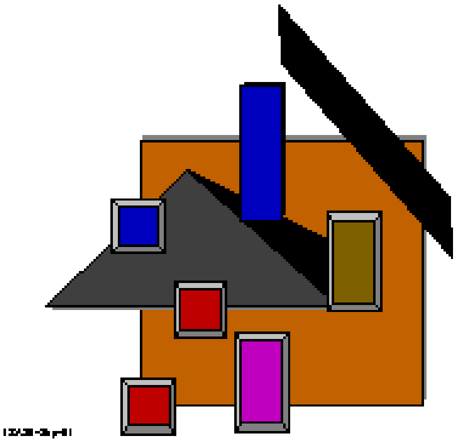


Figure 3.5: Prestructural
(Source: Atherton, 2013)

Unistructural

This response is based on a single relevant aspect of the task and, while the aspect is correct, it may be unreliable and inconsistent with other unistructural responses. Simple connections are made which are obvious to the student but the overall significance is overlooked as shown in Figure 3.6.

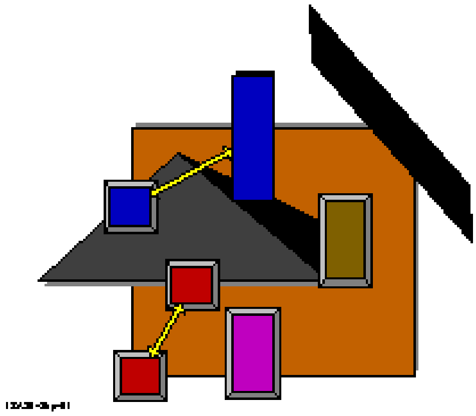


Figure 3.6: Unistructural
(Source: Atherton, 2013)

Multistructural

This response is based on multiple independent aspects of the task although interrelationship between components is not apparent. Individual aspects are correct but no connectedness between components is provided. Figure 3.7, shows the several links achieved but no interrelationship developed.

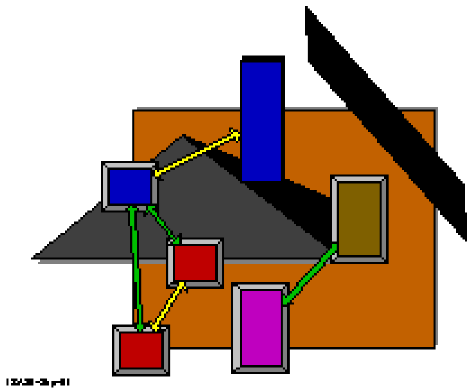


Figure 3.7: Multistructural
(Source: Atherton, 2013)

Relational

This response is based on multiple aspects becoming integrated with clear structure and meaning. The student is able to appreciate the significance of the parts as a whole entity, with a concentrated focus on the relationships amongst the aspects as shown in Figure 3.8. Whilst a correct response is given for the context provided, it is not usually generalisable to other contexts.

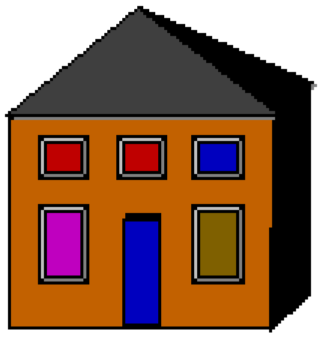


Figure 3.8: Relational
(Source: Atherton, 2013)

Extended abstract

This response goes beyond the thinking of the task and exhibits a higher level of abstraction. The student makes connections to be able to generalise and transfer to other situations as illustrated in Figure 3.9. This type of response may be considered to be a transitioning to a higher mode.

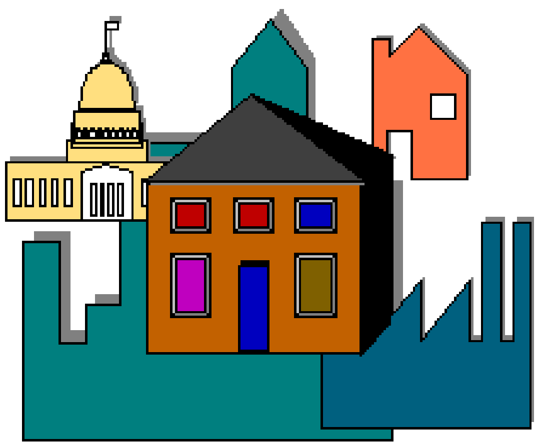


Figure 3.9: Extended abstract
(Source: Atherton, 2013)

In summary, within each mode, there exist five different levels of abstraction that classify the degree of sophistication for the response. The middle three levels represent the level of abstraction within the target mode, whereas the first and last levels suggest a previous mode and transition to a higher mode. The hierarchy of levels – namely, prestructural, unistructural, multistructural, relational and extended abstract – define not only the type of response provided but also what is required to achieve the next mode.

Cycles of levels

The concrete symbolic mode represents the target mode for instruction during secondary school years. Concerns have been raised that a single unistructural (U) – multistructural (M) – relational (R) learning cycle does not do justice to the variety of responses that can be provided for an extensive range of questions within a mode (Levins & Pegg, 1994). Studies have identified the existence of at least two cycles of levels within the concrete symbolic mode (Campbell, et al., 1992; Levins & Pegg, 1994; Pegg & Davey, 1998; Reading, 2002) and the formal mode (Serow, 2007), with the presence of more cycles possible. The cycle of levels associated with the concrete symbolic mode is illustrated in Figure 3.10.

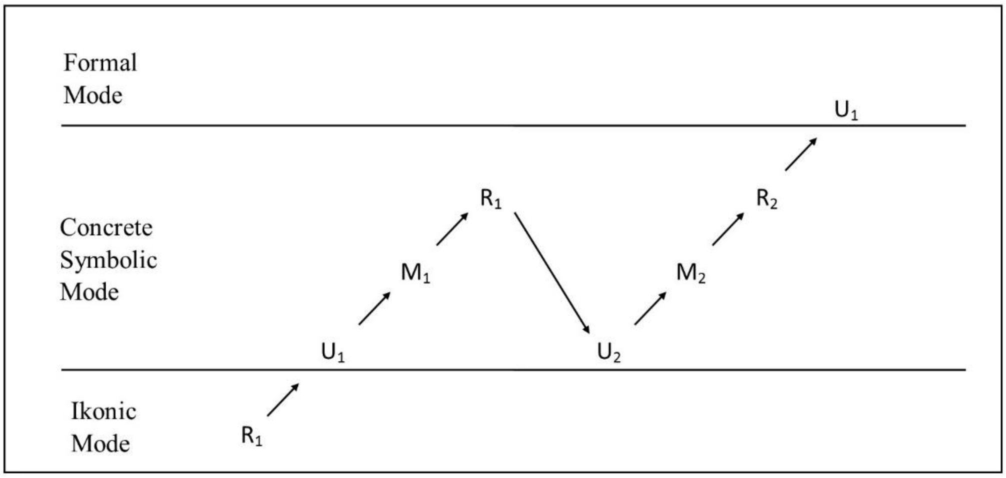


Figure 3.10: Diagrammatic representation of levels associated with the concrete symbolic mode
(Source: Pegg & Tall, 2005, p.470)

The extension of the SOLO model to address at least two cycles signifies an additional level of growth within modes. Pegg identified that for the concrete symbolic mode a "heavy reliance on imagery and visually presented stimuli" (1992a, p. 371) was present with many of the primary and early secondary students involved in his study. While a hierarchy of growth was noted, he found it was best explained through considering two cycles of levels. This enabled "credit" to be given to a broader range of responses within the concrete-symbolic mode, particularly for more complex and involved problems (Pegg & Tall, 2005). In the conclusion of his 1992 paper, Pegg acknowledges that the number of U-M-R cycles was dependent on "how many essential features make up the concept or how detailed the investigations into the growth in understanding need to be" (1992a, p. 383).

A noticeable characteristic of the cycles within modes is that the top level, the relational level response (R), is equivalent to the bottom level, or the unistructural (U) response of the next cycle. Figure 3.10 illustrates this point for two cycles in the concrete symbolic mode. Two other levels of response are missing from Figure 3.10: these are the prestructural level, indicating that the response provided is not meaningful to the question posed; and, extended abstract, indicating that the response provided includes information from outside the question demonstrating a higher level of abstraction. This latter level, when considered with respect to a response in a particular mode, can be equated with the unistructural response of the next mode.

3.3.4. Overview

In summary, there are five levels associated with the modes of the SOLO model: prestructural, unistructural, multistructural, relational and extended abstract. These levels describe the complexity of the structure of the response and provide a hierarchical description of the nature of the response to the stimulus question. The nature of the levels is dependent on the stimulus item of the targeted mode, and, hence, careful consideration is required for the preparation of assessment items. Cycles of levels have been identified that assist with categorising the range of responses within the concrete symbolic mode.

3.4. Conclusion

This chapter commenced with a quote by Pegg and Davey (1998) linking two theoretical and pedagogical frameworks that have been effectively used to examine student understanding for Geometry. These frameworks have enabled teachers to understand where students' learning is at, and how to approach the delivery of content in order to increase students' knowledge. The first framework, the van Hiele Theory, provides teachers with an understanding of how students learn mathematics through the identification of hierarchical levels of thinking and sequential Teaching Phases that assist students to move from one level to the next. Central to these levels and phases, are the development of language, the importance of insight and the role of the teacher within the classroom environment. Although initially predominantly used in Geometry, it has been more recently associated with other strands of mathematics. The van Hiele Teaching Phases, in particular, provide teachers with a tool that assists with the design of lesson activities, promoting students' progression through the levels. The other framework, the SOLO model, assists teachers by providing a useful structure with which to analyse the student's responses. The combination of the van Hiele Theory and Teaching Phases with the SOLO model provides a system such that student learning can be monitored and analysed to investigate and explore the understanding of Linear Relationships when using technology.

Through the considerations of the issues raised in this and the preceding chapter, the theoretical and pedagogical frameworks provided by the van Hiele Theory, and its Teaching Phases, and the SOLO model, are used in the present study to explore three themes concerning students' understandings of Linear Relationships when using dynamic mathematics software GeoGebra.

3.4.1. Research Theme 1

To explore the SOLO model and van Hiele Teaching Phases as frameworks to assist teachers when using technology as a teaching tool.

1.1 How does the van Hiele Teaching Phases offer a framework for designing a lesson sequence incorporating technology as a teaching tool?

1.2 How does the SOLO model offer a framework to explain the categories of responses concerning students' understandings of Linear Relationships?

3.4.2. Research Theme 2

To examine the responses of the Google Form tests in order to gain insight into students' understandings of Linear Relationships.

2.1 Can an analysis of the results offer insights into students' understandings of Linear Relationships?

2.2 Which response categories within the tests had a relatively larger increase in complexity from the prior response category, and how does this increase reflect upon students' growth in understanding Linear Relationships?

3.4.3. Research Theme 3

To investigate students' understandings of Linear Relationships concepts when using dynamic mathematics software GeoGebra.

3.1 What are the characteristics of students' responses when exploring concepts of Linear Relationships using dynamic mathematics software?

3.2 What is the nature of student interaction when using GeoGebra as an exploration tool?

3.3 What are the observed developmental hurdles and technical knowledge issues encountered by students when exploring Linear Relationships concepts utilising dynamic mathematics software?

These three themes have guided each stage of this study into Linear Relationships. The next chapter outlines the research design and methodology implemented to explore these themes.

CHAPTER 4:
RESEARCH METHODOLOGY

4.1. Introduction

This chapter outlines the methodology and considerations that define this research study, detailing the theory behind the approaches and methods employed. For clarity, it has been divided into six major sections. The first section sets the context of the study, stating the geographical setting and outlining the Mathematics courses of the Australian Curriculum and those targeted by the study. The second section outlines the research design and is divided further into two sub-groups: an overview of the design structure, detailing the type of qualitative methods employed, the targeted participants and selection process; and the teaching sequence which describes in detail the lesson plans and data collection structures used. The third section considers methodological issues, such as justification of test design as a research tool for the study. The next section discusses the plan for data analysis and strategies engaged for analysis. This is followed by an evaluation section that presents the strengths and weaknesses of the research design through addressing the issues of validity and reliability. Finally, ethical considerations of the research design are discussed, with the conclusion completing the chapter.

4.2. Setting the context

This section provides a detailed description of the background issues related to the study. It is divided into two sub-groups that detail the background of the chosen sample: the geographical setting; and the secondary Mathematics courses.

4.2.1. Geographical Setting

The sample population who agreed to participate in this project attended a systemic Catholic secondary school in the inland city of Griffith, within the Diocese of Wagga Wagga. The high school is one of three secondary schools in the small rural city of 24,000 (including surrounding areas), with the other two being public government high schools. Griffith is renowned for its Italian heritage, although in recent years its population has diversified to be a multi-cultural kaleidoscope that boasts a significant Sikh Indian community. Its socioeconomic status is predominantly working class, with 60 per cent of employed people over the age of 15 hired as labourers, trades workers, sales workers, machinery operators and administrative workers (Australian Bureau of Statistics, 2011).

4.2.2. Secondary Mathematics Courses

This study aligns with the Australian Curriculum, which, at the time of data collection – 2013, was to be implemented the following year for target participants of Year 9 (Stage 5 approximately 14–16 years old). Despite being on the political agenda for several decades, the Rudd Government was successful in commencing the development of a national curriculum in 2008. They established the National Curriculum Board with the idea of creating a world-class curriculum for all states and territories, containing a continuum of learning encompassing Kindergarten to Year 12 (approximately 5–18 years old). The development process was overseen by the newly appointed Australian Curriculum, Assessment and Reporting Authority (ACARA), "an independent statutory authority that aims to improve the education outcomes of all young Australians" (ACARA, 2013). ACARA continues to be responsible for the overall management of the national curriculum.

The basic structure of Australian Curriculum remains the same as the previous NSW Mathematics Syllabus 7–12 with three stages as illustrated in Figure 4.1 (NSW Board of Studies, 2012f). The Stage 6 syllabi continue to be in consultation at the time of writing; however, the outline of the suggested structure has been incorporated into Figure 4.1.

Acronyms and terms used in the figure relate to the categorisation of courses by the NSW Board of Studies. Board Developed Courses (BDC) are subjects that have a syllabus written by the Board of Studies syllabus committees and may be used to count towards an Australian Tertiary Admissions Rank (ATAR). Each BDC is examined externally at the end of the HSC course. Content Endorsed Courses (CEC) are subjects that are written by the teachers within the school into which the subjects are introduced or by other people, and which are then accredited by the Board of Studies becoming known as Board Endorsed Courses (BEC). While these subjects may be used to count towards HSC accreditation, they are not included in ATAR calculations and are not examined externally at the end of the HSC course.

The Life Skills course provides a program of study suitable for students with special educational needs, such as an intellectual disability. For students attempting the Life Skills courses, specific outcomes are selected that are achievable for the individual student from any stage within the continuum of learning. These outcomes may be completed independently, with adjustments or with support. Permission from the NSW Board of Studies is not required to access the Life Skills outcomes and content for a particular student, and planning documentation is not required. Consultation regarding whether or not a student requires a Life Skills course and what outcomes are selected is best decided from a collaboration of parents or guardians, teacher and special education teachers or coordinators.

Extension Mathematics courses are higher-level courses designed for students with a special interest or aptitude for Mathematics. Two extension courses are offered for Mathematics by the Board of Studies NSW: Extension 1 and Extension 2. To undertake the Extension 1 course students must have completed or be studying concurrently, Mathematics “2 unit”. Students wishing to study Extension 2 Mathematics have an understanding of the major concepts of Mathematics “2 unit” and hence drop it to study Extension 1 and Extension 2 concurrently.

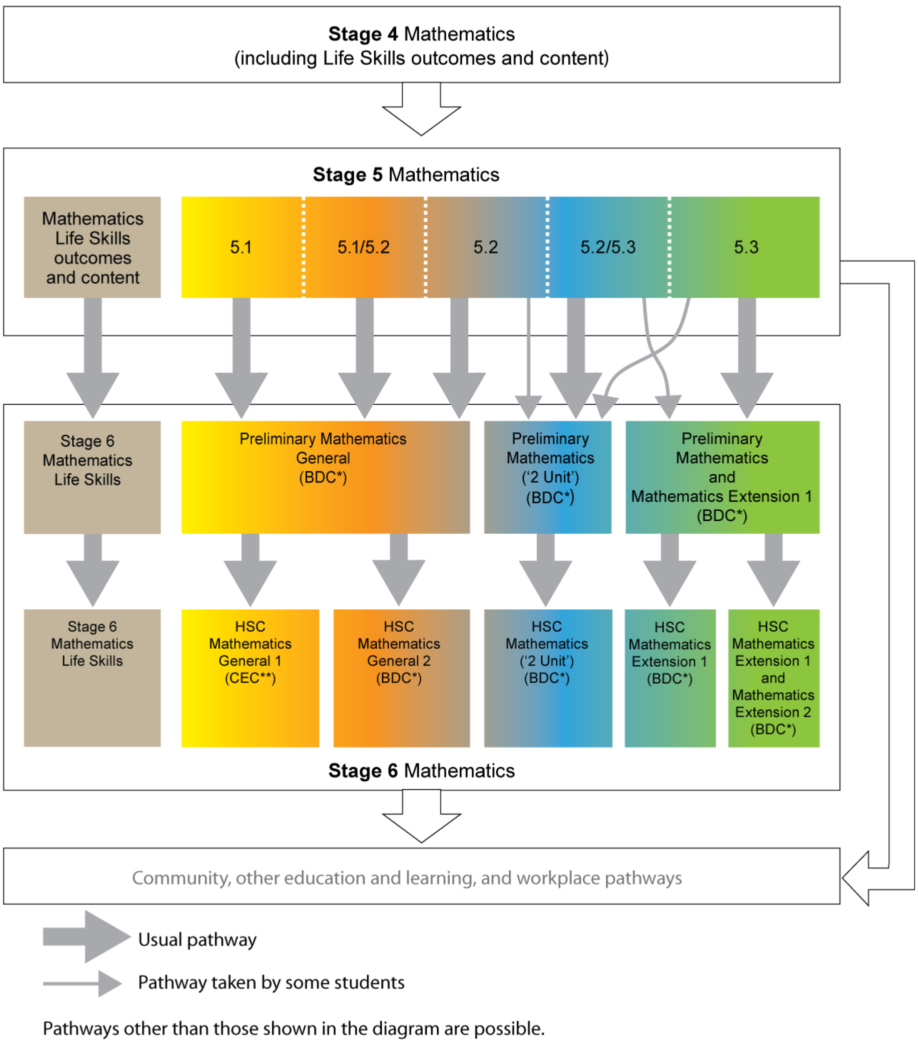


Figure 4.1: Place of the Syllabus

(NSW Board of Studies, 2012f)

For the Australian Curriculum, Mathematics is divided into three content strands: Number and Algebra; Measurement and Geometry; and Statistics and Probability, as shown in Figure 4.2 (NSW Board of Studies, 2012e). Throughout Foundation to Year 10, these three content strands develop from concrete ideas to more abstract concepts as students' understanding and knowledge progresses. Linear Relationships, the focus of this study, is a sub-strand that belongs to the Number and Algebra content strand. It is an important part of developing algebraic understanding in secondary mathematics, introduced in Stage 4: Year 7–8 (approximately 12–14 year olds) and it continues through to Stage 6: Year 11–12 (approximately 16–18 year olds).

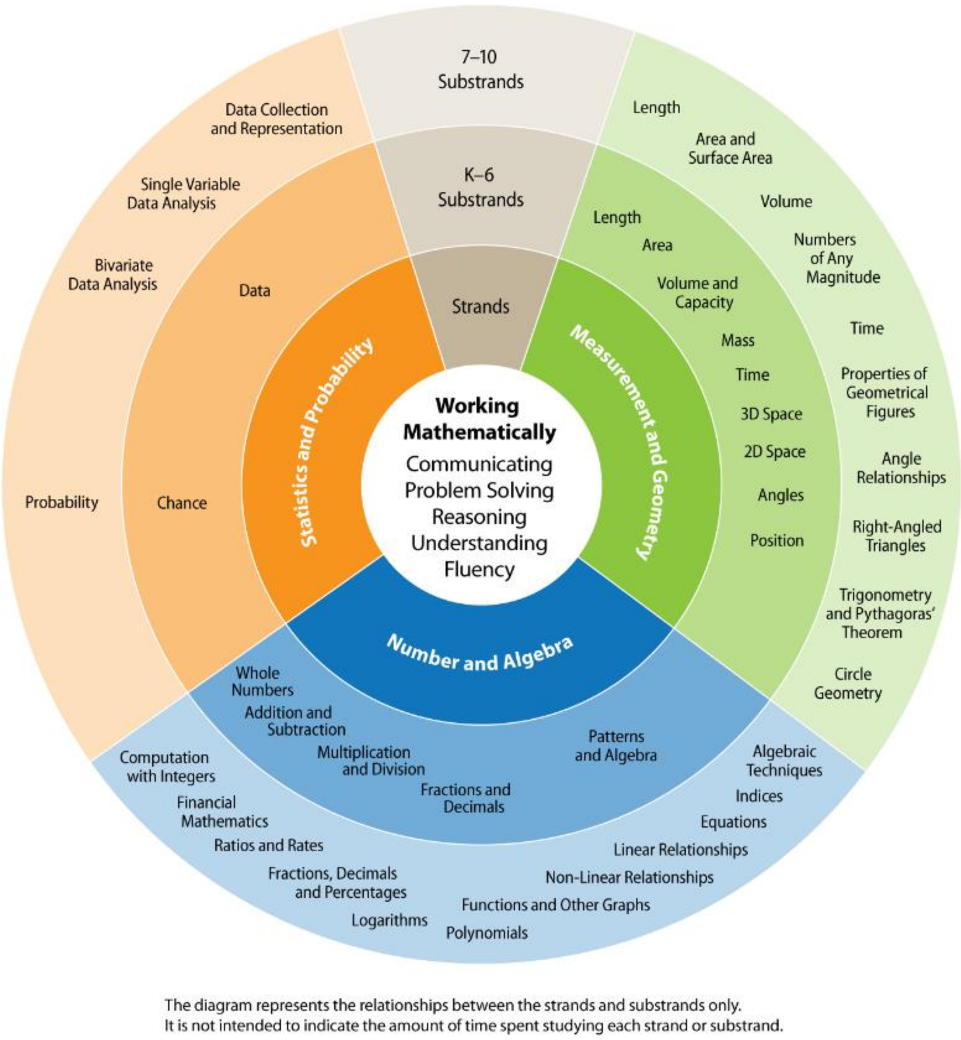


Figure 4.2: Strands and sub-strands
(NSW Board of Studies, 2012e)

The sub-strands were designed to be explored through the use of at least one of the five working mathematically components: communicating, problem solving, reasoning, understanding and fluency. Each component involves specific language and terminology that supports students' understanding of the content. In summary, the components are (NSW Board of Studies, 2012d):

- Communicating – using a range of methods in written, oral or graphical form to describe, represent and explain mathematical situations, concepts, methods and solutions to problems;
- Problem Solving – involves fostering and developing the ability to use mathematics when interpreting and expressing problems, and seeking to solve them through choosing designs and strategies which provide reasonable solutions;
- Reasoning – explores the logical thought processes which explain and justify choices, strategies and conclusions;
- Understanding – fosters the connection between what is known and adapting it to develop new ideas. Students grasp the relationship between "why" and "how" of Mathematics;
- Fluency – the consolidation of concepts and ideas is maintained through recall and efficiently completed solutions.

Stage 4: Year 7–8 (approximately 12–14 years old)

The Stage 4 Mathematics syllabus comprises common content for all students. Depending on the abilities and needs of students, the depth with which topics are covered may vary within classrooms and schools.

At this Stage, the topic Linear Relationships extends pattern activities covered in Stage 3, linking them to plotting points on a graph. Simple terminology is presented relating to the Cartesian plane and students familiarise themselves with the properties and features of the number plane such as coordinate pairs (x, y) the x-axis, y-axis, quadrants and extending patterns using coordinate axes. The outcomes and content related to the topic of Linear Relationships for Stage 4 is outlined in Table 4.1 (NSW Board of Studies, 2012g).

Table 4.1: Stage 4 outcomes and content

Outcomes	MA4-1WM	Communicates and connects mathematical ideas using appropriate terminology, diagrams and symbols
	MA4-3WM	Recognises and explains mathematical relationships using reasoning
	MA4-11NA	Creates and displays number patterns; graphs and analyses linear relationships; and performs transformations on the Cartesian plane
Content	ACMNA178	Given coordinates, plot points on the Cartesian plane, and find coordinates for a given point
	ACMMG181	Describe translations reflections in an axis, and rotations of multiples of 90° on the Cartesian plane using coordinates
	ACMNA193	Plot linear relationships on Cartesian plane, with and without digital technologies
	ACMNA194	Solve linear equations using graphical techniques

(Source: NSW Board of Studies, 2012g)

Stage 5: Year 9–10 (approximately 14–16 years old)

Stage 5 acknowledges that there exists a diverse range of levels of understanding reached by the end of Stage 4. To accommodate the differing levels of conceptual understanding, the syllabus is separated into three sub-stages, Stage 5.1, Stage 5.2 and Stage 5.3. These three sub-stages have been described in the syllabus as follows (NSW Board of Studies, 2012c):

- Stage 5.1 is intended to accommodate students who have not yet achieved but are continuing to work towards Stage 4 outcomes when entering Year 9 and have only a basic or limited understanding of content covered;
- Stage 5.2 is intended to accommodate those students who have achieved Stage 4 outcomes (by the end of Year 8) and have adequate understanding of content covered;
- Stage 5.3 is intended to accommodate those students who have successfully achieved Stage 4 outcomes (by the end of Year 8) and have a sound understanding of its content.

The Stage 5 structure remains similar to its predecessor syllabus with the only change being the addition of the new course Year 10A to the Stage 5.3 course. This is an optional course that contains content to enrich mathematical knowledge, and is intended for those students and classes who require more of a challenge while completing the common Year 10 content.

In Stage 5, Linear Relationships is developed further, building on Stage 4 content, and it continues with the study of patterns that result in the formation of a straight line. All three sub-stages; Stage 5.1, 5.2 and 5.3, contain the same content descriptors for Linear Relationships; however, their outcomes differ, reflecting the degree of difficulty and depth covered by the three different courses. For this study, Stage 5.3 is the course studied by the target sample, thus the outcomes and content specified for this sub-stage only will be discussed here.

Within the Stage 5.3 course, straight lines are explored as patterns of coordinate pairs and students develop an understanding of describing and extending patterns through the use of algebraic symbols. Features of straight lines (and line segments) are investigated; these include gradient, midpoint, distance, resulting in determining the equation of a straight line. Students use generalisable strategies and formulas that enable them to calculate the concepts mentioned without drawing a diagram. Terms associated with Linear Relationships, such as coefficient, constant, intercept, gradient and slope become more familiar when used in context. Parallel and perpendicular lines are also investigated and their properties are defined both visually and algebraically. The content and outcomes for Stage 5.3 have been detailed in Table 4.2 (NSW Board of Studies, 2012a).

Table 4.2: Stage 5.3 outcomes and content

Outcomes	MA5.3-1WM	Uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures
	MA5.3-2WM	Generalises mathematical ideas and techniques to analyse and solve problems efficiently
	MA5.3-3WM	Uses deductive reasoning in presenting arguments and formal proofs
	MA5.3-8NA	Uses formulas to find midpoint, gradient and distance on the Cartesian plane, and applies standard forms of the equation of a straight line
Content	ACMNA214	Find the distance between two points located on the Cartesian plane
	ACMNA294	Find the midpoint and gradient of a line segment (interval) on the Cartesian plane
	ACMNA215	Sketch linear graphs using the coordinates of two points
	ACMNA238	Solve problems involving parallel and perpendicular lines

(Source: Board of Studies, 2012a)

Stage 6: Year 11–12 (approximately 16–18 years old)

Linear Relationships is extended further in Stage 6, with all current 2 Unit Mathematics courses containing Linear Relationships content. For General Mathematics, it forms part of the Algebra and Modelling strand in both Preliminary and HSC courses; and for

Mathematics, it remains part of the Preliminary Course as Linear Functions and Lines. The following tables list the outcomes and content as stated in the respective syllabi. General Preliminary is shown in Table 4.3 (NSW Board of Studies, 2012b) with General 1 HSC in Table 4.4 (NSW Board of Studies, 2012b), General 2 HSC in Table 4.5 (NSW Board of Studies, 2012b) and Mathematics in Table 4.6 (NSW Board of Studies, 1982).

Table 4.3: General preliminary interpreting linear relationships

Outcomes	MGP- 1	Uses mathematics and statistics to compare alternative solutions to contextual problems
	MGP- 2	Represents information in symbolic, graphical and tabular form
	MGP- 9	Uses appropriate technology to organise information from a limited range of practical and everyday contexts
	MGP- 10	Justifies a response to a given problem using appropriate mathematical terminology
Content	•	Generate tables of values from a linear equation
	•	Graph linear functions with pencil and paper, and with technology, given an equation or a table of values
	•	Calculate the gradient of a straight line from a graph
	•	Determine the *y*-intercept for a given graph
	•	Identify independent and dependent variables in practical contexts
	•	Establish a meaning for the intercept on the vertical axis in a given context
	•	Sketch graphs of linear functions expressed in the form $y = mx + b$ without the use of tables
	•	Sketch the graphs of a pair of linear equations to find the point of intersection
	•	Find the solution of a pair of simultaneous linear equations from a given graph
	•	Solve practical problems using graphs of simultaneous linear equations
	•	Use stepwise linear functions to model and interpret practical situations, e.g. parking charges, taxi fares, tax payments and freight charges
	•	Use graphs to make conversions, e.g. Australian dollars to euros
	•	Use linear equations to model practical situations, e.g. simple interest
	•	Describe the limitations of linear models in practical contexts.

(NSW Board of Studies, 2012b)

Table 4.4: General 1 HSC

Outcomes	MG1H- 3	Uses Makes predictions about everyday situations based on simple mathematical models
	MG1H-9	Chooses and uses appropriate technology to organise information from a range of practical and everyday contexts
	MG1H-10	Uses mathematical argument and reasoning to evaluate conclusions drawn from other sources, communicating a position clearly to others
Content	• Generate tables of values for linear functions (including for negative values of x) • Graph linear functions for all values of x with pencil and paper, and with graphing software • Interpret the point of intersection and other important features of given graphs of two linear functions drawn from practical contexts, eg break-even point • Generate tables of values for quadratic functions of the form $y = ax^2$ and $y = ax^2 + c$ (including negative values of a and x) • Graph quadratic functions with pencil and paper, and with graphing software • Explain the effect of changing the magnitude of a and changing the sign of a • Explain the effect of changing the value of c • Identify the maximum and minimum values of a quadratic function from a prepared graph based on a practical context • Recognise the limitations of models when interpolating and/or extrapolating • Use linear and quadratic functions to model physical phenomena.	

(NSW Board of Studies, 2012b)

Table 4.5: General 2 HSC Modelling Linear Relationships

Outcomes	MG2H- 3	Uses makes predictions about everyday situations based on simple mathematical models
	MG2H-9	Chooses and uses appropriate technology to organise information from a range of practical and everyday contexts
	MG2H-10	Mathematical argument and reasoning to evaluate conclusions drawn from other sources, communicating a position clearly to others
Content	• generate tables of values for linear functions (including for negative values of x) • graph linear functions for all values of x with pencil and paper, and with graphing software • develop graphs of linear equations of the form $y = mx$ from descriptions of situations in which one quantity varies directly with another • use the graph in the previous dot point to establish the value of m (the gradient) and to solve problems related to the given variation context • interpret linear functions as models of physical phenomena • establish the meaning of the gradient and the y-intercept for a given practical context • develop linear equations from descriptions of situations in which one quantity varies directly with another • solve contextual problems involving linear models • interpret the point of intersection of the graphs of two linear functions drawn from practical contexts • solve contextual problems using a pair of simple linear simultaneous equations • develop and use linear functions to model physical phenomena • recognise the limitations of models when interpolating and/or extrapolating • apply break-even analysis to simple business problems that can be modelled with linear and quadratic functions.	

(NSW Board of Studies, 2012b)

Table 4.6: Mathematics linear functions and lines content

Content	6.1	The linear function $y = mx + b$ and its graph.
	6.2	The straight line: equation of a line passing through a given point with given slope; equation of a line passing through two given points; the general equation $ax + by + c = 0$; parallel lines; perpendicular lines.
	6.3	Intersection of lines: intersection of two lines and the solution of two linear equations in two unknowns; the equation of a line passing through the point of intersection of two given lines.
	6.4	Regions determined by lines: linear inequalities.
	6.5	Distance between two points and the (perpendicular) distance of a point from a line.
	6.7	The mid-point of an interval.
	6.8	Coordinate methods in geometry

(NSW Board of Studies, 1982)

In summary, this section explored the context of the study. Firstly, the sample population was examined to assist in providing a background for the geographical setting. Next, a thorough explanation of the Secondary Mathematics courses in NSW was presented, which detailed the outcomes and content of the Linear Relationship's topic from Stage 4 through to Stage 6. It is evident that Linear Relationships remains a significant component of the Number and Algebra content strand throughout secondary mathematics, making it worthy of further research.

4.3. Research Design

Research design refers to the rationale that explains the inner workings of the execution of a research project in order to avoid confusion and misinterpretations of results. The design of any research study provides the link which aligns the research questions to the data collected (Punch, 2009). This section details the methodology employed to investigate the research questions that were stated at the end of Chapter 3. It is divided into two main sub-groups: overview of design structure; and teaching sequence and data collection structure.

4.3.1. Overview of Design Structure

This study began as a teacher's desire to explore how technology could be better utilised within today's mathematical classrooms. The teacher was searching for a practical solution to a commonly experienced problem. Since it was the teacher who instigated the research, action research methodology was recognised as the most appropriate method of inquiry, and the teacher became the principal researcher for the project. To supplement the action research as well as obtain a more in-depth perspective of the effects of technology specific to students learning, a case study methods was also employed. This focussed on exploring the individual learning journey of two students throughout the teaching sequence. The case study method suits educational research since it requires understanding a specific case in depth, a unique example relying on data within a real-life context (Cohen, Manion, & Morrison, 2000, 2011; Denzin & Lincoln, 2005; Punch, 2009). This section, therefore, details the principles and background information of action research and case study methods as relevant to the research project, and discusses the participants of the project.

4.3.1.1 Action Research

Within the field of education, action research aims at improving teaching practices in collaboration with teachers. A method to attune specific strategies to students' learning requirements such that planned actions within the approaches are specific to the context required (Stringer, 2008; Tesch, 1990). It is broadly considered as research that is conducted by teachers for teachers (Craig, 2009; Mertler, 2012). Kurt Lewin, who is credited with coining the descriptive title of *action research*, was dissatisfied with completing research solely for the benefit of academic publishing and believed that those making decisions were the best to execute them (O'Brien, 2001). Since its beginnings in the 1940's, action research has experienced varying degrees of acceptance: declining in the 1950's, rising in the 1970's, declining again in the 1980's and re-surfacing in the 1990's to its current status where it remains a popular method for educational research. The declines were rooted in the belief, from academics, that teachers were not capable of being credible qualitative researchers. Action research is acknowledged as a prominent development for education, particularly for teacher education programs where it is implemented to improve practice, it "offers a process by which current practice can be changed toward better practice" (Mertler, 2012, p. 14). For this study, action research was seen as most appropriate because the researcher was interested in investigating how technology could be better implemented in the classroom and how frameworks such as van Hiele Theory and the SOLO model would support technology in the mathematical classroom.

Action research is a practical form of rigorous inquiry that encourages collaboration and participation. It provides a systematic approach to understanding and resolving a specific problem within a given framework or, in simple terms, it is "learning by doing" (O'Brien, 2001, p. 2). This project provided a set of lesson plans for the entire teaching sequence of Linear Relationships for the classroom teacher (see Appendix A). Using team teaching strategies with the researcher, these lessons were implemented for the duration of the Linear

Relationship unit. As deemed necessary, the lesson plans and associated activities were adjusted and/or modified to suit the needs of the students and their learning within the curriculum guidelines.

Action research is usually represented as a recursive cycle with a spiralling nature, as shown in Figure 4.3 (CELT, 2014). Despite differences in terminology provided by various authors, action research comprises of four main stages:

- Plan – identifying the issue and how to address it;
- Act – putting the plan into action;
- Observe – noticing how the plan has worked;
- Reflect – critically reviewing the plan then back to the plan stage to try it out again.

The four-stage cycle can continue until reflection indicates that the outcomes of the plan have been realised and satisfied.

Action research differs from traditional methods, since it is a "field-intensive process" (Craig, 2009, p. 3) that involves immersing the researcher into the context of the object or problem being researched. As previously mentioned, the researcher implemented team teaching strategies with the classroom teacher to deliver the content. This involved the researcher being present for every lesson during the teaching of the Linear Relationships unit, actively occupied within the classroom environment. The benefits of this situation being, firstly, that the intended teaching sequence was adhered to as detailed for validity and, secondly, the researcher was able to investigate the classroom's *natural* environment. Students also benefited by having more opportunities to be engaged in discussion since two teachers were actively monitoring their learning. Through this involvement, multiple views of the sample population were obtained and interconnectedness formed between the researcher and the environment being researched (Cohen, et al., 2011; Mertler, 2012; Schmuck, 1997).

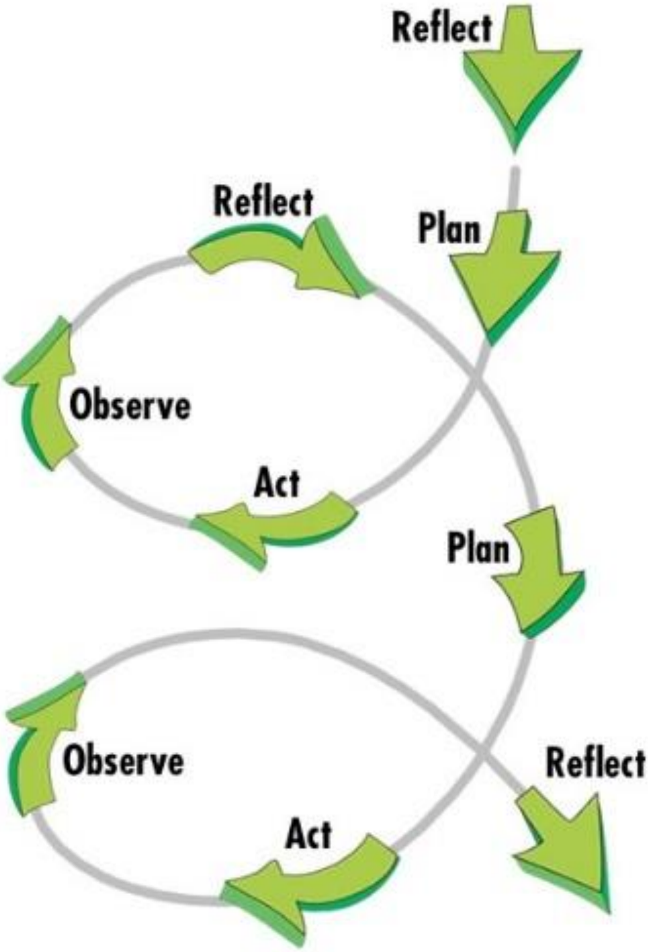


Figure 4.3: Action research cycle
(Source: CELT, 2014)

Action research in education relates to the constructivist theory that suggests learners construct their knowledge out of their experiences. Throughout action research, teachers are assigned with the ongoing task of facilitating learning experiences through which students can actively engage themselves in understanding. This role similarly corresponds to the role of the teacher when implementing technology; that is, teachers are required to "act more like team leaders, coordinators, or facilitators" (Stringer, 2008, p. 25), working in conjunction with students to achieve a learning outcome. While the teacher guides and monitors students' progress, it is the individual student who must develop their knowledge. Through ownership of their learning, students become effective learners gaining the deeper knowledge and understanding that promotes the development of insight, something that should be encouraged in today's learners (van Hiele, 1986; Westwell, Gibbins, & Costello, 2011). This method of constructivist learning and the development of insight is also supported by the van Hiele Theory of learning and Teaching Phases, as discussed in Chapter 3.

Apart from being a collaborative and practical form of inquiry, action research also differentiates itself from other approaches by its use of multi-methods. The employment of multi-methods for gathering data enriches the research process. Throughout the study, various forms of data were collected. While the primary sources of data collection for the project were the three Google Form tests, as part of pre-experimental design, data was also collected via student workbook samples, photographs of student activities and numerous hours of video footage collected from individual student's computers. Through the triangulation of data gathered from multi-method procedures, depth and rigor are increased.

At the heart of action research are the reflection and self-reflection stages (Cohen, et al., 2011). Self-reflection involves critically exploring the "what" and "why" of the decision making process (Mertler, 2012). The "what" comprises asking, defining and examining the situation and the effects along with the implications that the situation offers. The "why" critically explores and justifies the reasons behind certain decisions in the process. Throughout data collection, critical reflection occurs by the participants and researchers, who have actively observed the learning process. This enables revisions to be made to the data collected or collection strategies, ensuring rigor during the repetition of the action research cycle. Unlike its definition with respect to quantitative research, the term rigor does not limit itself solely to aspects of the data and its findings. It refers to "the quality, validity, accuracy and credibility" (Mertler, 2012, p. 29) of the whole action research process and its findings.

4.3.1.2 Case Study Methodology

The study of a case aims to fully understand the bounded context which defines the whole case in detail, it "is not a methodological choice but a choice of what is to be studied" (Denzin & Lincoln, 2005, p. 443). "Properly conducted case studies, especially in situations where our knowledge is shallow, fragmentary, incomplete or non-existent, have a valuable contributions to make in education research" (Punch, 2009, p. 123).

A case study can take any one of three forms: intrinsic, instrumental and collective, with the most suitable type of case study depending on the purpose of the inquiry (Stake, 2005). The collective case study, or multiple case study, is chosen when a "number of cases may be studied jointly in order to investigate a phenomenon, population, or general condition" (Stake, 2005, p. 445). The remaining two, instrumental and intrinsic, both refer to a singular case. The instrumental case study is best suited when a detailed case study is the only way of offering understanding to a new or challenging area. The intrinsic case study is most appropriate when a particular case can be of interest in its own right. It is this type of case study, the intrinsic case study, that best describes the approach used to explore the educational journey of a particular student pair during the teaching sequence. Detailing their journey as an intrinsic case study complements the action research methodology employed, as the various stages of the action research cycle become more evident.

A case study approach gains reliability and validity through the triangulation of data. Triangulation outlines a process of using multiple data collection strategies to gain a more holistic sense within the same data set (Cohen, et al., 2011; Mertler, 2012). It assists in identifying the reality within the data through continually and thoroughly interpreting the data set. Mertler (2012) uses the term "polyangulation" to reduce the ambiguity related to the number of methods suggested by "triangulation", since the prefix poly- indicates more than one. The primary sources of data collected for the case study approach were numerous hours of video footage and the three Google Form tests.

4.3.1.3 Participants of the Project

Determining the appropriate sample for the research was influenced by a number of factors. The data to be collected relied on the participation of the population sample to openly write, type and discuss problem-solving techniques. The extent to which students felt comfortable enough and were given adequate opportunity to discuss would also be determined by the teacher and their prior rapport with students. Students from Stage 5.2 and 5.3 (approximately 14–16 years old) were considered a suitable population for the study, as students from these levels demonstrated an understanding of Stage 4 (approximately 12–14 years old) content and would most likely be preparing for some form of Stage 6 (approximately 16–18 years old) mathematics. It was also considered important that the teacher chosen was an experienced and qualified mathematics teacher. This maintained validity and reliability for the delivery of the lessons. It also ensured that any modifications or suggestions regarding lesson and activities came from a competently trained character, drawing on their experience from teaching a range of levels of mathematics. Hence, these factors were all taken into consideration regarding the choice of teacher.

After discussions outlining the factors mentioned previously with the Leader of Learning for Mathematics of the high school, a target class and teacher were agreed to as possible participants pending assent and consent of all involved. The high school offered two Year 9

Stage 5.3 and two Year 9 Stage 5.2 classes with the top 30 students streamed into the first class of both stages. The second of the Year 9 Stage 5.3 (approximately 14–16 years old) classes was selected as most valuable for this study since the class teacher was a qualified mathematics teacher with over 20 years' experience in teaching all levels of mathematics from Year 7–12, as well as being a HSC marker. The class consisted of 26 students, with 13 males and 13 females.

Once consent of the teacher had been obtained, purposive sampling was used to place students into pairs. These were to be the pairs that students worked in for the duration of the research unit: Linear Relationships. When selecting students, the following points were considered as major criteria for a successful pairing:

- Were the students similar in abilities? In order to ensure that students would comfortably discuss problem-solving strategies at the same level of understanding, it was preferable that both students were of similar mathematical ability; and
- Would the students work effectively together? The collection of intelligible data required a productive discussion to occur between the students. For this, students needed to be socially suited, without either dominating the discussion or refusing to participate with the other person.

This section explored the design structure of the research project. It detailed key principles and information related to action research methodology and case study methodology since both methods of inquiry were chosen to effectively investigate the research questions posed. Important issues related to determining the appropriate sample participants was also outlined. Combined, this information provides the foundation for the structure of the design.

4.3.2. Teaching Sequence and Data Collection Structure

For clarity, the following section has been divided into two main sections. The first outlines the teaching sequence, detailing its planning and subsequent features of the activities chosen for the teaching sequence. This is provided in Table 3.9, which outlines the teaching sequence, mapping activities to the relevant Teaching Phase addressed. The next section explains the methods of data collection and types of data gathered during the course of the research project.

The teaching sequence and data collection was performed over a four-week period at the end of the third term (August-September) of the school year (a school year in NSW comprises of four ten week terms). In the Scope and Sequence issued by the school, the Linear Relationships sub-strand is allocated four weeks instruction, which equates to 15 lessons of their timetable (each of duration 63 minutes). The schools' timetable operates in a four-week cycle. The timetable for the sample Year 9 Stage 5.3 class (known as 9.2 Mathematics) can be seen in Tables 4.7 and 4.8, which show the spread of lessons over the cycle. The class teacher declared that, in his opinion, the timetable was far from being ideal, in particular noting the gap between period 2 of Wednesday Week 2 and period 5 of Tuesday Week 3 which resulted in students having almost a week without Mathematics.

Table 4.7: Timetable for 9.2 Mathematics (Week 1 and 2)

	Mon1	**Tue1**	**Wed1**	**Thu1**	**Fri1**	**Mon2**	**Tue2**	**Wed2**	**Thu2**	**Fri2**
1		Maths								
2					Maths	Maths		Maths		
Lunch										
3						Maths				
4			Maths							
Recess										
5	Maths						Maths			

Table 4.8: Timetable for 9.2 Mathematics (Week 3 and 4)

	Mon3	**Tue3**	**Wed3**	**Thu3**	**Fri3**	**Mon4**	**Tue4**	**Wed4**	**Thu4**	**Fri4**
1										
2							Maths			Maths
Lunch										
3					Maths		Maths			
4			Maths							Maths
Recess										
5		Maths								

4.3.2.1 Teaching Sequence

The credibility of the study relied significantly on the considerations involved when developing the teaching sequence. Three elements defined the development and structure of the teaching sequence: the Australian Curriculum, the GeoGebra environment and the van Hiele Teaching Phases. Aligning the teaching sequence to the Australian Curriculum for Year 9 Stage 5.3, as outlined in Table 4.2 previously, ensured the findings would make a valuable contribution to the recent body of research and the wider community since the new curriculum was to be implemented for Year 9, the year following data collection (2013). Thus, the findings would be relevant to current educational content. The only concern that was noted regarded two topics that were omitted from the Australian Curriculum for Linear

Relationships: regions of the Cartesian plane and the general form of a line ($ax + by + c = 0$). It was decided that the regions content would be excluded from the teaching sequence; however, the general form of a line would be covered since it was the default form used within the GeoGebra environment, thus considered necessary for student understanding. The outcomes associated with Linear Relationships were separated into five topics: Real Life Relationships, Midpoint, Distance, Further Graphing and Geometric Problems. An estimated allocation of time (number of periods) for delivery was issued for each of the five topics that kept them within the Scope and Sequence provided by the school, as shown in Figure 4.4.

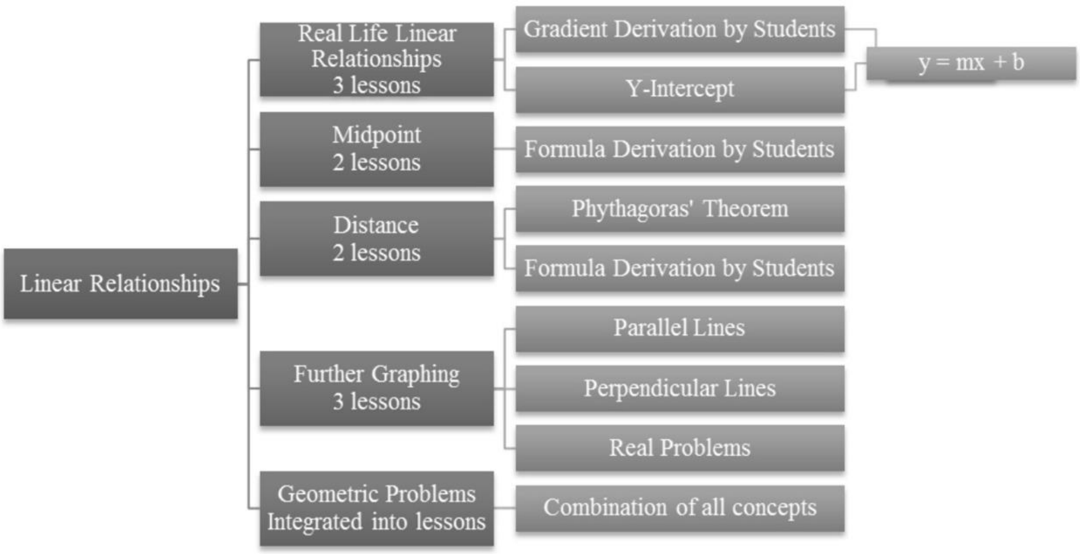


Figure 4.4: Teaching sequence structure

GeoGebra was integrated throughout the lesson sequence because it was considered to be a learning environment that could provide students with a dynamic, practical and concrete method of exploring concepts. Being a technological environment that the students had not worked with previously, made it a suitable choice. With GeoGebra, students would be able to investigate properties of concepts and calculate solutions, providing a checking tool for their handwritten work. GeoGebra had the potential to also be used as a demonstration tool for use with open class discussion.

The most important element contributing to development of the teaching sequence was the van Hiele Teaching Phases. These provided a framework underpinning the entire structure and choice of activities for delivering content. In developing the teaching sequence, considerable time was taken to ensure the van Hiele Teaching Phases were evident during each structured activity. While the Teaching Phases, as shown in Figure 4.5 and detailed previously in Chapter 3, are consecutively ordered, their progression during the teaching sequence often occurred as cycles of phases. rather than a direct pathway. This enabled students to develop concepts and understanding, strengthen their ideas and attempt different activities to consolidate content.

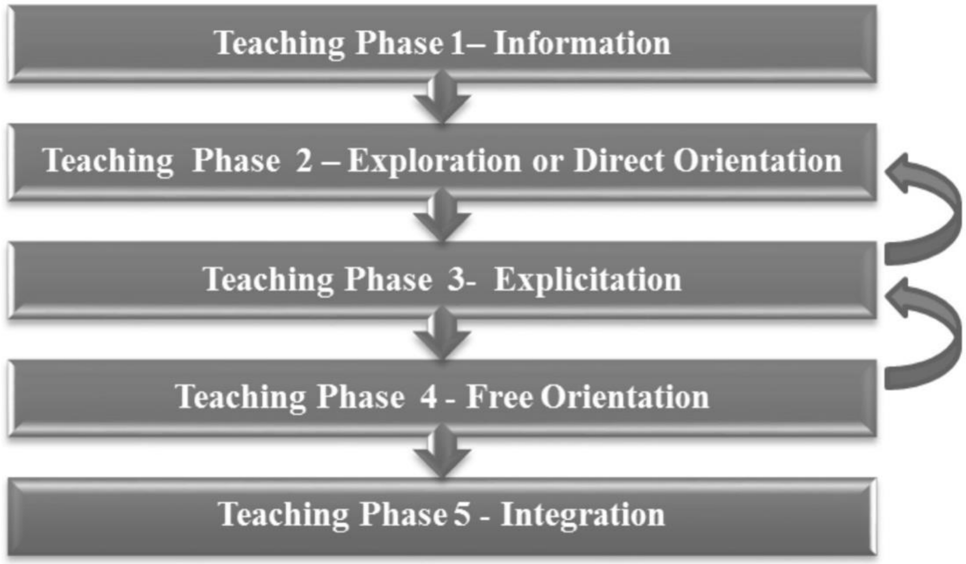


Figure 4.5: Teaching Phase Structure

Phase 1 (Information) involved the introduction of topics through simple activities aimed to introduce the lesson content and promote discussion of what was known. The next three phases represented the repeated cyclic phases of the van Hiele Teaching Phases structure, with Phase 2 (Exploration or Direct Orientation) involving teacher-guided activities that included student discussion through appropriate probing and questioning techniques employed by the teacher. Student questioning occurs in this phase as they attempt to grasp ideas and concepts, in simple terms. The discussions developed into Phase 3 (Explicitation), in which students become able to describe concepts with appropriate terminology relevant to the content they are exploring, able to make the connections required. These two phases contributed to most of the activities as students developed a sense of what was required and how to achieve understanding, though still with teacher guidance. Teaching Phase 4 (Free Orientation) and 5 (Integration) suggest that students have become more independent of the teacher, finding their own way towards solving the problem. Phase 4 occurred when students were confident of finding their way of completing activities on their own, using the conclusions developed from their own explorations of the previous Teaching Phases.

Although students were more independent, the teacher still monitored students' work. Phase 5 occurred at the end of the topics as students developed methods for memorising as a technique to overview what they had learnt.

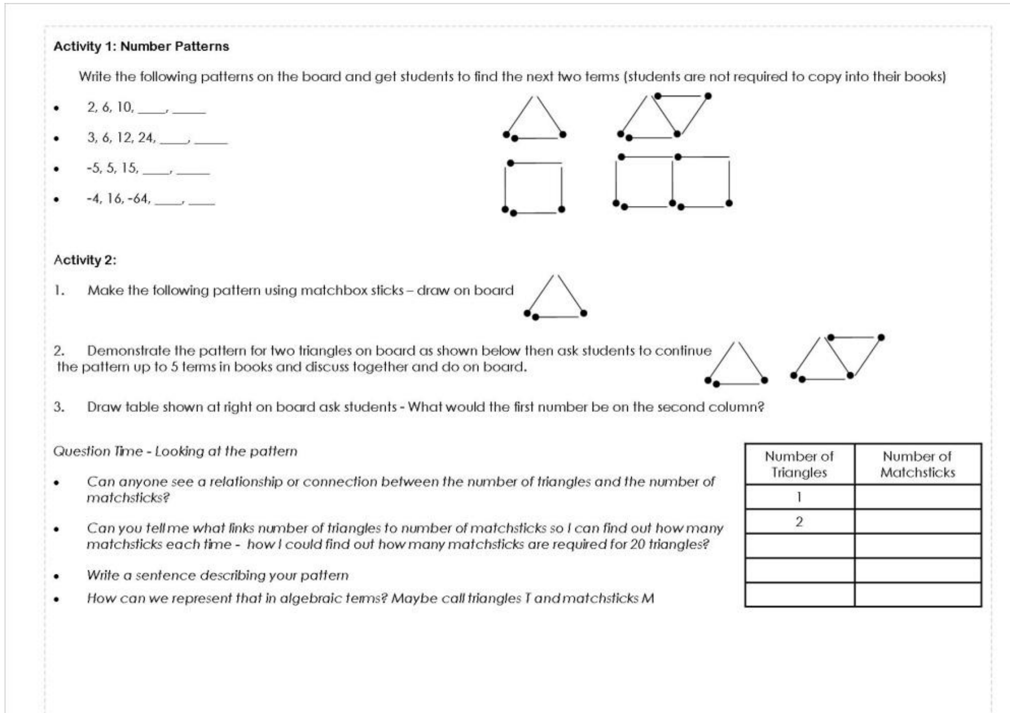
To provide further details regarding how the van Hiele Teaching Phases were utilised within the lesson plan structure, Table 4.10 lists a comprehensive explanation of each lesson with the activities linked to the teaching phase being addressed. The collection of lesson plans is provided in Appendix A, with the associated worksheets in Appendix B. Table 4.9 maps the teaching sequence structure to the outcome and content codes for Stage 5. The Google Form tests: the Pre-test, Post-test and Delayed Post-test, contained questions relevant to the outcomes and content of the syllabus, these tests along with the Extended Response sheets are presented in Appendices J, K, L and M respectively.

Table 4.9 Teaching sequence structure mapped to outcome and content codes for Stage 5.3

Teaching Sequence Structure	Key Learning Concepts	Outcome Code	Content Code
Real Life Linear Relationships	Gradient y-intercept $y = mx + b$	MA5.3-1WM MA5.3-2WM MA5.3-3WM MA5.3-8NA	ACMNA294 ACMNA215
Midpoint	Derivation of formula	MA5.3-1WM MA5.3-2WM MA5.3-3WM MA5.3-8NA	ACMNA294 ACMNA215
Distance	Pythagoras' Theorem Derivation of Distance formula	MA5.3-1WM MA5.3-2WM MA5.3-3WM MA5.3-8NA	ACMNA214 ACMNA215
Further Graphing	Parallel lines Perpendicular lines Problems	MA5.3-2WM MA5.3-3WM ACMNA214	ACMNA214 ACMNA294 ACMNA215 ACMNA238
Geometric Problems	Combination of concepts linked to geometry	MA5.3-8NA MA5.3-2WM MA5.3-3WM	ACMNA214 ACMNA294 ACMNA215 ACMNA238

Table 4.10: Example of lesson content with Teaching Phases

Lesson Title	Content	Teaching Phases
Pre-test	**Week 1 Lesson 1**	
Real Life Linear Relationships	**Week 1 Lesson 2** • Introduction Activity – Introduces 5 basic number and shape patterns, where teacher discussion involving direct questioning and brainstorming familiarises students with the working domain and vocabulary. Discussions revolve around how relationships develop from patterns and how to recognise and extend a pattern with the teacher extracting valuable information regarding the student's prior knowledge, shown in Figure 4.6.	Phase 1 Information



Activity 1: Number Patterns

Write the following patterns on the board and get students to find the next two terms (students are not required to copy into their books)

- 2, 6, 10, ____, _____
- 3, 6, 12, 24, ____, _____
- -5, 5, 15, ____, _____
- -4, 16, -64, ____, ____

Activity 2:

1. Make the following pattern using matchbox sticks – draw on board
2. Demonstrate the pattern for two triangles on board as shown below then ask students to continue the pattern up to 5 terms in books and discuss together and do on board.
3. Draw table shown at right on board ask students - What would the first number be on the second column?

Question Time - Looking at the pattern

- *Can anyone see a relationship or connection between the number of triangles and the number of matchsticks?*
- *Can you tell me what links number of triangles to number of matchsticks so I can find out how many matchsticks each time - how I could find out how many matchsticks are required for 20 triangles?*
- *Write a sentence describing your pattern*
- *How can we represent that in algebraic terms? Maybe call triangles T and matchsticks M*

Number of Triangles	Number of Matchsticks
1	
2	

Figure 4.6: Example from lesson plan of phase 1

Lesson Title	Content	Teaching Phases
	• Matchstick Pattern activity – Shape patterns using matchsticks were investigated, recording the number of shapes and matchsticks in a table. Students were then asked a series of questions, monitored by the teacher, with the aim to identify the focus of the activity to connect the pattern to an algebraic equation explaining the relationship between the number of shapes and matchsticks. GeoGebra was used as a checking tool to plot the points from the table and inspect if its solution for the equation corresponded to the one derived through the questioning process. This activity was modelled twice, with students developing an understanding to explain and express the pattern in the problem being explored and then the students completed more on their own to consolidate the ideas previously established. The teacher monitoring student pairs as different ability students take different times to grasp language and concepts.	Phase 2 Exploration Phase 3 Explicitation
	Week 1 Lesson 3	
	• Introduction Activity – (Algebra Walk) This Maths 300 Activity involved students placing themselves at various intervals upon the x-axis of a painted Cartesian Plane. Initial discussions include direct questioning and brainstorming to clarify language that reconnects students with the working domain of the Cartesian plane and identifies prior knowledge content. The practical activity progressed into the next phase with teacher guided instruction as students were required to find the corresponding y-value when given a sequence of operations by the teacher, to be performed to their respective x-value. The teacher monitoring student calculations through questioning and observations and students were able to visually identify their positioning with respect to other students. Students exploring the properties of the sequence and pattern provided by the teacher visualising the totality of the structure as a relationship resulting in straight line. The instructions advanced using more complicated rules and algebraic equations with the terminology and structure further developed. This entered the third phase with students as they openly discuss and interchange ideas consolidating the previous phase concepts and knowledge. Figure 4.7 demonstrates the students at the activity.	Phase 1 Information Phase 2 Exploration Phase 3 Explicitation
	• Revision of Matchstick Pattern activity – This activity consolidated the previous lesson activity (without the practical aspect of using matchsticks) with the teacher ensuring connections where recognised and correctly identified. The first question for this activity was typical of Phase 2 where student thinking was initiated but carefully monitored. This developed to the next phase as students extended shape patterns, making a table of the number of shapes and matchsticks, then describing the patterns in words and trying to make a rule for the pattern using algebra. During this, students openly discussed ideas for the structure they were exploring using GeoGebra as a tool to confirm their findings. Language continued to develop as familiarity with the working domain increased. The teacher watched each group probing to ensure students understanding to explain the structure was using acceptable language.	Phase 2 Exploration Phase 3 Explicitation

Lesson Title	Content	Teaching Phases
	 **Figure 4.7: Students on number line during Algebra walk activity**	
	• Garden Bed activity – this presented a more difficult pattern based on a practical problem where students used the relations and connections from the previous three phases during the matchstick pattern activity to explore to find their own way to solve the problem. Students recognised cues to bring together information assisting in quicker solution. • Worksheets given as homework A1, A2.	Phase 4 Free Orientation Phase 3 Explicitation

Lesson Title	Content	Teaching Phases
	Week 2 Lesson 4	
	• Introduction Activity – this involved the drawing of two graphs on the board, giving students the stimulus for brainstorming of knowledge related to the diagrams. Time was given for students to brainstorm in their working pairs, then class discussions brought together student ideas. Visual features were noted and language was clarified with further probing enabling the teacher to identify prior content knowledge, consolidating previous work. This activity progressed into Phase 2 as students commence visualising the line as a structure and investigating relations guided by the teacher. Concepts revisited included x and y-axis, y-intercept, gradient, slope. Students were invited to a practical approach tracing the graphs on the board to develop the concept of direction of slope.	Phase 1 Information Phase 2 Exploration
	• Worksheets A1, A2 (available in Appendix B) – GeoGebra was used as a tool for drawing graphs that were then copied onto a sheet, looking for connections between the rule and the graph. Teacher direction commences this activity, a Phase 2 activity, where the teacher is crucial to ensuring students recognise and establish correction relations. This develops into Phase 3 as students discuss the explorations within their working pair and with the teacher who roams around monitoring language and connections for correct meaning.	Phase 2 Exploration Phase 3 Explicitation
	• Gradient, m – discussed as a visual concept and compared to a previous assessment question (that in introducing language to students through direct questioning and connecting to prior knowledge indicative of Phase 1). Concepts of rise and run established and developed indicating Phase 2 as students understand the relation through observation. Students are encouraged use GeoGebra to draw lines and use appropriate tools to calculate the gradient. Students openly discuss with teacher what has been explored and exchange ideas using appropriate language to develop structure of gradient progressing students to Phase 3. Students prompted to use GeoGebra as a tool to confirm their ideas about features of the graphs, in particular, to gradient and y-intercept. Parallel lines are realised through exploration of gradients indicative of Phase 4 as students language develops and systems of relations are investigated.	Phase 1 Information Phase 2 Exploration Phase 3 Explicitation Phase 4 Free Orientation
	• Students use technique of drawing a triangle on lines to calculate rise and run from visual teaching roaming to monitor correct language and development of gradient as a concept linking rise and run of a triangle.	Phase 3 Explicitation
	• Teacher demonstrates the dynamic feature of GeoGebra using sliders to change slope giving students a visual perspective indicative of Phase 2.	Phase 2 Exploration

Lesson Title	Content	Teaching Phases
	Week 2 Lesson 5	
	• Students continue to plot graphs commencing with the equation using a table of values. Again, observations guide the learning and are indicative of Phase 2. Students consolidating simple concepts through teacher questioning such concepts including drawing lines including arrows on the end of the lines.	Phase 2 Exploration
	• A teacher guided activity enables students to recognise the links between graph features y-intercept and gradient and gradient-intercept form of an equation $y = mx + b$, indicative of Phase 2, Exploration. Teacher questioning promotes students' discussion without providing solutions. Consolidation of these links occurs with correct language developing as students investigate and explore equations progressing to Phase 3. After a number of examples students recognise that graphing is not required to determine y-intercept and gradient when given the equation. Connections develop and language is used in context.	Phase 2 Exploration Phase 3 Explicitation
	• Concept of slope is investigated in more depth with teacher questioning and brainstorming comparing to how GeoGebra calculates slope (using a diagram with triangle and rise and run distances shown) connecting prior knowledge indicative of Phase 1. Students recognise slope commencing with idea of rise over run through teacher guided activity, Phase 2, and attempting to connect each with particular coordinates in order to establish some relationship so that slope can be calculated without plotting points and drawing a triangle for rise over run. Teacher constantly monitoring language and ideas whilst allowing students to discuss strategies and consolidate concepts. Students continue to plot points, draw triangle until they are confident with moving rise and run to recognising coordinates, progressing to Phase 3.	Phase 1 Information Phase 2 Exploration Phase 3 Explicitation
	• Students establish how to calculate rise and run using coordinates given, then develop this further by using general notation consistent for use within a formula $(x_1, y_1)$and (x_2, y_2). Initially full class participation with teacher providing prompts that foster student understanding of connections between rise, run and how to calculate using coordinates then generalising using notation. Students then given time to investigate with another set of coordinates with teacher's role diminished and students attempt to find their own way. Derivation of the gradient demonstrates learning consistent with Phase 4. This was attempted at the end of lesson.	Phase 3 Explicitation Phase 4 Free Orientation
	Week 2 Lesson 6	
	• Introduction Activity – students given two points and asked to explore features identified in previous lessons from those two points, namely, identifying the y-intercept, gradient and equation of line using GeoGebra, this then demonstrated on board with pen and paper techniques. Here students investigate connection through technological medium resulting in whole class discussions to clarify concepts explored moving into Phase 3. Revising concepts through class discussion ensures connections are sustained and open exchange of ideas	Phase 2 Exploration Phase 3 Explicitation

Lesson Title	Content	Teaching Phases
	enable appropriate language to be used. Whole class discussion explores gradient form of the line – with students recognising that the same values, gradient and y-intercept are used to create an equation.	
	• Activity is further developed using more difficult points with students encouraged to use pen and paper techniques to consolidate concepts. Students become more proficient with completing the task themselves with minimal teacher intervention finding their own way in the network of relations and using GeoGebra as a checking tool to confirm their working. Development of the gradient form of an equation $y = mx + b$ indicative of Phase 4.	Phase 3 Explicitation Phase 4 Free Orientation
	• Worksheet C – diagrams of graphs requiring students to work out the equations of the line. Consolidates previous activities work of finding y-intercept, gradient and resulting in an equation, Phase 3. Teacher monitors students work moving throughout class as required. GeoGebra used as a checking tool to confirm students solutions. Phase 4 reached as students encounter negative slopes and more difficult graphs. Students discuss subject matter recognising relations thus becoming more proficient. Students complete sheet for homework.	Phase 3 Explicitation Phase 4 Free Orientation
	Week 2 Lesson 7	
	• Activity attempting to establish a link between coordinates and how to manipulate coordinates to obtain formula for gradient in terms of $(x_1, y_1)$and (x_2, y_2). Teacher guided activity with whole class discussion recognising connections between distance of rise and run from coordinates. Students assist in determining a formula without the need for diagrams. Diagrams used to develop students understanding of where formula derived from. Teacher role crucial to establishing correct relations. Phase 3 reached when upon repeat of activity students consolidate ideas openly in class discussion and clarify concepts.	Phase 2 Exploration Phase 3 Explicitation
	• Students given opportunity to work on a number of different problems using GeoGebra to confirm solutions. Teacher monitoring progress by circulating through class and providing assistance if and where needed. Students encouraged to continue to draw graphs, developing formula from first principles until confident of moving from coordinates to gradient formula directly. Students able to move from coordinates to formula have reached Phase 4. Teacher brings class together at various intervals allowing students to express solution strategy thus confirming cues used to bring information to solve problem.	Phase 3 Explicitation Phase 4 Free Orientation
	• Equation of line activity reverts back to finding the equation of a line. GeoGebra is used as a tool to determine equation of line. More difficult coordinates are provided whereby students are required to think about how GeoGebra presents solutions, consolidating gradient and y-intercept concepts. Teacher continues to circulate monitoring student work. Class discussions are used to demonstrate issues and enable students to explain	Phase 3 Explicitation Phase 4 Free Orientation

Lesson Title	Content	Teaching Phases
	structure. Students proficient with symbols and connections of gradient and y-intercept with gradient form of a line have reached Phase 4.	Phase 3 Explicitation
	• Students draw graph from a given equation indicating that they understand the concept of gradient and y-intercept. Class discussion, Phase 3, leads this activity as students demonstrate understanding consistent with requiring a total rethinking of the application of gradient and y-intercept, Phase 4. Students can distinguish connections and are confident with subject matter drawing a graph starting with y-intercept and using gradient to establish other points.	Phase 4 Free Orientation Phase 4 Free Orientation
	• Worksheet B – given as homework where students continue to draw graphs from equation using knowledge of y-intercept and gradient.	
Midpoint	**Week 3 Lesson 8**	
	• Introduction Activity – introduces basic terminology including dissected and bisected, through teacher discussion and story thus familiarising students with the working domain and vocabulary. Discussions culminate with defining bisect as being half, leading to concept of midpoint.	Phase 1 Information
	• Practical Activity – students given three separate pieces of graph paper with the same line segment and asked to find middle of the line segment, guided and monitored by the teacher, using three separate folding techniques namely, point to point, horizontal and vertical. The aim to demonstrate to students that all three methods find the same point and through teacher led discussions linking the axes to the midpoint. GeoGebra was used as a checking tool to plot the end points, draw a line and find the midpoint. Students completed more on their own, encouraged to find ways to calculate the midpoint without drawing or folding. Through more discussion students realised that half way of a distance was found by dividing the length by two. This was described as being similar to the concept of average.	Phase 2 Exploration Phase 3 Explicitation
	• Activity develops with students given two coordinates and asked to find how they could determine the midpoint without drawing. Students are encouraged to discuss and exchange their methods and reasoning, closely monitored by the teacher, calculating the midpoint informally without writing, thereby developing language and connections before formalising the concept. Students recognise that they need to understand which parts of the coordinate pair needs to be added together. The teacher constantly monitors students, as different ability students take different times to grasp connections. Students come to the realisation that they need to find the average of the x coordinates and then average the y coordinates. Students continue to develop ideas, drawing points on a Cartesian plane and attempting to find the average/midpoint then using GeoGebra as a checking tool to confirm their working.	Phase 2 Exploration Phase 3 Explicitation

Lesson Title	Content	Teaching Phases
	• Open class discussion enables all students to explain their ideas and results with developing language and connections. Students link the coordinates to determining the midpoint, similar to how they determined gradient using general notation in terms of (x_1, y_1) and (x_2, y_2). Through prompting and a teacher guided activity, Phase 2, the connections between midpoint and coordinates are further developed with students assisting in determining a formula without the need for diagrams. Diagrams used to support students understanding of where the formula derived from. Teacher role crucial to establishing correct relations and ensuring that language is used in context of working domain, progressing to Phase 3. Phase 4 reached when upon repeat of activity students consolidate ideas openly in class discussion and clarifying concepts, they are more proficient at finding the midpoint. This repeat activity includes finding midpoint by using formula with teacher fostering understanding through repeatedly asking students to name concept, describe how they find it and use GeoGebra to confirm their working, thus bringing together all information.	Phase 2 Exploration Phase 3 Explicitation Phase 4 Free Orientation
	• Final activity involves asking students to find an endpoint given the midpoint and the startpoint, without drawing if possible. Students encouraged to openly discuss their ideas on how to approach this problem as is evident in Phase 3. Teacher ensures connections that were established remain with the current structure and prompting encourages backtrack operations to solve the problem. Students share ideas on how to calculate the end point, moving to Phase 4, as it requires a rethinking on the application of midpoint. Teacher role is diminished but monitors students thinking about average and treating each coordinate separately, solving problem through linking formula to diagram. Students realising visual is easy method.	Phase 3 Explicitation Phase 4 Free Orientation
	Week 3 Lesson 9	
	• Introduction – review of terminology used in previous lesson and two points given with students asked to graph points and find midpoint using formula and check solution using GeoGebra.	Phase 3 Explicitation
	• Class discussion concludes activity bringing all students together again examining visual and combining it with algebra connection of finding midpoint. Teacher instrumental in guiding students to revisit the link between coordinates and midpoint formula, encouraging students to identify the connections. Students are left to find the midpoint of a given example then use GeoGebra to check answers and give students visual connection and consolidates concept. Teacher monitoring language but it is evident that students are becoming more proficient at solving midpoint problems.	
	• Teacher gives student the midpoint and the starting point and asks students how they could find the end point, discussion indicating Phase 3. Students are encouraged to discuss methods and strategies in their working pairs how to find the other point. This involves a rethinking of the original calculation of midpoint problem, typical of a Phase 4 activity. Teacher role diminished although suggestions provided to various students of differing	Phase 3 Explicitation Phase 4 Free Orientation

Lesson Title	Content	Teaching Phases
	abilities, such as to draw points, as stimulus. Once a solution is obtained, students encouraged to use algebra through the manipulation of the midpoint formula to solve and asked to work with partners to see if they can achieve the same answer as provided from visual.	
	• Discussion brings together all information to see what students have done to use the formula with coordinates given to find other point. Students are encouraged to think of each coordinate individually. Students explain that visually you can see how much distance between the given points and create the same distance to find the other point whilst others compared the values. Through full class participation and exchange of ideas language is developed and interchanged as is evident in Phase 3. Looking at each coordinate separately teacher prompts students to see if they can connect information given. Through teacher direction all students are able to insert the values into one side of the midpoint formula equating each coordinate separately using an example given. Students recognise that simple equation rules can be applied to this structure. More questions are given and students are encouraged to use the formula to find the end point given a midpoint and start point, using GeoGebra to help confirm their answer. Students are encouraged to draw and discuss their methods of obtaining an answer. Students bring together all available information and use the midpoint formula in a different application indicative of Phase 4.	Phase 3 Explicitation Phase 4 Free Orientation
Distance	**Week 3 Lesson 10**	
	• Introduction Activity – Using MAP (from Worksheet I) students visually explore the map to find the obstacle that represents the midpoint between two other obstacles. They then must prove their observation using the coordinates within the midpoint formula. Students apply the midpoint concept to a practical situation indicative of Phase 3 openly discussing what they have explored visually. Teacher discussion monitors correct terminology, notation and concept development and introduces new term collinear.	Phase 3 Explicitation
	• Activity extends into introducing the context of the new topic where students are asked to find the distance between two obstacles through open class discussion including brainstorming. The working domain is explored and vocabulary is clarified for the topic. Teacher role crucial in carefully ensuring students recognise and establish correct relations. Thinking initiated for students, some identifying that a practical approach such as creating a right-angled triangle could assist the understanding by observing that the hypotenuse represents the distance to be calculated. Through guided questioning, students recall that Pythagoras theorem is used for right-angled triangles and students identify its formula $a^2 + b^2 = c^2$. Careful probing by the teacher to ensure that students recognise and establish the correct relations students identify the sides associated with the theorem. With teacher guidance through full class participation students successfully calculate the lengths of the sides of the triangle drawn then substitute into the formula. Direct questioning by the teacher encourages students to	Phase 1 Information Phase 2 Exploration

Lesson Title	Content	Teaching Phases
	participate in the solution and enables them to visualise the totality of the structure. Through this practical approach students observe the focus of the topic, that is, how to find the distance between two points. Students then asked to identify the coordinates of the two obstacles carefully monitored by teacher and then told to plot points into GeoGebra to see if they can find a distance tool and get the same result as calculated using the right-angled triangle approach. Students continue the structure of determining the distance or length using Pythagoras Theorem consolidating the concept and connections between distance and Pythagoras' Theorem.	Phase 3 Explicitation
	• Building on this the next activity (no 4 page 34 of Lesson Plans) consolidates Pythagoras' Theorem through finding the smaller side of a right-angled triangle. This involves rethinking the method of applying the theorem, indicative of Phase 4. Teacher allows students to attempt this question, for those capable of solving it independently enabling them to become more proficient. To ensure all students make necessary connections, class discussion is used to confirm methods, re-spiralling back to Phase 3. Discussions include rearranging the equation to find the shorter side. Through this discussion students explore and further their understanding, developing their language and consolidating their ideas with those students capable of solving the problem leading the discussion.	Phase 4 Free Orientation Phase 3 Explicitation
	• This leads to the next activity (no 5 page 34 of lesson plans), where students are given two coordinates and asked to find the distance using whatever method they like to find the solution and then check their solution on GeoGebra, with some students still needing more guidance Phase 3 and others approaching the activity as a Phase 4 being more independent. Students are encouraged to use what they know, with teacher carefully monitoring, suggesting and assisting students to recognise that drawing a triangle will assist with determining the distance between the two points. Further consolidation of concept occurs through a number of similar questions of finding the distance through two coordinates. Students recognising important cues necessary for bringing together information and making connections.	Phase 3 Explicitation Phase 4 Free Orientation
	• Open class discussion enables students to explain their results for finding the distance and then they link the coordinates to how they determined gradient using general notation in terms of (x_1, y_1) and (x_2, y_2).	
	Week 4 Lesson 11	
	• Introduction activity – Students initiate thinking of how to find the distance between two points. They reflect on previous work by drawing a right-angled triangle then finding the length of the sides of the triangle and using Pythagoras' theorem to find the distance. The teacher crucial in an advisory role ensuring the connections established previously are correctly expressed and hence consolidates concept, Phase 2.	Phase 2 Exploration
	• The teacher encourages students to recall the method used previously for deriving a formula for the midpoint and gradient in order to remember a general way that works each and every time. Students are asked to then	

Lesson Title	Content	Teaching Phases
	use this to assist in determining a formula for finding the distance. Students identify to use general notation in terms of (x_1, y_1) and (x_2, y_2). Teacher questioning encourages students to work out ways to establish how to calculate the lengths of the sides of the triangle and this progresses into the formula. Students connect general notation to a formula from previous work encountered and formula is derived, Phase 4.	Phase 3 Explicitation Phase 4 Free Orientation
	• Students attempt using the formula with an example, under teacher direction, and then use GeoGebra to check the distance is correct. The aim to use the formula to calculate the distance which is applicable to any two coordinates. Open discussion used to ensure all students are able to complete calculation using formula. Through more examples association with formula occurs. Students are encouraged to draw the diagram to ensure connection between visual and algebra is established and the realisation occurs that both Pythagoras' Theorem and the formula achieve the same result.	Phase 2 Exploration Phase 3 Explicitation
	• MAP Activity 2 – Students use worksheet based on previous MAP activity that enables them to use MAP as a GeoGebra file to explore concept of distance using the distance tool. Teachers monitor progress and assist students to make the necessary connections. Worksheet contains questions revising simple concepts such as midpoint and extends students with reverse questions where distance from one point is given and students need to work out the other point, involving a rethinking of the distance application, consistent with Phase 4. In this way, students are able to work to their ability and progress through questions.	Phase 2 Exploration Phase 3 Explicitation Phase 4 Free Orientation
Further Graphing	**Week 4 Lesson 12**	
	• Introduction Activity – A guided revision activity of distance formula for students to clarify exactly what they know and consolidate the concept. Students given two points and asked how to find the distance between the points. Teacher guidance scaffolds the activity, Phase 2 and 3, for students of varying abilities, with suggestions of plotting the points, drawing a triangle, Pythagoras' Theorem, linking the length of sides to coordinates, then deriving the formula using general notation in terms of (x_1, y_1) and (x_2, y_2), Phase 4. Time is allocated for students to attempt on their own then teacher guidance ensures all students have correctly established relations, use correct language and connections are sustained. Consolidation occurs through completion of more examples. Students successfully able to describe structure and are proficient with concept of distance.	Phase 2 Exploration Phase 3 Explicitation Phase 4 Free Orientation
	• Further Graphing Introductory Activity – students brainstorm properties they can identify given a graph familiarising them with the working domain and vocabulary. Students recognise previously encountered features such as y-intercept as b, positive slope/gradient as m, gradient represented by rise over run, $y = mx + b$.	Phase 1 Information Phase 2 Exploration
	• Worksheet E and F – students draw a number of graphs aiming to visualise the properties of what they draw. This practical approach assists understanding through observation. Teacher crucial in questioning to ensure	

Lesson Title	Content	Teaching Phases
	correct relations are established with respect to how to draw graphs. To commence a guided example on the board is given ($y = 2x$), where students identified a start point that y-intercept is 0, then through further discussion recognising that gradient, rise over run represents up 2 and run 1 to find next point. Students given time to process this themselves, attempting other graphs on the sheet, the teacher vital monitoring the relations and connections between algebra and graph are correctly developed. Full class open discussion bringing all students together, to clarify vocabulary and consolidate concepts needed to graph. Sheets are to be finished for homework including Worksheet G and H also.	Phase 2 Exploration Phase 3 Explicitation
	Week 4 Lesson 13	
	• Worksheets E, F, G and H – Students asked to explain the structure visualised, hence developing vocabulary. Two features identified as significant for drawing graphs – y-intercept and gradient, leading into the rule $y = mx + b$.	Phase 1 Information
	• Parallel lines Introductory Activity – On board graphs are drawn on same coordinate plane with equations also given. Teacher directed task asks students to identify common features that can be either visual or algebraic. Students recognise that graphs are parallel then students are asked to look at equation to see if they can see the link between equations and graphs that represent parallel line. This introduces the language to students with the context of the topic involved. Students recognise that gradients (coefficients of x term) are the same. Students asked to confirm that this is true for graphs using homework sheets, a practical approach assisting understanding of parallel lines through observation. Students given opportunity to express examples of lines parallel to one given, recognising that the y-intercept is responsible for translating graphs. Students consolidate their findings by explaining how parallel lines occur using their own words, developing language and promoting literacy. Open discussion allowed students to share their definitions and enabled teachers to check that terminology and connections were correct, Phase 3.	Phase 2 Exploration Phase 3 Explicitation
	• Activity (no 3 page 42 of lesson plans) – requires students to make connections between equations and parallel lines. Students given a set of equations without graphs and required to find another equation that would be considered to be parallel to the given one simply through manipulating the equation. By drawing both graphs in GeoGebra enables a quick method of checking solutions. Full class discussion allows students to exchange ideas and offer their solutions. Teacher instrumental in leading discussion assisting students to identify that only the y-intercept needs to be changed to make examples of parallel lines, establishing the parallel line relationship $m_1 = m_2$. Further discussion explores the necessity of the y-intercept leading to realisation that no number after mx represents a y-intercept that is zero.	Phase 2 Exploration Phase 3 Explicitation

Lesson Title	Content	Teaching Phases
	• Perpendicular Lines Introductory Activity – Students draw pairs of lines on the same graph (with GeoGebra used as a checking tool) and investigate features of graph looking to see connections between graphs and equations, Phase 1. Teacher carefully monitors student discussion and drawings. Full class discussion is used and students express features using terms such as “opposite” (identifying gradients are positive and negative), “not parallel” and “perpendicular” (defined as crossing each other students unable to correctly articulate how they cross) with terminology developing, finally linking perpendicular to the idea of right-angles.	Phase 1 Information
	• Further discussion identifies gradients that are the same as being parallel with teacher questioning and probing to establish how to identify perpendicular lines merely by looking at equations. Responses such as “opposite gradients” indicate connections are developing, with students recognising numbers are switched, indicating the concept of “reciprocal”. Students recall what a perpendicular gradient could be verbally, without any formal definition and recognising that y-intercept is irrelevant to the perpendicular lines. The teacher instrumental in monitoring and developing the understanding through correct language usage. Students able to correctly identify perpendicular and parallel lines merely from observation of the equation.	Phase 2 Exploration Phase 3 Explicitation
	• Teacher questioning probes to identify a relationship for perpendicular lines to the equation $y = mx + b$ using general notation. Through a series of teacher guided calculator questions students investigate the solution when multiplying a number by its reciprocal, Phase 2. Students recognise the mathematical relationship that exists i.e. that a number multiplied by its reciprocal = 1. Class discussion leads to recall of condition for which parallel lines exist $m_1 = m_2$ and develops into condition for perpendicular lines $m_1 \times m_2 = -1$, Phase 3.	Phase 2 Exploration Phase 3 Explicitation
	• Concluding Activity – Students copy a cloze passage summarising the conditions for parallel lines and perpendicular lines to exist consolidating ideas of the two concepts, indicative of Phases 4 & 5. This is followed with open class discussion of students providing examples of parallel or perpendicular lines.	Phase 4 Free Orientation Phase 5 Integration
	Week 4 Lesson 14	
	• Matching Game – Students sit together in groups of 3-4 and are provided with a set of flash cards which they need match together. A card has on it either ○ an equation; ○ a graph; ○ a slope and y-intercept; ○ a table of values for a graph.	Phase 2 Exploration Phase 3 Explicitation
	• Students are encouraged to openly discuss their reasoning behind pairing any cards together. This enables students to consolidate terminology and concepts.	

In addition to the teaching sequence, team teaching strategies were employed with the delivery of content. Team teaching, also described as co-teaching, occurs when two or more educators work "collaboratively to deliver instruction to a heterogeneous group of students in a shared instructional space" (Conderman, 2011, p. 1). It has been used widely and is well documented in classes containing children with disabilities, English as a Second Language (ESL), and gift and talented students, where one educator is a general teacher and the other is a specialist teacher in a particular field that assists in delivering content (Friend, Cook, Hurley-Chamberlain, & Shamberger, 2010). Currently, as open classrooms gain popularity, team teaching and the strategies employed with team teaching are becoming more commonly practiced. Friend, et al. (2010, p. 12) identified six different approaches to team teaching. These are

- *one teach, one observe* – where one teacher instructs and the other gathers data, such as for teacher training or research;
- *station teaching* – where seating is arranged in stations (more than two) and teachers rotate between stations offering instruction;
- *parallel teaching* – where students are divided into two groups and each teacher presents the same content to a smaller group with the benefit of greater student participation;
- *alternative teaching* – where one teacher instructs to the majority of students' while the other provides support to a smaller group be it for remediation, enrichment, assessment or other purpose;
- *teaming* – where both teachers lead the instruction, for example offering different methods or explanations on a concept; and
- *one teach, one assist* – where one teacher instructs and the other circulates offering support to students individually.

For the research study, the classroom teacher agreed to team teach with the researcher. During the teaching sequence, combinations of the *one teach, one observe, teaming* and *one teach, one assist* approaches were used. Team teaching provided the researcher with a better perception of classroom dynamics, issues and concerns with the teaching sequence. It also enabled the classroom teacher to suggest modifications or additions to the teaching sequence in the presence of the researcher without comprising the teaching sequence.

4.3.2.2 Data Collection Structure

This section presents the data collection methods employed by the researcher throughout the study. It has been divided into sub-sections: Google Forms, Screen Capture Software and Other Data Collect, each providing more detail regarding the gathering of the data.

Multiple data collection methods were engaged during the study. Throughout the teaching sequence, various qualitative data sources were collected in order to capture a “thick description” (Punch, 2009, p. 261) of the context of the research, relevant to the problem and research questions (Wiersma, 2000). Methods of data collection included Google Form test results and their Extended Response worksheets, screen capture footage from individual students (using either Camstudio or Camtasia) and analysis of artefacts (e.g., photographs of class activities and teacher explanations, along with student workbook samples).

The primary source of data collected during the research study came from the three Google Form tests. In addition, each lesson of the teaching sequence was recorded, using Screen Capture Software (SCS). Despite attempts to eliminate issues with data collection, there were two that presented relating to technology. The first issue, concerned the students’ laptops being present and charged, a key responsibility of students. The second issue concerned technical aspects of the technology, including issues with the screen capture software, Internet connections and wireless networks.

Google Forms

Google Forms is a free, collaborative application within the Google Docs environment that provides a flexible form and survey builder interface, integrated within the Google Drive. It offers built-in reporting through Google Spreadsheets, as shown in Figure 4.8, available in the Google Docs suite that can be downloaded as a Microsoft Excel or Open Office file for further manipulation. To create a Google Form, it is necessary to have a Google account that provides the user with access to a Google drive and numerous Google Apps.

K	L	M	
6. Using Geogebra can you draw the graph of y = 4x + 8?	**a. How would you do this without Geogebra using pen and paper**	**b. Can you move the graph using Geogebra? Notice what changes on the graph and what changes in the equation. Explain these changes below.**	t
I typed y=4x+8 into the input bar	by using a calculator and a ruler	the equation remains y=4x but the thing we add changes	t
I typed the equation in at the bottom bar of geogebra	y=4x+8 8 is your starting point on your x axis 4/1 is your rise over run	its the same x axis different y axis	t
I typed in the equation?	idk	idk	p
by typing in the equation in the imput section at the bottom left corner of geogebra	it might be easy to complete a table first of the points	no i couldn't	t t t g
I wrote the equation	using a certain	the +8 changes its	t

Figure 4.8: Google Form responses spreadsheet

All staff at the chosen school were provided with a Google Gmail account and the Google Drive was used regularly for communication of documents. This provided the researcher with the appropriate tools to create the three tests for primary data collection. The Google Form provided an online environment where students were able to type in answers using a predefined form, shown in Figure 4.9. The only issue that presented was that formulas and equations were not easily transcribed into the Google Form environment because no specific mathematical editor was provided. Hence, students were required to consider ways of presenting their answers in a manner that reflected the mathematical expression or formula they wanted to convey. Once completed, students were required to submit the form and the results were then collated for further review.

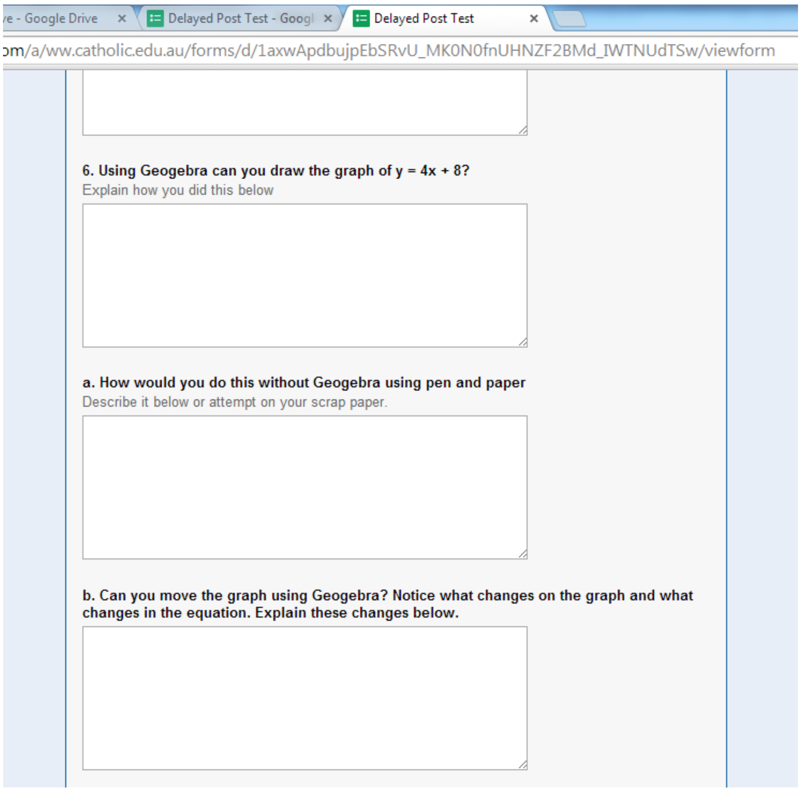


Figure 4.9: Live view of Google Form

Screen Capture Software

SCS delivers a background application with the ability to capture a window, a selected portion of the screen or the entire computer screen. The application records all of the actions occurring in the region in real-time video and audio. This includes, and is not limited to, mouse clicks and movements around the screen, typing, copying and pasting actions, and opening of additional windows. Screen capture software provides a non-biased and effective data collection tool capable of collecting rich, thick data (Chaney, Barry, Chaney, Stellefson, & Webb, 2013; Hider, 2005; Imler & Eichelberger, 2011).

For this study, Techsmith's Camtasia and open source Camstudio were used, with the latter software pre-loaded onto students' school laptops. Both packages have been used to produce video tutorials for e-learning and lectures and are widely recognised as a valuable tool for collecting data (Falloon, 2005). Only one pair of students were able to use Techsmith's Camtasia, which was loaded into the researcher's laptop, with the remaining students using Camstudio on their school laptops. Camtasia ($199 at time of buying), released in the early 2000's, was one of the first SCS programs on the market capable of powerful editing features, mixing and re-mastering of captured video and audio to produce professional quality videos. In comparison, Camstudio is a free download SCS having only limited basic features. Screenshots of both software is shown in Figure 4.10. The largest hurdle with the SCS was the size of the created data files, that were saved in .avi and .swf format. One class lesson (approximately 63 minutes) of full screen audio and visual data would cause the students laptops to crash because of the size of the file placed on the memory. To address this issue, a lossless compressor codec was loaded onto each laptop. This compressed the data into smaller, more manageable files (Camstudio, 2013).

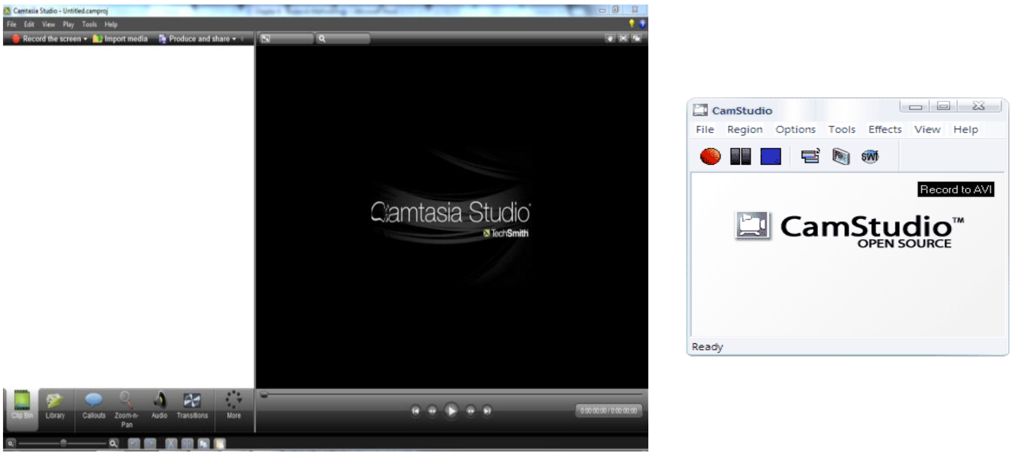


Figure 4.10: Screen shot of Camtasia vs Camstudio

Both SCS programs, Techsmith's Camtasia and Camstudio, were integral to the data collection process. They offered a means of probing into the thought processes of students undertaking the learning activities. The software was activated at the beginning of each lesson via a keystroke. The only noticeable difference between the two SCSs was a pop up screen for Camstudio and a small icon in the bottom right corner for Techsmith's Camtasia. Students were reminded at the end of the lesson to turn off the software, saving the file with their name and the current date. Files were then transferred to the researcher's hard drive, which provided a time line of lessons. The researcher's laptop was password protected as per Human Research Ethic Committee (HREC) requirements. Each lesson of the teaching sequence was recorded by at least one student, resulting with over 25 hours of video being captured. This provided an exclusive insider view of students' working methods during the structured lessons.

Other Data Collected

A variety of artefacts were collected during the teaching sequence, these included: photographs of activities, such as Algebra Walk and Matching Game activities shown in Figures 4.11 and 4.12 respectively; scanned pages from students work books, as shown in

Figure 4.13; video footage of practical activities (such as Algebra Walk and Matching Game) and photographs of teacher board work. These items supported the information gained from the three tests and the recordings of each lesson to assist with the subsequent data analysis.

Figure 4.11: Algebra walk activity

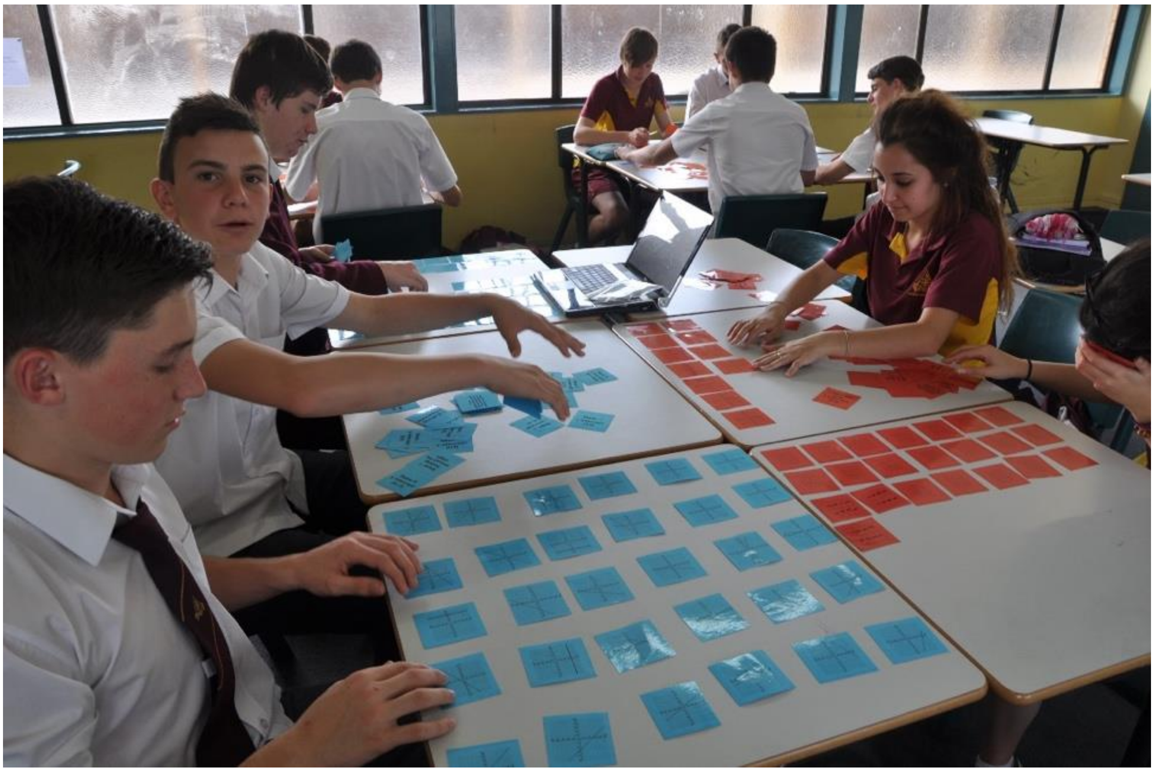


Figure 4.12: Matching pair activity

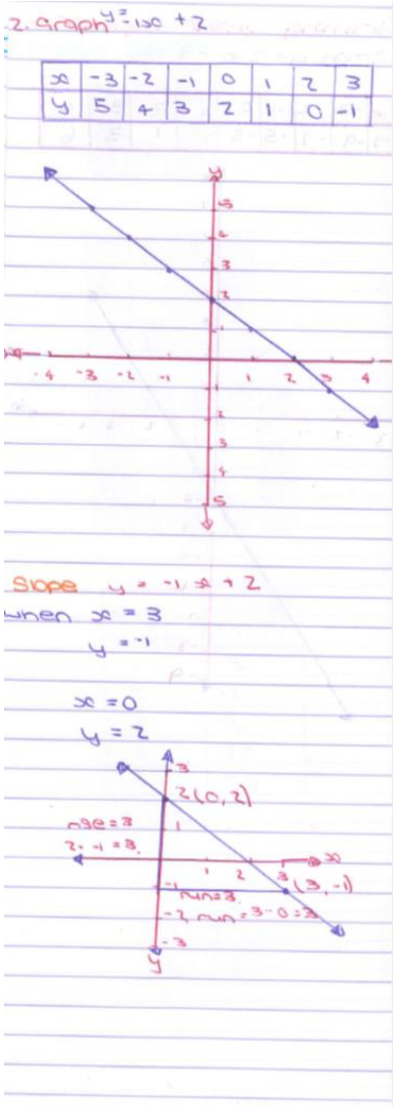


Figure 4.13: Work sample

This section detailed the teaching sequence and the data collection methods employed for the study. Activities in the teaching sequence were described and mapped to the van Hiele Teaching Phase being addressed. The data collection methods for the research study, Google Forms, SCS and other methods were described with examples provided for clarity.

In summary, this section explored matters relevant to the design of the research project. Firstly, it outlined the key principles for the methodologies chosen: Action Research Methodology and Case Study Methodology. It then discussed the participants chosen for the study and identified considerations made for choosing the appropriate sample. Next, the teaching sequence was detailed, with activities defined and mapped to the relevant van Hiele

Teaching Phase being addressed. Finally, the Data Collection Structure was described, itemising the tools used and type of data collected with examples provided.

4.4. Methodological Issues

This section considers the research design from a theoretical perspective in terms of the selection of the research instrument, and considerations throughout the study with the testing procedures implemented. This analysis provides justification for the appropriateness of the Google Form tests as a data collection tool.

4.4.1. Justification of Test Design

Punch (2009) stated that “qualitative research methods is a complex, changing and contested field – a site of multiple methodologies and research practise. ‘Qualitative research’ therefore is not a single entity, but an umbrella term that encompasses enormous variety” (p. 115). Hence, because of the generalised nature of qualitative methods, it is important to identify those qualitative methods chosen for the study and to carefully consider their appropriateness.

Testing was chosen as a suitable research tool since it offers researchers a powerful method of data collection (Cohen, et al., 2000). Three tests were considered for the research study: a Pre-test, End of Topic test and a Delayed Post-test. Cohen, et al., (2000) claim that certain considerations must be addressed to ensure the reliability and validity of the test as a research instrument. In summary (Cohen, et al., 2000, p. 321):

- identify the purposes of the test;
- identify the test specifications;
- select the contents of the test;
- consider the form of the test;
- write the test item;
- consider the layout of the test;
- consider the timing of the test; and
- plan the scoring of the test.

In addition to these Cohen, et al., (2000) emphasise specific guidelines for devising Pre-test and Post-tests. Of those mentioned, two that relate to the research study are (Cohen, et al., 2000, p. 334):

- The Pre-test may have questions which differ in form or wording from the post-test, though the two tests must test the same content; and
- The level of difficulty must be the same in both tests.

Together, these points provide a broad outline of the considerations of testing as a qualitative research tool. However, each characteristic mentioned above requires decisions to be made by the researcher based on the research questions involved. Each of these issues are discussed below in order to justify the adopted methodology.

4.4.1.1 Identify the Purposes of the Test

Cohen, et al. (2000) stated that there are several purposes for a test: "to *diagnose* a student's strengths, weaknesses and difficulties, to *measure* achievement, to *measure* aptitude and potential, to identify *readiness* for a programme" (p. 321). This research study utilised testing for a combination of these reasons. The Pre-test was used to identify prior knowledge of the Linear Relationships topic and create a base line from which to compare responses from subsequent tests. With the research focus being to analyse the improvement in responses after implementing a specific teaching sequence, the Pre-test was important as "one can only assess how much a set of educational experiences has added value to the student if one knows that student's starting point and starting abilities and achievements" (Cohen, et al., 2000, p. 322).

The End of Topic test was a form of summative testing given at the end of the teaching sequence, designed to measure the student's achievement concerning the outcomes and content. The Delayed Post-test was administered six weeks after the completion of the teaching sequence. The purpose of the Delayed Post-test was to ascertain how much understanding students had retained upon completion of the Linear Relationships unit. Six weeks was decided as a suitable period, as it included two weeks of holidays.

4.4.1.2 Identify the Test Specifications

An important aspect of the test specifications is in the identification of the objectives, student learning outcomes and content that the test will address. All three tests, the Pre-test, End of Topic test and Delayed Post-test, followed the Australian Curriculum outcomes and content for Stage 5.3 Linear Relationships as listed in Table 4.2. The number of items in the tests relied upon ensuring that the content and outcomes were fairly addressed, with an extended response item provided to enable written work to be included. Weighting of the test items was not an important consideration since it was the improvement in complexity of the responses provided that was the main feature in determining understanding, not the number of correct responses.

4.4.1.3 Select the Contents of the Test

According to Cohen et al. (2000), when selecting the contents of a test, there are two main considerations: that the test is subject to item analysis; and how each element of the test is operationalised. Item analysis refers to the independence, clarity, unambiguity and relationship of each item towards the learning outcome. These issues were addressed through thorough reviews made by the researcher and supervisor. Considering how each element of the test was operationalised refers to how the test indicates high, moderate and low achievement, what question types are used and identifying the content to be covered. The content of the test was based on the curriculum outcomes and content for Stage 5.3 Linear Relationships, and question types were chosen to be mainly short answer to enable students to provide justification for any answers provided.

4.4.1.4 Consider the Form of the Test

A major consideration in deciding the form of the test was the recording of responses for item analysis. Google Forms were considered a useful and easy way to present the test and collate the results, with an extended response sheet also given to enable written responses to be provided. Google Forms would automatically collate the responses into a spreadsheet, which could then be manipulated for clarity of viewing the responses. Students were able to complete the test in their working pairs for the Pre-test and End of Topic test. For the Delayed Post-test, students completed the test individually.

4.4.1.5 Write the Test Item

When writing the test items, a variety of question types were used, including multiple choice, single calculated answer and short answer. Cohen et al. (2000) summarised potential problems with several question types, which will now be discussed further.

For multiple choice questions, it was identified that the number of choices, location of the correct response, the sequence of items and the realism of the distractors all have potential to cause issues. Only one multiple choice question was used in the three tests, shown in Figure 4.14. The question contained five choices with three correct solutions. The three correct solutions were not positioned in order and students were required to provide justification as to how they came to their solution in the next question, hence the possibility of making the correct choice through incorrect reasoning could be explored. The question required a low level of readability and there was no clue in the stem as to which may be the correct response. The choices were presented in a format that related to GeoGebra, enabling students to use GeoGebra as a checking tool. The only ambiguity that existed was there was no indication to the students that more than one choice was correct.

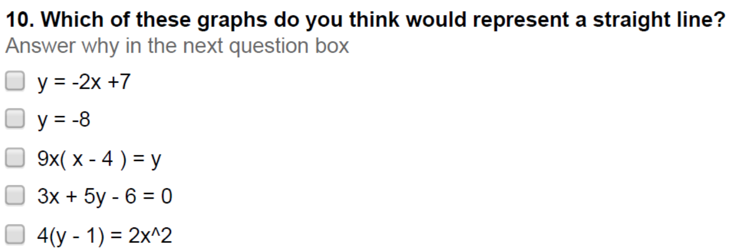
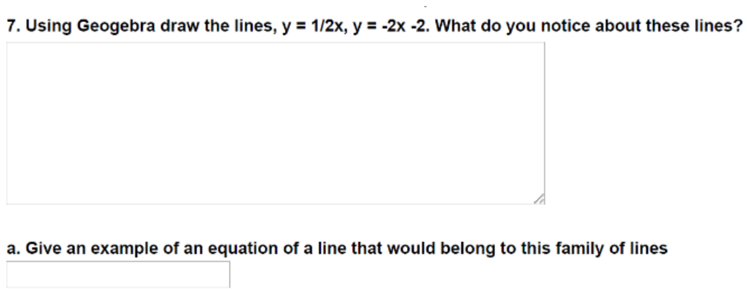

10. Which of these graphs do you think would represent a straight line?
Answer why in the next question box
- y = -2x +7
- y = -8
- 9x(x - 4) = y
- 3x + 5y - 6 = 0
- 4(y - 1) = 2x^2

Figure 4.14: Multiple choice question from End of Topic test

When constructing short answer items Cohen et al. (2000) listed several guidelines to reduce potential issues. Most of the listed guidelines make reference to the number of blank spaces allocated in short answer questions; however, this was eliminated with the questions included in the three tests used for the study because a short answer question was provided with a box in which to enter data, as shown in Figure 4.15. Two types of short answer question types were available on Google Forms, the larger box, as shown in the top of Figure 4.15 is the paragraph option, with the smaller box in the bottom of Figure 4.15 being the short answer option. The questions posed are simple, ensuring that the specificity of the answer required is clearly understood without ambiguity as demonstrated below.


7. Using Geogebra draw the lines, y = 1/2x, y = -2x -2. What do you notice about these lines?

a. Give an example of an equation of a line that would belong to this family of lines

Figure 4.15: Short answer question from End of Topic test

4.4.1.6 Consider the Layout of the Test

Cohen et al. (2000), suggests that the layout of a test can have an extreme effect on the test. Considerations for the layout of the test include the appropriateness and clarity of instructions on the test, readability of the items, the progression of difficulty and sequence of the items, the visual design or the page and the grouping of items.

Instructions were provided at the beginning of each test along with verbal instruction. For the Pre-test, being the first time students had used both Google Forms and GeoGebra, reiteration of instructions was required during the test because students were unfamiliar with the environments in which they were working. Instructions for each question were clear and concise, ensuring low level readability. Questions were sequenced such that the concepts were addressed individually at first and then progressed into problem-solving questions. Students were encouraged to work through and attempt as many questions as possible with the level of difficulty increasing with the question number. The default presentation style was used for all three Google Form tests, so that students were not distracted by any visual aspect of the form.

4.4.1.7 Consider the Timing of the Test

When considering the timing of the test, two areas must be addressed: when the test will take place and the allocated time for the test. The specific time that the test will take place, "is a matter of reliability, for the time of day, week, etc. might influence how alert, motivated, capable a student might be" (Cohen, et al., 2000, p. 331). With respect to this consideration, the tests occurred at the first available opportunity because of time restrictions by the Scope and Sequence. Each test was allocated one period (approximately 63 minutes), for completion.

4.4.1.8 Plan the Scoring of the Test

Scoring of tests was not considered a priority by the researcher since it was the quality of the responses that were valuable, not the number of correct or incorrect responses. This was made clear in the verbal instructions given at the beginning of each test.

The Google Form tests used are considered to be forms of non-parametric testing, a form of testing tailored to the specific circumstances by the teacher, and providing the opportunity for quick feedback on student performance.

In summary, this section examined the Google Form tests as an appropriate data collection tool for the research design structure proposed in the previous section. It identified a number of issues that Cohen et al. (2000) suggested should be considered to ensure the reliability and validity of testing as a method of data collection. Each of these considerations was defined and addressed with respect to the study.

4.5. Approach To Data Analysis

The design of this study, as described previously, involves the implementation of Google Form tests as the research instrument. The SOLO model was used to assist with data analysis and provides an effective framework for attaining deeper understanding of the van Hiele Teaching Phases. The data was analysed using the qualitative method of thematic content analysis. The following section details the application of this method when analysing the responses.

4.5.1. Thematic Content Analysis

Thematic Content Analysis (TCA), also known as Thematic Analysis (TA), is an inductive approach that is poorly defined and seldom recognized as analytical method, despite its widespread use, particularly in medical research. Based on the more traditional and well documented method of content analysis (CA), which establishes categories and bases the analysis on the frequency of attributes, TCA distinguishes itself by permitting the researcher to understand the context of the themes (Joffe, 2011). It is "a data reduction and analysis strategy by which qualitative data are segmented, categorised, summarised and reconstructed in a way that captures the important concepts within the data set" (Given, 2008, p. 868). Braun suggests that TCA should be the first qualitative method of analysis learnt by researchers since "it provides core skills that will be useful for conducting many other forms of qualitative analysis" (Braun & Clarke, 2006, p. 78). TCA is not affiliated with any one particular theoretical approach so can be applied to any research design. It is a systematic strategy that remains faithful to the data, aiming to "reflect a balanced view of the data, and its meaning within a particular context of thoughts, rather than attaching too much importance to the frequency of codes abstracted from their context" (Joffe, 2011, p. 219).

The first approach to analysis considered the responses submitted for each question of the three tests. The Google Spreadsheet that was automatically created for each Google Form test provided each question with the submitted responses. This was transcribed into a document where significant themes within the responses relevant to the question were identified. Marks & Yardley (2003) define a theme as "a specific pattern found in the data in which one is interested" (p. 57). "A theme captures something important about the data in relation to the research question, and represents some level of patterned response or meaning within the data set" (Braun & Clarke, 2006, p. 82). Identifying patterns and themes within the data collected assisted in the data analysis; the aim being, to make the data intelligible and relevant to the research questions (Fetterman, 1989; Hammersley, 1995).

The second approach to analysis considered the data obtained from two individual students. This enabled the researcher to examine the case in depth. Viewing the data of the two individual students captured a glimpse of their learning journeys throughout the entire Linear Relationships unit.

4.5.2. SOLO Model

SOLO is an acronym for **S**tructure of the **O**bserved **L**earning **O**utcome and defines a generalizable model, developed by Australian duo, John Biggs and Kevin Collis, that identifies the sequence of learning through the structure of student responses (Serow & Callingham, 2011). The model is a systematic way of describing how students' understanding develops both in complexity and level of abstraction when mastering tasks, through observing the quality of their responses. The SOLO model was discussed in depth in Chapter 3 and can be referred to there. The nature of the responses is presented in Chapters 5 and 6.

In summary, this section discusses the methods and approaches used for data analysis for this research project. It provides the background information required for the understanding of the coding used in the following results chapters. Each question of the Google Form test is analysed using a thematic coding followed by SOLO coding. Both methods provide valuable information to assist in analysing the development of understanding for the student.

4.6. Evaluation

This section discusses the strengths and weaknesses of the research design and analysis plan. The research findings and derived conclusions of a study are considered to only be as good as the data on which it is founded (Punch, 2009). "While reliability is concerned with the replicability of scientific findings, validity is concerned with the accuracy of scientific findings" (LeCompte & Goetz, 1982, p. 31). Although threats to validity and reliability of research can never be totally erased, solid research attends to the issues surrounding the credibility of its methods and findings. Cohen et al. (2000) suggest that "reliability is a necessary but insufficient condition for validity in research; reliability is a necessary precondition of validity" (p. 105).

Evaluating qualitative inquiry is considered to be elusive, with suggestions that the traditional criteria of validity and reliability are not the most suitable for qualitative action research (Stringer, 2008). Lincoln and Guba (cited in Stringer, 2008) propose that due to the subjectivity of qualitative research the most valuable method of demonstrating the success of the research study "is to identify ways of establishing *trustworthiness,* the extent to which we can trust the truthfulness or adequacy of a research project" (Stringer, 2008, p. 48). Hence, this section considers the research design and analysis plan in terms of its validity, reliability and trustworthiness.

4.6.1. Validity

The validity of a study focuses on how well the main components; namely, the questions, design and methods, fit together to provide quality research. Thus, increasing the confidence placed in the results and the credibility of the findings. The validity of a research study is not considered as an absolute state but rather in terms of varying degrees (Wiersma, 2000). A researcher's objective is to increase a study's validity through addressing issues that undermine the two concepts that simultaneously define validity: internal and external validity: "internal validity is the extent to which the results can be interpreted accurately and external validity is the extent to which the results can be generalised to populations, situations and conditions"(Wiersma, 2000, p. 4).

For qualitative inquiry, validity refers to the degree to which the research questions measure and represent what they claim they should. Concerns regarding invalidity generally occur because of some form of bias, "a systematic or persistent tendency to make errors in the same direction, that is, to overstate or understate the 'true value' of an attribute" (Cohen, et al., 2000, p. 120). Factors which contribute towards bias of qualitative data include "the subjectivity of respondents, their opinions, attitudes and perspectives" (Cohen, et al., 2000, p. 105). Bias is reduced through "careful formulation of questions so that the meaning is crystal clear" (Cohen, et al., 2000, p. 122).

For action research, Stringer (2008) suggests that procedures for assessing validity differ from those used for experimental studies. Rather than measuring the degree of validity, a new criteria proposed by Lincoln and Guba (cited in Stringer, 2008) identifies the trustworthiness of the action research project. This involves recognising procedures that establish "the extent to which we can trust the truthfulness or adequacy of a research project"

(Stringer, 2008, p. 48). Four criteria are stated that establish trustworthiness: credibility, transferability, dependability and confirmability (Stringer, 2008).

The following section will explore the threats to internal and external validity relative to qualitative research as well as include the criteria specific for action research.

4.5.1.1. Internal Validity

Internal validity establishes the extent that the outcomes project accurately and confidently describe that which is being researched (Wiersma, 2000). It “seeks to demonstrate that the explanation of a particular event, issue or set of data which a piece of research provides can actually be sustained by the data” (Cohen, et al., 2000, p. 107). Various factors affect the internal validity of a research project. These include history, testing, instrumentation, observer effects and differential attrition. Each of these is explained below.

History refers to “any event, other than a planned treatment event, that occurs between the pretest and posttest measurement of the dependent variable and influences the post measurement of the dependent variable” (Johnson & Christensen, 2004, p. 235). To reduce the effects of history in this research project, students completed the Google Form tests on the set day, any students who were absent did not complete the test at all.

Testing refers to “any change in scores obtained on the second administration of a test as a result of having previously taken the test” (Johnson & Christensen, 2004, p. 236). To reduce the effects of testing in this study, the End of Topic test, which occurred at the end of teaching sequence and four weeks after commencing the topic, used different numerical values for the coordinates and equations such that answers could not be memorised. The Delayed Post-test, which was an exact copy of the Pre-test, occurred six weeks after the End of Topic test, which included two weeks of school holidays. The time interval between tests and including the school holidays provided a buffer for students memorisation of Pre-test results, which was almost ten weeks prior.

Instrumentation refers to “any change that occurs in the measuring instrument” (Johnson & Christensen, 2004, p. 237). This threat occurs when an instrument used in pre-testing is not completely equivalent to that used in post-testing. To reduce the effects of instrumentation in this research project the same form was used for all three tests with numerical values changed for the End of Topic test as previously stated. Factors that increased the effects of instrumentation in this research project were the addition of an extra question in the End of Topic test and an error in the wording of one of the questions in the End of Topic test.

Observer effects refer to the threat to validity imposed by an observer – for this project the researcher – upon the natural setting. This threat was reduced by the fact that the researcher was well known to the students, having taught most of the students during their time at the school; hence a familiar setting was maintained.

Differential attrition refers to the fact that "some individuals do not complete outcome measures"(Johnson & Christensen, 2004, p. 241). This relates to the study in that not all students were present for each of the Google Form tests. Being absent from school on the scheduled day of the test resulted in a loss of data for that person and/or pair. For the End of Topic test some students chose to re-group, since their partner was away in order to complete the test.

4.6.1.2 External Validity

"External validity refers to the degree to which the results can be generalized to the wider population, cases or situations" (Cohen, et al., 2000, p. 109). Generalisability is difficult to achieve and "to a large extent, internal validity is a prerequisite for external validity because if results cannot be interpreted it is not likely that they can be generalised" (Wiersma, 2000, p. 6). External validity is maximised through providing "clear, detailed and in-depth description" (Cohen, et al., 2000, p. 109) of results, thus enabling others to decide on its generalisability. Three threats to external validity have been identified for this study that limit the degree to which generalisability can occur: population validity, ecological validity, and temporal validity.

Population validity refers to the "ability to generalize the study results to individuals who were not included in the study" (Johnson & Christensen, 2004, p. 242). It refers to the inability for research to be compared to "across groups because they are specific to a single group or because the researcher mistakenly has chosen groups for which the construct does not obtain" (LeCompte & Goetz, 1982, p. 51). The study involved students studying Stage 5.3 Mathematics.

Ecological validity refers to the "ability to generalize the results of a study across settings" (Johnson & Christensen, 2004, p. 245). It refers to the generalisability of the research to other environments or, in other words: is the research dependent on the setting in which the study occurred and were the results a function of their context? The research project was designed based on the Australian Curriculum for Stage 5.3. The results are independent of the setting as all students of Australia, whether in rural or city locales, private or public school, are required to complete the same outcomes.

One subtle factor that affects the generalisability of the study occurs when a participant alters their performance as a result of being aware that they are partaking in a research project; known as *reactivity effect*. The reactivity of the data "concerns the extent to which the process of collecting the data changes the data" (Punch, 2009, p. 313). It also occurs when "the presence of the researcher alters the situation as participants may wish to avoid, impress, direct, deny, influence the researcher" (Cohen, et al., 2000, p. 157); this is known as the Hawthorne effect.

Temporal validity refers to the "extent to which the results of a study can be generalized across time" (Johnson & Christensen, 2004, p. 245). The study was designed based on the Australian Curriculum, which was to be implemented the year following the data collection. Hence, data collected remains relevant while the Australian Curriculum is the current syllabus.

Through the consideration of the potential threats to both internal and external validity, various strategies were implemented or issues were addressed to ensure that the degree of validity of the research project was maximised and the invalidity minimised as much as possible.

4.6.2. Reliability

Reliability is concerned with maintaining consistency and accuracy, focussing "on the stability of the results across time, settings and samples" (Stringer, 2008, p. 47). It refers to the "replicability and consistency of the methods, conditions, and results" (Wiersma, 2000, p. 9). Validity and reliability may be considered related concepts since research cannot be valid without being reliable. Reliability is also a quantifiable concept that may be considered in terms of varying degrees.

4.6.2.1 Internal Reliability

Internal reliability refers to "the extent that the data collection, analysis, and interpretations are consistent given the same conditions" (Wiersma, 2000, p. 8). When internal reliability is found to be lacking, the data becomes "a function of who collects them rather than what actually happened" (Wiersma, 2000). Strategies used to reduce the threat of internal reliability for the study include low inference descriptors, peer examination and mechanically recorded data, each is discussed below (LeCompte & Goetz, 1982).

Low inference descriptors refer to the accuracy of the data presented. Through providing precise and descriptive notes rich in primary data, the reader can decide whether they accept or reject the findings. Because the main source of data was stored digitally, the data was able to be reviewed numerous times to ensure that the accuracy of transcriptions and intended meanings were captured.

Peer examination occurred because findings were discussed with the class teacher, who was an experienced mathematics educator, along with discussions with an experienced researcher in this field.

Through utilising the SCS Camstudio and Camtasia preloaded onto computers, data was *mechanically recorded*. The recording of each lesson using such software enabled the researcher to focus on lesson structure without having to make written notes. Both SCS programs, once activated, worked in the background, permitting the student to focus on using GeoGebra without being aware of recording software, thus fostering the recording of actual student responses to current work.

4.6.2.2 External Reliability

External reliability refers to "the issue of whether or not independent researchers can replicate studies in the same or similar settings" (Wiersma, 2000, p. 9) This involves providing sufficient descriptions of procedures and conditions to enable the research project to be replicated. To address this, a detailed description of the teaching sequence implemented including activities and the associated van Hiele Teaching Phases that the activities were mapped to, was provided within the Research Design section of this chapter, along with comprehensive lesson plans provided in Appendix A.

External reliability is improved through the use of multiple data-collection procedures (Wiersma, 2000, p. 260). As discussed earlier in the chapter, multiple data collection methods were employed, namely, the use of Google Forms, Screen Capture Software along with various other artefacts also improved the external reliability of the project.

Concepts affecting the reliability of the research project have been defined and examined in this section. Potential threats to internal and external reliability were identified and discussed with respect to the research study in order to ensure that the degree of reliability was increased.

4.6.3. Trustworthiness

Trustworthiness is the term Stringer (2008) uses to define the reliability and validity of action research. He claims that the traditional criteria used to evaluate the rigor of an experimental or survey research are not suitable for qualitative action research. Due to the subjective nature of qualitative research, Lincoln & Guba (cited in Stringer, 2008) suggest that “because there can be on objective measures of validity, the underlying issue is to identify ways of establishing *trustworthiness*, the extent to which we can trust the truthfulness or adequacy of a research project” (Stringer, 2008, p. 48). The extent to which trustworthiness is established is attributed to four procedures: credibility, transferability, dependability and confirmability, which are outlined below.

4.6.2.1 Credibility

Credibility refers to the “plausibility and integrity of the study” (Stringer, 2008, p. 48). Procedures that enhance the credibility of an action research project include prolonged engagement, persistent observation, triangulation, referential adequacy and member checks.

Prolonged engagement refers to spending extended time in the research environment facilitating relationships with the participants “allowing them to gain greater access to ‘insider’ knowledge rather than the often superficial or purposeful information given to strangers” (Stringer, 2008, p. 48). The researcher had been a member of the staff of the secondary school chosen as the sample population for over 10 years. As such, the relationship between the researcher and students, along with the research environment remained a natural classroom atmosphere promoting the credibility of the information collected.

Persistent observation refers to not only being present for extended periods but also to recording repeated observations to provide depth. Every lesson of the teaching sequence was recorded using a minimum of two devices, enabling the researcher to capture different conversation data for the same activities.

Triangulation refers to “the use of multiple and different sources, methods and perspective to corroborate, elaborate or illuminate the research problem and its outcomes” (Stringer, 2008, p. 49). A variety of data collection methods were employed for the study to provide triangulation. For example, lessons of the teaching sequence were recorded using SCS (with data from a minimum of two devices being retained), photos of the board work were obtained and students workbooks were scanned. “These multiple sources and methods provide the rich resources for building adequate and appropriate accounts and understandings that form the base for working toward the resolution of research problems” (Stringer, 2008, p. 49).

The triangulation of data also assisted ameliorate researcher bias. While the co-teacher researcher was known to the students, the mere presence of the researcher has the potential to influence trustworthiness of data. Through the use of multiple data collection methods this bias was reduced.

Referential adequacy refers to “the need for concepts and structures of meaning within the study to clearly reflect the perspective, perceptions, and language of participants” (Stringer, 2008, p. 50). Google Forms tests were in English and all students were able to read, understand and write in English. No students were of the ESL background.

Member Checks refers to a process of review whereby “the participants be given frequent opportunity to review the raw data, the analysed data, and reports that are produced” (Stringer, 2008, p. 50). Students were given the opportunity to view any data that was collected through their laptops at any point during the teaching sequence. Upon request, they could also have access to the data, either raw or analysed, and any reports that were created.

4.6.3.2 Transferability

The extent to which a research project is transferable refers to the possibility of the results being applicable or transferred to another context so others may take advantage of the findings from the project. The transferability of research is enhanced by thickly detailed descriptions that “contribute to the trustworthiness of a study by enabling other audiences to clearly understand the nature of the context and the people in the study” (Stringer, 2008, p. 50).

4.6.3.3 Dependability

Trustworthiness of a project relies also on the dependability of the research, which refers to the extent to which “observers are able to ascertain whether research procedures are adequate for the purposes of the study” (Stringer, 2008, p. 50). This is achieved through making the details of the research process openly available for scrutiny. It includes access to defining the research problem and describing the procedures for collecting and analysing the data. This is also known as an inquiry audit and forms the methodology chapter.

4.6.3.4 Confirmability

Confirmability refers to the extent with which the outcomes of the research are drawn from the data and includes the review of both raw and analysed data, plans and reports derived from data (Stringer, 2008, p. 51). This audit trail enables observers to confirm the research "accurately and adequately represents the perspectives presented" (Stringer, 2008, p. 51). The confirmability of this study relates to the review process whereby specific individuals are asked to evaluate the project, its findings and thesis.

In summary, the extent of trustworthiness of an action research project assist in defining the rigor of the research. Through defining and addressing issues of trustworthiness, such as the credibility, transferability, dependability and confirmability, the validity and reliability of the research project is increased, which enhances the credibility of the findings reported. While the details provided in this methodology chapter addresses many of the issues associated with the trustworthiness of the study, of particular note, the multiple and varied methods of data collection ensures that the findings of this project are enhanced and the extent of trustworthiness is improved.

4.6.4. Validity and Reliability in Tests

Research instruments must be administered such that the degree of reliability is maximised. For this study, three separate tests were used as the basis of the data collected. There exists a range of issues that affect the reliability of instruments such as tests; these include "the time of day, the time of the school year, the temperature in the test room, the perceived importance of the test, the degree of formality of the test situation, 'examination nerves', the amount of guessing of answers by the students (the calculation of *standard error* which the tests demonstrate feature here), the way that the test is administered, the way that the test is marked, the degree of closure or openness of test items" (Cohen, et al., 2000, p. 130).

Feldt and Brennan (as cited in Cohen, et al., 2000) suggest four types of potential sources that pose threats to the reliability of tests; these factors include influences based on:

- *Individuals*, such as their motivation, concentration, forgetfulness, health, carelessness, success from random guessing, mental competence, including reading and problem-solving ability. Because of the personal nature of individual factors, it is difficult for any research project to identify or eliminate these completely;

- *situational factors*, such as the working environment for the test, including psychological and physical conditions of the examinees. The working environment for the research project involved the same classroom for each lesson, which was the student samples' regular classroom for the school year;
- *test marker factors*, such as idiosyncrasy and subjectivity particularly with free response and open ended questions. The Google Form tests were predominantly examples of objectivity tests where each question had a specific answer. Students could obtain the answer using a variety of methods, however the process required for each was generally of a definite nature;
- *instrumental variables* such as mechanical and electronic equipment issues and measurement procedures. For this study, while technical issues arose with the computers and sometimes software, each was dealt with and resolved as quickly as possible.

Although often difficult to organise, in principle, data from tests should be collected in a manner such that error sources are acknowledged. This enables the test to reflect true score components. For the Google Form tests, data was automatically transferred to a spreadsheet by the Google software so that data was collected instantaneously once the student had completed the test and submitted their responses.

In summary, the Google Form tests provided an excellent instrument with which to collect data for the study. The validity and reliability of the instrument was enhanced through addressing issues as they arose. Of all the threats posed to the reliability of the tests, the instrumental variables influenced the research project most. However, these were resolved quickly with as little disruption as possible to the activity taking place.

4.7. Ethical Considerations

It is necessary within the design of a research project to carefully consider the balance between validity, reliability and trustworthiness, as have been discussed; but of equal importance to the research project are the ethical considerations of the design. Attention to ethical conduct is required to "protect the well-being and interests of research participants"

(Stringer, 2008, p. 44). Ethical issues often arise from the methods implemented while attempting to obtain reliable and valid data (Cohen, et al., 2000, p. 49).

An application addressing ethical issues such as informed consent, access and acceptance, privacy and confidentiality, was approved by the University of New England's Human Research Ethic Committee (HREC), which strictly adheres to the National Health and Medical Research Council (NHMRC) *National Statement on Ethical Conduct in Human Research 2007*. The HREC approval is provided in Appendix C. The hierarchy of ethical considerations is illustrated in Figure 4.16. Ethical issues pertinent to the research project are also discussed.

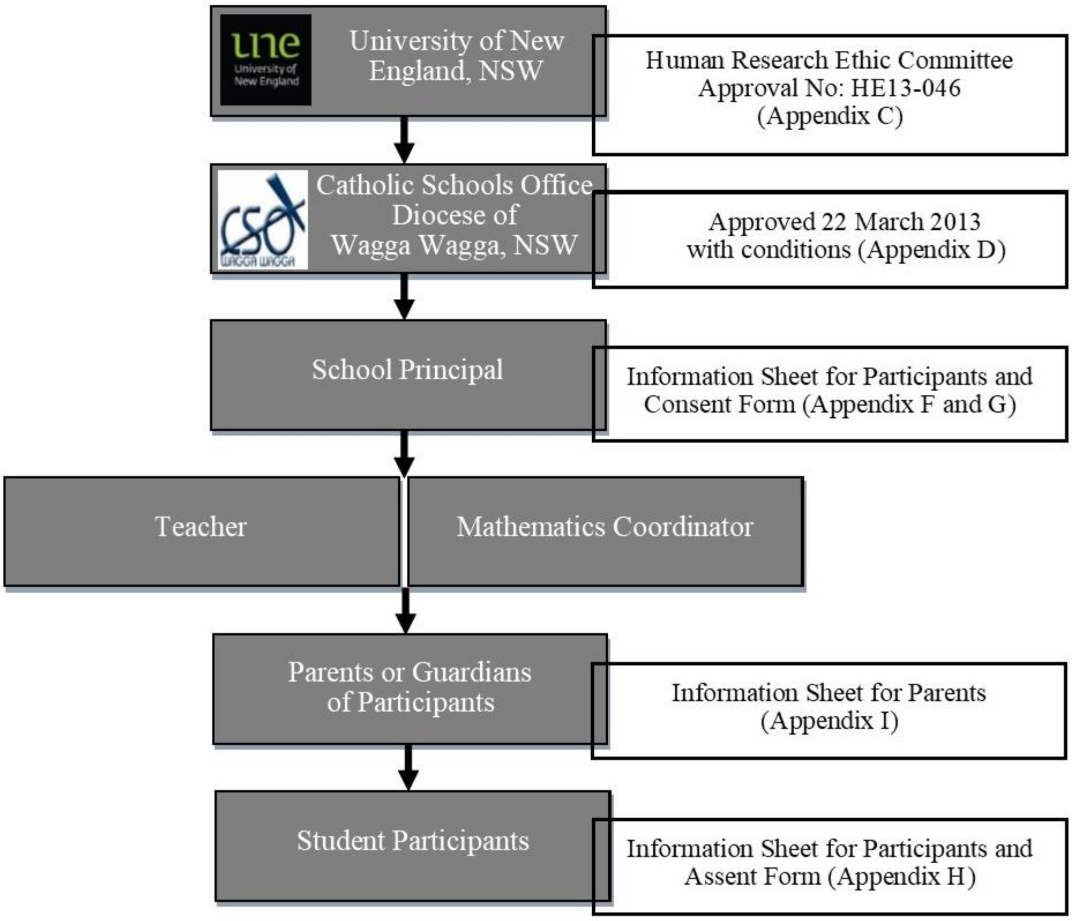


Figure 4.16: Ethical considerations

4.7.1. Informed Consent

Informed consent involves providing any prospective participants with "a description of all the features of the study that might reasonably influence his or her willingness to participate" (Johnson & Christensen, 2004, p. 102). This usually involves (Stringer, 2008, p. 46):

- Informing participants of the purpose and nature of the study;
- Asking whether they wish to participate;
- Asking permission to record information they may provide;
- Assuring them of confidentiality of information;
- Advising them that they may withdraw their consent at any time and have any recorded information returned;
- Advising them of contacts should they wish to make a complaint;
- Asking them to sign a document to confirm their permission.

When people give their informed consent, they indicate that they are "competent and legally free" (Johnson & Christensen, 2004, p. 105) to make the knowledgeable decision regarding their participation in the study. The success of any study requires the cooperation of many individuals and, as such, informed consent must be obtained prior to commencing the research. For this study, all interested parties, namely, the classroom teacher, parents or legal guardians of prospective participants and participants, were provided with an information sheet using plain language, explaining the points mentioned above. Information sheets for parents or legal guardians and participants have been provided in Appendices F and I respectively with the corresponding consent form in Appendix G.

The students involved in the project were under the age of 18 and legally considered minors, hence they are "presumed incompetent to make decisions and cannot give consent" (Johnson & Christensen, 2004, p. 105). As a result, prior consent must be obtained from parents or legal guardians to permit the student to participate. Once consent has been obtained from the parent or legal guardian, the minor may be asked for his or her assent. In order to provide assent, the minor must be "able to understand what is being asked, to realise that permission is being sought, and to make choices that are free from outside constraints" (Johnson & Christensen, 2004, p. 107). The information sheet and assent forms for students have been provided in appendix H. At regular intervals students were reminded that they could choose to withdraw consent at any time and would not be disadvantaged if they chose to do so since the entire class were completing the same unit of work.

4.7.2. Access and acceptance

Access and acceptance refers to "access to the institution or organization where the research is to be conducted, and acceptance by those whose permission one needs before embarking on the task" (Cohen, et al., 2000, p. 53). Prior to distributing information and consent letters to participants, access and acceptance must be obtained from the principal and any other authorities, as researchers "cannot expect access to a nursery, school, college, or factory as a matter of right" (Cohen, et al., 2000, p. 53). Accordingly, a meeting was arranged with the principal who explained that the Catholic Schools Office (CSO) of the Diocese also required to be notified and their consent be given.

The CSO of Wagga Wagga had their own policy and form for conducting research in a school of their Diocese. This required the researcher to provide the CSO with an overview of the research to be conducted, the benefits of the research, the design and methodology, and procedures for obtaining consent and maintaining privacy of participants. This form is provided in Appendix E. It also contained a section to be completed by the principal supervisor. Included in their conditions were that the final report or thesis was required to be presented to the CSO Wagga Wagga upon completion and, since the researcher was a staff member of the Diocese, the first copyright would be theirs. Upon obtaining the CSO approval, provided in Appendix D, the Principal gave his informed consent and the Leader of Learning for Mathematics was informed and discussions were entered into regarding suitable class, times and teachers.

4.7.3. Privacy and Confidentiality

Privacy refers to how data regarding an individual is governed with respect to other peoples' access to that data. Diener and Crandall, consider privacy to relate to three distinct perspectives: "the sensitivity of the information being given, the setting being observed, and dissemination of information" (quoted in Cohen, et al., 2000, p. 61). A consideration of each of these perspectives in light of this study yields the following results. The information collected was not of a personal or sensitive nature nor was it potentially threatening. Students were recorded in their natural classroom environment, hence the setting was completely public.

The dissemination of information concerns "the ability to match personal information with the identity of the research participants" (Cohen, et al., 2000, p. 61), and involves maintaining confidentiality and protecting anonymity. Confidentiality refers to the identity of people involved in the research, such that "the participant's identity, although known to the research group, is not revealed to anyone other than the research and his or her staff" (Johnson & Christensen, 2004, p. 112). Whilst anonymity refers to the information provided by the people involved in the research, such that the "information provided by participants should in no way reveal their identity" (Cohen, et al., 2000, p. 61). To address this dissemination of findings of both the participants and school, pseudonyms were used, as demonstrated in Table 5.1 of Chapter 5, hence protecting anonymity and maintaining the promise of confidentiality. All data gathered was kept in a locked filing cabinet with all possible identifiers removed from student samples. Digital video files were secured on a password protected computer.

In summary, "with respect to data subjects, researchers should be conscious of their intrusive potential, and should seek to minimise any intrusion; the confidentiality of data must be respected and protected by positive measures; and data subjects should be told of the purposes of the research and should have adequate opportunity to withhold their cooperation" (Burgess, 2005, p. 15). Ethical issues relevant to this study include, informed consent, access and acceptance and privacy and confidentiality. The nature of each issue was addressed appropriately as required before, during and after the study.

Providing a comprehensive evaluation of the research design and analysis plan is the foundation of good, valuable research. This section has aimed to identify and address the possible threats to validity, reliability and trustworthiness related to the study. A number of issues specific to the study were discussed and the means with which those issues were resolved or dealt with was explained. Further to this, ethical issues were considered and details were provided regarding how and when each issue was addressed for the study.

4.8. Conclusion

This chapter provided an outline of the methodologies and considerations that define this study. It was presented in six main sections that explained both the concerns and the factors affecting the design. The first section described the context of the project, stating the geographical setting and outlining the secondary mathematics courses of the Australian curriculum; this demonstrated the importance with which Linear Relationships is held within the Australian curriculum. The second section then outlined the research design, providing an overview of the design structure, detailing the type of qualitative methods employed, the targeted participants and selection process. It continued with a description of the teaching sequence employed, which included a table with a detailed lesson-by-lesson account of the activities implemented, and mapped the Teaching Phase addressed by each activity. This was followed by an explanation of the data collection structures used and examples of each were provided for further clarity.

The third section considered methodological issues and provided a justification of the Google Form test design as a research tool. It addressed each of the eight considerations proposed by Cohen, et al. (2000) that ensure the reliability and validity of the test as a research instrument.

The fourth section discussed the plan for data analysis and strategies engaged for analysis: thematic content analysis and the SOLO model. The next section assessed the evaluation of the project, investigating the validity, reliability and trustworthiness of the design. The section explored, in detail, all issues that affected validity, reliability and trustworthiness in order to maximise the integrity of the research project as being worthy of being qualitative research with valuable findings. The final section discussed the ethical considerations of the study.

The following chapters present the data and its analysis from the considerations and application of methodological issues outlined in this chapter. They begin with the presentation of results in Chapter 5 and Chapter 6, with Chapter 7 exploring a case study.

CHAPTER 5: LINEAR RELATIONSHIPS CLOSED CORE CONTENT RESPONSES

5.1. Introduction

The purpose of Chapters 5 and 6 is to present the student responses to the tasks designed to provide a tool to view students' understanding of Linear Relationships. Chapter 5 presents the closed core content of the Linear Relationships unit obtained from the three tests: the Pre-test, End of Topic test and Delayed Post-test. In particular, the focus of the results is on two research themes.

Research Theme 1

To explore the SOLO model and van Hiele Teaching Phases as frameworks to assist teachers when using technology as a teaching tool.

1.1 How does the SOLO model offer a framework to explain the categories of responses concerning students' understandings of Linear Relationships?

Research Theme 2

To examine the responses of the Google Form tests in order to gain insight into students' understandings of Linear Relationships.

2.1 Can an analysis of the results offer insights into students' understandings of Linear Relationships?

2.2 Which response categories within the tests had a relatively larger increase in complexity from the prior response category, and how does this increase reflect upon students' growth in understanding Linear Relationships?

For clarity, this chapter is divided into two sections. The first section, Background, details the preliminary information surrounding the Google Form tests. The next section, Response Results, presents each question, followed by the student responses, which are then analysed and coded using thematic and SOLO coding. For thematic coding, in-depth descriptions of code types are provided for each question then a table synthesises the response categories and provides examples of typical responses, and a second table provides statistics concerning the number and percentage of students that submitted each type of response. For the SOLO coding, in-depth descriptions of the types of responses categorised to the various SOLO levels are provided. Following this, a table provides statistics concerning the number and percentage of students that submitted responses of each SOLO level. All responses have been quoted directly from student work as submitted on the Google Form. Any irregularities with subscript notation presented in this chapter, resulted from student difficulties when entering formulas onto the Google Form.

5.2. Background

This section provides a brief overview of the methodological background to the research. A thorough and comprehensive explanation of the methodology can be found in Chapter 4. The responses from the three Google Form tests provide the data for this results chapter. Of the three tests, both the Pre-test and Delayed Post-test were identical, with the End of Topic test slightly modified to eliminate memorisation of previous results.

A consistent structure was maintained for all of the Google Form tests. The first question of the common component of each test asked for the student's name, for coding purposes. The next three questions related to the basic concepts within Linear Relationships: midpoint, distance and slope. For each question students were required to solve the problem, firstly using the GeoGebra environment and then using pen and paper techniques, that is, without GeoGebra. The remaining questions related to perpendicular and parallel lines, finishing with further equations of lines.

For the Pre-test, students were emailed a link to the online Google Form (see Appendix J) and given Extended Response questions on a sheet (see Appendix K). None of the students were able to fully complete the Google Form Test for the Pre-test.

For the End of Topic test, ten out of a possible 13 student pairs provided data for the results, due to absences when the test was conducted. Of these ten student pairs, five remained the same as the initial Pre-test; one pair contained only one student as the other student from the pair had relocated prior to the end of the topic; three pairs were mixed from the Pre-test, that is, students from different pairs combining to form a new pair due to absences on the day of the End of Topic task; and one group were present that were not originally included in the Pre-test results since both students were absent for the Pre-test. While students could discuss questions, solutions and issues within their pair, test conditions were maintained and they were not permitted to communicate between pairs. To ascertain whether or not students recall and knowledge was retained throughout the unit, students were not allowed to use any form of assistance from their textbook, workbook, teacher or the Internet. In addition to the common component of the Google Form test, an extension question was included to the form with a graph included, the results of this extension question are provided in Chapter 6. The Google Form questions for the End of Topic test are provided in Appendix L.

The Delayed Post-test was issued six weeks after the completion of the unit on Linear Relationships. During this time, the students also had two weeks of school holidays. The Delayed Post-test was an exact replica of the Pre-test except students were asked to complete this test individually rather than in pairs. Of the 26 students in the class, only 19 where present on the day the Delayed Post-test was conducted. Again, test conditions were maintained and students were not allowed to use any form of assistance from their textbook, workbook, teacher, student pair partner or Internet. In addition to the Google Form test and extension question sheet, students were also asked to complete four evaluation questions if they had time. As previously mentioned, the Google Form test questions are the same as the Pre-test provided in Appendix J.

5.3. Response Results

Question 1: Give the full names of the students in your pair.

Firstly, students were asked to provide their full names. These were then coded to colours to maintain confidentiality with letters A and B used to identify each of the students within the pair. Thirteen pairs were identified with 25 out of a possible 26 students who agreed to participate. Table 5.1 lists the pseudonym colours and gender of each student with each pair.

Table 5.1: Student pseudonyms and student gender

Pseudonym	Gender
Team Red A & B	F F
Team White A & B	F F
Team Blue A & B	M M
Team Yellow A & B	M M
Team Purple A & B	M F
Team Brown A & B	F F
Team Black A & B	M M
Team Orange A & B	M M
Team Indigo A & B	F F
Team Maroon A & B	M M
Team Lime A & B	F F
Team Cream A & B	F F
Team Lemon A & B	M M

Question 2: Questions 2 to 5 involve using the toolbar in GeoGebra and the points (2, 0) and (0, 5). First find the point which would represent the midpoint**.** [For End of Topic test points were (-1, 4) and (3, 6)].

Two types of responses were noted for the thematic coding, as well as a ceiling response, summarised in Tables 5.2 and 5.3.

Thematic Coding

Type A: This type of response indicated that the solution was a coordinate and, although could be representative of the midpoint, was incorrect. Some responses indicate that the midpoint tool may have been used. Typical responses included omitting the decimal place, using incorrect endpoint coordinates and transcription errors.

Two Type A responses were recorded in the Pre-test, Team Red stated "$(1{,}25)$", omitting the decimal place and Team White with "$(1.08, 2.5)$", indicating that their initial points were not correctly entered. This was attributed to freehand positioning their points (using the mouse) rather than entering them in the input bar, resulting in the precision of *x*-coordinate being slightly off. For the End of Topic test, only one pair, Team Lime, presented this type of response with "$(0.5, 7)$" instead of a correct solution of $(1, 5)$. For the Delayed Post-test, only Team Black B, submitted a Type A response when he submitted "$(1, 25)$", omitting the decimal place.

Type B: This type of response indicated that students used the correct GeoGebra tool to calculate the midpoint, obtaining a solution of $(1, 2.5)$ or $(1, 5)$ for End of Topic test.

From the 12 pairs in the Pre-test, ten correctly identified the midpoint as $(1, 2.5)$ with correct notation, presenting the coordinate within brackets with the *x*-coordinate followed by a comma and then the *y*-coordinate. From the ten pairs in the End of Topic test, nine correctly identified the midpoint as $(1, 5)$, again with correct notation. For the Delayed Post-test, from the 19 students who were present, 18 correctly identified the midpoint as $(1, 2.5)$.

Table 5.2: Response types for finding the midpoint with GeoGebra (Question 2)

Response Type	Explanation	Examples
A	Correct tool used but correct solution not obtained	$(1.08, 2.5)$ – indicating initial points were not correctly entered $(1, 25)$ – decimal place omitted
B	Correct tool used to obtain correct solution	$(1, 2.5)$ for Pre-test and Delayed Post-test $(1, 5)$ for End of Topic test

Table 5.3: Thematic coding statistics for finding the midpoint with GeoGebra (Question 2)

Response Type	A	B	Total
Pre-test	2 (17)	10 (83)	12 pairs (100)
End of Topic test	1 (10)	9 (90)	10 pairs (100)
Delayed Post-test	1 (5)	18 (95)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

There was insufficient information to provide a fine-grained coding for this particular question. It appears that students were responding in a manner consistent with concrete symbolic mode, however, more information would be required to classify the responses into cycles of levels.

Question 2a: How could you find the midpoint without using GeoGebra and using pen and paper?

This question encouraged students to think about how they could manually calculate the specific point that identified the midpoint. Five types of responses and six SOLO levels were found, summarised in Tables 5.4, 5.5 and 5.6.

Thematic Coding

Type A: This type of response indicated that there was no understanding of the concept of midpoint or was representative of a non-attempt. Typical responses ranged from a blank comment to the mention of a formula that was either not identifiable or related to the midpoint formula.

For the Pre-test and End of Topic test, none of this type of response was submitted. For the Delayed Post-test, eight submitted this type of response. Team White A and Team Lime B left their responses blank. Another two suggested formulas, with Team Lime A stating "use the midpoint formula" and Team Lemon A stating "use a certain formula", neither specifically explaining what the formula was or how to apply it. The remaining four students presented an incorrect formula with Team Red A and Team Red B both quoting "rise over run", Team Black A submitting "$M = d$" and Team Yellow A stating the gradient formula "$\frac{y2-y1}{x2-x1}$".

Type B: This type of response indicated that the concept of midpoint was found using practical methods with descriptions using keywords "midpoint" or "middle". Typical responses proposed that midpoint was a measurement, which could be found using tools, such as a ruler, but lacked detail regarding how the tool would help.

In the Pre-test, four student pairs submitted this type of response. Team Orange stated "draw a line from point to point and then find the mid point", Team Red stated "By using a ruler to measure the middle point", Team Blue stated "You could find the midpoint without using Geogebra [sic] or using pen or Paper by using the graph to guide you, whilst using a ruler"; and finally, Team Indigo stated "First step would be making the x and y axis Secondly, you would find the coordinates, mark them out and measure it. Thirdly, using a ruler you would find midpoint and mark it out". For the End of Topic test, no responses of this type were recorded. In the Delayed Post-test, two submitted this type of response. Team Black B suggested "use a ruler" indicating that the abstract idea of using coordinates to find the midpoint was not yet achieved and Team Orange B suggested "draw a line between two points and then find the middle".

Type C: This type of response contained keywords and phrases such as "half", "halfway", "halve", "divide by 2" or "average", indicating that some comprehension towards finding the midpoint was present. Each time these keywords were used, it was in relation to the distance between the points not to the individual coordinates. While it builds on the visual perspective considered to be a Type B response, no formula was given and it demonstrated a basic understanding that still required the assistance of measuring tools, such as a ruler, to calculate the distance since coordinates were not mentioned.

For the Pre-test, seven student pairs submitted this type of response with some responses being more detailed than others. Team Black stated "use a ruler to measure the distance between A and B then the half way mark is the answer measure the distance by using a ruler between 2 points and finding the middle of those points". Team White stated, "finding length between the two points and dividing by two" and Team Cream stated "you would first make the x and y axis. Then you would find the coordinates and mark them out, then you would connect them. Secondly you would grab a ruler and measure the line and mark halfway; this then would be the midpoint." Team Maroon stated "use a ruler to measure the length between the two points. Then divide the answer by 2 to find the mid point", and Team Yellow stated "you could use a ruler to measure the distance between the two points, and the answer would be half the length", Team Brown stated "find the length between two of the points and divide it by 2 and you will get the approximate answer of the midpoint.", and Team Lemon stated “you could use a ruler to measure the distance between the 2 points and then half it and then BAM! Midpoint”.

Only Team Black submitted this type of response for the End of Topic test when they stated “draw a Cartesian plane and using the formula midpoint (halve)” The keyword “halve” distinguishing this response from Type B. For the Delayed Post-test two students provided a Type C response. Team Orange A stating “half the distance measure the middle” and Team Maroon A who submitted the keyword “average” indicating that they understood some idea of the concept, relating to the average of either the distance or coordinates.

Type D: This type of response indicated that understanding of midpoint had shifted from it being a concrete measurement requiring tools, to a more abstract concept calculable by manipulating the coordinates. Each response made reference to using the coordinate points to calculate the midpoint. This was presented as either a description or a formula, but contained one element that was incorrect restricting the correct solution from being achieved.

In the Pre-test, only Team Purple provided this type of response stating, “We could halve each of the points to get the midpoint”. Only Team Indigo submitted this type of response in the End of Topic test, incorrectly stating the coordinates in the formula stating “$midpoint = \left(\frac{x1+y1}{2}, \frac{x2+y2}{2}\right)$”. For the Delayed Post-test, this type of response occurred twice, both incorrectly placing a “+” sign to add the coordinates together. This would provide a result that would be a number not a coordinate. Team Blue A stated “you can find the midpoint by using formulae $\frac{(x1+x2)}{2} + \frac{(y1+y2)}{2}$” and Team Indigo B stated, “$\frac{(x2+x1)}{2} + \frac{(y2+y1)}{2}$ = answer”.

Type E: This type of response contained a correct version of the midpoint formula using the coordinate points to calculate the midpoint. Discrepancies occurred in presentation of the formula depending on how students could represent fractions in the Google Form. All responses coded to this category used correct notation (x_1, y_1) and identified that the solution was a point which had two separate parts.

No one presented this type of response in the Pre-test. For the End of Topic test, eight submitted this type of response. Typical responses varied, ranging from the formula presented simply as “$\left(\frac{x1+x2}{2}, \frac{y1+y2}{2}\right)$”, as stated by Team Lime, to a description, such as provided from Team Cream: “You would plus x1 and x2 together and divide by 2, then you would do the same with the 2 y's.”; to a more detailed description by Team Brown who submitted: "We use the formula : M = (x1+ x2 over 2 , y1 + y2 over 2) (-1,4) represent : -1 = x1 4 = y1 (3,6) represent :   3 = x2 6 = y2".

In the Delayed Post-test, a Type E response was submitted five times. Typical responses again varied from stating the midpoint formula simply to a more detailed description. Team Indigo A provided the basic formula "$\left(\frac{x1+x2}{2}, \frac{y1+y2}{2}\right)$" with Team Cream A again providing a written response of: "You would first plus the $x1$ and the $x2$ co ordinates together and divide that by 2 (this will give you the x co ordinate) and then you would do the same thing for the y co ordinate", and, again, Team Brown A provided a more detailed explanation submitting: "Use the formula: $= x1 + x2/2$, $y1 + y2/2$ We use the two coordinates and label them as $x1$, $y1$, $x2$ and $y2$".

Table 5.4: Response types for finding midpoint without GeoGebra (Question 2a)

Response Type	Explanation	Examples
A	No understanding of midpoint Formulas given do not calculate midpoint	"use the midpoint formula" "rise over run" "$M = d$" "$y2 - y1/x2 - x1$"
B	Limited knowledge of midpoint definitions refers to visual perspective Use of ruler is suggested No specific details on how to find midpoint, use of midpoint term used in description	"draw a line from point to point and then find the midpoint" "by using a ruler to measure the middlepoint"
C	Basic understanding of midpoint demonstrated using keywords – half, halfway, divide by 2 or average No formula given Midpoint described with respect to distance rather than individual coordinates Presumes use of some form of tool to calculate midpoint	"use a ruler to measure the distance between A and B then the half way mark is the answer measure the distance by using a ruler between 2 points and finding the middle of those two points" "you would first make the x and y axis. Then you would find the coordinates and mark them out, then you would connect them. Secondly you would grab a ruler and measure the line and mark halfway; this then would be the midpoint"

Response Type	Explanation	Examples
D	Understanding of midpoint develops to being a concept calculable by manipulating coordinates Responses contained one element that was incorrect	"we could halve each of the points to get the midpoint" "$midpoint = \left(\frac{x1+y1}{2}, \frac{x2+y2}{2}\right)$" "$\frac{(x2+x1)}{2} + \frac{(y2+y1)}{2}$ = answer"
E	Correct midpoint formulas Correct notation presented Presentations of formula range from attempts to input the formula to a detailed description	"$\left(\frac{x1+x2}{2}, \frac{y1+y2}{2}\right)$" "you would plus $x1$ and $x2$ together and divide by 2, then you would do the same with the 2 y's." "we use the formula : $M = (x1 + x2\ over\ 2, y1 + y2\ over\ 2)$ $(-1,4)$ represent : $-1 = x1$ $4 = y1$ (3,6) represent : $3 = x2$ $6 = y2$" "you would first plus the x1 and the x2 co ordinates together and divide that by 2 (this will give you the x co ordinate) and then you would do the same thing for the y co ordinate" [sic]

Table 5.5: Thematic coding statistics for finding midpoint without GeoGebra (Question 2a)

Response Type	A	B	C	D	E	Total
Pre-test	0 (0)	4 (33)	7 (61)	1 (8)	0 (0)	12 pairs (100)
End of Topic test	0 (0)	0 (0)	1 (10)	1 (10)	8 (80)	10 pairs (100)
Delayed Post-test	8 (42)	2 (11)	2 (11)	2 (11)	5 (26)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Prestructural: Responses were below those expected for the question. Examples coded to this level were a blank response.

The following cycle of levels were identified in the Concrete Symbolic (CS) mode:

Unistructural (U_1): Responses indicated uncertainty of the requirements of the question, focussing on one specific aspect that was usually visual and unrelated to the question. Examples coded to this level included "rise/run".

Multistructural (M_1): Responses focussed on more than one aspect with no attempt to make connections between them. Responses included using a specific formula with no further explanation, or using a device or tool that would assist to find the midpoint but without links connecting how this could be achieved. Examples coded to this level included "use the midpoint formula".

Unistructural (U_2): Responses focussed on one particular aspect with less reliance on visual cues. Examples coded to this level included "half the distance measure the middle".

Multistructural (M_2): Responses focussed on more than one isolated aspect and use more technical language. Responses may also take the form of a formula with incorrect elements. Examples coded to this level included "Find the length between two of the points and divide it by 2 and you will get the approximate answer of the midpoint".

Relational (R_2): Responses indicated a number of connections have become apparent in the understanding of the question. The response is usually a correct response. Examples coded to this level included correct formulas requiring no manipulation such as "You would plus $x1$ and $x2$ together and divide by 2, then you would do the same with the 2 y's.".

Table 5.6: SOLO coding statistics for finding the midpoint without GeoGebra (Question 2)

SOLO Coding	**Number**		
	Pre-test	End of Topic test	Delayed Post-test
Prestructural	0	0	2
U_1 (CS)	0	0	3
M_1 (CS)	4	0	5
U_2 (CS)	0	0	2
M_2 (CS)	8	2	2
R_2 (CS)	0	8	5
Total	12	10	19

Question 3: Find the distance between these two points when connected**.**

This question required students to find the correct tool in order to calculate the distance between the two given points. This question was similar to the previous question concerning the midpoint, and students were required to navigate their way through GeoGebra to find the correct tool. Three types of responses were noted after thematic coding, as well as a ceiling response, summarised in Table 5.7 and Table 5.8.

Thematic Coding

Type A: This type of response indicated no knowledge of how to find the distance using GeoGebra, or was representative of a non-attempt. Typical responses were either not representative of a distance or blank comments.

In the Pre-test, only Team Orange submitted this type of response. They stated "$(1, 2.5)$", which was a replica of the solution they provided for the midpoint. For the End of Topic test, no responses of this type were recorded. In the Delayed Post-test, two Type B responses were recorded, with Team Lime B providing a blank comment and Team Indigo A providing a solution of "$5x + 2y = 10$".

Type B: This type of response provided a solution that could represent the distance but it was incorrect for the points given.

In the pre-test, three Type B responses were provided. Both Team Cream and Team Indigo submitted a solution of "2.69" with Team White offering "5.33" (note: Team White was the pair who incorrectly placed their coordinates on the Cartesian Plane). Only one pair, Team Lime, submitted this type of response in the End of Topic test, submitting a solution of "7". In the Delayed Post-test, only Team Lime A submitted this type of response with "2.5".

Type C: This type of response provided a correct response of 5.39 for the Pre-test and Delayed Post-test and 4.47 for the End of Topic test. It demonstrates that the correct tool was used to calculate the distance and it was accurately recorded into the Google Form.

In the Pre-test, eight Type C responses were recorded. For the End of Topic test, nine of the ten student pairs presented this type of response. In the Delayed Post-test 16 Type C responses were submitted.

Table 5.7: Response types for finding the distance between two given points with GeoGebra (Question 3)

Response Type	Explanation	Examples
A	No understanding of what was required Incorrect tool chosen	blank comment "$5x + 2y = 10$" "$(1, 2.5)$"
B	Answer represented a solution possible of being the distance but incorrect	"2.69" "5.33" "7" "2.5"
C	Correct solution obtained Indicates correct tool used	5.39 for Pre-test and Delayed Post-test 4.47 for End of Topic test

Table 5.8: Thematic coding statistics for finding the distance between two given points with GeoGebra (Question 3)

Response Type	A	B	C	Total
Pre-test	1 (8)	3 (25)	8 (67)	12 pairs (100)
End of Topic test	0 (0)	1 (10)	9 (90)	10 pairs (100)
Delayed Post-test	2 (11)	1 (5)	16 (84)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Responses to Question 3 do not provide enough detail about the quality of the student response to attempt a fine-grained SOLO coding. Those responses classified as Type A could be broadly coded as Prestructural with Type B responses coded as Concrete Symbolic cycle 1 or early cycle 2. It is not possible to distinguish further without knowing the strategy employed by the student. The use of the distance tool in GeoGebra to obtain the correct answer does not provide sufficient information to code response Type C as relational in the second cycle of the Concrete Symbolic mode. In the light of these results, the responses provided to Question 3a do offer a window to view the students' understanding of the distance between two points and this analysis follows.

Question 3a: How could you find the distance without using GeoGebra and using pen and paper?

Five different types of responses and six SOLO levels were recognised in the coding of the responses for this question, summarised in Tables 5.9, 5.10 and 5.11.

Thematic Coding

Type A: This type of response indicated no understanding of how to find the distance without the GeoGebra environment or was representative of a non-attempt. Typical responses were blank comments or "I don't know".

In the Pre-test, Team Cream provided the only Type A response with "I don't know". No responses of this type were recorded in the End of Topic test. For the Delayed Post-test, four Type A responses were recorded. Of these Team Orange B submitted "I don't know" while Team Indigo B, Team Cream A and Team Lime B, all recorded blank comments.

Type B: This type of response indicated a basic understanding of the concept of distance as a measurement. Typical responses described practical methods for its calculation using measuring tools such as a ruler.

In the Pre-test, ten Type B responses were submitted. A typical response was presented by Team Maroon who stated: “Use a ruler to measure the length between the two points”. Interestingly, the response from Team Orange included a protractor in their explanation stating: “You could find the distance by using a protractor and a ruler so you can measure the distance”. No Type B responses were presented in the End of Topic test. Team Orange A, Team Red B, Team Red A and Team Black B all submitted this type of response in the Delayed Post-test. Team Orange A simply stated: “measure it”, providing no explanation as to how to do this while the remaining three pairs all suggested using a ruler in their explanations. A typical response was provided by Team Red B who stated: “by using a ruler and measuring the distance between the points”.

Type C: This type of response indicated a basic understanding towards being able to calculate distance without tools. It demonstrated a developing progression from moving beyond practical concrete methods to abstract concepts such as using a formula. Typical responses had explanations that suggested using “a formula” but fail to define the formula. While the formula was lacking, the intention of linking the coordinates to the concept of distance was emerging.

No Type C responses were recorded in the Pre-test. Only one response of this type was presented in the End of Topic test. This was provided by Team Black who stated, “Cartesian plane use the formula to work out the distance between the two” making mention of a formula without any detail. This team also presented with a similar reason when asked to find the midpoint, suggesting a formula without any detail; in essence, re-wording the question. In the Delayed Post-test, five Type C responses were submitted. Three of these, made mention of using a formula without any specifics; namely, Team Lime A, who stated: “use the distance formula”, Team Lemon A who stated: “use a certain formula”, and Team Maroon A, who stated: “don’t remember the formula”. The remaining two responses were incorrect formulas. Team Black A submitted: “$D = m2 + b2$”, which resembled Pythagoras’ Theorem, and Team Yellow A, who provided the midpoint formula “$\frac{x1+x2}{2}\ \frac{y1+y2}{2}$”. Unfortunately, Team Yellow A provided the incorrect formula for the midpoint in the previous question.

Type D: This type of response indicated a developing knowledge of how to calculate the concept of distance without using measurement tools or ICT tools such as GeoGebra. Typical responses suggested distances calculated using a specific distance formula with incorrect elements. Replicating the distance formula into Google Forms proved difficult and representations varied depending on how students were able to notate powers and square roots on the computer.

Team Brown provided the only Type D response in the Pre-test submitting “Pythagoras’ Theorem”. Despite not expanding on how to use Pythagoras’ Theorem, this was a very interesting result for the Pre-test and demonstrated a high order of thinking. In the End of Topic test, this type of response was recorded by four pairs, each provided formulas representative of the distance formula with incorrect elements. Team Red A, who teamed with Team White B for this task, forgot that each bracket needed to be squared and stated “square rout of $(x1 - x2\ ) + (y1 - y2\ )$” [sic]. Two student pairs, Team Indigo and Team Blue B, who teamed with Team Purple A for this test, both forgot the square root with Team Indigo stating: “$distance = (x2 - x1)2 + (y2 + y1)2$” (also placing a “+” in the second bracket instead of a “–”), and Team Blue B stating: “distance formula $d = x2 - x2\ squared + y2 - y1\ squared$”. Finally, Team Yellow A, who combined with Team Blue A for this task, forgot to add the brackets together stating: “$\sqrt{(x1 - x1)2 - (y2 - y1)2}$”. For the Delayed Post-test, four Type D responses were recorded. Various elements were omitted or incorrect within each of the formulas. Team Blue B stated “$d = square\ route\ of\ x1 + x2 + y1 + y2\ square\ route$” [sic] forgetting to subtract each of the coordinates and square their result, and Team Indigo A, who stated: “$x1 + y1 + x2 + y2$”, which was incomplete in many ways. Team White A stated: “$(x2 - x1)2 - (y2 - y1)2$”, forgetting the square root over the whole formula, and Team Brown A stated: “We use the formula: $D = square\ root\ of\ (x - y1)squared + (x2 - y2)squared$ We use the two coordinates and label them as $x1$, $y1$, $x2$ and $y2$”, mistakenly placing an x and y coordinate in each bracket.

Type E: This type of response correctly stated the distance formula as a method of calculating the distance between two given points. All of these responses contained correct notation or used a combination of description and algebra to produce a solution equivalent to $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

In the Pre-test, no Type E responses were recorded. For the End of Topic test, five submitted this type of response indicating recall of the correct formula to calculate the distance. Team Brown contained the most detail stating: “We use the formula: $D = square\ root\ of\ (x2 - x1)squared + (y2 - y1)squared$ (-1,4) represent: $-1\ =\ x1\quad 4\ =\ y1$ (3,6) represent:

$3 = x2\ \ 6 = y2$". In the Delayed Post-test, two Type E responses were submitted. While presentations differed due to how the students could manage to correctly type the formula into the Google Form, each one contained the correct notation and operations. Team White B stated: "$root: (x2 - x1)sqared + (y2 - y1)sqared$" [sic] with Team Blue A submitting "$D = \sqrt{(x2 - x1)2 + (y2 - y1)2}$".

Table 5.9: Response types for finding the distance between two given points without GeoGebra (Question 3a)

Response Type	Explanation	Examples
A	A non-attempt No understanding of calculating distance	"I don't know" blank comment
B	Basic understanding of concept of distance as a measurement Practical methods using some form of measuring tool to calculate distance described	"use a ruler to measure the length between the two points" "you could find the distance by using a protractor and a ruler so you can measure the distance"
C	Basic understanding of concept of distance without the use of tools Progressing from concrete to abstract idea of formula Suggests formula without correct definition of what is required for the formula	"Cartesian plane use the formula to work out the distance between the two" "use the distance formula" "use a certain formula" "$D = m2 + b2$" "$\frac{x1+x2}{2}\ \frac{y1+y2}{2}$"
D	Developing knowledge of calculating distance through using Pythagoras Theorem Links distance to coordinates Responses contained a formula representative of distance formula with elements incorrect	"$sqaure\ rout\ of\ (x1 - x2) + (y1 - y2)$" "Pythagoras theorem" "$distance = (x2 - x1)2 + (y2 + y1)2$" "$distance\ formula\ d = x2 - x2\ squared + y2 - y1\ squared$" "$\sqrt{(x2 - x1)2 - (y2 - y1)2}$" "$d = square\ route\ of\ x1 + x2 + y1 + y2\ square\ route$" "$x1 + \text{y}1 + x2 + y2$" "$(x2 - x1)2 - (y2 - y1)2$" "we use the formula:$D = square\ root\ of\ (x1 - y1)squared + (x2 - y2)squared$ We use the two coordinates and label them as x1, y1 , x2 and y2"

Response Type	Explanation	Examples
E	Correct distance formula with correct notation presented Varying representations ranging from formula to description	"$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$" "we use the formula: $D = square\ root\ of\ (x2 - x1)squared + (y2 - y1)squared$ (-1,4) represent : -1 = x1 4 = y1" (3,6) represent : 3 = x2 6 = y2" "$root: (x2 - x1)sqared + (y2 - y1)sqared$"[sic]

Table 5.10: Thematic coding statistics for finding the distance between two given points without GeoGebra (Question 3a)

Response Type	A	B	C	D	E	Total
Pre-test	1 (8)	10 (83)	0 (0)	1 (8)	0 (0)	12 pairs (100)
End of Topic test	0 (0)	0 (0)	1 (10)	4 (40)	5 (50)	10 pairs (100)
Delayed Post-test	4 (21)	4 (21)	5 (26)	4 (21)	2 (11)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Prestructural: Responses were below those expected. Examples coded to this level are a blank response or "I don't know".

The following cycle of levels were identified in the CS mode:

Unistructural (U_1): Responses indicated uncertainty regarding the specific requirements of the question, stating only one physical method or strategy triggered by visual cues, using a measuring instrument. Examples coded to this level included "using a ruler" or "measure the distance with a ruler".

Multistructural (M_1): Responses focussed on more than one method or strategy that could be used to calculate the distance, using physical measuring instruments, without the ability to link them together. The only example coded to this level mentions was: "You could find the distance by using a protractor and a ruler so you can measure the distance".

Unistructural (U_2): Responses focussed on one particular idea that involved less reliance on visual cues, and demonstrated the development of abstract thought. In both cases, the use of a specific formula was stated without presenting the formula, its requirements or its application. Examples coded to this level included "Pythagoras Theorem" and "use the distance formula".

Multistructural (M_2): Responses contained more than one feature that would assist in finding the distance between two points. These responses identified that the *x* and *y* values of the coordinates must be addressed individually and often presented as a formula with incorrect elements. Examples coded to this level included "(x2-x1)2 – (y2-y1)2" and "distance formula d=x2-x2 squared + y2-y1squared" [sic].

Relational (R_2): Responses indicated a number of connections had become apparent in the understanding of the question. The response was usually a correct response. Examples coded to this level included correct formulas that require no manipulation, such as "square root (x2-x1)2 +(y2-y1)2" [sic].

Table 5.11: SOLO coding statistics for finding the distance between two given points without GeoGebra (Question 3a)

SOLO Coding	Number		
	Pre-test	End of Topic test	Delayed Post-test
Prestructural	1	0	5
U_1 (CS)	9	0	5
M_1 (CS)	1	0	0
U_2 (CS)	1	0	1
M_2 (CS)	0	5	7
R_2 (CS)	0	5	1
Total	12	10	19

Question 4: Find the slope between the two points.

This question required students to find the gradient or slope of the two points given. During the lesson sequence, the two terms were used interchangeably so that the students became familiar with both terms. However, the term, "slope", was chosen for this question as that is the term used in the GeoGebra environment. Similar to the previous two questions, GeoGebra contains a tool to calculate the slope and students were now accustomed to searching through the menus and buttons to search for the correct tool. Three different types of responses were noted with the thematic coding of this question, as well as a ceiling response, summarised in Table 5.12 and Table 5.13.

Thematic Coding

Type A: This type of response was an incorrect solution. It indicated that the correct tool had not been used or the points used with the tool were incorrect.

In the Pre-test, four Type A responses were recorded. Team Red stated that the slope was "8" and Team White stated: "-2.72" (note: Team White plotted their initial points incorrectly which previously resulted with incorrect midpoint and distance). Team Indigo and Team Cream both stated: "5.39cm", which was the correct answer for the distance between the two given points yet, interestingly, both failed to submit this for the distance, obviously confusing the solutions or tools. Only one Type A response was submitted in the End of Topic test; namely, Team Blue B, who combined with Team Purple A for this task, when they stated: "rise 3 run 5". While incorrect numbers were used and not presented as a fraction, the mention of rise and run was indicative of some knowledge towards calculating the gradient. No responses of this type were recorded for the Delayed Post-test.

Type B: This type of response indicated knowledge of finding the slope using the correct numbers but incorrect direction. This would indicate that the students used rise over run to calculate the slope, forgetting to take into account the negative symbol required to indicate the line started from the left going down to right, hence, the GeoGebra tool was not used.

In both the Pre-test and End of Topic test, no Type B responses were recorded. In the Delayed Post-test, four submitted this type of response. Three of these students, namely, Team Red A, Team Lime B, and Team Blue B submitted: "2.5" with the remaining student, Team Lime A, submitting "5/2".

Type C: This type of response provided a correct response of -2.5 for the Pre-test and Delayed Post-test and 0.5 for the End of Topic test. It indicated that students competently found the correct tool for the slope and were able to accurately present that result in the Google Form.

In the Pre-test, eight pairs correctly stated the slope as "-2.5" with Team Lemon stating "-2.5cm" including the cm in the measurement. For the End of Topic test, nine pairs correctly stated the slope was 0.5 with Team Lime stating "2/4". In the Delayed Post-test, fifteen responses correctly identified the slope as "-2.5".

Table 5.12: Response types for finding the slope between two points with GeoGebra (Question 4)

Response Type	Explanation	Examples
A	Incorrect tool or points chosen Incorrect answer	"8" "-2.72" "5.39cm" "rise 3 run 5"
B	Knowledge of slope Correct magnitude but incorrect direction	"2.5" "5/2"
C	Correct solution	-2.5 for Pre-test and Delayed Post-test 0.5 for End of Topic test

Table 5.13: Thematic coding statistics for finding the slope between two points with GeoGebra (Question 4)

Response Type	A	B	C	Total
Pre-test	4 (33)	0 (0)	8 (67)	12 pairs (100)
End of Topic test	1 (10)	0 (0)	9 (90)	10 pairs (100)
Delayed Post-test	0 (0)	4 (21)	15 (79)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Responses to Question 4 do not provide enough detail about the quality of the student response to attempt a fine-grained SOLO coding. Those responses classified as Type A could be broadly coded as Prestructural with Type B responses coded as Concrete Symbolic cycle 1 or early cycle 2. It is not possible to distinguish further without knowing the strategy employed by the student. The use of the slope tool in GeoGebra to obtain the correct answer does not provide sufficient information to code response Type C as relational in the second cycle of the Concrete Symbolic mode. In the light of these results, the responses provided to Question 4a do offer a window to view the students' understanding of the gradient/slope of two points and this analysis follows.

Question 4a: How could you find the slope without using GeoGebra and using pen and paper***?***

When GeoGebra calculates the slope, it first requires a line to be drawn between the two points then, after finding the correct tool, a click on the line calculates the slope. Its solution is presented not only as a value but also includes a right-angled triangle, drawn demonstrating where the slope originates, as shown in Figure 5.1. This provides students with a clue to developing the concept of slope. The class teacher did reveal that students had been exposed to the concept of slope in a previous unit of work. With this question, five types of responses and six SOLO levels were identified, summarised in Tables 5.14, 5.15 and 5.16.

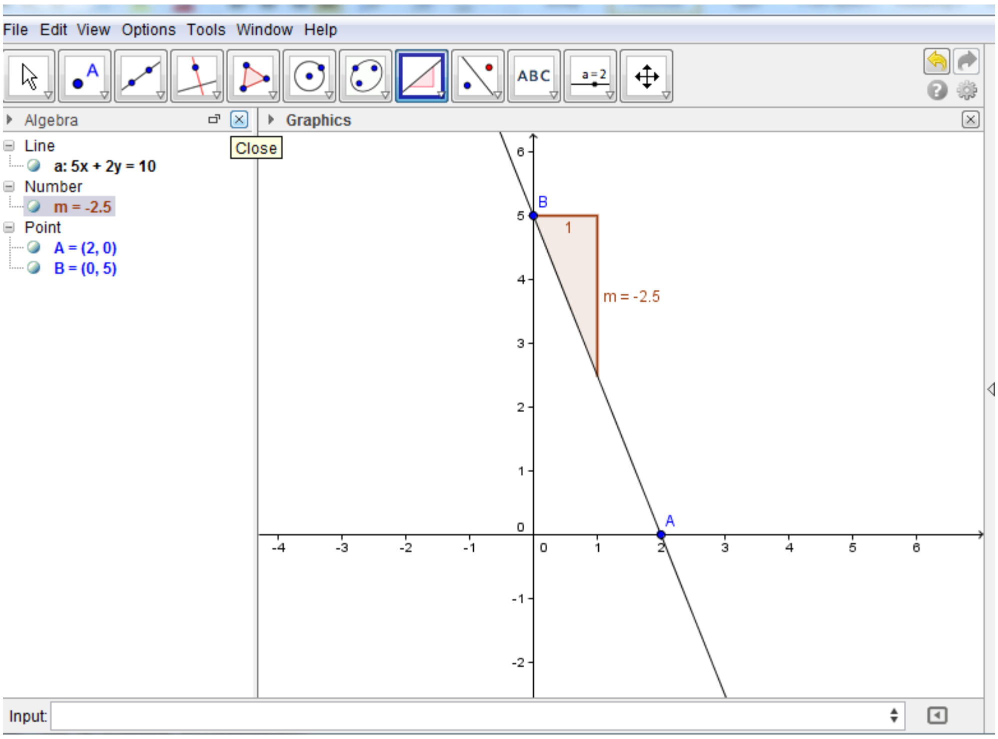


Figure 5.1: Screenshot when finding the slope with GeoGebra

Thematic Coding

Type A: This type of response indicated no understanding of what was required to find the slope without GeoGebra or was representative of a non-attempt. Either GeoGebra was unable to be used successfully to calculate the slope initially or the clue provided by GeoGebra was not used to assist with the calculation. Typical responses were “I don’t know”, “no idea” or a blank comment.

In the Pre-test, five Type A responses were presented. While Team Cream submitted a blank comment, the remaining student pairs; that is, Teams Black, White, Maroon and Indigo, all submitted a comment suggesting they didn’t know. For the End of Topic test, no responses of this type were submitted. In the Delayed Post-test, only Team Lime B and Team Black B submitted blank comments.

Type B: This type of response indicated a basic understanding that the slope was a measurement that could be calculated with practical tools such as a ruler or protractor. While the concept of slope was not fully understood, the concrete idea of measuring with a tool was well established. Typical responses described calculating the slope in practical terms. The suggestion of a protractor indicated that the slope was considered to be related to angles. This connection between slope and angles could be related to their previous encounter of slope, which was a problem involving a wheelchair being pushed up a ramp and the slope was explained as a way of describing the steepness of the incline.

The Pre-test contained five type B responses. Each one identified that a protractor could be used to measure the slope. Team Yellow stated: "you could measure the angle of the slope with a protractor". This indicated that the concept of slope was directly related to an angle the line formed, but no further explanation was made as to what angle they were measuring. No Type B responses were presented in the End of Topic test. In the Delayed Post-test, only Team Orange A submitted this type of response stating: "using the measurement" without any expansion on what measurement or how to find it.

Type C: This type of response shifted from the concept of slope being a measurement that required tools. These responses begin to incorporate right-angled triangles into the explanations and descriptions are simple, attempting to link the triangles to calculating the slope. The idea of using a formula was also established as being a method that calculated the slope although an actual formula was not presented. Typical responses included an example of a formula, although it was not one which would correctly calculate the slope.

Two Type C responses were submitted in the Pre-test. Both gave descriptions involving a right-angled triangle. Team Purple submitted: "using the midpoint and adding a right-angled triangle from point b", and Team Brown provided a more detailed response stating:

> To find the slope, start from point A to B measure the distance and find the midpoint between the two. As the answer would be $(1, -2.5)$, from point A measure 1 away and then measure downwards -.2,5 towards the midpoint. Connect the points into a triangle.

While the idea of using the midpoint is incorrect, it is obvious that the students involved are making an attempt to link the diagram given by GeoGebra (shown previously in Figure 5.1) to a method for calculating the slope. They are challenging themselves to understand how the slope is calculated.

Only Team Black recorded this type of response in the End of Topic test when they submitted: "using the Cartesian plane and the formula". No further information on what formula or what was required for the formula was provided. In the Delayed Post-test, five responses were of this type. A wide range of descriptions were provided, with Team Lemon A stating: “using a certain formula”, which was expanded on by Team Lime A, who stated: “use the slope formula”. Team Cream A identified that “you would make a small triangle” but failed to provide any more information. Team Black A gave the response: “$G = m2 + b2$”, which is a combination of the gradient form of the equation of a line (the letters m representing slope and b representing y-intercept) and Pythagoras’ Theorem, $c^2 = a^2 + b^2$, and finally Team Orange B, who submitted: “mx/b”, again using elements of the gradient-intercept form of the equation of a straight line.

Type D: This type of response extended on a Type C response by being able to provide a simple formula for slope, namely, “rise over run”. This was a common mantra used to define and calculate the slope. It provided a basic formula that could be used with any right-angled triangle, comparing the rise (change in y coordinates) to the run (change in x coordinates) in order to calculate the slope of the line. It was not considered a Type E response, as it does not establish whether the slope would be positive or negative; so, while it assisted in calculating the numerical value of the slope, it was not a complete solution.

In the Pre-test, no Type D responses were submitted. Only Team Maroon submitted this type of response for the End of Topic test. For the Delayed Post-test, eight responses were presented of this type. Team Yellow A provided detail in their response with: “find the points where the line intercepts the x and y axis and then divide the rise by the run” [sic].

Type E: This type of response extended on the correct response provided as a Type D response with a formula that provides the link between rise over run to the coordinate pairs. The formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$, correctly calculates the slope of a line between any two points. Once again, using the Google Form did not make it easy to type in such a formula and subscripts, which contains a fraction, hence different representations of the formula were presented.

In the Pre-test, no Type E responses were submitted. Eight submitted this type of response in the End of Topic test. While five of these stated the slope formula, the remaining three, namely, Team Indigo, Team Cream and Team Red A, who combined with Team White B for this task, first stated rise over run before also providing the formula. This indicates that the initial reaction to the concept of gradient was rise over run, which was then expanded on to obtain the formula. In the Delayed Post-test, three Type E responses were submitted. Team Blue A correctly stated the formula swapping the coordinates around when stating: "using the formula, $\frac{y_1-y_2}{x_1-x_2}$", and Team Brown A provided a more detailed response with "$Slope = rise/run$ Formula: $G = y2 - y1\ over\ x2 - x1$".

Table 5.14: Response types for finding the gradient/slope between two points without GeoGebra (Question 4a)

Response Type	Explanation	Examples
A	No understanding of what was required to calculate slope A non-attempt	"I don't know" "no idea" blank comment
B	Limited understanding of concept of slope as a measurement to be calculated Practical methods using some form of tool, such as a ruler or protractor, to calculate slope described	"you could measure the angle of the slope with a protractor" "using the measurement"
C	Basic understanding of concept of slope found using a formula Links emerging between slope and right-angled triangles	"using the midpoint and adding a right-angled triangle from point b" "to find the slope, start from point A to B measure the distance and find the midpoint between the two. As the answer would be (1,-2.5), from point A measure 1 away and then measure downwards -.2,5 towards the midpoint. Connect the points into a triangle" "using the Cartesian plane and the formula" "using a certain formula" "you would make a small triangle and" "$G = m2 + b2$" "mx/b"
D	Developing understanding of calculating distance through using simple formula rise over run No explanation as to how to decide if negative or positive slope	"rise over run" "find the points where the line interceps the x and y axis and then divide the rise by the run"
E	Correct formula using coordinate pair notation	"using the formula, $\frac{y_1-y_2}{x_1-x_2}$" "$Slope = rise/run$ Formula: $G = y2 - y1\ over\ x2 - x1$"

Table 5.15: Thematic coding statistics for finding the gradient/slope between two points without GeoGebra (Question 4a)

Response Type	A	B	C	D	E	Total
Pre-test	5 (42)	5 (42)	2 (17)	0 (0)	0 (0)	12 pairs (100)
End of Topic test	0 (0)	0 (0)	1 (10)	1 (10)	8 (80)	10 pairs (100)
Delayed Post-test	2 (11)	1 (5)	5 (26)	8 (42)	3 (16)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Prestructural: Responses were below those expected for the question. Examples coded to this level were a blank response or responses such as “I don’t know”, “no idea” or “using a certain formula”.

The following cycle of levels were identified in the CS mode:

Unistructural (U_1): Responses indicated uncertainty regarding the specific requirements of the question, stating only one strategy triggered by visual cues. The only example coded to this level was an incomplete response that stated: “You would make a small triangle and”.

Multistructural (M_1): Responses focussed on more than one isolated aspect without any attempt to link these together. Responses may also take the form of an incorrect formula. Examples coded to this level included: “using the midpoint and adding a right-angled triangle from point b”, “using the Cartesian plane and the formula”, “mx/b” or “G=m2+b2”.

Relational (R_1): Responses represented an educated guess taking into account all the data using visual cues, such as the concept of measuring something. Examples coded to this level included: “you could measure the angle of the slope with a protractor” and “using the measurement”.

Multistructural (M_2): Responses focussed on more than one isolated aspect described using more mathematical language. Examples coded to this level included:

> To find the slope, start from point A to B measure the distance and find the midpoint between the two. As the answer would be $(1, -2.5)$, from point A measure 1 away and then measure downwards -.2,5 towards the midpoint. Connect the points into a triangle.

Relational (R_2): Responses indicated a number of connections had become apparent in the understanding of the question. The response was usually a correct response. Examples coded to this level included correct formulas that require no manipulation such as “rise over run”, "find the points where the line intercepts the x and y axis and then divide the rise by the run" [sic], or "$Slope = rise/run$ Formula: $G = y2 - y1\ over\ x2 - x1$".

Table 5.16: SOLO coding statistics for finding the gradient/slope between two points without GeoGebra (Question 4a)

SOLO Coding	Number		
	Pre-test	End of Topic test	Delayed Post-test
Prestructural	5	0	3
U_1 (CS)	0	0	1
M_1 (CS)	1	1	3
R_1 (CS)	5	0	1
M_2 (CS)	1	0	0
R_2 (CS)	0	9	11
Total	12	10	19

Question 5: What is the equation of the line you drew in part four?

This question required students to find a tool in GeoGebra, which would be able to provide the equation of the line drawn in the previous question, or use information already obtained to work out the equation of the line. By default, GeoGebra presented equations in general form and during the lesson sequence students were shown how to switch presentation of the equations to gradient-intercept form. Five types of responses were identified, as well as a ceiling response, summarised in Table 5.17 and Table 5.18.

Thematic Coding

Type A: This type of response indicated that the elements required for an equation were not recognisable, signifying no understanding of an equation or represented a non-attempt. Responses were either blank comments or single values representing a measurement.

In the Pre-test, four Type A responses were submitted. Team Black and Team Maroon both presented with a blank comment, while the remaining two pairs, Team Cream and Team Indigo, both submitted the solution they provided for the slope, "5.39", which was, in fact, the distance. Incidentally, both did not correctly provide this solution for the distance when asked in Question 3, demonstrating confusion between distinguishing the different concepts. For the End of Topic test, three pairs submitted this type of response. While Team Lime presented with a blank comment, Team Orange and Team Purple both submitted a solution representative of the slope. Team Orange submitted "$m = 0.5$" and Team Purple "6-4 over

3- -1". Interestingly, Team Purple did not provide this solution when asked to find the slope in Question 4. In the Delayed Post-test, 12 Type A responses were presented, with eight of these returning a blank comment. Team Maroon provided a response which was the slope of the line "$m = -2.5$", with Team Red A and Team Lime B both presenting responses that would represent the slope but were incorrect for the line given.

Type B: This type of response was an expression, which contained two or three terms. While some of these responses omitted the most important part of an equation that being the equals "=" sign, it indicated that students identified that equations had x and y terms. On the students' laptops "+" and "=" are on the same key, while "=" is default, "+" requires the shift key to be pressed at the same time as the key, thus keystroke errors could easily occur.

In the Pre-test, three Type B responses were submitted. Team Red and Team Yellow both presented expressions, submitting "$4.17x + 1.67y$" and "$5x + 2y + 10$" respectively, and Team White submitted: "$5x = 1.84y = 10$". Team White consistently obtained incorrect results because of the initial manual placement of points, making their x-coordinate inaccurate. Both Team Yellow and Team White provided solutions that would indicate an incorrect keystroke was the determining factor preventing them from obtaining a Type C or D response. Only one Team Indigo submitted this type of response in the End of Topic test with " $-x + 2y$". For the Delayed Post-test, no students provided a Type B response.

Type C: This type of response was a solution that was an equation with one or more incorrect numbers. Typical responses appeared to be a result of incorrect keystrokes either omitting digits or adding more digits to the number. It was possible that students were able to find the correct equation but were unable to replicate it.

In the Pre-test, Team Blue was the only pair that submitted a Type C response. They submitted "$5x + 2y = 0$", omitting the "1" required to make the right hand side of the equation 10. No student pairs presented this type of response in the End of Topic test. For the Delayed Post-test, four Type C responses were recorded. Team Red B submitted "$y = 1\frac{1}{2}x - 2½ \; y = 1x - 2.5$", which was the correct presentation of an equation but used the information from the slope to provide the numbers in the equation. Interestingly, Team Blue A submitted the exact same solution as in the Pre-test, namely: "$5x + 2y = 0$". Team White A included an extra number typing 120 instead of 10, submitting "$5x + 2y = 120$" and

Team White B submitted: "$y = 2.5x + 5$", which omitted the negative sign required for the slope.

Type D: This type of response completed a Type C response by providing the correct equation of $5x + 2y = 10$ for the Pre-test and Delayed Post-test, and $x + 2y = 9$ for the End of Topic test. By default, GeoGebra presented equations in general form, hence, students presenting their solution in this form indicated that they had simply transcribed what GeoGebra produced.

In the Pre-test, four Type D responses were submitted with three, Team Lemon, Team Brown and Team Purple, submitting: "$5x + 2y = 10$". Team Orange, however, submitted a solution of "d: $2.5x + y = 5$", directly copying all the information provided by GeoGebra. The "d:" indicated the identifier used in the GeoGebra environment to distinguish between multiple lines. For the End of Topic test, only Team Yellow A, who combined with Team Blue A for this task, provided the GeoGebra default solution of "$x + 2y = 9$". In the Delayed Post-test, only Team Lime A provided this type of response.

Type E: This type of response converted a Type D response, the default general form into the gradient-intercept form for the equation of a straight line. For the Pre-test and Delayed Post-test the solution was $y = -2.5x + 5$ and for the End of Topic test the solution was $y = 0.5x + 4.5$. This demonstrated students either recalled how to choose the right menu in order to convert the equation to gradient-intercept form, which was demonstrated during the lesson sequence for the unit, or were able to change the subject of the formula manually.

No Type E responses were submitted in the Pre-test. For the End of Topic test, five responses were coded as this type of response, namely: Team White, Team Maroon, Team Brown, Team Black and Team Cream. All provided the solution which required them to change the default format to "$y = 0.5x + 4.5$". In the Delayed Post-test, two Type E responses were recorded, namely: Team Brown A and Team Yellow A.

Table 5.17: Response types for determining the equation of a line (Question 5)

Response Type	Explanation	Examples
A	No understanding of equation Unable to recognise connect elements required in an equation A non-attempt	blank comments single values representing some form of measurement "5.39" "$m = 0.5$" "6-4 over 3—1"
B	Limited understanding of elements of an equation Recognising that equations contain expressions with x and y terms but not including correct signs	"$4.17x + 1.67y$" "$5x + 2y + 10$" "$5x = 1.84y = 10$" "$-x + 2y$"
C	Correct tool used Understanding of elements of an equation containing correct format with one or more numbers incorrect Incorrect transcription	"$5x + 2y = 0$" "$y = 1½\, x - 2½\ \ y = 1x - 2.5$" "$5x + 2y = 120$" "$y = 2.5x + 5$"
D	Correct tool used Demonstrates understanding of equation Accurate transfer of information from GeoGebra General form of equation	"$5x + 2y = 10$" "d: $2.5x + y = 5$" "$x + 2y = 9$"
E	Correct tool used Demonstrates knowledge of converting equation from general form (type D response) to gradient-intercept form	"$y = 0.5x + 4.5$" "$y = -2.5x + 5$"

Table 5.18: Thematic coding statistics for determining the equation of a line (Question 5)

Response Type	A	B	C	D	E	Total
Pre-test	4 (33)	3 (25)	1 (8)	4 (33)	0 (0)	12 pairs (100)
End of Topic test	3 (30)	1 (10)	0 (0)	1 (10)	5 (50)	10 pairs (100)
Delayed Post-test	12 (63)	0 (0)	4 (21)	1 (5)	2 (11)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Responses to Question 5 do not provide enough detail about the quality of the student response to attempt a fine-grained SOLO coding. Those responses classified as Type A could be broadly coded as Prestructural with Type B responses coded as Concrete Symbolic cycle 1 or early cycle 2. It is not possible to distinguish further without knowing the strategy employed by the student. In the light of these results, the responses provided to Question 5a do offer a window to view the students' understanding of the how they established the equation of the line, and this analysis follows.

Question 5a: How did you work this out?

This question required students to consider how they achieved their solution in Question 5. While some responses did improve after the teaching sequence, the majority of students found it difficult to explain how to find the equation of a line. Five different types of responses and six SOLO levels were noted based on the understanding and reasoning given, summarised in Tables 5.19, 5.20 and 5.21.

Thematic Coding

Type A: This type of response indicated no understanding of how to find the equation of the line or was representative of a non-attempt. Typical responses were blank comments or "I don't know". Any responses other than these were identified as not being able to assist in determining the equation of a line, illustrating that there was no understanding present.

In the Pre-test, four Type A responses were submitted. These were Team White, who stated: "I don't know", and Team Black, Team Maroon and Team Purple all left their comments blank. Interestingly, Team Purple submitted a correct solution for their equation but could not explain how they achieved this. For the End of Topic test, four pairs presented with this type of response. Team Lime and Team Indigo both submitted blank comments with Team Orange and Team Purple A, who combined with Team Blue A for this task, submitting "the gradient formula". In the Delayed Post-test, ten Type A responses were recorded. Nine of these presented with a blank comment or "I don't know" while the remaining student, Team Red B, stated the incorrect formula of "rise over run".

Type B: This type of response provided a description that implied GeoGebra was a main reason behind finding the solution. Typical answers involved keywords such as "GeoGebra" or "tool" without any explanation as to what was done in GeoGebra or which tool was used. This demonstrated limited or no conceptual understanding of the equation of a line or the elements required to obtain it.

In the Pre-test, five Type B responses were submitted. Team Indigo and Team Cream both stated: "I used the tool", without any mention of which tool. Team Brown stated "The GeoGebra provided the answer" [sic] without any mention of how the GeoGebra environment performed this. Team Orange and Team Lemon had similar responses stating, respectively: “we clicked on the equation and it showed us the line that it was on”, and “I looked to the left side of the screen and I noticed the answer was there already”. Each response indicated that the solution appeared and the students were either unaware of how and why, or unable to articulate how they achieved it. For the End of Topic test, three pairs recorded this type of response. Team Maroon stated: “we used geogebra, by clicking on the equation” [sic] and Team Yellow A, who combined with Team Blue A for this task, stated “we got the answer of geogerbra” [sic]. Team Black continued their response pattern, providing the same response each time when asked to explain how they did things by stating with little detail: “drawing a Cartesian plane and using the formula”. In the Delayed Post-test, six Type B responses were submitted. All stated the keyword, GeoGebra, by itself or in short sentences, such as: “Geogebra told me so” [sic], as stated by Team Blue A.

Type C: This type of response provided a correct response detailing the process followed to achieve an equation using GeoGebra. Typical responses involved explanations that a line needed to be drawn and naming the specific tool used in GeoGebra.

In the Pre-test, three Type C responses were submitted. Team Yellow and Team Blue provided similar simple statements with: “i drew a line between the two points” [sic] and “By using the 'line through points' on GeoGebra.”, respectively. Team Red accurately named the tool used stating: “By using the "line through two points" icon on Geogbra” [sic]. Interestingly, despite providing the best explanations for this question in the Pre-test, none of these three pairs provided a correct answer for the equation previously in Question 5. In both the End of Topic test and Delayed Post-test, no responses of this type were recorded.

Type D: This type of response expanded on a Type C response, demonstrating a clear understanding of how to obtain an equation of a straight line without using GeoGebra. The individual elements necessary were recognised, namely, the slope and y-intercept, along with the gradient-intercept form “$y = mx + b$”. Typical responses contained the equation “$y = mx + b$”.

In the Pre-test, no Type D responses were submitted. For the End of Topic test, three responses of this type were recorded. While Team Red A, who combined with Team White B for this task, presented with only “$y = mx + b$”, both Teams Brown and Cream gave more detailed responses, with Team Brown stating: “ $y = mx + b$ to find the y intercept: the interval will pass a point on the y axis.", and Team Cream stating: "$y = mx + b$ y=gradient (rise over run) x+the y intercept". Both provided some explanation to further clarify the meaning of the $y = mx + b$. In the Delayed Post-test, three Type D responses were submitted. Team White A providing only "$y = mx + b$" and Team White B expanded slightly stating: "$y = mx + b$ (the y intercept then the gradient)" – although, it is interesting to note that if the order of what is in brackets is supposed to correspond to the equation it is in fact wrong! Team Brown A provided an excellent response stating: "$y = mx + b$ m = the slope b = y intercept (it touches the y axis)", clearly defining each variable.

Table 5.19: Response types explaining how to find the equation of a line (Question 5a)

Response Type	Explanation	Examples
A	No understanding of how to find an equation or the elements required A non-attempt	blank comments "I don't know" "the gradient formula" "rise over run"
B	Limited understanding usually claiming GeoGebra or a tool was required to find the equation No conceptual understanding of what was required for an equation of a line Lacking detail explaining how GeoGebra or tool was used	"I used the tool" "the GeoGebra provided the answer" "we clicked on the equation and it showed us the line that it was on." "I looked to the left side of the screen and I noticed the answer was there already." "drawing a Cartesian plane and using the formula"
C	Developing understanding of how to find the equation of a line using GeoGebra	"I drew a line between the two points" "by using the 'line through points' on GeoGebra." "by using the "line through two points" icon on GeoGebra"
D	Students recognise elements required to construct an equation Connection between elements and gradient-intercept form established $y = mx + b$	"$y = mx + b$" "$y = mx + b$ to find the y intercept: the interval will pass a point on the y axis" "$y = mx + b$ y=gradient (rise over run) x+the y intercept" "$y = mx + b$ m = the slope b = y intercept (it touches the y axis)"

Table 5.20: Thematic coding statistics for explaining how to find the equation of a line (Question 5a)

Response Type	A	B	C	D	Total
Pre-test	4 (33)	5 (42)	3 (25)	0 (0)	12 pairs (100)
End of Topic test	4 (40)	3 (30)	0 (0)	3 (30)	10 pairs (100)

Response Type	A	B	C	D	Total
Delayed Post-test	10 (53)	6 (32)	0 (0)	3 (16)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Prestructural: Responses were below those expected. Examples coded to this level are a blank response or responses such as "I don't know".

The following cycle of levels were identified in the CS mode:

Multistructural (M_1): Responses focussed on more than one isolated aspect without being able to link these together. This response may also take the form of an incorrect formula. Examples coded to this level included a combination of "$-x + 2y$" for the previous question with a blank response for Question 5a indicating the different aspects identified without any understanding of how this was connected.

Unistructural (U_2): Responses indicated attempts to start the problem in a mathematical way but only focused on one, usually incorrect, aspect. Examples coded to this level included a combination of "AB=5.39" for the previous question with "I used the tool" for Question 5a, indicating that only one aspect was thought about, which coincidentally was incorrect for the question.

Multistructural (M_2): Responses focussed on more than one isolated aspect attempted sequentially and described using more mathematical language. Equations provided for the line contain incorrect elements in notation. Examples coded to this level included: "I drew a line between the two points" combined with "$5x + 2y + 10$" from the previous question or "the gradient formula" combined with "6-4 over 3- - 1".

Relational (R_2): Responses indicated a number of connections have become apparent in the understanding of the question. The response was usually a correct response. Examples coded to this level included the correct equation, "$5x + 2y = 0$", combined with "Geogebra told me so" for Delayed Post-test and "$5x + 2y = 10$" combined with "I looked to the left side of the screen and I noticed the answer was there already" from the Pre-test.

Formal (F): Responses focussed on the interrelationships between the equation and its different forms. It demonstrates effective and confident use of GeoGebra to convert the equation into gradient-form (default format in GeoGebra is general form of an equation).

Examples coded to this level included: “$y = 0.5x + 4.5$”, combined with: “$y = mx + b$ to find the y intercept: the interval will pass a point on the y axis” [sic], or: “$y = 0.5/1x + 4.5$” combined with “$y = mx + b$ $y =$ gradient (rise over run) x $+$ the y intercept” [sic].

Table 5.21: SOLO coding statistics for finding the equation of a line (Question 5 and 5a)

SOLO Coding	Number		
	Pre-test	End of Topic test	Delayed Post-test
Prestructural	2	2	8
M_1 (CS)	1	1	0
U_2 (CS)	2	0	3
M_2 (CS)	3	1	4
R_2 (CS)	4	1	2
F	0	5	2
Total	12	10	19

Question 6: Using GeoGebra can you draw the graph of $y = 4x + 8$?

This question was self-explanatory, asking students if they could draw the graph. While the question was worded such that responses could be a simple yes or no, most students provided further details as to how this was achieved, such that three types of responses were noted with thematic coding, summarised in Table 5.22 and Table 5.23.

Thematic Coding

Type A: This type of response indicated limited to no understanding of how to input an equation into GeoGebra or was representative of a non-attempt. Typical responses answered a simple yes or no as to whether or not the graph could be drawn and any procedures described were not coherent.

For the Pre-test, five pairs submitted this type of response. Team Indigo simply stated, “yes”, with no further explanation, and Teams Black, Blue and Purple all left their comments blank. Team Brown produced an interesting statement when they declared: “it is too hard to extend the numbers” with no further details. It is possible this pair had scaling issues with GeoGebra. No response of this type was coded in the End of Topic test. For the Delayed Post-test, four Type A responses were recorded. Team Yellow A and Lime B both stated: “yes”, while Team White A wrote “input”. Although the input bar is where you would type the equation into the GeoGebra environment, the term, “input”, without further detail would not enable someone to draw the graph. Team Orange B was also coded as a Type A response stating: “I typed the equation?”, the question mark suggesting uncertainty.

Type B: This type of response indicated recognition that the equation needed to be typed into GeoGebra but no specific direction as to how to do this. Typical responses indicated that the equation was typed in with either incorrect or not enough detail.

For the Pre-test, Team Orange, presented a Type B response, stating: “by typing in the equation GeoGebra automatically does it for you”. In the End of Topic test, four pairs recorded this type of response. Team Orange, Team Brown, Team Black and Team Yellow A, who combined with Team Blue A for this task, were all unable to articulate the correct place to type the equation, stating: “I typed it in the equation on GeoGebra”, “Type in the equation”, “we typed it in the toolbar bar” and “by typing the equation into GeoGebra” respectively. For the Delayed Post-test, this type of response was submitted five times. Responses ranged from simple statements as submitted by Team Black A, who stated “wrote it in the tool bar”, incorrectly naming where they wrote it, and Team Brown A who stated, “Type in the equation and press enter”, providing no detail as to where to type in the equation, with Team Blue B stating: “I put the equation into GeoGebra as the question says”, and Team Orange A stated: “yes I can by typing it into my computer”, to Team Indigo A who provided a more detailed response but included incorrect elements, stating: “yes, you put 8 on the y intercept and put 4 on the x axis”.

Type C: This type of response correctly indicated that the equation needed to be typed into the input bar of GeoGebra. Explanations included keywords, such as “bottom of the page” and “input bar and bottom bar”, to provide clarification on how to perform the required operation.

For the Pre-test, six pairs submitted this type of response. A typical response was provided by Team Lemon, who stated: “Yes, I wrote the equation in the input bar”. For the End of Topic test, this type of response was recorded six times. Responses ranged from simple statements as submitted by Team Cream, who stated: “type it into the input bar”, to more detailed explanations, such as from Team Lime who stated: “wrote in the bottom bar of GeoGebra $y = 4x + 8$ and it plotted it for us”. In the Delayed Post-test, ten Type C responses were submitted. Again, responses ranged from simple statements, such as provided by Team Red A who stated: “I typed $y = 4x + 8$ into the input bar”, and Team Black B who stated: “typed it into the input bar”, to more the detailed description, as submitted by Team Maroon A, who clearly articulated: "To create the graph of $y = 4x + 8$, I wrote the equation into the input bar which then automatically created the graph".

Table 5.22: Response types for drawing the graph of $y = 4x + 8$ using GeoGebra (Question 6)

Response Type	Explanation	Examples
A	Limited to no understanding	"it is too hard to extend the numbers" "input" "yes" blank comment
B	Recognition of how without specific direction	"type in the equation", "write it in the tool bar" "I put the equation into GeoGebra as the question says"
C	Correct description of typing into the input bar	"to create the graph of $y = 4x + 8$, I wrote the equation into the input bar which then automatically created the graph" "I typed $y = 4x + 8$ into the input bar" "yes I wrote the equation in the input bar"

Table 5.23: Thematic coding statistics for drawing the graph of $y = 4x + 8$ using GeoGebra (Question 6)

Response Type	A	B	C	Total
Pre-test	5 (42)	1 (8)	6 (50)	12 pairs (100)
End of Topic test	0 (0)	4 (40)	6 (60)	10 pairs (100)
Delayed Post-test	4 (21)	5 (26)	10 (53)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Responses to Question 6 do not provide enough detail about the quality of the student responses to attempt a fine-grained SOLO coding. Those responses classified as Type A could be broadly coded as Prestructural, with Type B and C responses coded as Concrete Symbolic cycle 1 or early cycle 2. It is not possible to distinguish further as the question itself did not ask for more information on how the drawing of the line was achieved. In the light of these results the responses provided to Question 6a and 6b do offer a window to view the students understanding of the how they drew the line and this analysis follows.

Question 6a: How would you do this without GeoGebra using pen and paper?

This question required students to reflect on how they would draw the previous graph if they had to produce it in their books. Five types and six different SOLO levels were noted for the responses, summarised in Tables 5.24, 5.25 and 5.26.

Thematic Coding

Type A: This type of response indicated no knowledge of how to draw a graph in their workbooks or was representative of a non-attempt. Typical responses were a blank comment or words of uncertainty.

In the Pre-test, ten pairs submitted this type of response. Four of these, namely: Team Indigo, Team Lemon, Team Cream and Team White, provided some form of comment such as "I don't know", "no idea" or "you couldn't", and the remaining six left their comments blank. For the End of Topic test, only Team Orange coded to Type A with a blank response. In the Delayed Post-test, six Type A responses were submitted, with two students, Team Orange B and Team Cream A, admitting they didn't know, while the remaining four were blank comments.

Type B: This type of response indicated a limited understanding of what was required to draw the graph. No specific instructions were offered in the response and it would be difficult to draw a graph using the information provided.

In the Pre-test, Team Red submitted the only Type B response with, "use a ruler". For the End of Topic test, two responses of this type were recorded, stating: "use a formula", but failed to provide any description of what formula to use and how to apply it. In the Delayed Post-test, four Type B responses were submitted with incomplete explanations, Team Orange A stated: "work out the sum" with no clarification as to what "sum". Team Red A stated: "by using a calculator and a ruler", without mentioning what calculation should be performed. Team Lemon stated: "using a certain formula", without explicitly stating what formula. Team Lime B simply stated: "draw it".

Type C: This type of response extended on the Type B response with basic understanding being demonstrated, specific elements were either incorrect, or not enough information was provided to enable the reader to follow the instructions to draw the graph.

For the Pre-test, only Team Brown provided this type of response stating: "draw a table with x and y", although no detail was provided to explain how values for the table should be chosen or calculated for drawing a graph. In the End of Topic test, three Type C responses were submitted. Team Cream lacked detail with "start at the y intercept which is 8". Team Indigo had incorrect information stating: "draw a Cartesian plane and plot the y intercept on the y axis and put the gradient on the x axis". While Team Lime provided correct facts with their response, "$y = 4x + 8$, rise = 4, run = 1, y intercept = 8", more detail was necessary to demonstrate how to use the information to produce the graph. For the Delayed Post-test, five Type C responses were recorded. Three were varied responses with incorrect elements, such as Team Black A, who stated: "go to the 4 on the x axes and draw a line on the 8 on the x axes", Team White A who stated: "8 is the midpoint rise over run is 4", and Team Blue A who stated: "by finding the y-interval then going down with how many x's and across with the y". The two remaining students, Team Blue B and Team Red B, made reference to a table of values and plotting points, when they stated, respectively: "put the x and the y values in a table and then plot the points" and "it might be easy to complete a table first of the points by creating a table first".

Type D: This type of response indicated a developing understanding of explaining how to draw a graph, although details relating to the graph in question were missing. Typical responses provide more detail than a Type C response but not enough to enable the reader to construct the graph themselves.

In the Pre-test, no responses of this type were recorded. For the End of Topic test, two Type D responses were submitted, with Team Brown stating: "$y = mx + b$, m = gradient, b = y – intercept. To find m use: rise over run. To find y use: the interval and it will pass through a point on the y axis." Unfortunately, despite the in-depth description, the students did not link the information to the specific graph or explain how to use it to draw the line. Team Red A, who combined with Team White B for this task, responded with: "plot down the y intercept number and work out the rise over run". In the Delayed Post-Test Team White B produced a similar response stating "start at the y-intercept and go up by the rise and the run". Again, correct information was presented, but finishing details to provide a Type E response were missing.

Type E: This type of response extended on the Type D response and provided clear instructions of how to draw the graph $y = 4x + 8$.

For the Pre-test, no Type E responses were recorded. In the End of Topic test, two pairs submitted this type of response. Team Maroon stated: "8 in the y intercept, then it's 4 over 1 so you rise by 4 and run by 1", and Team Yellow A, who combined with Team Blue A for this task, stated: "(1) from the point 0 move upwards 8 points along the y axis and plot a point there. (2) move downwards 4 points and to the left one point and plot the second point (3) join the points with a continuous line". Only one response of this type was submitted in the Delayed Post-test, Team Maroon A, stating: "I would start at the zero mark and then plus the eight then move along to either the plus or minus one and times the number by four then adding eight I would do this at different intervals to get the graph". While this response is verbose it does explain how to calculate the points necessary for the graph.

Table 5.24: Response types for drawing the graph $y = 4x + 8$ without GeoGebra (Question 6a)

Response Type	Explanation	Examples
A	No understanding of how to draw a graph	"I don't know" "no idea" blank comment
B	Limited understanding, no specific direction, difficult to draw using instructions provided	"use a ruler", "use a formula" "using a certain formula"
C	Elements of understanding present requiring more information to be correct	"draw a Cartesian plane and plot the y intercept on the y axis and put the gradient on the x axis" "start at the y intercept which is 8", "$y = 4x + 8$, rise = 4, run = 1, y intercept = 8" "put the x and the y values in a table and then plot the points" "draw a table with x and y"
D	Knowledge of graphs present lacking details to the graph in question	"$y = mx + b$, m = gradient, $b = y$ – intercept. To find m use rise over run. To find y use the interval and it will pass through a point on the y axis." "plot down the y intercept number and work out the rise over run"
E	Clear instructions regarding drawing graph $y = 4x + 8$	"8 in the y intercept, then it's 4 over 1 so you rise by 4 and run by 1" "I would start at the zero mark and then plus the eight then move along to either the plus or minus one and times the number by four then adding eight I would do this at different intervals to get the graph"

Table 5.25: Thematic coding statistics for drawing the graph $y = 4x + 8$ without GeoGebra (Question 6a)

Response Type	A	B	C	D	E	Total
Pre-test	10 (83)	1 (8)	1 (8)	0 (0)	0 (0)	12 pairs (100)
End of Topic test	1 (10)	2 (20)	3 (30)	2 (20)	2 (20)	10 pairs (100)
Delayed Post-test	6 (32)	4 (21)	5 (26)	3 (16)	1 (5)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Prestructural: Responses were below those expected for the question. Examples coded to this level are a blank response or responses such as "I don't know".

The following cycle of levels were identified in the CS mode:

Unistructural (U_1): Responses indicated uncertainty of the question requirements, focussing on one specific aspect that was usually visual or unrelated to the question. Examples coded to this level included "using a ruler" or "use a formula" or "draw it".

Relational (R_1): Responses reflected an educated guess that attempted to link all the information and was usually assisted by visual cues. Examples coded to this level included: "by finding the y interval then going down with how many x's and cross with the y".

Unistructural (U_2): Responses focussed on one aspect, with less assistance from visual cues that may be incorrect for the question. Examples coded to this level included: "start at the y intercept which is 8" or "it might be easy to complete a table first of the points by creating a table first".

Multistructural (M_2): Responses focussed on more than one isolated aspect of the equation. While elements of the response were correct, explanations were incomplete for drawing a line. Examples coded to this level included: "$y = 4x + 8$ rise =4 run=1 y intercept=8".

Relational (R_2): Responses indicated a number of connections had become apparent in the understanding of the question, with explanations that outlined steps to drawing the line. The response may contain incorrect elements but demonstrated developing understanding of procedures of drawing a line. Examples coded to this level included: "plot down the y intercept number and work out the rise over run" and "I would start at the zero mark then plus the eight then move along to either plus or minus one and times the number by four then adding eight I would do this at different intervals to get the graph".

Table 5.26: SOLO coding statistics for drawing the graph $y = 4x + 8$ without GeoGebra (Question 6a)

SOLO Coding	Number		
	Pre-test	End of Topic test	Delayed Post-test
Prestructural	10	1	6
U_1 (CS)	1	2	4
R_1 (CS)	0	0	2
U_2 (CS)	0	1	1
M_2 (CS)	1	3	4
R_2 (CS)	0	3	2
Total	12	10	19

Question 6b: Can you move the graph using GeoGebra? Notice what changes on the graph and what changes in the equation. Explain these changes below**.**

This question was an exploratory question linking the graph to the equation. When moving the graph, students were encouraged to recognise changes in the equation and attempt to establish if some, or any, connections were present between the graph and its associated equation. Four types and seven SOLO levels of responses were noted, summarised in Tables 5.27, 5.28 and 5.29.

Thematic Coding

Type A: This type of response indicated that no understanding regarding how to work out how to move the graph or was representative of a non-attempt. Typical responses were blank comments and statements such as “I don’t know”.

For the Pre-test, ten Type A responses were recorded. While Team White commented, “nothing changes”, indicating their inability to either properly move the graph or have the equation properly written in the algebra section, the remaining nine all submitted blank comments. In the End of Topic test, three pairs, namely Team Black, Team Indigo and Team Lime, all presented blank comments. For the Delayed Post-test, seven Type A responses were submitted ranging from Team Red B and Team Indigo B who both admitted to not being able to move the graph, to Team Orange B who stated “idk” an acronym for “I don’t know”, and the remaining four, namely, Team Black A, Team Lime B, Team Black B and Team Indigo A all submitted blank comments. Interestingly, Team Indigo submitted a Type A response for this question in all three Google Form tests, indicating inability to understand moving graphs within the GeoGebra environment.

Type B: This type of response indicated that, while a change was noticed, it could not be correctly articulated, reflecting limited understanding of the Linear Relationships concepts.

For the Pre-test, only Team Red provided this type of response when they stated, “yes you can move it. The points move and the numbers change”. Specific mention of which numbers and how they changed were not described. In the End of Topic test, four Type B responses were recorded. Reponses varied; Team Brown stated: “the gradient and y intercept changes while moving the graph”, Team Maroon stated: “the line increases according to the y intercept”, Team Red A, who combined with Team White B for this task, stated: “the gradient changes”, and Team Yellow A, who combined with Team Blue A for this task, stated: “the y axis changes and the x axis stays the same”. For the Delayed Post-test, only Team Lime A and Team Blue B submitted this type of response when they stated, respectively: “it’s the same x axis different y axis” and “the x value in the graph changes but still keeps it a straight line”. Both students were unable to describe the change in terms of concepts of Linear Relationships.

Type C: This type of response correctly suggested changes that occur without reference to terminology used in Linear Relationships. Typical responses used simple language to describe changes.

In the Pre-test, only Team Cream submitted this type of response with “Yes, the first part of the equation stayed the same but after the + the number changed”. In the End of Topic test, no responses of this type were recorded. For the Delayed Post-test, eight Type C responses were submitted. All recognised the y-intercept changed with three stating that the gradient remained the same. The three responses that mentioned the gradient were: Team Red A, who stated: “the equation remains $y = 4x$ but the thing we add changes”, Team White A, who stated: “the slope stays the same but the interval changes”, and Team Yellow A, who stated: “in the equation the y intercept changes but the x stays the same”. The remaining six responses mention the change as affecting the y-intercept only, namely, Team Orange A, who stated: “the number changes at the end”, Team Lemon A, who stated: “the +8 changes its value”, Team White B, who stated: “y-intercept”, Team White A, who stated: “the slope stays the same but the interval changes”, Team Blue A, who stated: “Yes you can. The y interval changes”, and the verbose response from Team Maroon A who stated: “I do not know what you mean but I am guessing that you mean moving it by I think parallels you do this by changing the number (8) to any other number and it should move cross. Sorry but

I can't remember the proper terminology". Team Maroon A acknowledging language was an important part of the learning process but was beyond his recall.

Type D: This type of response indicated a sound understanding of the change reflected in the graph and the equation. Descriptions made reference to the change in the y-intercept and the constant gradient. Typical responses used correct terminology, including keywords "slope", "gradient" and "y-intercept".

None of this type of response was submitted in the Pre-test. In the End of Topic test, three pairs presented this type of response. Team Orange stated: "yes the slope stays the same and the y-intercept changes", while Team Cream stated: "The $4x$ stays the same and the +8 changes", and Team Purple A, who combined with Team Blue B for this task, stated: "to move the graph all you need to do is change the y-intercept and keep the same gradient". For the Delayed Post-test, two Type D responses were submitted, namely, Team Brown A, who stated: "I notice that when I move the line the gradient stays the same but the y intercept changes and it is still a positive slope", and Team Cream A who stated the same response as provided for the End of Topic test.

Table 5.27: Response types for noticing changes in a graph (Question 6b)

Response Type	Explanation	Examples
A	Unable to move graph	"nothing changes" blank comment
B	Change noticed but unable to be explained	"yes you can move it. The points move and the numbers change" "the gradient and y intercept changes while moving the graph" "the line increases according to the y intercept"
C	Explanations demonstrate more understanding lacking terminology	"yes, the first part of the equation stayed the same but after the + the number changed" "the number changes at the end" "the equation remains $y = 4x$ but the thing we add changes" "the slope stays the same but the interval changes" "I do not know what you mean but I am guessing that you mean moving it by I think parallels you do this by changing the number (8) to any other number and it should move cross. Sorry but I can't remember the proper terminology"

Response Type	Explanation	Examples
D	Descriptions contain reference to y-intercept and gradient	"yes the slope stays the same and the y-intercept changes" "the 4x stays the same and the +8 changes" "I notice that when I move the line the gradient stays the same but the y intercept changes and it is still a positive slope"

Table 5.28: Thematic coding statistics for noticing changes in a graph (Question 6b)

Response Type	A	B	C	D	Total
Pre-test	10 (83)	1 (8)	1 (8)	0 (0)	12 pairs (100)
End of Topic test	3 (30)	4 (40)	0 (0)	3 (30)	10 pairs (100)
Delayed Post-test	7 (37)	2 (11)	8 (42)	2 (11)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Prestructural: Responses were below those expected for the question. Examples coded to this level are a blank response or responses, such as "idk".

The following cycle of levels were identified in the CS mode:

Unistructural (U_1): Responses indicated uncertainty of the requirements, with focus on one specific, usually visual aspect that may be unrelated to the question. Examples coded to this level included: "I wasn't able to move the graph".

Multistructural (M_1): Responses focussed on more than one aspect with no evident connections between each aspect, usually characterised by separate statements. Examples coded to this level included: "the y axis changed and the x axis stays the same", "the line increases according to the y intercept", "yes you can move it. The points move and the numbers change" and "it's the same x axis different y axis".

Relational (R_1): Responses reflected an educated guess using visual cues to connect information. Examples coded to this level included: "the x value in the graph changes but still keeps it a straight line" and "I do not know what you mean but I am guessing that you mean moving it by I think parallels you do this by changing the number (8) to any other number and it should move across. Sorry but I can't remember the proper terminology".

Unistructural (U_2): Responses focussed on one aspect, that may be incorrect, but less reliance on visual cues was evident and language had developed to be more conceptual. Examples coded to this level included: “the gradient changes”, “y-intercept” and “the number changes at the end”.

Multistructural (M_2): Responses focussed on more than one isolated aspect of the changes, often as separate statements. Information that connected the changes was missing or incomplete. Examples coded to this level included: “the slope stays the same but the interval changes”, “in the equation the y intercept changes but the x stays the same”.

Relational (R_2): Responses indicated a number of connections had become apparent in the understanding of the question, with explanations demonstrating an understanding of changes that occur. The response was usually a correct response. Examples coded to this level included “I notice that when I move the line the gradient stays the same but the y intercept changes and it is still a positive slope”, “the equation remains $y = 4x$ but the thing we add changes” and “to move the graph all you need to do is change the y-intercept and keep the same gradient”.

Table 5.29: SOLO coding statistics for noticing changs in a graph (Question 6b)

SOLO Coding	Number		
	Pre-test	End of Topic test	Delayed Post-test
Prestructural	9	3	6
U_1 (CS)	1	0	1
M_1 (CS)	1	3	1
R_1 (CS)	0	0	2
U_2 (CS)	0	1	4
M_2 (CS)	0	0	2
R_2 (CS)	1	3	3
Total	12	10	19

Question 7: Using GeoGebra draw the lines $y = -3x$, $y = -3x - 2$, $y = -3x + 4$. What do you notice about these lines?

This question addressed the concept of parallel lines. For the Pre-test and Delayed Post-test this was Question 7 and for the End of Topic test this was presented as Question 8. Students were invited to use GeoGebra to assist them to illustrate the graphs, thus enabling them to draw conclusions from their observations. Responses indicated that some students were able to distinguish noticeable changes and features between the graphs simply by analysing their equations. Four types of responses and seven SOLO levels were identified for this question, summarised in Tables 5.30, 5.31 and 5.32.

Thematic Coding

Type A: This type of response indicated no knowledge of properties of graphs or was representative of a non-attempt. Those who struggled to draw the graphs using the GeoGebra environment were unable to draw conclusions and coded this type of response. Typical responses were blank comments or incorrect descriptions.

In the Pre-test, 11 Type A responses were submitted; ten of these were a blank comment. Team Red, the only team to attempt an explanation, submitted: "they cross over each other", indicating that the wrong graphs were drawn or that the statement was made without any basis. Type A responses were submitted by three pairs in the End of Topic test. Teams Black and Indigo both submitted blank comments, while Team Yellow A, who combined with Team Blue A for this task, submitted: "They are the exact same equation, therefore they are the same, and you can only see one of the lines" [sic], since they did not listen to the instructions at the beginning of the lesson which clearly asked students to change one of the equations (due to a typing error that had the equations as being the same). In the Delayed Post-test, three Type A responses were recorded. Each response was incorrect, with Team White B and Team Indigo B both stating that the lines were "perpendicular" and Team Black A stating "there all going one way" [sic], clearly unable to articulate what they visualised.

Type B: This type of response indicated a correct response using simple language. Students recognised that the slope was the same but were unable to correctly label them as parallel. It demonstrated GeoGebra was correctly used; recognition of the common features of the equations were evident but language had not been sufficiently developed.

In both the Pre-test and End of Topic tests, no Type B responses were evident. In the Delayed Post-test, two Type B responses were recorded. Team Lemon A stated: "they all have the same slope", and Team Lime B stated: "the all have $y = 3x$". Team Lime B provided an interesting response that indicated it was possible that they may not have drawn the graphs but discovered a common theme in the algebra of the equations.

Type C: This type of response extended on a Type B response where students recognised that the lines have the same slope and were capable of labelling it using correct terminology. Typical responses of this type contained the keyword "parallel". Again, this indicated the ability to draw the graphs or identify that the graphs were parallel through observations involving the coefficient of x. Unfortunately, none of the responses provided an explanation of how they reached their conclusions – whether it was the visual or the algebraic perspective that determined they were parallel.

In the Pre-test, Team White provided the only Type C response stating: “they are parallel”. For the End of Topic test, five Type C responses were recorded, namely, Team Lime, Team Orange, Team Cream, Team Blue B, who combined with Team Purple A for this task, and Team Red A, who combined with Team White B for this task. All stated similar responses, with simple statements such as: “they’re parallel”. In the Delayed Post-test, 11 Type C responses were recorded, all with simple statements involving the word “parallel”.

Type D: This type of response indicated that the lines were recognised as being parallel not only from a visual perspective but also from an algebraic perspective. Explanations connected parallel lines to the gradient/slope, identifying that the lines with the same gradient were parallel. Responses demonstrated a higher level of thinking because students not only stated what they noticed but also provided justification.

In the Pre-test, no Type D responses were submitted. For the End of Topic test, two Type D of responses were recorded, namely, Team Maroon who stated: “they are parallel to each other because they have the same gradient”, and Team Brown who stated: “They are parallel where the gradient stays the same and the y intercept changes”. In the Delayed Post-test, three Type D responses were noticed. Team Red B stated: “they are all parallel to each other and they all have a gradient of -3”, Team White A stated: “because the rise over run is the same all the lines are parrallel” [sic], with Team Brown A providing a clear definition submitting: “They are parallel which means the gradient stays the same but the y-intercept changes $m1 = m2$”.

Table 5.30: Response types for Identifying parallel lines (Question 7)

Response Type	Explanation	Examples
A	No knowledge of properties of graphs Incorrect comments derived from incorrect drawing of graphs Non-attempt	“they cross over each other” “they are the exact same equation, therefore they are the same, and you can only see one of the lines” “perpendicular” “there all going one way” blank comment

Response Type	Explanation	Examples
B	Students recognise graphs have a common feature but unable to link it to the terminology "parallel"	"they all have the same slope" "the all have $y = 3x$"
C	Students recognise and describe graphs as parallel No explanation is provided to determine whether conclusion was due to visual or algebra	"they are parallel" "parallel"
D	Thorough understanding of parallel lines presented Students recognise and describe the graphs as being parallel Connection is made to gradient/slope in equation	"they are parallel to each other because they have the same gradient" "they are parallel where the gradient stays the same and the y intercept changes" "they are all parallel to each other and the all have a gradient of -3" "because the rise over run is the same all the lines are parrallel" [sic] "they are parallel which means the gradient stays the same but the y-intercept changes $m1 = m2$"

Table 5.31: Thematic coding statistics for identifying parallel lines (Question 7)

Response Type	A	B	C	D	Total
Pre-test	11 (92)	0 (0)	1 (8)	0 (0)	12 pairs (100)
End of Topic Test	3 (30)	0 (0)	5 (50)	2 (20)	10 pairs (100)
Delayed Post-test	3 (16)	2 (11)	11 (58)	3 (16)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Prestructural: Responses were below those expected for the question. Examples coded to this level are blank responses.

The following responses were identified in the CS mode:

Unistructural (U_1): Responses indicated uncertainty of the question requirements, with a primary focus on one specific aspect that was usually visual and unrelated or specific to the question. Examples coded to this level included: "they cross over each other".

Unistructural (U_2): Responses indicated an attempt to start the problem using more mathematical language but focus on only one aspect. Usually reference was made to the fact that the lines were parallel or an incorrect concept. No explanation was provided as to why or how the conclusion was made. Examples coded to this level included simple statements such as: “they are parallel” and “perpendicular”.

Multistructural (M_2): Responses focussed on more than one isolated aspect of the equations, but no connection was established between the aspects. An example coded to this level included: “they are all parallel to each other and they all have a gradient of -3”. This response falling short of R_2 because of the use of the word “and”, indicating no connection was established between the gradient and the parallel lines.

Relational (R_2): Responses indicated a number of connections had become apparent in the understanding of the question. The response was usually correct and linked parallel lines with the gradient. Examples coded to this level included: “they are parallel which means the gradient stays the same but the *y*-intercept changes m1=m2” and “because the rise over run is the same all the lines are parallel”.

Table 5.32: SOLO coding statistics for identifying parallel lines (Question 7)

SOLO Coding	Number		
	Pre-test	End of Topic test	Delayed Post-test
Prestructural	10	2	0
U_1 (CS)	1	1	3
U_2 (CS)	1	5	13
M_2 (CS)	0	1	1
R_2 (CS)	0	1	2
Total	12	10	19

Question 7a: Give an example of an equation of a line that would belong to this family of lines.

While the previous question presented equations where students were invited to use the GeoGebra environment and comment on what they noticed, this question aimed to see if the students recognised the direct link between the parallel lines and gradient so that they could create a new equation of a line parallel to those previously given. It removes the visual perspective and required students to look closely at the equations to determine the algebraic association. Four types and five SOLO levels of responses were identified for this question, summarised in Tables 5.33, 5.34 and 5.35.

Thematic Coding

Type A: This type of response indicated that students had no idea of a line that would belong to the family of parallel lines given in the first part of the question, or was representative of a non-attempt. It was possible that students may have been able to correctly identify parallel lines previously but were unable to recognise that the gradient/slope (represented by the coefficient of x in the gradient form of an equation) were the same for each equation.

In the Pre-test, a total of eleven Type A responses were coded for this question with only one pair that submitted a response for the previous part of this question. Team White, submitted: “hmmm”, although, interestingly, they identified the graphs as parallel previously, but could not produce another equation that would be parallel, the remaining responses were blank. For the End of Topic test, Team Indigo and Team Black presented blank comments. In the Delayed Post-test, no Type A responses were submitted.

Type B: This type of response provided an answer that was incorrect. It indicated no awareness of any link between parallel lines and the gradient. It is quite possible that students guessed the solutions submitted.

In both the Pre-test and End of Topic test, no Type B responses were presented. For the Delayed Post-test, three responses of this type were recorded. Team Blue A stated: “$y = -5x$”, and Team Black B stated: “$x - y = -1$”, while Team Orange A submitted a huge constant with more than 300 digits. All three contained no direct link to the correct solution despite two students providing equations in their responses.

Type C: This type of response indicated developing links between parallel lines and the gradient, although the responses provided were not correct. Typical responses involved what would be expected as a correct Type D response with an element missing or incorrect.

In both the Pre-test and End of Topic test, no Type C responses were submitted. In the Delayed Post-test, five of this type of response were recorded. Three of these, Team Yellow A, Team Lime B and Team Blue B all submitted an equation that omitted the negative symbol in front of the gradient. Team Yellow A submitted a typical response stating “$y = 3x - 5$”. Team Lime A submitted an expression rather than an equation, presenting “$-3x + 100$”. This was still considered a Type C response as the coefficient of x was correct, the only element missing was the “$y =$” to make it into an equation. Team Orange B submitted “$y = 3x = 3329863257563257325986325986953829863526835286326896$” which contained only one incorrect element, the second “=” symbol used instead of a “+” or, possibly indicative of a keystroke error, due to “+” and “=” being on the same key for students' laptops, as mentioned previously.

Type D: This type of response indicated that the students understood the direct link between the gradient and the coefficient of the x term when presented as an equation in gradient-intercept form. That is, for equations presented in gradient-intercept form, the coefficient of the x terms of parallel lines were equal. Responses coded as a Type D response commence with $y = -3x + constant$ for the Pre-test and Delayed Post-test, and $y = \frac{1}{2}x + constant$ for the End of Topic test.

In the Pre-test, Team Red provided a Type D response, submitting: “$y = -3x + 6$”. While they recognised that the coefficient of x in each of the examples provided in the previous question were the same; interestingly, they did not recognise that the lines were parallel – it demonstrates algebraic awareness without the visual understanding. For the End of Topic test, eight pairs submitted this type of response. Seven of these provided one or two examples of lines parallel to the original examples given in the previous part of this question. A typical response was provided by Team Maroon, who stated: “$y = 1/2x + 2$”. Team Blue B, who combined with Team Purple A for this task, presented an interesting response. They submitted: “$y = 40x - 1000000$ $y = 40x + 2$”, which could, at first glance, be presumed as an incorrect response. However, the question required an example of an equation of a line that would belong to this family of lines. This student pair interpreted the question as asking for equations of lines that were parallel to each other, identifying the family of lines as parallel rather than parallel to the examples given. Viewed from this lens, their response represents a Type D response, demonstrating that they understood that to be parallel requires two (or more) lines which have the same gradient, determined by the coefficient of x. In the Delayed Post-test, 11 Type D responses were submitted, correctly identifying that the coefficient of the x term in the equation of a line must be equal to represent parallel lines. A typical response was given by Team Red A, who stated: “$y = -3x + 200$”.

Table 5.33: Response types for finding an equation of line parallel to another line (Question 7a)

Response Type	Explanation	Examples
A	Unable to recognise the elements of an equation that relate to parallel lines Non attempt	"hmmm" blank comment
B	No understanding present regarding link between parallel lines and gradient Possible guess	"$y = -5x$" "$x - y = -1$"
C	Developing understanding between gradient and parallel lines One element incorrect	"$y = 3x -$ 5" (should be $-3x$) "$-3x\ +\ 100$" (missing $y =$) "$y = 3x =$ 33298632575632573259863259.." (= should be +)
D	Correct solution demonstrating understanding of link between gradient, co-efficient of x and parallel lines	"$y = -3x + 6$" "$y = 1/2x + 2$" "$y = -3x + 200$" "$y = 40x - 1000000\ y\ =\ 40x + 2$"

Table 5.34: Thematic coding statistics for finding an equation of line parallel to another line (Question 7a)

Response Type	A	B	C	D	Total
Pre-test	11 (92)	0 (0)	0 (0)	1 (8)	12 pairs (100)
End of Topic test	2 (20)	0 (0)	0 (0)	8 (80)	10 pairs (100)
Delayed Post-test	0 (0)	3 (16)	5 (26)	11 (58)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Responses to Question 7a do not provide enough detail about the quality of the student response to attempt a fine-grained SOLO coding. When viewed with the responses provided to Question 7, they do offer a window to view the students understanding of the how they established the equation of the line and this analysis follows.

Prestructural: Responses were below those expected for the question. Examples coded to this level are a blank response.

The following cycle of levels were identified in the CS mode:

Unistructural (U_1): Responses indicated uncertainty of the requirements of the question, focussing on one specific aspect that was usually visual and unrelated or specific to the question. Examples coded to this level included the combination of: “they are parallel”, for Question 7 and: “hmmm” for Question 7a.

Unistructural (U_2): Responses focussed on only one aspect, demonstrating a development in the conceptual language that may be incorrect. Examples coded to this level included a combination of: “they are parallel”, for Question 7 and then a huge constant provided for Question 7a.

Multistructural (M_2): Responses focussed on more than one isolated aspect of the equation. In most cases, students responded that lines were parallel but were not able to produce an equation representing a line parallel to the one given. The responses provided usually contained an element of notation that was incorrect. Examples coded to this level included the combination of, “they are all parallel” for Question 7 and then “$y = 3x - 585748$” for Question 7a, which failed to have the gradient as -3.

Relational (R_2): Responses indicated a number of connections had become apparent in the understanding of the question. Responses were a correct response for the equation (Question 7a) with various responses for the identification of the lines being parallel (Question 7). These responses also demonstrated understanding of the connection between parallel lines; however, expressing this understanding using correct terminology was difficult when required in Question 7. Examples coded to this level included: “they are all parallel”, combined with “$y = -3x + 200$”, “that they are all perpendicular”, combined with “$y = -3x + 2$”, and “there all going one way”, combined with “$y = -3x + 6$”.

Table 5.35: SOLO coding statistics for finding an equation of line parallel to another line (Question 7a)

SOLO Coding	Number		
	Pre-test	End of Topic test	Delayed Post-test
Prestructural	10	2	0
U_1 (CS)	1	0	0
U_2 (CS)	0	0	1
M_2 (CS)	0	0	7
R_2 (CS)	1	8	11
Total	12	10	19

Question 8: Using GeoGebra draw the lines, $y = 1/3x + 4$, $y = -3x + 4$. What do you notice about these lines?

This question addressed the concept of perpendicular lines. For the Pre-test and Delayed Post-test, this was Question 8 and for the End of Topic test, this was as Question 7. Students were invited to use GeoGebra to assist with drawing the graphs, thus, enabling them to focus on conclusions drawn from their observations. Five types and five SOLO levels were identified for the responses provided in this question, summarised in Tables 5.36, 5.37 and 5.38.

Thematic Coding

Type A: This type of response indicated no knowledge of how to correctly draw lines or was representative of a non-attempt. Typical responses of this type were blank comments or irrelevant and incorrect statements.

In the Pre-test, 11 Type A responses were submitted. Of these, ten were non-attempts, with only Team White submitting a response with: "didn't come up but I typed then in "input""[sic]. For the End of Topic test, only Team Indigo, submitted this type of response with: "On the first one it crosses on the 0 for the y axis". While this was a correct statement for the first equation, $y = 1/2x$, there was no description given for the second line, $y = -2x - 2$. In the Delayed Post-test, no responses were coded to this type.

Type B: This type of response indicated a basic understanding of graphs. Typical responses used simple language, with keywords such as "cross", "cross over" and "intercept". No distinguishing features that would identify where or how they cross were noted.

In both the Pre-test and End of Topic test, no Type B responses were submitted. For the Delayed Post-test, eight responses were coded to this type. Six of these made simple statements using the keywords "cross" or "cross over". Team Maroon A identified that there was a specific term stating: "I notice that in these lines they cross over to each other but I can't remember what it is called (crossover?, intersection?, horizontal?, meeting at a point????)" [sic]. Interestingly, Team Maroon A submitted a similar response for Question 6b, indicating that recall of concepts was not sufficiently developed. Team Indigo B stated "they both cross on (4, 0)", but, unfortunately, the point was inaccurately labelled, with the x and y coordinates in the incorrect order, hence coded as a Type B response.

Type C: This type of response extended on the basic understanding demonstrated in Type B responses by the recognition of *where* the lines cross; that is, the point of intersection, namely, (0,4).

In the Pre-test, only Team Red submitted this type of response, stating: "They meet at the coordinates 0,4". No responses of this type were presented in the End of Topic test. For the Delayed Post-test, three Type C responses were submitted. While Team Red A identified a specific point, stating: "they cross over at (0, 4)", Team Orange B and Team Cream A submitted general comments about the point of intersection rather than stating the coordinate. Their responses, respectively, were: "they cross over at the same y axis" and "that they cross over at the y –intercept".

Type D: This type of response indicated understanding that considered the relationship between the two lines. Typical responses identified the lines as perpendicular to each other without any clear justification. Responses demonstrated understanding about *how* the lines crossed, this required more abstract thinking since perpendicular lines are not clearly distinguished purely from the visual perspective.

In the Pre-test, no Type D responses were submitted. For the End of Topic test, eight responses of this type were submitted. Each of these responses was a simple statement: "They are perpendicular". In the Delayed Post-test, eight Type D responses were coded. While seven of these contained simple statements similar to those presented in the End of Topic test, Team Brown A provided a little more detail, stating: "These lines are perpendicular and the reciprocal changes". Although a new term, reciprocal, was identified there was no explanation of how it related to the perpendicular lines.

Type E: This type of response extended the Type D response by providing an explanation as to *why* the lines cross perpendicularly from an algebraic perspective, demonstrating an understanding of the relationship between perpendicular lines and gradients.

For the Pre-test and Delayed Post-test, no Type E responses were submitted. In the End of Topic test, only Team Maroon provided this type of response stating: "they are perpendicular because the equations are reciprocal to each other and one equation has a negative". While they use the term "equation" instead of "gradient", they demonstrate understanding of finding the gradient of the perpendicular algebraically.

Table 5.36: Response types for identifying perpendicular lines (Question 8)

Response Type	Explanation	Examples
A	Demonstrates no understanding of how to draw lines both lines Non-attempt	“didn't come up but I typed then in "input"” “On the first one it crosses on the 0 for the *y* axis” blank comment
B	Basic understanding of visual features of graphs No features noted such as where or how they cross Simple keywords “cross” or “cross over” or “intercept”	“I notice that in these lines they cross over to each other but I can't remember what it is called (crossover?, intersection?, horizontal?, meeting at a point????)” “they both cross on (4, 0)”
C	Further identifying the point where graphs meet (0, 4) Recognises *where* they cross	“they meet at the coordinates 0,4” “they cross over at (0, 4)” “they cross over at the same *y* axis”
D	Developing understanding of intersection of lines Uses correct terminology without any explanation Recognises *how* the lines cross	“they are perpendicular” “these lines are perpendicular and the reciprocal changes”
E	Further recognising *why* the lines are perpendicular Linking perpendicular lines to algebra and gradient	“they are perpendicular because the equations are reciprocal to each other and one equation has a negative”

Table 5.37: Thematic coding statistics for identifying perpendicular lines (Question 8)

Response Type	A	B	C	D	E	Total
Pre-test	11 (92)	0 (0)	1 (8)	0 (0)	0 (0)	12 pairs (100)
End of Topic test	1 (10)	0 (0)	0 (10)	8 (80)	1 (10)	10 pairs (100)
Delayed Post-test	0 (0)	8 (42)	3 (16)	8 (42)	0 (0)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Prestructural: Responses were below those expected. Examples coded to this level were a blank response.

The following cycle of levels were identified in the CS mode:

Unistructural (U_1): Responses focussed on one visual aspect from the graphs explained using simple language. Examples coded to this level included: “they meet at the coordinates 0,4”, “they cross over each other”, “they cross over at the same y axis”, and “the lines run through each other”.

Unistructural (U_2): Responses indicated focus on one aspect, using specific language for Linear Relationships, usually involving the term “perpendicular”. Examples coded to this level included: “they are perpendicular”.

Multistructural (M_2): Responses focussed on more than one isolated aspect of the equation, without any statements to link the aspects to each other. The example coded to this level included: “these lines are perpendicular and the reciprocal changes”.

Relational (R_2): Responses indicated connections had become apparent in the understanding of the question. The relationship between perpendicular lines and the gradient was established, although the explanation may not be accurate. The only example coded to this level was: “they are perpendicular because the equations are reciprocal to each other and one equation has a negative”.

Table 5.38: SOLO coding statistics for identifying perpendicular lines (Question 8)

SOLO Coding	Number		
	Pre-test	End of Topic test	Delayed Post-test
Prestructural	10	0	0
U_1 (CS)	2	1	11
U_2 (CS)	0	8	7
M_2 (CS)	0	0	1
R_2 (CS)	0	1	0
Total	12	10	19

Question 9: Rearrange the equation $4x + 2y - 16 = 0$ such that y is the subject of the formula (that is y = ...)

The responses to this question proved challenging to code since only a final answer was provided by the vast majority of students despite being given appropriate paper to facilitate working, which was collected as data. Hence, it was difficult to understand the thinking involved with the final solution, which may have changed the type a response was coded to. Four types and four SOLO levels were identified in the coding of the responses for this question, summarised in Tables 5.39, 5.40 and 5.41.

Thematic Coding

Type A: This type of response reflected no understanding of what was required or was representative of a non-attempt. Typical responses included blank comments and "?".

In the Pre-test, only Team White submitted a response stating: "not sure", with the remaining responses left blank. Thus, all 12 pairs were coded with this type of response. For the End of Topic test, only Team Blue B, who combined with Team Purple A for this task, submitted a blank comment. In the Delayed Post-test, five blank comments were submitted.

Type B: This type of response represented an attempt to rearrange the equation, although elements of the manipulation were incorrect.

In the Pre-test, no student pairs submitted a Type B response. Two Type B responses were submitted in the End of Topic test. Team Maroon submitted: "$y = 6x - 6$", providing their working on scrap paper, as shown in Figure 5.2, and Team White B, who combined with Team Red A for this task, stated: "$y = -6x - 12$", incorrectly, removing the 2 from the y by subtracting it from the 12 rather than dividing it through. In the Delayed Post-test, ten incorrect attempts at rearranging the equation were submitted. A wide range of solutions were presented and inaccuracies occurred at various stages of the rearrangement process. Team Red B and Team White A both submitted the same solution of "$2y = 4x - 16$", which failed to correctly move terms to the opposite side of the equals sign, and did not remove the 2 from the y term. Team Lime A, Team Maroon A and Team Indigo A all submitted the same solution of, "$y = 4x - 16 + 2$", incorrectly adding the 2 instead of dividing by -2. The remaining teams submitted a Type B response although it was difficult to work out exactly where they went wrong. Team Orange B stated their solution as: "$y = 4x + 2y - 16 = 0$", Team Lemon A as: "$y = 16 + 2y + 4x$", Team Blue A as "$8x + 2y - 16 = 0$", Team White B as "$2y - 4x + 16$" and finally Team Yellow A stated "$y = 4x - 11$".

Type C: This type of response indicated a rearrangement with only one element incorrect. The majority of inaccuracies concerned operations involving directed numbers.

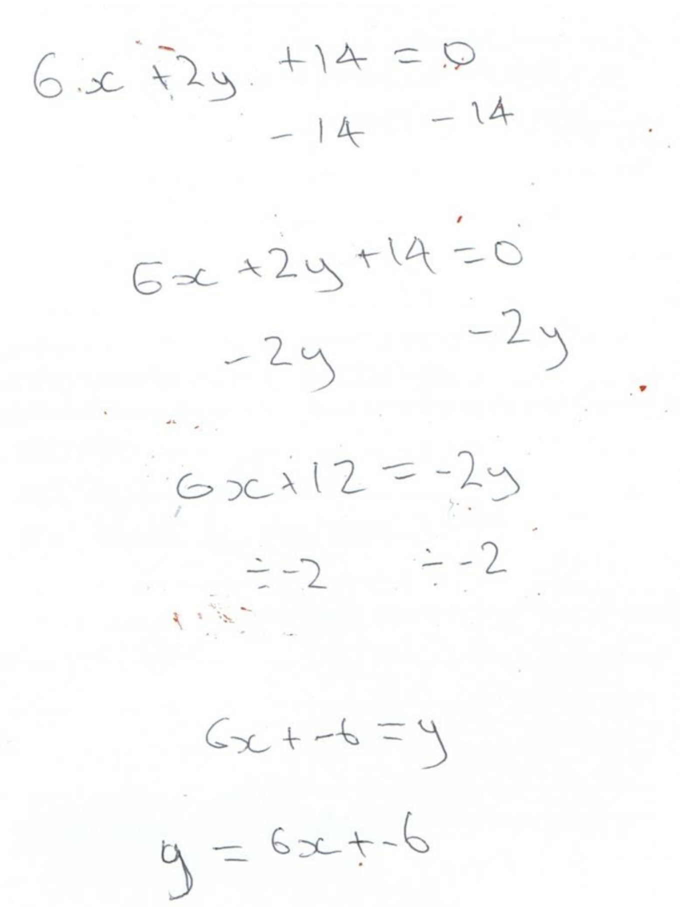


Figure 5.2: Type B response provided by Team Maroon for Question 9

In the Pre-test, no responses of this type were submitted. For the End of Topic test, four Type C responses were submitted. Team Orange and Team Indigo both submitted: "$y = 6x + 14$", indicating that, while the response stated: $y =$, the right-hand side of the equation, $6x + 14$, was not divided by -2. Team Lime submitted: "$y = -14 - 6x$" once again, demonstrating that the 2 was not divided to both sides of the equation. Team Yellow A, who combined with Team Blue A for this task, submitted "$y = 3x - 7$", which did not include the negative symbol required for the $3x$. In the Delayed Post-test, three responses of this type were submitted. Team Red A and Team Cream A both stated: "$y = 2x + 8$", which omitted the negative symbol required for $2x$. Team Indgio B stated: "$y = -4x + 16$", which removed the 2 from the left hand side without actually dividing it to both sides of the equation.

Type D: This type of response demonstrated correct rearrangement of the equation from general form to gradient-intercept form. For the Pre-test and Delayed Post-test, the solution was $y = 8 - 2x$, and for the End of Topic test, the solution was $y = -3x - 7$.

In the Pre-test, no Type D responses were submitted. For the End of Topic test, three correct responses were provided. Team Brown and Team Black stated: "$y = -3x - 7$", with Team Cream presenting their solution as: "$y = -6x - 14\ over\ 2$". This was still considered a Type D response, since it was possible that it could simplify to the correct solution (despite no brackets being used, which could have been an oversight due to the difficulty when entering answers into the form) and was presented as: $y =$. In the Delayed Post-test, only Team Brown A submitted a Type D response with the correct solution of "$y = -2x + 8$".

Table 5.39 Response types for changing the subject of the formula (Question 9)

Response Type	Explanation	Examples
A	No understanding of what was required Non-attempt	"not sure" blank comment
B	Attempt to rearrange with more than one element incorrect	"$y = 6x - 6$" "$y = -6x - 12$" "$2y = 4x - 16$" "$y = 4x - 16 + 2$" "$y = 4x + 2y - 16 = 0$" "$y = 16 + 2y + 4x$" "$8x + 2y - 16 = 0$" "$2y - 4x + 16$" "$y = 4x - 11$"
C	Attempt in rearranging with only one element incorrect Generally issues with directed numbers	"$y = 6x + 14$" "$y = -14 - 6x$" "$y = 3x - 7$" "$y = 2x + 8$" "$y = -4x + 16$"
D	Correct rearrangement of equation from general form to gradient-intercept form	"$y = -3x - 7$" "$y = -6x - 14\ over\ 2$" "$y = -2x + 8$"

Table 5.40: Thematic coding statistics for changing the subject of the formula (Question 9)

Response Type	A	B	C	D	Total
Pre-test	12 (100)	0 (0)	0 (0)	0 (0)	12 pairs (100)
End of Topic test	1 (10)	2 (20)	4 (40)	3 (30)	10 pairs (100)
Delayed Post-test	5 (26)	10 (53)	3 (16)	1 (5)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Prestructural: Responses were below those expected for the question. Examples coded to this level were a blank response or “not sure”.

The following levels were identified in the CS mode:

Unistructural (U_2): Responses indicated an attempt to start the problem in a mathematical way focussing on performing only one operation towards the manipulation of the equation. Examples coded to this level included: “$2y = 4x - 16$”.

Multistructural (M_2): Responses focussed on more than one operation attempted sequentially. These responses were usually incorrect. Examples coded to this level included: “$y = -14 - 6x$”, with the student pair removing the coefficient of y without performing the operation to both sides, “$y = 3x - 7$”, with the student pair performing all necessary numerical calculations by dividing by 2 rather than -2 in the final operation.

Relational (R_2): Responses indicated a number of connections had become apparent with all necessary operations performed sequentially and accurately to provide a correct response. Examples coded to this level included: “$y = -3x - 7$” for the End of Topic test, and “$y = -2x + 8$” in the Delayed Post-test.

Table 5.41: SOLO coding statistics for changing the subject of a formula (Question 9)

SOLO Coding	Number		
	Pre-test	End of Topic test	Delayed Post-test
Prestructural	12	1	5
U_2 (CS)	0	0	7
M_2 (CS)	0	6	6
R_2 (CS)	0	3	1
Total	12	10	19

Question 10: Which of these graphs do you think would represent a straight line?

- ☐ $y = -2x + 7$
- ☐ $y = -8$
- ☐ $9x(x - 4) = y$
- ☐ $3x + 5y - 6 = 0$
- ☐ $4(y - 1) = 2x^2$

This question was presented as a multiple selection question, where students were able to click on one, all or none of the equations to denote that which were identified as straight lines. It was difficult to ascertain whether students used their own intuition, guessed or specifically chosen their selections for their response. Hence, this question must be viewed alongside the next, Question 10a, which asks for further explanation regarding the selections. Four types of responses were identified for the thematic coding of the responses for this question, summarised in Table 5.42 and Table 5.43.

Thematic Coding

Type A: This indicated a non-attempt with typical responses being an all or nothing approach; that is, no equation marked or all equations marked as straight lines.

In the Pre-test, all responses were blank and, hence, coded as Type A. For the End of Topic test, no Type A responses were recorded. In the Delayed Post-test, three responses of this type were coded. Of these, Team Lime B selected an incorrect equation with $9x(x-4) = 0$, as the only solution, and the remaining two, Team Orange A and Team Lemon A, selected all equations.

Type B: This type of response indicated that only one of the equations was correctly recognised as representing a straight line. It is possible that these students thought that there was only one solution to this question, and the chosen equation represented the most recognisable equation of a straight line.

In the End of Topic test, eight Type B responses were recorded. Seven of these chose $y = -8$ as the single example, which represented a straight line with Team Blue A, who combined with Team Purple A for this task, choosing $3x + 5y - 6 = 0$ as their representation of a straight line. In the Delayed Post-test, 13 selected only one equation as their response. Of these, ten selected $y = -8$ as the equation of straight line with Team Blue B selecting $3x + 5y - 6 = 0$, and Team Indigo A and Team Maroon A, both selecting $y = -2x + 7$.

Type C: This type of response selected two equations that were correctly recognised as representing straight lines. It demonstrated a developing understanding of the properties of the equation of a straight-line.

In the End of Topic test, only Team Brown chose two equations which represented straight lines, choosing $y = -2x + 7$ and $3x + 5y - 6 = 0$. In the Delayed Post-test, three Type

C responses were recorded. Team Indigo B and Team Black A both selected $y = -2x + 7$ and $y = -8$, while Team Brown A selected $y = -2x + 7$ and $3x + 5y - 6 = 0$.

Type D: This type of response indicated that all three of the equations that were straight lines were correctly recognised. This demonstrated an understanding of the properties present in an equation of a straight line.

For the End of Topic test, only Team Maroon correctly identified all three straight lines, namely, $y = -2x + 7$, $y = -8$ and $3x + 5y - 6 = 0$. In the Delayed Post-test, no Type D responses were recorded.

Table 5.42: Response types for recognising equations representing a straight line (Question 10)

Response Type	Explanation	Examples
A	Guess with all equations selected as straight lines No understanding or conscious thought regarding properties of a straight lines Non-attempt	one incorrect equation selected all equations selected none selected
B	Only one equation recognised as being a straight line	$y = -8$ $3x + 5y - 6 = 0$ $y = -2x + 7$
C	Two equations recognised as being straight lines	$y = -2x + 7$ and $3x + 5y - 6 = 0$ $y = -2x + 7$ and $y = -8$ $y = -2x + 7$ and $3x + 5y - 6 = 0$
D	All straight lines correctly recognised	$y = -2x + 7$, $y = -8$ and $3x + 5y - 6 = 0$

Table 5.43: Thematic coding statistics for recognising equations representing a straight line (Question 10)

Response Type	A	B	C	D	Total
Pre-test	12 (100)	0 (0)	0 (0)	0 (0)	12 pairs (100)
End of Topic test	0 (0)	8 (80)	1 (10)	1 (10)	10 pairs (100)
Delayed Post-test	3 (16)	13 (68)	3 (16)	0 (0)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Responses to Question 10 do not provide enough detail about the quality of the student response to attempt a fine-grained SOLO coding. Those responses classified as Type A could be broadly coded as Prestructural with Type B, C and D responses coded as Concrete Symbolic cycle 1 or early cycle 2. It is not possible to distinguish further as the question itself did not ask for more information on how to distinguish the equation of a straight line. In the light of these results, the responses provided to Question 10a do offer a window to view the students understanding of the how they chose their selections for the equation of a straight line.

Question 10a: Why do you think they would be straight lines?

This question enabled students to provide some justification for their selections in the previous part of the question. It provided mixed results and again proved difficult to code. Three types and six SOLO levels of responses were noted, summarised in Tables 5.44, 5.45 and 5.46.

Thematic Coding

Type A: This type of response indicated a non-attempt or the reason had no foundation or link with understanding the correct presentation of an equation of a straight line.

In the Pre-test, all responses were blank and hence coded as Type A. For the End of Topic test, six Type A responses were recorded. Explanations varied and demonstrated no real understanding of the properties that define an equation of a straight line. For those who previously stated $y = -8$ as the only straight line, the following responses were supplied as their reasoning: Team Lime stated: "the y intercept is 0"; Team Black stated: "there's no x coordinate"; Team Indigo: "as it doesn't have a y intercept"; Team Red A, who combined with Team White B for this task, stated: "because there is no x value"; and Team Yellow A, who combined with Team Blue A, stated: "because it has no x intercept". Team Maroon, who previously chose three correct straight lines now stated their reason as: "because the y value is in its proper place". In the Delayed Post-test, 13 Type A responses were recorded. Team Indigo A, who selected $y = -2x + 7$ as a straight line, and Team Indigo B, who selected two straight lines previously, both had blank comments. Team Orange A, who previously selected all the lines as being straight, justified this using one word, "coz". Team Lemon A, who also selected all the lines to be straight previously, stated that: "all these lines are straight because lines can go in any direction and be straight". Team

Maroon A stated his logic for finding the straight lines as: “first I ruled out the ones without a y at the front or back (last two) then I guessed that the equation which I am most familiar with in straight lines”, after selecting $y = -2x + 7$ previously. The remaining students all selected $y = -8$ as the only line that was straight and their reasons ranged from: Team Black A, who stated: “cause they cross”; Team Yellow A who stated: “because it has no x intercept”; Team Cream A, who stated: “Because their y co ordinate is zero” [sic]; Team Red B, who stated: “because it doesn’t have a y intercept” [sic]; Team Red A, who stated “because you aren't changing any factor”; Team Orange B, who stated: “because it only crosses the y axis”; Team Blue A, who stated: “because they do not have an x-axis so they all sit on the y interval in a striaght line” [sic]; and, finally, Team Black B, who stated: “because there is no x axis”.

Type B: This type of response indicated a correct statement based on the equation chosen previously. However, the statement provided didn’t demonstrate understanding towards correctly defining the equation of a straight line.

For the End of Topic test, three Type B responses were coded. Team Orange stated: “because the slope = 0” and Team Cream offered a similar reason stating: “Because it hasn’t got a gradient”, since both these student pairs submitted the equation $y = -8$ as their solution for the previous question. Team Blue A, who combined with Team Purple A for this task, submitted: “BECAUSE IT EQUALS 0” [sic], after selecting $3x + 5y - 6 = 0$ for the previous question. These demonstrate that none of the submitted responses would correctly identify an equation of a straight line but the statements provided were accurate with respect to the previous selection and indicate that some thought was made to try and determine the link between the equation and a straight line. In the Delayed Post-test, five Type B responses were submitted. Team Blue B stated, “because the equation equals 0” as their selection previously was $3x + 5y - 6 = 0$. Interestingly, both Team Lime A and Team Lime B stated that GeoGebra assisted them; however, both only selected one equation as the solution, with Team Lime B selecting $9x(x - 4) = y$, which wasn’t even a correct solution. Team White A and White B both made comments regarding no slope or gradient as both had previously selected $y = -8$.

Type C: This type of response provided a statement that could be used to distinguish between the equations to assist with selecting straight line graphs.

For the End of Topic test, only Team Brown presented this type of response when they stated: "They would be straight lines because they have no brackets or squares. A straight line is when point lie on the same line called the collinear." [sic] The first sentence demonstrated an understanding that a straight line must be a direct relationship. Although the second sentence does not relate to the question and is incorrect, this was still considered a Type C response because it would clearly sort the graphs listed. Interestingly, Team Brown A provided the only Type C response in the Delayed Post-test providing a more accurate response of: "They would be straight lines because it is in a form of $y = mx + b$" given with the previous selection of "$y = -2x + 7$, $3x + 5y - 6 = 0$".

Table 5.44: Response types for selecting straight lines (Question 10a)

Response Type	Explanation	Examples
A	Reason has no foundation or link with understanding correct presentation of straight line Non-attempt	"the y intercept is 0" "because the y value is in its proper place" "there's no x coordinate" "as it doesn't have a y intercept" "because there is no x value" "because it has no x intercept" "coz" "first I ruled out the ones without a y at the front or back (last two) then I guessed that the equation which I am most familiar with in straight lines" "cause they cross" "because it has no x intercept" "because their y co ordinate is zero" "because it doesn't have a y intercept" "because you aren't changing any factor" "because it only crosses the y axis" "all these lines are straight because lines can go in any direction and be straight" "because they do not have an x-axis so they all sit on the y interval in a striaght line" "because there is no x axis"
B	Simple correct statement made based on equation selected previously No understanding demonstrated of properties of a straight line in statement Would not identify straight line	"because the slope =0" "because it hasn't got a gradient" "BECAUSE IT EQUALS 0" "because the equation equals 0"

Response Type	Explanation	Examples
C	Statement provided correctly distinguishes features of equations that would indicate straight lines graphs	“they would be straight lines because they have no brackets or squares A straight line is when point lie on the same line called the collinear” “they would be straight lines because it is in a form of $y = mx + b$”

Table 5.45: Thematic coding statistics for selecting straight lines (Question 10a)

Response Type	A	B	C	Total
Pre-test	12 (100)	0 (0)	0 (0)	12 pairs (100)
End of Topic test	6 (60)	3 (30)	1 (10)	10 pairs (100)
Delayed Post-test	13 (68)	5 (26)	1 (5)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Prestructural: Responses were below those expected for the question. Examples coded to this level are a blank response.

The following cycle of levels were identified in the CS mode:

Unistructural (U_1): Responses indicated uncertainty of what was being asked and used visual cues to assist in determining the equations, with focus on one particular aspect to identify the lines to be straight. These responses were incorrect. Examples coded to this level included the combination of selecting all the equations as straight lines and stating “all these lines are straight because lines can go in any direction and be straight”.

Relational (R_1): Responses indicated an educated guess, taking into account the data available. These responses gave an equation with little to no justification (Question 10a) as to how it was selected. Examples coded to this level included the combination of “$y = -2x + 7.\ Y = -8$” with a blank comment, the combination of “$y = -2x + 7$” with “first I ruled out the ones without a y at the front or back (last two) then I guessed that the equation which I am most familiar with in straight lines”.

Unistructural (U_2): Responses focussed on one aspect to identify the equation as a straight line, with more mathematical language used, these were the largest group of responses identified. Examples coded to this level included the combination of “$y = -8$” with “because it only crosses the y axis”, “$y = -8$” with “because no gradient”, “$y = -2x + 7$,

$y = -8, 3x + 5y - 6 = 0$" with "because the y value is in its proper place" and "$3x + 5y - 6 = 0$" with "because it equals 0".

Multistructural (M_2): Responses focussed on more than one aspect to identify the equation as a straight line. The responses provided were not always correct. Examples coded to this level included the combination of "$y = -2x + 7$, $3x + 5y - 6 = 0$" with "they would be straight lines because they have no brackets or squares. A straight line is when point lie on the same line called collinear".

Relational (R_2): Responses indicated a number of connections had become apparent in the understanding of the question and students were able to articulate why an equation may be considered a straight line. The only example coded to this level included the combination of "$y = -2x + 7$, $3x + 5y - 6 = 0$" with "they would be straight lines because it is in a form of $y = mx + b$".

Table 5.46: SOLO coding statistics for identifying perpendicular lines (Question 10 & 10a)

SOLO Coding	Number		
	Pre-test	End of Topic test	Delayed Post-test
Prestructural	12	0	1
U_1 (CS)	0	0	1
R_1 (CS)	0	0	4
U_2 (CS)	0	9	12
M_2 (CS)	0	1	0
R_2 (CS)	0	0	1
Total	12	10	19

5.4. Conclusion

In summary, this chapter reported the responses provided by students for the closed core content questions of the three Google Form Tests. Each question was examined and student responses coded for further analysis using both thematic coding and the SOLO model. Comprehensive explanations of the coding categories for each question were included to provide clarity for the reader. The findings of this chapter reveal that a range of response types were provided by the students throughout the Linear Relationships unit, demonstrating varying degrees in their levels of learning.

In particular, the three research questions posed at the beginning of the chapter were addressed formally.

Research Question 1.1. states*:*
How does the SOLO model offer a framework to explain the categories of responses concerning students' understandings of Linear Relationships?

The SOLO model when considered from a multiple cycle perspective, provided a structure to describe the students' understanding of the particular core content addressed by a question. It also offers a deeper interpretation to investigate the characterisations of developmental growth. A number of levels were identified, within the Concrete Symbolic (CS) mode, owing to the range of responses recorded. Biggs and Collis (1989) identified that the CS mode was the main mode that secondary students operate within as it involved using a system of symbols. A detailed summary of the response groupings is provided in Table 5.47 offering an overview of the developmental pathway with respect to Linear Relationships.

Table 5.47: The SOLO model and overview of developmental pathway for Linear Relationships

SOLO modes and levels	Description
U_1(CS)	Provides facts unrelated to question Demonstrates no understanding of question requirements Single feature based on a visual cue on GeoGebra or graph
M_1(CS)	Focusses on more than one feature without any connection between features Explanations to calculate distance, midpoint or gradient involve disconnected statements regarding the use of a measuring tool, or incorrect formula Suggestion of using a specific formula, such as, the distance, midpoint or gradient formula without any explanation Distinct statements based on visual cues from graphs or GeoGebra
R_1(CS)	Educated guess taking into account visual data available on graph or GeoGebra usually incorrect response A description of the method required for finding a gradient from a visual perspective using measuring tools Attempts to link features and establish relationships
U_2(CS)	Less reliance on visual cues with descriptions involving only one operation Explanations or justifications using only one property with more accurate terminology A single property nominated as justification for a geometrical shape drawn on graphs, or why an equation represents a straight line

SOLO modes and levels	Description
M_2(CS)	More than one unrelated property used to identify straight lines or describe drawing a straight line Description of operations or instructions provided sequentially to calculate a concept such as distance, midpoint or gradient Disconnected statements regarding justification of method Correctly identifying equations as lines using unrelated aspects as justification Stating a formula or mathematical concept with incorrect elements
R_2(CS)	Relationships are established based on similar properties not relying solely on visual cues All necessary operations performed sequentially and correctly Specifying or describing the correct formula for calculating midpoint, gradient, distance Justifying why an equation represents a straight line, or lines are perpendicular using terminology and/or algebra formulas
Formal	Clear overview of what the question requires Steps in working may be skipped Formulas used required manipulations (including when using GeoGebra)

Research Question 2.1 states:

Can an analysis of the results offer insights into students' understandings of Linear Relationships?

Through using the SOLO categorisations, the developmental pathway of students understanding of Linear Relationships can be monitored and used by teachers as a tool to assist them in preparing lessons. While these students were mainly operating in the concrete symbolic mode, evident from their use of a system of symbols, recognising whether they were on the first or second cycle of U-M-R can provide valuable information.

Movement between the two U-M-R cycles within the concrete symbolic (CS) mode can be mapped to the student's ability to use GeoGebra as a tool for investigating problems. Through the investigation process, they are provided with the opportunity to increase their mathematical understanding. Responses coded within the first cycle of CS mode indicated that there was reliance on the visual features of GeoGebra. These responses indicated various levels of support from the ikonic mode, where language is determined primarily through images. The majority of first cycle CS mode responses were evident in the Pre-test (PT), indicating the limited ability using GeoGebra and the Linear Relationships concepts. With responses making reference to familiar mathematical ideas, such as hands on measuring tools, or simple visual features to calculate and justify concepts, rather than exploring the potential of GeoGebra.

The second U-M-R cycle of the CS mode indicated that GeoGebra was used as a tool to support the understanding of concepts rather than as a direct tool to provide an answer. This was evident through the complexity of the responses provided; namely, the choice of terminology and articulation of ideas associated with Linear Relationships. The responses contained less reliance on visual triggers, using algebra and formulas rather than explanations. Results indicated that once the second U-M-R cycle of the CS mode was achieved, it was generally maintained.

A number of responses were recorded as returning to the first cycle CS U-M-R mode after the teaching sequence. This regression would indicate that the students were not able to establish relationships between concepts of Linear Relationships and were operating with more visual signals in order to assist them in solving problems. It indicates a limited ability to use GeoGebra as a supportive tool, rather using it as a tool to provide an answer.

The prestructural responses offer an interesting insight into the developmental progression of the students over the teaching sequence. After the teaching sequence, there were a small number of prestructural responses still evident. These responses indicated that these students may have been following procedures without understanding the underlying mathematical concepts. These procedures were stored in short term memory and not retained.

Research Question 2.2 stated:

Which response categories within the tests had a relatively larger increase in complexity from the prior response category, and how does this increase reflect upon students' growth in understanding Linear Relationships?

Movement between the SOLO levels was not uniform, indicating a developmental hurdle. The transition from U to M was observed to be easier than the transition from M to R. The shift from M to R requiring links to become evident as relationships to the students, whether it be visually through the use of GeoGebra, as would be the case of a R_1 response, or through correctly articulating or performing operations to produce a solution to the question, as would be a R_2 response; making connections between properties or individual aspects of the question being the turning point for students understanding of the Linear Relationships concept. This was an integral part of the Exploration and Explicitation phases of the lesson structure when sequenced with the van Hiele Teaching Phases. The spiralling between the Exploration, Teaching Phase 2, and Explicitation, Teaching Phase 3, consolidating the student's ability to make the necessary connections towards solving a problem. Activities associated with these phases used GeoGebra as an integral, supportive and investigative tool, fostering students ability to explore and experiment to develop their understanding of Linear Relationships concepts.

CHAPTER 6: PROBLEM-SOLVING RESPONSES FOR LINEAR RELATIONSHIPS

6.1. Introduction

This chapter continues to report on responses provided for all three tests, concentrating on the problem-solving investigations component that was provided as an extended response. In particular, the focus of the results is on two research themes.

Research Theme 1

To explore the SOLO model and van Hiele Teaching Phases as frameworks to assist teachers when using technology as a teaching tool.

1.2 How does the SOLO model offer a framework to explain the categories of responses concerning students' understandings of Linear Relationships?

Research Theme 2

To examine the responses of the Google Form tests in order to gain insight into students' understandings of Linear Relationships.

2.1 Can an analysis of the results offer insights into students' understandings of Linear Relationships?

For clarity, the chapter is divided into two sections, Background, which details the preliminary information, and Response Results, presented in the same manner as in the previous chapter, listing each question then separately addressing the thematic coding and SOLO model coding of the responses. The thematic coding provides an in-depth description of the responses summarised in two tables. The first table synthesises the response categories with examples of each, and the second table provides statistics concerning the number and percentage of students that submitted each type of response. For the SOLO model coding, an in-depth description of the responses is coded into the various levels and a table follows showing the number of responses coded to each level. All responses have been quoted directly from student work as submitted on the Google Form. Any irregularities with subscript notation presented in this chapter, resulted from student difficulties when entering formulas onto the Google Form.

6.2. Background

This section provides a brief overview of the methodology for the extended response questions. There were two types of problem-solving questions. For the End of Topic test, an extra question, Question 11, was inserted into the Google Form test. The remaining problem-solving questions were provided on paper to enable students to demonstrate the strategies and techniques used for solving Linear Relationships extended response questions. Also, it offered the researcher an opportunity to pose questions based on a line segment of a graph, rather than linking individual points.

For the Pre-test and Delayed Post-test, the extended response sheet used was identical and is provided in Appendix K. The first question was a straightforward question enabling students to demonstrate their strategies for finding the length of the line segment. The following question required deeper understanding of concepts to find the equation of the perpendicular bisector of the interval drawn on the graph. The last question had four parts involving the previous answer to form a quadrilateral from the line segment already provided on the graph. Students were required to name the quadrilateral formed, describe ways they could prove that it was the quadrilateral they named, demonstrate the proof and, finally, label the missing vertices of the quadrilateral.

For the End of Topic test, there were two types of extended response questions. The first was an extra question that was added to the Google Form, as previously mentioned. Question 11 consisted of five parts, all based on the graph embedded into the Google Form, shown in Figure 6.1. This question can also be seen in Appendix L. The second was an extended response problem-solving sheet similar to the Pre-test and Delayed Post-test sheet modified slightly to discourage memorisation of results. For this sheet, the first question again was a straight-forward question enabling students to demonstrate their strategies for finding the length of the line segment. The following question required students to find the gradient of the line segment. The third question required students to use the line segment and a given coordinate to prove it formed a right-angled triangle. And, finally, the last question had four parts involving properties of a quadrilateral and the results formed in the previous question. Students were asked to name the quadrilateral formed given specific conditions, describe ways they could prove that it was the quadrilateral they named, then demonstrate the proof and label the missing vertex of the quadrilateral.

6.3. Response Results

6.3.1. Extension Question on Google Form

In the End of Topic test, an extra question, Question 11, was added to the Google Form test. It had five parts associated with it and all were based on the graph shown in Figure 6.1, which was embedded into the Google Form.

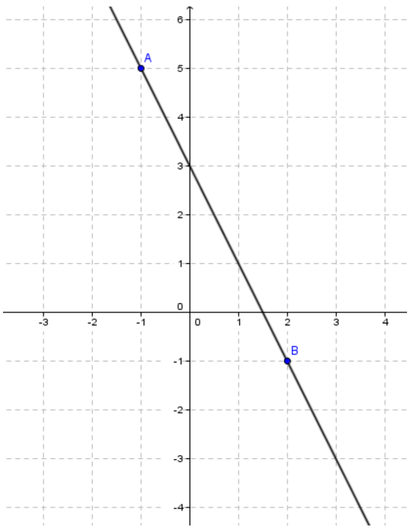


Figure 6.1: Graph on Google Form Question 11

Question 11a: What are the coordinates of point A and B?

This was a simple question that provided a suitable introduction and awareness of the working domain for the written component of the task. Only two types and levels of responses were noted for both thematic and SOLO model coding, and there was a ceiling response, summarised in Tables 6.1, 6.2 and 6.3.

Thematic Coding

Type A: This type of response indicated that students could not correctly identify the points on the Cartesian plane or was representative of a non-attempt.

Only Team Cream submitted this type of response, leaving their comment blank.

Type B: This type of response represented correct presentation of both points as A(-1, 5) and B(2, -1).

Nine of the ten pairs correctly identified these points for the End of Topic test.

Table 6.1: Response types for finding the coordinates of A and B (Question 11a)

Response Type	Explanation	Examples
A	Unable to identify point on Cartesian plane Non-attempt	blank comment
B	Correct presentation of points	A(-1, 5) B(2, -1)

Table 6.2: Thematic coding statistics for finding the coordinates of A and B (Question 11a)

Response Type	A	B	Total
End of Topic test	1 (10)	9 (90)	10 pairs (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Prestructural: Responses indicated no understanding of the requirements of the question. The only response coded to this level was a blank response.

The following level was identified in the CS mode:

Relational (R_2): Responses indicated a relationship between x and y values in forming a coordinate pair, with all responses identifying the points correctly as A(-1, 5) and B(2, -1).

Table 6.3: SOLO coding statistics for finding the coordinates of A and B (Question 11a)

SOLO Coding	Number End of Topic test
Prestructural	1
R_2 (CS)	9
Total	10

Question 11b: Identify the y-intercept for the line

This was another simple question identifying a basic feature of a graph. Two types and three levels of responses were identified and there was a ceiling response, summarised in Tables 6.4, 6.5 and 6.6.

Thematic Coding

Type A: This type of response demonstrated no understanding of what the y-intercept identified or was representative of a non-attempt. Typical responses were an incorrect number or a blank comment.

Both Team Lime and Team Cream submitted this type of response. Team Lime suggested that the y-intercept was 1.5, which was, in fact, the x-intercept, and Team Cream left their comment blank.

Type B: This type of response indicated knowledge of how to identify the y-intercept on the graph, providing a correct response.

Eight responses were coded to this type. A typical response was made by Team Brown who stated: "$b = 3$".

Table 6.4: Response types for identifying the y-intercept for the line (Question 11b)

Response Type	Explanation	Examples
A	No understanding of y-intercept incorrect answer provided Non-attempt	1.5 blank comment
B	Correctly identifies y-intercept	"$b = 3$"

Table 6.5: Thematic coding statistics for identifying the y-intercept for the line (Question 11b)

Response Type	A	B	Total
End of Topic test	2 (20)	8 (80)	10 pairs (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Prestructural: Responses were below those expected for the question. The only response coded to this level was a blank response.

The following cycle of levels were identified in the CS mode:

Unistructural (U_1): Responses indicated a singular visual feature recognised from the graph. The only responses coded to this level include "1.5". The response, "1.5", represented the x-intercept in the graph provided.

Unistructural (U_2): Responses indicated focus on one feature, the y-intercept. Examples of the responses coded to this level include "$b = 3$".

Table 6.6: SOLO coding statistics for identifying the y-intercept for the line (Question 11b)

SOLO Coding	Number End of Topic test
Prestructural	1
U_1 (CS)	1
U_2 (CS)	8
Total	10

Question 11c: What would the equation of the line be?

This builds on the previous two questions. Obtaining the equation was achievable either by copying the line from the Google Form test into the GeoGebra environment and using the correct tool in GeoGebra or by manually calculating the gradient, having previously found the y-intercept, then substituting into the gradient-intercept formula, or by a combination of both. Interestingly, three types and two levels of responses were noted and there was a ceiling response, summarised in Tables 6.7, 6.8 and 6.9.

Thematic Coding

Type A: This type of response indicated an inability to provide an equation or was representative of a non-attempt. Typical responses were blank comments.

Team Indigo and Team Cream both submitted blank comments.

Type B: This type of response contained an equation with elements incorrect or not specific for the line in question.

Both Team Lime and Team Black submitted this type of response. Team Lime was not specific, providing the gradient-intercept form of an equation of a straight line, "$y = mx + b$", without relating it to the graph in question, while Team Black forgot to include the negative symbol for the slope stating: "$y = 4/2x + 3$".

Type C: This type of response provided a correct response for the equation of the line. Typical solutions included the gradient-intercept form of an equation of a line $y = -2x + 3$ and the general form of the equation of a line with $2x + y = 3$, which was default form for the GeoGebra environment.

Six Type C responses were recorded. Five of these presented their equation in gradient-intercept form, with Team Brown providing the most comprehensive response stating: "y =

$mx + b$ m = $gradient = rise\ over\ run$, so rise 2 run -1 after all the gradient should = -2 $b = 3$ $y = -2x + 3$", while Team Blue B, who combined with Team Purple A for this task, were the only pair to submit a response in general form.

Table 6.7: Response types for finding the equation of the line (Question 11c)

Response Type	Explanation	Examples
A	Unable to provide an equation Non-attempt	blank comment
B	Equation provided with some elements incorrect or missing	"$y = mx + b$" (not related to graph in question) "$y = 4/2x + 3$" (negative missing from gradient)
C	Correct presentation of equation of the line	"$y = mx + b$ m = gradient = rise over run , so rise 2 run -1 after all the gradient should = -2 $b = 3$ $y - 2x + 3$" "$y = -2x + 3$" "$2x + y = 3$"

Table 6.8: Thematic coding statistics for finding the equation of the line (Question 11c)

Response Type	A	B	C	Total
End of Topic test	2 (20)	2 (20)	6 (60)	10 pairs (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Prestructural: Responses indicated no understanding of the question. Examples coded to this level were a blank response.

The following levels were identified in the CS mode:

Multistructural (M_2): The response indicated that more than one property of the line was evident using mathematical terminology. The example coded to this level was the response "$y = mx + b$", presenting the form of the gradient intercept formula without being able to link it to the line in question.

Relational (R_2): Responses indicated an understanding of what was required using algebraic notation. While responses may have been incorrect the relationships between the gradient, y-intercept and equation of a line were clearly evident. Examples of the responses coded to this level include points "$y = -2x + 3$" and "$2x + y = 3$".

Table 6.9: SOLO coding statistics for finding the equation of the line (Question 11c)

SOLO Coding	Number End of Topic test
Prestructural	2
M_2 (CS)	1
R_2 (CS)	7
Total	10

Question 11d: Give the equation of the line that is perpendicular to this line but passes through the same *y*-intercept.

This question continues to extend the basic concepts of Linear Relationships, with a correct solution requiring understanding the gradient of a perpendicular, the *y*-intercept (obtained from Question 11b) and forming the equation of the line from this information. While GeoGebra has capabilities of finding the equation of a line and drawing a line perpendicular to any given line, it was not explicitly demonstrated during the learning sequence. Four types of responses, along with three SOLO levels were identified and there was a ceiling response, summarised in Tables 6.10, 6.11 and 6.12.

Thematic Coding

Type A: This type of response indicated an incorrect solution, or was representative of a non-attempt. It demonstrated that no understanding was present regarding either the *y*-intercept, gradient or how to manipulate the gradient in order to calculate the perpendicular.

Five Type A responses were recorded, with both Team Indigo and Team Cream submitting blank comments. Team Blue B, who combined with Team Purple A for this task, stated: "$(-6,0)$ $(4,5)$", as their response, clearly not understanding that the question required an equation as a solution. Team Lime stated their equation as: "$y = 1.5x + 2$", where 1.5 was the answer they previously submitted as the *y*-intercept, demonstrating no understanding of the fact that the constant of the equation represents the *y*-intercept in gradient-intercept form.

Type B: This type of response indicated that students were able to correctly identify the *y*-intercept but unable to perform the necessary manipulation to calculate the gradient of the perpendicular.

Team Black were the only student pair who submitted this type of response with "$y = 3x + 3$". Team Red A, who combined with Team White B for this task, submitted: "$y = -4 + 3$", although the *x* was omitted from the equation the 3 was correct for the *y*-intercept.

Type C: This type of response indicated an understanding of perpendicular lines and how to calculate the gradient of perpendicular line; however, the *y*-intercept was incorrect.

Team Maroon and Team Yellow A, who combined with Team Blue A for this task, submitted this type of response stating: "$y = 0.5 + 3.01$" and "$y = 1/2x - 2$", respectively. Both were unable to correctly provide the *y*-intercept required to be coded as a Type D response.

Type D: This type of response provided a correct answer of $y = ½\, x + 3$. In the End of Topic test two student pairs, Team Orange and Team Brown both submitted this type of response.

Table 6.10: Response types for finding the equation of a perpendicular line passing through the same *y*-intercept (Question 11d)

Response Type	Explanation	Examples
A	Incorrect solution indicating no understanding regarding elements of creating a perpendicular line Non-attempt	blank comment "(−6,0) (4,5)" "$y = 1.5x + 2$"
B	Incorrect solution containing correct y-intercept but incorrect gradient	"$y = 3x + 3$" "$y = -4 + 3$"
C	Incorrect solution containing correct gradient but incorrect *y*-intercept	"$y = 0.5 + 3.01$" "$y = 1/2x - 2$"
D	Correct solution	"$y = ½\, x + 3$"

Table 6.11: Thematic coding statistics for finding the equation of a perpendicular line passing through the same *y*-intercept (Question 11d)

Response Type	A	B	C	D	Total
End of Topic test	4 (40)	2 (20)	2 (20)	2 (20)	10 pairs (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Prestructural: Responses demonstrated no understanding of the requirements of the question. Examples coded to this level were a blank response.

The following cycle of levels were identified in the CS mode:

Multistructural (M_2): Responses focussed on more than one isolated feature. The only M_2 response gave two coordinates, both of which would lie on the perpendicular line, $(-6,0)$ and $(4,5)$, however no attempt was made to establish how the points were related to the equation of the line.

Relational (R_2): Responses indicated an understanding of the requirements, and, although responses may be incorrect, the connections were evident. In each response, the gradient was changed reflecting the understanding that the perpendicular gradient was not equal to the gradient of the line given. Responses usually contained at least one correct element, either the y-intercept or gradient. Examples of the responses coded to this level include points "$y = 0.5x + 3.01$", "$y = 1/2x + 3$" and "$y = 3x + 3$".

Table 6.12: SOLO coding statistics for finding the equation of a perpendicular line passing through the same *y*-intercept (Question 11d)

SOLO Coding	Number End of Topic test
Prestructural	2
M_2 (CS)	1
R_2 (CS)	7
Total	10

Question 11e: Find the missing coordinate C of the triangle (show all working remembering it is a right-angled triangle).

The final part of this question asked students to find the missing point C on the graph such that points A, B and C would create a right-angled triangle. They were asked to submit all working to ascertain the underlying justification for their final answer. Three types of responses, along with three SOLO levels were identified and there was a ceiling response, summarised in Tables 6.13, 6.14 and 6.15.

Thematic Coding

Type A: This type of response indicated an answer that was not a coordinate or was representative of a non-attempt.

Four Type A responses were recorded of these, Team Black, Team Indigo and Team Cream all provided blank comments. Team Orange submitted: "$c2 = a2 + b2$", connecting Pythagoras' theorem to question, obviously from the keywords "right-angled triangle", given in the question.

Type B: This type of response provided a coordinate that was not possible as a correct solution.

Two responses of this type were recorded. Team Blue B, who combined with Team Purple A for this task, suggested: "$(-2, 2)$", and Team Yellow A, who combined with Team Blue A for this task, stated a detailed explanation that was not coherent submitting: "is $y - 1/2x - 2$, that cuts through the other sequence of lines, making it right-angle, so it is perpendicular, so if you connect the points to create the triangle, C would have to plot to (3,2) (2,4)". Both points stated were incorrect.

Type C: This type of response provided a correct coordinate that would successfully create a right-angled triangle with points A and B. It indicates that students thought carefully regarding the requirements of the question. Correct solutions included (-1, 1) or (2, 5) depending on the orientation of the right-angled triangle.

Four pairs submitted this type of response. Team Brown, Team Lime and Team Red A, who combined with Team White A for this task, all stated: (-1, 1), while Team Maroon stated: (2, 5).

Table 6.13: Response types for finding the missing coordinate C of the triangle (Question 11e)

Response Type	Explanation	Examples
A	Incorrect response not representative of a coordinate Non-attempt	blank comment "$c2 = a2 + b2$"
B	Coordinate pair provided not possible as a correct solution	"(-2, 2)" "is $y - 1/2x - 2$, that cuts through the other sequence of lines, making it right-angle, so it is perpendicular, so if you connect the points to create the triangle, C would have to plot to (3,2) (2,4)"
C	Coordinate pair provided would successfully create a right-angled triangle	(-1, 1) or (2, 5)

Table 6.14: Thematic coding statistics for finding the missing coordinate C of the triangle (Question 11e)

Response Type	A	B	C	Total
End of Topic test	4 (40)	2 (20)	4 (40)	10 pairs (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Prestructural: Responses were below those expected for the question. Examples coded to this level were a blank response.

The following levels were identified in the CS mode:

Multistructural (M_2): Responses focussed on more than one property related to the question, using mathematical terms within the explanation. The only response coded to this level was: "c2=a2+b2" [sic]. The response identifying Pythagoras' Theorem with the concept of right-angled triangle (stated in the question).

Relational (R_2): Responses may be incorrect but attempts to find multiple connections were evident, with each response providing a coordinate as a solution. Examples of the responses coded to this level include points: "C(-1,-1)" and "(2,5)" being correct responses; and "(-2,2)" and "is $y - 1/2x - 2$, that cuts through the other sequence of lines, making it right-angle, so it is perpendicular, so if you connect the points to create the triangle, C would have to plot to: (3,2) (2,4)", both incorrect but establishing relationships.

Table 6.15: SOLO coding statistics for finding the missing coordinateC of the triangle (Question 11e)

SOLO Coding	Number End of Topic test
Prestructural	3
M_2 (CS)	1
R_2 (CS)	6
Total	10

6.3.2. Extended Response Sheet – Pre-test and Delayed Post-test

All three tests had extended response sheets, with the Pre-test and Delayed Post-test both using the same sheet, provided in Appendix K. The extended response sheet was used to record pen and paper techniques to solve problems including open-ended investigations based on the graph shown in Figure 6.2.

Question a: Find the length of this line segment – show your working.

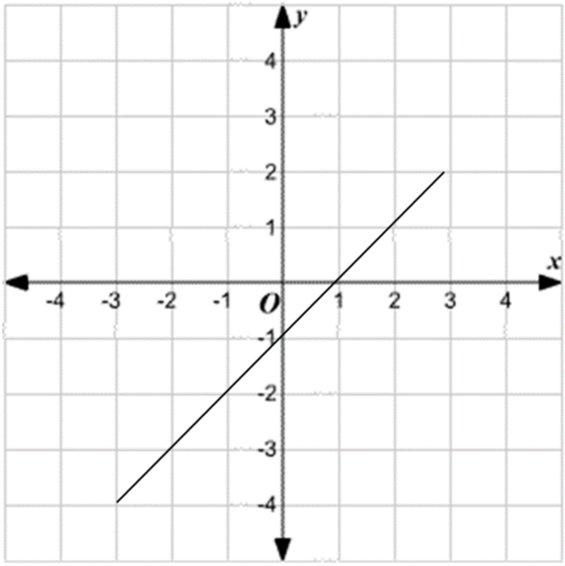


Figure 6.2: Graph on extended response sheet

For the Pre-test, no student pairs completed any part of the extended response sheet and, hence, all responses have been coded as a Type A response for thematic coding and prestructural level for SOLO coding. For this Question a six types and three SOLO levels were identified, summarised in Tables 6.16, 6.17 and 6.18.

Thematic Coding

Type A: This type of response indicated no recall on how to calculate the distance between two given points or was representative of a non-attempt. Typical responses were blank comments or answers that could not possibly represent the distance.

For the Pre-test, all twelve responses were coded as Type A, indicative of a non-attempt. For the Delayed Post-test, six Type A responses were recorded. While five of these, namely, Team Cream A, Team Orange B, Team Lime B, Team Indigo B and Team Blue B, presented a blank comment, Team Red A submitted a coordinate pair of $(0, -1)$ that could not be representative of the distance between two points.

Type B: This type of response made an attempt at solving the problem but the solution is incorrect. Typical responses were answers that could represent distance but are incorrect or showed working applying an incorrect formula.

In the Delayed Post-test, two Type B responses were recorded. Team Blue A presented a solution of 8.69 units without any working, making it difficult to understand where the errors occurred. Team Lime A submitted an incorrect formula that resembled the gradient formula, as shown in Figure 6.3.

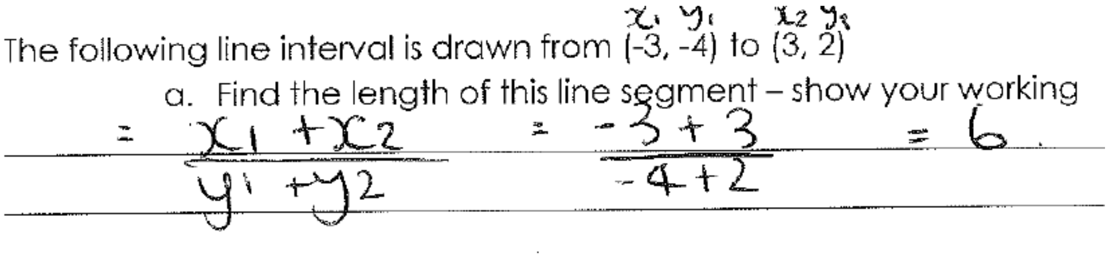


Figure 6.3: Type B response provided by Team Lime A for Question a

Type C: This type of response indicated recall of a method or formula capable of calculating the distance, without the ability to apply it. Typical responses presented the correct distance formula with incorrect substitution or solving strategy, resulting in an incorrect solution.

In the Delayed Post-test, two responses of this type were submitted. Team White B completed the substitution correctly then attempted to square root each number first before adding, as seen in Figure 6.4. Team Brown A was also coded a Type C response, despite recalling the correct distance formula. This coding was because the solution was presented as a coordinate, despite the coordinates being correctly labelled, as shown in Figure 6.5. As no working was shown, it is difficult to ascertain where errors occurred.

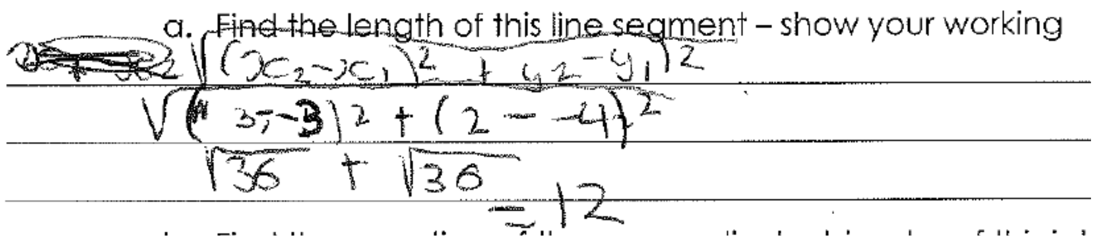
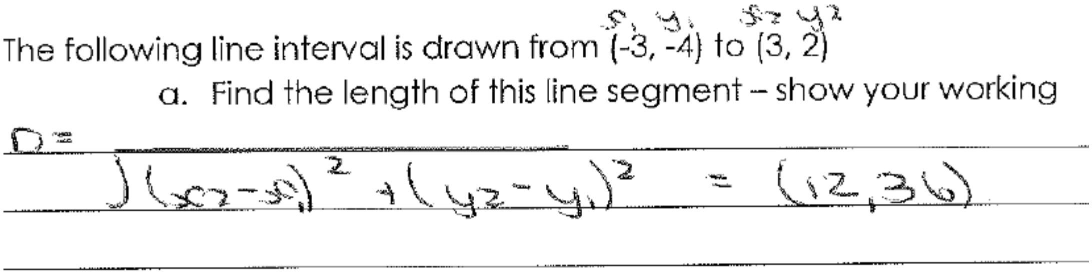


Figure 6.4: Type C response provided by Team White B for Question a



The following line interval is drawn from (-3, -4) to (3, 2)

a. Find the length of this line segment – show your working

D = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = (12, 36)$

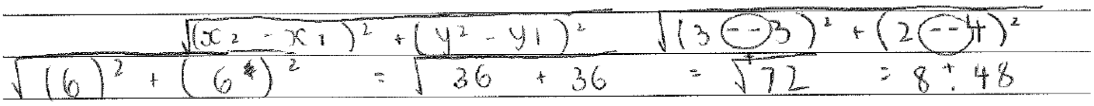
Figure 6.5: Type C response provided by Team Brown A for Question a

Type D: This type of response presented a correct response without providing any working, suggesting that the GeoGebra environment was used. Typical responses contain the distance as 8.49 and, in some instances, one word was offered to justify the answer.

In the Delayed Post-test, seven Type D responses were recorded. Team Red B, stated: "8.49 Geogebra" [sic], with two other students, Team Black B and Team Orange A, stating: "8.49 ruler" (by chance, and not purposely, it was found possible to achieve the correct result using a ruler). The remaining four students simply submitted their answer with no comments.

Type E: This type of response contained one element wrong in the solution process, resulting in an incorrect final answer.

For the Delayed Post-test, only Team White A submitted this type of response. The formula, substitution and subsequent calculation were accurately implemented; however, the final result was incorrectly rounded to 8.48 instead of 8.49, as shown in Figure 6.6.



$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $\sqrt{(3 - -3)^2 + (2 - -4)^2}$

$\sqrt{(6)^2 + (6)^2} = \sqrt{36 + 36} = \sqrt{72} = 8.48$

Figure 6.6: Type E response provided by Team White A for Question a

Type F: This type of response resulted in a correct solution, demonstrating knowledge, substitution and application of the formula to calculate the distance.

In the Delayed Post-test, only Team Yellow A submitted this type of response, substituting the correct coordinates into the formula and calculating the distance as 8.49.

Table 6.16: Response types for finding the length of line segment (Question a)

Response Type	Explanation	Examples
A	Solution provided could not be representative of a distance No understanding of how to calculate distance Non-attempt	blank comment $(0, -1)$
B	Attempt made at solving problem solution could be considered a distance Wrong formula used Incorrectly calculated	8.69 see Figure 6.3
C	Recall of how to calculate distance without correctly implementing it Correct distance formula with incorrect solution	see Figure 6.4 and 6.5
D	Correct solution of distance obtained without any working provided	8.49 "8.49 Geogebra" "8.49 ruler"
E	Working provided demonstrating ability to calculate distance using pen and paper but one element incorrect resulting in incorrect solution	see Figure 6.6
F	Correctly able to recall and calculate distance using formula to provide correct solution	

Table 6.17: Thematic coding statistics for finding the length of line segment (Question a)

Response Type	A	B	C	D	E	F	Total
Pre-test	12 (100)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	12 pairs (100)
Delayed Post-test	6 (32)	2 (11)	2 (11)	7 (37)	1 (5)	1 (5)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets**.**

SOLO Coding

Prestructural: Responses were below those expected for the question. Examples coded to this level were blank responses.

The following cycle of levels were identified in the CS mode:

Unistructural (U_1): Responses focused on one visual feature identifiable from the graph. The only example coded to this level included a response that provided $(0, -1)$, which was the *y*-intercept.

Relational (R_2): Responses indicated a number of patterns had become apparent in the understanding of the question. Responses were not always correct; however, relationships were evident. Examples coded to this level were the correct application of the distance formula to calculate distance, the correct distance formula with incorrect application and an incorrect formula applied correctly.

Table 6.18: SOLO coding statistics for the length of line segment (Question a)

SOLO Coding	Number	
	Pre-test	**Delayed Post-test**
Prestructural	12	5
U_1 (CS)	0	1
R_2 (CS)	0	13
Total	12	19

Question b: Find the equation of the perpendicular bisector of this interval

This question required a number of concepts to be connected to produce the desired result, including the midpoint, that would provide the bisecting point for the line to pass through; the gradient of the points given, to assist in calculating the perpendicular gradient: and combining the elements to form the equation of the line. Five types of responses and four SOLO levels were identified for the responses of this question, summarised in Tables 6.19, 6.20 and 6.21.

Thematic Coding

Type A: This type of response indicated no understanding of what was required or was representative of a non-attempt. Typical responses were blank comments.

All twelve responses were coded a Type A response for the Pre-test, indicative of a non-attempt. In the Delayed Post-test, ten blank responses were coded as Type A.

Type B: This type of response consisted of individual elements that would be used to find the equation of the perpendicular bisector required.

In the Delayed Post-test, Team Lime A and Team Lemon A both submitted this type of response. Team Lemon A submitted two coordinates: "$(-3, 2)$ and $(3, -4)$", consistent with being end points of a line segment that would produce the perpendicular bisector, (working demonstrates the perpendicular bisector was drawn in); while Team Lime A submitted the *y*-intercept: "$(0, -1)$", which was the midpoint of the segment in the question, representing only a very small part of the working required to solve the question.

Type C: This type of response provided an equation for the solution although no elements of the equation were correct for the perpendicular bisector required. In the Delayed Post-test, only Team Blue A submitted this type of result when stating: "$y = -2x + 4$".

Type D: This type of response builds on a Type C response, presenting an equation that has one element correct with respect to the requirements from the question. Typical responses provided an equation of a straight line that has either the gradient or y-intercept correct.

In the Delayed Post-test, three responses of this type were recorded. Team White B and Team Brown A both stated that the perpendicular bisector was "$y = 1x - 1$", correctly stating the y-intercept with the gradient missing a negative sign, restricting them from both achieving a Type E response. Team Black B stated the correct gradient: "$- x - y = -1$".

Type E: This type of response provided a correct solution, consisting of an equation of a straight line that would be the perpendicular bisector of the line presented in the graph, namely: "$y = -x - 1$". It was possible to have navigated GeoGebra to find this result, although no direct instruction was provided on this.

For the Delayed Post-test, Team Indigo A, Team Red A and Team White A all submitted this type of response.

Table 6.19: Response types for finding the equation of perpendicular bisector (Question b)

Response Type	Explanation	Examples
A	No understanding of what was required Non-attempt	blank comment
B	States elements that would be used to find the perpendicular bisector	$(0, -1)$ $(-3, 2)$ and $(3, -4)$
C	Knowledge of equation present but no elements are correct for perpendicular bisector	"$y = -2x + 4$"
D	Equation presented with one element correct for the perpendicular bisector	"$y = 1x - 1$" "$- x - y = -1$"
E	Correctly identifies perpendicular bisector	"$y = -x - 1$"

Table 6.20: Thematic coding statistics for finding the equation of perpendicular bisector (Question b)

Response Type	A	B	C	D	E	Total
Pre-test	12 (100)	0 (0)	0 (0)	0 (0)	0 (0)	12 pairs (100)
Delayed Post-test	10 (53)	2 (11)	1 (5)	3 (16)	3 (16)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Prestructural: Responses were below those expected for the question. Examples coded to this level included blank responses.

The following levels were identified in the CS mode:

Unistructural (U_2): Responses focussed on one property that could be used to find the perpendicular bisector but was not clearly evident from the graph. The only example coded to this level included a response that provided the y-intercept of the perpendicular bisector $(0, -1)$, which was the midpoint of the line segment provided on the graph.

Multistructural (M_2): Responses focussed on more than one property that could be used to find the perpendicular bisector but was not clearly evident from the graph. For this question, the only response coded to this level provided two coordinates, both of which would lie on the perpendicular bisector, $(-3,2)$ and $(3, -4)$; working demonstrated that the student drew the perpendicular bisector but was unable to present the findings as an equation.

Relational (R_2): Responses indicated that relationships had become apparent in the understanding of the requirements of the question while not always correct; all responses presented an equation. Examples coded to this level are: "$y = -2x + 4$", containing no correct elements; "$y = 1x - 1$" and "$-x - y = -1$", containing one correct element; and "$y = -x - 1$", the correct response for the question.

Table 6.21: SOLO coding statistics for the equation of perpendicular bisector (Question b)

SOLO Coding	Number	
	Pre-test	Delayed Post-test
Prestructural	12	10
U_2 (CS)	0	1
M_2 (CS)	0	1
R_2 (CS)	0	7
Total	12	19

Question c.i: If the interval drawn and its perpendicular bisectors are diagonals of a quadrilateral what could the quadrilateral be?

This question extends the previous one, requiring recall of the properties of quadrilaterals; although not part of the Linear Relationships unit, it provided an opportunity to demonstrate how Linear Relationships assists with problem solving. Three types of responses were identified, summarised in Table 6.22 and Table 6.23.

Thematic Coding

Type A: This type of response indicated no understanding of what was required or was representative of a non-attempt. Typical responses were a blank comment.

All twelve student pairs were coded a Type A response for the Pre-test, indicative of a non-attempt. In the Delayed Post-test, ten responses of this type were submitted, also indicative of a non-attempt.

Type B: This type of response indicated a limited understanding of what was required, although a shape or points were submitted as a response, they were incorrect.

For the Delayed Post-test, Team Red B, Team Brown A and Team Lime A submitted this type of response. Team Red B stated: "Rhombus", not recognising the angle as being a right-angle, which would have resulted in a Type C response; and Team Lime A stated a 3-dimensional object: "rectangular prism". Team Blue A stated: "a 4 squared figure".

Type C: This type of response indicated a correct response, being a square, or indicated, on the diagram, a square. Students were either confident in drawing the shape on their graph, allowing them to name it accordingly, or guessed the correct answer.

In the Delayed Post-test, six responses of this type were submitted, namely, Team White B, Team Red A, Team Indigo A, Team Lemon A and Team Maroon A. Team Brown A did not provide the name of a quadrilateral, instead presented the coordinates $(-3, 2)$, $(3, -4)$ as a solution, which provided coordinates for the quadrilateral required despite not naming it.

Table 6.22: Response types for determining the type of quadrilateral (Question c.i)

Response Type	Explanation	Examples
A	No understanding of what was required Non-attempt	blank comment
B	Demonstrate an understanding of what was required providing a solution that could be linked to the question Incorrect solution	"rhombus" "rectangular prism" "4 squared figure"
C	Correct quadrilateral identified	"square" $(-3, 2)$ $(3, -4)$ (identifies the missing vertices of square despite not naming it)

Table 6.23: Thematic coding statistics for determining the type of quadrilateral (Question c.i)

Response Type	A	B	C	Total
Pre-test	12 (100)	0 (0)	0 (0)	12 pairs (100)
Delayed Post-test	10 (53)	3 (16)	6 (32)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Responses for Question c.i do not provide enough detail about the quality of the student response to attempt a fine-grained SOLO coding. Those responses classified as Type A could be broadly coded as Prestructural with Type B and C responses coded as Concrete Symbolic; however, it is not possible to distinguish further as the question itself did not ask for more information on how they came to the conclusion of what quadrilateral the shape would be. In the light of these results, the responses provided to the following question, Question c.ii, offer a window to view the students' understanding towards how they made this decision. Hence, a SOLO coding is presented following the next question.

Question c.ii: Describe all the different ways you could prove that it's that type of quadrilateral.

This part of the question required students to think about properties of quadrilaterals and what properties distinguish and categorise the quadrilateral they stated in previous part of the question. Four types of responses and five SOLO levels were identified for this question, summarised in Tables 6.24, 6.25 and 6.26.

Thematic Coding

Type A: This type of response indicated no understanding of what was required or was representative of a non-attempt. Typical responses were a blank comment or comments that would not prove the quadrilateral previously stated.

All twelve student pairs were coded a Type A response for the Pre-test indicative of a non-attempt. In the Delayed Post-test, ten Type A responses were recorded with nine blank comments. Team Red A stated: "measuring the sides, ruler and geogebra" [sic], which does not coherently justify or prove anything.

Type B: This type of response indicated some understanding of what was required, the sole comment coded to this type referred to a single obvious feature, namely four sides, without further explanation. It is possible that the feature was derived from the term quadrilateral mentioned in the question. This type of response does not provide any details particular to the specific quadrilateral of the previous question, namely a square. In the Delayed Post-test, Team Red B, Team Black B and Team Maroon A all stated four sides as their response.

Type C: This type of response identified a single property of a square, but the answer would not be enough to prove a square.

In the Delayed Post-test, both Team White B and Team Indigo A stated this type of response with: "right-angled" and "all sides equal up", respectively. While both stated a square as the quadrilateral in the previous question, the single property alone was not sufficient as proof.

Type D: This type of response identified more than one property or feature, although descriptions required more detail to be considered as a proof.

In the Delayed Post-test, four Type D responses were recorded. Team Blue A stated: "four sides, parallel side, equal", failing to make connections between the three statements to properly define a square. Team Lime A stated: "4 sides, 2 even sides, can be evenly cut in half" and, although symmetry is implied, it is not accurately explained. Finally, Team Brown A stated: "they are equal, bisect each other and perpendicular", although it is not made clear what "they" represented. Team Lemon A provided the best Type D response stating: "equal sides and 90° angles on each angle", but failed to number how many equal sides or angles, either of which would have provided a higher response.

Table 6.24: Response types for describing strategies for quadrilateral proof (Question c.ii)

Response Type	Explanation	Examples
A	No understanding of what was required No properties given relate to a quadrilateral Non-attempt	blank comment "measuring the sides, ruler and geogebra"
B	A single property or feature is identified from a visual cue No thought about other features	"It would be a quadrilateral because it has 4 sides" "It has four sides" "4 sides"
C	A single property or feature identified that could be used as part of a proof	"right-angled" "all sides equal up"
D	More than one feature or property identified that could be used as part of a proof Missing elements details	"four sides, Parallel side, equal" "4 sides, 2 even sides, can be evenly cut in half" "they are equal, bisect each other and perpendicular" "equal sides and 90° angles on each angle"

Table 6.25: Thematic coding statistics for describing strategies for quadrilateral proof (Question c.ii)

Response Type	A	B	C	D	Total
Pre-test	12 (100)	0 (0)	0 (0)	0 (0)	12 pairs (100)
Delayed Post-test	10 (53)	3 (16)	2 (11)	4 (21)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Prestructural: Responses were below those expected. Examples coded to this level were a blank response.

The following cycle of levels were identified in the CS mode:

Unistructural (U_1): Responses indicated uncertainty of the specific requirements, focussing on one feature from some form of stimulus, either visual or from the question terminology. The only example coded to this level was a combination of a blank response for naming of the quadrilateral and with the description of how it was the quadrilateral simply stating: "four sides". It is possible that motivation for the response was taken from the question wording, namely, "quad" meaning "four".

Relational (R_1): Responses indicated an educational guess, taking into account all information available, including the visual aspect of the diagram. The only example coded to this level included the combination of "a rhombus?" with "it has four sides". The diagram provided by this student, clearly showed a four-sided figure; however, uncertainty in determining the type of a quadrilateral was evident.

Unistructural (U_2): Responses focus on one feature of the quadrilateral, namely a square, as justification. Examples coded to this level included: the combination of "square" for c.i, and "right-angled" for c.ii; "a square" for c.i, and "all sides equal up" for c.ii; and "a square" for c.i, and "it would be a quadrilateral because it has 4 sides" for c.ii.

Relational (R_2): Responses indicated a number of properties to justify the choice of answer stated in c.i. While the naming of quadrilateral may not be accurately articulated, the justification demonstrates that relationships are evident. Examples coded to this level included a combination of: "4 squared figure" with "4 sides, parallel sides, equal"; and, "rectangular prism" combined with "4 sides, 2 even sides 2 even sides, can be evenly cut in half" (a diagram was provided to demonstrate the 2 even sides explanation).

Table 6.26: SOLO coding statistics for describing strategies for quadrilateral proof (Question c.ii)

SOLO Coding	Number	
	Pre-test	Delayed Post-test
Prestructural	12	9
U_1 (CS)	0	1
R_1 (CS)	0	2
U_2 (CS)	0	3
R_2 (CS)	0	4
Total	12	19

Question c.iii: Use one of your strategies listed above to prove that it's that type of quadrilateral.

This question required students to justify their previous answer by providing methods and/or strategies employed to complete their proof. Four types of responses and four SOLO levels were identified, summarised in Tables 6.27, 6.28 and 6.29.

Thematic Coding

Type A: This type of response indicated no understanding of what was required or was representative of a non-attempt.

All twelve student pairs were coded a Type A response for the Pre-test, indicative of a non-attempt. In the Delayed Post-test, 13 Type A responses were submitted, which were blank comments.

Type B: This type of response indicated that some visual stimulus was involved to identify the type of quadrilateral. Explanations involved a single statement.

In the Delayed Post-test, two Type B responses were recorded. Type Red B stated: "by drawing the shape"; and Team Maroon A stated: "my strategies when working it out is that 2 know quad=4". Team Maroon A's response indicated that the terminology used in the question provided a valuable clue, this was also reflected in their previous answer of four sides.

Type C: This type of response attempted to state a strategy without detail.

In the Delayed Post-test, two responses of this type were submitted. Team Lime A stated: "can be evenly cut in half", implying the concept of symmetry without further explanation, basically repeating the previous answer. Team Blue A stated: “measure the top and bottom sides if they are different it’s not parallel so not quadrilateral”, indicating that parallel sides were considered the main feature, a similar response was provided for the previous question.

Type D: This type of response developed on a Type C response, stating a strategy involving a single property that could be used as part of a proof. It focussed on a single aspect but falls short of providing information to complete the strategy.

In the Delayed Post-test, two responses of this type were recorded. Team Lemon A stated: “all angles are 90°”, repeating his previous response, and Team Red A, identified that all the sides were equal, stating: “Point C to D = 6 units, Point B to C = 6 units, Point D to A = 6 units, Point B to A = 6 units”. This team provided a Type A response in the previous question when stating their proof strategy was to measure the sides.

Table 6.27: Response types for using strategies from question c.iii (Question c.iii)

Response Type	Explanation	Examples
A	No understanding of what was required No properties given relate to a quadrilateral Non attempt	blank comment “the perpendicular”
B	A single property or feature is identified from a visual cue No thought about other features	“by drawing the shape” (with picture drawn) “my strategies when working it out is that 2 know quad =4”
C	A single property or feature identified that requires more detail to provide a strategy that could be used to assist in a proof	“measure the top and bottom sides if they are different its not parallel so not quadrilateral” “can be evenly cut in half”
D	A single property or feature identified that could be used as part of a proof No link to other properties	“point C to D = 6 units, Point B to C = 6 units, Point D to A = 6 units, Point B to A = 6 units” “all angles are 90°”

Table 6.28: Thematic coding statistics for using strategies from question c.iii (Question c.iii)

Response Type	A	B	C	D	Total
Pre-test	12 (100)	0 (0)	0 (0)	0 (0)	12 pairs (100)
Delayed Post-test	13 (68)	2 (11)	2 (11)	2 (11)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Prestructural: Responses were below those expected for the question. Examples coded to this level were a blank response.

The following cycle of levels were identified in the CS mode:

Relational (R_1): Responses indicated an educated guess, taking into account all information available, focussing on a form of measurement as a strategy. Examples coded to this level included: "measure the top and bottom sides if they are different, its not parallel, so not quadrilateral", and "can be evenly cut in half".

Unistructural (U_2): Responses focussed on one feature as a strategy, not necessarily identified from a visual perspective. Examples coded to this level are: "all angles are 90°" and: "my strategies when working it out is that 2 know that quad=4".

Multistructural (M_2): Responses focussed on more than one isolated feature with no links to connect the information with the question. The only multistructural response provided was: "Point C to D = 6 units, Point B to C = 6 units, Point D to A = 6 units, Point B to A = 6 units".

Table 6.29: SOLO coding statistics for using strategies from question c.iii (Question c.iii)

SOLO Coding	**Number**	
	Pre-test	**Delayed Post-test**
Prestructural	12	12
R_1 (CS)	0	3
U_2 (CS)	0	3
M_2 (CS)	0	1
Total	12	19

Question c.iv: What are the coordinates of the missing vertices of the quadrilateral?

This question required students to label the missing coordinates of the quadrilateral previously suggested. Three types and two levels of responses were identified, summarised in Tables 6.30 and 6.31.

Type A: This type of response indicated no understanding of what was required or was representative of a non-attempt. Typical responses were a blank comment.

All twelve student pairs were coded a Type A response for the Pre-test, indicative of a non-attempt. In the Delayed Post-test, 12 blank comments were submitted.

Type B: This type of response indicated an attempt to identify a set of coordinates that would form the missing vertices of the quadrilateral; however, the coordinates submitted were not able to form the quadrilateral as described.

In the Delayed Post-test, Team Blue A, Team Red B and Team Lime A all submitted this type of response, claiming: "$(-4, 2)$ $(-3, -1)$", "$(2, 4)$ $(-3, 0)$" and "$(2, -3)$ $(5, 0)$", respectively. The point suggested by Team Blue A would form a rectangle, although he previously suggested the quadrilateral was a "4 squared figure". The point suggested by Team Red B, as shown in Figure 6.7, didn't form any regular shape; and the point suggested by Team Lime A, as shown in Figure 6.8, would create a rectangle but uses the midpoint as a vertex.

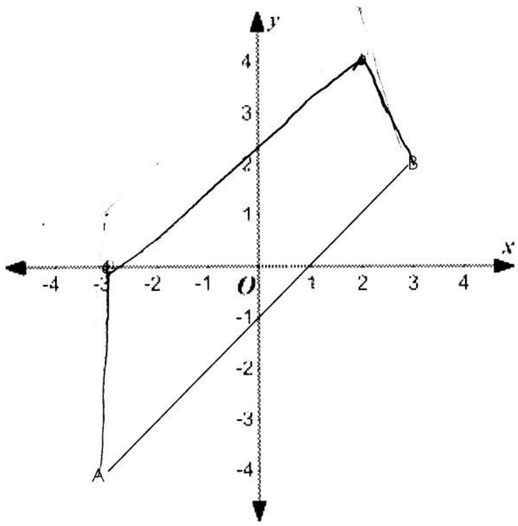


Figure 6.7: Type C response provided by Team Red B for Question c.iv

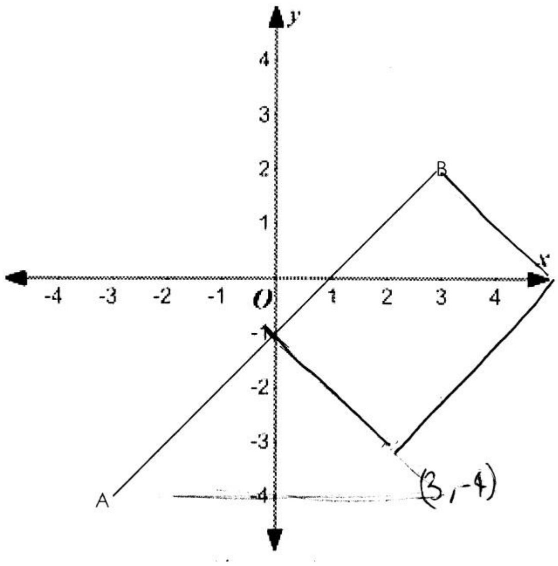


Figure 6.8: Type C response provided by Team Lime A for Question c.iv

Type C: This type of response indicated a correct solution of $(3,-4)$ and $(-3,2)$, which would form a square with the graph given.

In the Delayed Post-test, Team White B, Team Red A, Team Lemon A and Team Brown A all submitted the correct coordinates. Three of these, namely, Team White B, Team Red A and Team Lemon A, all previously stated that the quadrilateral was a square, however Team Maroon A was the student who, although failing to name the quadrilateral, submitted the coordinates which, incidentally formed a square.

Table 6.30: Response types for finding the coordinates of missing vertice of quadrilateral (Question c.iv)

Response Type	Explanation	Examples
A	No understanding of what the missing vertices were Non-attempt	blank comment
B	Coordinates submitted could not represent the quadrilateral as required	"$(-4,2)$ $(-3,-1)$" "$(2,4)$ $(-3,0)$" "$(2,-3)$ $(5,0)$"
C	Correct solution which would form a square given	$(3,-4)$ and $(-3,2)$

Table 6.31: Thematic coding statistics for finding the coordinates of missing vertice of quadrilateral (Question c.iv)

Response Type	A	B	C	Total
Pre-test	12 (100)	0 (0)	0 (0)	12 pairs (100)
Delayed Post-Test	12 (63)	3 (16)	4 (21)	19 students (100)

Percentages of the sample for each response type in each test are included in brackets.

6.3.3. Extended Response Question Sheet – End of Topic test

A separate extended response sheet was provided for the End of Topic test, requiring students to complete an open-ended task based on Linear Relationships concepts. The graph, shown as Figure 6.9, was provided and the questions associated with it are as follows. While students were still working in pairs, each student completed their own Extended Response Sheet.

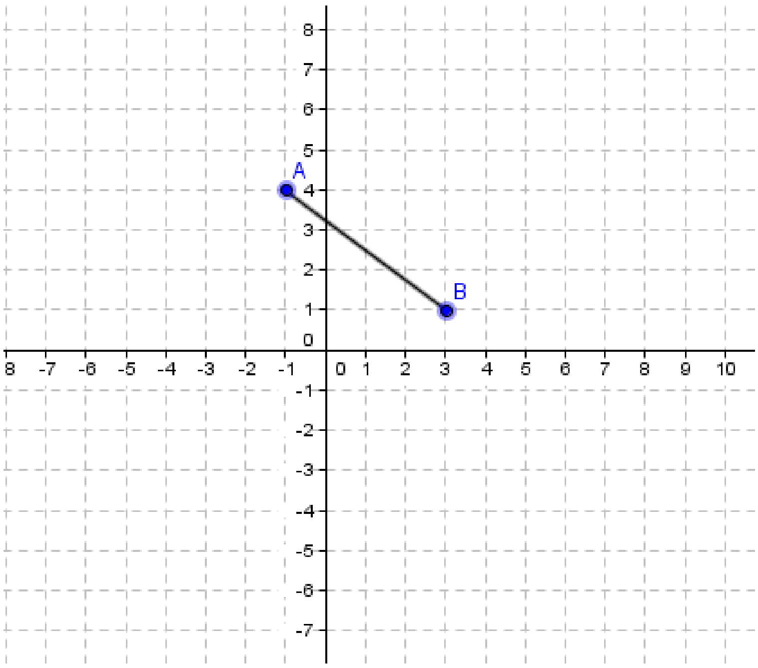


Figure 6.9: Graph on extended response sheet used End of Topic test

The following line interval is drawn from A(-1, 4) to B(3, 1).

Question a: Find the length of this line segment AB – show your working**.**

This question recalls the concept of distance with students able to use either the distance formula, Pythagoras' Theorem or GeoGebra to calculate the answer. Five types and three SOLO levels of responses were noted and there was a ceiling response, summarised in Tables 6.32, 6.33 and 6.34.

Thematic Coding

Type A: This type of response indicated no accurate recall of the distance formula, or any other method able to calculate the distance, or was representative of a non-attempt. Typical responses attempted to use a formula that was incorrect, resulting in an incorrect solution.

In the End of Topic test, Team Lemon A, Team Purple A, Team Blue A and Team Indigo B all recorded this type of response. While Team Lemon A failed to include a "+" sign between brackets, as shown in Figure 6.10, Team Purple A used the incorrect formula, as shown in Figure 6.11, Team Blue A struggled with the formula and its working, as shown in Figure 6.12; and Team Indigo B omitted the square root sign in the formula then substituted incorrectly, as shown in Figure 6.13.

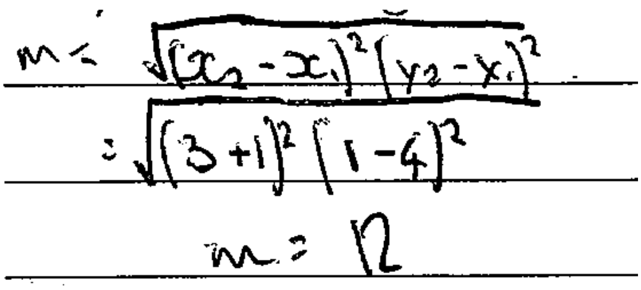


Figure 6.10: Type A response provided by Team Lemon A for Question a

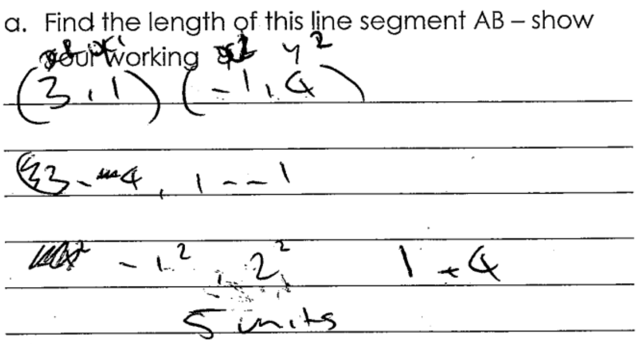


Figure 6.11: Type A response provided by Team Purple A for Question a

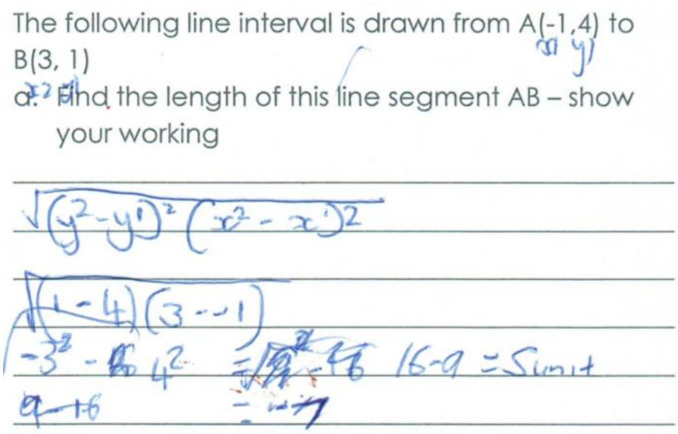


Figure 6.12: Type A response by Team Blue A for Question a

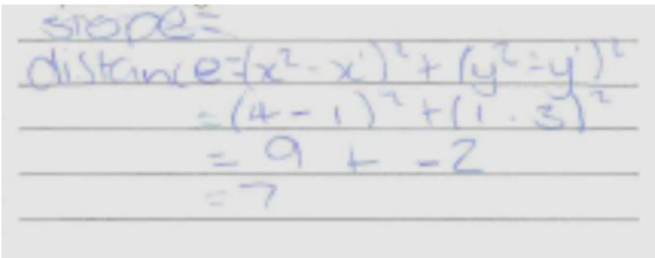


Figure 6.13: Type A response provided by Team Indigo B for Question a

Type B: This type of response indicated recall of the distance formula only, without any attempt to use it.

In the End of Topic test, only Team Indigo A submitted a Type B response, correctly stating the distance formula with no other working shown.

Type C: This type of response builds on a Type B response, with accurate recall of the distance formula, but an element preventing the correct solution to be realised. This type of response was broken into two sub groups. Group I calculated the distance as being 2.6, a direct result of incorrectly calculating $(-3)^2$ as -9 not +9 as required, and Group II incorrectly substituted the coordinates into the formula.

In the End of Topic test, ten Type C responses were submitted. Eight of these presented a Group I response, namely: Team Maroon A, Team Maroon B, Team Lime A, Team Lime B, Team Brown A, Team Orange B and Team Red A, who combined with Team White B for this task. An example of a typical response working is shown in Figure 6.14.

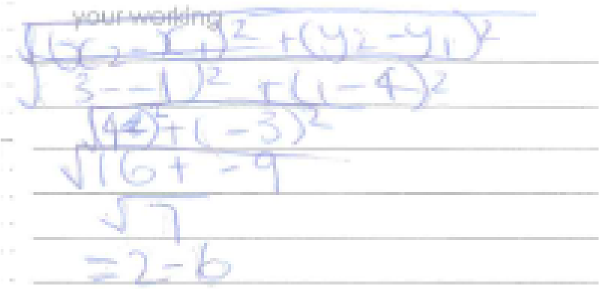


Figure 6.14: Type C Group I response provided by Team White B for Question a

Team Brown A and Team Orange B both submitted a Type C Group II response, both incorrectly substituting the coordinates into the formula. Team Brown A inserting -4 as x_1 instead of -1, despite labelling points correctly in the question, as shown in Figure 6.15.

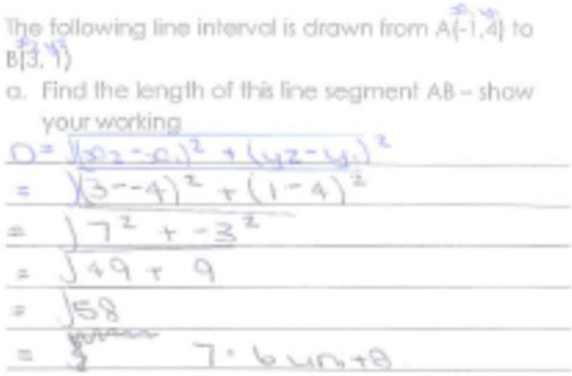


Figure 6.15: Type C Group II response provided by Team Brown A for Question a

Type D: This type of response indicated that, despite little or no working being provided, the correct distance was calculated. It was possible that GeoGebra was used to find the solution, with attempts to replicate the result using a formula provided on the worksheet.

For the End of Topic test, Team Orange A and Team Yellow A submitted this type of response. An example of this type of response is provided in Figure 6.16, submitted by Team Yellow A.

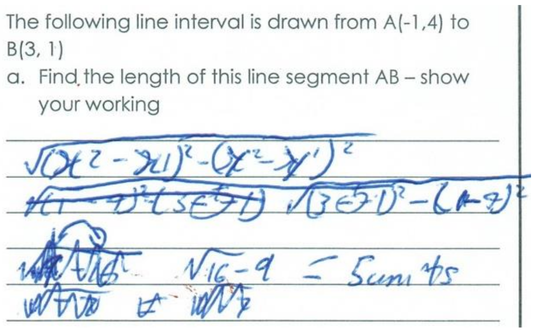


Figure 6.16: Type D response provided by Team Yellow A for Question a

Type E: This type of response was a correct response that accurately stated the distance formula, correctly substituted the coordinates to obtain a solution of 5 units. Both Team Cream A and Team Cream B, correctly submitted a Type E response.

Table 6.32: Response types for finding the length of this line segment AB (Question a)

Response Type	Explanation	Examples
A	No accurate recall of distance formula or any other method capable of calculating the distance Non-attempt	blank comment see Figure 6.10, 6.11, 6.12 and 6.13
B	Recall of distance formula without any attempt to substitute points	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
C	Correct recall of distance formula One element of calculation incorrect resulting in incorrect solution	Group I – see Figure 6.14 Group II – see Figure 6.15
D	Correct solution without working or demonstrating incorrect working	see Figure 6.16
E	Correct solution a result of accurately stating the distance formula and then correct substitution and calculation	5 units

Table 6.33: Thematic coding statistics for finding the length of this line segment AB (Question a)

Response Type	A	B	C	D	E	Total
End of Topic test	4 (21)	1 (5)	10 (53)	2 (11)	2 (11)	19 (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

The following cycle of levels were identified in the CS mode:

Multistructural (M_2): Responses identified more than one feature without relating the features to calculate the distance. Examples coded to this level provided points and then used them in a calculation without any formula or links to tie the calculation and points together or provided the distance formula without any substitution, as seen in Figure 6.11, or link to the coordinates in the question "$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$".

Relational (R_2): Responses indicated relationships were evident between formulas and coordinates. While most responses contained incorrect elements, all contained something resembling a formula with coordinates from the question. Examples of responses coded to this level include Figures 6.10, 6.12, 6.13, 6.14, 6.15 and 6.16.

Table 6:34: SOLO coding statistics for finding the length of this line segment AB (Question a)

SOLO Coding	Number End of Topic test
M_2 (CS)	2
R_2 (CS)	17
Total	19

Question b: Find the gradient of the line segment AB.

This question consolidated the gradient concept for a given line segment. Students were free to use any method or strategy, including using the GeoGebra environment. Five types of responses and four SOLO levels of complexity were noted and there was a ceiling response, summarised in Tables 6.35, 6.36 and 6.37.

Thematic Coding

Type A: This type of response was representative of a non-attempt or provided a solution that was incorrect without any justification as how the solution was reached. Only Team Purple A submitted a Type A response with a blank comment.

Type B: This type of response either presented a formula that was incorrect, and hence provided an incorrect result, or provided an incorrect result with little working to demonstrate the thinking leading to the solution.

Four Type B responses were submitted. Three, namely Teams Black A, Black B and Indigo A, all presented the gradient formula in reverse, with the *x*-coordinates in the numerator and the *y*-coordinate in the denominator, as shown in Figure 6.17, submitted by Team Black A. While Team Lemon A stated: "$m = -1.5$", providing only a right-angled triangle drawn on the Cartesian plane as a working towards the solution.

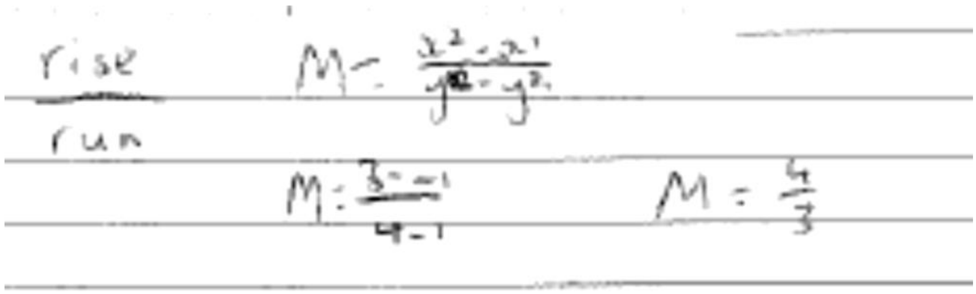
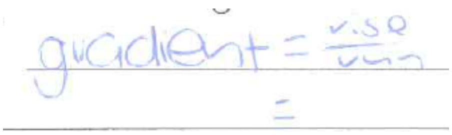


Figure 6.17: Type B response provided by Team Black A for Question b

Type C: This type of response provided a formula with no working or indication of how to



complete the question. While there was recognition of the gradient formula, the understanding and skills to apply it were lacking. Only Team Indigo B submitted a Type C response, as shown in Figure 6.18.

Figure 6.18: Type C response provided by Team Indigo B for Question b

Type D: This type of response contained the correct formula with one incorrect element affecting the final solution.

Six responses of this type were recorded. Typical responses are presented in the figures below. Team Maroon A, Team Maroon B and Team Yellow A all implemented the correct formula and substitution however, overlooked the negative sign, as demonstrated in Figure 6.19.

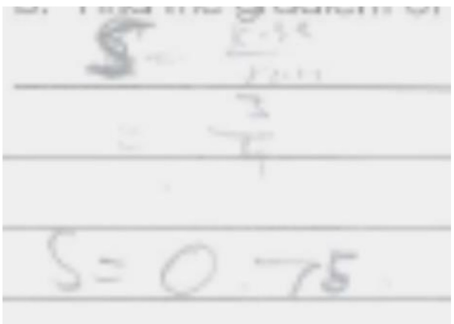


Figure 6.19: Type D response provided by Team Maroon A for Question b

Team Brown A stated the appropriate formula and substituted coordinates correctly however, presented the final solution as a coordinate. This is shown in Figure 6.20.

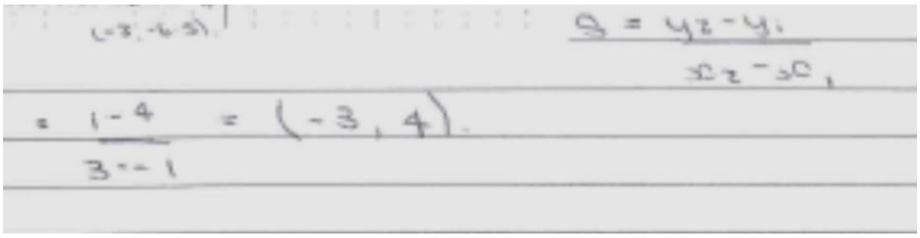


Figure 6.20: Type D response provided by Team Brown A for Question b

Two students, Team Lime A and Team Lime B, correctly stated the formula but substituted incorrectly, despite accurately labelling the coordinates in the wording of the question. This can be seen in Figure 6.21, submitted by Team Lime A.

Figure 6.21: Type D response provided by Team Lime A for Question b

Type E: This type of response demonstrated knowledge of the gradient formula along with a clear understanding of how to substitute the appropriate coordinates to obtain a correct solution of $\frac{-3}{4}$. This type of response was divided into two groups, Group I, who provided an accurate solution of $\frac{-3}{4}$ or -0.75 or both, and Group II who provided $\frac{-3}{4}$ as their solution followed by 0.75. It is possible that the incorrect decimal was simply a typo and oversight.

Seven Type E responses were submitted. Of these, five were coded as Group I, namely, Team Orange B, Team Cream A, Team Red A, Team White B and Team Blue A. A typical Type E Group I response is shown in Figure 6.22, provided by Team Orange B. The remaining two students, Team Orange A and Team Cream B, submitted a Group II response, working out the correct solution then omitting the negative sign when converting to a decimal.

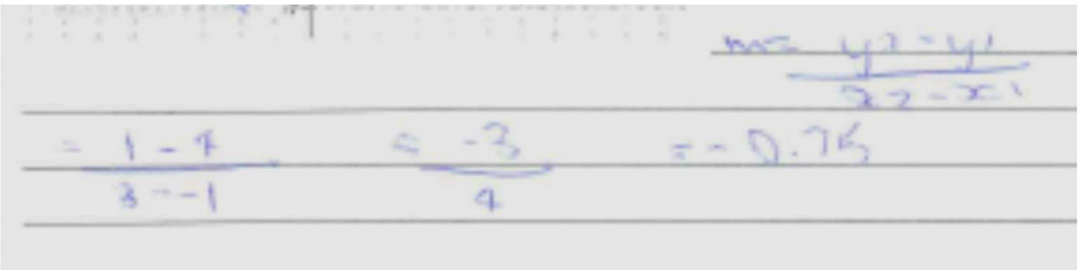


Figure 6.22: Type E Group I response provided by Team Orange B for Question b

Table 6.35: Response types for finding gradient of the line segment AB (Question b)

Response Type	Explanation	Examples
A	Limited understanding of how to calculate gradient Non-attempt	blank comment
B	Provided an incorrect formula resulting in incorrect solution	see Figure 6.17 $m = -1.5$
C	Provided a formula that could be used to calculate the gradient without any further substitution	see Figure 6.18
D	Provided correct formula with one element of calculation affecting final solution	see Figure 6.19 (direction not considered) see Figure 6.20 (solution presented as coordinate) see Figure 6.21 (coordinate labelled incorrectly)
E	Correct formula providing a correct solution	Group I - -3/4 or -0.75 or both provided see Figure 6.22 Group II - -3/4 followed by 0.75 (first solution correct second missing negative sign)

Table 6.36: Thematic coding statistics for finding gradient of the line segment AB (Question b)

Response Type	A	B	C	D	E	Total
End of Topic test	1 (5)	4 (21)	1 (5)	6 (32)	7 (37)	19 (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Prestructural: Responses provided are below those expected for the question. Examples coded to this level were a blank response.

The following cycle of levels were identified in the CS mode:

Relational (R_1): Responses reflect an educated guess based on all the information available, including the Cartesian plane visual. The only example coded to this level used a right-angled triangle drawn onto the Cartesian plane then stated as a single solution of: "$m = -1.5$", without any calculation or further working shown.

Unistructural (U_2): Responses focussed on a singular isolated feature, such as a formula. Examples coded to this level submitted a formula without any substitution or link to the coordinates in the question. This is demonstrated by Figure 6.18.

Relational (R_2): Responses indicated a number of relationships had become apparent in the understanding of the question. While most responses contained incorrect elements, all included some type of formula and linked it to the coordinates stated. Examples of the responses coded to this level include Figure 6.19, 6.20, 6.21 and 6.22.

Table 6.37: SOLO coding statistics for finding gradient of the line segment AB (Question b)

SOLO Coding	**Number** End of Topic test
Prestructural	1
R_1 (CS)	1
U_2 (CS)	1
R_2 (CS)	16
Total	19

Question c: Prove that ΔABC is a right-angled triangle if the coordinates of C $(-7, -4)$.

This question required students to consider what concepts they would use to prove the triangle was right-angled. Responses for this question were coded into four thematic types and five SOLO levels, summarised in Tables 6.38, 6.39 and 6.40.

Thematic Coding

Type A: This type of response indicated no understanding or idea on how to solve the problem or was representative of a non–attempt.

Seven responses were Type A. Team Blue A and Team Yellow A, provided working that resembled finding the equation of a line, as shown in Figure 6.23. This had no relevance to the question.

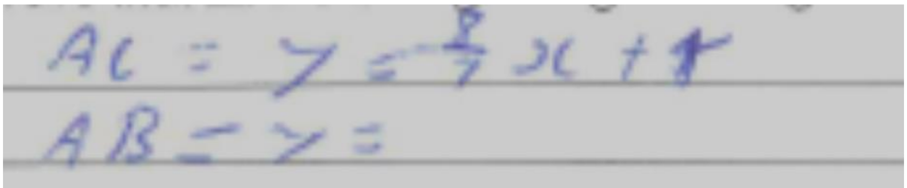


Figure 6.23: Type A response provided by Team Yellow A for Question c

The other two students, Team Maroon A and Team Maroon B, claimed that the triangle in question was not a right-angle. Unfortunately, they provided no justification to support the statement, as demonstrated in Figure 6.24.

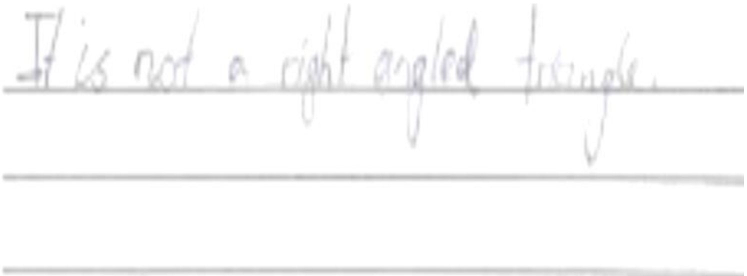


Figure 6.24: Type A response provided by Team Maroon A for Question c

Type B: This type of response provided a written description using keywords "right-angle" or "perpendicular" without further demonstrating any proof or justification. It is possible that the stimulus came from the wording within the question itself.

Seven Type B responses were submitted. Of these, six: Team Black A, Team Black B, Team Lemon A, Team Red A, Team White B and Team Lime A, all provided a simple statement with one of the keywords. Team Lime B submitted the most detailed response, shown in Figure 6.25.

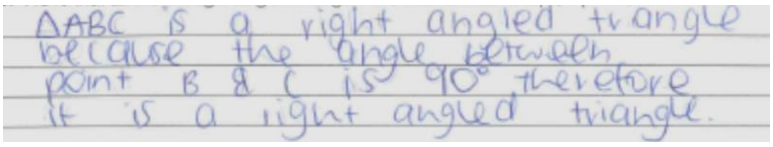
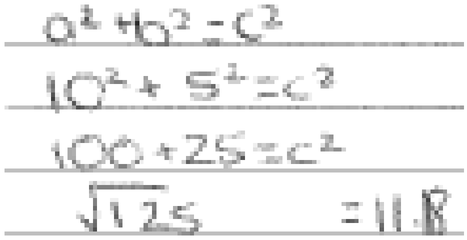


Figure 6.25: Type B response provided by Team Lime B for Question c

Type C: This type of response recognised a connection between right-angled triangles and Pythagoras' Theorem. While the relationship was identified it required more justification in order to complete the proof.

Five responses stated Pythagoras' theorem. Two of these, Team Orange A and Team Orange B, simply stated the theorem with no other working or calculation. One student, Team Brown A, stated: "it has a 90° angle, find the equation of the intervals use Pythagoras theorem to find the missing side $a^2 + b^2 = c^2$." The remaining responses from Team Cream A and Team Cream B, used Pythagoras theorem to calculate the hypotenuse of the triangle but could not complete the proof, as shown in Figure 6.26. Interestingly, the values used in Pythagoras' theorem were correct; however, there was no justification provided to



demonstrate that the solution was the actual length of BC.

Figure 6.26: Type C response provided by Team Cream B for Question c

Type D: This type of response identified that the triangle was perpendicular by comparing the gradients of two sides of the triangle, demonstrating that they satisfy the condition for right-angled lines. Unfortunately, no one submitted a Type D response.

Table 6.38: Response types for proving ∆ABC is a right-angled triangle (Question c)

Response Type	Explanation	Examples
A	No understanding demonstrated of what was required Non-attempt	blank comment see Figure 6.23 see Figure 6.24
B	Written response using keywords right-angle or perpendicular without any proof Re-wording of the question	see Figure 6.25
C	Link between right-angle and Pythagoras theorem identified No justification provided	Pythagoras theorem stated see Figure 6.26 "it has a 90° angle, find the equation of the intervals use Pythagoras theorem to find the missing side $a^2 + b^2 = c^2$"
D	Correctly calculates gradients of two sides of the triangle and demonstrates	non provided

Response Type	Explanation	Examples
	they are perpendicular through relationship between gradients	

Table 6.39: Thematic coding statistics for proving ΔABC is a right-angled triangle (Question c)

Response Type	A	B	C	D	Total
End of Topic test	7 (37)	7 (37)	5 (26)	0 (0)	19 (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Prestructural: Responses provided were below those expected for the question. Examples coded to this level were a blank response.

The following cycle of levels were identified in the CS mode:

Unistructural (U_1): Responses indicated uncertainty regarding the question requirements, with attempts to determine whether the triangle is right-angled or not assisted from visual cues on the graph, or rewording of the question using another mathematical term. Examples coded to this level included the statement: "it is not a right-angle triangle", "if its perpindicular"[sic], and "ΔABC is a right-angled triangle because the angle between point B & C is 90° therefore it is a right-angled triangle".

Multistructural (M_2): Responses focus on more than one feature that could be used to determine if the triangle was right-angled without connections being established. Examples coded to this level included: "$a^2 + b^2 = c^2$", the Pythagoras' theorem substituted with values not related to the triangle in the question.

Relational (R_2): Responses indicated attempts to establish links between various properties of the graph to the question. Both responses coded to this level were incorrect but demonstrated correct format for the equation of a line for AC, recognising that the right-angle occurs between the lines AB and AC and attempts to find equations that represent those line. Examples coded to this level included: "$AC = y = -8/7x + 4\ AB = y =$" and "$ac = y = x + 4\ ab = y$".

Table 6.40: SOLO coding statistics for proving ∆ABC is a right-angled triangle (Question c)

SOLO Coding	Number End of Topic test
Prestructural	3
U_1 (CS)	9
M_2 (CS)	5
R_2 (CS)	2
Total	19

Question d: If ABCD is a quadrilateral such that AB is parallel to CD and AB = CD. What could the quadrilateral be?

Despite not being part of the Linear Relationships unit, this question required students to think about the properties of quadrilaterals enabling them to recognise how the concepts learned for Linear Relationships can relate to the properties of quadrilaterals. Three types of responses were noted for this question and there was a ceiling response, summarised in Table 6.41 and Table 6.42.

Thematic Coding

Type A: This type of response indicated a non-attempt and was represented by a blank comment. Seven blank comments were submitted from Team Indigo A, Team Indigo B, Team Maroon A, Team Maroon B, Team Purple A, Team Yellow A and Team Blue A.

Type B: This type of response described the quadrilateral as a parallelogram. It is possible that this was chosen from keywords present in the question, namely "quadrilateral" and "parallel". Five submitted this type of response, namely: Team Orange A, Team Orange B, Team Black A, Team Black B and Team Brown.

Type C: This type of response correctly described the quadrilateral as a rectangle. Seven correct responses were submitted, namely: Teams Orange A, Team Orange B, Team Cream A, Team Cream B, Team Red A, Team White B, Team Lime A, and Team Lime B.

Table 6.41: Response types for naming type of quadrilateral formed (Question d)

Response Type	Explanation	Examples
A	No understanding of what was required Non-attempt	blank comment
B	Identifies quadrilateral based on features given in question No links to previous part of question	parallelogram

Response Type	Explanation	Examples
C	Relates previous part of question which states that one angle is 90° Links angle to features given in question Correct solution	rectangle

Table 6.42: Thematic coding statistics for naming type of quadrilateral formed (Question d)

Response Type	A	B	C	Total
End of Topic test	7 (37)	5 (26)	7 (37)	19 (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Responses to Question d do not provide enough detail about the quality of the student response to attempt a fine-grained SOLO coding. Those responses classified as Type A could be broadly coded as Prestructural with Type B and C responses coded as Concrete Symbolic. It is not possible to distinguish further as the question itself did not ask for more information on how they came to the conclusion of what quadrilateral the shape would be. In the light of these results, the responses provided to Question d.ii offer a window to view the students' understanding of how this decision was made and this follows.

Question d.ii: Describe all the different ways you could prove that it's that type of quadrilateral**.**

This question required students to detail what strategies they used to arrive at the quadrilateral suggested in the previous question. The correct solution for the previous question was a rectangle and the following categories are based on proving a rectangle, namely, opposite sides are equal and parallel with four right-angles. There were five students who, in the previous question, stated that the shape was a parallelogram and their responses for this question will be categorised based on proving it was a parallelogram, namely, opposite sides equal and parallel and opposite angles are equal or consecutive angles are supplementary.

The coding used for this question was the same as that of the corresponding question from the Pre-test and Delayed Post-test extended response sheets. That is, the thematic coding categorised the responses for this question, d.ii, with respect to the previous responses for question d; and the SOLO coding uses one code that incorporates responses from both questions, d and d.ii. Five types of responses and six SOLO levels were identified, summarised in Tables 6.43, 6.44 and 6.45.

Thematic Coding

Type A: This type of response indicated no understanding of what was required or was representative of a non-attempt. Typical responses were a blank comment or statements that missed vital details and would not be able to be used to prove a quadrilateral.

Nine students submitted this type of response with five being blank responses. Team Orange A and Team Orange B both suggested the quadrilateral as a parallelogram in the previous question and now stated their justification as: “4 points connect, They never meet if they are parallel”, and “4 points content, Parallel never meet, Line never meet”, respectively. Both responses lacked links connecting the statements made, switching focus from points to lines without any reference to sides or angles. Another two responses that were considered to be a Type A response were: Team Yellow A, who stated: “Parallel lines”, and Team Blue A who stated: “Parallel lines, Same *x* axis”. Again, neither could be used to prove a type of quadrilateral, and both these students did not submit a response for the previous question, hence, had no idea what type of quadrilateral they were supposed to prove.

Type B: This type of response indicated that students attempted the question and had some understanding of what was required. Comments reflected a single property that could be derived from the meaning of the term quadrilateral, namely: it has four sides or four angles with no link to other features or further explanation. This type of response does not provide any details unique to the specific quadrilateral of the previous question, namely, a rectangle. No responses were submitted of this type.

Type C: This type of response builds on a Type B response, identifying a single property that is unique to the quadrilateral of the previous question. Unfortunately, the single property was not enough to prove a rectangle. Again, no responses were submitted of this type.

Type D: This type of response contained more than one single property but was not specific in describing the property correctly. More detail was required to define each property or links needed to be established between the properties.

Nine responses of this type were submitted. Three of these, namely: Team Brown A, Team Black A and Team Black B, all stated that the quadrilateral was a parallelogram in the previous question. In this question, however, Team Brown A stated: “Never meet each other, Has 90° angles, Has a positive slope, Have 2 right-angle triangles", and Team Black A stated "4 sides, Opposite sides never meet, 4 points", while Team Black B stated: "4 sides, Opposite sides never meet". Each response had elements that could assist in forming a proof if more depth was provided.

The remaining responses submitted were from students who submitted "rectangle" for the previous question, yet none could correctly articulate what was required to prove the quadrilateral. Responses varied from Team Cream B, who stated: "Two side are parallel, Both equal triangles, that make a rectangle", to Team Cream A, who stated: "2 of the sides are parallel, There are 2 equal triangles that make it up", and Team Lime A who stated, "Has 4 sides, Has 2 sides of equal length, Parallel lines, Can be bisected into 2". Team Black A stated: "4 sides, Opposite sides never meet, 4 points", with Team Black B, who stated: "4 sides, Opposite sides never meet", and Team Red A, who stated: "Because AB is parallel to BC, AB = CD, They are the same distance, All right angls" [sic]. Lastly, Team White B stated: "Because AB is parallel to CD, AC is = BD, They are same distance, All right-angles" and Team Lime B, who stated: "two set of equal lines, 4 sides, 2 set of parallel lines, Equal when bisected". All contained elements that were correct but not enough to complete a proof.

Type E: This type of response contains more than one single property that would be consistent with proving a rectangle. Only Team Lemon A submitted this type of response stating: "Lines are parallel, All angles are right-angle, All sides are equal with the opposite".

Table 6.43: Response types for describing strategies to prove quadrilateral type (Question d.ii)

Response Type	Explanation	Examples
A	No understanding of what was required No properties given relate to a quadrilateral Non-attempt	"4 points connect, They never meet if they are parallel" "4 points connect, Parallel never meet, Line never meet" "parallel lines" "parallel lines, Same x axis"
B	A single property or feature is identified from a visual cue No thought about other features	none provided
C	A single property or feature identified that could be used as part of a proof	none provided

Response Type	Explanation	Examples
D	More than one feature or property identified that could be used as part of a proof Missing specific details	"4 sides, Opposite sides never meet, 4 points" "4 sides, Opposite sides never meet" "because AB is parallel to BC, AB = CD, They are the same distance, All right angls" "because AB is parallel to CD, AC is = BD, They are same distance, All right-angles" "two set of equal lines, 4 sides, 2 set of parallel lines, Equal when bisected" "two side are parallel, Both equal triangles, that make a rectangle" "2 of the sides are parallel, There are 2 equal triangles that make it up" "never meet each other, Has 90° angles, Has a positive slope, Have 2 right-angle triangles" "has 4 sides, Has 2 sides of equal length, Parallel lines, Can be bisected into 2"
E	More than one feature or property identified that would be sufficient for proving a rectangle	"lines are parallel, All angles are right-angle, All sides are equal with the opposite"

Table 6.44: Thematic coding statistics for describing strategies to prove quadrilateral type (Question d.ii)

Response Type	A	B	C	D	E	Total
End of Topic test	9 (60)	0 (0)	0 (10)	9 (10)	1 (10)	19 (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

Prestructural: Responses are below those expected for the question. Examples coded to this level are a blank response.

The following cycle of levels were identified in the CS mode:

Unistructural (U_1): Responses indicated uncertainty of the question's requirements, with a focus on one specific property that was evident from visual cues or was recognised as being relevant to the quadrilateral mentioned. Examples coded to this level included a combination of a blank response for naming of the quadrilateral and for the description of its proof simply stating: "Parallel lines".

Multistructural (M_1): Responses contained more than one separate statement as proof without any connection between statements established. The only example coded to this level included a combination of a blank response for naming of the quadrilateral (d) and for the description of its proof (d.ii), simply stating: “Parallel lines, Same x axis”.

Relational (R_1): Responses reflect an educated guess that takes into account all information available, including the visual aspect of the diagram. Examples coded to this level included a combination of identifying the quadrilateral as a parallelogram (d) and for the description of its proof (d.ii), stating, “4 points connect, They never meet if they are parallel” and for another “4 points connect, Parallel never meet, Line never meet”, and “Never meet each other, Has 90° angles, Has a positive slope, Have 2 right-angle triangles”. Neither response provided substantial proof for previously stating the quadrilateral was a parallelogram.

Multistructural (M_2): Responses involved more than one separate and disjointed statements regarding the properties of the quadrilateral named in the previous part of the question. Statements were not sufficient for establishing a proof. Examples coded to this level included a combination of naming the quadrilateral as a rectangle and stating: “Two side are parallel, Both equal triangles, that make a rectangle”; “2 of the sides are parallel, There are 2 equal triangles that make it up”; “Because AB is parallel to CD, AC is = BD, They are same distance, All right-angles”; and “Has 4 sides, Has 2 sides of equal length, Parallel lines, Can be bisected into 2”, and “two set of equal lines, 4 sides, 2 set of parallel lines, Equal when bisected”. Examples coded to this level also includes a combination of naming the quadrilateral as a parallelogram and stating: “4 sides, Opposite sides never meet, 4 points” and “4 sides, Opposite sides never meet”.

Relational (R_2): Responses indicated that connections had become apparent in the understanding of the question, with sufficient statements to provide a proof for the choice of the quadrilateral stated in Question d, with the diagram reflecting this. The only example coded to this level was a combination of naming the quadrilateral correctly as a rectangle and stating: “Lines are parallel, All angles are right-angle, All sides are equal with the opposite”.

Table 6.45: SOLO coding statistics for describing strategies for quadrilateral proof (Question d.ii)

SOLO Coding	**Number** End of Topic test
Prestructural	5
U_1 (CS)	1
M_1 (CS)	1
R_1 (CS)	3
M_2 (CS)	8
R_2 (CS)	1
Total	19

Question d.iii: Use one of your strategies listed above to prove it's that type of quadrilateral.

Four types of responses were identified for this question, summarised in Tables 6.46 and 6.47.

Thematic Coding

Type A: This type of response indicated limited understanding of the requirements and was representative of a non-attempt or a comment that was not relevant to establishing a proof. Typical responses were blank comments and random statements.

Eleven responses were coded to this type of response. Team Orange A and Team Orange B stated: "All 4 points connect", with Team Cream A and Team Cream B making reference to triangles which were equal without indicating how the triangles related to proving the quadrilateral, stating: "Both triangles are equal, they then make a rectangle", and "Both triangles are equal, which make a rectangle ~~two sides are parallel to each other~~" respectively.

Type B: This type of response relied on visual cues to find the quadrilateral type.

Two responses of this type made reference to a drawing. Both used terminology consistent with Linear Relationships, not with quadrilaterals and use a form of drawing to support their statement. Team Black A stated: "Its parallel because the *y* intercept are the same" with their respective diagram, as shown in Figure 6.27, and Team Black B stated: "Parallel Y intercept is the same" with their diagram, shown in Figure 6.28.

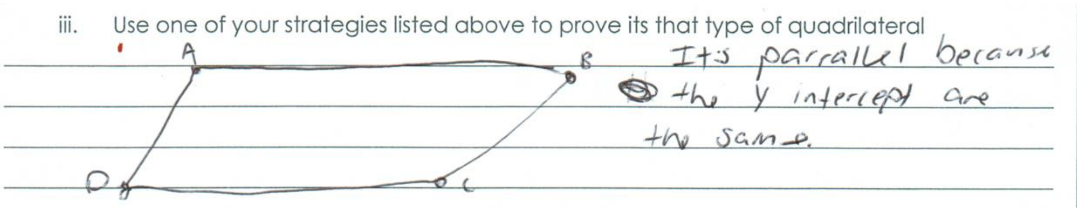


Figure 6.27: Type B response provided by Team Black A for Question d.iii

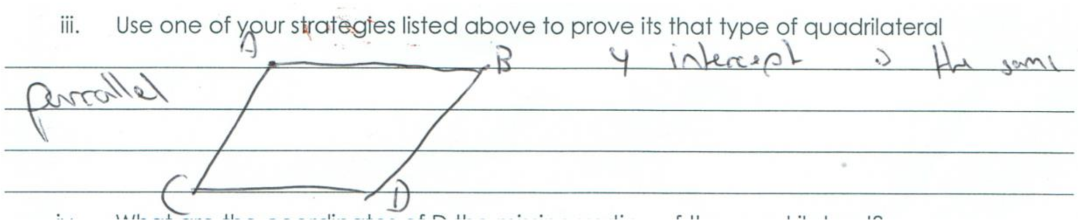


Figure 6.28: Type B response provided by Team Black B for Question d.iii

Type C: This type of response stated a single property of the shape, stated without making connections necessary to formalise the proof.

Six of the responses submitted were this type of response. Of the six, five were simple statements: Team Lemon A, who stated: "All sides are parallel", Team Red A, who stated: "AB= CD", Team Lime A, who stated: "It has 4 sides", Team Lime B, who stated: "It has two sides and two sets of equal sides", and Team White B, who stated: "$m1 = m2$ parallel". Team Brown A provided more detail in her explanation, falling short of a better response due to the inclusion of the term "parallelogram" in the response, stating: "Find the distance of the intervals if they are equal, then yes it is a parallelogram, or by using, phythagoras theorem find the angle of the hypotenuse when the y intercept changes, gradient stays the same" [sic].

Type D: This type of response develops a Type C response as the property stated could be used as part of a proof, although it fails to link any other properties to complete the strategy. No responses of this type were submitted.

Table 6.46: Response types for using strategies to prove quadrilateral type (Question d.iii)

Response Type	Explanation	Examples
A	No understanding of what was required No properties given relate to a quadrilateral Non-attempt	"all 4 points connect" "all 4 point connect" "both triangles are equal, which make a rectangle (crossed out)- two sides are parallel to each other"

Response Type	Explanation	Examples
		“both triangles are equal, they then make a rectangle”
B	A single property or feature is identified from a visual cue No thought about other features	“(drawing) Its parallel because the y intercept are the same” “(drawing) Parallel Y intercept is the same”
C	A single property or feature identified that requires more detail to provide a strategy that could be used to assist in a proof	“all sides are parallel” “find the distance of the intervals if they are equal, then yes it is a parallelogram, or by using, phythagoras theorem find the angle of the hypotenuse when the y intercept changes, gradient stays the same” “AB= CD” “It has 4 sides” “It has two sides and two sets of equal sides” “m_1=m_2 parallel”
D	Single property or feature identified that could be used as part of a proof No link to other properties	none provided

Table 6.47: Thematic coding statistics for using strategies to prove quadrilateral type (Question d.iii)

Response Type	A	B	C	D	Total
End of Topic test	11 (58)	2 (10.5)	6 (31.5)	0 (0)	19 (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

This question was difficult to provide a fine-grained coding for since most responses replicated one of the statements made for the previous question. Hence, no further detail was provided with which to code the quality of the response. Those responses classified as Type A could be broadly coded as Prestructural with the remaining responses coded as Concrete Symbolic.

Question d.iv: What are the coordinates of D, the missing vertice of the quadrilateral?

Four types of responses were identified for this question, summarised in Tables 6.48 and 6.49.

Thematic Coding

Type A: This type of response was an incorrect solution demonstrating no understanding of how to obtain the coordinate by either diagram or working out, or was representative of a non-attempt. Eleven of the responses submitted were of this type. Nine were blank comments while the remaining two provided incorrect responses. Team Cream B, provided a statement claiming: “You don’t need it because you’ve already got the same triangle there”, and Team Cream A, submitted a coordinate: “(-6, -2)” that could not possibly be a point on the rectangle. Interestingly, both correctly identified the quadrilateral as a rectangle but were unable to determine the missing point.

Type B: This type of response contained either an *x* coordinate as -3 or *y* coordinate as -7, indicating that one value of the coordinate pair was correct. Six students submitted responses of this type, with two, Team Lime A and Team Lime B, claiming D as $(-3, -6)$, Team Brown A stating D as $(-3, -6.5)$ and the remaining three, Team Red A, Team White A and Team Orange B stating D as $(-2, -7)$.

Type C: This type of response was similar to a Type B response, with only one value within the coordinate pair being correct, although the diagram demonstrates a different value. This occurred with Team Orange A, who submitted a solution of $(-2, -7)$; however their diagram clearly showed a solution of $(-3, -7)$, which was the correct solution, as shown in Figure 6.29.

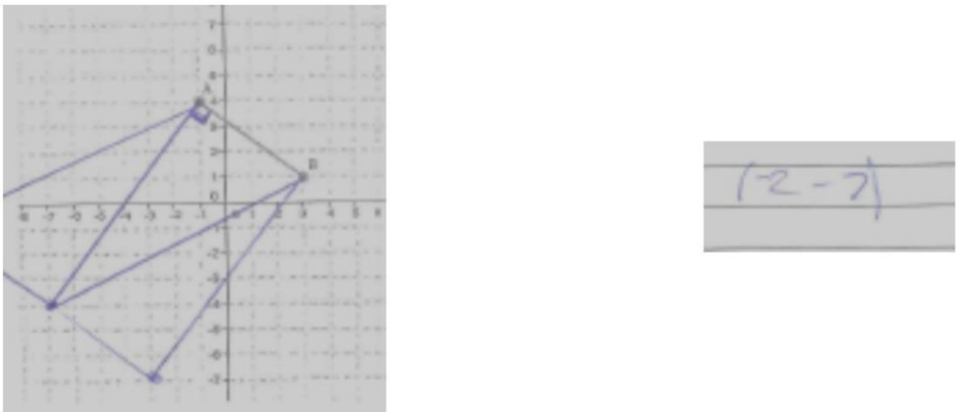


Figure 6.29: Type C response provided by Team Orange A for Question d.iv

Type D: This type of response contained the correct coordinate of $(-3, -7)$. Only Team Lemon A submitted this solution.

Table 6.48: Response types for finding the coordinates of D the missing vertice of the quadrilateral (Question d.iv)

Response Type	Explanation	Examples
A	Incorrect solution demonstrating no understanding of how to obtain coordinate Non-attempt	(-6, -2)
B	Either x or y coordinate correct but not both	(-3, -6) (-3, -6.5) (-2, -7)
C	Only one coordinate correct but working suggests correct solution	see Figure 6.29
D	Correct solution	(-3, -7)

Table 6.49: Thematic coding statistics for finding the coordinates of D the missing vertice of the quadrilateral (Question d.iv)

Response Type	A	B	C	D	Total
End of Topic test	11 (58)	6 (32)	1 (5)	1 (5)	19 (100)

Percentages of the sample for each response type in each test are included in brackets.

SOLO Coding

The context of the question allowed students to use visual cues to assist them in finding a solution. Therefore, a fine-grained SOLO coding was not attempted for this question. Those responses classified as Type A could be broadly coded as Prestructural, with the remaining responses as Concrete Symbolic. It is not possible to distinguish further as the question itself did not ask for more information on how they came to the final coordinate.

6.4. Conclusion

In summary, this chapter reported the responses provided by the students for the problem-solving component of the three Google Form Tests. Similar to the previous chapter, each question was examined and student responses coded for analysis using both thematic coding and the SOLO model. For clarity, comprehensive explanations of the coding categories for each question were included. Once again, the findings of this chapter reveal that a range of response types were provided by the students when completing written responses for problem solving, demonstrating varying degrees in their levels of learning. Written responses provided an opportunity to view the working underlying the solution process when completing Linear Relationships problems.

In particular, the two research questions posed at the beginning of the chapter were able to be addressed formally.

Research Question 1.2. states*:*

How does the SOLO model offer a framework to explain the categories of responses concerning students' understandings of Linear Relationships?

In a similar vein to the analysis provided in the previous chapter, the SOLO model, when considered from a multiple cycle perspective, provided a structure to describe the students' understanding of problem solving within Linear Relationships. It also offers a deeper interpretation to explore the characterisations of developmental growth, with a number of levels being identified, due to the range of responses recorded. A detailed summary of the response groupings is provided in Table 6.50, offering an overview of the developmental pathway with respect to Linear Relationships.

Table 6.50: The SOLO model and overview of developmental pathway for Linear Relationships

SOLO modes and levels	Description
U_1(CS)	Recognising a single visual feature using basic language in explanations Identifying an intercept from a graph Identifying one obvious feature of a quadrilateral such as four sides
M_1(CS)	Focus on more than one feature using separate distinct statements No relationship between statements is established or evident Strategies to prove a quadrilateral use disconnected statements that are obvious from visual cues but not related to each Strategies involve practical methods such as using measuring tools to calculate concepts such as distance An incorrect formula stated Describing changes when moving a graph using separate statements for elements
R_1(CS)	Educated guess from a visual perspective, or using measurement tools, such as stating a quadrilateral in a diagram is a rhombus because it has four sides Justification for a quadrilateral linking a number of ideas evident from visual cues either from the graph or question Explanations use less formal language
U_2(CS)	Recognising a single feature using more technical language Focus has less reliance on visual cues and is based on a more abstract approach Identifying a property and labelling it with correct symbolism Focus on one property that assists in finding a concept Recognising Pythagoras' Theorem as an attempt to finding the missing vertex of a right-angled triangle

SOLO modes and levels	Description
	Identifying a quadrilateral as a square and using stating one feature such as right-angled for justification Performing one operation towards rearranging an equation Justifying an equation of a line as being straight based on one feature not necessarily using visual cues Stating to use a specific formula
M_2(CS)	Disconnected statements Identifying points that belong on a line without the ability to calculate the equation Labelling two coordinates that would lie on a perpendicular bisector without being able to establish the equation of the perpendicular bisector Focussing on more than one property without the ability to link them together Stating a formula that may have incorrect elements
R_2(CS)	Usually a correct response Responses demonstrate that relationships are established Understanding using algebraic notation that relates all elements Multiple connections and their relationship are evident within response Identifying a number of connected properties to justify a specific quadrilateral Recognising the coordinates involved in a coordinate pair, finding the equation of a line using notation Recognising relationships when attempting to calculate the equation of a perpendicular line Providing an equation for the perpendicular bisector of an interval Incorrect formula applied correctly Correct application of formulas

Research Question 2.1 states:

Can an analysis of the results offer insights into students' understandings of Linear Relationships?

Through using the SOLO categorisations, the developmental pathway of students' understanding of problem-solving within the Linear Relationships topic can be monitored and used by teachers as a tool to assist them in preparing lessons. While these students were mainly operating in the concrete symbolic mode, identifying which U-M-R cycle they are operating at can provide valuable information.

Movement between the two U-M-R cycles within the concrete symbolic (CS) mode can be mapped to the student's ability to use GeoGebra as a tool for investigating problems. Responses coded within the first cycle of the CS mode indicated that there was a reliance on the visual features either provided in the question or in GeoGebra. These responses indicated various levels of support from the ikonic mode, where language is determined primarily through images. The majority of first cycle CS mode responses were evident in the End of Topic problem solving sheet, indicating the limited ability to use GeoGebra for problem solving with Linear Relationships concepts. Typical responses stating familiar mathematical ideas from diagrams on the sheet as justification for proofs, or stating that something couldn't be proven based on visual features unable to be seen on the diagram rather than exploring the potential of GeoGebra.

The second U-M-R cycle of the CS mode indicated that GeoGebra was used as a tool to support the understanding of concepts rather than as a direct tool to provide an answer. This was evident through the complexity of the responses provided, with less reliance on visual features, using algebra and formulas, and with justification of proofs using statements demonstrating connections with Linear Relationships concepts. Results indicate that once the second cycle concrete symbolic mode was achieved, it was generally maintained.

A number of responses were recorded as being in the first cycle of the concrete symbolic mode after the teaching sequence and for the Delayed Post-test. This was not seen as a regression since, for the Pre-test, the problem-solving component presented with prestructural responses. It demonstrated that students relied on individual elements and visual cues present either in the diagram or able to be determined using GeoGebra, thus indicating a limited ability to use GeoGebra as a supportive and investigative tool.

After the teaching sequence, the number of prestructural responses evident for the Delayed Post-test increased. These responses indicate that, while students may have been able to focus on individual elements of the task, they were not able to connect multiple concepts of Linear Relationships to assist them in solving more complex problems involving proofs.

The next chapter reports on the case study of one student pair, Team Brown. Their journey throughout the Linear Relationships unit is examined in more detail and complements the class results discussed in this chapter. Together, the class results and case study provide a rich and detailed account of the student responses when learning Linear Relationships with technology such as GeoGebra.

CHAPTER 7: CASE STUDY

7.1. Introduction

The purpose of this chapter is to present the findings of a student pair's responses and their learning journey through the Linear Relationships unit. Team Brown was the student pair chosen due to the amount of data available for the pair, including screen shots and transcriptions obtained from recordings, providing an in-depth description of their journey. To assist the investigation, the following research questions are addressed.

Research Theme 1

To explore the SOLO model and van Hiele Teaching Phases as frameworks to assist teachers when using technology as a teaching tool.

1.1 How do the van Hiele Teaching Phases offer a framework for designing a lesson sequence incorporating technology as a teaching tool?

Research Theme 3

To investigate students' understanding of Linear Relationships concepts when using dynamic mathematics software GeoGebra.

3.1 What are the characteristics of students' responses when exploring concepts of Linear Relationships using dynamic mathematics software?

3.2 What is the nature of student interaction when using GeoGebra as an exploration tool?

3.3 What are the observed developmental hurdles and technical knowledge issues encountered by students when exploring Linear Relationships concepts utilising dynamic mathematics software?

This chapter is divided into two main sections, an introduction to Team Brown then the case study, followed by the conclusion section that ties together findings that have emerged from an analysis of the case in the context of answering the research questions stated at the beginning of this chapter.

7.2. Team Brown – Rhonda & Narelle

Team Brown consisted of two female students, Rhonda and Narelle, who were paired because of their similar quiet personalities which meant that neither was likely to attempt to dominate the other. The girls were not social friends nor had they previously worked closely with each other, so this union was considered an appropriate mix. Unfortunately, Narelle left the school before the end of the Linear Relationships unit, leaving Rhonda to complete the End of Topic test (ETT) and Delayed Post-test on her own.

A summary of Rhonda and Narelle's responses to the Google Form tests in terms of SOLO categorisations is presented in Table 7.1.

Table 7.1: Team Brown SOLO Coding for Google Form Tests

	SOLO CODING		
Question	**Pretest**	**End of Topic Test**	**Delayed Post-Test**
2a (finding midpoint)	M_2	R_2	R_2
3a (finding distance)	M_2	R_2	M_2
4a (finding gradient)	M_2	Formal	Formal
5a (equation of line)	R_2	Formal	Formal
6a (draw a graph)	M_2	M_2	R_2
6b (can you move graph)	Prestructural	M_1	R_2
7a (parallel lines)	Prestructural	R_2	R_2
8 (perpendicular)	Prestructural	U_2	M_2
9 (changing subject of eqn)	Prestructural	R_2	R_2
10a	Prestructural	M_2	R_2
11a	n/a	R_2	n/a
11b	n/a	U_2	n/a
11c	n/a	R_2	n/a
11d	n/a	R_2	n/a
11e	n/a	R_2	n/a
Extended a	Prestructural	n/a	R_2
Extended b	Prestructural	n/a	R_2
Extended c.i	Prestructural	n/a	n/a
Extended c.ii	Prestructural	n/a	R_2
Extended c.iii	Prestructural	n/a	U_2
Extended c.iv	Prestructural	n/a	R_2
Extended a ETT	n/a	R_2	n/a
Extended b ETT	n/a	R_2	n/a
Extended c ETT	n/a	M_2	n/a
Extended d.ii ETT	n/a	R_1	n/a
Extended d.iii ETT	n/a	R_2	n/a

It was appropriate to examine Team Brown with a closer lens because of their predominance of second cycle responses in the concrete symbolic mode at the end of the teaching sequence. In most cases, the second cycle responses were retained for the Delayed Post-test. Rhonda and Narelle engaged in open class discussion that demonstrated their growing understanding of the content. Of the two, Rhonda engaged most in discussion throughout the teaching sequence, hence her responses are the ones most referred to.

In Google Form tests, Rhonda and Narelle consistently provided responses that had little change in their SOLO categorisation. The majority of their responses were coded to the second U-M-R cycle of the concrete symbolic mode. This indicates that the methods employed during the learning sequence provided them with long-term retention of the content.

Their responses demonstrated that their understanding of the Linear Relationship's topic increased during the teaching sequence. From the beginning, Team Brown demonstrated a high level of understanding, despite never having used GeoGebra previously. Their responses, submitted at the beginning of the unit for the Pre-test (Lesson 1), were considered Concrete Symbolic (CS) second cycle U-M-R. These questions addressed basic concepts of Linear Relationships, such as the midpoint, distance and slope between two given points with and without using the GeoGebra environment. Relying on their navigation of GeoGebra as a supportive tool with which to investigate these concepts, Rhonda and Narelle were able to articulate a number of sequential instructions or state a formula that could be used to calculate the concept required. Their responses for all three concepts were coded as CS M_2, and, while not all three responses were correct, they did indicate understanding beyond the visual cues provided by GeoGebra. Responses for the questions attempted in the Pre-test are provided in Table 7.2.

When exploring and investigating lines, Rhonda and Narelle were able to provide responses that, over time, increased in quality. This demonstrated that their understanding of the gradient, y-intercept and equation of a line was developing. The pairs first response towards drawing a line without GeoGebra suggested: "Draw a table with x and y. The equation will be". This response drew on previous knowledge of Stage 4 content where lines are drawn using a table of values. It demonstrates that algebraic thinking was involved with an understanding of the requirements of the question although no relationship had been identified as to how to substitute values or choose the initial values for x. The response revealed thinking beyond using visual cues, thereby representing a second cycle M_2 response.

Table 7.2: Pre-test responses for Team Brown

Question	Response
Questions 2 to 5 involve using the toolbar in GeoGebra and the points (2, 0) and (0, 5). First find the point which would represent the midpoint.	(1,2.5)
a. How could you find the midpoint without using GeoGebra and using pen and paper?	Find the length between two of the points and divide it by 2 and you will get the approximate answer of the midpoint.
3. Find the distance between these two points when connected.	5.39
a. How could you find the distance without using GeoGebra and using pen and paper?	Pythagoras Theorem
4. Find the slope between the two points	-2.5
a. How could you find the slope without using GeoGebra and using pen and paper?	To find the slope, start from point A to B measure the distance and find the midpoint between the two. As the answer would be (1,-2.5), from point A measure 1 away and then measure downwards -.2,5 towards the midpoint. Connect the points into a triangle.
5. What is the equation of the line you drew in part four?	$5x + 2y = 10$
a. How did you work this out?	The GeoGebra provided the answer.
6. Using GeoGebra can you draw the graph of $y = 4x + 8$?	It is too hard to extend the numbers.
a. How would you do this without GeoGebra using pen and paper	Draw a table with x and y. The equation will be

As both Rhonda and Narelle continued using GeoGebra (Lesson 3), to complete chosen activities aligned with van Hiele Teaching Phase 2, Directed Orientation, they began to explore the algebra and geometry windows of the software, drawing lines to distinguish key features that could assist them when using pen and paper techniques. The first of these features to become apparent was the y-intercept. When presented with the equation of the line, $y = 4x + 13$, Rhonda stated: “it will pass and touch the 13”, then later continued to identify that the line $y = 2x + 3$, was “going to touch the 3”. Both responses single out where the graph passes through the y-axis; although in the descriptions provided, the word “touch” does not clearly explain the y-intercept from visual cues but relies more on the feature being identified from the equation, thus a U_2 response. The progression to Phase 3 of the van Hiele Teaching Phases, Explicitation, was achieved when the term "*y*-intercept" was attached to the idea by Rhonda, thus developing the language associated with the concept.

An issue that caused concern for Rhonda during the Directed Orientation phase of this activity was with the reading and interpretation of graphs within the GeoGebra environment. Depending on the nature of the graph and how GeoGebra presented it in the geometry window, certain features such as the *y*-intercept were not always visible. These scaling issues made it necessary for students to understand how to manipulate technical aspects of the GeoGebra environment to ensure the information they viewed was relevant to what they required. Often, this involved manipulating the settings to ensure a major part of the graph could be seen in the view window. This was not always a simple task as there were many options available in the settings window to be changed, including the maximum and minimum values for each axis and the ratio of *x*-axis: *y*-axis. In Rhonda's case, she was reluctant to attempt this on her own and sought teacher assistance regarding what to do. This supports research regarding the importance of teachers in their role of assisting students to overcome such dilemmas by identifying "important internal processes of the technology and its limitations" (Cavanagh, 2005, p. 83).

With the *y*-intercept concept established, further investigations recognised that the slope of the line changed for different graphs. Initial activities chosen were consistent with Phase 2, Directed Orientation, thus the progression of Teaching Phases spiralled back from previously being Phase 3 with the *y*-intercept. Through class discussion, changing slope was related to the number directly in front of the *x* term in the equation of the line. An integral association for the van Hiele Teaching Phases is the development of language combined with a concept, hence the value in front of the *x* term was introduced as the coefficient of *x* and the slope was used interchangeably with the term gradient, a natural progression to Phase 3 of the Teaching Phases, Explicitation. Despite no explicit conversations being heard from Rhonda and Narelle on the recordings during this time, it remains an introduction to the concept of gradient for the girls, since they were active listeners.

Concepts were continually revisited through activities designed to further the exploration of features of graphs. Both Rhonda and Narelle became more confident with their explorations, as was evident from the conversations heard on the recordings. For example, with an introductory activity (Lesson 4) that brainstormed features of two graphs drawn by the teacher, Rhonda provided distinct responses as part of open class discussion, stating: “the gradient”, “one graph is steeper than the other”, and “y-intercept”, which in terms of the SOLO model would be considered M_1 level responses, since they demonstrated a heavy reliance on the visual cues provided by the graph. This Phase 2 activity of exploring the graphs progressed into Phase 3 with the development of terminology specific to Linear Relationships. While visual cues assisted to provide stimulus for Rhonda’s responses, her terminology indicated a developing understanding of the concepts involved. She continued her explorations using GeoGebra to assist in the drawing of graphs, whose equations were provided on a worksheet. During this activity, which again spirals between Phases 2 and 3, conversations with her partner, Narelle, demonstrated that Rhonda was able to further classify the gradient of lines as being negative or positive through the visual appearance provided by GeoGebra when graphing the lines. This was reflective of an M_1 response because multiple features were identified through the visual presentation of the lines with no evident connection between features.

Having seen the y-intercept and gradient from a visual perspective (Lesson 6), Rhonda and Narelle then concentrated their focus on the equation of the line. However, when calculating the equation of a line, issues with the presentation of the equation in the algebra window of GeoGebra caused confusion. The activity commenced with the two points $(0,3)$ and $(4,6)$, that Rhonda and Narelle entered into GeoGebra. In this situation, GeoGebra was used as a supportive tool to assist with drawing the line connecting the two points and subsequently presented its equation in the algebra window of the software, consistent with teaching Phase 2, Directed Orientation. Upon creating the line, the default form for the equation was presented in GeoGebra’s algebra window, as shown in Figure 7.1; the confusion for Rhonda being that the presentation of the equation was not in the expected gradient form of $y = mx + b$. Despite having the graph in the geometry window to refer to for visual cues, when asked to identify the y-intercept, Rhonda responded with “four”. When other students responded with “three”, she repeated her answer twice, somewhat puzzled as she waited for some form of justification, which did not eventuate as her responses were not heard by the teacher. Rhonda’s response demonstrated a concrete symbolic U_2 response, as she concentrated on the y term from the equation (provided in the algebra window) rather than the graph in the geometry window. This blind acceptance of what GeoGebra provided indicative of what Geiger (2009) classifies as *Technology as Master* in his MSPE framework. Rhonda was obviously unaware that the presentation of the equation was in a different form, general form, the default format for GeoGebra. The *y*-intercept can be clearly seen in Figure 7.1 as 3. It was interesting that Rhonda did not look at the graph for visual cues, as with the introductory activities that explored graphs from a visual perspective looking for features, in earlier lessons.

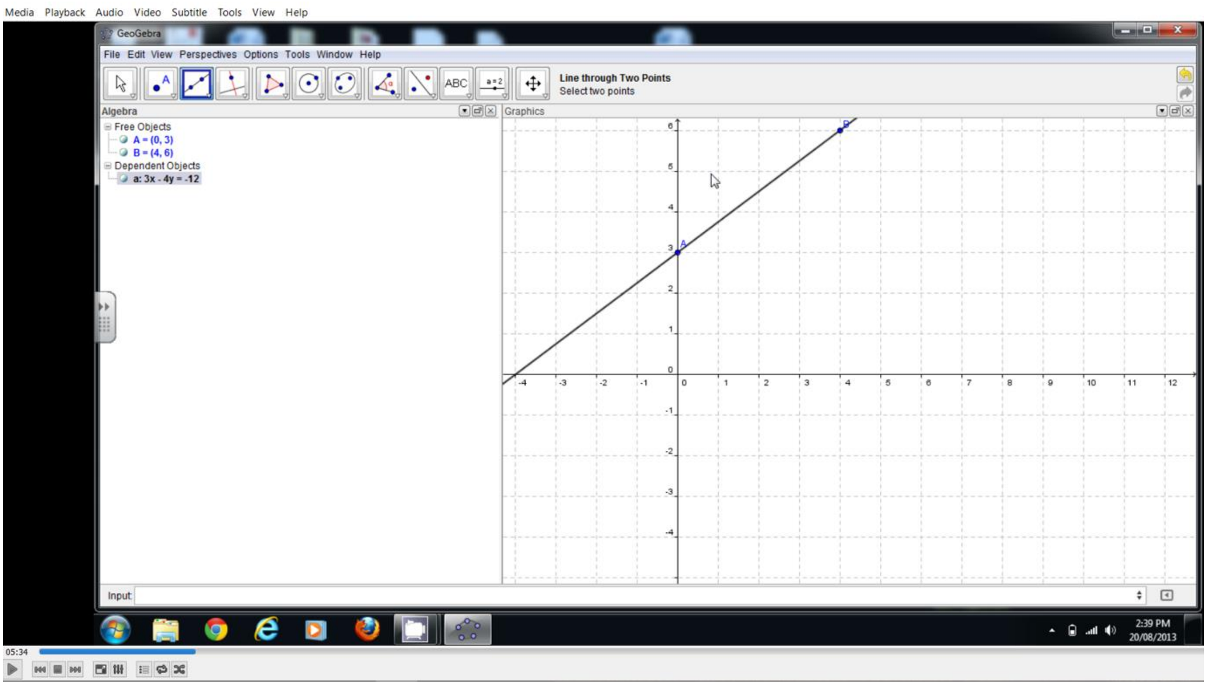


Figure 7.1: Rhonda and Narelle finding the *y*-intercept

Next, the teacher modelled how to calculate the equation of the line, with the same coordinates using pen and paper techniques and without any support from GeoGebra. Together, in open class discussion, students were able to suggest features, strategies and methods that would assist in determining the equation of the line. Language was monitored, as the teacher facilitated the conversation rather than directed it, and the activity spiralled between teaching Phases 2 and 3 as conversations progressed. Rhonda was heard to state that the *y*-intercept was required, which could be found “on the *y*-axis”, along with the gradient which could be found “using rise over run”. She continued that the rise was “3 minus 6”; this algorithm indicated the use of the correct elements of the coordinates, in incorrect order, the solution of which would give a negative value as a solution. After calculating the gradient as ¾, the teacher continued to probe students in order to establish the relationship between concepts, *y*-intercept and gradient, and equation.

> Teacher: If I know that the *y*-intercept is 3 and I know my slope is ¾ … I can then say that the equation is *y* equals something *x* plus something, … the number in front of the *x* is always the … ?

Rhonda: Gradient.

Teacher: And the number by itself at the back is the ….?

Rhonda: *y*-intercept.

These responses demonstrated that Rhonda understood the relationship between the variables of the equation of a line, $y = mx + b$, and the concepts represented by those variables. In terms of the SOLO model, her responses represented a R_2 response.

The concepts of gradient, y-intercept, drawing the graphs of lines and determining the equation of lines are revisited throughout the teaching sequence. Activities were selected such that the language associated with the concepts was developed and consolidated with the cycle of phases extending to Phase 4, Free Orientation, where students were able to find their own way towards solving problems. Commencing firstly with Phase 3 (Lesson 7), class conversation occurred which explored the gradient concept further as students were guided through a process of substitution. Calculating the gradient using rise over run, then substituting the values from the coordinates lead to the development of a general formula for the gradient in terms of coordinates (x_1, y_1) and (x_2, y_2). The progression to deriving the formula was considered as Teaching Phase 4, Free Orientation.

Deriving the formula from first principles was not a requirement for this topic, however, it was determined that it would assist the students understanding of how the formula evolved. Interestingly, the teacher stated that the derivation of formulas from first principles was generally reserved for Stage 6 (approximately 16–18 years old) as from his experience it was considered too difficult to comprehend in Stages 4 and 5 (approximately 12–16 years old). Rhonda and Narelle, along with the rest of the class, through the choice of appropriate activities were able to successfully derive the gradient formula from first principles; this dispelled the myth previously considered by the teacher. Following this, activities, consistent with Teaching Phase 3, were attempted that consolidated the use of the formula.

Rhonda and Narelle discussed the use of the formula and the substitution of values while continuing to use GeoGebra as a tool to assist with the checking of solutions obtained. As a concluding activity (Lesson 7) for the concept of gradient, designed as a progression to Phase 4, Free Orientation, the following problem was posed: If the slope of a line is 4 and the *y*-intercept is -2 what does the line look like? While class discussion established that the starting point was -2 on the y-axis, it was Rhonda who suggested: "to go one across and 4 up" this demonstrated the practical sense of the gradient concept and how it could be used to determine points on a graph given a specific starting point, signifying a rethinking of the application of the gradient by Rhonda. This R_2 response combined the value of the slope with the concept of rise over run and explained it in graphical terms, revealing that her understanding of the concept of gradient had been strengthened through the choice of activities. This was also evident in Rhonda's responses for the End of Topic and Delayed Post-test questions on the gradient concept. In both cases her responses were of the type R_2, this demonstrated relationships between the concept and elements required to calculate the concept were well developed, with both responses stating the formula: "G = y2-y1/x2-x1" [sic].

The concept that was addressed next was "midpoint" (Lesson 8). This was introduced by the teacher through a fictional story that incorporated vocabulary associated with midpoint, such as middle, centre, average, dissect and bisect, to develop understanding of terminology within context, consistent with Teaching Phase 1 of providing information. A practical activity followed, consistent with Teaching Phase 2, Directed Orientation, which investigated the middle of two given points using three different folding strategies. GeoGebra was used as a checking tool for students to compare their solutions from the folding activity, indicative of what Geiger (2009) claims as the second category of his MSPE framework, *Technology as Servant*. Through these activities and open class discussion, the midpoint concept was clearly defined, this demonstrated a progression to Teaching Phase 3 and, in a similar method to the gradient concept, a general formula for midpoint was derived, indicative of Teaching Phase 4, Free Orientation. The midpoint concept was easily visualised on graphs and using GeoGebra, hence the general formula was easily achieved. With a specific midpoint tool available in GeoGebra, students were easily able to check solutions. Rhonda's work, as shown in Figure 7.2, demonstrated how questions were attempted that consolidated the midpoint concept: activities consistent with Teaching Phase 3, Explicitation. As can be seen, Rhonda continually labelled points to ensure she substituted correctly into the formula, with diagrams that can be compared using GeoGebra.

Activities were used that extended students' understanding of midpoint, indicative of a progression from Phase 3 to Phase 4, Free Orientation, as shown in Question 2 from Figure 7.2 (Lesson 9). Rhonda's initial working used visual strategies that included drawing the given points on a Cartesian plane, in order to gain understanding of the solution through visual cues. Just visible on the edge of the Cartesian Plane in Figure 7.2, was a point marked that could represent the solution to the problem. In Figure 7.3, Rhonda used an algebraic approach combined with the midpoint formula to calculate the endpoint, which was previously found visually. She attempted another similar problem underneath, this time without drawing a Cartesian plane, representative of the progression to Phase 4. Following this, a problem involving properties of geometrical figures was explored; again the activity was targeted at Phase 3 with the possible progression to Phase 4. While GeoGebra was used to visualise the shape, unfortunately Rhonda calculated the midpoint of BC instead of BD. Had she found the correct midpoint she would have discovered that the diagonals crossed at the same point, also indicating that the diagonals were bisected for the quadrilateral drawn, namely, a parallelogram. Such problems provided a context that enabled students to develop an appreciation for the concept in terms of its application to another topic of mathematics, namely Geometry.

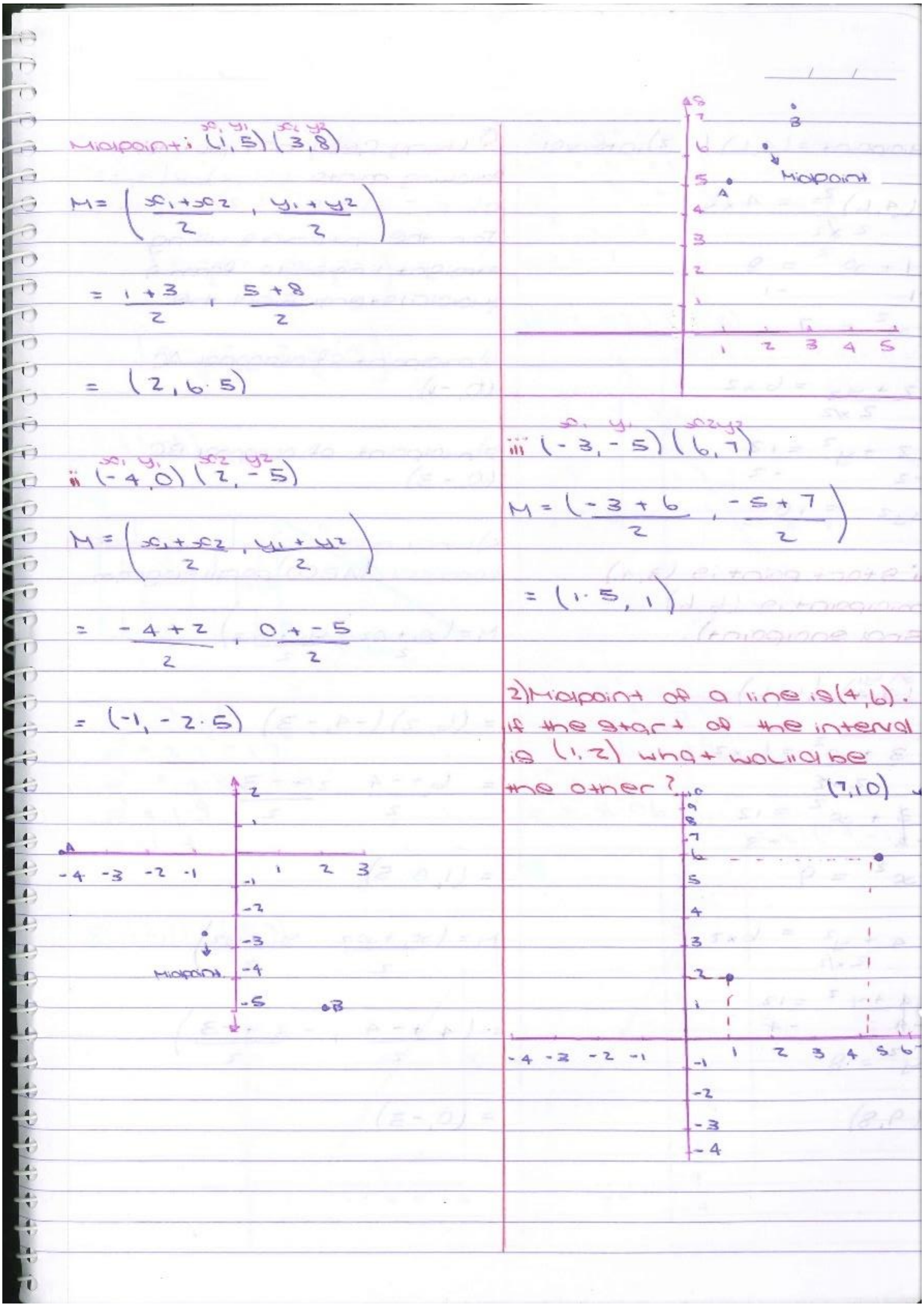


Figure 7.2: Rhonda workbook sample for midpoint

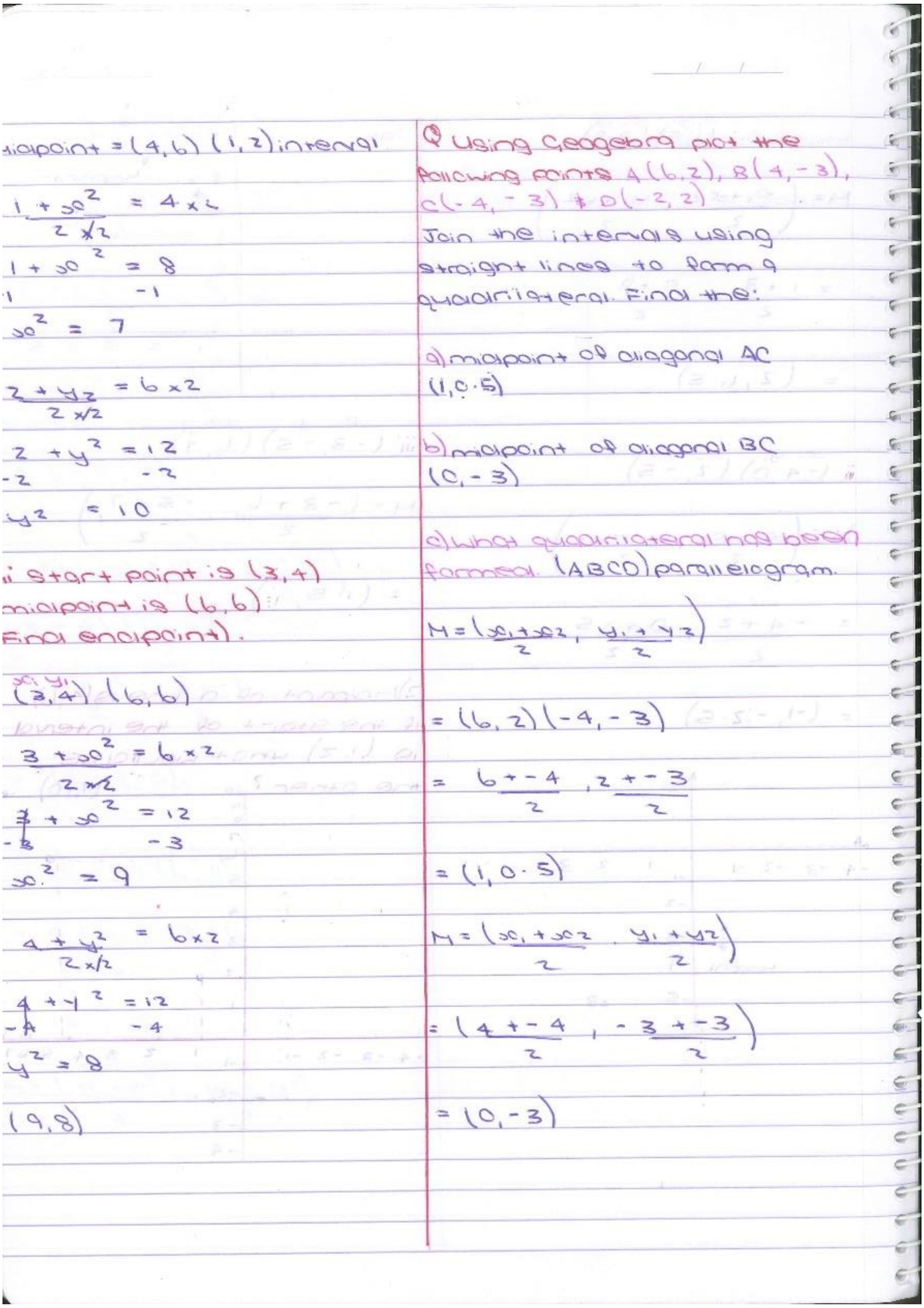


Figure 7.3: Rhonda workbook sample for midpoint problem solving

The concept of distance was approached in a similar manner as the previous concepts, through exploration and investigation using GeoGebra (Lesson 10). Commencing with a Phase 2 activity that involved the construction of a right-angled triangle, using the given points, class discussion established that the familiar stage 4 concept of Pythagoras' Theorem would be a suitable method that could be used to calculate the distance between the two points, with the hypotenuse of the triangle representing the required length. With the identification and application of the Pythagoras' Theorem, the activity progressed into Phase 3. Consolidating this idea with the points (2,5) and (6,8), Rhonda identified that the difference between 8 and 5 would find the vertical length of the triangle and the difference between 6 and the 2 calculated the horizontal length, exploration consistent with a Phase 2 activity. Recognising which element of the coordinates was required to establish the length of the sides demonstrated that Rhonda comprehended the information of both coordinates, with the assistance of visual cues, indicative of a R_1 type response of the SOLO model. Her understanding continued to develop and was demonstrated by the responses she provided when deriving the distance formula. Accustomed to the idea of deriving the formula with the labelling of coordinates as (x_1, y_1) and (x_2, y_2), from the previous two concepts Rhonda engaged in class discussion that lead to the formation of a general equation for the distance between two points.

> Rhonda: Is it … do you have to takeaway from $x_2 - x_1$.
>
> Teacher: Good, so which one will that give me the 4 or the 3.
>
> Rhonda: The 4.
>
> Teacher: So what about the other length?
>
> Rhonda: $y_2 - y_1$
>
> Teacher: Ok … so now we know how to find those distance but we need to get to the answer … so this finds us the 3 and this finds us the 4 …
>
> Other student: Three squared and 4 squared.
>
> Teacher: Well we don't want to use 3 squared and 4 squared … do we? … we want to come up with a general formula.
>
> Rhonda: Is it do you square the …
>
> Teacher: Ok tell me, start me off.
>
> Rhonda: So would it be distance equals $x_2 - x_1$ bracket 2.

Conversation continues and Pythagoras' Theorem is completed in general terms. Probing by the teacher continued and Rhonda again leads the conversation,

> Teacher: What do you do at the end of it to actually find the distance.
>
> Rhonda: Square root.

The process of substituting coordinate values into Pythagoras' Theorem, then replacing those values for more generalised coordinates, (x_1, y_1) and (x_2, y_2), progressed the activity to Phase 4. In terms of the SOLO model, Rhonda's responses for the entire conversation were indicative of a type R_2 response; this demonstrated her understanding of converting coordinates into generalised algebra that could then be substituted into Pythagoras' Theorem to construct a formula for distance. Figure 7.4 demonstrates the board work modelled by the teacher during the activity.

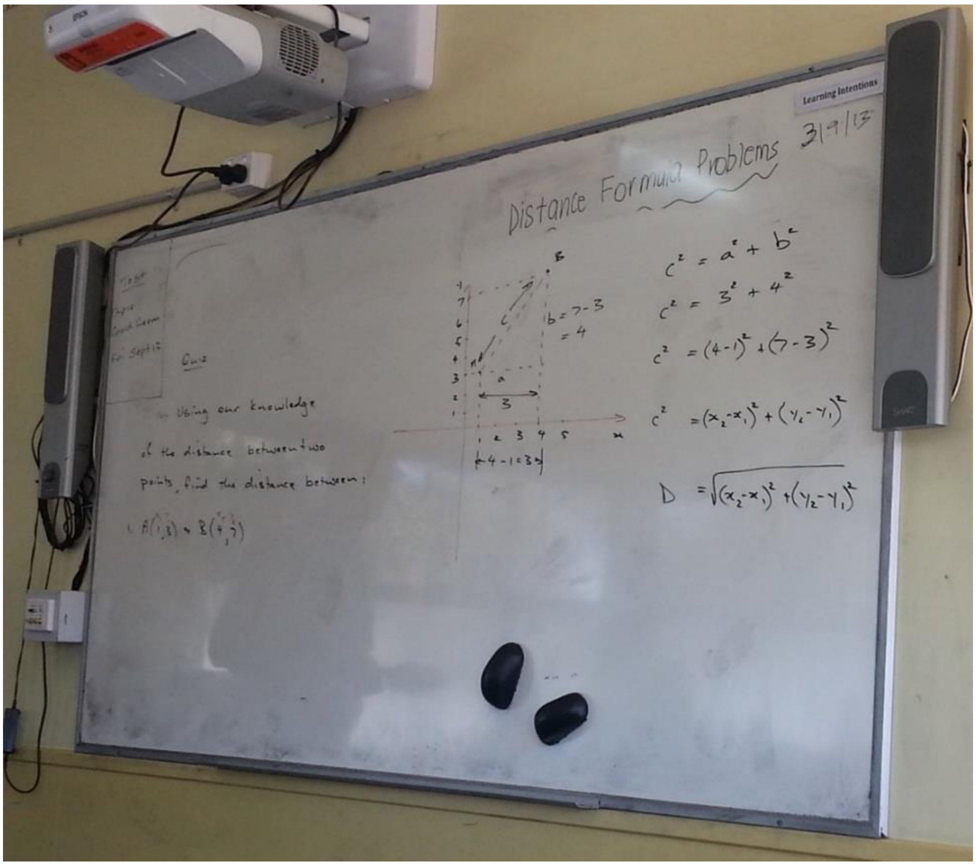


Figure 7.4: Teacher working for introductory activity

The GeoGebra environment lends itself to the creation of many different types of activities. These can be aligned with the transformation phase of the SAMR model (Puentedura, 2010). One such activity constructed within the GeoGebra environment presented with real-life problems that involved the concepts of distance and midpoint (Lesson 11). In Figure 7.5 the screenshot demonstrates the map and coordinates as were presented in GeoGebra. Exploration of the map, and the tools provided by GeoGebra fostered the Phase 2 activity, which progressed into Phase 3 as the problems increased in complexity. This reflected *Technology as Partner*, from the MSPE framework (Geiger 2009). In particular, when the problem was reversed, where the answer was given and the question required to be found, this progressed into Phase 4, because students were required to rethink the application of formulae or concepts. One particular problem asked students to find the endpoint given the midpoint obstacle: Flying Fox $(18{,}7)$ and another obstacle Rope Climb $(21{,}5)$. Problems similar to this had been previously attempted when the concept of midpoint was explored. Rhonda was able to recall the method used to calculate the endpoint and tried to explain this to Narelle, who was quite confused. Part of the conversation between the girls is presented below and demonstrates how Rhonda articulated her thoughts in an attempt to explain it to Narelle.

Rhonda: $x_2 + 21$ over $2 = 18$ and then you do the other one.

Rhonda: So x_2 is 15 and then we have to do with the other one.

Narelle: I am so confused … that's 36 then you take away, you takeaway 21 from both sides … yeah takeaway 21 from here.

Rhonda: Yep and then you get 15, x_2 equals 15.

Narelle: So x_2 equals 15.

Rhonda: Then you check and for the other one $y_2 + 5 = 14$ then you minus 5, y equals 9 … yep I got it … do you get it?

Narelle: Yep.

Rhonda: You gotta times 2 is 14.

Narelle: Oh yeah.

Rhonda: Because …

Narelle: So the coordinates are …

Rhonda: $(15, 9)$

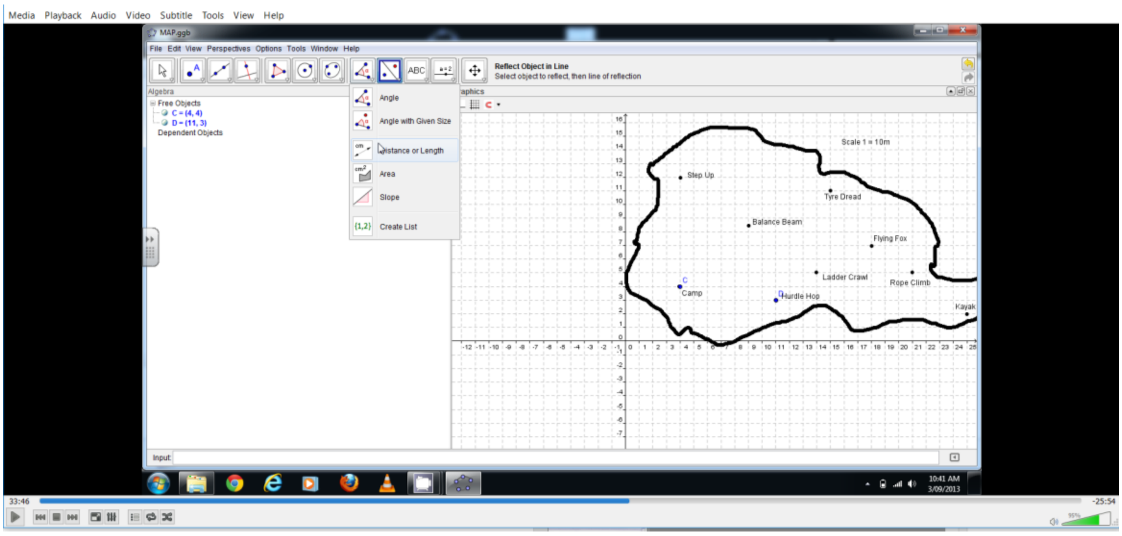


Figure 7.5: Map activity on GeoGebra demonstrating distance tool

Rhonda's responses revealed an understanding of the sequential operations required to manipulate the midpoint equation in order to solve the problem. Her responses were indicative of type R_2 as she realised what was required and supported Narelle through the individual steps to the solution. For Rhonda, being able to explain the process strengthened her own understanding of the problem and concept of midpoint. GeoGebra was also available as a checking tool with which to confirm the solution was correct. This demonstrated *Technology as Servant*, of the MSPE framework (Geiger, 2009), where the student controls the technology for the benefit of efficiently checking answers.

Rhonda, in particular, was result driven and wanted to achieve high results so she would be able to study Advanced Mathematics in her senior secondary years. An interesting conversation occurred between Rhonda and Narelle while consolidating the concept of distance:

> Rhonda: Do we have to do this in the test?
>
> Narelle: I don't know because that's two different ways of doing it.
>
> Rhonda: Yeah.
>
> Narelle: It isn't hard but it takes a long time just to get this one …
>
> Rhonda: I like this one better than … I don't know.
>
> Narelle: I like this one.

Unfortunately, the recording does not specify which particular method they were referring to when they said which one they like: the formula or Pythagoras' Theorem. Rhonda confirmed with the teacher if indeed they had learnt two methods for the concept of distance. She then appeared confused with the idea that multi-path solutions were possible, demonstrating discussion consistent with activities aligned with Phase 4, Free Orientation. Her confusion reflects previous learning experiences when she used a specific method to find a concept. Although, the teacher reassured her that, if asked to find the distance, she could use whichever method, Pythagoras' Theorem or the Distance formula, to calculate the distance between two points, she sounded annoyed. It was evident to the teacher and researcher that she would rather be told to use a particular method knowing that would calculate the correct solution rather than learning multiple methods for the sake of developing understanding of the concept.

It was interesting to note that despite the high level of responses demonstrated by Rhonda during the teaching sequence for the distance concept, retention of the formula was not maintained for the Delayed Post-test. Rhonda correctly submitted the formula for the distance in the End of Topic test when she stated: "We use the formula: D = square root of (x2 - x1) squared + (y2-y1) squared (-1,4) represent : -1 = x1 4 = y1 (3,6) represent : 3 = x2 6 = y2" [sic]; however in the Delayed Post-test the coordinates were misplaced: "We use the formula: D = square root of (x1-y1) squared + (x2 -y2) squared. We use the two coordinates and label them as x1, y1, x2 and y2".

Next, an activity that reviewed the properties of graphs was revisited, a Phase 2 activity (Lesson 12). Responses provided by Rhonda and Narelle during this exercise demonstrated that retention of a majority of the content from the beginning of the unit was maintained. While it was presented as a whole class open discussion, Narelle was heard in the recordings to easily identify the *y*-intercept in the graph, and both Rhonda and Narelle recognised the gradient of the line as being positive and represented by the letter m. Through further discussion it was evident that neither Rhonda nor Narelle were able to remember how to calculate the gradient. Rhonda responded with the suggestion of using "$y = mx + b$". In terms of the SOLO model, this was indicative of a R_1 response since she had an educated guess based on the information provided but was unable to provide the correct relationship towards how to find the gradient itself. Rhonda was able to continue her line of thinking when she clarified that the *m* was the gradient and *b* was the *y*-intercept, the use of language progressed the activity to Phase 3. When probed further by the teacher, and using the information provided by other students, she was able to state to the class that the equation of the given line was "y $equals$ $1x$ $plus$ 1". Despite having issues and requiring assistance to calculate the gradient, in terms of the SOLO model, her responses were consistent with R_2 because she was able to see the relationship between the concepts and unite them to form an equation that correctly identified the line.

As an overview of the unit, a matching activity that included a set of cards where each line had a separate card with an equation, a graph, a table of values, a description of the rule in real-life context and the gradient with a point or y-intercept to match up was completed (Lesson 14). The sheets with individual card setup for this activity can be found in Appendix O. Students worked in pairs, discussing various concepts that would link cards and once cards were deemed to be the same line, they were piled together. Rhonda and Narelle found the activity enjoyable at first but, with the huge number of cards to categorise, as shown in Figure 7.6, it became difficult for them to sort through and they found it challenging and, at times, overwhelming. This activity represented Phase 2, Directed Orientation, but progressed to Phase 3, as language developed and students found relationships between cards. It completed the teaching sequence for all students and was followed by the End of Topic Google Form test and subsequently Delayed Post-test.

Figure 7.6: Matching Activity

It was interesting that for two of the concepts explored within the tests, the complexity of the responses improved between the End of Topic and Delayed Post-test. This was fascinating, considering no further teaching intervention for Linear Relationships occurred during this time frame. Responses were all considered concrete symbolic second cycle and demonstrated that while procedural knowledge was evident in the responses provided for the End of Topic test, responses for the Delayed Post-test displayed a deeper and more developed understanding of the procedures involved.

The first concept where the difference was noted was with the drawing of a graph without the assistance of GeoGebra; that is, by using pen and paper techniques. For the End of Topic test, Rhonda was able to correctly detail the gradient form of an equation of a line and identified the separate elements and what each meant, as can be seen in the green section of Figure 7.7. No relationship was established as to how this information would assist in drawing the graph, and each statement was a separate identity, hence, in terms of the SOLO model, demonstrated a M_2 type of response. Later, for the Delayed Post-test, Rhonda's response provided more detail and related the specific elements to the line given in the question. She suggested how to draw the graph, incorporating the starting point as the *y*-intercept then articulated sequential steps that found the next point. The response, in terms of the SOLO model, was coded as type R_2, since it explained how the *y*-intercept and gradient concept could be used together to draw the graph, hence a relationship between the concepts and graph was established. This response is shown in purple in Figure 7.7.

a. How would you do this without Geogebra using pen and paper	**a. How would you do this without Geogebra using pen and paper**
y = mx + b m = gradient b = y - intercept To find m use: rise over run To find y use: the interval will pass through a point on the y axis.	Formula = rise/run Put a dot on the 8 at the y-intercept Then we use the formula rise over run to find the gradient start from 8 rise 4 up and run 1 right

Figure 7.7: Question 6a End of Topic test and Delayed Post-test response for Rhonda

Another instance where the response improved between End of Topic and Delayed Post-testing occurred when Rhonda described what happened when a graph was moved within the GeoGebra environment. For the End of Topic test, her response was: “The gradient and y intercept changes while moving the graph”. While she was able to express that change occurred to the two concepts, no further explanation was provided as to how the change presented in the graph or how the changes related to each other. In terms of the SOLO model, this was coded as a M_1 response. The complexity of the response improved, for the Delayed Post-test, when she stated: “I notice that when I move the line the gradient stays the same but the y intercept changes and it is still a positive slope”. This response was more specific in its explanation of the changes that occurred and, in terms of the SOLO model, was coded as a R_2 response.

The third occasion where her response improved over time arose with the identification of perpendicular lines. For both the End of Topic test and Delayed Post-test, Rhonda identified that the lines were perpendicular from the equations provided in the question. However, for the Delayed Post-test Rhonda expanded her response further by adding: “the reciprocal changes”. This demonstrated a more developed understanding of terminology and presentation of gradient associated with perpendicular lines and subsequently upgraded her initial response, from U_2, to a M_2 response, signifying the increased complexity observed.

At the completion of the unit, after the End of Topic test, students were asked four questions to reflect and comment on. The four questions along with Rhonda’s comments are provided in Table 7.3.

Table 7.3: Rhonda’s evaluation responses

How would you rate this topic?	Did you think it was delivered better than you normally learn maths?	What would you like to see continued?	What would you change?
8.5/10	Using technology is a good thing but whenever there is a test it makes it difficult because we can’t check the answer. I like remembering formulas and using papers to work out	Practise more in books	I would use books more than technology

As mentioned, although Rhonda liked the delivery of the content she would have preferred more examples and consolidation exercises in her workbook because, in her mind, this better prepared her for examinations. Rhonda’s workbook was neatly presented, as has been previously shown, and she used it as a study tool to assist her preparation for summative tasks.

7.3. Conclusion

This chapter described the educational journey of Team Brown during the teaching sequence for Linear Relationships. The changes in conceptual understandings, viewed through the SOLO model as an analytical lens, are presented using multiple artefacts including screen shots, photos, workbook samples and transcriptions obtained from recordings. Activities that shaped understanding, aligned with the van Hiele Teaching Phases, are considered in the light of their respective responses that provide a window to view their development of concepts related to Linear Relationships.

In particular, the three research questions posed at the beginning of the chapter were able to be addressed:

Research Question 1.1*:*

How do the van Hiele Teaching Phases offer a framework to explain the categories of responses concerning students' understandings of Linear Relationships?

The van Hiele Teaching Phases offers a framework that supports the inclusion of technology as a tool to improve learning. Through the selection of targeted student activities that address specific phases, the use of technology as a teaching tool is enhanced. This utilisation facilitates conceptual development through explorations and investigations.

The Teaching Phases allow opportunities for integrating technology in ways that enable students to develop a deeper understanding of the Linear Relationships concepts, as evidenced in the use of more formal language. In particular, the spiralling of phases fosters the consolidation of Linear Relationships concepts and the strengthening of language involved with each phase prior to progressing to the next phase. The improvement of language reflects an increase in conceptual understanding, as students are able to articulate the requirements necessary to complete more complex tasks by finding solutions to non-routine problems in familiar situations.

Sequencing lessons with the van Hiele Teaching Phases ensured that the instructional activities and experiences employed facilitated students' cognitive development for Linear Relationships. Although students spiralled through Phases 2 and 3 while their understanding of a concept developed, phases were not skipped and students passed through each phase at some stage throughout the teaching sequence. This is a critical component of the teaching sequence, since it is a common practice for teachers to skip Phase 4, Free Orientation, as it is often difficult to facilitate.

During the teaching sequence implemented, students completed tasks designed to incorporate the technology as a tool to support learning rather than as a tool that instantly provides an answer. Through the spiralling of Phase 2 and 3, students were able to take ownership of their mathematical ideas, leading them to discover relationships that enabled a progression to Phase 4. This was evident with the development of generalised formulas for concepts derived from first principles.

Progression to Teaching Phase 4 offered the opportunity for students to develop insight. Insight is an invaluable ability that van Hiele considers to be the main purpose of instruction (van Hiele, 1986). Phase 4 activities, in the teaching sequence, required students to find their own way, being already familiar with the Linear Relationships content, and exploring further, combining all the information to solve the problems. These problems were "not simply 'hard' questions they are questions in which multi-path solutions are possible" (Pegg, 1995, p. 99).

Research Question 3.1 states:

What are the characteristics of students' responses when exploring concepts of Linear Relationships using dynamic mathematics software?

Students responses when exploring concepts of Linear Relationships using GeoGebra generally improved in complexity and quality. This was evident from the Google Form test responses along with the responses provided during class discussions. As their understanding developed they were able to articulate concepts and link information more appropriately. Initially, responses focussed on individual elements associated to Linear Relationships concepts. These elements were evident from visual cues that were easily recognisable with GeoGebra. Through investigations using GeoGebra, a gradual improvement in responses was observed. Further explorations enabled students to recognise the relationships between elements, defining concepts in such a way that the visual aspect was not a necessity. These relationships assisted with calculation of the concepts, which developed into individual formulas for concepts. Upon consolidation of the formula and its application, it was further developed to offer a generalised formula derived from first principles.

Research Question 3.2:

What is the nature of student interaction when using GeoGebra as an exploration tool?

Using GeoGebra as an exploration tool promoted discussions between students as they investigated features of GeoGebra that would assist them when checking the solutions to problems. GeoGebra facilitated the exploration of Linear Relationships such that students were learning without the intention to learn, using the visual capabilities of the software to explore features and properties of points, lines and shapes to assist with the development of conceptual understanding. This removed the need for repetitive procedural knowledge from activities as GeoGebra enabled students to become more involved with the investigations and explorations involved with the activities.

In particular, it was observed that conversations between students regarding how to find concepts contributed to the development of language that was specific to Linear Relationships. As phases progressed, more elements became associated to specific terms, such that there was an increased growth in the language used when solving problems. Students were able to define concepts in more depth using a number of elements related to Linear Relationships. This growth and development of language was assisted by GeoGebra because students could use the dynamic software to support their problem-solving strategies and not only work towards finding a solution, but also work the solution to see if it presented them with the same question.

The nature of student interactions provided an avenue for the teacher to assess student learning as a seamless component of the teaching and learning sequence. This is often described as Assessment *for* Learning (AFL). Classroom conversations and observations enabled the teacher to assess individual students level of understanding of Linear Relationships concepts. Through prompting and probing techniques, students were required to explain or justify aspects of their working or thinking. This enabled the teacher to gain a deeper awareness into the developmental understanding of concepts and assess if students were developing insight, an invaluable ability often overlooked in today’s classrooms. It also offered an opportunity for the teacher to provide instant feedback to assist and improve student understanding.

Research Question 3.3:

What are the observed developmental hurdles and technical knowledge issues encountered by students when exploring Linear Relationships concepts utilising dynamic mathematics software?

A few viewing limitations were observed with GeoGebra that presented unexpected results for students. Similar to problems identified with graphics calculators by Mitchelmore & Cavanagh (2000) and Kemp et al. (1996), the scaling of graphs in the view window of GeoGebra caused confusion for students. There were a number of occasions where the geometry view window needed to be expanded, or axes needed to be rescaled in order to visualise the features of the graph. This process required manipulation of the settings and became frustrating for many students.

Another technical contradiction observed was with the presentation of the equation of lines within the GeoGebra environment. By default, GeoGebra presented equations in general form. The presentation in general form distracted students, resulting in misinterpretations of what was visualised, which is another common issue for technology also identified by Cavanagh (2005).

The algebra window of GeoGebra was found not to display fractions; this provided another confusing limitation for students. This was noticed when calculating the gradient of lines and with the equation of lines when converted into gradient form. It does explain why the default presentation of equations of lines for GeoGebra mentioned previously was general form.

The final conflict was observed to cause problems for students was with the implementation of the concept tools offered by GeoGebra. GeoGebra assists users by providing a short explanation of how to find a concept when selecting the concept tool. In many cases, this explanation was overlooked, leaving the students unaware of how to use the tool appropriately such that it would provide an answer.

Chapter 8 presents the overall findings of the research study, the study limitations and will consider the implications of these findings in relation to the practice of teaching, the van Hiele Theory and the SOLO model, and future research.

CHAPTER 8: CONCLUSIONS

In the final chapter of this thesis, the overall findings of this study, which investigated students' understanding of Linear Relationships when using the dynamic software GeoGebra, are considered. The chapter is divided into five main sections. Firstly, the limitations imposed by the study are discussed. Then, an overview is provided of the main findings with respect to the three research themes addressed in Chapters 5 to 7. This is followed by a consideration of the implications of the findings in relation to the van Hiele Teaching Phases, the SOLO model and for the practice of teaching. Finally, recommendations for possible future research are presented, followed by a conclusion that completes the chapter.

8.1. Limitations

It is important that the previous results be viewed in light of possible limitations imposed by the study. These potential weaknesses of design and methodology impact the interpretation of the findings and are constraints on the generalisability of the study. This section reviews four aspects of the limitations including: the number of students in the sample, difficulties when investigating a single class that ran parallel with another class of the same stage, time frame of the study and the case study approach.

The first possible concern relevant to the limitations of this study was related to the sample chosen. The small number of participants from one class limited the amount of data available. As a consequence, the attendance rate during the research period was important, as it also affected students' responses. Although many concepts were revisited throughout the teaching sequence, students who were absent missed important discussions that could not be replicated because of the nature of the conversations being student-centered and not teacher driven.

The second possible concern significant to the limitations of this study was the unforeseen difficulties associated with organising multiple classes of the same stage. In particular, these difficulties related to the time allocated for the Linear Relationships unit. While the sample class focussed on implementing the teaching sequence with the spiralling of Teaching Phases including the addition of activities that supported students' understanding of Linear Relationships concepts using GeoGebra; the other class worked through the textbook, with direct instruction of concepts. As a consequence, the other class covered the content in the time specified by the Scope and Sequence, whereas extra time was required for the sample class such that understandings developed. With a formal summative school assessment scheduled for the end of the Linear Relationships unit, pressure mounted to finish the unit in order to complete the assessment to subsequently finalise marks. This is a realistic concern for many teachers that the structure of documents, such as the Scope and Sequence, or pressure from senior teachers dictates instructional techniques. It is also why teachers ultimately resort to chalk and talk and heavy reliance on a single textbook as teaching strategies.

The third possible concern relevant to the limitations of this study was the time frame of the study. An extended time-frame that tested more classes would have increased the data available. Subsequently, the developmental pathway of students' understanding of Linear Relationships would have been even richer in detail. Unfortunately, this was not practical for the researcher at the time.

The final possible concern relevant to the limitations of this study related to the case study approach used to detail Rhonda and Narelle's journey. This case study was not aimed to investigate features of their case in order to produce generalised results. The particular case of Rhonda and Narelle was investigated as an intrinsic case study since it provided interest concerning their educational journey in its own right. Reliability and validity were maintained through the triangulation of multiple data collection, including numerous hours of video and audio footage, workbook samples and the three Google Form tests.

Overall, despite the possible limitations imposed on the study by the nature of the research design, this discussion demonstrates that the effects of these factors were considered carefully during the design phase. The design of this study allowed for the collection of detailed qualitative data concerning students' understanding of Linear Relationships concepts when utilising GeoGebra as an explorative tool.

8.2. Main Findings

The focus of this thesis was to explore students' utilisation of technology when learning Linear Relationships and investigate what skills enhanced their understanding of Linear Relationships when using technology. This qualitative study was designed using the van Hiele Teaching Phases and SOLO model as frameworks to support the research themes. Activities during the teaching sequence were successful catalysts that initiated detailed discussion regarding students' understandings of Linear Relationships concepts. As a result, a range of categorised responses were identified and described that provide a window with which to assess and view students' understandings.

Consequently, there are 11 major findings originating from this study:

1. A developmental pathway leading to an understanding of Linear Relationships was identified. This pathway characterises an understanding of the Linear Relationships concepts in terms of the SOLO levels, which have not been considered previously in such depth.
2. It has been established by this study that the van Hiele Teaching phases was an effective design framework with which to sequence lessons. Targeting activities that were sequenced to the Teaching Phases facilitated students' cognitive development for Linear Relationships concepts.
3. This study validated the importance of language within the van Hiele Teaching Phases. Within all levels of the Teaching Phases, student-centred activities provided a catalyst for discussion, which enabled teachers to monitor the progression from the use of informal to formal technical language. This development assisted with the assessment of students' understanding of Linear Relationships concepts.
4. It has been established by this study that the developmental pathway that leads to the derivation of formulas from first principles is easier than generally perceived by teachers. With exploration and teacher guidance, students were able to draw on familiar situations investigated using GeoGebra to determine formulas that were generalised for use by pen and paper techniques without any visual cues required.
5. It has been established that the formulas for concepts of Linear Relationships should not be considered the main learning outcome, rather that learning leads to the outcome of the development of a formula for the concept. Formulas for Linear Relationships concepts are the final result obtained from explorations and investigations, not the starting point as demonstrated by many mathematical texts and pedagogical practices.
6. This study supports the importance of the development of insight for students, as suggested by van Hiele (1986). The aim for all teachers is to impart knowledge to students for their benefit in the future. By this definition, it is important for mathematical students to be able to apply Linear Relationships concepts appropriately to new and unfamiliar problems to those previously encountered.

7. This study supports the SOLO model as being an efficacious framework for assisting in the interpretation of data (Panizzon & Pegg, 2008; Sriraman & English, 2009). The SOLO model offers a rich interpretation of students' developmental growth. Using the SOLO categorisations, a developmental pathway of students' understandings of a particular topic can be determined that can be used as a pedagogical tool for teachers when preparing lessons.
8. Through the use of GeoGebra as a pedagogical tool, this study established that the understanding of Linear Relationships was enhanced. The visualisation capabilities and dynamic changeability that GeoGebra provides enabled students to focus on the conceptual understanding of Linear Relationships rather than procedural knowledge. This supports and adds to previous research that showed that GeoGebra improved students' understandings of Fractions (Thambi & Eu, 2013), Trigonometry (Zengin, et al., 2012), Coordinate Geometry (Saha, et al., 2010) and Functions (Gómez-Chacón & Prieto, 2011; Hohenwarter, 2006).
9. It was established that using technology as a supportive tool, students were learning without the intention to learn. Technology facilitated explorations and investigations that stimulated interest in concepts and problems, such that students were intrigued with what the technology could deliver.
10. The results obtained in this study found students to be more engaged and motivated through the use of technology embedded into the teaching sequence. This supports previous research by Bate et al (2013), Raines & Clark (2011), Kissane (2008) and Bobis, et al. (2011), amongst others.
11. It was established that exploring with technology promoted familiarity that assisted students to use GeoGebra as a tool for learning mathematics. It intrigued them as they attempted to investigate different properties of a graph that could not be achieved using pen and paper.

8.3. Implications for Theoretical Frameworks

Two frameworks underpinned this study; namely, the van Hiele Teaching Phases and the SOLO model. These frameworks and their implications to the study are discussed below.

8.3.1. The van Hiele Teaching Phases

The van Hiele Teaching Phases provided a pedagogical framework that assisted with sequencing student activities and, subsequently, lessons for this study. They supported the incorporation of the dynamic software, GeoGebra, as a tool to improve the understanding of Linear Relationships concepts.

Together with the van Hiele Theory, the Teaching Phases acknowledge the importance of the teacher and their role in guiding the students' learning process; the importance of language along with the importance of developing insight. The teacher is considered a crucial component and although tasks are student-centred, appropriate teacher guidance is required to assist, monitor, observe and ultimately assess the student's language and subsequent conceptual understanding of Linear Relationships.

Sequencing lessons with the Teaching Phases, adequately addressed the cognitive developmental needs of the students for Linear Relationships. Selecting activities that targeted specific Teaching Phases prevented students from relying on the memorisation of formulas and/or working, and enhanced the use of GeoGebra as a tool to support learning. These activities also provided a catalyst for discussions; discussions between students and discussions between students and teachers. Discussions were instrumental in monitoring and assessing the language used as well as providing a window with which to view the developmental pathway of students' understanding of Linear Relationships concepts. In particular, this study supports previous research that found the structure and design of activities associated with the technology are as important as the technology itself (Healy, et al., 2010).

For this study, the first of the Teaching Phases, Information, set the background for the working domain, with activities that facilitated discussion enabling the introduction of terms relevant to the context of the topic. These activities involved direct questioning or brainstorming to utilise those ideas considered prior knowledge connected to the main component of the lesson. Activities that targeted the next two phases, Directed Orientation and Explicitation, often spiralled repeatedly before progression to activities for Phase 4, Free Orientation were attempted.

The spiralling between Phases 2 and 3 assisted with the consolidation of Linear Relationships concepts. The second phase, Directed Orientation, involved a series of teacher guided activities that included investigations and explorations using GeoGebra, identifying features of GeoGebra that were relevant to the Linear Relationships concept being studied. Through teacher-guided student-centred activities, students were encouraged to discuss what they observed while exploring, providing an opportunity for them to establish relations, while being monitored by the teacher. With the third phase, Explicitation, activities were aimed at ensuring connections and meanings developed in the second phase were maintained. The teacher continued to monitor language, introducing more formal, technical terms associated with the Linear Relationships concept being explored. Activities of this phase often involved whole class discussion fostering the exchange of ideas, which subsequently developed language. Hence, activities targeting Teaching Phases 2 and 3, were crucial to the development of students' understanding of Linear Relationships concepts. Students who were absent for these activities missed important learning experiences that subsequently affected the quality of their responses, not only in subsequent lessons but in the Google Form tests.

Once the teacher was satisfied that students' understanding of Linear Relationships using GeoGebra had developed through the spiralling of Phases 2 and 3, activities that focussed on Phase 4, Free Orientation, were attempted. This phase is widely misused in many of today's classrooms because of external pressures to produce high performing mathematical students. Most external and formal testing consists of activities aimed at phase 4 problem solving. Phase 4 tasks presume students are familiar with content, are able to recognise essential cues necessary for solving the problem, and subsequently can solve the problem through combining all the information available. As a result, teachers, who are constantly under pressure to cover all the content in order to prepare students for high stake tests, teach directly to this level, presenting short-cuts, memorisation and explicit teaching strategies. Unfortunately for students, with this type of instructional experience, the teacher has initiated the understanding – or, as termed by van Hiele (1986) the "crisis of thinking" (p. 43), that has resulted in solving the problem. Hence, the students have no ownership over the ideas, concepts or understanding. When problems are presented differently, these students are unable to reflect on the necessary cues from familiar situations that would have been explored, investigated and consolidated, from Phases 2 and 3, to assist them in determining the pathway to finding the solution. Hence, the time expended in providing activities targeting Phase 2 and 3 is well spent when considering the students' understanding of concepts.

Phase 4 activities chosen for the teaching sequence involved problems that eventuated in the development of generalised formulas for Linear Relationships concepts. The derivation of formulas, from first principles, is another task often overlooked in today's classrooms, being considered too difficult and beyond the capability of students. This study demonstrated that not only is deriving formulas with Year 9 achievable, but it is also a natural development when activities are selected that progress through the van Hiele Teaching Phases. Other Phase 4 activities chosen during this study, used the concepts of Linear Relationships to solve problems involving other mathematical topics, such as Geometry. These topics worked well together, especially since GeoGebra merges these two topics in its dynamic environment, providing a supportive tool that supplemented students' conceptual understanding.

The final Teaching Phase, Integration, involved students' reflections of their findings. Activities targeting this phase summarised what had been previously discovered, providing an overview of the content. This included memorisation of formulas and rules that assisted them with further study. This study supported the van Hiele's idea that formulas and rules should be the end result from a consolidation of investigative and explorative activities that progress to develop students' conceptual understanding.

Addressing students' levels of thinking during the teaching sequence enables them to have ownership of the mathematical ideas, subsequently leading students to the development of insight. However, the assessment of insight is difficult and forms part of an ongoing process of the formative assessment method, AFL, including teacher observation and questioning. Through the organisation of instructional activities based on the Teaching Phases, this study provided ideal opportunities for the teacher to monitor and assess the development of students' insight - the development of which van Hiele considered to be the main reason of instruction (Pegg & Davey, 1998). The capability of insight enables students to solve non-routine problems using Linear Relationships concepts from previously encountered situations.

Overall, the Teaching Phases provided an effective framework that assisted with the sequencing of lessons. Carefully selecting activities targeting the various phases ensured that the instructional needs of the students were adequately addressed. Furthermore, with the assistance of the SOLO model to categorise responses students' understanding of Linear Relationships concepts using GeoGebra developed.

8.3.2. SOLO Model

The SOLO model, also known as the SOLO taxonomy, provided a framework to assist in qualifying the student responses in this study. Its application offered a means of categorising students' level of understanding of Linear Relationships concepts through classifying their responses.

The SOLO model provided a useful tool that assisted and enlightened teaching decisions. The ability to isolate responses provided a deeper interpretation of response categories. Through these categories, the characterisation of developmental growth of Linear Relationships was established. The categories also depicted different language use and the progression of language when describing mathematical ideas. This proved invaluable when monitoring students' understanding and supported the van Hiele Theory's assertion that language is an important element of learning and instructional experiences.

While students were mainly operating within the concrete symbolic mode, two cycles of SOLO levels were identified. Students providing responses categorised to the first cycle, indicated that they were operating using visual cues to assist them in solving problems, with various levels of support from the ikonic mode. These students relied heavily on GeoGebra to provide an answer rather than as a supportive tool; or followed procedures with limited understanding of the underlying mathematical concepts. This reflected little ownership of the mathematical ideas involved with concepts. The language used in first cycle responses referenced to simple features stored in short-term memory.

The second cycle of responses demonstrated that GeoGebra was used as a supportive tool to supplement understanding of Linear Relationships concepts. This was evident from the language used, which was articulated with less reliance on visual cues, often involving algebra and formulas rather than worded explanations.

Developmental hurdles were evident through the uneven movement between levels in the SOLO model. In particular, the transition from M to R was observed to be a complex transition. Students' ability to make connections in order to establish relationships represented an important shift in their thinking. It also demonstrates the strength of the SOLO model, as monitoring students' responses enabled the teacher to select appropriate activities that addressed the needs of the students, ensuring consolidation of conceptual understanding for long-term retention.

The SOLO model enabled an overview of the developmental pathway for Linear Relationships to be identified. Its application provided valuable information about the cognitive processes and hurdles met by students when studying Linear Relationships concepts with GeoGebra. Together with the van Hiele Teaching Phases, this study has identified its benefits.

8.4. Implications for Teaching

As a consequence of this study, a number of important implications for teaching Linear Relationships with technology in the secondary school setting have emerged. A number of difficulties associated with students' understandings of Linear Relationships and graphing technology have been previously documented when looking at each as separate identities. Challenges with Linear Relationships often relate to issues involving algebra with contextual connections and meanings (Bardini & Stacey, 2006; Beatty & Bruce, 2012; Ellis, 2007) and challenges with graphing technology relating to a range of issues from problems posed by the technology, such as viewing difficulties, to the application of technology by both the teacher and student, such as blind acceptance of solutions (Cavanagh, 2005; Kemp, et al., 1996; Mitchelmore & Cavanagh, 2000). While this study supports previous research, it also extends research findings by identifying a number of challenges for understanding Linear Relationships, with particular focus on the dynamic graphing technology, GeoGebra. The following discussion delivers pedagogical recommendations to assist with students' progression along the Linear Relationships developmental pathway, leading to an increased conceptual understanding of the unit using GeoGebra as a supportive tool.

With regard to assisting students' growth along the Linear Relationships developmental pathway, this study identified that the van Hiele Teaching phases combined with the SOLO model were an essential pedagogical tool. The combination of these two frameworks encourages the development of higher-order thinking skills, through carefully selected activities, aligned with the Teaching Phases, that address the needs of the students as assessed through monitoring of the quality of students' responses in terms of the SOLO model.

As a result, it would be of benefit that teachers utilise these two frameworks when programming, developing teaching sequences and lesson plans. Through understanding the frameworks, teachers can adapt, modify, select and design activities that incorporate dynamic software such as GeoGebra to promote higher-order understanding. When implementing and facilitating activities through assessing students' responses using the SOLO model, the teacher can identify where along the developmental pathway of Linear Relationships the student is operating using Tables 5.47 and 6.50 as guidelines.

The process of carefully selecting and designing activities often requires the spiralling of Teaching Phases to promote higher-order thinking. This spiralling consolidates understanding for students, promoting them to develop their own connections of mathematical ideas while being closely monitored by the teacher. The spiralling of Teaching Phases, in particular in Phase 2, Directed Orientation, and Phase 3, Explicitation, assist with addressing technological developmental hurdles presented by technology, such as GeoGebra. Students benefit from this spiralling of Teaching Phases and this recommendation for teaching practice consolidates their conceptual understanding. It is important that students explore and investigate problems using technology, such as GeoGebra, thus promoting familiarity of the technological environment, using it as a tool to assist understanding, not just providing an answer. This supports the comprehension of technological inconsistencies presented, such as viewing and scaling issues.

Since the development of language is an important consideration within the van Hiele Theory and Teaching Phases, activities targeting mathematical literacy involving language specific to Linear Relationships would be of benefit to students. Early in the Teaching Phases, students' conversations involve using their own language and terminology to explain Linear Relationships concepts. This is refined and developed through the use of GeoGebra and targeted activities to include terms more specific to Linear Relationships. Hence, the inclusion of activities targeting mathematical literacy would improve spelling, grammar and comprehension of Linear Relationships concepts. Possible activities could include, comprehension tasks involving skimming or scanning strategies, find-a-word, or cloze passages. This would also address cross curricula literacy requirements.

This study demonstrates that utilising dynamic software, such as GeoGebra, assists with the development of students' understanding of Linear Relationships concepts. These findings contribute to the case for increasing technology use in the mathematics classroom (Drijvers, et al., 2016; Hopper, 2009), in particular, it contributes to the widely-documented range of research related to the success of GeoGebra (Arbain & Shukor, 2015; Zulnaidi & Zakaria, 2012). The dynamic nature of GeoGebra assists the development of understanding of mathematical concepts, through providing instant feedback to students as well as offering an investigative and explorative environment with which to examine mathematical concepts.

With regards to assessment techniques, this study demonstrates how the use of formative assessment techniques to evaluate individual student understanding benefits teaching practice. The SOLO model effectively enables teachers to assess students' understanding of Linear Relationships, through the complexity of their responses. Rather than comparing students, it enables teachers to assess students against the content and understanding of concepts. In this way, activities can be designed that address the needs of students. Hence, encouraging the use of these assessment techniques through in-service programs and professional learning programs would be of benefit to teachers and for students, who, in particular, are anxious about summative tasks.

Finally, the discussion provided above has one central theme that is essential to the success of teaching; namely, the teacher. It is the teacher who must select and facilitate activities appropriate for the improvement of students' understanding of Linear Relationships; assess student understandings through analysing responses to Linear Relationships problems; incorporate GeoGebra, or other technology, such that it is implemented in a way that supports understanding of Linear Relationships concepts; and monitor students use and development of language specific to Linear Relationships. Through the exploration of cognitive processes, an understanding of the developmental pathway for understanding Linear Relationships concepts, teachers are provided with a tool from which to view students' thinking, and a starting point to design activities using GeoGebra that offer students opportunities to develop insight.

8.5. Future Research

Opportunities for further research into a range of issues arise from this study. In particular, five research directions stand out as worthy of investigation.

Firstly, as an extension to this study, a future research direction relates to an investigation using the same sample with prolonged testing time frame, commencing at secondary school Year 7 up to Year 12. This would explore the retention of content and provide richer, more detailed data on the developmental pathway of Linear Relationships concepts that emerged from this study.

The second research direction relates to the development and trialling of more teaching materials and activities that could be used for Linear Relationships incorporating GeoGebra across secondary school. As highlighted earlier, the selection and design of activities is an important aspect of the teaching sequence. Hence, designing a range of activities that do not base themselves solely on a single textbook, aligned with the Australian Curriculum, would be beneficial to teachers throughout Australia.

Thirdly, a possible area of research that would be an extension to this study, involves a more varied sample of students with which to explore students' understandings of Linear Relationships concepts using GeoGebra. In particular, it would be advantageous to use a sample representing a wider cross-section of students. This would enable generalisations regarding developmental pathways for Linear Relationships to be made based on culture, ESL, socioeconomic status or location. This information would provide valuable for teachers where these matters are prevalent.

A fourth research direction investigates the teachers' role in implementing technology with other various strands of mathematics throughout primary and secondary school. The role of the teacher was identified as critical in the van Hiele Teaching Phases and SOLO model. This would involve similar qualitative procedures, investigating teacher-student interactions and teacher pedagogical practices.

The final research direction relates to the structure of scope and sequence and time requirements of Stage 5.3 Mathematics Courses. This could be achieved through an investigation that explores a number of different schools and the time currently used to cover the specific topics of the Stage 5.3 Mathematics Course. Possible suggestions of how time could be allocated to foster explorations and implementing technology with these topics would prove invaluable to teachers and coordinators when programming for future years.

8.6. Conclusion

This study investigated the nature of student responses to Linear Relationships activities and the nature of the student interaction while completing interactive tasks using GeoGebra as an exploration tool. GeoGebra is often used by teachers in an impromptu manner; however, many teachers have difficulty designing a complete unit of work that embeds GeoGebra throughout the lesson sequence. The findings that emerged from this study depict a developmental path leading to an understanding of Linear Relationships concepts.

The characterisation of developmental growth leading to an understanding of Linear Relationships concepts when using GeoGebra, highlights some hurdles that must be addressed throughout the learning and teaching process. Hurdles encountered concern:

- movement between SOLO levels which were not uniform, indicative of developmental hurdles;
- viewing and scaling issues displayed with GeoGebra;
- default presentation of equations within the GeoGebra environment; and,
- correct implementation of GeoGebra tools.

This study aimed to examine the student exchange of ideas when utilising dynamic mathematics software GeoGebra as an educational tool for exploring Linear Relationships. The results indicate characteristic changes in the nature of the students' responses with a shift to more formal and technical language use as the teaching sequence progressed. The dynamic environment was an important component of the learning environment that fostered the development of language leading that enhanced the understanding of Linear Relationships concepts.

The findings suggest that there are benefits to using GeoGebra in the Linear Relationships strand in the middle years of secondary school and that it is essential to build familiarity of the tools to enable the students to focus on the exploration at hand rather than the tools they are using to explore it. In addition, through the identification of hurdles encountered when incorporating GeoGebra, a number of implications for teaching have emerged. This study highlighted the potential of the SOLO model as a tool that assists teachers in determining students' understanding of Linear Relationships, which, when combined with the van Hiele Teaching Phases, supports teachers to select and design activities which aim to investigate Linear Relationships concepts using GeoGebra as an exploration tool.

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Appendix A: Lesson Plans Linear Relationships

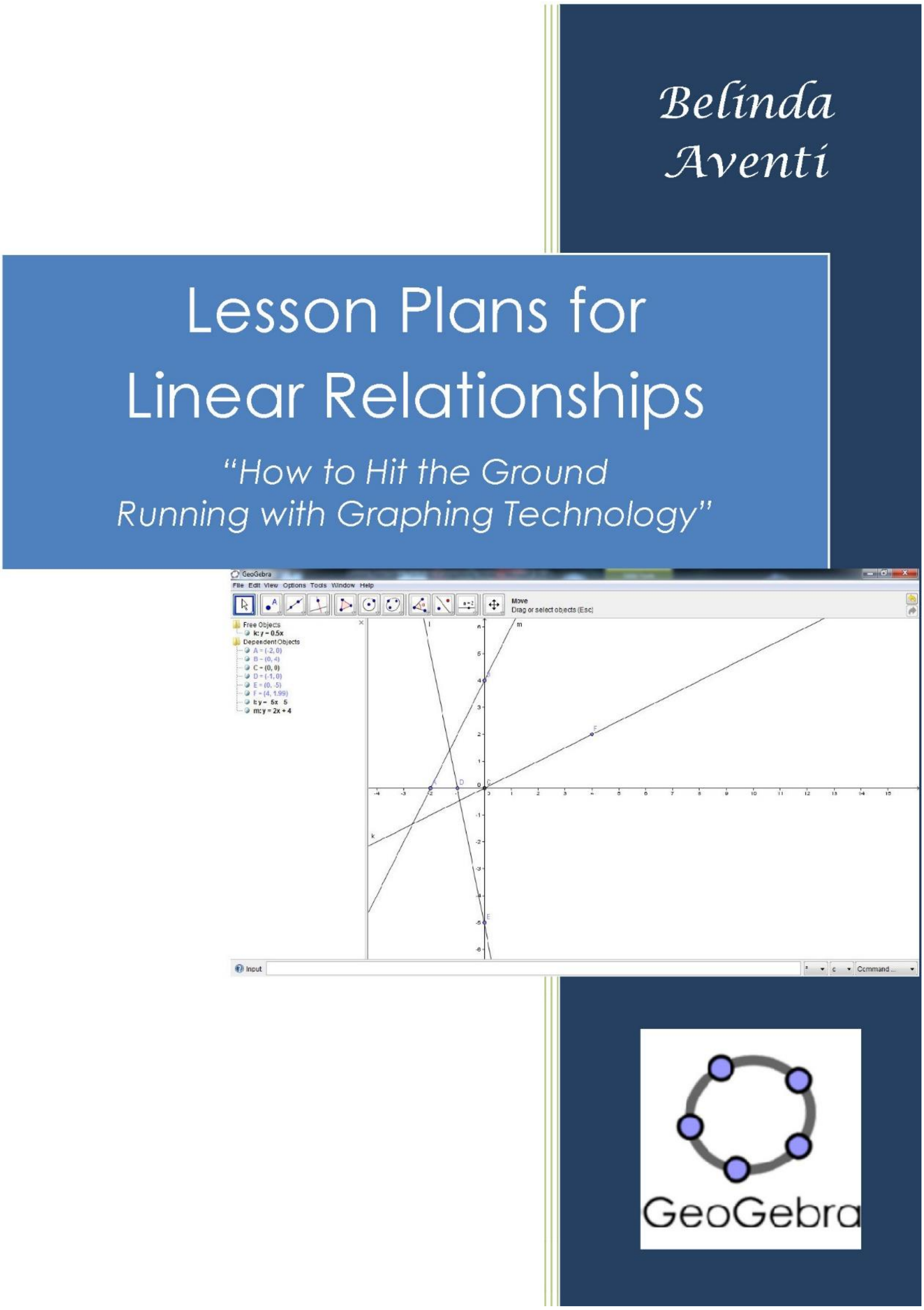


Table of Contents

Lesson 1: Pre-Test Task

Syllabus Outcomes:

- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- uses formulas to find midpoint, gradient and distance on the Cartesian plane, and applies standard forms of the equation of a straight line MA5.3-8NA

Materials Required:	**Learning Intention:**
• Laptops with Geogebra installed	What do you know about linear relationships?

Syllabus Content: Students:

Find the midpoint and gradient of a line segment (interval) on the Cartesian plane (ACMNA294)

- use the concept of an average to establish the formula for the midpoint, *M*, of the interval joining two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane: $M(x, y) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
- use the formula to find the midpoint of the interval joining two points on the Cartesian plane
- use the relationship $gradient = \frac{rise}{run}$ to establish the formula for the gradient, *m*, of the interval joining two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane: $m = \frac{y_2-y_1}{x_2-x_1}$
- use the formula to find the gradient of the interval joining two points on the Cartesian plane explain why the formula $m = \frac{y_2-y_1}{x_2-x_1}$ gives the same value for the gradient as $m = \frac{y_2-y_1}{x_2-x_1}$ (Communicating, Reasoning)

Find the distance between two points located on the Cartesian plane (ACMNA214)

- use Pythagoras' theorem to establish the formula for the distance, *d*, between two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
  - → explain the meaning of each of the pronumerals in the formula for distance (Communicating)
- use the formula to find the distance between two points on the Cartesian plane

Sketch linear graphs using the coordinates of two points (ACMNA215)

Solve problems using various standard forms of the equation of a straight line

- describe the equation of a line as the relationship between the *x*- and *y*-coordinates of any point on the line
  - → recognise from a list of equations those that can be represented as straight-line graphs (Communicating, Reasoning)
- rearrange linear equations in gradient-intercept form ($y = mx + b$) into general form $ax + by + c = 0$
- find the equation of a line passing through two points

Solve problems involving parallel and perpendicular lines (ACMNA238)

- find the equation of a line that is parallel or perpendicular to a given line
- determine whether two given lines are perpendicular
  - → use gradients to show that two given lines are perpendicular (Communicating, Problem Solving)

Content:

Teacher Notes: Each student must have a

- **"Camstudio Codeless Compressor" installed on their laptops**
- **A new folder on their desktop called Linear Relationships – this is to store every Geogebra file in.**

1. Place students into working pairs (of similar ability) which they are to work together for this unit
2. Each student is to receive a piece of SCRAP PAPER – in Appendix
3. Students are instructed to put their name clearly marked at the top of the paper which will be collected at the end of the lesson
4. Students are instructed to work in pairs SELECTED BY TEACHER using their laptops
   - one student opens their laptop onto Geogebra
   - the other student opens their laptop onto link of Google forms
   - only one copy of solutions needs to be submitted
   - Link to get students there is emailed to students?

https://docs.google.com/a/ww.catholic.edu.au/forms/d/1yTR4ZGKMkiqCp8to8s2XOTvhUr-

94E4JxUnrO4NFpXY/edit?usp=sharing

5. Students who have approved to be recorded **MUST BE INSTRUCTED TO RECORD THEIR WORK**
6. *Pre-test needs little teacher intervention students will engage in the learning themselves.*
7. *It is a task based on what students should know about Linear Functions in year 9 – hence some students may be able to complete all, some or none.*
8. Encourage students to attempt as much as possible

Extra Activity:

Handout Fill In Activity Pre-Test – 1, Fill In Activity Pre-Test – 2, Fill In Activity Pre-Test – 3 as needed.

Teacher Notes: This is for early finishers or people not willing to have a go!

Lesson 2: Real Life Linear Relationships - 1

Syllabus Outcomes:

- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- uses formulas to find midpoint, gradient and distance on the Cartesian plane, and applies standard forms of the equation of a straight line MA5.3-8NA

Materials Required:

- Laptops
- Graph Paper
- Teacher has Geogebra loaded ready to use

Learning Intention:

What are linear relationships used for?

Syllabus Content: Students:

Find the midpoint and gradient of a line segment (interval) on the Cartesian plane (ACMNA294)

- use the relationship $gradient = \frac{rise}{run}$ to establish the formula for the gradient, m, of the interval joining two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane: $m = \frac{y_2 - y_1}{x_2 - x_1}$
- use the formula to find the gradient of the interval joining two points on the Cartesian plane explain why the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ gives the same value for the gradient as $m = \frac{y_2 - y_1}{x_2 - x_1}$ (Communicating, Reasoning)

Sketch linear graphs using the coordinates of two points (ACMNA215)

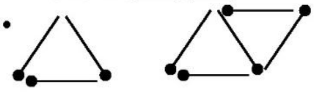
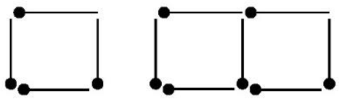
- sketch the graph of a line by using its equation to find the x- and y-intercepts

Solve problems using various standard forms of the equation of a straight line

- describe the equation of a line as the relationship between the x- and y-coordinates of any point on the line
  - → recognise from a list of equations those that can be represented as straight-line graphs (Communicating, Reasoning)
- rearrange linear equations in gradient-intercept form ($y = mx + b$) into general form $ax + by + c = 0$
- find the equation of a line passing through two points
- recognise and find the equation of a line in general form $ax + by + c = 0$
-

Heading in Books – **<u>Patterns and Linear Relationships</u>**

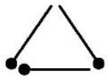
Activity 1: Number Patterns ***(LP 1 – Information)***

1. Write the following patterns on the board and get students to find the next two terms (students are not required to copy into their books)
   - 2, 6, 10, ____, _____
   - 3, 6, 12, 24, ____, _____
   - -5, 5, 15, ____, _____
   - -4, 16, -64, ____, ____
   - 
   - 

Activity 2:

Teacher Notes: An activity that starts students at a basic level of patterns and shows them that linear relationships replicate patterns in life. From creating a pattern a linear relationship can be formed by investigating the pattern. This activity is done together as a class.

2. Make the following pattern using matchbox sticks – draw on board *(LP 1 – Information)*



3. Demonstrate the pattern for two triangles on board as shown below then ask students to continue the pattern up to 5 terms in books and discuss together and do on board.

4. Draw table on board ask students - What would the first number be on the second column?

Number of Triangles	Number of Matchsticks
1	
2	

5. *Question Time - Looking at the pattern* *(LP 3 – New Ideas)*
   - *Can anyone see a relationship or connection between the number of triangles and the number of matchsticks?*

Teacher Notes: Students may not make the link across they may only see that number of triangles increases by one and number of matchsticks increases by 2. Encourage them to see the connection between across the row of the table – obviously it makes it difficult to work out for 20 triangles we would have to draw a bigger table?

   - *Can you tell me what links number of triangles to number of matchsticks so I can find out how many matchsticks each time - how I could find out how many matchsticks are required for 20 triangles?*

Teacher Notes: Eventually someone will discover that you need to multiply the number of triangles by 2 and add 1 then encourage them to write it in their own words

   - *Write a sentence describing your pattern*
   - *How can we represent that in algebraic terms? Maybe call triangles T and matchsticks M*

Teacher Notes: Get them to help you do it – converting their sentence into algera M = 2T + 1

6. Turn on projector and plot points from table into Geogebra –
7. *Teacher Talk :Now we are going to graph it by plotting points? What is the first point I have o plot?*

Teacher Notes: Reinforce (x, y) coordinates and their respective placement on axes. To plot individual points in Geogebra type the coordinate with brackets (2,4) into the input bar at the bottom of the screen and press ENTER.

8. Draw the line connecting the interval

Teacher Notes: To plot the line in Geogebra click on Line Through Two Points button (third from left) then click on Interval between Two Points then click on first point on graph and last point and interval will be drawn; or

9. *Teacher Talk : Notice our pattern is represented in algebraic terms in the Algebra window. Is it the same as the equation we derived?*

   This is given in a different form called general form – lets change it to how we know it by…(change it as directed below)

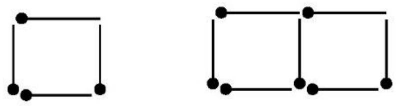
Teacher Notes: The equation of line is by default given in general form – this term can be introduced here, to convert to gradient intercept form right click on the line and select equation y = mx + b

10. Instruct students to copy the graph into their workbooks with the equation of the line.

Activity 2: ***(LP 1– Information)***

Teacher Notes: Repeat of Activity 1 with a different pattern to consolidate the concept of patterns that create linear relationships. Enabling students to understand what linear relationships represent. Again this activity is done on the board together to ensure students understand

1. Draw the following matchbox sticks pattern on the board – this is called a train



2. *Question Time - What is the perimeter of the square? What is the perimeter of the 2-train?*
3. Instruct students to copy into their books and continue with two squares
4. Draw the table on the board and talk students through completing the table.

Square Train	
Length of Train (# of Squares)	Perimeter of Train
1	
2	

(LP 3– New Ideas)

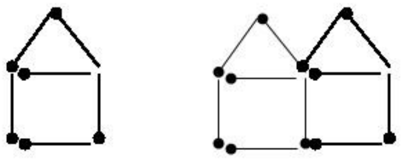
5. *Question Time - Look for patterns in the data and how you could describe it*
   a. *Do you see a pattern in how the number of matchsticks increases as the pattern increases by one train?*
   b. *What do you think the number of matchsticks would be for a 20- train pattern?*
   c. *Write a sentence to describe this pattern*
   d. *Convert that sentence into algebraic terms like last activity – L for Length and P for Perimeter.*

6. Plot points on Geogebra using projector.
7. Draw line connecting interval.
8. Check rule using Geogebra.

Activity 3:

Teacher Notes: A more complicated pattern to consolidate the concept of patterns that create linear relationships – start students off with this and give instructions. Hopefully students will be able to continue this activity themselves and start to make some connections between pattern and linear relationship.

1. Draw a house pattern on the board *(LP 1 – Information)*



2. Instruct students to continue the pattern up to 4 houses and then complete the following table comparing the number of houses with the number of matchbox sticks used

Number of Houses	Number of Matchsticks
1	
2	
3	
4	
5	

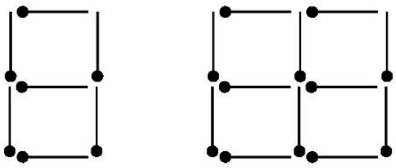
(LP 3– New Ideas)

3. *Question Time - Look for patterns in the data and how they could be described*:
   e. *What is the pattern?*
   f. *What do you think the number of matchsticks would be for a 20 house pattern?*
   g. *Write a sentence to describe this pattern*
   h. *Convert that sentence into algebraic terms.*
4. Instruct students to plot information in the table into a graph in their books and then use Geogebra as in previous activities to plot points and check the graph and equation.
5. Check to see who gets same equation in book and Geogebra.

Activity 4:

Teacher Notes: Hopefully students can complete by themselves

1. Draw following pattern on the board *(LP 1 – Information)*



2. Instruct students to continue the pattern up to 4 terms (the word term may need explanation) and then complete the following table comparing the number of houses with the number of matchbox sticks used
3.

Pattern Number	Number of Matchsticks
1	
2	
3	
4	
5	

(LP 3– New Ideas)

4. *Question Time - Look for patterns in the data and how they could be described*:
   i. *What is the pattern?*
   j. *What do you think the number of matchsticks would be for a 20 house pattern?*
   k. *Write a sentence to describe this pattern*
   l. *Convert that sentence into algebraic terms.*
5. Instruct students to plot information in the table into a graph in their books and then use Geogebra as in previous activities to plot points and check the graph and equation.
6. Check to see who gets same equation in book and Geogebra.

Teacher Talk: Linear relationships model real life phenomena. We use them to make find rules and formulas to explain the phenomenon and then make predictions based on what we have found. Let's change it up a little…….

Activity: Algebra Walk

1. Move students outside to use Cartesian Plane on cement
2. Acting Out the Rules: Observation and Discussion

   m. Select nine students to stand on the numbers from $^{-}4$ to $^{+}4$. Give an instruction like:

Multiply the number between your feet by two. Now add one. Now step out that number.

n. Quite a simple, achievable task, but instantly, for all to see, pupils form lines as on a graph - in this case a concrete example of y = 2x + 1.
o. Discussions occur regarding how to multiply negative numbers and the group self-correct
p. Ask this set of nine students to remember what happened to each other – where each stood record into books if necessary.
q. Invite another nine students to step up and ask another rule. This time ask the observers to predict what will happen. These students also record what happened to each other.
r. Continue inviting children to step out rules until all have had at least one turn.

Multiply the number between your feet by negative one. Now add two. Now step out that number.

Multiply the number between your feet by negative two. Now add one. Now step out that number.

Halve the number between your feet. Now add two. Now step out that number.

Multiply the number between your feet by one. Now add three. Now step out that number.

3. Transferring the results to the classroom –

Teacher Notes: The use of imagery is very powerful. It is spectacular to observe and very non-threatening for students.

s. Go back into classroom and ensure there are blocks, graph paper, or other things that students can use to model what happened outside.

t. Ask the students to group themselves according to the rules they walked. If students were with more than one rule ask check that each group has approximately the same number of students.

Teacher Talk: Now we can't always go outside to make pictures of these rules, so I want your group to use anything you can find in the classroom to model what your group had to do outside and recreate the result.

Teacher Notes: Interestingly, graph paper may not always be the first option.

4. Inventing and testing rules

u. Once students have developed facility with their model, invite them to invent their own walking rules, carry them out on the model and investigate what happens.

5. Finishing the problem

v. Review what has been learned, and in particular focus on:

i. the need for a starting line
ii. the need for equally spaced starting numbers
iii. the need for equal stepping
iv. the connection between a number pattern governed by the rule and the visual pattern

Teacher Talk : So it is a good thing to remember that if you have a visual pattern there is likely to be a rule - and if there is a rule there is likely to be a visual pattern.

Extension A

6. Distinguish two sets of nine students by, for example, sleeves up and sleeves down.
7. Ask each group to step out different rules.
8. The intersection of straight line graphs and a representation of simultaneous equations.
9. If you choose the rules appropriately (for example $y = 2x + 1$ and $y = -x + 7$), two students will actually be standing on the same spot.

Lesson 3: Real Life Linear Relationships - 2

Syllabus Outcomes:

- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- uses formulas to find midpoint, gradient and distance on the Cartesian plane, and applies standard forms of the equation of a straight line MA5.3-8NA

Materials Required:

- Laptops with Geogebra installed
- Worksheet A – 1 and Worksheet A - 2

Learning Intention:

Equations, Y-intercept and Gradient

Syllabus Content: Students:

Find the midpoint and gradient of a line segment (interval) on the Cartesian plane (ACMNA294)

- use the relationship $gradient = \frac{rise}{run}$ to establish the formula for the gradient, m, of the interval joining two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane: $m = \frac{y_2-y_1}{x_2-x_1}$
- use the formula to find the gradient of the interval joining two points on the Cartesian plane explain why the formula $m = \frac{y_2-y_1}{x_2-x_1}$ gives the same value for the gradient as $m = \frac{y_2-y_1}{x_2-x_1}$ (Communicating, Reasoning)

Sketch linear graphs using the coordinates of two points (ACMNA215)

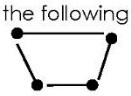
- sketch the graph of a line by using its equation to find the x- and y-intercepts

Solve problems using various standard forms of the equation of a straight line

- describe the equation of a line as the relationship between the x- and y-coordinates of any point on the line
  - → recognise from a list of equations those that can be represented as straight-line graphs (Communicating, Reasoning)
- rearrange linear equations in gradient-intercept form ($y = mx + b$) into general form $ax + by + c = 0$
- find the equation of a line passing through two points
- recognise and find the equation of a line in general form $ax + by + c = 0$
-

Activity 1: Short Introduction and Revision of Last Lesson *(LP 1 – Information)*

1. Draw the following pattern on the board and ask students to copy into their books.



2. *Question Time - What is the perimeter of the hexagon?*
3. Instruct students to continue the pattern adding trapeziums up to 3 trapeziums then complete the table that follows.

Number of Trapesiums	Number of Matchsticks
1	
2	

(LP 3– New Ideas)

4. *Question Time - Look for patterns in the data and how you could describe it - what rule describes this pattern? Remember to write it in your own words then convert to algebraic terms?*

Teacher Notes: Hopefully students can recall from last lesson how to describe the rule and put into an equation using algebraic terms.

5. Instruct students to plot the information from the table into a graph in their books and then use Geogebra to check their solutions.

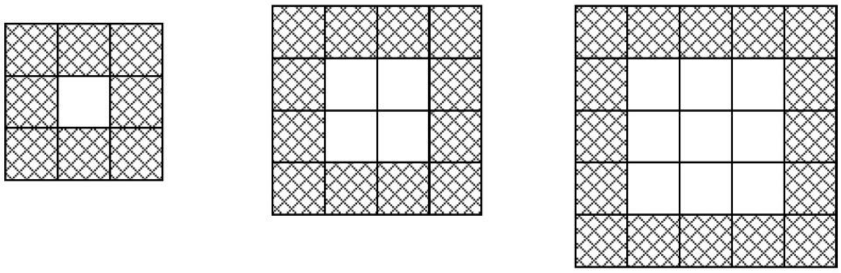
Teacher Talk: Here we are going to look at patterns but in another context …….

Activity 2: Garden Beds

Heading in Books : **Garden Beds** ***(LP 4 - Ownership)***

1. Copy the following problem (with diagrams) onto the board and instruct students to copy it into their books

A gardener creates square garden beds and then decorates them by tiling the edges. As seen in the diagrams below. The relationship between the length of the garden bed and the number of tiles required to decorate around the garden bed is represented by a linear relationship.



2. Students need to construct and complete a table for the length of the garden bed and the number of tiles required to decorate the edges.

Length of Garden Bed	Number of Tiles
1	
2	

3. *Question Time: Again look for patterns in the data and how you could describe it*
   a. *Do you see a pattern in how the number of tiles increases as the length of the garden bed increases by one?*
   b. *What do you think the number of tiles would be for garden bed of length 10?*
   c. *Find the equation which describes the garden bed problem?*
4. Instruct students to plot the information from the table into a graph in their books.
5. They then can plot this into Geogebra and check their solutions.

Activity 3:

Heading in Books: **Polygons and Symmetry** ***(LP 4 - Ownership)***

1. Draw a triangle, square, pentagon and hexagon on the board.
2. *Question Time:*
   - *I have drawn four different regular polygons? What does that mean?Identify that each is a regular polygon where all sides are of equal length and all angles are equal.*
   - *How many lines of symmetry are there*
     - → *in the triangle? – and demonstrate on diagram*
     - → *in the square? – and demonstrate on diagram*
     - → *in the pentagon? – and demonstrate on diagram*
     - → *in the hexagon? – and demonstrate on diagram*
3. Instruct students to copy the diagrams into their books with the lines of symmetry
4. Draw the table below on the board to compare the regular polygons to the number of lines of symmetry they have.

Number of Sides in Regular Polygon	Name of Polygon	Number of Lines of Symmetry
3		
4		
5		
6		
7		
8		
9		
10		

5. Instruct students to complete the data with the shapes already done.
6. *Question Time*
   - *Can anyone recognize a pattern?*
   - *Continue the pattern for the rest of the table*
7. Instruct students to plot the data from the table into a graph with the number of sides on the horizontal axis and the number of lines of symmetry on the vertical axis.
8. Ask students to find the rule or equation which describes this pattern as they have in previous activities.
9. They can check their answer using Geogebra and get equation.

Overview: Finding the connections ***(LP 3– New Ideas)***

Teacher Notes: Review yesterday and attempt to get students to recognize the connections that exist between the graph and the equation – linking y-intercept and forming some concept for gradient. Class discussion occurs for most of this with the Geogebra ready to go on projector.

Teacher Talk: Let's go back to our problems yesterday – have a look in your book open to the page …..the triangle pattern was M = 2T + 1 lets have look at what that looks like in Geogebra (put up on projector)

- *Does anyone notice anything about the rule and what it looks like?*

Teacher Notes: We want students to recognize it crosses at 1 and the equation has 1 once they do proceed

Teacher Talk: Lets have a look at a another one – lets see if your right let's make it y = 2x + 3 (type this in with the other graph still on screen)

Ok lets have a prediction ……….what do you think its going to do if I put in y = 2x – 1? …….wait for student response …………….then check using Geogebra

What was another we did yesterday? (get students to give another equation – the square one y = 3x + 1)

Anyone know what it's going to do? ……..wait for student response y-intercept should be recognised………check using Geogebra

Ok it will go through 1 but it will be interesting to see what it does?

Teacher Notes: Students will begin to realize that in the equation y = mx + b the b is where it crosses the y axis and the m represents something also. Starting to investigate slope.

Teacher Talk: What is different?

Teacher Notes: Students may recognize the steeper slope

Teacher Talk: For interest sake lets make it y = -2x + 1.

Teacher Notes: This should capture their interest – they now should be able to work out how to graph lines mentally without doing a table – they know the y-intercept and realize that the cooefficient of x determines the slope of the line – positive or negative.

Teacher Talk: Do you think you can graph without a table?

How?

Activity 4:

1. Hand out Worksheet A
2. Students are to sketch equations on Geogebra than draw onto Worksheet A.

Teacher Notes: Students draw equations by typing each equation straight into the input bar

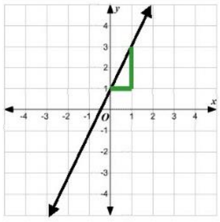
3. *Question Time: Pose the following question before they start - What connections are noticed between drawings and equations?*
4. Allow time for students to complete
5. Get their attention back for a class discussion

Teacher Talk: What did you notice about the graphs?

Teacher Notes: Connection between rule and y-intercept should be established. It is expected (and hoped) that students will come back with

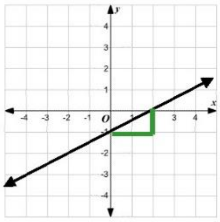
- **The +1 or -2 at the end of the equation is where it cuts the axis – clarify that this is the y-intercept also known as 'b'**
- **Positive coefficient slopes left up**
- **Negative coefficient slopes left down**
- **If number is bigger its steeper**
- **They won't know how to get or calculate gradient - need to investigate rise/run**
- **Two things identify a linear relationship the coefficient of x and the y-intercept.**

Teacher Talk: So what does the 2 mean in y = 2x + 1….lets draw it on the board (draw the graph with a few points) If we draw a triangle here what does the 2 represent….. (see green lines)



If we look at the triangle we have gone 2 units up or we can say our rise is 2units and our run is 1 unit so 2/1 is the same as 2

Lets look at the one which is a ½ - notice when we draw the triangle the rise is 1 and the run is 2

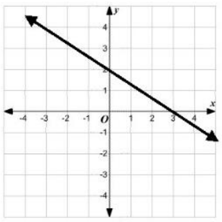


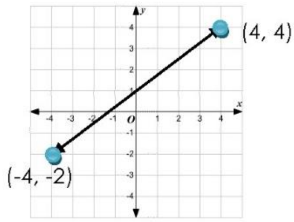
Slope is also known as the mathematical term 'gradient' or 'm'

Teacher Notes: The big picture here is to consolidate the concept of gradient and y-intercept – the formula will come if the students have a sound understanding of what they mean.

The two things which identify a linear relationship are the gradient 'm' and the y-intercept 'b'

6. *Question Time: How can we calculate the slope? What helps to define the slope or gradient?*
7. Listen to any responses before continuing
8. Give students a new graph on the board and instruct students to find the gradient





9. *Question Time: How can rise and run be identified using only the coordinates?*

Teacher Talk: We don't always draw the graphs how can we work out gradient without drawing the graph?

a. Using the graph above write in the coordinates of x-intercept and y-intercept (3, 0) and (0, 2)

Teacher Talk: Is there any connection that you can see between the coordinates and the rise and run?

b. Do the same for (4, 4) and (-4, -2) Using the coordinates how can you work out the rise and run?

Teacher Talk: How can we generalize so that we know how to calculate the rise and run no matter what the coordinates are?

c. Establish that each coordinate can be identified using (x_1,y_1) and (x_2,y_2)

d. Rise = $x_1 - x_2$ and Run = $y_1 - y_2$

Teacher Talk: Does it matter what order the coordinates are? $x_1 - x_2$ *or* $x_2 - x_1$

Teacher Notes: The distance between two points is not dependent on which point you start and end with. No need to formalize the formula get the understanding solid.

Extra Activity: (if needed) ***(LP 1 – Information)***

1. Copy the following problem onto the board for students to copy into their books:

Jodie earns $2 pocket money each week. For every chore she does at home she receives $3 on top of her pocket money.

2. Draw the following table on the board and instruct students to complete the following table for Jodie ***(LP 2 – Direction)***

Number of Chores Jodie Does Per Week	Total Pocket Money Jodie Receives Per Week
0	
1	
2	
3	
4	

Teacher Talk: There is a linear relationship which defines the total pocket money Jodie receives each week compared to the number of chores she does. Find the rule for the total money Jodie receives.

2. Instruct students to plot the points in the table onto a graph in their books –allow time for this
3. Instruct students to plot their rule on Geogebra and check if its correct.

Teacher Notes: Students type in their rule using x and y into the input bar and check to see if it displays the same graph that they have drawn in their books.

Lesson 4: Real Life Linear Relationships - 3

Syllabus Outcomes:

- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- uses formulas to find midpoint, gradient and distance on the Cartesian plane, and applies standard forms of the equation of a straight line MA5.3-8NA

Materials Required:

- Laptops
- Worksheet B
- Worksheet C

Learning Intention:

y = mx + b

Syllabus Content: Students:

Find the midpoint and gradient of a line segment (interval) on the Cartesian plane (ACMNA294)

- use the relationship $gradient = \frac{rise}{run}$ to establish the formula for the gradient, *m*, of the interval joining two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane: $m = \frac{y_2-y_1}{x_2-x_1}$
- use the formula to find the gradient of the interval joining two points on the Cartesian plane explain why the formula $m = \frac{y_2-y_1}{x_2-x_1}$ gives the same value for the gradient as $m = \frac{y_2-y_1}{x_2-x_1}$ (Communicating, Reasoning)

Sketch linear graphs using the coordinates of two points (ACMNA215)

- sketch the graph of a line by using its equation to find the *x*- and *y*-intercepts

Solve problems using various standard forms of the equation of a straight line

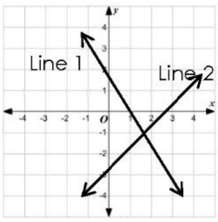
- describe the equation of a line as the relationship between the *x*- and *y*-coordinates of any point on the line
  - → recognise from a list of equations those that can be represented as straight-line graphs (Communicating, Reasoning)
- rearrange linear equations in gradient-intercept form ($y = mx + b$) into general form $ax + by + c = 0$
- find the equation of a line passing through two points
- recognise and find the equation of a line in general form $ax + by + c = 0$
-

Activity 1: Revision from previous lessons

1. Draw the Cartesian plane below on the board and brainstorm what they can identify about the lines.

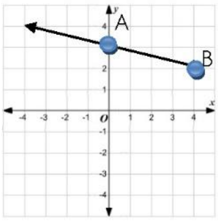
   Expected answers:

   1. y-intercept or b
   2. Positive slope/gradient
   3. Negative slope/gradient
   4. y-axis
   5. x-axis
   6. y= x -2
   7. y= -2x + 1



Activity 2:

1. Draw the following cartesian plane and line on the board and get students to copy into their workbooks



Teacher Talk:

2. What are the coordinates of points A and B? Answer: A(0, 3) and B(4, 2)
3. What is the y-intercept? Answer: 3
4. How can we work out the gradient? Answer: 1/4

Teacher Notes: Students may come up with various methods

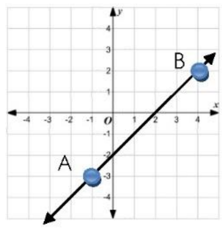
- **Rise/run – through creating a triangle**
- **Using the coordinates and then doing rise/run**

5. What is the equation of the line? Answer: $y = -1/4x + 3$
6. What can we say about any line?
7. What two identifying features must we know in order to come up with the rule?

Teacher Notes: Here we confirm the y = mx + b and present it as the gradient-intercept form of a line we can also introduce general form as a different way of representing the line given that a ≠ 0 and a, b and c cannot be fractions.

Activity 3:

1. Draw the following cartesian plane and line on the board and get students to copy into their workbooks



Teacher Talk:

2. What are the coordinates of points A and B? Answer: A(-1, -3) and B(4, 2)
3. What is the y-intercept? Answer: -2
4. How can we work out the gradient? Answer: 5/5 = 1

Teacher Notes: Students may come up with various methods

- **Rise/run – through creating a triangle**
- **Using the coordinates and then doing rise/run**
- **Using the formula as derived in previous lesson**

5. What is the equation of the line? Answer: $y = 1x - 2$

Activity 3:

Teacher Talk : Remembering how we calculated gradient yesterday – who can recall how we did it without drawing the graph?

1. Given the coordinates (2, 3) and (5, 7) how can we calculate the gradient with drawing the graph
2. Allow time for a solution
3. Check your solution using the slope button on Geogebra

Activity 4:

1. Handout Worksheet B
    a. Students are to draw the lines when given the equation on the Cartesian planes
    b. Instruct student to then plot the equations into Geogebra to check that their graphs are correct

Teacher Notes: To plot equation into Geogebra - Type in the equation straight into the input bar at the bottom of the screen

2. Handout Worksheet C
    a. Instruct students to find the equation of the line given the graphs
    b. Allow time to complete then discuss as a class – encouraging students to justify how they got their equations

Activity 5:

Teacher Talk: A different way of representing the line is to manipulate the terms such that we put each one on the left hand side to make the equation = 0. This is how Geogebra represents equations in its Algebra window.We know how to change it in Geogebra but how can we do it using pen and paper……. Remember we want y = mx + b

1. Start with y = -3x + 5 and demonstrate on the board
2. Give students another example e.g the given equation y = -1/4x + 2 (and show on the board) noting that no fractions can be used and a must be zero.
3. Allow students to attempt to write some into general form
    i. y = 4x -1
    ii. y = -5x + 2
    iii. y = ½ x + 1

    Copy these onto the board and ask for students to complete

    b. Convert the following equations into General Form
        i. y = 3x + 7
        ii. y = -4x -2
        iii. y = 5 x
        iv. y = -2x + 8
    c. Convert the following from general form to gradient- intercept form
        i. 2x + y – 9 = 0
        ii. 3x - y – 4 = 0
        iii. 10x + 5y – 15 = 0
4. Discuss answers – allowing different students to come forward and perform there conversions on the board.

Lesson 5: Midpoint - 1

Syllabus Outcomes:

- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- uses formulas to find midpoint, gradient and distance on the Cartesian plane, and applies standard forms of the equation of a straight line MA5.3-8NA

Materials Required:	**Learning Intention:**
• Laptops • Worksheet D – Copy 2 to a page	Midpoint

Syllabus Content: Students:

Find the midpoint and gradient of a line segment (interval) on the Cartesian plane (ACMNA294)

- use the concept of an average to establish the formula for the midpoint, M, of the interval joining two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane: $M(x, y) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
- explain the meaning of each of the pronumerals in the formula for midpoint (Communicating)
- use the formula to find the midpoint of the interval joining two points on the Cartesian plane

Sketch linear graphs using the coordinates of two points (ACMNA215)

Activity 1: *(LP 1 – Information)*

Teacher Notes: Initial aim is to discover what students know about the language of midpoint & bisect

1. Explain the scenario of the Murderer to students

Teacher Talk: Over a weekend there was a gruesome murder where a man was bludgeoned in the streets of Leeton and his remains were found in the park behind the main street. On Monday morning the article on the front page of Leeton Irrigator described the murder saying "the man had been dissected". In the article on the front page of The Area News it described the murder saying "the man had been bisected". Do they mean the same? If not what is the difference?

a. Wait for student responses – conclusion should be that to bisect means to cut in half and dissect means to cut in into pieces of different sizes

Teacher Talk: In mathematics, we often talk of bisecting things- in particular lines, which simply means to cut the line in half. Although we don't require a line we might want to find the half-way point between two sets of coordinates this is called the "mid-point"

So we have three different terms to define the middle most point of an interval or two points

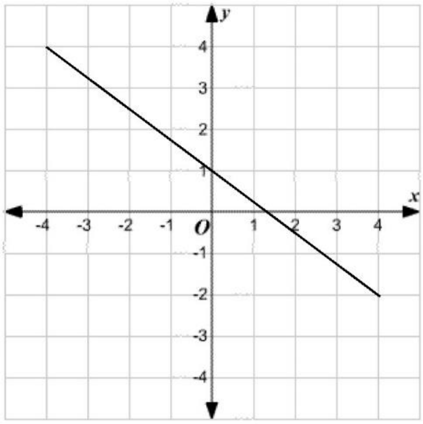
- *Bisect*
- *Half-way*
- *Mid-point*

Activity 2: *(LP 2 – Direction)*

1. Handout Worksheet D

Teacher Talk: Using the following graph we are going to find the mid-point of the line segment or interval drawn in three different ways.

2. Instruct students to cut the three graphs out so you have them all separate.



Teacher Notes: The following youtube clip may help to understand the three methods

http://www.youtube.com/watch?v=bPtQij6E0Zk

3. Instruct the students using a new graph for each method

Method 1 – Point to Point

a. Match the endpoints of the line segment up so that the points meet and fold the paper so that the points remain matched up.
b. Open the paper up and mark the point where the paper fold and the line on the Cartesian plane meet
c. Instruct students to write that point down on the graph.

Method 2 – Horizontal

a. Match the endpoints of the line segment up so that the points match up horizontally and fold the paper.
b. Open the paper up and mark the point where the paper fold and the line on the Cartesian plane meet
c. Write that point down on the graph.

Method 3 – Vertical

a. Using the endpoints of the line segment match them up vertically and fold the paper so that the points remain matched up vertically.
b. Open the paper up and mark the point where the paper fold and the line on the Cartesian plane meet?
c. Write that point down on the graph

Teacher Talk: Let's look at what we have found

1. *Put the three graphs infront of you*
2. *What do you notice about the points you marked on the graphs ?*
3. *Why would each method produce the same answer? How is it that they are the same?*
4. *What does that say about how you might find the midpoint?*

Teacher Notes: Looking to get them to see that horizontal fold related to the x coordinates and vertical fold the y coordinates

5. *So if we look at the end points which are (-4, 4) and (4, -2) can we see some connection between the coordinates and the midpoint?*

Teacher Talk: Lets check with another set of coordinates……

Activity 3: ***(LP 2 – Direction)***

1. Have Geogebra open on projector give students two easy points (1, 2) and (5, 6) plotted on Geogebra with line segment drawn
2. Instruct students to work with the person beside them (in pairs) for 2 minutes, to see if they can find the mid-point **without** paper folding

(LP 3 – New Ideas)

3. Discuss with students how they achieved their solutions and why they think they are correct?

Teacher Notes: Allow time for students to discuss their thoughts to explain their methods

4. Check the answer using Geogebra – plotting the points and using the mid-point button to check
5. *Question Time –*
   a. *Who can explain the method for obtaining the mid-point without folding?*

Activity 4: ***(LP 2 – Direction)***

Teacher Talk: Lets try and predict what the midpoint of the coordinates (1, 5) and (7, 4) might be without Geogebra or folding?

Who thinks they know how to do it?

1. Give students time to have a go at it
2. If someone knows let them try to explain it
3. Check the answer using Geogebra – plotting the points and using the mid-point button to check
4. *Question Time –*
   a. *Who can explain the method for obtaining the mid-point without folding?*

Activity 5: ***(LP 2 – Direction)***

1. Repeat for another set of points (-2, 6) and (4, 10)
2. Get students to copy points and insert line segment in books
3. Students are to work with the person beside them (in pairs) for 2 minutes, to find the mid-point without paper folding

4. Discuss with students how they achieved their solutions and why they think they are correct?
5. Instruct students to plot solution in their books
6. Instruct students to check their answer through using Geogebra – plotting the points and using the mid-point button to check
7. Question Time -
   a. Can anyone see a method for obtaining the mid-point ?
   b. A connection between the values in the coordinate pair and the mid-point?
   c. Is there a way to calculate the mid-point without drawing the points and line interval?
8. If need – do another to establish the connections (-1, 5) and (7, -3)

Activity 6: ***(LP 3 – New Ideas)***

1. Ask students to determine the midpoint of (3, 4) and (1, 2) without drawing it

Teacher Talk: Maths is about communication, communicating ideas and patterns to do this properly can we find a rule, a generalization that we can use for any points without having to draw a diagram?

   a. Establish a method of working it out in words
      i. get students to discuss and explain how to do it in normal terminology
      ii. Next write it in their books as a sentence
   b. Reintroduce the concept of (x_1,y_1) and (x_2,y_2)
   c. Ask them to work out the rule
      i. work out rule using (x_1,y_1) and (x_2,y_2) think about the horizontal and vertical
   d. Discuss whether (x_1,y_1) first in formula or (x_2,y_2)

2. Write formula correctly $M(x,y) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Lesson 6: Midpoint - 2

Syllabus Outcomes:

- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- uses formulas to find midpoint, gradient and distance on the Cartesian plane, and applies standard forms of the equation of a straight line MA5.3-8NA

Materials Required:	**Learning Intention:**
• Laptops with Geogebra •	Midpoint

Syllabus Content: Students:

Find the midpoint and gradient of a line segment (interval) on the Cartesian plane (ACMNA294)

- use the concept of an average to establish the formula for the midpoint, M, of the interval joining two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane: $M(x, y) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
- explain the meaning of each of the pronumerals in the formula for midpoint (Communicating)
- use the formula to find the midpoint of the interval joining two points on the Cartesian plane

Sketch linear graphs using the coordinates of two points (ACMNA215)

Short Question Revision for Last Lesson:

1. How can we find the midpoint of the interval between (1, 5) and (3, 8) using the formula used last lesson?
2. Draw the interval on a graph and check your answer
3. Then check your answer by plotting the points on Geogebra and using the midpoint tool
4. How can we find the midpoint of the interval between (-4, 0) and (2, -5) using the formula?
5. Draw the interval on a graph and check your answer
6. Then check your answer by plotting the points on Geogebra and using the midpoint tool
7. How can we find the midpoint of the interval between (-3, -5) and (6, 7) using the formula?
8. Then check your answer by plotting the points on Geogebra and using the midpoint tool

Activity 1: *(LP 4 - Ownership)*

Write the following problems on the board and allow students to work through them in their pairs

1. If an interval starts at (1, 4) and the midpoint is (7, -2) what are the coordinates of the endpoint?
2. The midpoint M of a line segment AB is (4, 6). If the co-ordinate of A is (1, 2) what is the coordinate of B?

Teacher Notes: Continue the use of terminology such as 'BISECT'

Activity 2:

1. Using Geogebra plot the following points A(6, 2), B(4, -3), C(-4, -3) and D(-2, 2). Join the intervals using straight lines to form a quadrilateral. Find the
   a. midpoint of the diagonal AC
   b. midpoint of the diagonal BD
   c. what quadrilateral has been formed?

Activity 3: *(LP 5 – Integration)*

1. Students are to design 3 questions of different levels of difficulty regarding midpoint within the Geogebra environment.
2. Using the textbox button they are to write the problem and solution In the Geogebra
3. They can plot the points and check their working using Geogebra
4. Any additional information or ideas (such as the formula for mid-point) is to be placed in a separate text box on the same Geogebra page.

Lesson 7: Distance - 1

Syllabus Outcomes:

- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- uses formulas to find midpoint, gradient and distance on the Cartesian plane, and applies standard forms of the equation of a straight line MA5.3-8NA

Materials Required:

- Laptops
- Load Geogebra page with MAP on it
- Worksheet I - MAP

Learning Intention:

Distance between two points

Syllabus Content: Students:

Find the distance between two points located on the Cartesian plane (ACMNA214)

- use Pythagoras' theorem to establish the formula for the distance, d, between two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
  → explain the meaning of each of the pronumerals in the formula for distance (Communicating)
- use the formula to find the distance between two points on the Cartesian plane
  → explain why the formula $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ gives the same value for the distance as $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (Communicating, Reasoning)

Solve a variety of problems by applying coordinate geometry formulas

→ derive the formula for the distance between two points (Reasoning)
→ show that three given points are collinear (Communicating, Reasoning)

1. Handout Worksheet I – MAP
2. Open Geogebra MAP on projector
3. Talk about the scenario then commence with Questions below

Activity 1: *(LP 1 – Information)*

1. *Question Time:*
   a. *There is one obstacle point that is midpoint between two other obstacles – which obstacle represents that midpoint?*
   b. *How can you prove that it is the midpoint? Do this on the worksheets*

Teacher Talk: Now you notice that if you were to join the three points they look as though they would lie in a line ………… there is a mathematical term for this – collinear…….that is when three of more coordinate points all lie on the same line

   c. *The three landmark points are 'collinear'…. How can you prove three points are collinear? First try using Geogebra then prove it on your worksheets using pen and paper.*

Teacher Notes: The next activity is teacher guided however the students will work along on their worksheet.

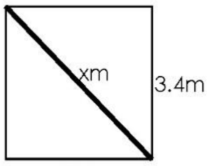
Activity 2: *(LP 1 – Information: This revises the topic Pythagoras Theorem)*

1. Let's see how we can calculate the distance between two obstacles say Step Up and Tyre Dread
2. Using printed Geogebra MAP let's construct a right angled triangle from Step Up to Tyre Dread
3. How did you construct it?

4. The distance from Step UP to Tyre Dread is known as the ___________________________ of a right angled triangle.
5. How can you find the value of the hypotenuse? What theorem do we associate with right angled triangles?
6. Can anyone explain how it would work to find the distance from Step Up to Tyre Dread
7. Use Pythagoras' Theorem to find the distance
8. Complete Activity 3 on the worksheet (if necessary)

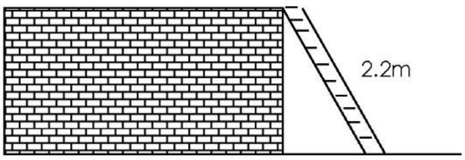
Activity 3:

1. A builder is constructing a wall in a house which is 2.6m long and 3.4m high. He needs extra strength in his wall and must build a diagonal support in the framework as indicated in the diagram below. What length of wood is required to construct the diagonal support?



Activity 4:

1. A house has a blocked gutter and so the owner decides to get a ladder and place it against the house to reach the gutter as shown in the diagram below.



If the ladder is 2.2m long and the owner has placed it 1.7 m from the wall – how high does the ladder reach up the wall?

Teacher Talk: Today we are going take all this information we have available to us… you know how to graph points, you know how to do the distance between points and you know [phythagoras points we are going to use this to find the distance between two coordinates….

Activity 5: ***(LP 2 – Direction)***

1. In your books plot the points (2, 1) and (5, 7)
2. How could you find the distance between the two points?

Teacher Notes: Hoepfully someone will make the connection that Pythagoras Theorem will help tp find the solution

3. By drawing a right angled triangle find the hypotenuse which represents the distance between the two points using Pythagoras Theorem
4. Plot the points on Geogebra then find the distance using the distance tool to check your answer.

Activity 6: *(LP 2 – Direction)*

1. In your books plot the points (-1, 5) and (3, -2)
2. Find the distance between the two points

5. Plot the points on Geogebra then find the distance using the distance tool to check your answer.

Activity 7: *(LP 2 – Direction)*

1. In your books plot the points (2, 8) and (-4, -3)
2. Find the distance between the two points
3. Plot the points on Geogebra then using the distance tool check your answer.

Activity 8: *(LP 2 – Direction)*

1. In your books plot the points (-1, -5) and (-4, -9)
2. Find the distance between the two points
3. Plot the points on Geogebra then using the distance tool check your answer.

Teacher Talk: We are spending a lot of time doing the line and drawing the right angled triangle on the graph – is there a way we can do it that doesn't involve drawing it?

Activity 9: *(LP 3 –Formalising the language and developing the rule)*

1. Question Time - *How can we find the distance between the two points without drawing a triangle?*
2. *What connection is there between the method of finding the distance and the coordinates?*
3. *A generalization that we can use for any points without having to draw a diagram?*
   a. Establish a method of working it out in words
   b. Reaffirm the concept of (x_1,y_1) and (x_2,y_2)
   c. Get them to work out rule using (x_1,y_1) and (x_2,y_2)
   d. Discuss whether (x_1,y_1) first in formula or (x_2,y_2)
4. Write formula correctly $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Lesson 8: Distance - 2

Syllabus Outcomes:

- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- uses formulas to find midpoint, gradient and distance on the Cartesian plane, and applies standard forms of the equation of a straight line MA5.3-8NA

Materials Required:

- Laptops
- Load Geogebra page with MAP on it
- Worksheet MAP

Learning Intention:

Distance without drawing a graph

Syllabus Content: Students:

Find the distance between two points located on the Cartesian plane (ACMNA214)

- use Pythagoras' theorem to establish the formula for the distance, d, between two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
  - → explain the meaning of each of the pronumerals in the formula for distance (Communicating)
- use the formula to find the distance between two points on the Cartesian plane
  - → explain why the formula $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ gives the same value for the distance as $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (Communicating, Reasoning)

Solve a variety of problems by applying coordinate geometry formulas

- → derive the formula for the distance between two points (Reasoning)
- → show that three given points are collinear (Communicating, Reasoning)

Write the following problems on the board

Heading in books – **Distance Formula Problems**

Activity 1: *(LP 3 – New Ideas)*

1. Find the distance between (2, 5) and (6, 8) using the distance formula you have derived last lesson without drawing a graph.
2. Check the solution by plotting the points on Geogebra then using the distance tool to calculate the distance between the points.

Activity 2: *(LP 3 – New Ideas)*

1. Find the distance between (-3, 1) and (5, 13) using the distance formula without plotting it on a graph
2. Check the solution by plotting the points on Geogebra then using the distance tool to calculate the distance between the points.

Activity 3: *(LP 3 – New Ideas)*

1. Find the distance between (-4, -2) and (8, 7) without plotting it on a graph
2. Check the solution by plotting the points on Geogebra then using the distance tool to calculate the distance between the points.

Activity 4: *(LP 4 – Ownership)*

1. Handout the Worksheet – MAP (which is also available for students as a ggb file in the Handouts folder)
2. Read the scenario on the sheet and then the following instructions will be
3. Using Worksheet - MAP handout find the distance the following landmark points using the

distance formula in your book between

a. Camp and Hurdle Hop
b. Rope Climb and Flying Fox
c. Ladder Crawl and Tyre Dread

4. Allow students time to complete this
5. Instruct students to check their answers using the Geogebra MAP.ggb file and using the distance tool
6. Using the MAP handout find the obstacle point which is 5 units away from Rope Climb in your books and prove that it is this distance.
7. Check your answer using the Geogebra MAP.ggb and using the distance tool
8. There is one obstacle that is precisely mid-point f two other obstacles – if the mid-point obstacle and prove that it is mid-point using appropriate formulas.
9. Allow students time to complete this
10. Instruct students to check their answer using the Geogebra MAP.ggb file and using the mid-point tool

Activity 5: ***(LP 4 – Ownership)***

1. Find the possible coordinates of point B if its distance from point A (2, 5) is 8 units using the distance formula you have derived

Teacher Notes: This is giving the students the distance and students need to find the missing point. Remembering there are different solutions depending on which way the students travels.

Activity 6:

Heading in books - **y = -1/2 x + 2** ***(LP 4 – Ownership)***

1. Instruct students to draw the line y = -1/2x + 2 in their books
2. Does the point (1, 1.5) lie on the line? Show if it does.
3. Plot the line and the point in Geogebra and confirm your findings
4. Question Time: If you travel from the point (1, 1.5) 5 units where could you land?
5. Students can use Geogebra to prove this and then use appropriate formulas to prove it in their books

Activity 7 (if necessary): ***(LP 5 - Integration)***

1. Students are to design 3 questions of different levels of difficulty regarding distance within the Geogebra environment.
2. Using the textbox button they are to write the problem and solution In the Geogebra
3. They can plot the points and check their working using Geogebra
4. Any additional information or ideas (such as the formula for mid-point) is to be placed in a separate text box on the same Geogebra page.

Lesson 9: Further Graphing - 1

Syllabus Outcomes:

- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- uses formulas to find midpoint, gradient and distance on the Cartesian plane, and applies standard forms of the equation of a straight line MA5.3-8NA

Materials Required:

- Laptops
- Worksheet E, F, G and H

Learning Intention:

Parallel and Perpendicular Lines

Syllabus Content: Students:

Find the midpoint and gradient of a line segment (interval) on the Cartesian plane (ACMNA294)

- use the relationship $gradient = \frac{rise}{run}$ to establish the formula for the gradient, m, of the interval joining two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane: $m = \frac{y_2 - y_1}{x_2 - x_1}$

Sketch linear graphs using the coordinates of two points (ACMNA215)

Solve problems using various standard forms of the equation of a straight line

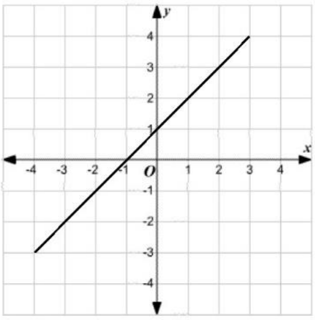
- describe the equation of a line as the relationship between the x- and y-coordinates of any point on the line
  - → recognise from a list of equations those that can be represented as straight-line graphs (Communicating, Reasoning)

Solve problems involving parallel and perpendicular lines (ACMNA238)

- find the equation of a line that is parallel or perpendicular to a given line
- determine whether two given lines are perpendicular
  - → use gradients to show that two given lines are perpendicular (Communicating, Problem Solving)

Activity 1: Short Introduction and Revision of Last Lesson *(LP 1 – Information)*

1. Brainstorm everything that can be identified on the following graph



2. Identify the equation of the line

Activity 2: *(LP 2 - Direction)*

Teacher Talk: Lets take a closer look at straight line graphs

1. Handout Worksheets E and F
2. Instruct students to draw the graphs (in the second column) using their own methods
3. Then they are to check their graphs by plotting the equations into the input bar of Geogebra and then draw that graph on the worksheet in the last column
4. Allow time for this to be completed

Teacher Notes: Students may get bored with the number of graphs they are required to draw however it should enable then to become quicker and more effective at graph drawing. Encourage students to draw the whole page then check the whole page on Geogebra as their confidence grows (and boredom sets in!)

5. *Question Time: Is there any connections that are evident between the graphs from worksheet E and F*

Teacher Notes: Possibly some student will identify parallel lines if so encourage students to continue with next worksheets to confirm their ideas. If not continue to next worksheets monitoring the drawing of graphs continuing to ask the same question – what connections are evident?

6. Handout Worksheets G and H
7. Instruct students to draw the graphs (in the second column) using their own methods
8. Then they are to check their graphs by plotting the equations into the input bar of Geogebra and then draw that graph on the worksheet in the last column
9. Allow time for this to be completed

Teacher Notes: Students may get bored with the number of graphs they are required to draw however it should enable then to become quicker and more effective at graph drawing. Encourage students to draw the whole page then check the whole page on Geogebra as their confidence grows (and boredom sets in!)

10. *Question Time:* ***(LP 3 – New Ideas)***
    a. *What connections or relationships are evident between the graphs we have drawn?*
        i. *Y-intercept*
        ii. *Parallel lines*
            - *Lines with the same slope*
            - *The number in front of the x is the same (explain this is called the coefficient)*
        iii. *Perpendicular lines*
            - *Lines are at right angles to each other*
            - *The number in front of the x in the equation are*
                → *Opposite in sign*
                → *Upside down (explain this as reciprocal)*
    b. *How can we explain these in our own words?*

Teacher Notes: Encourage students to use their own words and write down in their books if necessary. The importance of developing correct language is vital. By offering these questions students are initiating the learning themselves.

    c. *Can we put this ideas into mathematical terms*
        i. *The concept that gradients must be the same for parallel lines –* $m_1 = m_2$
        ii. *The y-intercept is different for parallel lines*
        iii. *Perpendicular lines occur when the gradients are negative reciprocal*

Teacher Notes: To plot a specific point type the point into the input bar at the bottom of the page in

this format (x,y).

d. Can we predict certain properties if given equations?

Activity 3: ***(LP 3 – New Ideas)***

1. Copy out the following problems on the board
2. Give an example of an equation of a line that is parallel to
   a. y = 2x + 4
   b. y = -5x -1
   c. y = ½ x - 6
3. Instruct students to check their predictions using Geogebra

Teacher Notes: To find a parallel line students may either

1. **Plot the original line – by typing directly into the input bar then click on the Parallel Line tool which requires you to click a point and the line you want to be parallel to (remember to click on equation to rearrange into gradient-intercept formula – equation y = mx + b)**
2. **Plot both lines by typing directly into the input bar and move one line to directly superimpose it onto the other demonstrating it has the same slope.**

4. Give an example of an equation of a line that is perpendicular to
   a. y = 2x + 4
   b. y = 2/3x + 5
   c. y = -4x – 3
   d. y = -1/2x – 2
5. Instruct students to check their predictions using Geogebra

Teacher Notes: To find a perpendicular line students may either

3. **Plot the original line – by typing directly into the input bar then click on the Perpendicular Line tool which requires you to click a point and the line you want to be perpendicular to (remember to click on equation to rearrange into gradient-intercept formula – equation y = mx + b)**
4. **Plot both lines by typing directly into the input bar and using the Angle tool find the angle between the two lines.**

Lesson 10: Further Graphing - 2

Syllabus Outcomes:

- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- uses formulas to find midpoint, gradient and distance on the Cartesian plane, and applies standard forms of the equation of a straight line MA5.3-8NA

Materials Required:

- Laptops
- Matching Game Card Sets

Learning Intention:

Consolidating Graph Features

Syllabus Content: Students:

Find the midpoint and gradient of a line segment (interval) on the Cartesian plane (ACMNA294)

- use the relationship $gradient = \frac{rise}{run}$ to establish the formula for the gradient, m, of the interval joining two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane: $m = \frac{y_2-y_1}{x_2-x_1}$
- use the formula to find the gradient of the interval joining two points on the Cartesian plane explain why the formula $m = \frac{y_2-y_1}{x_2-x_1}$ gives the same value for the gradient as $m = \frac{y_2-y_1}{x_2-x_1}$ (Communicating, Reasoning)

Sketch linear graphs using the coordinates of two points (ACMNA215)

Solve problems using various standard forms of the equation of a straight line

- describe the equation of a line as the relationship between the x- and y-coordinates of any point on the line
  - → recognise from a list of equations those that can be represented as straight-line graphs (Communicating, Reasoning)
- rearrange linear equations in gradient-intercept form ($y = mx + b$) into general form $ax + by + c = 0$
- find the equation of a line passing through two points

Solve problems involving parallel and perpendicular lines (ACMNA238)

- find the equation of a line that is parallel or perpendicular to a given line
- determine whether two given lines are perpendicular
  - → use gradients to show that two given lines are perpendicular (Communicating, Problem Solving)

Activity 1: Matching Games *(LP 4 – Ownership)*

1. Students are put into pairs.
2. Each pair is given a set of cards (there are two separate games) which include a mixture of rule, tables, graphs and slope/y-intercept cards.
3. Students must match the cards together that represent the same equation.
4. Students swap games and see if they can match the cards into the correct groups.

Teacher Notes: Students can work on the floor spreading out the cards to sort into categories

Activity 2: *(LP 5 - Integration)*

1. Students are to summarise parallel and perpendicular lines within the Geogebra environment.
2. Open a new window in Geogebra - FILE → NEW WINDOW
3. Draw two parallel lines and two perpendicular lines in Geogebra using any method
4. Using the textbox button they are to write the how they can identify that these lines are parallel and perpendicular lines
5. Any additional information or ideas (such as any formulas required) is to be placed in a separate text box on the same Geogebra page.
6. Save this file in your Linear Relationship Folder as "Parallel and Perpendicular Lines"

Lesson 11: Geometric Problems - 1

Syllabus Outcomes:

- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- uses formulas to find midpoint, gradient and distance on the Cartesian plane, and applies standard forms of the equation of a straight line MA5.3-8NA

Materials Required:	**Learning Intention:**
• Laptops	Coordinate Geometry Methods with Triangles

Syllabus Content: Students:

Find the midpoint and gradient of a line segment (interval) on the Cartesian plane (ACMNA294)

- use the relationship $gradient = \frac{rise}{run}$ to establish the formula for the gradient, m, of the interval joining two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane: $m = \frac{y_2 - y_1}{x_2 - x_1}$
- use the formula to find the gradient of the interval joining two points on the Cartesian plane explain why the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ gives the same value for the gradient as $m = \frac{y_2 - y_1}{x_2 - x_1}$ (Communicating, Reasoning)

Find the distance between two points located on the Cartesian plane (ACMNA214)

- use Pythagoras' theorem to establish the formula for the distance, d, between two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
  - → explain the meaning of each of the pronumerals in the formula for distance (Communicating)
- use the formula to find the distance between two points on the Cartesian plane
  - → explain why the formula $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ gives the same value for the distance as $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (Communicating, Reasoning)

Solve a variety of problems by applying coordinate geometry formulas

- → use coordinate geometry to investigate and describe the properties of triangles and quadrilaterals (Communicating, Problem Solving, Reasoning)
- → use coordinate geometry to investigate the intersection of the perpendicular bisectors of the sides of acute-angled triangles (Problem Solving, Reasoning)
- → prove that a particular triangle drawn on the Cartesian plane is right-angled (Communicating, Reasoning)

Heading in Books – **Geometric Problems**

1. Write the following problems on the board and ask students to follow the instructions provided – annotate as necessary.
2. Students work in pairs.
3. You need to use what you know about
   a. Straight lines
   b. Mid-point
   c. Distance between two points
   d. Gradient - Parallel and Perpendicular Lines

   To solve the following problems

Teacher Notes: Students work in pairs to enable discussion so that correct language may be used and monitored by the teacher and assists each student to solve the problems. Students must use the concepts listed above not Geogebra tools such as angles to prove the following problems.

Activity 1: *(LP 4 - Ownership)*

1. Open Geogebra

2. Plot points A(0, -3), B(-2, -1) and C(4, 3)

Teacher Notes: To plot a specific point type the point into the input bar at the bottom of the page in this format (x,y).

3. Use the Geogebra tools to prove that Δ ABC is Isosceles
4. Anything you have done using Geogebra show on paper in your books using formulas that are appropriate to the proof
5. Using the diagram you already have in Geogebra now create a textbox on the same page which outlines how you proved that Δ ABC is Isosceles

Teacher Notes: Textbox could be as simple as AC = AB as measured and in their books would be distance formula used for both AC and AB.

6. Save this file in your Linear Relationship Folder as "Isosceles Triangle"

Activity 2: ***(LP 4 - Ownership)***

1. Open a new window in Geogebra - FILE → NEW WINDOW
2. Plot points D(5, 6), E(9, 3) and F(5, 3)
3. Use Geogebra tools to prove that Δ DEF is Right Angled
4. Next prove what you have done using Geogebra in your books using formulas that are appropriate to the proof.
5. Using the diagram you already have in Geogebra now create a textbox on the same page which outlines how you proved Δ DEF was Right Angled
6. Save this file in your Linear Relationship Folder as "Right Angled Tringle"

Activity 3: ***(LP 4 - Ownership)***

1. Open a new window in Geogebra - FILE → NEW WINDOW
2. Plot points L(1, 4), M(1, 2) and N$(1+\sqrt{3},3)$

Teacher Notes: Point N may need to be input as a decimal. Students will need to convert this first.

3. Use Geogebra tools to prove that Δ LMN is an equilateral triangle
4. Next prove what you have done using Geogebra in your books using formulas that are appropriate to the proof.
5. Using the diagram you already have in Geogebra now create a textbox on the same page which outlines why Δ LMN is equilateral and how you proved it
6. Save this file in your Linear Relationship Folder as "Equilateral Triagnle"

Extra Activity (if needed):

Activity 3: ***(LP 4 - Ownership)***

1. Open a new window in Geogebra - FILE → NEW WINDOW
2. Do the same but students must create their own scalene triangle
3. Next prove what you have done using Geogebra in your books using formulas that are appropriate to the proof.
4. Using the diagram you have already in Geobgebra now create a textbox on the same page which outlines how proved it was scalene

Lesson 12: Geometric Problems - 2

Syllabus Outcomes:

- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- uses formulas to find midpoint, gradient and distance on the Cartesian plane, and applies standard forms of the equation of a straight line MA5.3-8NA

Materials Required:	**Learning Intention:**
• Laptops	Coordinate Geometry Methods with Quadrilaterals

Syllabus Content: Students:

Find the midpoint and gradient of a line segment (interval) on the Cartesian plane (ACMNA294)

- use the relationship $gradient = \frac{rise}{run}$ to establish the formula for the gradient, m, of the interval joining two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane: $m = \frac{y_2 - y_1}{x_2 - x_1}$
- use the formula to find the gradient of the interval joining two points on the Cartesian plane explain why the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ gives the same value for the gradient as $m = \frac{y_2 - y_1}{x_2 - x_1}$ (Communicating, Reasoning)

Find the distance between two points located on the Cartesian plane (ACMNA214)

- use Pythagoras' theorem to establish the formula for the distance, d, between two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
  - → explain the meaning of each of the pronumerals in the formula for distance (Communicating)
- use the formula to find the distance between two points on the Cartesian plane
  - → explain why the formula $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ gives the same value for the distance as $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (Communicating, Reasoning)

Solve a variety of problems by applying coordinate geometry formulas

- → derive the formula for the distance between two points (Reasoning)
- → show that three given points are collinear (Communicating, Reasoning)
- → use coordinate geometry to investigate and describe the properties of triangles and quadrilaterals (Communicating, Problem Solving, Reasoning)
- → use coordinate geometry to investigate the intersection of the perpendicular bisectors of the sides of acute-angled triangles (Problem Solving, Reasoning)
- → show that four specified points form the vertices of particular quadrilaterals (Communicating, Problem Solving, Reasoning)
- → prove that a particular triangle drawn on the Cartesian plane is right-angled (Communicating, Reasoning)

Heading in Books – **Geometric Problems**

1. Write the following problems on the board and ask students to follow the instructions provided – annotate as necessary.
2. Students work in pairs.
3. These problems can be solved
   a. Straight lines
   b. Mid-point
   c. Distance between two points
   d. Gradient - Parallel and Perpendicular Lines

Teacher Notes: Students work in pairs to enable discussion so that correct language may be used and monitored by the teacher and assists each student to solve the problems.

Activity 1: ***(LP 4 - Ownership)***

4. Open a new window in Geogebra - FILE → NEW WINDOW
5. Use Geogebra tools to find the equation of the perpendicular bisector of the line joining the points (-2, 9) and (4, 0).
6. Now prove what you have done using Geogebra in your books using formulas that are appropriate to the proof.

7. Using the diagram you already have in Geogebra now create a textbox on the same page which outlines how you found the equation of the perpendicular bisector of the line joining the points (-2, 9) and (4, 0).
8. Save this file in your Linear Relationship Folder as "Perpendicular Bisector"

Teacher Notes: The idea here is that the students create a summary sheet within the Geogebra environment so they consolidate the concepts behind how they achieved their goal of find the perpendicular bisector.

Activity 2: *(LP 4 - Ownership)*

9. Open a new window in Geogebra - FILE → NEW WINDOW
10. Plot points A(-1, 2), B(3, 0), C(4, 6) and D(0, 4)
11. Use Geogebra tools to prove that the quadrilateral ABCD is a parallelogram – how many ways can you do it?

Teacher Notes: You may give as many hints and ideas you want so long as the using the concepts and how many ways can they do it?

12. Now prove what you have done using Geogebra in your books using formulas that are appropriate to the proof.
13. Using the diagram you already have in Geogebra now create a textbox on the same page which outlines how you proved that ABCD is a parallelogram
14. Save this file in your Linear Relationship Folder as "Parallelogram Proof"

Activity 3: *(LP 4 - Ownership)*

15. Open a new window in Geogebra - FILE → NEW WINDOW
16. Plot points L(0, 0), M(4, 2), N(3, 3) and O(1, 2)
17. Use Geogebra tools to prove that the quadrilateral ABCD is a trapezium – how many ways can you do it?
18. Now prove what you have done using Geogebra in your books using formulas that are appropriate to the proof.
19. Using the diagram you already have in Geogebra now create a textbox on the same page which outlines how you proved LMNO is a trapesium
20. Save this file in your Linear Relationship Folder as "Trapesium Proof"

Activity 4: *(LP 4 - Ownership)*

21. Open a new window in Geogebra - FILE → NEW WINDOW
22. Plot points S(-2, 0), T(1, 4), U(6, 4) and V(3, 0)
23. Use Geogebra tools to prove that the quadrilateral ABCD is a rhombus – how many ways can you do it?
24. Now prove what you have done using Geogebra in your books using formulas that are appropriate to the proof.
25. Using the diagram you already have in Geogebra now create a textbox on the same page which outlines how you proved STUV is a rhombus
26. Save this file in your Linear Relationship Folder as "Rhombus Proof"

Extra Activity (if needed): *(LP 4 - Ownership)*

1. Open a new window in Geogebra - FILE → NEW WINDOW
2. Plot points A(2, 1), B(4, 4), C(6, 3) and D(6, 1)
27. Use Geogebra tools to prove that the quadrilateral ABCD is a kite – how many ways can you

do it?

3. Now prove what you have done using Geogebra in your books using formulas that are appropriate to the proof.
4. Using the diagram you already have in Geogebra now create a textbox on the same page which outlines how you proved ABCD is a kite
5. Save this file in your Linear Relationship Folder as "Kite Proof"

Lesson 13: End of Topic Task

Syllabus Outcomes:

- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- uses formulas to find midpoint, gradient and distance on the Cartesian plane, and applies standard forms of the equation of a straight line MA5.3-8NA

Materials Required:	**Learning Intention:**
• Laptops with Geogebra installed	• Geogebra

Syllabus Content: Students:

Find the midpoint and gradient of a line segment (interval) on the Cartesian plane (ACMNA294)

- use the concept of an average to establish the formula for the midpoint, *M*, of the interval joining two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane: $M(x, y) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
- use the formula to find the midpoint of the interval joining two points on the Cartesian plane
- use the relationship $gradient = \frac{rise}{run}$ to establish the formula for the gradient, *m*, of the interval joining two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane: $m = \frac{y_2-y_1}{x_2-x_1}$
- use the formula to find the gradient of the interval joining two points on the Cartesian plane explain why the formula $m = \frac{y_2-y_1}{x_2-x_1}$ gives the same value for the gradient as $m = \frac{y_2-y_1}{x_2-x_1}$ (Communicating, Reasoning)

Find the distance between two points located on the Cartesian plane (ACMNA214)

- use Pythagoras' theorem to establish the formula for the distance, *d*, between two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
  - → explain the meaning of each of the pronumerals in the formula for distance (Communicating)
- use the formula to find the distance between two points on the Cartesian plane

Sketch linear graphs using the coordinates of two points (ACMNA215)

Solve problems using various standard forms of the equation of a straight line

- describe the equation of a line as the relationship between the *x*- and *y*-coordinates of any point on the line
  - → recognise from a list of equations those that can be represented as straight-line graphs (Communicating, Reasoning)
- rearrange linear equations in gradient-intercept form ($y = mx + b$) into general form $ax + by + c = 0$
- find the equation of a line passing through two points

Solve problems involving parallel and perpendicular lines (ACMNA238)

- find the equation of a line that is parallel or perpendicular to a given line
- determine whether two given lines are perpendicular
  - → use gradients to show that two given lines are perpendicular (Communicating, Problem Solving)

1. Each student is to receive a piece of scrap paper
2. Students are instructed to put their name clearly marked at the top of the paper which will be collected at the end of the lesson
3. Students are instructed to work in pairs using their laptops (STUDENT OR TEACHER SELECTED PAIRS ???)
   - i. one student opens their laptop onto Geogebra
   - ii. the other student opens their laptop onto link of Google forms
   - iii. only one copy of solutions needs to be submitted
   - iv. Link to get students there is emailed to students?

https://docs.google.com/a/ww.catholic.edu.au/forms/d/1yTR4ZGKMkiqCp8to8s2XOTvhUr-94E4JxUnrO4NFpXY/edit?usp=sharing

4. Students who have approved to be recorded can record their work
5. *Pre-test needs little teacher intervention students will engage in the learning themselves.*
6. *It is a task based on what students should know about Linear Functions in year 9 – hence some students may be able to complete all, some or none.*

Lesson 14: Delayed Post Test

Syllabus Outcomes:

- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- uses formulas to find midpoint, gradient and distance on the Cartesian plane, and applies standard forms of the equation of a straight line MA5.3-8NA

Materials Required:	**Learning Intention:**
• Laptops with Geogebra installed	• Geogebra

Syllabus Content: Students:

Find the midpoint and gradient of a line segment (interval) on the Cartesian plane (ACMNA294)

- use the concept of an average to establish the formula for the midpoint, *M*, of the interval joining two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane: $M(x, y) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
- use the formula to find the midpoint of the interval joining two points on the Cartesian plane
- use the relationship $gradient = \frac{rise}{run}$ to establish the formula for the gradient, *m*, of the interval joining two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane: $m = \frac{y_2-y_1}{x_2-x_1}$
- use the formula to find the gradient of the interval joining two points on the Cartesian plane explain why the formula $m = \frac{y_2-y_1}{x_2-x_1}$ gives the same value for the gradient as $m = \frac{y_2-y_1}{x_2-x_1}$ (Communicating, Reasoning)

Find the distance between two points located on the Cartesian plane (ACMNA214)

- use Pythagoras' theorem to establish the formula for the distance, *d*, between two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
  - → explain the meaning of each of the pronumerals in the formula for distance (Communicating)
- use the formula to find the distance between two points on the Cartesian plane

Sketch linear graphs using the coordinates of two points (ACMNA215)

Solve problems using various standard forms of the equation of a straight line

- describe the equation of a line as the relationship between the *x*- and *y*-coordinates of any point on the line
  - → recognise from a list of equations those that can be represented as straight-line graphs (Communicating, Reasoning)
- rearrange linear equations in gradient-intercept form ($y = mx + b$) into general form $ax + by + c = 0$
- find the equation of a line passing through two points

Solve problems involving parallel and perpendicular lines (ACMNA238)

- find the equation of a line that is parallel or perpendicular to a given line
- determine whether two given lines are perpendicular
  - → use gradients to show that two given lines are perpendicular (Communicating, Problem Solving)

Content:

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2. Students are instructed to put their name clearly marked at the top of the paper which will be collected at the end of the lesson
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   i. one student opens their laptop onto Geogebra
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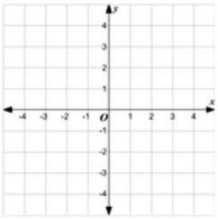
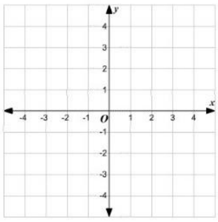
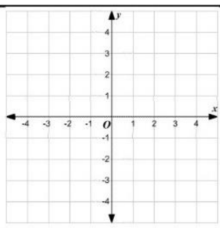
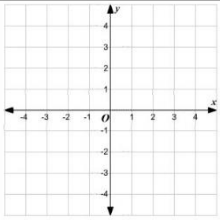
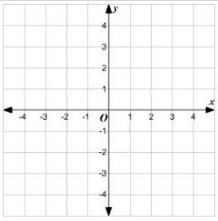
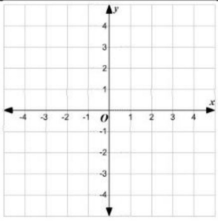
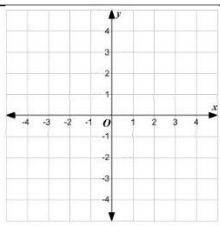
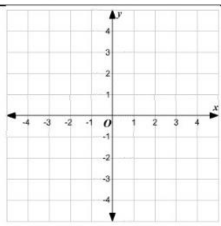
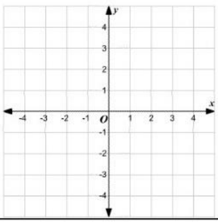
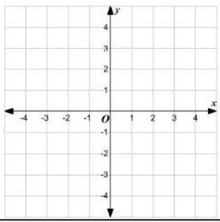
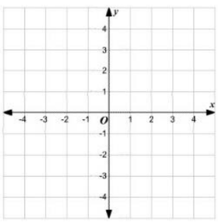
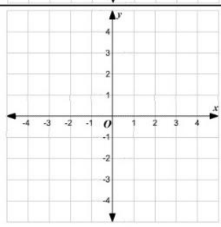
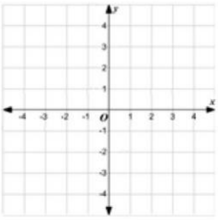
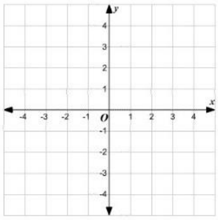
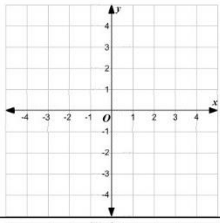
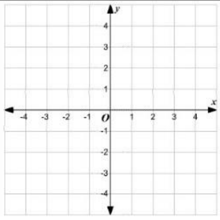
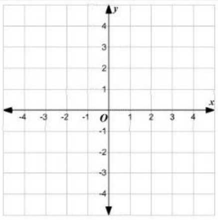
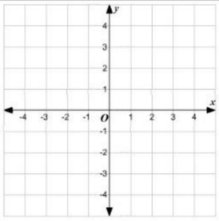
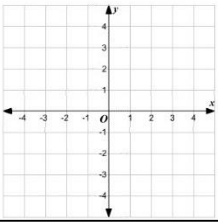
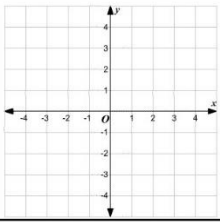
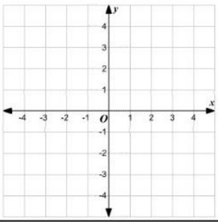
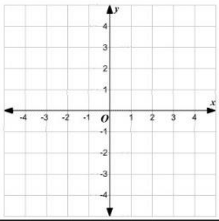
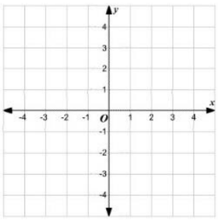
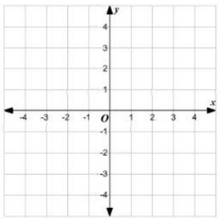
https://docs.google.com/a/ww.catholic.edu.au/forms/d/1yTR4ZGKMkiqCp8to8s2XOTvhUr-94E4JxUnrO4NFpXY/edit?usp=sharing

4. Students who have approved to be recorded can record their work

5. *Pre-test needs little teacher intervention students will engage in the learning themselves.*
6. *It is a task based on what students should know about Linear Functions in year 9 – hence some students may be able to complete all, some or none.*

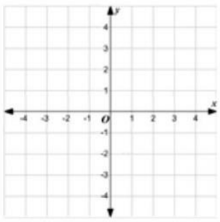
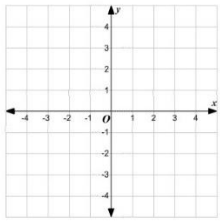
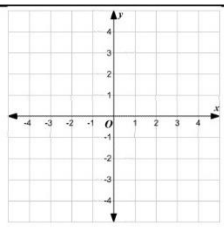
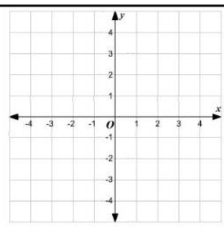
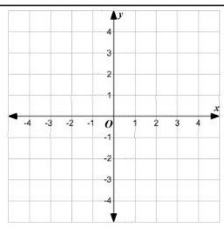
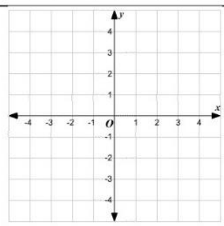
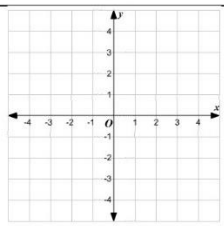
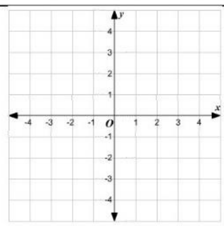
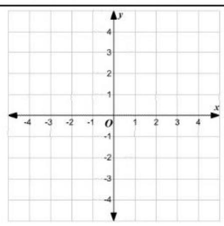
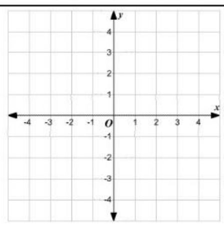
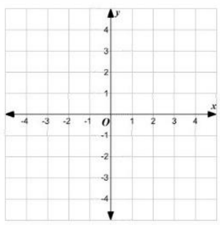
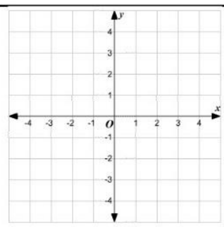
Appendix B: Lesson Worksheets Linear Relationships

WORKSHEET A - 1		Name: ____________________	
y = 2x + 1		y = ½ x + 1	
y = -2x + 1		y = ½ x + 2	
y = 2x + 3		y = -½ x - 1	
y = 2x - 1		y = -½ x + 1	
y = x + 2		y = x + 3	
y = -x + 2		y = x - 1	

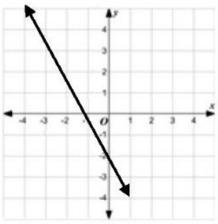
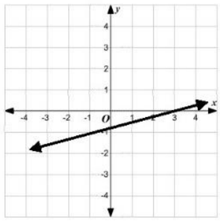
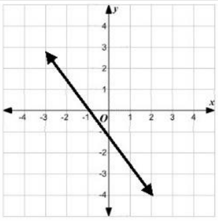
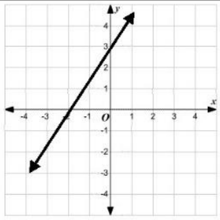
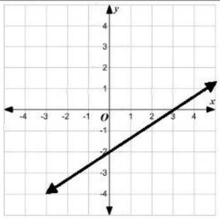
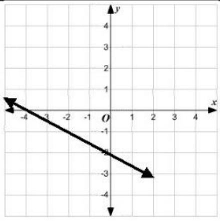
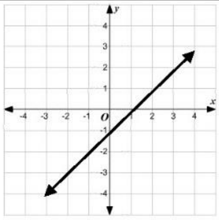
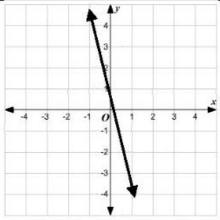
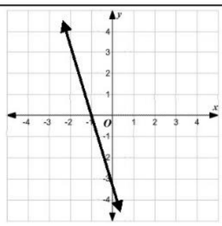
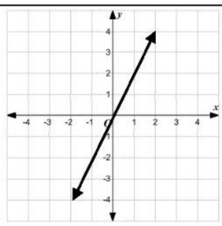
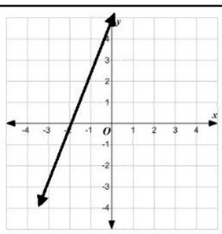
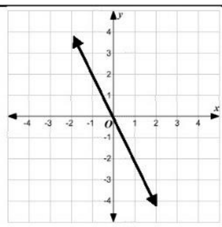
WORKSHEET A - 2		Name: ____________________	
y = ¼ x - 4		y = 3 x + 1	
y = ¼ x		y = 3 x + 2	
y = ¼ x +1		y = -3 x - 1	
y = - ¼ x - 1		y = -3 x + 1	
y = - ¼ x + 1		y = 3x - 2	
y = - ¼ x + 2		y = -3 x + 2	

WORKSHEET B

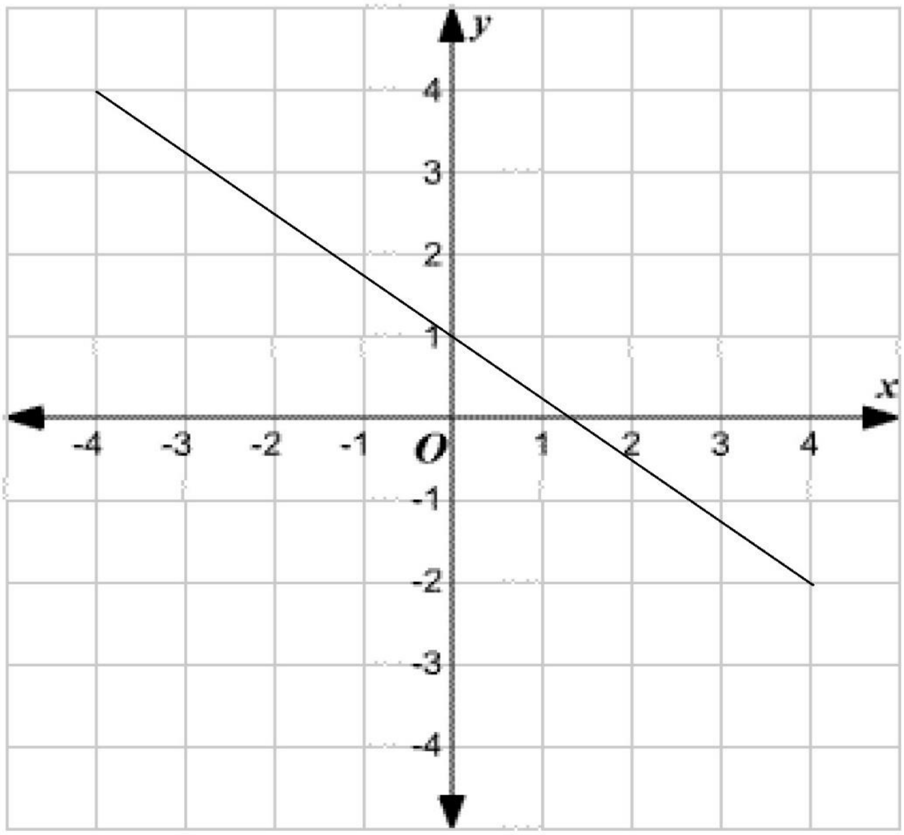
Name: ____________________

y = 3x - 4		y = 5 x - 4	
y = 2x - 3		y = ¼ x - 1	
y = 3x -1		y = -½ x - 2	
y = -2x + 2		y = -½ x + 3	
y = x -4		y = 4x -2	
y = -6x - 4		y = 3x - 3	

WORKSHEET C **Name: ____________________**

m = b = Equation =		Equation =	
m = b = Equation =		Equation =	
m = b = Equation =		Equation =	
m = b = Equation =		Equation =	
m = b = Equation =		Equation =	
m = b = Equation =		Equation =	

WORKSHEET D Name: ____________________



WORKSHEET E	Name: ____________________	
$y = 2x$		
$y = 2x + 1$		
$y = 2x - 1$		
$y = 2x - 3$		
$y = -2x + 2$		
$y = -2x - 1$		

WORKSHEET F **Name:** ____________________

$y = \frac{1}{2}x + 1$		
$y = -\frac{1}{2}x - 1$		
$y = \frac{1}{2}x - 2$		
$y = \frac{1}{2}x + 3$		
$y = -\frac{1}{2}x + 2$		
$y = -\frac{1}{2}x - 1$		

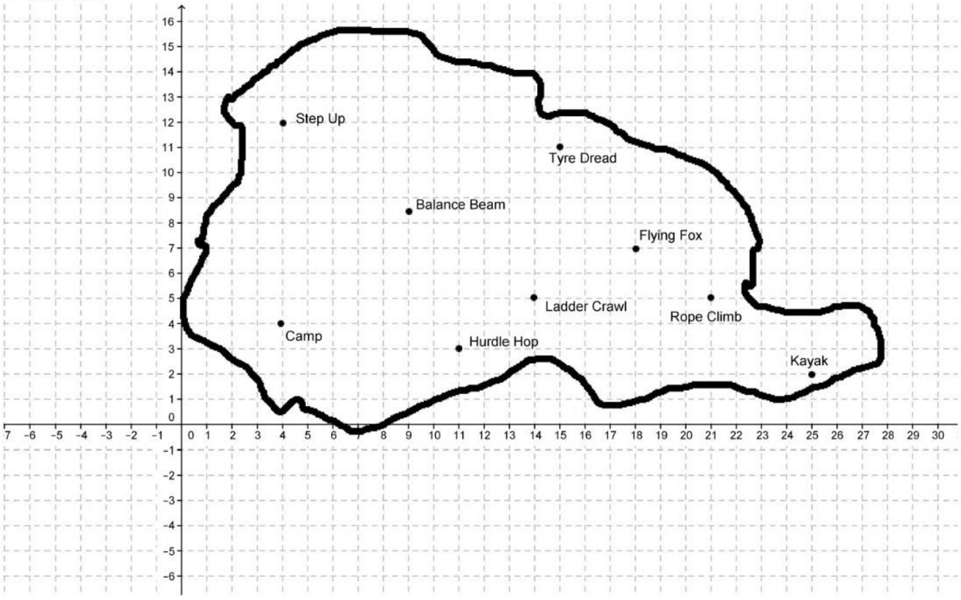
WORKSHEET G	**Name: ____________________**	
$y = \frac{1}{4}x + 2$		
$y = -4x - 1$		
$y = 4x - 2$		
$y = -\frac{1}{4}x - 1$		
$y = -4x + 2$ $y = -4x - 2$		
$y = \frac{1}{4}x + 1$ $y = -\frac{1}{4}x + 3$		

WORKSHEET H	Name: ____________________	
y = 3x + 3		
y = -3x - 1		
y = 3x - 2		
y = -3x - 1		
y = -3x + 2		
y = 3x + 1		

WORKSHEET I - MAP

Name: ____________________

The map below is the picnic area at Lake Wyangan and the obstacle course setup for a Year 9 Excursion. There are eight different elements of the obstacle course which starts and ends at the camp site.



Activity 1:

1. There is one obstacle that is the midpoint between two other obstacles - what is the midpoint obstacle?

___

2. Prove it is the midpoint using the midpoint formula you derived.

___

___

___

3. How can you prove three points are collinear – lie on the same line?

___

___

Activity 2:

1. Construct a right angled triangle from Step Up to Tyre Dread
2. How did you construct it?

___

3. The distance between Step Up to Tyre Dread is called the ______________________ of a right angled triangle?
4. What can you do to find the distance between Step Up to Tyre Dread?

___

___

___

Name: ____________________

Activity 3:

1. Using the same method as in Activity 2 find the distance between
   a. Camp and Hurdle Hop

____

____

____

____

   b. Rope Climb and Flying Fox

____

____

____

____

   c. Ladder Crawl and Tyre Dread

____

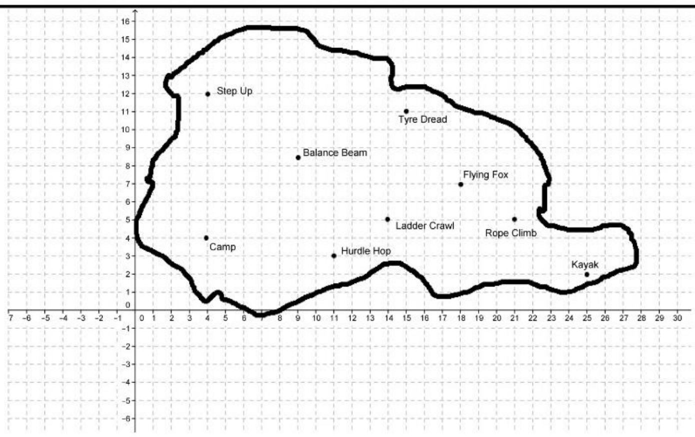
____

____

____

WORKSHEET J – MAP (2) Name: ____________________

Find the distance the following obstacles using the distance formula



a. Camp and Hurdle Hop

____

____

____

b. Rope Climb and Flying Fox

____

____

____

c. Ladder Crawl and Tyre Dread

____

____

____

2. Check your answers on the Geogebra MAP.ggb file using the distance tool
3. Find the obstacle that is 5 units from Rope Climb and prove it using the distance formula

____

____

____

4. Check your answer on the Geogebra MAP.ggb using the distance tool
5. An obstacle called the FALL SWING was left out from the map by mistake. The Flying Fox is mid-point between the Rope Climb and Fall Swing. Where could the Fall Swing be? Show all calculations and formulas.

____

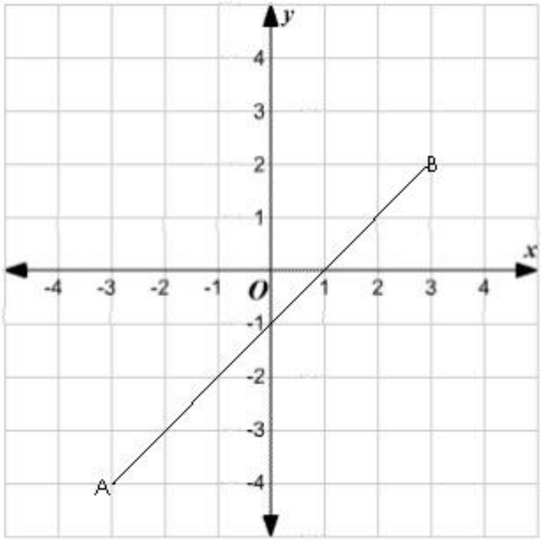
____

____

____

6. Check your answer using the Geogebra MAP.ggb and using the midpoint tools

EXTENDED RESPONSE **Name:** ____________________



The following line interval is drawn from (-3, -4) to (3, 2)

a. Find the length of this line segment – show your working

b. Find the equation of the perpendicular bisector of this interval

c. If the interval drawn and its perpendicular bisector are diagonals of a quadrilateral

   i. What could the quadrilateral be?

   ii. Describe all the different ways you could prove that its that type of quadrilateral

   iii. Use one of your strategies listed above to prove its that type of quadrilateral

   iv. What are the coordinates of the missing vertices of the quadrilateral?

Appendix C: UNE Ethics Approval

Ethics Office
Research Development & Integrity
Research Division
Armidale NSW 2351
Australia
Phone 02 6773 3449
Fax 02 6773 3543
jo-ann.sozou@une.edu.au
www.une.edu.au/research-services

HUMAN RESEARCH ETHICS COMMITTEE

MEMORANDUM TO: Dr Penelope Serow, Prof Stephen Tobias & Mrs Belinda Aventi

School of Education

This is to advise you that the Human Research Ethics Committee has approved the following:

PROJECT TITLE:	How to Hit the Ground Running with Graphing Technology
APPROVAL No.:	HE13-036
COMMENCEMENT DATE:	29 April, 2013
APPROVAL VALID TO:	29 April, 2014
COMMENTS:	Nil. Conditions met in full

The Human Research Ethics Committee may grant approval for up to a maximum of three years. For approval periods greater than 12 months, researchers are required to submit an application for renewal at each twelve-month period. All researchers are required to submit a Final Report at the completion of their project. The Progress/Final Report Form is available at the following web address: http://www.une.edu.au/research-services/researchdevelopmentintegrity/ethics/human-ethics/hrecforms.php

The NHMRC National Statement on Ethical Conduct in Research Involving Humans requires that researchers must report immediately to the Human Research Ethics Committee anything that might affect ethical acceptance of the protocol. This includes adverse reactions of participants, proposed changes in the protocol, and any other unforeseen events that might affect the continued ethical acceptability of the project.

In issuing this approval number, it is required that all data and consent forms are stored in a secure location for a minimum period of five years. These documents may be required for compliance audit processes during that time. If the location at which data and documentation are retained is changed within that five year period, the Research Ethics Officer should be advised of the new location.

[redacted]

Jo-Ann Sozou
Secretary/Research Ethics Officer

19/03/2013 A13/2282

Appendix D: CSO Approval 22-3-13

Diocese of Wagga Wagga

Catholic Schools Office

A.B.N. 36 345 537 994

CSO WAGGA WAGGA

P.O. Box 1012 (205 Tarcutta St.) 2650 ❖ Telephone: (02) 6937 0000 ❖ Fax: (02) 6921 2986 ❖ E-mail: office@csoww.catholic.edu.au

22 March 2013

Belinda Aventi
PO Box 893
GRIFFITH NSW 2680

Dear Belinda

I refer to your application seeking permission to contact Marian Catholic College, Griffith in order to conduct research for your study "Identify the necessary skills which are essential to grasping the concept of graphing within a technological environment, in particular Geogebra".

I wish to advise that approval is granted for you to approach Marian Catholic College in order to seek their willingness to participate in this research. However, the decision to participate is the prerogative of the Principal.

Should you require further details, please do not hesitate to contact Rosemary Clarke at this office on (02) 69370048.

Yours sincerely

[redacted]

Alan Bowyer
Director of Schools

Committed to being a Community of Faith, Learning, Care and Service

Appendix E: CSO Form for Approval to perform Research

CONDUCT RESEARCH POLICY

DIOCESE OF WAGGA WAGGA

Policy Number	05/06
Policy Name	Conduct Research Poilicy
Applicability	Anyone who wishes to conduct research in diocesan schools
Contact Person	Director of Schools
Policy Status	Policy
Date of Approval	2004
Date Last Amended	2006
Related Policies/Documents	• Privacy Policy
Review Period:	2008

Guidelines for Applicants Wishing to Conduct Research in Catholic Schools in the Diocese of Wagga Wagga

Introduction

These guidelines are intended for researchers wishing to conduct research within Catholic schools in the Diocese of Wagga Wagga. The Catholic Schools Office of the Diocese of Wagga Wagga welcomes research projects in our schools and education centres where it can be demonstrated that such research will assist in maintaining and improving the provision of quality Catholic education. Any approvals granted are in principle only, and are subject to the provision of satisfactory evidence of the nature and standard of the research as well as the approval of the participating school principal.

Approval Procedures of Applications to Conduct Research in Catholic System Schools in the Diocese of Wagga Wagga.

Details to be included in the application:

All applications should be addressed to: **Director of Schools** **Mr Alan Bowyer** **Catholic Schools Office** **Diocese of Wagga Wagga** **PO Box 1012** **WAGGA WAGGA NSW 2650** **Email:** office@csoww.catholic.edu.au
Principal Researcher Contact Details Name: *Belinda Aventi* Address: *81 Verri St* Telephone:*02 69646388* Fax: *n/a* E-mail: *belinda1@iinet.net.au*
The supervisor(s) of your research programme (where relevant) Name: *Dr Pep Serow* Address: *School of Education, University of New England, ARMIDALE NSW 2351* Telephone: *02 6773 2378* Fax: *02 6773 2445* E-mail: *pserow2@une.edu.au*

Name: Associate Professor Steve Tobias Address: *School of Education, University of New England, ARMIDALE NSW 2351* Telephone: 02 6773 2573 Fax: *02 6773 2445* E-mail: *stobias@une.edu.au*
Are you currently employed in the Diocese of Wagga Wagga? ☐ ✓ **Yes** ☐ **No**
The title of your research proposal *How to Hit the Ground Running with Graphing Technology*

A **brief overview of the research project**, describing in particular the research procedures and the extent of student, teacher and parental involvement in the project.

This research project aims to identify the necessary skills which are essential to grasping the concept of graphing within a technological environment, in particular Geogebra.

Unfortunately, when students require technology in senior years, to further develop their understanding of conceptual ideas which will be assessed in final examinations, they are unable to use it to their advantage. They lack the required skills which would enable them to competently advance with the technology. In essence they are unable to 'hit the ground running' with technology, instead they need to revise basic concepts and familiarise themselves with the task of simultaneously, learning graphing and learning the technology. In this project a focus group from a class of mathematics students will be observed during the instruction of the graphing topic.

In consultation with the class teacher, a focus group of 6 students will placed into working pairs. All students will complete a set unit of work on graphing prepared by the student researcher. Within this unit of work 4 of the activities completed by the entire class (one at beginning, one half way through, one at end and one test about month after completion) will be the main source of data collection for the study. Whilst the focus group will complete these 4 activities, using the software Camtasia their audio and keystrokes will be recorded. In this way, the focus group will complete the same unit of work and activities as the main class, only that their work will be what is recorded and kept for data analysis. This will then be reviewed to identify how students respond and process the information given to them and what ideas they draw from. Focus group students will not miss out on any learning opportunities within their mathematics class and will not be advantaged for being a part of the group.

Activities will be designed to align with the Van Hiele Teaching Phases to see if any progression is made during and after a two week teaching unit which is designed by the student researcher is implemented. It is anticipated that specific skills will become apparent that are crucial to the development of TGK (Technological Graphical Knowledge). These will be skills that are considered essential to the learning of graphing with technology and skills that will be able to be refined and consolidated in junior secondary mathematics classes.
A **brief description outlining the benefits of the research** to the participants. For example, how will teachers and students benefit from your research? Please specify any long term and more general benefits for Catholic schools in the Diocese of Wagga Wagga. *It is anticipated that the research will support teachers by identifying possible skills and areas that students need extra help with, to better use the graphing tools to aid their understand of the graphing topic. It may assist teachers in choosing suitable activities which will promote the consolidation of these necessary skills.* *For the participants, hopefully they will be enlightened by knowing that they are contributing to mathematical research relevant for future mathematical students.*
A **brief description of the research design and methodology** and any strategies to be employed to ensure validity and reliability. Copies of data collection instruments, where available, should be attached. *A thematic content analysis using the van Hiele framework in geometry, in the context of technology use in mathematics education will be employed for this research project.* *The phases are specific and will provide a solid structure and basis to compare the skills learnt and those needed.*
A **letter outlining the nature of the research** must accompany approaches to school principals seeking approval to conduct research in respective schools for which they are responsible and the commitments required of the school staff. A copy of *this letter* must be included with the proposal. *Please find letter attached*

If appropriate to your research project, specify how you intend to obtain **personal approval** for contact with students in our schools and include examples of permission letters/consent forms. *An information sheet will be distributed to all student participants, parents or guardians and teachers associated with the research study. This information sheet will detail all key issues regarding the research and data collected. It will contain contact details of all researchers involved so that participants may be able to contact someone if they have questions or complaints. Upon reading this information sheet and agreeing, participants, parents and guardians will confirm their consent (and students assent) by signing the consent form.* *All information sheets and consent and assent forms are attached*
Provide details of procedures for establishing confidentiality and procedures for protecting privacy of the participants including information management practices. Information should only be collected for the purpose of this research application. Any subsequent use of information must be clearly outlined in your application and must have ethical approval from a university ethics committee. *Data storage and security will be the responsibility of Dr Pep Serow. All data will be kept in a locked filing cabinet in the School of Education UNE and any computer files will be available by password only. Only the named researchers Dr Pep Serow, Associate Professor Steve Tobias and Belinda Aventi will have access to data. This complies with Australian Code for Responsible Conduct of Research (Research Practice for the Management of Research Data & Primary Records). All data will be kept for a minimum of 5 years after which time it will be disposed of by Dr Pep Serow. All paper files will be shredded and all digital material will be deleted.*
List the schools or groups that will be asked to participate in the research. Include the name of the school and the suburb. *Marian Catholic College Griffith*

Indicate the period of the year during which the research activity will commence and be concluded. Also indicate the **estimated amount of time required** of the school and any individual participants in the research. *Term 3 2013 it is estimated that a maximum of 4 weeks*
<u>All applications must be signed and dated by the Principal researcher.</u> Please attach a copy of the Ethical Clearance approval from your university/institution. *Please find letter attached*

Approval by the Catholic Schools Office

Please note that any submissions to conduct research in schools in the Diocese of Wagga Wagga require approval from the Catholic Schools Office.

Reporting to the Catholic Schools Office

It is a condition of approval that, upon completion of a project, the researcher will provide the Catholic Schools Office, Diocese of Wagga Wagga, with a copy of the research findings and provide the schools in which the research was carried out with a summary of the research findings, along with permission for the Catholic Schools Office, Diocese of Wagga Wagga, to disseminate reports to its personnel.

Please refer to **Form C** *"Agreement to provide research findings to the Catholic Schools Office, Diocese of Wagga Wagga"*.

Research not directly related to education

The Catholic Schools Office will only give approval for non-educational research projects to have access to schools where there is a demonstration of significant public benefit outweighing the inconvenience to school communities.

Copyright

Staff employed in Catholic schools and the Catholic Schools Office in the Diocese of Wagga Wagga who wish to conduct research need to be aware that, where a publication is made by an employee in the course of employment and as part of the employee's usual duties, the first owner of copyright will usually be the Catholic Schools Office, Diocese of Wagga Wagga, as the employer.

Any enquiries in this regard should be forwarded to the Director of Schools for consideration.

Commercial Gain

It is not the intention of the Catholic Schools Office, Diocese of Wagga Wagga to provide approval for research that is undertaken primarily for commercial or material gain.

Form A

Confidential Declaration by Principal Researcher

(Where a research project involves any contact with a school in the Diocese of Wagga Wagga)

a) I am aware of and will comply with the special responsibilities associated with undertaking research with children and young people, specifically, my responsibilities and obligations under the *Child Protection (Prohibited Employment) Act 1998*.

b) I declare that there are no other circumstances or reasons that might preclude my undertaking research with children and young people.

c) In relation to assistants conducting research with children and young people with me, and/or on my behalf, I will ensure that they will be made aware of the special responsibilities associated with undertaking research with children and young people, specifically, their responsibilities and obligations under the *Child Protection (Prohibited Employment) Act 1988* (See **Form B** for assistant researchers).

____ __________________

Signature of Principal Researcher Date

Form B

Confidential Declaration by Assistant Researcher(s)

(Where a research project involves any contact with a school in the Diocese of Wagga Wagga)

a) I am aware of and will comply with the special responsibilities associated with undertaking research with children and young people, specifically, my responsibilities and obligations under the *Child Protection (Prohibited Employment) Act 1998*.

b) I declare that there are no other circumstances or reasons that might preclude my undertaking research with children and young people.

____ ____________________

Signature of Assistant Researcher Date

Form C

Agreement to provide Research findings to

the Catholic Schools Office, Diocese of Wagga Wagga

As Principal Researcher:

- I agree to provide the Catholic Schools Office, Diocese of Wagga Wagga, with a copy of the research findings of the proposed study upon completion.

- I agree to provide participating schools with a summary of the research findings.

- I understanding that, if the Catholic Schools Office, Diocese of Wagga Wagga, wishes to disseminate the report more widely, this will be done in consultation with me.

____ __________________

Signature of Principal Researcher Date

Form D

Supervisor's Report

Name of Principal Researcher: *Belinda Aventi*	
Title of research proposal: *How to Hit the Ground Running With Graphing Technology*	
Name of Referee: Dr Pep Serow	
Referee's position: Senior Lecturer	Institution: University of New England
Referee's address: School of Education, University of New England, Armidale, 2350	
Telephone: 0428 022 935 E-Mail address: pserow2@une.edu.au	Fax: 02 6773 2445
Relationship to researcher: Academic Principal Supervisor	
Please comment on the following aspects of the proposal, in relation to the submitted applications.	
• Significance, purpose and value of the research This research is significant as it specifically targets the affordances of dynamic geometry software as a teaching tool in the secondary mathematics classroom. Whilst this software is freely available, most classrooms in Australia do not utilise this technology in a meaningful manner. This is timely research in the light of the rollout of the new national curriculum and subsequent unit design.	
• Appropriateness of the research design The design of the project is unique in that specific mathematical tasks utilising the technology are used as catalysts for student discussion concerning the mathematical content and technological tools. The students will be completing the mandatory curriculum content adopting a range of innovative and motivating teaching strategies. The students are catered for appropriately should they choose not to be a participant in the research and they will be included in the classroom tasks without being a participant.	

• Methodological adequacy and viability The methodological strategies are suitable to the classroom context and do not require the participants or non-participants to enter into any tasks that are not considered as routine in

the mathematics classroom. The strategies are replicable in other contexts and will be of benefit to the students in the class generally and future classes should the techniques be shared.
• Ethical considerations All ethical issues have been considered and addressed through the UNE Human Research Ethics Committee. The HREC has provided this research with an approval number acknowledging that the project meets the high ethical standards expected.
To what extent do you consider the principal researcher to be capable of undertaking the research described in the attached proposal? Belinda is an experienced teacher who has demonstrated excellent research skills and ethical standards. I have complete confidence in her ability to complete this project professionally and ethically.
Is this proposal exempt from ethical approval? No
If exempt, for what reasons?

[redacted]

Supervisor's Signature

Date: March 21 2013

Appendix F: Information Sheet for Participants

INFORMATION SHEET for PARTICIPANTS (CSO)

I wish to invite a school within your Diocese, Marian Catholic College, to participate in my research project, described below.

My name is Belinda Aventi and I am conducting this research as part of my Masters of Education (Honours) in the School of Education at the University of New England. My supervisors are Dr Pep Serow and Associate Professor Steve Tobias.

Research Project	**How to Hit the Ground Running with Graphing Technology**
Aim of the research	The research aims to explore how students learn graphing with respect to technology. We hope to identify skills that can be consolidated in early secondary school so that by senior school they are more equipped to embrace graphing with technology.
Focus Group Sessions	I would like to select 6 students from a Year 9 Stage 5.3 Class at Marian Catholic College to form a focus group, in consultation with the class teacher. This group will meet in total four times, during a mathematics period (63 minutes) to complete the same activities which are being done by the rest of their class. However, with your permission, the focus group will be recorded, both audio and computer keystrokes to ensure that I accurately recall the information provided by the students regarding their solution solving techniques and strategies. There will be no disruption to the student's mathematical learning; they will not be doing anything different from their classmates, other than being recorded. The recordings will occur in an open area in the library in full view of other students and teachers. The research project should take approximately 3-4 weeks maximum.
Confidentiality	Any information or personal details gathered in the course of the study will remain confidential. No individual will be identified by name in any publication of the results. All names will be replaced by pseudonyms; this will ensure that participants are not identifiable.
Participation is Voluntary	Please understand that student involvement in this study is voluntary and I respect the right to withdraw from the study at any time. Anyone of the students may discontinue the focus group at any time without consequence and they do not need to provide any explanation if they decide not to participate or withdraw at any time.
Focus Group Activities	The focus group activities will not be of a sensitive nature: rather they are mathematical, aiming to enable you to enhance my knowledge of the challenges and learning difficulties when graphing with technology. The focus group activities will be part of the class program so all students within the class will complete these activities. The only difference being that only the focus group (6 students) will be recorded whilst completing these activities.

Use of information	I will use information from the focus group as part of my master's thesis, which I expect to complete in February 2014. Information from the focus group may also be used in journal articles and conference presentations before and after this date. At all times, I will safeguard the identities of participants by presenting the information in way that will not allow them to be identified.
Upsetting issues	It is highly unlikely that this research will raise any personal or upsetting issues but if it does you may wish to contact your local Community Health Centre 02 6966 900.
Storage of information	I will keep hardcopy recordings and notes of the focus group sessions in a locked cabinet at the researcher's office at the University of New England's School of Education. Any electronic data will be kept on a password protected computer in the same School. Only the research team will have access to the data.
Disposal of information	All the data collected in this research will be kept for a minimum of five years after successful submission of my thesis, after which it will be disposed of by deleting relevant computer files, and destroying or shredding hardcopy materials.
Approval	This project has been approved by the Human Research Ethics Committee of the University of New England (Approval No HE13-036, Valid to 29/4/2014).
Contact details	Feel free to contact me with any questions about this research by email at bzanotto@myune.edu.au or aventib@ww.catholic.edu.au by phone on 02 69692400. You may also contact my supervisors. My Principal supervisors name is Dr Pep Serow and she can be contacted at pserow2@une.edu.au or 02 6773 2378 and my Co-supervisors name is Associate Professor Steve Tobias and he can be at stobias@une.edu.au or 02 6773 2573.
Complaints	Should you have any complaints concerning the manner in which this research is conducted, please contact the Research Ethics Officer at: Research Services University of New England Armidale, NSW 2351 Tel: (02) 6773 3449 Fax: (02) 6773 3543 Email: ethics@une.edu.au Thank you for considering this request and I look forward to further contact with you. regards, [redacted] Belinda Aventi

INFORMATION SHEET for PARTICIPANTS (MCC STAFF)

I wish to invite your school, Marian Catholic College, to participate in my research project, described below.

My name is Belinda Aventi and I am conducting this research as part of my Masters of Education (Honours) in the School of Education at the University of New England. My supervisors are Dr Pep Serow and Associate Professor Steve Tobias.

Research Project	**How to Hit the Ground Running with Graphing Technology**
Aim of the research	The research aims to explore how students learn graphing with respect to technology. We hope to identify skills that can be consolidated in early secondary school so that by senior school they are more equipped to embrace graphing with technology.
Focus Group Sessions	I would like to select 6 students from a Year 9 Stage 5.3 Class at Marian Catholic College to form a focus group, in consultation with the class teacher. This group will meet in total four times, during a mathematics period (63 minutes), to complete the same activities which are being done by the rest of their class. However, with your permission, the focus group will be recorded, both audio and keystrokes to ensure that I accurately recall the information provided by the students regarding their solution solving techniques and strategies. There will be no disruption to the student's mathematical learning; they will not be doing anything different from their classmates, other than being recorded. The recordings will occur in an open area in the library in full view of other students and teachers. The research project should take approximately 3-4 weeks maximum.
Confidentiality	Any information or personal details gathered in the course of the study will remain confidential. No individual will be identified by name in any publication of the results. All names will be replaced by pseudonyms; this will ensure that participants are not identifiable.
Participation is Voluntary	Please understand that participant involvement in this study is voluntary and I respect the right to withdraw from the study at any time. Anyone of the students may discontinue the focus group at any time without consequence and they do not need to provide any explanation if they decide not to participate or withdraw at any time.
Focus Group Activities	The focus group activities will not be of a sensitive nature: rather they are mathematical, aiming to enable you to enhance my knowledge of the challenges and learning difficulties when graphing with technology. The focus group activities will be part of the class program so all students within the class will complete these activities. The only difference being that only the focus group (6 students) will be recorded whilst completing these activities.
Use of information	I will use information from the focus group as part of my master's thesis, which I expect to complete in February 2014. Information from the focus group may also be used in journal articles and conference presentations before and after this date. At all times, I will safeguard the identities of participants by presenting the information in way that will not allow them to be identified.

Upsetting issues	It is highly unlikely that this research will raise any personal or upsetting issues but if it does you may wish to contact your local Community Health Centre 02 6966 900.
Storage of information	I will keep hardcopy recordings and notes of the focus group sessions in a locked cabinet at the researcher's office at the University of New England's School of Education. Any electronic data will be kept on a password protected computer in the same School. Only the research team will have access to the data.
Disposal of information	All the data collected in this research will be kept for a minimum of five years after successful submission of my thesis, after which it will be disposed of by deleting relevant computer files, and destroying or shredding hardcopy materials.
Approval	This project has been approved by the Human Research Ethics Committee of the University of New England (Approval No HE13-036, Valid to 29/4/2014).
Contact details	Feel free to contact me with any questions about this research by email at bzanotto@myune.edu.au or aventib@ww.catholic.edu.au by phone on 02 69692400. You may also contact my supervisors. My Principal supervisors name is Dr Pep Serow and she can be contacted at pserow2@une.edu.au or 02 6773 2378 and my Co-supervisors name is Associate Professor Steve Tobias and he can be at stobias@une.edu.au or 02 6773 2573.
Complaints	Should you have any complaints concerning the manner in which this research is conducted, please contact the Research Ethics Officer at: Research Services University of New England Armidale, NSW 2351 Tel: (02) 6773 3449 Fax: (02) 6773 3543 Email: ethics@une.edu.au Thank you for considering this request and I look forward to further contact with you. regards, [redacted] Belinda Aventi

Appendix G: Consent Form for Participants

CONSENT FORM
for
MCC STAFF

Research Project: *How to Hit the Ground Running with Graphing Technology*

I,……………………………………………., have read the information contained in the Information Sheet for Participants and any questions I have asked have been answered to my satisfaction. Yes/No

I agree to participate in this research, realising that I may withdraw at any time .Yes/No

I agree that research data gathered for the study may be published using a pseudonym Yes/No

I agree that I may be quoted using a pseudonym Yes/No

I agree to have my audio recorded and transcribed Yes/No

I am older than18 years of age Yes/No

…………………………….. ……………………………

Participant Date

[redacted]

…………………………….. ……………………………

Belinda Aventi Date

CONSENT FORM
for
MCC PRINCIPAL

Research Project: *How to Hit the Ground Running with Graphing Technology*

I,………………………………………………………………………, have read the information contained in the Information Sheet for Participants and any questions I have asked have been answered to my satisfaction. Yes/No

I agree to permit Marian Catholic College to participate in this research, realising that I may withdraw at any time. Yes/No

I agree that research data gathered for the study may be published using a pseudonyms Yes/No

I agree that students (under 18) may be quoted using pseudonyms Yes/No

I agree to have students (under 18) audio recorded and transcribed Yes/No

…………………………….. ……………………………

Marian Catholic College Principal Date

[redacted]

……………………………….. ……………………………

Belinda Aventi Date

CONSENT FORM
for
CSO WAGGA WAGGA

Research Project: *How to Hit the Ground Running with Graphing Technology*

I, ………………………………………………………………….., have read the information contained in the Information Sheet for Participants and any questions I have asked have been answered to my satisfaction. Yes/No

I agree to permit Marian Catholic College within the Diocese of Wagga Wagga to participate in this research, realising that they may withdraw at any time. Yes/No

I agree that research data gathered for the study may be published using a pseudonyms Yes/No

I agree that students (under 18) may be quoted using pseudonyms Yes/No

I agree to have students (under 18) audio recorded and transcribed

Yes/No

……………………………….. ……………………………

CSO Date

[redacted]

… .. ……………………………

Belinda Aventi Date

Appendix H: Information Sheet and Assent for Students

INFORMATION SHEET
for
PARTICIPANTS (aged 12-16)

Research Project: *How to Hit the Ground Running with Graphing Technology*

Dear Students,

We wish to invite you to participate in my research on above topic. Dr Pep Serow and Associate Professor Steve Tobias have been teachers in schools and now teach pre-service teachers at the University of New England. Belinda Aventi is conducting this research in order to complete her Masters of Education (Honours) through the University of New England. We are currently doing a research study that is trying to find out more about what skills students need to better use graphing technology. We are hoping that mathematical students will help us by taking part in this study.

This Information Sheet has the answers to many of the questions that you and your parent(s) may have about the study. There is a lot of information in here so don't worry if it is too much for one read. Just read through a bit at a time if you want.

1) What is the study for and why is it being done?

We hope that by doing this study, we will learn more about how students use technology in particular Geogebra. There is a lot of research regarding mathematics and graphics calculators but no other technology. This information will help us to assist teachers when writing programs so that they can do the best job in helping students to achieve a better understanding of technology in the graphing topic.

2) What would I be asked to do if I took part in the study?

You will not be asked to do anything different from your class mates. You will learn the same work and participate in the same activities as the rest of the class. A focus group of six students from your class will be selected to work in pairs and complete 4 of the class activities on laptops which will record audio and keystrokes. This will form the basis of our data collection.

You will be asked at the beginning of each session if you are happy to be recorded. If you are not feeling well, or do not want to participate, that's OK, and you won't have to do it.

These audio recordings will NOT be seen by anyone except Pep, Steve and Belinda. They will be kept for 5 years to allow us to gather all the information we need and then will be destroyed. Until they are destroyed they will be kept in a locked program on a private computer. This is something we will be very careful about, as we must follow special rules set down by the university to protect you.

3) Will my parents have to do anything?

Apart from making sure you are happy to participate in this research project and signing the consent form, your parents will not need to do anything.

4) When and where would the focus group sessions take place?

Each focus group session will be filmed in an open area in the school library in full view of other staff and students. The focus group sessions will take place over 3 - 4 weeks with a single session one month after.

5) What information will be recorded?

If you are chosen in the focus group you will be required to complete four class activities on a specific laptop which will record your conversations and how you solve the activity. You will not be required to tell the researcher anything you don't want to.

6) Why do you need to record audio?

We record these sessions because it allows us to go back over and check things. If you do not feel comfortable at any time we can stop recording.

7) What will be done with this information that I give?

First, Belinda will take the digital recording to her office and will type out every word that is recorded so that we know we haven't missed anything. This is called the 'transcript'.

When this is being done, Belinda makes sure that no-one's real names are used in the 'transcript' of the focus group that is printed out. This makes sure that nobody reading it can tell who has said what, and so what you say stays confidential.

Then, the researchers read through these typed-out 'transcripts' very carefully, making notes and trying to pick out all of the most interesting and important things that the participants are doing. We are looking for things that will help us and other people understand more about what struggles students face with graphing and technology.

The researchers will write a report at the end of the study so that we can share the information from this study with other researchers and teachers who are interested and involved in children and young people's education.

We also try to write articles about the study and publish these and also talk about the study at meetings and conferences so that what we have found out actually gets to people who might be able to use the information to help. If we didn't do this, then the students who helped us might feel that they had done this for nothing.

Remember again though, that in any of the articles or reports, your name will not appear as what you tell us is confidential and private. What we would do is perhaps say that "A 14 year old girl chose to calculate..." or write that "John (not his real name), said he could change the way the graph was positioned by…."

8) Who will be told about the information collected?

Each focus group session is strictly confidential. What is recorded stays within the research team; apart from when we report the study as explained in point **(7).** None of the information will become part of any school records or notes. All information remains confidential.

9) Do I have to take part in this study?

Not at all. You should only take part if you want to and are happy to be recorded.

10) What will happen if I don't want to take part?

Nothing at all. You have every right to say that you would rather not take part.

11) Can I change my mind if I decide to participate?

Yes. You can choose to leave the study at any time and nothing at all will be said, apart from 'Thank you very much for thinking about taking part'.

12) Will the study benefit me in any way?

We can't promise that you will get any benefit from taking part. However, you might feel that by being in the focus group, if chosen, you may be helping other mathematics students and teachers to get a better understanding of skills that are needed with graphing technology.

13) Have you got permission to do this study?

Yes. We have permission from the Human Research Ethics Committee of the University of New England. They have looked carefully at this study and have 'passed' it.

14) What if I have other questions about the study?

Please contact the Principal Researcher, Dr Pep Serow at any time. Her office phone number is 02 6773 2378. You can also call Assoicate Professor Steve Tobias on 02 6773 2573 or Belinda Aventi by phoning 02 6969 2400.

If you have any complaints about the way this research is conducted, please contact the Research Ethics Officer at the following address:

Research Services
University of New England
Armidale, NSW 2351.
Telephone: (02) 6773 3449 Facsimile (02) 6773 3543
Email: ethics@une.edu.au

15) What if I feel that I would like to talk to someone after the focus group about any thoughts, feelings or problems that I have?

You may contact any member of the research team, or you may prefer to speak with your classroom teacher or parents.

16) The formal 'stuff':

This project has been approved by the Human Research Ethics Committee of the University of New England (Approval No. HE13-036, valid to 29/4/2014).

Please keep this information sheet as you might want to discuss it with friends, family or relatives.

Thanks a lot for taking the time to read this and for any help that you are able to give us with this study.

Dr Pep Serow, Assoc Professor Steve Tobias & Belinda Aventi

ASSENT FORM for STUDENTS

Research Project: *How to Hit the Ground Running with Graphing Technology*

Please write your full name after 'I,', and circle the yes/no answer you want.

I, ………………………………………., have read the Information Sheet for Students and any questions I asked have been answered and I understand them. Yes/No

I agree to take part in this research project. Yes/No

I know that I can change my mind at any time. Yes/No

I agree that any work taken and anything we talk about will be written about using an invented name. .Yes/No

I agree in allowing the focus group sessions to be recorded and transcribed. Yes/No

……………………………….. ……………………………

Student Date

[redacted]….. ……………………………

Belinda Aventi Date

Appendix I: Information Sheet for Parents

INFORMATION SHEET
for
PARENTS/GUARDIANS

Dear Parents & Guardians,

I wish to invite your child to participate in my research project, described below.

My name is Belinda Aventi and I am conducting this research as part of my Masters of Education (Honours) in the School of Education at the University of New England. My supervisors are Dr Pep Serow and Associate Professor Steve Tobias.

Research Project	**How to Hit the Ground Running with Graphing Technology**
Aim of the research	The research aims to explore how students learn graphing with respect to technology. We hope to identify skills that can be consolidated in early secondary school so that by senior school they are more equipped to embrace graphing with technology.
Focus Group Sessions	I would like to select 6 students from your child's Year 9 Stage 5.3 Mathematics class at Marian Catholic College to form a focus group, in consultation with the class teacher. This group will meet in total four times, during a mathematics period (63 minutes) to complete the same activities which are being done by the rest of their class. However, with your permission, whilst the focus group complete these four activities their audio and keystrokes will be recorded to ensure that I accurately recall the information provided by the students regarding their solution solving techniques and strategies. There will be no disruption to the student's mathematical learning as they will not be doing anything different from their classmates, other than being recorded whilst completing four of the class activities. The recordings will occur in an open area in the library in full view of other students and teachers. The research project should take approximately 3-4 weeks maximum.
Confidentiality	Any information or personal details gathered in the course of the study will remain confidential. No individual will be identified by name in any publication of the results. All names will be replaced by pseudonyms; this will ensure that participants are not identifiable.
Participation is Voluntary	Please understand that student involvement in this study is voluntary and I respect the right to withdraw from the study at any time. Anyone of the students may discontinue the focus group at any time without consequence and they do not need to provide any explanation if they decide not to participate or withdraw at any time.
Focus Group Activities	The focus group activities will not be of a sensitive nature: they are mathematical, aiming to enhance my knowledge of the challenges and learning difficulties students face when graphing with technology. The focus group activities will be part of the class program so all students within the class will complete these activities. The only

difference being that only the focus group (6 students) will be recorded whilst completing these activities.

Use of information

I will use information from the focus group as part of my master's thesis, which I expect to complete in February 2014. Information from the focus group may also be used in journal articles and conference presentations before and after this date. At all times, I will safeguard the identities of participants by presenting the information in way that will not allow them to be identified.

Upsetting issues

It is highly unlikely that this research will raise any personal or upsetting issues but if it does you may wish to contact your local Community Health Centre 02 6966 9000.

Storage of information

I will keep hardcopy recordings and notes of the focus group sessions in a locked cabinet at the researcher's office at the University of New England's School of Education. Any electronic data will be kept on a password protected computer in the same School. Only the research team will have access to the data.

Disposal of information

All the data collected in this research will be kept for a minimum of five years after successful submission of my thesis, after which it will be disposed of by deleting relevant computer files, and destroying or shredding hardcopy materials.

Approval

This project has been approved by the Human Research Ethics Committee of the University of New England (Approval No HE13-036, Valid to 29/4/2014).

Contact details

Feel free to contact me with any questions about this research by email at bzanotto@myune.edu.au or aventib@ww.catholic.edu.au by phone on 02 6969 2400.

You may also contact my supervisors. My Principal supervisors name is Dr Pep Serow and she can be contacted at pserow2@une.edu.au or 02 6773 2378 and my Co-supervisors name is Associate Professor Steve Tobias and he can be at stobias@une.edu.au or 02 6773 2573.

Complaints

Should you have any complaints concerning the manner in which this research is conducted, please contact the Research Ethics Officer at:
Research Services
University of New England
Armidale, NSW 2351
Tel: (02) 6773 3449 Fax: (02) 6773 3543
Email: ethics@une.edu.au
Thank you for considering this request and I look forward to further contact with you.

regards,

[redacted]

Belinda Aventi

CONSENT FORM
for
PARENTS/GUARDIANS

Research Project: *How to Hit the Ground Running with Graphing Technology*

Please write your full name after 'I,', and circle the yes/no answer you want.

I ………………………………………, have read the Information Sheet for Parents/Guardians and any questions I asked have been answered and I understand them Yes/No

I agree to my child …………………………………………….. (insert name of child) taking part in this research project. Yes/No

I know that I can change my mind at any time. Yes/No

I understand that my child may be quoted in the final report published. I know that all information will be coded so that my child will remain anonymous to all but the researcher. Yes/No

I agree in allowing the focus group sessions to be recorded and transcribed. Yes/No

……………………………….. ……………………………

Parent Signature Date

[redacted]

……………………………

Belinda Aventi Date

Appendix J: Pre-test Google Form

Edit this form

Pre-Test Task

Pre-Test task for all students in Year 9 class.

Instructions:
1. In pairs one person is to open Geogebra and together you are to work through the following questions.
2. Attempt all questions you may skip a question and come back to it later
3. Do not use the internet to Google how to do something try and navigate your way through by trying things on the Geogebra program
4. You may use your scrap paper to do any working but remember that all scrap paper must have your name on it and be collected at the end of the lesson.

1. Give the full names of the students in your pair

2. Questions 2 to 5 involve using the toolbar in Geogebra and the points (2, 0) and (0, 5). First find the point which would represent the midpoint.

a. How could you find the midpoint without using Geogebra and using pen and paper?
Describe it below or attempt on your scrap paper.

3. Find the distance between these two points when connected.

a. How could you find the distance without using Geogebra and using pen and paper?
Describe it below or attempt on your scrap paper.

4. Find the slope between the two points

a. How could you find the slope without using Geogebra and using pen and paper?
Describe it below or attempt on your scrap paper.

5. what is the equation of the line you drew in part four?

a. How did you work this out?
Describe it below or attempt on your scrap paper.

6. Using Geogebra can you draw the graph of y = 4x + 8?
Explain how you did this below

a. How would you do this without Geogebra using pen and paper
Describe it below or attempt on your scrap paper.

b. Can you move the graph using Geogebra? Notice what changes on the graph and what changes in the equation. Explain these changes below.

7. Using Geogebra draw the lines, y = -3x, y = -3x -2, y = -3x + 4. What do you notice about these lines?

a. Give an example of an equation of a line that would belong to this family of lines

8. Using Geogebra draw the lines, y = 1/3x + 4 , y = -3x + 4. What do you notice about these lines?

9. Rearrange the equation 4x + 2y - 16 = 0 such that y is the subject of the formula (that is y = ...). Write you answer below

Use paper provided to do your working - only insert your answer below. Paper will be collected at the end.

10. Which of these graphs do you think would represent a straight line?

Answer why in the next question box

- [ ] y = -2x +7
- [ ] y = -8
- [ ] 9x(x - 4) = y
- [ ] 3x + 5y - 6 = 0
- [ ] 4(y - 1) = 2x2

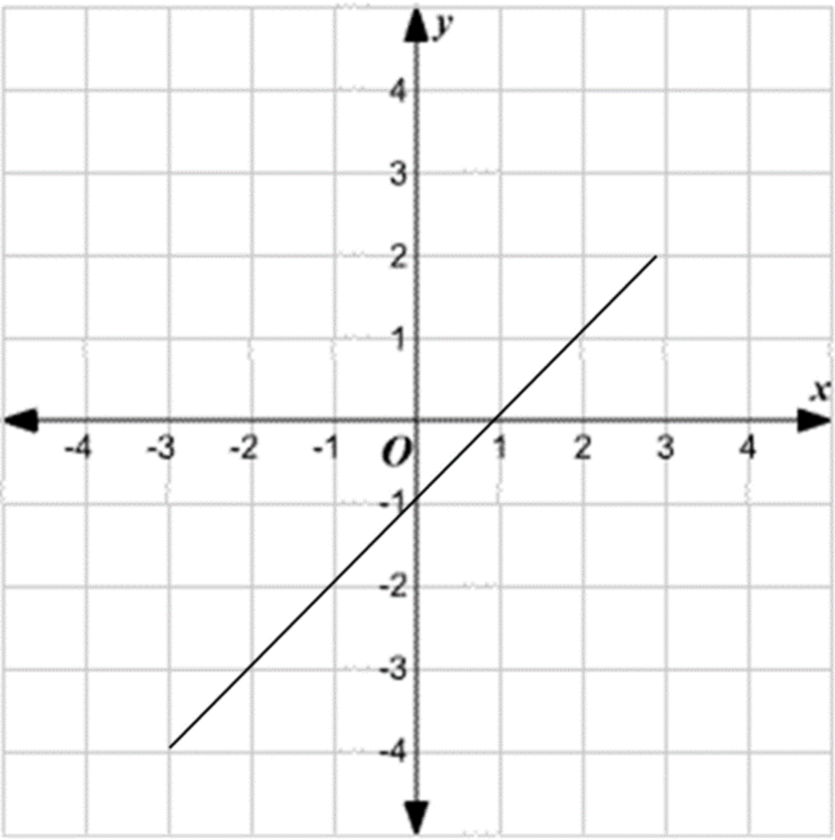
a. Why do you think they would be straight lines?

Write your answer for the why part of the question above in the box below

Submit

Never submit passwords through Google Forms.

Appendix K:
Extended Response for Pre-test and Delayed Post-test



The following line interval is drawn from (-3, -4) to (3, 2)

a. Find the length of this line segment – show your working

____

____

____

____

b. Find the equation of the perpendicular bisector of this interval

____

____

____

____

c. If the interval drawn and its perpendicular bisector are diagonals of a quadrilateral

   i. What could the quadrilateral be?

ii. Describe all the different ways you could prove that its that type of quadrilateral

iii. Use one of your strategies listed above to prove its that type of quadrilateral

iv. What are the coordinates of the missing vertices of the quadrilateral?

Appendix L: End of Topic test Google Form

Edit this form

End of Topic Task

End of Linear Relationships Topic task for all students in Year 9 class.

Instructions:
1. In pairs one person is to open Geogebra and together you are to work through the following questions.
2. Attempt all questions you may skip a question and come back to it later
3. Do not use the internet to Google how to do something try and navigate your way through by trying things on the Geogebra program
4. You may use your scrap paper to do any working but remember that all scrap paper must have your name on it and be collected at the end of the lesson.

1. Give the full names of the students in your pair

2. Questions 2 to 5 involve using the toolbar in Geogebra and the points (-1, 4) and (3, 6). First find the point which would represent the midpoint.

a. How could you find the midpoint without using Geogebra and using pen and paper?
Describe it below or attempt on your scrap paper.

3. Find the distance between these two points when connected.

a. How could you find the distance without using Geogebra and using pen and paper?
Describe it below or attempt on your scrap paper.

4. Find the slope between the two points

a. How could you find the slope without using Geogebra and using pen and paper?
Describe it below or attempt on your scrap paper.

5. what is the equation of the line you drew in part four?

a. How did you work this out?
Describe it below or attempt on your scrap paper.

6. Using Geogebra can you draw the graph of y = 4x + 8?
Explain how you did this below

a. How would you do this without Geogebra using pen and paper
Describe it below or attempt on your scrap paper.

b. Can you move the graph using Geogebra? Notice what changes on the graph and what changes in the equation. Explain these changes below.

7. Using Geogebra draw the lines, y = 1/2x, y = -2x -2. What do you notice about these lines?

a. Give an example of an equation of a line that would belong to this family of lines

8. Using Geogebra draw the lines, y = 1/2x + 4 , y = 1/2x + 4. What do you notice about these lines?

a. Give an example of an equation of a line that would belong to this family of lines

9. Rearrange the equation 6x + 2y + 14 = 0 such that y is the subject of the formula (that is y = ...). Write you answer below

Use paper provided to do your working - only insert your answer below. Paper will be collected at the end.

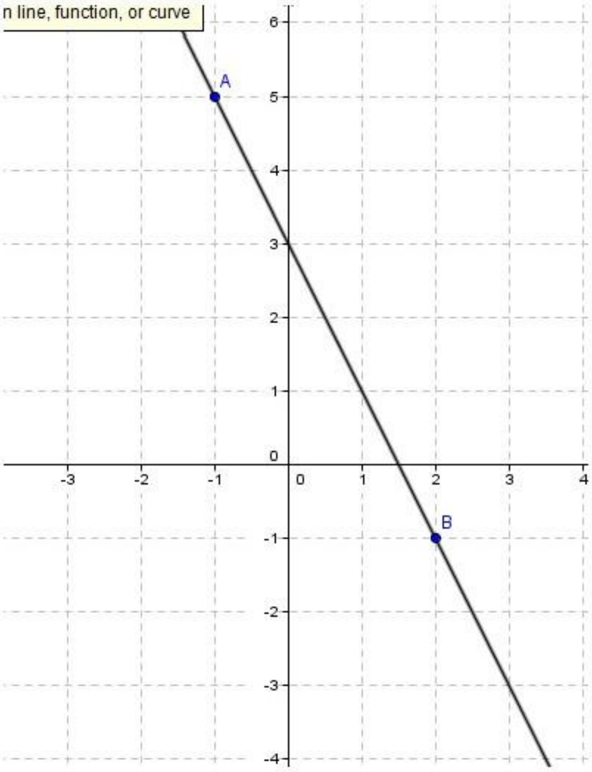
10. Which of these graphs do you think would represent a straight line?

Answer why in the next question box

- ☐ y = -2x +7
- ☐ y = -8
- ☐ 9x(x - 4) = y
- ☐ 3x + 5y - 6 = 0
- ☐ 4(y - 1) = 2x^2

a. Why do you think they would be straight lines?

Write your answer for the why part of the question above in the box below



11. The following questions are based on the graph above

a. What are the coordinates of the points A and B

b. Identify the y-intercept for the line

c. What would the equation of the line be?

Show all your working below

d. Give the equation of the line that is perpendicular to this line but passes through the same y-intercept

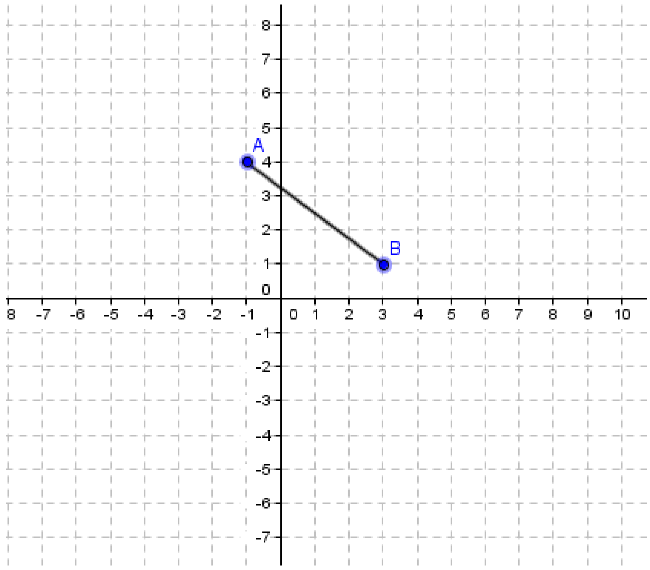
b. Find the missing coordinate C of the triangle

Show all working remembering it is a right angle triangle

Submit

Never submit passwords through Google Forms.

Appendix M: Extended Response for End of Topic test



The following line interval is drawn from A(-1,4) to B(3, 1)

a. Find the length of this line segment – show your working

b. A and B are two of the vertices of a right-angled triangle find the third coordinate C if AB = AC

c. If ABCD is a quadrilateral such that AB is parallel to CD
    a. What could the quadrilateral be?
    b. Describe all the different ways you could prove that its that type of quadrilateral

c. Use one of your strategies listed above to prove its that type of quadrilateral

d. What are the coordinates of D the missing vertice of the quadrilateral?

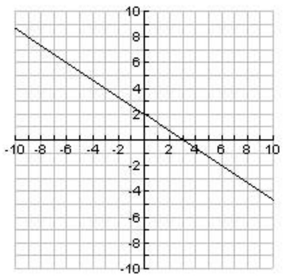
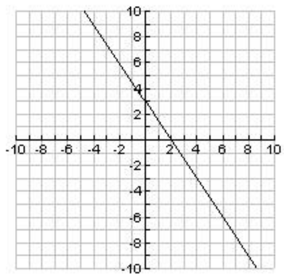
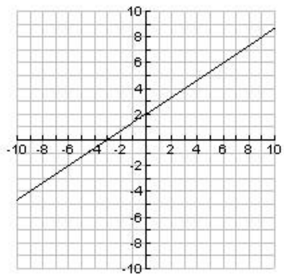
Appendix N: Student Evaluation Questions

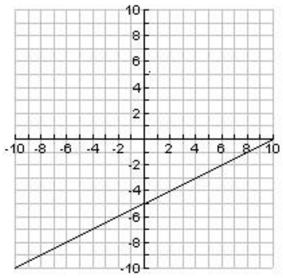
1. How would you rate this topic?
2. Did you think it was delivered better than you normally learn mathematics?
3. What would you like to see continued?
4. What would you change?

Appendix O: Matching Game

x-intercept: (3,0) y-intercept: (0,2)	x-intercept: (2,0) y-intercept: (0,3)	Slope: $\frac{2}{3}$ x-intercept: (-3,0)
Slope: $\frac{1}{2}$ y-intercept: (0,–5)	Slope: $-\frac{4}{3}$ y-intercept: (0,6)	Slope: $\frac{4}{3}$ y-intercept: (0,6)
Horizontal line through (0,7)	Vertical line through (7,0)	Slope: 3 y-intercept: (0,-6)

Slope: $\frac{1}{3}$ y-intercept: (0,-6)	**Slope: 1 Passing through the Origin**	**Slope: –1 y-intercept: (0,0)**
Slope: – 4 x-intercept: (0,0)	**Slope: $\frac{1}{4}$ y-intercept: (0,0)**	**Slope: –2 y-intercept: (0,–1)**
Slope: $-\frac{3}{2}$ y-intercept: (0,2)	**Slope: $\frac{3}{5}$ y-intercept: (0,6)**	**x-intercept: (–8,0) y-intercept: (0,4)**

Slope: 1 **y-intercept:** **(0,1)**	**Slope: –1** **x-intercept:** **(–1,0)**	**Slope: 0** **y-intercept:** **(0,–7)**
Slope: **undefined** **x-intercept:** **(–7,0)**	**Slope:** $-\frac{1}{2}$ **y-intercept:** **(0,8)**	**Slope:** $\frac{1}{2}$ **y-intercept:** **(0,8)**
		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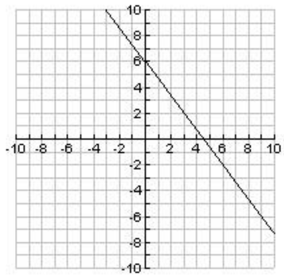
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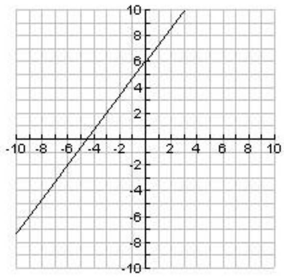
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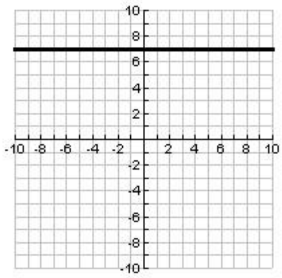
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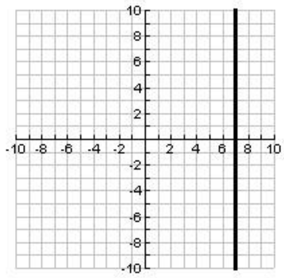
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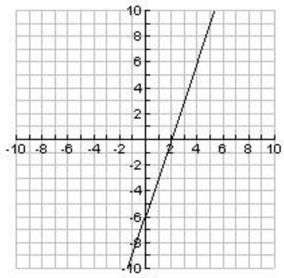
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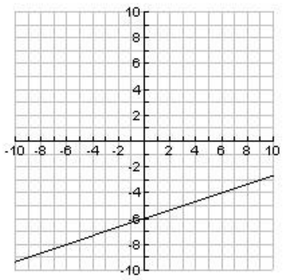
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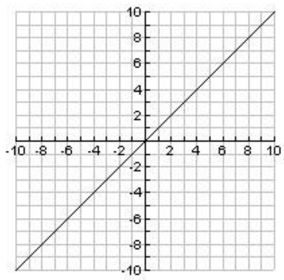
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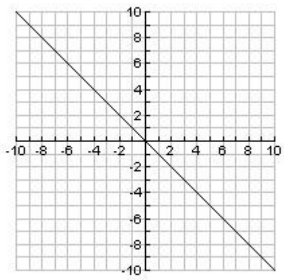
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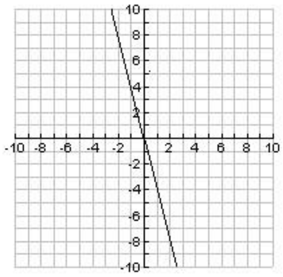
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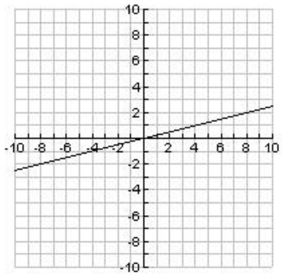
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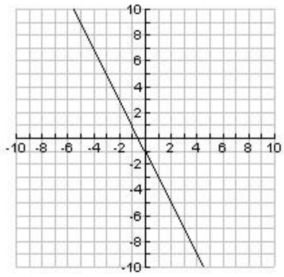
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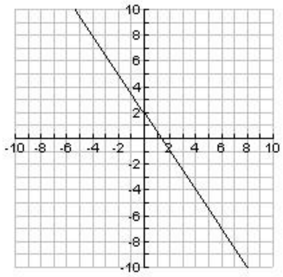
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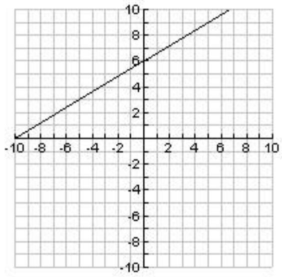
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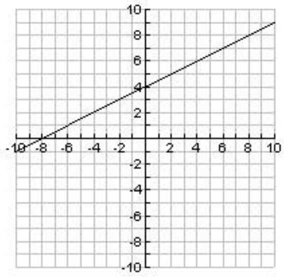
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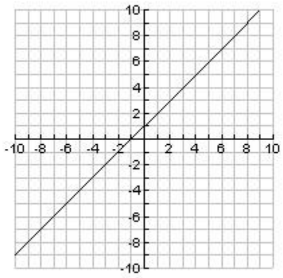
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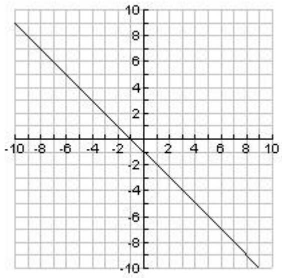
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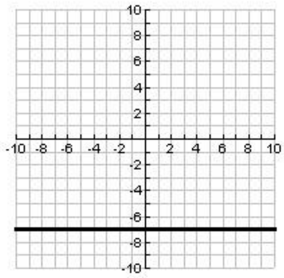
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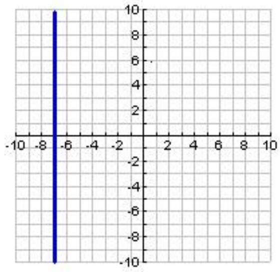
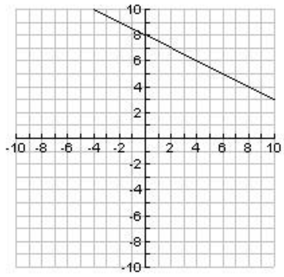
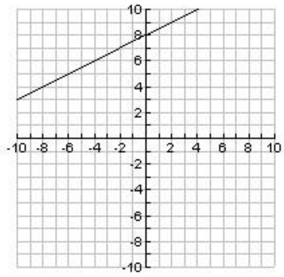






x	***y***
–3	4
0	2
3	0

x	***y***
–4	9
0	3
4	–3

x	***y***
–3	0
0	2
3	4

x	***y***
–2	–6
0	–5
2	–4

x	***y***
–3	10
0	6
3	2

x	***y***
–3	2
0	6
3	10

x	y
–4	7
0	7
5	7

x	y
7	–2
7	0
7	8

x	y
0	–6
2	0
4	6

x	y
–3	–7
0	–6
3	–5

x	y
–4	–4
0	0
4	4

x	y
–3	3
0	0
3	–3

x	y
–2	8
0	0
2	–8

x	y
–8	–2
0	0
8	2

x	y
–1	1
0	–1
1	–3

x	y
–4	8
0	2
4	–4

x	y
–5	3
0	6
5	9

x	y
–2	3
0	4
2	5

x	y
–1	0
0	1
1	2

x	y
–1	0
0	–1
1	–2

x	y
–6	–7
0	–7
6	–7

x	y
–7	–6
–7	0
–7	6

x	y
–2	9
0	8
2	7

x	y
–2	7
0	8
2	9

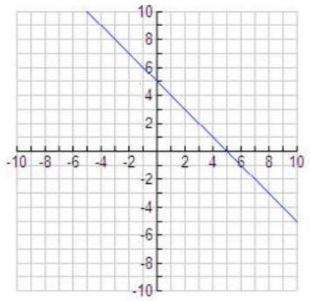
$2x+3y=6$	$3x+2y=6$	$-2x+3y=6$
$y=\frac{1}{2}x-5$	$y=-\frac{4}{3}x+6$	$y=\frac{4}{3}x+6$
$y=7$	$x=7$	$y=3x-6$

$y=\frac{1}{3}x-6$	$y=x$	$y=-x$
$y=-4x$	$y=\frac{1}{4}x$	$y=-2x-1$
$y=-\frac{3}{2}x+2$	$3x-5y=-30$	$2y-x=8$

$y = x + 1$	$y = -x - 1$	$y = -7$
$x + 7 = 0$	$y = -\frac{1}{2}x + 8$	$y = \frac{1}{2}x + 8$
The cost for adults to swim is \$3. The cost for children is \$2. I spent \$6 total.	**The cost for adults to swim is \$3. The cost for children is \$2. I spent \$6 total.**	**I pay \$2 w, wholesale for each book I buy. I then sell them for \$3 each. I have made \$6 so far.**

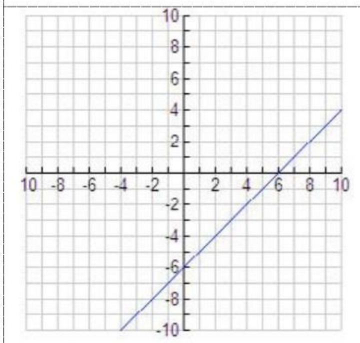
The water was ½ cm above the maximum fill line then dropped by 5 cm every hour.	**I was saving up to buy a bike. I started off owing my mom 4/3 of a dollar, and then gave her \$6 of my new allowance each week until I had saved enough.**	**The jar had 4/3 of a centimeter of water when I started, and I added 6 centimeters each day.**
No matter how much work I do, I always get \$7.	**I always work for 7 hours, but my pay varies.**	**The temperature started out at 3 degrees Celsius and dropped by 6 degrees each hour.**
My handicap on a game is 6 point and each time I score I gain 1/3 of a point.	**My daughter gets one dollar for each chore she does.**	**I pay (lose) one dollar for each chore my daughter does.**

I was climbing down an extremely steep mountain and for each foot forward I went down four metres.	For each four cups of flour I use when making bread, I use 1 egg.	The temperature starts at -1 degrees and decreases another 2 degrees each hour.
I start a movie with two cups of popcorn and eat at a rate of 1 and ½ cups per hour.	My suitcase is 30 kg too heavy. I can take out some large books that are 5 kg each. I also might put in some smaller books, which are 3 kg each.	I sold some hair bows at a craft fair and only made $8. Each bow sold for $2 and cost me $1 to make.
On the first day of my new workout I do one pushup. I add one more pushup to my routine each consecutive day.	I always pay my daughter one penny each morning, plus one additional penny for each chore. (hint: I am losing money.)	The temperature of a freezer is maintained at -7 degrees.



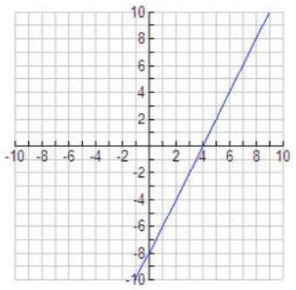
x	y
0	5
2	3
7	-2

$$y = -x + 5$$



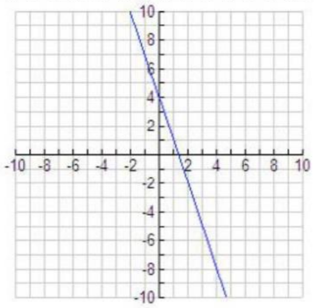
Slope: 1
y-intercept:
(0,-6)

$$x - y - 6 = 0$$



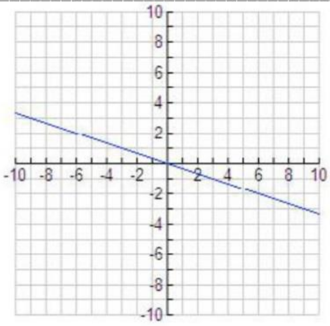
Slope: 2
y-intercept:
(0,-8)

$$y = 2x - 8$$



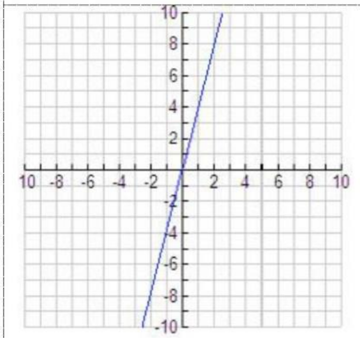
x	y
0	4
4	-8
2	-2

$$3x - y + 4 = 0$$



x	y
-6	2
-3	1
9	3

$$y = -\frac{1}{3}x$$



Slope: 0
y-intercept: the origin

$$4x - y = 0$$