CHAPTER 4. The $K(n,d)$ problem.

To determine the value of $K(n,d)$ given n and d, we can use all the results of Chapter 3. Indeed, if $N(d,k) = n$ then $K(n,d) = k$. However, we still need to consider all n's which are not in the range of $N(d,k)$.

It is obvious that if $n \leq d+1$ then $K(n,d) = 1$ since $G(n,d,1)$ is nonempty for $n \leq d+1$. We will assume from now on that $n > d+1$.

• **Theorem 6.**

If $n \leq d^s$ then $G(n,d,k)$ is nonempty for some $k \leq s$.

Proof.

Given $n \leq d^s$ we shall construct a digraph $G \in G(n,d,k)$ such that $k \leq s$.

Labelling the n vertices of G by $0, 1, 2, \ldots, n-1$ let there be arcs from vertex i to vertices

$$(-di + t) \bmod n, \qquad t = 1, 2, \ldots, d.$$

Then G is a diregular digraph with degree d.

We will show that the diameter k of G is at most s, i.e., that there is a directed path of length s or less from each vertex to all the other vertices.

From vertex i we can reach the following vertices (mod n)

$-di+1, -di+2, \ldots, -di+d$ in 1 step

$d^2i - 0, d^2i - 1, \ldots, d^2i - (d^2-1)$ in 2 steps

$-d^3i+1, -d^3i+2, \ldots, -d^3i+d^3$ in 3 steps

and so on.

In this way we reach in s steps the vertices

$$-d^s i + 1, -d^s i + 2, \ldots, -d^s i + d^s \pmod{n}$$

if s is odd;

or the vertices

$$d^s i - 0, d^s i - 1, \ldots, d^s i - (d^s - 1) \pmod{n}$$

if s is even.

In either case, if $n \leq d^s$ then in s steps we reach from the vertex i all the n vertices of G (including vertex i itself).

Hence G has diameter at most s ◇

Corollary 1.

If $n \leq d^s$ then $K(n, d) \leq s$.

Corollary 2.

If $n = d^s + 1$ and s odd then for some $k \leq s$, $G(n, d, k)$ is nonempty.

Proof.

Using the same construction as in the proof of Theorem 6, if s is odd then i and

$$-d^s i + 1, -d^s i + 2, \ldots, -d^s i + d^s \pmod{n}$$

are all distinct vertices, $d^s + 1$ of them ◇

The next result was also found independently by Imase and Itoh (1983).

Corollary 3.

If $n = d^{s-b}(d^b + 1)$ and b ($\leq s$) is odd then $G(n, d, k)$ is nonempty for some $k \leq s$.

Proof.

Using the same construction as in the proof of Theorem 6,

(i) If s is odd then from vertex i we reach the vertices (mod n)

$$d^{s-b}i - 0, d^{s-b}i - 1, \ldots, d^{s-b}i - (d^{s-b} - 1) \quad \text{in } s-b \text{ steps}$$

$$-d^{s}i + 1, -d^{s}i + 2, \ldots, -d^{s}i + d^{s} \quad \text{in } s \text{ steps.}$$

All numbers in each row are different.

For two numbers from different rows we have

$$d^{s-b}i - t - (-d^{s}i + t') = (d^{s} + d^{s-b})i - (t + t') = ni - (t + t') \mod n$$

and $1 \leq t + t' \leq n - 1$.

(ii) If s is even then from vertex i we reach the vertices (mod n)

$$-d^{s-b}i + 1, -d^{s-b}i + 2, \ldots, -d^{s-b}i + d^{s-b} \quad \text{in } s-b \text{ steps}$$

$$d^{s}i - 0, d^{s}i - 1, \ldots, d^{s}i - (d^{s} - 1) \quad \text{in } s \text{ steps.}$$

All numbers within each row are different.

For two numbers from different rows we have

$$(d^{s}i - t) - (-d^{s-b}i + t') = (d^{s-b} + d^{s})i - t - t' = ni - (t + t') \mod n$$

and $1 \leq t + t' \leq n - 1$.

Hence in each case all the numbers in $(s-b)^{th}$ and s^{th} rows are different and there are altogether n numbers in the two rows. Hence the diameter k of G is at most s $\diamondsuit$

Corollary 4.

If s is odd then $d^{s-1} + 1 \leq n \leq d^{s} + 1$ implies $s - 1 \leq K(n,d) \leq s$;

if s is even then $d^{s-1} + 2 \leq n \leq d^{s}$ implies $s - 1 \leq K(n,d) \leq s$.

Corollary 5.

(i) If s is odd and $d^s + 1 \geq n \geq (d^s - 1)/(d - 1)$ then $K(n, d) = s$;

if s is even and $d^s \geq n \geq (d^s - 1)/(d - 1)$ then $K(n, d) = s$.

(ii) Further, if $n = d^{s-b}(d^b + 1)$ and b ($\leq s$) is a positive odd integer

then $K(n, d) = s$.

Note that part (ii) of Corollary 5 is in fact a generalisation of Theorem 1.

Note also that in the proof of Theorem 6 we could have used constructions other than

$i \to (-di + t) \bmod n$ $(t = 1, 2, \ldots, d)$.

In fact, any of the following three construction schemes could have been used instead.

(a) $i \to (-di - t) \bmod n$ $(t = 1, 2, \ldots, d)$

(b) $i \to (di - t) \bmod n$ $(t = 1, 2, \ldots, d)$

(c) $i \to (di + t) \bmod n$ $(t = 1, 2, \ldots, d)$

However, the digraph resulting from construction (a) is isomorphic to the digraph in the proof of Theorem 6; while the constructions (b) and (c) also give isomorphic digraphs. It is easy to show that the constructions (a) and (b) do not give isomorphic digraphs.

Thus the above four construction schemes provide two essentially different possibilities.

However, had we used scheme (b) or (c) we would not have been able to deduce Corollaries 2 and 3 in the same straightforward way.

The following theorem, due to Fiol, Alegre and Yebra (1983) gives us a way of determining the values of $K(n, d)$ for certain values of n and d.

Theorem 7.

If $d > 1$ and $G(n, d, k)$ is nonempty then $G(dn, d, k + 1)$ is also nonempty.

Proof.

Suppose $d > 1$ and $G \in G(n, d, k)$ exists, with vertices labelled $1, 2, \ldots, n$.

We construct the line digraph of G, $L(G)$ as follows.

If $i \to j$ in G then there is a vertex denoted by (ij) in $L(G)$. If G contains a multiple arc (i, j) with multiplicity m then there are m distinct vertices denoted by $(ij)_1, (ij)_2, \ldots, (ij)_m$ in $L(G)$.

Thus $L(G)$ has dn distinct vertices, one for each arc of G.

To construct arcs in $L(G)$, we connect vertex (ij) (with or without a subscript) to vertex (mn) (with or without a subscript) if and only if $j = m$.

That is, if $i \to j$ and $j \to n$ in G then $(ij) \to (jn)$ in $L(G)$.

It is obvious that the degree of $L(G)$ is d.

To prove that the diameter of $L(G)$ is $k + 1$, note that to go from vertex $u = (ij)$ to the vertex $v = (mn)$, $u \neq v$, in $L(G)$ is equivalent to going from the arc (i, j) to the arc (m, n) in G.

Now, the directed path of minimum length with these two terminal arcs contains the shortest path from vertex j to vertex m.

Therefore, the length of the shortest path from vertex u to vertex v is

$$l_{L(G)}(u, v) = l_G(j, m) + 1$$

provided $u \neq v$.

If $d > 1$ then such vertices u and v exist in $L(G)$ and so the diameter of $L(G)$ is $k + 1$ $\diamondsuit$

Corollary.

If $n = dm$ and $K(m,d) = k$ then $K(n,d) \le k+1$.

Theorem 7 gives us an infinite sequence of line digraph iterations (if $d > 1$)

$$G, L(G), L^2(G) = L(L(G)), \ldots, L^n(G) = L(L^{n-1}(G)), \ldots$$

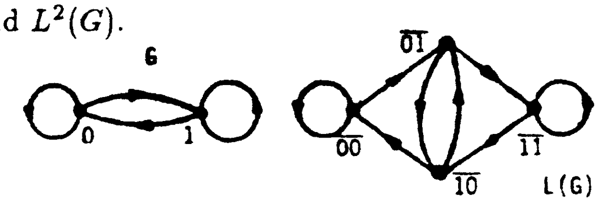
Note that each vertex x in $L^2(G)$ represents an arc (u,v) of $L(G)$ and that the vertices of this arc correspond to two adjacent arcs of G, say (i,j) and (j,k) (i,j,k not necessarily all distinct).

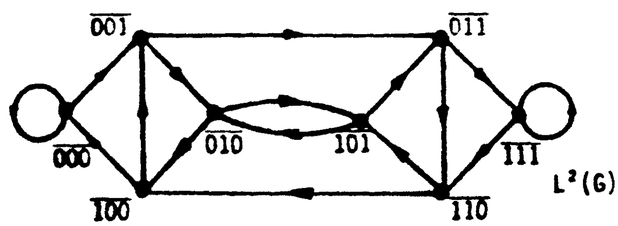
Thus the vertex x of $L^2(G)$ represents a directed path of length 2 in G and we may write $x = \overline{ijk}$.

Further, the vertex x will be adjacent to another vertex y if and only if y repesents an arc of the form (v,w) in $L(G)$. Therefore y must be equal to $\overline{jkl}$ with $(k,l) \in G$.

More generally, each vertex x in $L^n(G)$ represents a directed path $(i_0, i_1, \ldots, i_n)$ of length n in G and $x = \overline{i_0 i_1 \ldots i_n}$ is adjacent to the vertices of the form $y = \overline{i_1 i_2 \ldots i_n i_{n+1}}$ with $(i_n, i_{n+1}) \in G$.

As an illustration, starting with $G \in G(2,2,1)$, the line digraph iteration gives the following digraphs $L(G)$ and $L^2(G)$.

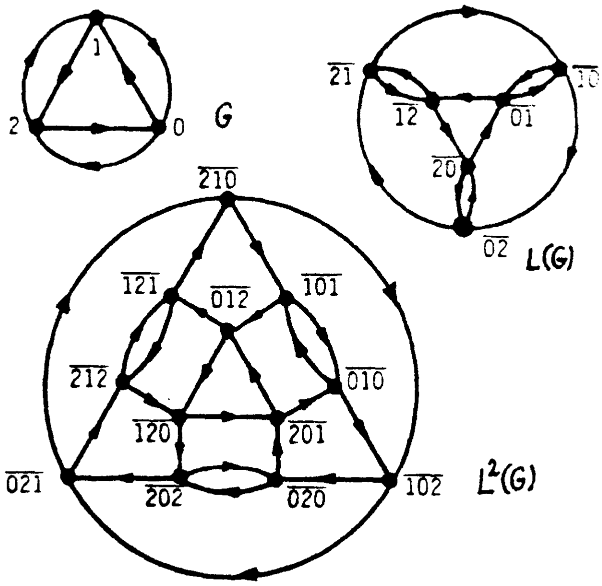




The above digraphs are called the *perfect shuffle*.

It is interesting to note that the digraphs of Theorem 1 may be obtained by line digraph iterations, starting with the complete digraph $G \in G(d+1, d, 1)$.

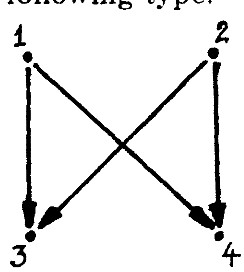
For example, if $d = 2$ then starting from $G \in G(3, 2, 1)$ the line digraph iteration gives us the following digraphs $L(G)$ and $L^2(G)$.



If $G_1 \in G(d+1, d, 1)$ then $G_2 = L(G_1), \ldots, G_k = L(G_{k-1}), \ldots$

where $G_k = L^{k-1}(G)$ has $n = (d+1)d^{k-1}$ vertices and diameter k.

Note that a digraph F is a line digraph of some digraph if and only if whenever $(v, w), (u, w)$ and (u, x) are arcs of F then so is (v, x). This result is due to Heuchenne (1964). Thus every line digraph of a diregular digraph of degree $d > 1$ consists of a set of subdigraphs (not necessarily disjoint) of the following type.



Equivalently, a digraph F is a line digraph of some digraph if and only if any two rows (columns) of its adjacency matrix are either identical or orthogonal (Richards, 1967).

Another interesting point about the line digraph iteration scheme is that if G contains p digons then it is easy to show that $L(G)$ also contains p digons. We will denote the set of all diregular digraphs of order n, degree d, diameter k and p digons by $G(n, d, k) < p >$. Let us now summarize our present state of knowledge of the $K(n, d)$ problem. We have

$K(n, 1) = n - 1 \quad (n \geq 2)$

$K(1, d) = 0$

$K(d - 1, d) = 1$

$K(d + d^2, d) = 2$

$K(d^{k-b}(d^b + 1), d) = k \qquad$ (for b $(\leq k)$ odd)

$K(n, d) = k$ whenever $d^k + 1 \geq n \geq (d^k - 1)/(d - 1)$ (if k odd)

or $d^k \geq n \geq (d^k - 1)/(d - 1)$ (if k even).

Otherwise, if $d^k + 2 \leq n \leq (d^{k+1} - 1)/(d - 1) - 1$ (if k odd)

or $d^k + 1 \leq n \leq (d^{k+1} - 1)/(d - 1) - 1$ (if k even) then

$$k \leq K(n, d) \leq k + 1.$$

In the remainder of this chapter we will turn our attention to the case when $d = 2$.

Since $N(2, 0) = 1$, $N(2, 1) = 3$, $N(2, 2) = 6$ and $N(2, 3) = 12$

we have $K(1, 2) = 0$, $K(3, 2) = 1$, $K(6, 2) = 2$ and $K(12, 2) = 3$.

Theorem 8 (Culik, 1984).

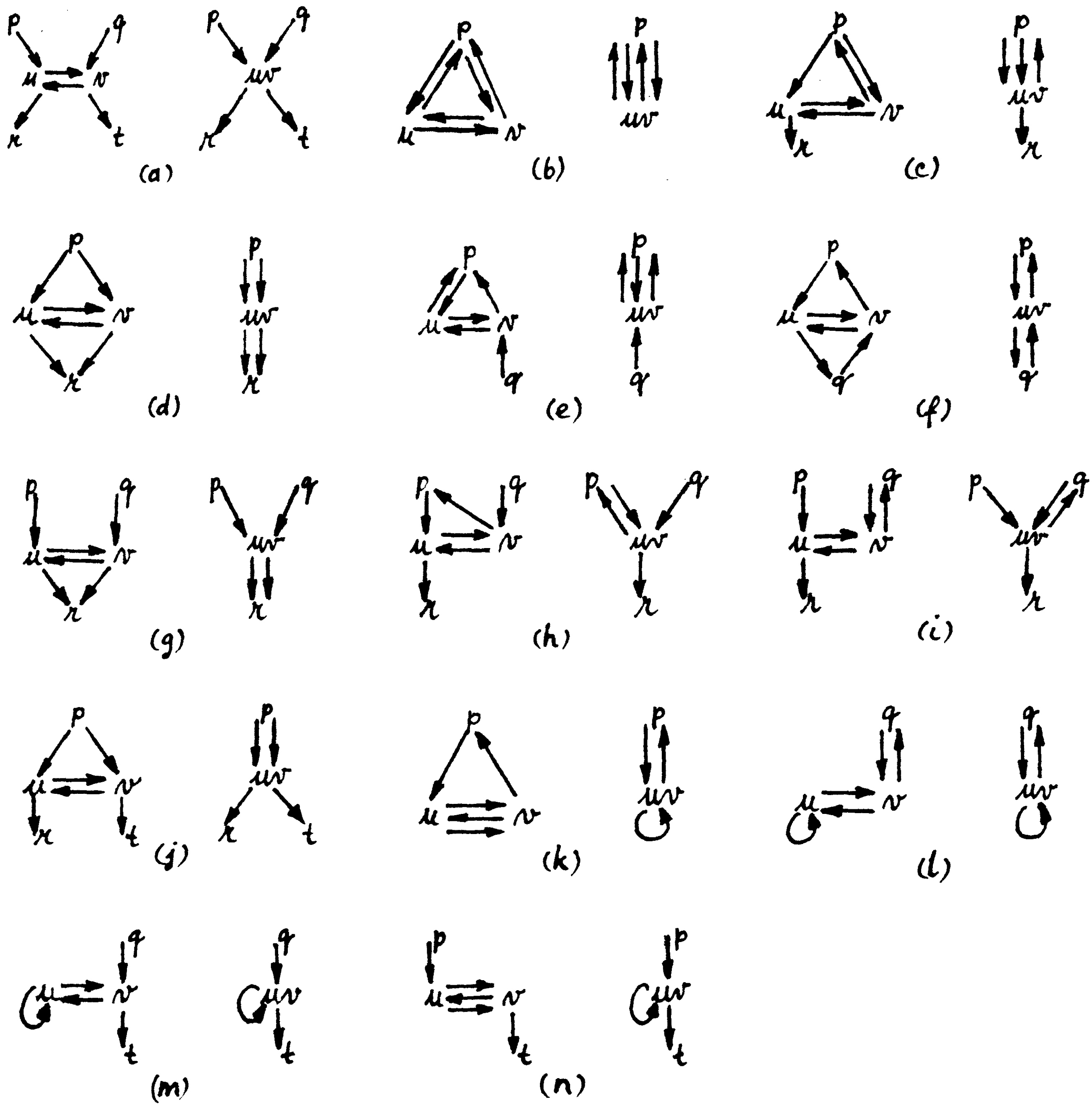
If $n > 2$ and $G \in G(n, 2, k) < p >$ exists then also $G' \in G(n - 1, 2, k') < p' >$ exists with $k' \leq k$ and $p' \geq p - 1$.

Proof.

Suppose $G \in G(n, 2, k)\langle p\rangle$, $p \geq 1$. Then for some vertices u, $v (u \neq v)$ there exist arcs (u, v) and (v, u) in G.

Suppose also that $p \to u$, $q \to v$ and $u \to r$, $v \to t$ (u, v, p, q, r, t not necessarily all distinct).

Construct G' from G by "gluing" vertices u and v together, so that instead of vertices u and v we have a vertex uv in G'. The following cases can occur.



It is obvious that G' is a diregular digraph of degree 2, $n-1$ vertices, diameter either k or $k-1$ and if $n > 2$, G' has at least $p-1$ digons $\diamond$

Corollary.

If $n \geq 2^k + 2$ then $p' = p - 1$.

Proof.

If $n \geq 2^k + 2$ then cases (f),(h) and (k) cannot happen in $G \in G(n, 2, k)$ $\diamond$

Theorem 8 together with Theorem 7 gives us a "digon reduction" scheme (Fris, 1985). If $G \in G(n, 2, k) < p >$ $(p \geq 1)$ then we can use digon reduction p times as follows.

$$G \in G(n, 2, k) < p > \Longrightarrow G_1 \in G(n-1, 2, k_1) < p-1 >$$

$$\Longrightarrow G_2 \in G(n-2, 2, k_2) < p-2 >$$

$$\Longrightarrow \ldots$$

$$\Longrightarrow G_p \in G(n-p, 2, k_p) < 0 >= G(n-p, 2, k_p)$$

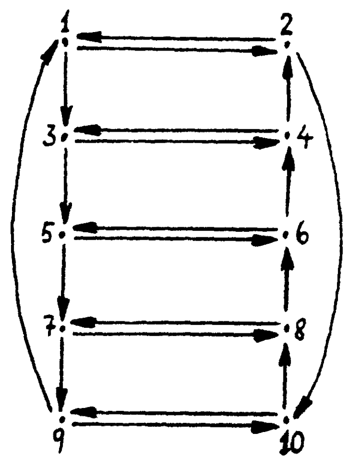
where $k \geq k_1 \geq k_2 \geq \ldots \geq k_p$.

Hence if $K(n, 2) = k$ and $G(n, 2, k) < p >$ is nonempty with $p \geq 1$ then $K(n-1, 2) \leq k$.

We will call the iteration scheme of Theorem 7 Rule I, or I; and the iteration scheme of Theorem 8 Rule II, or II for brevity.

We can use these rules to deduce the minimum diameter $K(n, 2)$ for many values of n.

For example, starting with $G \in G(10,2,3) < 5 >$ i.e.,



we can proceed as follows

$G_1 \in G(10,2,3) < 5 > \Longrightarrow$

$\Longrightarrow G_2 \in G(20,2,4) < 5 >$ by I and since $4 \leq K(20,2) \leq 5$

$\Longrightarrow G_3 \in G(19,2,4) < 4 >$ by II and since $4 \leq K(19,2) \leq 5$

$\Longrightarrow G_4 \in G(18,2,4) < 3 >$ by II and since $4 \leq K(18,2) \leq 5$

$\Longrightarrow G_5 \in G(17,2,4) < 2 >$ by II and since $4 \leq K(17,2) \leq 5$

$\Longrightarrow G_6 \in G(16,2,4) < 1 >$ by II and since $4 \leq K(16,2) \leq 5$

$\Longrightarrow G_7 \in G(15,2,4) < 0 >$ by II and since $4 \leq K(15,2) \leq 5$

Further,

$G_1 \in G(20,2,4) < 5 > \Longrightarrow$

$\Longrightarrow G_2 \in G(40,2,5) < 5 >$ by I and since $5 \leq K(40,2) \leq 6$

$\Longrightarrow G_3 \in G(39,2,5) < 4 >$ by II and since $5 \leq K(39,2) \leq 6$

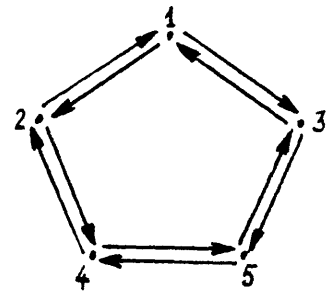
$\Longrightarrow G_4 \in G(38,2,5) < 3 >$ by II and since $5 \leq K(38,2) \leq 6$

etc.

Hence we deduce that

$K(20,2) = 4$, $K(19,2) = 4$, $K(18,2) = 4$. $K(17,2) = 4$. $K(16,2) = 4$, $K(15,2) = 4$ and $K(40,2) = 5$, $K(39,2) = 5$, $K(38,2) = 5$ etc.

Note that $G_1 \in G(10,2,3) < 5 >$ can be deduced from $G_2 \in G(5,2,3) < 5 >$, i.e., from



by using Rule I.

In the following table we give the values of $K(n,2)$ for $n \leq 100$.

n	$K(n,2)$	n	$K(n,2)$	n	$K(n,2)$
1	0	34	5	67	6
2	1	35	5	68	6
3	1	36	5	69	6
4	2	37	5	70	6
5	2	38	5	71	6
6	2	39	5	72	6
7	3	40	5	73	6
8	3	41	5	74	6
9	3	42	5	75	6
10	3	43	5	76	6
11	3	44	5	77	6
12	3	45	5	78	6
13	4	46	5	79	6
14	4	47	5	80	6
15	4	48	5	81	6 or 7
16	4	49	5 or 6	82	6
17	4	50	5	83	6
18	4	51	5 or 6	84	6
19	4	52	5 or 6	85	6
20	4	53	5 or 6	86	6
21	4	54	5 or 6	87	6
22	4	55	5 or 6	88	6
23	4	56	5 or 6	89	6
24	4	57	5 or 6	90	6
25	4	58	5 or 6	91	6
26	4 or 5	59	5 or 6	92	6
27	4 or 5	60	5 or 6	93	6
28	4 or 5	61	5 or 6	94	6
29	4 or 5	62	6	95	6
30	5	63	6	96	6
31	5	64	6	97	6 or 7
32	5	65	6	98	6 or 7
33	5	66	6	99	6 or 7
				100	6

CONCLUSION.

In this thesis we have proved some new results about the relationship between the degree, diameter and order of diregular digraphs. In particular, for degree 2 we have improved upon the bounds for the order (given diameter); and for certain values of order we have improved upon the bounds for diameter (given order). However, the $N(2,k)$ and the $K(n,2)$ problems still remain open, as do the $N(d,k), K(n,d)$ and $D(n,k)$ problems in general. To get nearer to the solutions we could try to answer some simpler questions, such as

1. Is $K(n,d)$ monotonic in n?
2. Is $D(n,k)$ monotonic in n?
3. Is $D(n,k)$ monotonic in k?
4. Does $K(n,d) = k$ imply $K(nd,d) = k+1$?

Certainly, $K(n,d) = k$ implies $K(nd,d) \leq k+1$ using the line digraph iteration scheme of Theorem 7. On the other hand, a similar implication for $N(d,k)$ does not hold as for example $N(2,3) = 12$ and $N(2,4) \geq 25$.

5. Is $K(n,d)$ monotonic for some intervals of n? In particular,
   (a) Is $K(n,d)$ monotonic for all n such that $(d^k - 1)/(d-1) \leq n \leq N(d,k))$?
   (b) Is $K(n,d)$ monotonic for all n such that $(d^k - 1)/(d-1) \leq n \leq (d+1)d^{k-1}$?

An affirmative answer to any of these questions would much advance the solution of the $N(d,k), K(n,d)$ and $D(n,k)$ problems.

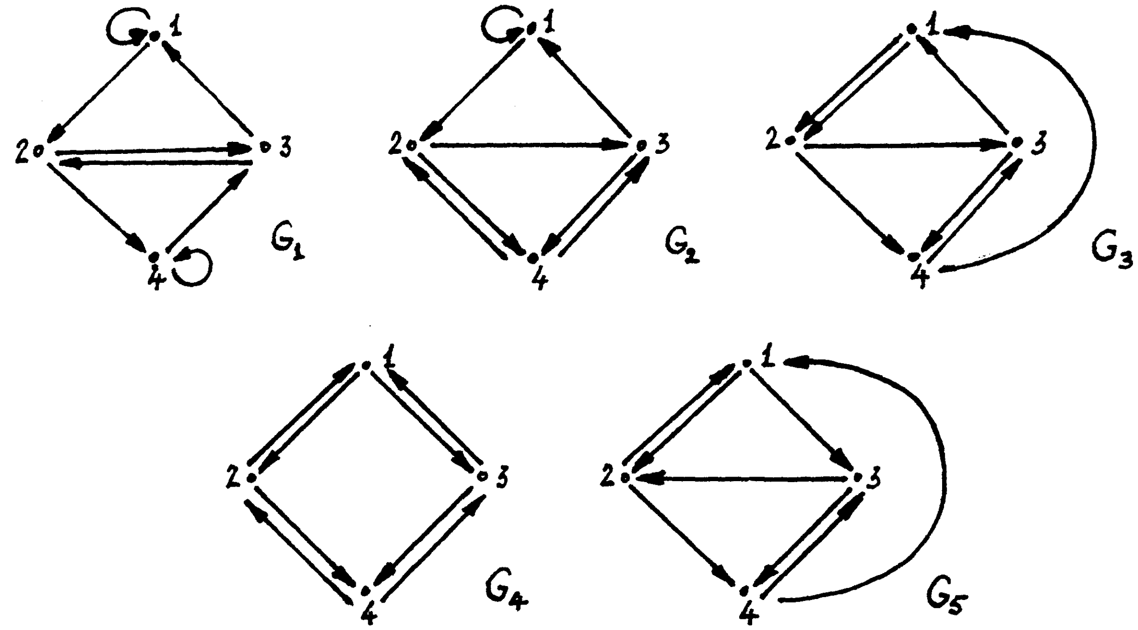
Apart from the three problems treated in this thesis, there are many other related open problems. We will mention just a few of these.

6. The problem of finding the *minimum average diameter* $\bar{K}(n,d) = \min_{G(n,d,k)} \bar{k}$ given n, d and k, where $\bar{k}$ is the *average diameter* of G,

$$\bar{k} = \bar{k}(G) = \frac{\sum_{i,j \in G} d_{ij}}{n(n-1)}$$

where d_{ij} is the length of the shortest path from vertex i to vertex j in the digraph G.

For example, there are five nonisomorphic digraphs $G_1, G_2, G_3, G_4, G_5 \in G(4,2,2)$.



The average diameters of digraphs G_1, G_2, G_3, G_4 and G_5 are

$\bar{k}(G_1) = 1.5$, $\bar{k}(G_2) = \bar{k}(G_3) = 1.413$ and $\bar{k}(G_4) = \bar{k}(G_5) = 1.3$.

7. The problem of finding all the optimal nonisomorphic digraphs of a given order and degree.

Bowen (1985) found (with the use of a computer) all the optimal nonisomorphic diregular digraphs for degree 2, and order up to 12.

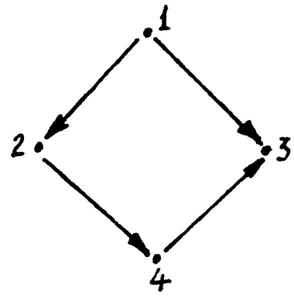
The following table gives the number of nonisomorphic diregular digraphs $I(n, k)$ for given order n and diameter k.

n	k	$I(n,k)$
1	0	1
2	1	2
3	1	1
4	2	5
5	2	5
6	2	3
7	3	158
8	3	295
9	3	413
10	3	203
11	3	24
12	3	3

APPENDIX.

• Lemma 5.

If $G \in G(13,2,3)$ then G does not contain the following subdigraph S_4.



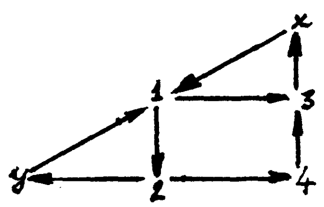
S_4

Proof.

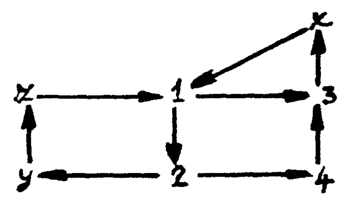
Suppose $S_4 \subset G \in G(13,2,3)$. Then there must be a path of length 2 or 3 from vertex 2 to vertex 1; and a path of length 2 or 3 from vertex 3 to vertex 1. Thus vertex 1 must lie on at least two 3- or 4-circuits.

Suppose the two arcs going to vertex 1 are $x \to 1$ and $y \to 1$. Obviously, $x \neq y$. Then there must be a path of length 2 or 3 from vertex 1 to vertex x and a path of length 2 or 3 from vertex 1 to vertex y. Hence x lies on at least one 3- or 4-circuit containing 1; and y lies on at least one 3- or 4-circuit containing 1.

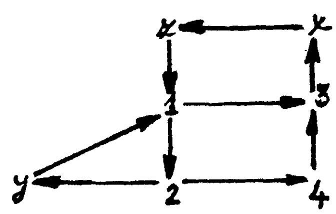
Thus we have the following possibilities.



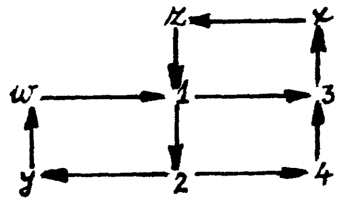
(a)



(b)



(c)



(d)

where $1, 2, 3, 4, x, y, z, w$ need not be all distinct vertices. We will use Lemma 4 throughout this proof.

Case (a).

Impossible as we can reach at most 11 vertices from vertex 1.

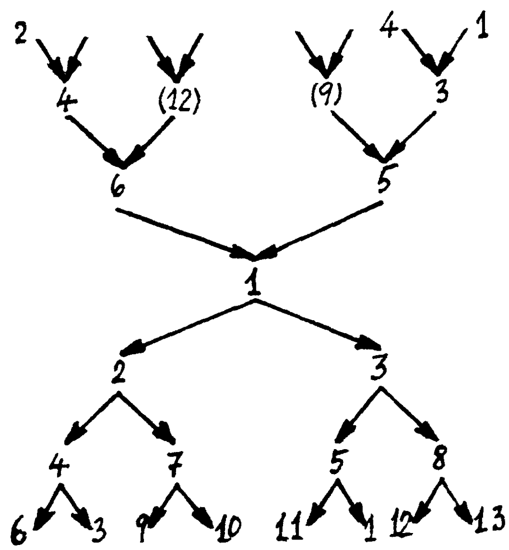
Case (b).

Obviously, $x \neq 1, 2, 3, 4, y, z$; $y \neq 1, 2, 3, x, z$; $z \neq 1, 2, 3, 4, x, y$. This gives the following possible cases.

(i) $y = 4$ and vertices $1, 2, 3, 4, x, z$ all distinct.

(ii) Vertices $1, 2, 3, 4, x, y, z$ all distinct.

(i) We have



where we have arbitrarily labelled the vertices 5,6,7,8,9,10,11,12,13 and where placing (9) and (12) is explained below. Since vertex 1 has to reach all the other 12 vertices in at most 3 steps, any other labelling of vertices would produce an isomorphic digraph.

Now, to reach 1 from 7 we must have (9 or 10) $\rightarrow$ (6 or 5).

To reach 1 from 8, we need (12 or 13) $\rightarrow$ (6 or 5).

Now $9, 10 \not\to 6$ (else we cannot reach 6 from 3) so (say) $9 \to 5$ and $12 \to 6$.

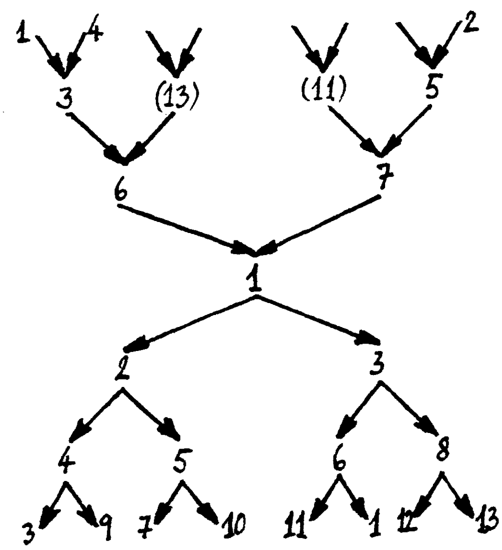
To reach 1 from all vertices we have to have arcs from vertices 10,11,13 to vertices 4,12,9, i.e., $10 \to$ (4 or 12 or 9) and $11 \to$ (4 or 12 or 9) and $13 \to$ (4 or 12 or 9). We will write this as $\{10, 11, 13\} \to \{4, 12, 9\}$ for brevity.

Moreover, to reach 12 from 2, we need $10 \to 12$.

Now to reach 3 from 7 we need (9 or 10) $\to 4$; and to reach 3 from 8 we need (12 or 13) $\to 4$.

As we already have $2 \to 4$, this is not possible.

(ii) We have



Now, to reach 3 from 8, we need (12 or 13) $\to 4$, say $12 \to 4$.

To reach 3 from all vertices we need $\{9, 10, 11, 13\} \to \{6, 7, 2, 12\}$.

To reach 1 from 8, we must have (12 or 13) $\to$ (6 or 7); and so $13 \to$ (6 or 7).

Now $11 \not\to 2$ and $11 \not\to 12$ (else we cannot reach 12 from 2) and $11 \not\to 6$.

Hence $11 \to 7$ and then also $13 \to 6$ and $\{9, 10\} \to \{2, 12\}$.

To reach 1 from all, we need $\{9, 10, 12\} \to \{13, 5, 11\}$.

Now $12 \not\to 13$, $12 \not\to 11$ (else we cannot reach 11 from 2) and so $12 \to 5$ and

$\{9, 10\} \to \{13, 11\}$.

To reach all from 3, we must have $\{11, 13\} \to \{9, 10\}$.

That leaves $7 \to 8$ as the only possibility.

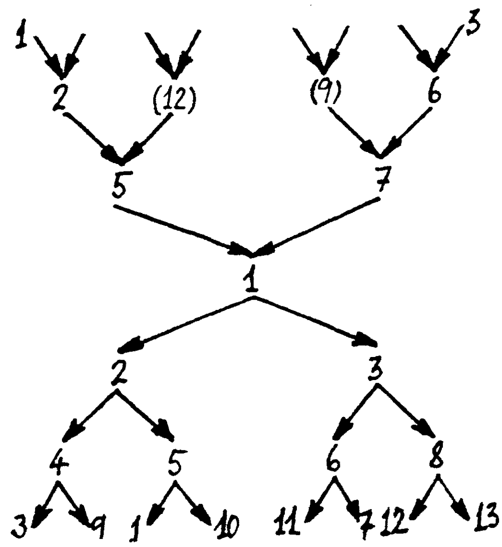
But then we cannot reach all vertices from vertex 7.

Case (c).

Obviously, $x \neq 1, 2, 3, 4, y, z$; $y \neq 1, 2, 3, 4, x, z$; $z \neq 1, 2, 3, 4, x, y$.

Hence the vertices $1, 2, 3, 4, x, y, z$ are all distinct.

We have



To reach 1 from 4, we must have $9 \to (5 \text{ or } 7)$.

Now $9 \not\to 5$ (else we cannot reach 5 from 3) so $9 \to 7$.

To reach 1 from 8, we must have $(12 \text{ or } 13) \to 5$, say $12 \to 5$.

To reach all from 2, we need $\{9, 10\} \to \{11, 12, 13\}$, i.e., $10 \to$ two of $\{11, 12, 13\}$.

But now we cannot reach 7 from 5.

Case (d).

Obviously, $x \neq 1, 2, 3, 4, y, z, w$; $y \neq 1, 2, 3, x, z, w$; $z \neq 1, 2, 3, 4, x, y, w$;

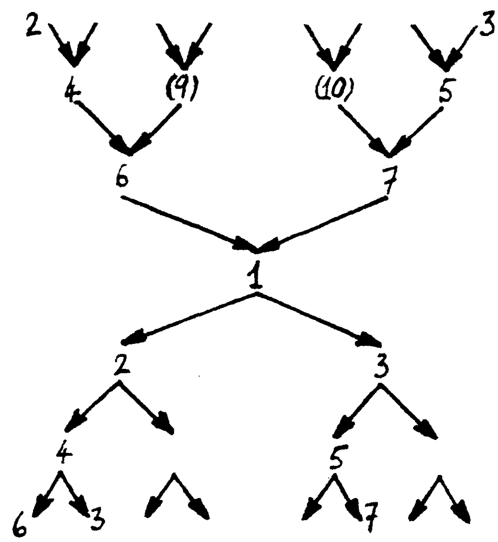
$w \neq 1, 2, 3, 4, x, y, z$.

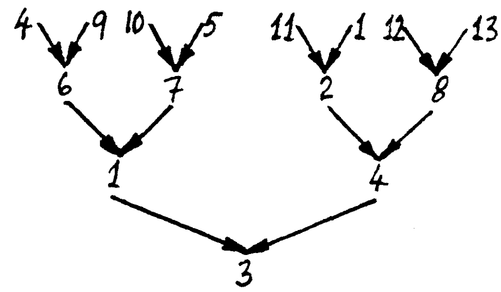
This gives the possible cases

(i) $y = 4$ and the vertices $1, 2, 3, 4, x, z, w$ all distinct.

(ii) Vertices $1, 2, 3, 4, x, y, z, w$ all distinct.

(i) We have





In this case we can immediately label all the vertices going to vertex 3 in up to 3 steps as there is only one possibility (up to isomorphism).

Now, to reach 4 from 3, we must have $3 \to$ (11 or 12 or 13).

Suppose $3 \to 11$.

To reach 2 from all, we need $8 \to 11$.

To reach 4 from 5, we need $5 \to$ (12 or 13), say $5 \to 12$.

To reach 6 from 3, we need $11 \to 9$ (as $11 \not\to 4$).

But then we cannot reach all from 8.

Hence $3 \not\to 11$ so $3 \to 12$ (say).

To reach 2 and 6 from 3, we must have $\{5, 12\} \to \{11, 9\}$.

To reach 4 from 5, we need $5 \to$ (11 or 13) (since $5 \not\to 1, 12$).

Hence $5 \to 11$ and $12 \to 9$.

To reach 6 from 5, we need (7 or 11) $\to$ 9.

To reach 2 from 12, we must have (8 or 9) $\to$ 11.

Suppose $9 \to 11$. Then $7 \to 9$ (as $11 \not\to 9$).

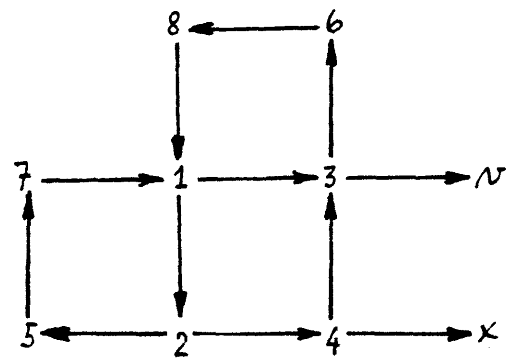
Then also $11 \to 10$ (to reach 1 from 11) and $13 \to 5$ (to reach 2 from 13).

But then we cannot reach all from 5.

Hence $8 \to 11$. Then $11 \to 10$ (to reach 7 from 8).

But then we cannot reach 9 from 8 as $6 \not\to 9$, $2 \not\to 9$ and $10 \not\to 9$.

(ii) We have

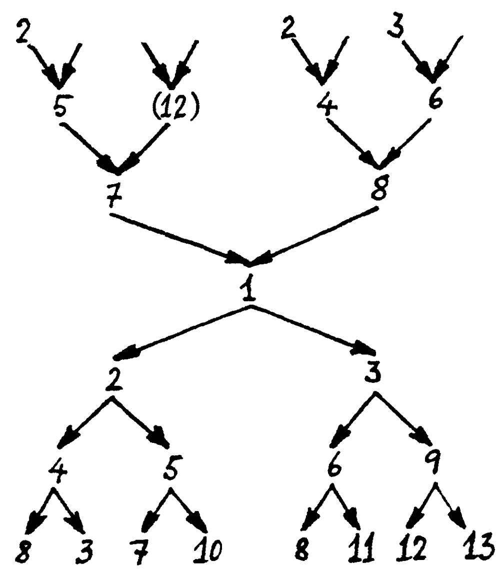


Obviously, $v \neq 1, 2, 3, 4, 5, 6, 7, 8$ so $v = 9$ (say).

Obviously, $x \neq 1, 2, 3, 4, 5, 6, 7, 9$.

Suppose $x = 8$.

Then we have



Then to reach 1 from 9, we need $12 \to 7$ (say).

To reach 3 from 9, we must have $13 \to 4$ (as $12 \not\to 4$ -else we cannot reach 3 from all).

To reach 4 from 5, we need $10 \to 13$ (since $7, 10 \not\to 2$ - else we cannot reach 2 from 3; and $7 \not\to 13$ - else we cannot reach 3 from all).

To reach 3 from 11, we need $11 \to 2$.

To reach all from 4, we must have $8 \to 5$.

But then we cannot reach 5 from 9.

Hence $x = 10$ (say).

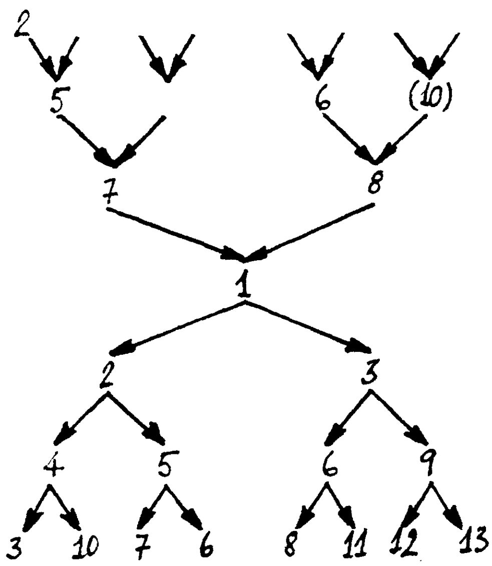
Now we have

$$
\begin{tikzcd}
& & 8 \arrow[d] & 6 \arrow[l] & \\
& 7 \arrow[r] & 1 \arrow[r] \arrow[d] & 3 \arrow[r] \arrow[u] & 9 \\
y & 5 \arrow[l] \arrow[u] & 2 \arrow[l] \arrow[r] & 4 \arrow[r] \arrow[u] & 10
\end{tikzcd}
$$

Obviously, $y \neq 1, 2, 3, 4, 5, 7, 8, 10$.

Suppose $y = 6$.

Now we have



Then to reach 6 from 9, we must have $12 \to 5$ (say).

To reach 6 from all, we must have $\{10, 11, 13\} \to \{2, 4, 12\}$.

Now $11, 13 \not\to 12$ (else we cannot reach 12 from 2) so $10 \to 12$.

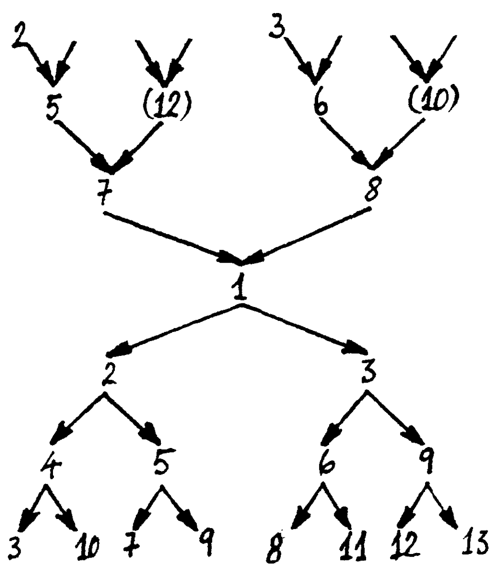
To reach 1 from 4, we need $10 \to$ (7 or 8).

Now $10 \not\to 7$ (else we cannot reach 7 from 3) and so $10 \to 8$.

But then we cannot reach 2 from 4. Hence $y \neq 6$.

Suppose $y = 9$.

Then we have



Then $10 \to 8$ (to reach 8 from 2 and since $7 \not\to 8$).

To reach 1 from 9, we must have $12 \to 7$ (say).

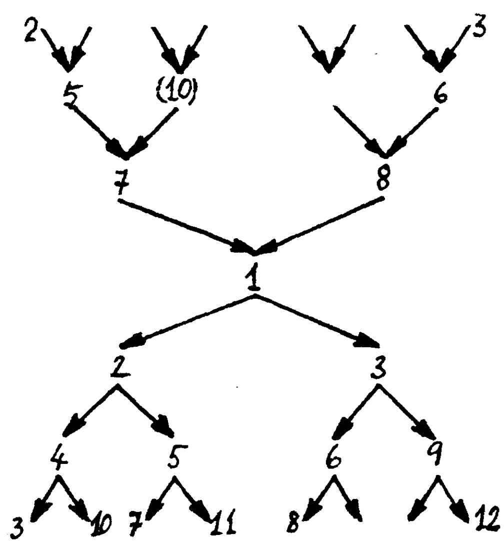
To reach 7 from 4, we need $10 \to 12$ ($10 \not\to 5$ -else we cannot reach 5 from 3).

Then $7 \to 11$ (to reach 11 from 2).

But then we cannot reach 8 from 5.

Hence $y \neq 9$ and so $y = 11$ (say).

Now we have



As vertex 3 is already a repeat from 1, we can only have one more repeat.

Thus we can assume $9 \to 12$.

To reach 1 from 4, we need $10 \to (7 \text{ or } 8)$.

Supose $10 \to 7$. To reach 13 from 1, we must have $(6 \text{ or } 9) \to 13$.

If $10 \to 7$ then $(6 \text{ or } 9) \to (5 \text{ or } 10)$ (to reach 7 from 3).

To reach 8 from 2, we need $11 \to 8$ (as $10 \not\to 8$ and $7 \not\to 8$).

Then $9 \to (5 \text{ or } 10)$ (to reach 1 from 9) and so $6 \to 13$.

To reach 2 from 4, we must have $10 \to x \to 2$ (as $7 \not\to 2$ and $9 \not\to 2$ and $10 \not\to 2$).

Now $x \neq 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$.

If $x = 13$ then $9 \to 10$ (to reach 2 from 9) and $12 \to 6$ (to reach 2 from 12).

But then we cannot reach 13 from all.

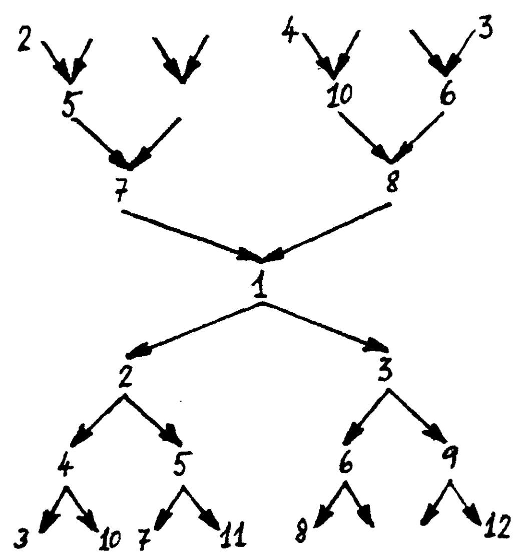
Hence $x = 12$ and $10 \to 12, 12 \to 2$.

Then $9 \not\to 10$ (as $9 \to 12$) and so $9 \to 5$.

But then we cannot reach 3 from 9 (as $12 \not\to 1, 12 \not\to 4$).

Hence $10 \not\to 7$ and so $10 \to 8$.

Now we have



Suppose $6 \to$ repeat from 1.

Then $9 \to 13$.

To reach 3 from 9, we must have (12 or 13) $\to 4$, say $12 \to 4$.

To reach 4 from 5, we must have (7 or 11) $\to$ (12 or 2).

To reach 3 from all, we need $\{11, 13\} \to$ two of $\{2, 12, 7\}$.

To reach 1 from 9, we need (12 or 13) $\to 7$ and so $11 \to$ (2 or 12).

If $11 \to 2$ then $6 \to 11$ (to reach 2 from 3).

But then we cannot reach 2 from all.

Hence $11 \not\to 2$ and so $11 \to 12$.

Now if $12 \to 7$ then $10 \to 5$ (to reach 7 from 4).

To reach 5 from 9, we need (12 or 13) $\to$ (10 or 2).

To reach 5 from 3, we must have $6 \to$ (10 or 2).

To reach 13 from 2, we must have (7 or 11) $\to 13$.

To reach 8 from 5, we need (7 or 11) $\to 6$.

Now (12 or 13) $\to$ (10 or 6) (to reach 8 from 9) so (12 or 13) $\to 10$ and $6 \to 2$.

Then $8 \to 9$ (to reach all from 6).

But then we cannot reach 4 from 10.

Hence $12 \not\to 7$ and so $13 \to 7$.

Then $10 \to$ (5 or 13) (to reach 7 from 4).

If $10 \to 5$ then $6 \to 2$ (to reach 2 from 4) and $12 \to 6$ (to reach 2 from all).

But then we cannot reach all from 12.

Hence $10 \not\to 5$ and so $10 \to 13$.

Now to reach 13 from 5, we need (7 or 11) $\to$ (9 or 10).

To reach 12 from 10, we must have (8 or 13) $\to$ (9 or 11).

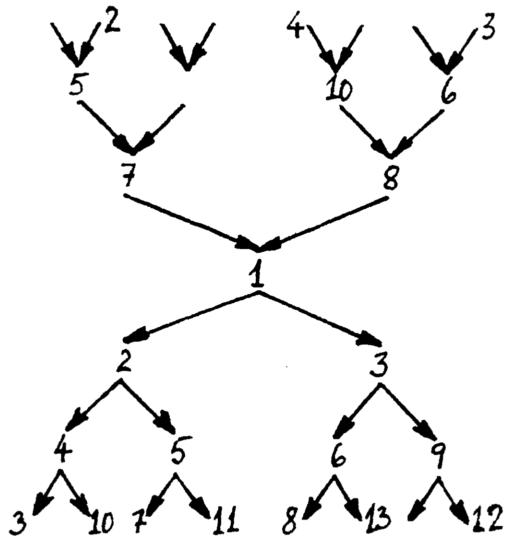
To reach 4 from 10, we need (8 or 13) $\to 2$.

Since $8 \not\to 2$ we must have $13 \to 2$.

But now we cannot reach all from 13.

Hence $6 \not\to$ repeat from 1 and so $6 \to 13$.

Now we have



To reach 3 from 9, we need $9 \to 7$ or $9 \to 2$ or $9 \to 4$ or $9 \to t \to 4$.

To reach 1 from 9, we need $9 \to 5$ or $9 \to 10$ or $9 \to 6$ or $9 \to 7$ or $9 \to s \to 7$.

That is, $9 \to 7$ or $9 \to 10 \to 4$ or $9 \to t, t \to 4, t \to 7$.

Suppose $9 \to t, t \to 4, t \to 7$.

Then $t \neq 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13$.

Hence $t = 12$.

Then $13 \to 2$ (to reach 3 from 13 and since $13 \not\to 12$ - else we cannot reach 12 from 2) and $11 \to 12$ (to reach 3 from 12).

To reach 2 from 4, we must have $10 \to 13$.

To reach 7 from 4, we need $10 \to 5$.

But then we cannot reach 2 from 4.

Hence $9 \to 7$ or $9 \to 10 \to 4$.

Suppose $9 \to 10 \to 4$.

Then we cannot reach 8 from all.

Hence $9 \to 7$.

To reach 2 from 3, we need (12 or 13) $\rightarrow$ 2 (as $7, 8 \not\rightarrow 2$).

Suppose $12 \rightarrow 2$.

To reach 2 from 4, we must have $10 \rightarrow 12$.

Then $11 \rightarrow 9$ (to reach 2 from 11 and since $11 \not\rightarrow 10$ - else we cannot reach 10 from 3) and $13 \rightarrow 10$ (to reach 2 from 13).

To reach 8 from 5, we must have (7 or 11) $\rightarrow$ 6.

To reach 8 from 9, we need (12 or 7) $\rightarrow$ 6.

Hence $7 \rightarrow 6$.

But now we cannot reach 12 from 7.

Hence $12 \not\rightarrow 2$ and so $13 \rightarrow 2$.

Now to reach 2 from 4, we need $10 \rightarrow 13$.

To reach 2 from 11, we must have $11 \rightarrow 6$ (as $11 \not\rightarrow 10$ - else we cannot reach 10 from 3).

To reach 2 from 12, we need $12 \rightarrow 10$.

To reach 6 from 9, we must have $12 \rightarrow 11$ (as $7 \not\rightarrow 11$).

To reach 3 from 12, $11 \rightarrow 4$.

But now we cannot reach all from 12.

Hence if $G \in G(13, 2, 3)$ then it cannot contain S_4.

REFERENCES.

ALEGRE, I. 1983

Personal communication via I. Fris, K. Culik and C. Choffrut.

BERGE, C. 1973

Graphs and Hypergraphs. *North-Holland*

BONDY, J.A. and MURTY, U.S.R. 1976

Graph Theory with Applications. *MacMillan*

BOWEN, E. 1985

Personal communication.

BRIDGES, W. G. and TOUEG, S. 1980

On the impossibility of directed Moore graphs. *J. Combinatorial Theory* Series B29, No.3, 339-341

CAMERON, P.J. 1978

Strongly regular graphs in **Selected Topics in Graph Theory** (ed. Beineke, L.W. and Wilson, R.J.) *Academic Press*, 337-360

CULIK, K. 1984

Personal communication via I.Fris.

DAMERELL, R.M. 1973

On Moore graphs. *Proc. Cambridge Phil. Soc.* 74, 227-236

FIOL, M. A., ALEGRE, I and YEBRA, J. L. A. 1983

Line digraph iteration and the (d,k) problem for directed graphs. *Proc. 10th Symp. Comp. Architecture, Stockholm* 174-177

FRIS, I. 1985

Personal communication.

GOODMAN, J.R. and SEQUIN, C.H. 1981

Hypertree : A multiprocessor interconnection topology. *IEEE Transactions on Computers* Vol.C-30, No.12, 923-933

HARARY, F. 1967

Graph Theory and Theoretical Physics. *Academic Press*

HARARY, F. 1969

Graph Theory. *Addison-Wesley*

HEUCHENNE, C. 1964

Sur une certaine correspondance entre graphs. *Bull. Soc. Roy. Sci. Liege* 33, 743-753

HOFFMAN, A.J. and SINGLETON, R.R. 1960

On Moore graphs with diameter 2 and 3. *IBM Journal* Nov. 1960, 497-504

IMASE, M. and ITOH, M. 1981

Design to minimize diameter on building-block network. *IEEE Transactions on Computers* Vol.C-30, No.6, 439-442

IMASE, M. and ITOH, M. 1983

A design for directed graphs with minimum diameter. *IEEE Trans. on Computers* Vol. C-32, No.8, 782-784

MEMMI, G. and RAILLARD, Y. 1982

Some new results about the (d,k) graph problem. *IEEE Transactions on Computers* Vol.C-31, No.8, 784-791

RICHARDS, P.I. 1967

Precedence constraints and arrow diagrams. *SIAM Rev.* 9, 548-553