INTRODUCTION.

In recent years many advances in very large scale integrated circuit technology have been made. These have resulted in a growing interest in the design of microprocessor networks. In such networks it is obviously desirable to minimize the distance (i.e., the number of times a message has to be relayed to reach its destination) between any two microprocessors, and thus to reduce the delays and also the load on the interconnecting lines.

Ignoring other factors, the smaller the distance between any two microprocessors, the more efficient the network. On the other hand, the cost of the interconnections between the microprocessors increases with the number of lines in the network. Thus it is desirable to have networks with both the minimum distance between any two microprocessors, and the minimum total number of lines. However, if the number of lines is decreased then the distance between the microprocessors tends to increase and vice versa. There seems to be some kind of an inverse relationship between these two criteria of a network, although it is possible that there might be some anomalies.

When designing a network, we may consider other criteria as well. For example, we can require an overall balance of the system: given that all the microprocessors have the same status, the flow of information and exchange of data between microprocessors will be on average faster if there is a similar number of interconnections going in and out of each microprocessor, i.e., if there is a balance (or regularity) in the network.

Next, instead of (or as well as) aiming at the minimum distance between any two microprocessors of the network we may require that the network be of minimum average distance.

Other factors may also be taken into account when designing a network. For example, we may wish to produce a network which is reducible, that is, given a network N, we wish to find partial networks $N_1, N_2, \ldots, N_t = N$ such that $N_1 \subset N_2 \subset \ldots \subset N_t$ and each N_i keeps certain given desired properties (e.g., minimum distance or minimum average distance). Conversely, we could require a network to be extensible.

One important aspect of reducibility of a network is fault tolerance. Specifically, a system that is fault tolerant should continue to work correctly (although possibly with reduced performance) when one or more connections or when one or more microprocessors have failed – as long as that particular component of the network is not involved in the computation.

Given any combination of requirements for a network such as above, there might be more than one essentially different (not isomorphic) network which satisfy the requirements. This gives rise to the problem of finding all nonisomorphic networks satisfying the given requirements. These are just some examples of unsolved problems in the design of microprocessor networks.

We are interested here in those network design problems that can be translated into graph theoretical problems. We can represent each microprocessor by a point. If all the interconnections in the network are two-way, then a connection between point A and a point B can be represented by a line joining A and B. In this way we obtain an undirected graph. Otherwise we represent a connection from point A to point B by an oriented arc (a line with an arrow pointing in the direction of the flow) from A to B and we obtain a directed graph. The kind of properties of a network that we are interested in, such as the distances between microprocessors, or the regularity of the network, will be preserved in such representations.

In this thesis we will consider mainly directed networks which are balanced (that is, regular). To deal with such networks formally we introduce our basic terminology in Chapter 1. In Chapter 2 we discuss the following three problems:

(i) The problem of optimizing the number of microprocessors given the number of connections going in and out of each microprocessor and given that the distance between any two microprocessors should not be more than some given value.

(ii) The problem of optimizing the distance between any two microprocessors given the number of microprocessors in the network and given the number of interconnections going in and out of each microprocessor.

(iii) The problem of optimizing the number of connections in and out of each microprocessor given the number of microprocessors in the network and given that the distance between any two microprocessors is not more than some given value.

In Chapters 3 and 4 we deal with the first two problems in more detail and we consider especially the case when there are 2 in and 2 out connections from each microprocessor. Finally, in the Conclusion of this thesis we present some open problems which follow from this work or are related to it.

All original results in this thesis are indicated by •. These are Lemma 1, Lemma 2, Theorem 2 and Lemma 3 in Chapter 2; Theorem 3*, Theorem 3', Theorem 4, Lemma 4, Lemma 5 and Theorem 5 in Chapter 3; and Theorem 6 with Corollaries 1,2,3,4 and 5 in Chapter 4.

CHAPTER 1. Basic concepts.

A graph, or an undirected graph, is a set of points and a set of lines, with each line joining two points together. The points are often called the vertices (or nodes) of the graph and the lines are also called the edges of the graph.

More formally, we have

Definition 1.

A *graph* $G = (V, E)$ where

(i) V is a nonempty set $V = \{v_1, v_2, \ldots, v_n, \ldots\}$ of distinct elements, called *vertices*;

(ii) E is a bag $E = \{e_1, e_2, \ldots, e_m, \ldots\}$ of unordered pairs $\{v_i, v_j\}, v_i, v_j \in V$, called *edges*.

Thus an edge $\{v, w\}$ is the same as the edge $\{w, v\}$. A bag is a collection of elements, possibly with repetitions, and so an element $\{v, w\}$ of the cartesian product $V \times V$ can appear more than once in E. If an edge appears more than once in E, it is called a *multiple edge*.

An edge of the form $\{v, v\}$ is called a *loop*. Some authors (e.g., Harary, 1969) do not admit the existence of multiple edges and loops in their definition of a graph. They then call a graph with multiple edges a *multigraph* and a graph with multiple edges and loops a *pseudograph*.

In this thesis we will not make such restrictions; we will allow both multiple edges and loops in a graph.

If an edge e occurs m times in a graph G then we say that e is of *multiplicity* m.

The number of vertices in $G = (V, E)$ is called the *order* of G. If the order of G is finite and if also E is a finite bag then G is a *finite graph.* Two vertices u, $v \in G$ are called *adjacent* if there is an edge $\{u, v\}$ in G. Two edges $\{u, v\}$ and $\{w, x\}$ are called *adjacent* if $u = w$ or $u = x$ or $v = w$ or $v = x$.

The *degree* of a vertex $v \in G$ is the number of edges in G of the form $\{u, v\}$, $u \in G$. If the multiplicity of all edges of G is 1, then the degree of a vertex $v \in G$ is the number of vertices adjacent to v.

If in a graph G every vertex has the same degree d then G is said to be a *regular graph*, or a *regular graph of degree d.*

A *chain* is a sequence $\mu = (e_1, e_2, \ldots, e_q)$ of edges of G such that each edge after the first in the sequence has one point in common with its predecessor in μ and its other point in common with its successor in μ. The number of edges in μ is the *length* of the chain. Alternatively, we can define a chain as a sequence $\mu = (v_0, v_1, \ldots, v_q)$ of vertices $v_i \in G$ such that for all i, $0 \leq i \leq q - 1$, the edge $\{v_i, v_{i+1}\}$ exists in G. A chain that does not encounter the same vertex twice is called *elementary.* A chain that does not contain the same edge twice is called *simple.*

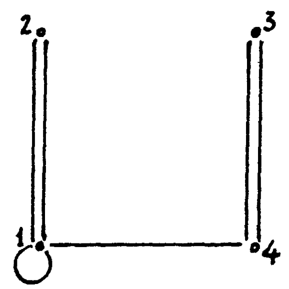
A *cycle* is a simple chain $\mu = (v_0, v_1, \ldots, v_q)$ such that $v_0 = v_q$. Thus a cycle is a "closed" chain. A loop is then a cycle of length 1.

A graph $G = (V, E)$ is said to be *connected* if for each pair of distinct vertices $x, y \in V$, G contains a chain $\mu = (v_0, v_1, \ldots, v_q), v_i \in G$, where $v_0 = x$ and $v_q = y$.

The *distance* $d(x, y)$ of two vertices x and y is the length of the shortest chain with x and y as its end points.

The *diameter* k of a graph G is the longest distance between any pair of vertices of G, i.e., $k = \max_{x,y \in G} d(x, y)$.

We can represent a graph $G = (V, E)$ by a *geometric diagram*, or a *diagram*, in which the vertices are indicated by small circles or dots, while any two vertices u, $v \in V$ are joined by a continuous curve if and only if $\{u, v\} \in E$. For example, the graph $G = (V, E)$ where $V = \{1, 2, 3, 4\}$ and $E = \{\{1,1\}, \{2,1\}, \{2,1\}, \{3,4\}, \{4,3\}, \{1,4\}\}$ can be represented by the following diagram.



A graph $G_1 = (V, E)$ is a ***subgraph*** of a graph $G = (W, F)$, denoted $G_1 \subset G$, if $V \subset W$ and $E \subset F$.

Many applications of graph theory require a direction to be associated with each edge. For example, flow through a program is directed as is the traffic flow through one-way streets. In these cases, an edge from a point u to a point v is not the same as an edge from v to u. We will call a directed edge an arc. If a is an arc from vertex u to vertex v then we call u the *start point* of a, and v the *end point* of a. Alternatively, we may say that the vertex u *goes* to the vertex v. We will also say in such a case that a *joins* vertex u to vertex v. Graphs with directed edges are called directed graphs.

More formally, we have

Definition 2.

A *digraph*, or a *directed graph*, $G = (V, E)$ where

(i) V is a nonempty set $V = \{v_1, v_2, \ldots v_n, \ldots\}$ of distinct elements called *vertices*;

(ii) E is a bag $E = \{e_1, e_2, \ldots, e_m, \ldots\}$ of ordered pairs (v_i, v_j), $v_i, v_j \in V$, called *arcs*.

The number of vertices n in a digraph G is called the *order* of the digraph. If V is finite and if E is a finite bag then G is called a *finite digraph.*

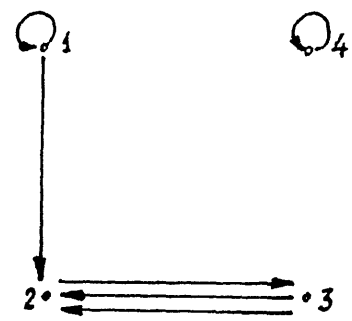
If G is a digraph then two vertices $x, y \in G$ are called *adjacent* if there is an arc (x, y) or an arc (y, x) in G; two arcs (u, v) and (w, z) of G are called *adjacent* if $v = w$ or $u = z$. The *indegree* of a vertex $v \in G$ is the number of arcs of the form (u, v) in G. Similarly, the *outdegree* of a vertex $u \in G$ is the number of arcs of the form (u, v) in G. The *degree* of a vertex v is the sum of its indegree and outdegree.

If in a digraph G every vertex has the same degree d then G is said to be a *regular digraph*, or a *regular digraph of degree d*. However, if G is a regular digraph in which

indegree $=$ outdegree $= d$ for every vertex in G then G is called a *diregular digraph* of degree d .

We can represent a digraph $G = (V, E)$ by a *geometric diagram* differing from a diagram of an undirected graph only in that we draw arrows on the lines representing directed edges of G. For example, if $G = (V, E)$

where $V = \{1, 2, 3, 4\}$ and $E = \{(1, 2), (1, 1), (3, 2), (3, 2), (2, 3), (4, 4)\}$ then the diagram of G is as follows.

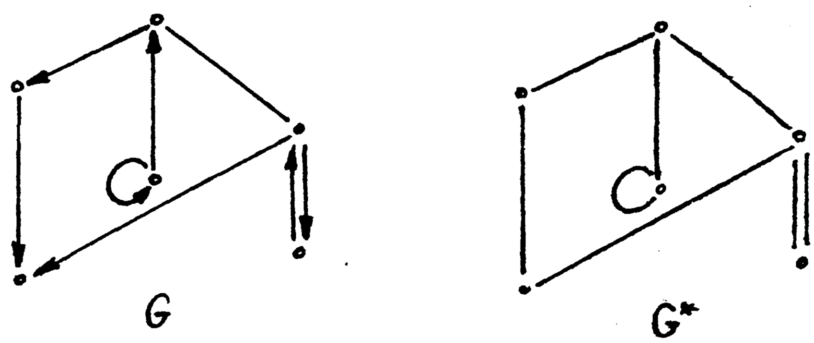


A digraph is called *complete* if for every two distinct vertices $x, y \in G$ there is an arc (x, y) or an arc (y, x) in G (or both). A digraph G is *symmetric* if whenever $(x, y) \in G$ then $(y, x) \in G$.

A digraph $G_1 = (V, E)$ is a *subdigraph* of a digraph $G = (W, F)$, denoted $G_1 \subset G$, if $V \subset W$ and $E \subset F$.

The *underlying graph* $G^* = (V, E^*)$ of a digraph $G = (V, E)$ is a graph which contains all the vertices of G, and an edge $\{u, v\}$ whenever there exists an arc (u, v) in G.

Below is an illustration of a digraph G and its underlying graph G^*.



A *directed path of length* q, or a *path of length* q, is a sequence of arcs of a digraph G, $\mu = (a_1, a_2, \ldots, a_q)$ in which the start point of a_i is the end point of a_{i-1} for all i, $1 < i \leq q$. We will denote a path also by $\mu = (v_0, v_1, \ldots, v_q)$ where (v_{i-1}, v_i) is an arc in G. If $\mu = (v_0, v_1, \ldots, v_q)$ we will sometimes say that vertex v_0 *reaches* vertex v_q in q steps.

A *circuit* is a path $\mu = (v_0, v_1, \ldots, v_q)$ such that $v_0 = v_q$. A circuit of length 1 is called a *loop*, a circuit of length 2 is called a *digon.* For any $p > 0$ we will call a circuit of length p a *p-circuit.*

Note that path and circuit in digraphs are concepts analogous to chain and cycle respectively in undirected graphs. Analogously, we also have the concepts of *elementary path*, *simple path*, *elementary circuit* and *simple circuit* in digraphs.

A digraph $G = (V, E)$ is said to be *connected* if the underlying graph of G is connected. A digraph $G = (V, E)$ is said to be *strongly connected* if for each pair of distinct vertices $x, y \in V$, G contains a path $\mu = (v_0, v_1, \ldots, v_q)$ where $v_0 = x$ and $v_q = y$.

Two digraphs $G = (V, E)$ and $G^* = (W, F)$ are *isomorphic* if there exists a one-to-one correspondence between V and W, $f : V \to W$ such that there is an arc of multiplicity m joining $f(u)$ to $f(v)$ in G^* if and only if there is an arc of multiplicity m joining u to v in G.

With every digraph G we can associate various "adjacency" matrices. We call a matrix A the *vertex-adjacency* matrix of a digraph G of order n, if A is a $n \times n$ matrix such that

$A_{ij} = 1$ if there is an arc from vertex i to vertex j in G;

$A_{ij} = 0$ otherwise.

Note that two nonisomorphic digraphs may have the same vertex-adjacency matrix as in the definition of a vertex-adjacency matrix of a digraph G we ignore the multiplicity of multiple arcs in G.

The elements of A can be regarded as elements of the simple Boolean algebra B_2 with

Boolean addition $\cup$ and Boolean multiplication $\cap$ defined as follows.

$\cup$	0	1
0	0	1
1	1	1

$\cap$	0	1
0	0	0
1	0	1

Thus if the order of G is n then A is a $n \times n$ Boolean matrix. We define multiplication of two $n \times n$ Boolean matrices A and B as follows.

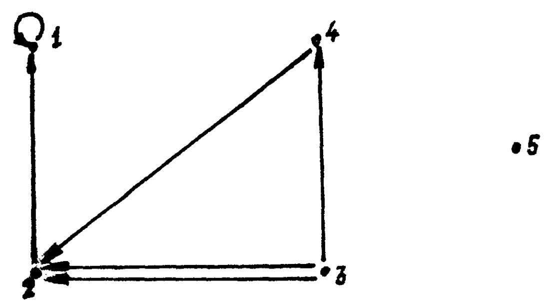
$$(AB)_{ij} = \bigcup_{k=1}^{n} (A_{ik} \cap B_{kj})$$

Thus the product of two $n \times n$ Boolean matrices is again a $n \times n$ Boolean matrix. Further, we can define *Boolean powers* of a $n \times n$ Boolean matrix A iteratively by

$$A^2 = A \times A,\ A^3 = A \times A^2\ ,\ldots, A^k = A \times A^{k-1}, \ldots$$

For example, if the digraph $G = (V, E)$

where $V = \{1,2,3,4,5\}$ and $E = \{(1,1),(2,1),(3,2),(3,4),(4,2)\}$ then the diagram of G is as follows



and the vertex-adjacency matrix of G is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and further,

$$A^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and in this case $A^t = A^3$ for $t \geq 3$.

Note that if A is a vertex-adjacency matrix of a digraph G of order n, then $A^t_{ij} = 1$ if and only if there exists a path of length t from vertex i to vertex j in G.

If G is a digraph of order n and with m arcs, then we could similarly define the $m \times m$ *arc-adjacency* matrix of G and the $n \times m$ *vertex-arc-adjacency* matrix of G.

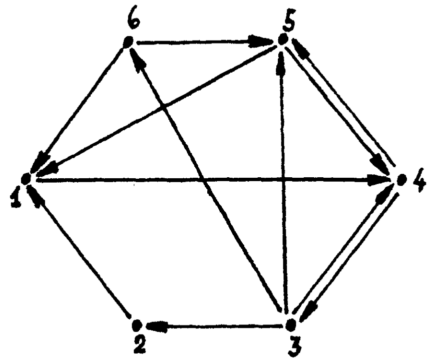
However, since in this thesis we will need to use only the vertex-adjacency matrix of a digraph G, we will call it simply the *adjacency* matrix of G.

Next, we define the *directed distance* $d(x, y)$ from a vertex x to a vertex y $(x, y \in G)$ as the length of the shortest path from x to y. We define $d(x, x) = 0$, and if x and y are distinct vertices of G and there does not exist a path from x to y we define $d(x, y) = \infty$.

The *associated number* $a(v)$ of a vertex v is defined to be $a(v) = \max_{w \in G} d(v, w)$. That is, from the vertex v we can reach any other vertex in $a(v)$ or less steps.

We define the *diameter* of G to be the maximum of the associated numbers of all the vertices of G, that is, $k(G) = \max_{v \in G} a(v)$.

The concept of a diameter is important in the design of interprocessor communication of multicomputers. The diameter represents the maximum number of times that a message must be relayed before it reaches its destination. The diameter of a digraph G is finite if and only if G is strongly connected. For example, if $G = (V, E)$ where $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{(1,4), (2,1), (3,2), (3,4), (3,6), (3,5), (4,3), (4,5), (5,1), (5,4), (6,1), (6,5)\}$



G

then the diameter of G is 4.

The sort of problems we shall deal with in this thesis follows from the so called *Moore graph problem* (Cameron, 1978). This problem was originally posed for finite undirected graphs which are connected and regular. If G is such a graph with diameter k and degree d then G can contain at most

$$N_m = 1 + d\sum_{i=1}^{k}(d-1)^{i-1}$$

vertices. N_m is called the *Moore bound for undirected graphs* and the graphs which attain this bound are called *Moore graphs*. The problem of finding all Moore graphs has been practically solved (Cameron, 1978). If G is a Moore graph with diameter k and degree d then

(i) If $k = 1$ then G exists for any d; it is the complete graph on $d + 1$ vertices.

(ii) If $k = 2$ then $d = 2, 3, 7$ or (possibly) 57; in each of the cases $d = 2, 3, 7$ G is unique up to isomorphism.

(iii) If $k > 2$ then $d = 2$ and G is a cycle of length $2k + 1$.

Thus apart from the fact that it is not known whether or not a Moore graph with diameter 2 and degree 57 exists, the Moore graph problem has been solved.

Next, the so called (d, k) *problem for undirected graphs* follows naturally from the Moore graph problem. The (d, k) problem is : given d and k, find the maximum possible number of vertices in a regular graph of degree d and diameter k. That is, if we cannot attain the Moore bound N_m for a given d and k, we wish to find a graph whose order is as close to N_m as possible. This problem has been studied elsewhere (e.g., Memmi and Raillard, 1982). Alternatively, it is possible to try to minimize the diameter given the order and the degree of a graph G; or to try to minimize the degree given the order and the diameter of G. At present all these problems remain open extremal problems in graph theory.

In this thesis we will study analogous problems for directed graphs.

CHAPTER 2. Discussion of the three problems.

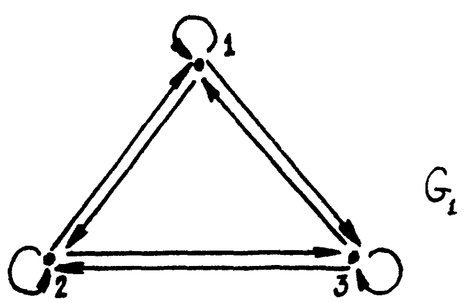
We will consider a diregular digraph G of order n, degree d, and diameter $k < \infty$. G has n vertices; there are d arcs going in and out of each vertex, and any vertex can be reached in at most k steps. This implies that G is a finite digraph which is strongly connected. We will study the relationships between n, d and k. In particular we wish to find the bounds for n given d and k; the bounds for k given n and d; and the bounds for d given n and k.

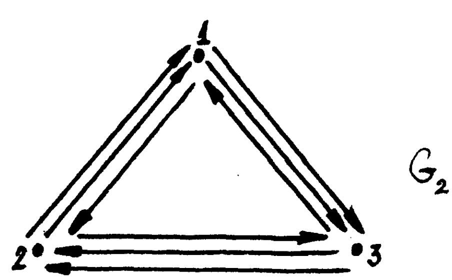
In the following $G(n,d,k)$ will denote the set of all diregular digraphs of order n , degree d, and diameter k. We shall investigate for which triples (n,d,k) this set is nonempty. Note that if $G(n,d,k)$ is nonempty then it does not necessarily identify a unique (up to isomorphism) digraph; indeed it is possible to have two digraphs $G_1 \in G(n,d,k)$ and $G_2 \in G(n,d,k)$ such that G_1 and G_2 are not isomorphic.

For example, if $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, where $V = \{1,2,3\}$ and

$E_1 = \{(1,2),(1,3),(1,1),(2,1),(2,3),(2,2),(3,1),(3,2),(3,3)\}$ and

$E_2 = \{(1,2),(1,2),(1,3),(2,1),(2,3),(2,3),(3,1),(3,1),(3,2)\}$





then $G_1 \in G(3,3,1)$, $G_2 \in G(3,3,1)$ but G_1 and G_2 are not isomorphic.

If at least one of the numbers n, d, k is 1 then it is easy to determine whether $G(n,d,k)$ is empty or not . In particular,

(i) If $n = 1$ then $G(1, d, 0)$ is nonempty and for $k > 0$ $G(1, d, k)$ is empty.

(ii) If $d = 1$ then $G(n, 1, k)$ is nonempty if and only if $k = n - 1$. $G \in G(n, 1, n - 1)$ is then an elementary circuit on n vertices.

(iii) If $k = 1$ then $G(n, d, 1)$ is nonempty if and only if $d \geq n - 1$. If $d = n - 1$ then $G \in G(n, n - 1, 1)$ is a complete symmetric digraph on n vertices. If $d > n - 1$ then $G \in G(n, d, 1)$ contains a subdigraph which is a complete symmetric digraph on n vertices.

Let G be a digraph of diameter k. This implies that there is a directed path in G of length k on which all vertices are distinct, that is, there are at least $k + 1$ distinct vertices in G. Thus we have $n \geq k + 1$ for any $d \geq 1$.

Moreover, a diregular digraph $G \in G(k + 1, d, k)$ exists for any $d \geq 1$.

If $d = 1$ then $G \in G(k + 1, 1, k)$ is an elementary circuit on $k + 1$ vertices.

If $d > 1$ then we can construct $G_1 \in G(k + 1, d, k)$ from $G_2 \in G(k + 1, 1, k)$ simply by inserting $d - 1$ loops at each vertex of G_2.

Equivalently, given n and d, the maximum possible diameter of $G \in G(n, d, k)$ is $k = n - 1$.

Next we will consider an upper bound for the degree d of a digraph G, given n and k. Since we allow loops and multiple edges in G, it is obvious that if $G_1 \in G(n, d, k)$ exists then $G_2 \in G(n, d', k)$ also exists for every $d' \geq d$. We can construct G_2 from G_1 for example by inserting $d' - d$ loops at each vertex of G_1. Thus no upper bound for d exists, i.e., the best we can say is $d < \infty$.

To sum up, we have the following bounds.

(i) Given d and k, $n \geq k + 1$

(ii) Given n and d, $k \leq n - 1$

(iii) Given n and k, if $n \geq k + 1$ then $d < \infty$;

if $n < k + 1$ then $G(n, d, k)$ is empty for any d, that is, a diregular digraph G of diameter k and order less than $k + 1$ does not exist for any degree d.

Since $G(k + 1, d, k)$ and $G(n, d, n - 1)$ are nonempty and $G(n, d, k)$ is nonempty for all sufficiently large d (when $n \geq k + 1$), the above bounds cannot be improved upon.

Next we wish to find an upper bound for n given d and k; and lower bounds for k (given n and d) and d (given n and k). We will use the adjacency matrix A of a digraph $G \in G(n, d, k)$ to obtain such bounds.

Labelling the n vertices of $G \in G(n, d, k)$ by 1,2,..., n we have

$A_{ij} = 1$ if there is an arc from vertex i to vertex j in G;

$A_{ij} = 0$ otherwise.

Further, we can form Boolean powers of the matrix A and we get

$(A^2)_{ij} = 1$ if there is a directed path of length 2 from vertex i to vertex j in G;

$(A^2)_{ij} = 0$ otherwise.

In general,

$(A^t)_{ij} = 1$ if there is a directed path of length t from vertex i to vertex j in G;

$(A^t)_{ij} = 0$ otherwise.

Now, if the diameter of G is k we must be able to reach from every vertex of G all the other vertices of G in at most k steps. Thus for a $G \in G(n, d, k)$ to exist we must have

$$I \cup A \cup A^2 \cup \ldots \cup A^k \supseteq E$$

where I is the $n \times n$ identity matrix and E is the $n \times n$ unit matrix (that is, the $n \times n$ matrix all of whose entries are 1).

Now, for every $i \in \{1, \ldots, n\}$, the ith column of I contains one 1; the i^{th} column of A contains at most d 1's; the i^{th} column of A^2 contains at most d^2 1's; and in general, the i^{th} column of A^t contains at most d^t 1's. On the other hand, the i^{th} column of E contains exactly n 1's.

Thus if $G(n, d, k)$ is nonempty then

$$n \leq 1 + d + d^2 + \ldots + d^k$$

or, equivalently,

$$n \leq k + 1 \qquad \text{if } d = 1$$

$$n \leq (d^{k+1} - 1)/(d - 1) \quad \text{if } d > 1.$$

This relationship between n, d and k gives an upper bound for n given d and k; and lower bounds for d (given n and k) and k (given n and d).

Denoting $N(d, k)$ to be $N(d, k) = \max\{n : G(n, d, k) \neq \emptyset\}$ we have

$$k + 1 \leq n \leq N(d, k) \leq 1 + d + d^2 + \ldots + d^k.$$

If $d = 1$ then $N(1, k) \leq k + 1$, that is, $N(1, k) = k + 1$ and $G(n, 1, k)$ is nonempty if and only if $n = k + 1$.

If $k = 1$ then $2 \leq n \leq N(d, 1) \leq 1 + d$.

Now, $G(d+1, d, 1)$ is nonempty for every $d \geq 1$; $G \in G(d+1, d, 1)$ is a complete symmetric digraph on $d + 1$ vertices.

Thus $N(d, 1) = d + 1$ and given $d \geq 1$ and $k = 1$, $G(n, d, 1)$ is nonempty if and only if $2 \leq n \leq d + 1$.

Similarly, let us write $K(n,d) = \min\{k : G(n,d,k) \neq \emptyset\}$.

Then if $d = 1$ we have $n-1 \leq K(n,1) \leq k \leq n-1$, that is $K(n,1) = n-1$ and $G(n,1,k)$ is nonempty if and only if $k = n-1$.

If $n = 1$ then $K(1,d) = 0$ and $G(1,d,k)$ is nonempty if and only if $k = 0$.

If $n > 1$ and $d > 1$ then

$$\log_d(n(d-1)+1) - 1 \leq K(n,d) \leq k \leq n-1.$$

Lastly, writing $D(n,k) = \min\{d : G(n,d,k) \neq \emptyset\}$ we have (given n and k, $k \leq n-1$)

$$d^* \leq D(n,k) \leq d < \infty$$

where d^* is the minimum d for which $n \leq (d^{k+1}-1)/(d-1)$ holds.

If $n = 1$ then $k = 0$ and $D(1,0) = 0$ and $G(1,d,0)$ is nonempty for any $d < \infty$.

If $k = 1$ then $d^* = n-1$. Since $G(n,n-1,1)$ is nonempty for all n ($G \in G(n,n-1,1)$ is a complete symmetric digraph on n vertices) we have $D(n,1) = n-1$ and $G(n,d,1)$ is nonempty if and only if $d \geq n-1$.

Note that $N(d,k)$ is defined for any $d > 0$ and $k \geq 0$ as well as in the case $d = 0$ and $k = 0$.

$K(n,d)$ is defined for any $d > 0$ and $n > 0$ as well as in the case $d = 0$ and $n = 1$.

On the other hand, $D(n,k)$ is not defined whenever for given n and k, $G(n,d,k)$ is empty for any d. That is, for $D(n,k)$ to be defined we must have $0 < k \leq n-1$, or $n = 1$ and $k = 0$.

Let us call the problems of finding $N(d,k)$, $K(n,d)$ and $D(n,k)$ the *$N(d,k)$ problem*, the *$K(n,d)$ problem* and the *$D(n,k)$ problem* respectively.

The following theorem, due to Fiol, Alegre and Yebra (1983) will be used in further discussion.

Theorem 1.

If $d > 1$ then $G((d+1)d^{k-1}, d, k)$ is nonempty.

Proof.

If $k = 1$ then $G(d+1, d, 1)$ is nonempty.

Assume $k \geq 2$.

Let G be a digraph with $n = (d+1)d^{k-1}$ vertices ($d > 1$), labelled 0,1,2,..., $n-1$.

For each vertex i let there be arcs from i to the vertices

$$(-di + t) \bmod n, \ t = 1, 2, \ldots d.$$

We will show that G is a diregular digraph of degree d and diameter k.

Firstly, for every $i \in \{0, 1, \ldots, n-1\}$ there is an arc from i to vertices

$$-di + 1, -di + 2, \ldots, -di + d \pmod{n}.$$

As $n > d$, every vertex i goes to d distinct vertices of G so that G is a diregular digraph of degree d.

Secondly, to show that the diameter of G is k, we will show that there is a directed path of length $k-1$ or k from each vertex of G to all other vertices of G.

From vertex i we arrive successively at the following vertices (mod n)

$$\begin{array}{ll} -di+1, -di+2, \ldots, -di+d & \text{in 1 step} \\ d^2i - 0, d^2i - 1, \ldots, d^2i - (d^2 - 1) & \text{in 2 steps} \\ -d^3i + 1, -d^3i + 2, \ldots, -d^3i + d^3 & \text{in 3 steps} \\ d^4i - 0, d^4i - 1, \ldots, d^4i - (d^4 - 1) & \text{in 4 steps} \end{array}$$

and so on.

If k is even then we reach in $k-1$ or k steps the vertices (mod n)

$$-d^{k-1}i+1, -d^{k-1}i+2, \ldots, -d^{k-1}i+d^{k-1} \quad \text{in } k-1 \text{ steps}$$

$$d^k i-0, d^k i-1, \ldots, d^k i-(d^k-1) \quad \text{in } k \text{ steps}.$$

All the numbers in each of the above rows are different. For numbers in different rows we have

$$(-d^{k-1}i+t)-(d^k i-t') = -d^{k-1}(d+1)i+t+t' \pmod n$$

and $0 < t+t' < d^{k-1}(d+1) = n$.

Therefore they are all different and since we have $d^{k-1}+d^k = n$ vertices, we have reached all vertices in at most k steps.

If k is odd then we reach in $k-1$ or k steps the vertices (mod n)

$$d^{k-1}i-0, d^{k-1}i-1, \ldots, d^{k-1}i-(d^{k-1}-1) \quad \text{in } k-1 \text{ steps}$$

$$-d^k i+1, -d^k i+2, \ldots, -d^k i+d^k \quad \text{in } k \text{ steps}.$$

All these vertices are different and there are $d^{k-1}+d^k = n$ vertices in G, so again we have reached all vertices in at most k steps.

Thus in either case, the diameter of G is at most k.

Since $n = (d+1)d^{k-1} > 1+d+d^2+\ldots+d^k-1$ for $d > 1$ it follows that the diameter of G is more than $k-1$, that is, the diameter of G is k $\diamondsuit$

Corollary.

If $k > 0$ then $N(d,k) \geq (d+1)d^{k-1}$.

Note that the digraphs used in the proof of Theorem 1 always contain $\binom{d+1}{2}$ digons. These are the solutions of the transition scheme

$$i \to -di + t_1 \to d^2 i - dt_1 + t_2 \equiv i \pmod{n}.$$

For example, if $t_1 = 1$, $t_2 = d$ we obtain the solutions

$$i = md^k, \quad m = 0, 1, 2, \ldots, d.$$

Note also that the construction of $G \in G((d+1)d^{k-1}, d, k)$ used in the proof of Theorem 1 is not the only one possible. Instead of joining vertex i to vertices

$$-di + t \pmod{n}, t = 1, 2, \ldots, d$$

we could have equally well used

$$i \to -di - t \pmod{n}, t = 1, 2, \ldots, d$$

and thus obtained a digraph $G^* \in G((d+1)d^{k-1}, d, k)$. However, it is easy to show that the digraphs G and G^* are isomorphic.

Next we will show in Theorem 2 that $K(N(d,k), d) = k$ and $D(N(d,k), k) = d$. This means that if we have the maximum possible number of vertices N in a digraph of degree d and diameter k then d is the minimum possible degree of a digraph with N vertices and diameter k; and k is the minimum possible diameter of a digraph with N vertices and degree d.

To prove Theorem 2 we will need the following two Lemmas.

- **Lemma 1.**

  If $d_1 < d_2$ then $N(d_1, k) < N(d_2, k)$ for all $k > 0$.

Proof.

Suppose $d_1 < d_2$.

If $k = 1$ then $N(d_1, k) = d_1 + 1 < d_2 + 1 = N(d_2, k)$.

If $k > 1$ then using Theorem 1,

$$N(d_1,k) \leq 1 + d_1 + d_1^2 + \ldots + d_1^k = (d_1^{k+1} - 1)/(d_1 - 1)$$

$$< (d_2 + 1)d_2^{k-1} \leq N(d_2,k).$$

Hence for any $k > 0$, $N(d_1,k) < N(d_2,k)$ whenever $d_1 < d_2$ $\diamondsuit$

- **Lemma 2.**

If $k_1 < k_2$ then $N(d,k_1) < N(d,k_2)$ for all $d > 0$.

Proof.

Suppose $k_1 < k_2$.

If $d = 1$ then $N(1,k_1) = k_1 + 1 < k_2 + 1 = N(1,k_2)$.

If $d > 1$ then using Theorem 1,

$$N(d,k_1) \leq (d^{k_1+1} - 1)/(d - 1) < (d + 1)d^{k_2-1} \leq N(d,k_2).$$

Hence for any $d > 0$, $N(d,k_1) < N(d,k_2)$ whenever $k_1 < k_2$ $\diamondsuit$

- **Theorem 2.**

If $k > 0$ then $K(N(d,k),d) = k$ and $D(N(d,k),k) = d$.

Proof.

Let $N(d,k) = n$.

Then $K(n,d) \leq k$ since there exists a digraph $G \in G(n,d,k)$.

Now, if $K(n,d) = k_1 < k$ then there exists a digraph $G \in G(n,d,k_1)$.

Hence $N(d,k_1) \geq n$.

Then by Lemma 2 we have $k_1 < k$ implies $N(d,k_1) < N(d,k)$

i.e., $n \leq N(d,k_1) < N(d,k) = n$

which is a contradiction.

Thus $K(n,d) = k$.

Similarly, because $G \in G(n, d, k)$ exists we have $D(n, k) \leq d$.

If $D(n, k) = d_1 < d$ then $G \in G(n, d_1, k)$ exists. Then also $N(d_1, k) \geq n$ and by Lemma 1, $N(d_1, k) < N(d, k)$.

Thus we have $n \leq N(d_1, k) < N(d, k) = n$ which is a contradiction.

Hence $D(n, k) = d$ $\diamondsuit$

The next lemma shows that $K(n, d)$ is monotonic in d.

• **Lemma 3.**

If $d_1 < d_2$ then $K(n, d_1) \geq K(n, d_2)$.

Proof.

Suppose $d_1 < d_2$.

Let $K(n, d_1) = k_1$, $K(n, d_2) = k_2$.

Then there exist a digraph $G_1 \in G(n, d_1, k_1)$ and a digraph $G_2 \in G(n, d_2, k_2)$.

Suppose $k_1 < k_2$.

Then we can construct $G_3 \in G(n, d_2, k_1)$ from G_1 simply by inserting $d_2 - d_1$ loops at each vertex of G_1.

Thus $K(n, d_2) \leq k_1 < k_2 = K(n, d_2)$ which is a contradiction.

Hence if $d_1 < d_2$ then $K(n, d_1) \geq K(n, d_2)$ $\diamondsuit$

To summarize, Lemmas 1,2 and 3 give the following relationships.

(a) $d_1 < d_2$ implies $N(d_1, k) < N(d_2, k)$

(b) $k_1 < k_2$ implies $N(d, k_1) < N(d, k_2)$

(c) $d_1 < d_2$ implies $K(n, d_1) \geq K(n, d_2)$

On the other hand, we do not know whether or not any of the following implications hold.

(i) $k_1 < k_2$ implies $D(n, k_1) \geq D(n, k_2)$ $(k_1, k_2 \geq n - 1)$

(ii) $n_1 < n_2$ implies $K(n_1, d) \leq K(n_2, d)$

(iii) $n_1 < n_2$ implies $D(n_1, k) \leq D(n_2, k)$ $(n_1, n_2 \leq k + 1)$

Thus it can be seen that the three problems are related but (as far as we know at present) not equivalent.

Of course, if (i) were true then the solution of the $K(n,d)$ problem would give the solution of the $D(n,k)$ problem.

If (ii) were true then the solution of the $N(d,k)$ problem would give the solution of the $K(n,d)$ problem.

Finally, if (iii) were true then the solution of the $N(d,k)$ problem would give the solution of the $D(n,k)$ problem.

As the situation is at present, we can deduce the following from Lemmas 1, 2 and 3.

(A) The solution of the $K(n,d)$ problem would give the solution of the $N(d,k)$ problem.

(B) The solution of the $D(n,k)$ problem would give the solutions of both the $N(d,k)$ and the $K(n,d)$ problems.

Thus it would be best to concentrate on the $D(n,k)$ problem. Unfortunately, the $D(n,k)$ problem is the hardest one of the three problems to handle.

CHAPTER 3. The N(d,k) problem.

The problem of finding the maximum possible number of vertices in a diregular digraph of degree d and diameter k was the first of the three problems to be studied (Bridges and Toueg, 1980). It is often called the (d,k) *problem for directed graphs.*

Bridges and Toueg introduced the notion of (d,k) *Directed Moore Graphs*, (d,k) DMG for short. A (d,k) DMG is a diregular digraph $G \in G(N_m, d, k)$ which achieves the Moore bound $N_m = 1 + d + d^2 + ... + d^k$. The problem of finding all the (d,k) DMGs is called the *Moore graph problem for directed graphs.* The Moore graph problem for directed graphs has been completely solved.

For $k = 1$, $(d,1)$ DMGs are complete digraphs on $d+1$ vertices.

For $d = 1$, $(1,k)$ DMGs are circuits of $k+1$ vertices.

For $d > 1$ and $k > 1$ Bridges and Toueg showed that there are no DMGs.

Theorem 3.

If $d > 1$ and $k > 1$ then $N(d,k) \neq 1 + d + d^2 + \ldots + d^k$.

Proof.

Suppose $n = N(d,k) = 1 + d + d^2 + \ldots + d^k$.

Then $I \cup A \cup A^2 \cup \ldots \cup A^k = E$

and also $I + A + A^2 + \ldots + A^k = E$

where $+$ denotes the ordinary matrix addition.

Obviously, the eigenvalues of E are n (simple) and 0 ($n-1$ times).

Thus $(I + A + \ldots + A^k)Y = nY$

and $(I + A + \ldots + A^k)X = 0X$

for some eigenvectors X and Y.

Thus the eigenvalues of A are d and some of the roots of

$$1+\lambda+\lambda^2+\ldots+\lambda^k=0$$

i.e., $\lambda^{k+1}-1=0$ $(\lambda\neq 1)$

i.e., the $(k+1)^{st}$ roots of unity.

Now Trace $A^j=0,\ j=1,2,\ldots,k$ and so for $j=1,2,\ldots,k$ we have

$$d^j+\sum_{i=1}^{n-1}\lambda_i^j=0$$

where $\lambda_1,\lambda_2,\ldots,\lambda_{n-1}$ are the eigenvalues of A. That is,

$$\sum_{i=1}^{n-1}\lambda_i^j=-d^j.$$

Now

$$-d=\sum_{i=1}^{n-1}\lambda_i=\sum_{i=1}^{n-1}\bar{\lambda}_i=\sum_{i=1}^{n-1}\lambda_i^k=-d^k$$

since $\bar{\lambda}_i=\lambda_i^k$.

Thus $-d=-d^k$. This is true only if $d=1$ or $k=1$.◇

Combining Theorem 1 and Theorem 3 we have

Corollary.

If $d>1$ then $N(d,2)=d+d^2$ and for $k>2$ $N(d,k)\leq d+d^2+\ldots+d^k$.

Interestingly, if $n=1+d+d^2+\ldots+d^k,\ k>1,\ d>1$ and if k does not divide dn then we can prove that $G(n,d,k)$ is empty in a very simple way.

• **Theorem 3*.**

If $d>1$ and $k>1$ and k does not divide $d(1+d+d^2+\ldots+d^k)$ then $G(1+d+d^2+\ldots+d^k,d,k)$ is empty.

Proof.

Suppose $G \in G(1 + d + d^2 + \ldots + d^k, d, k)$, $d > 1$, $k > 1$.

Then every vertex of G reaches itself in $k + 1$ steps in d distinct ways.

Thus a vertex v of G lies on d circuits each consisting of k distinct vertices.

These d circuits have only the vertex v in common.

Thus G is composed of some x circuits, each consisting of k vertices.

Now, each vertex of G occurs exactly d times in these x circuits and so we have

$$kx = d(1 + d + d^2 + \ldots + d^k).$$

If k does not divide $d(1 + d + d^2 + \ldots d^k)$ then such a digraph G cannot exist $\diamondsuit$

Let us now summarize our present state of knowledge of the $N(d, k)$ problem.

We have

$$N(1, k) = k + 1$$

$$N(d, 1) = d + 1$$

$$N(d, 2) = d + d^2$$

$$(d + 1)d^{k-1} \leq N(d, k) \leq d + d^2 + \ldots + d^k \ , \quad d > 1, \ \ k > 2.$$

In the remainder of this chapter we will turn our attention to the case when $d = 2$.

For $d = 2$ we can prove $G(2^{k+1} - 1, 2, k)$ is empty (Theorem 3 for $d = 2$) in a more direct way as follows.

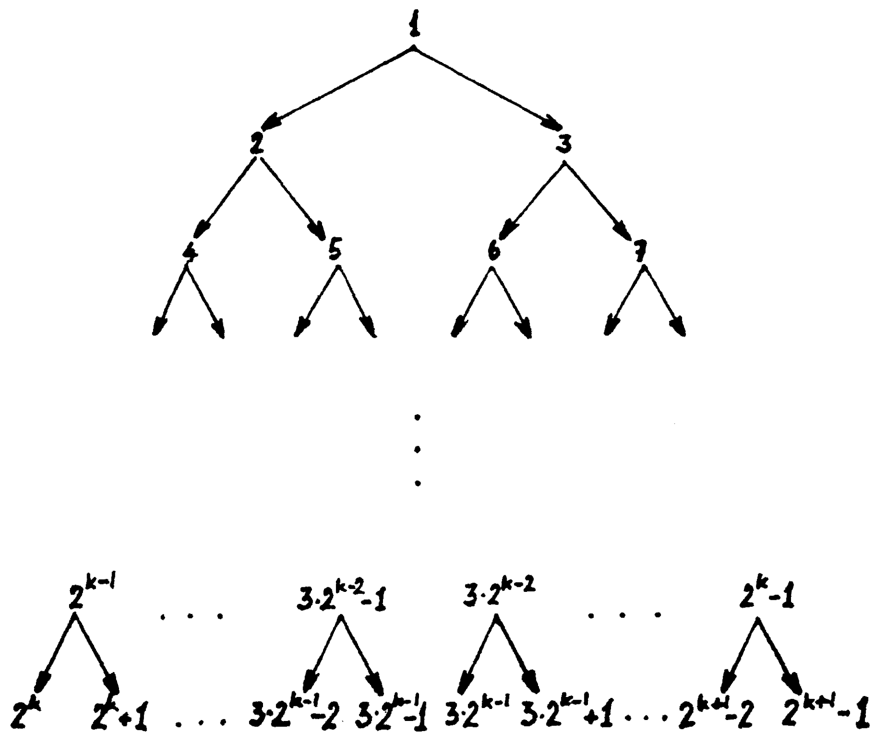
- **Theorem 3$'$.**

If $k > 1$ then $G(2^{k+1} - 1, 2, k)$ is empty.

Proof.

Suppose $k > 1$ and $G \in G(2^{k+1} - 1, 2, k)$.

Denoting the vertices of G by $1, 2, \ldots 2^{k-1}$-1 we can partly draw the digraph G as follows.



Now, to reach every other vertex from vertex 2 in at most k steps there must be arcs from the vertices $2^k, 2^k + 1, \ldots, 3 \times 2^{k-1} - 1$ to the vertices $1, 3, 6, 7, 12, 13, 14, 15, \ldots, 3 \times 2^{k-1}, \ldots, 2^{k+1} - 1$.

Similarly, to reach every other vertex from vertex 3 in at most k steps there must be arcs from the vertices $3 \times 2^{k-1}, 3 \times 2^{k-1} + 1, \ldots, 2^{k+1} - 1$ to the vertices $1, 2, 4, 5, 8, 9, 10, 11, \ldots, 2^k, 2^k + 1, \ldots, 3 \times 2^{k-1} - 1$.

Two arcs go to every vertex and two arcs come from every vertex.

Let $x \rightarrow 2$ $(x \neq 1)$ and let y be the end point of the second arc from x. Then

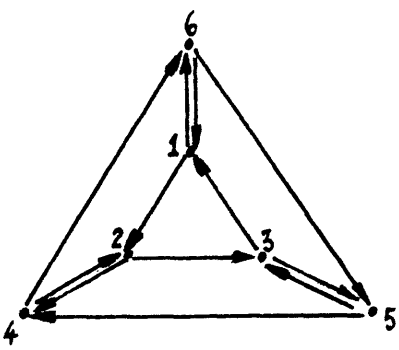
$y \in \{1, 2, 4, 5, 8, 9, 10, 11, \ldots, 2^k, 2^k + 1, \ldots, 3 \times 2^{k-1} - 1\}$. But then we reach the vertex y from x in two different ways in k steps or less. Then there must be a vertex $z \in \{1, 3, 6, 7, 12, 13, 14, 15, \ldots, 3 \times 2^{k-1}, \ldots, 2^{k+1} - 1\}$ which cannot be reached from x in k steps or less and so the diameter of G is not k.

Hence $G(2^{k+1} - 1, 2, k)$ is empty $\diamondsuit$

Corollary.

If $k > 1$ then $N(2, k) \leq 2^{k+1} - 2$.

Now, $G(6, 2, 2)$ is nonempty. For example, the following digraph belongs to $G(6, 2, 2)$.



Since by Corollary to Theorem 3, $N(2, 2) \leq 2^3 - 2 = 6$, it follows that $N(2, 2) = 6$. However, if $k > 2$ then we can prove that $G(2^{k+1} - 2, 2, k)$ is empty.

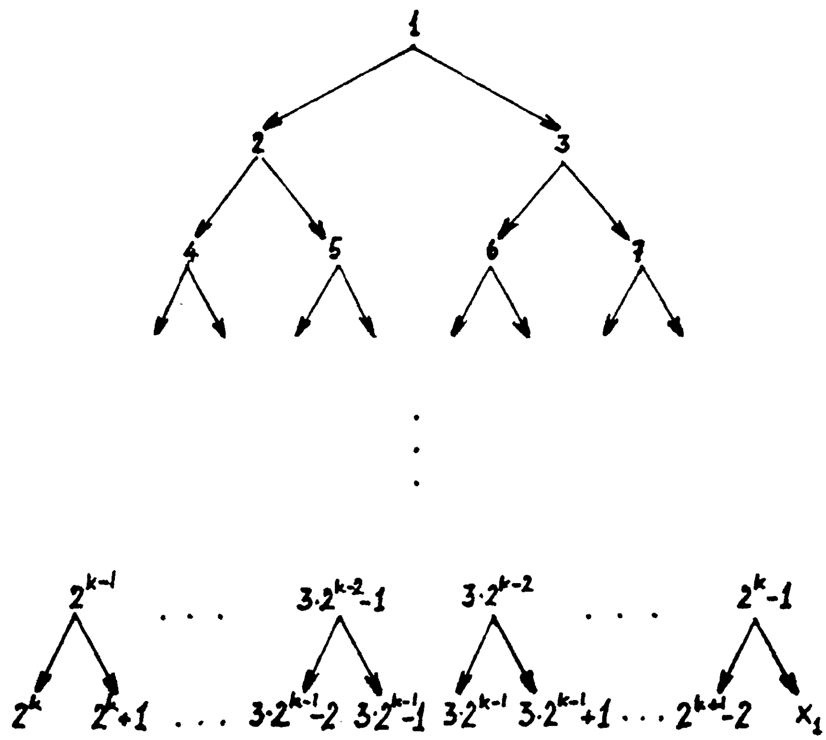
• **Theorem 4.**

If $k > 2$ then $G(2^{k+1} - 2, 2, k)$ is empty.

Proof.

Suppose $k > 2$ and $G \in G(2^{k+1} - 2, 2, k)$.

Denoting the vertices of G by $1, 2, \ldots, n$ we can partly draw G as follows.



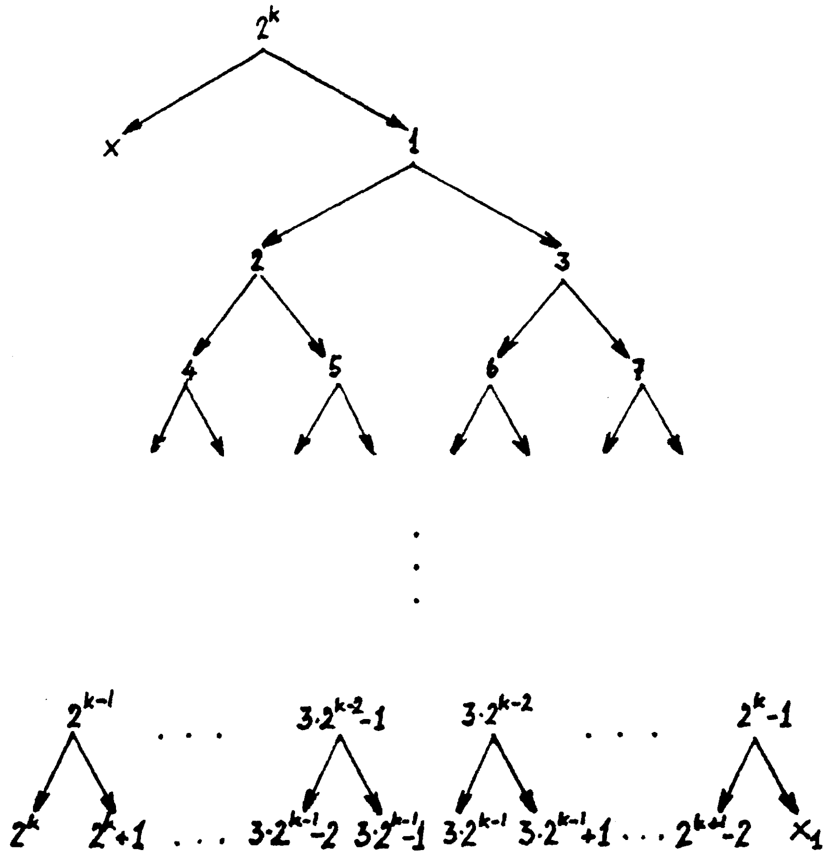
where x_1 is a duplicate of one of the nodes $1, 2, \ldots 2^{k+1} - 2$. We shall call such a node the *repeat from* 1, or just the *repeat* when the node from which it is a repeat is understood.

It is obvious that a repeat from the vertex v always occurs k steps from vertex v.

One of the elements $2^k, 2^k + 1, \ldots, 3.2^{k-1} - 1$ must go to 1 so that we can reach 1 from 2, say $2^k \to 1$.

Denote $L = \{2^k + 1, \ldots, 3.2^{k-1} - 1\}$ and $R = \{3.2^{k-1}, \ldots, 2^{k+1} - 2\}$.

Thus we have



Then we can reach from 2^k through 1 the vertices $1, 2, 3, \ldots, 2^k - 1$ in k or less steps.

The number of elements in L is $2^{k-1} - 1$ and the number of elements in R is also $2^{k-1} - 1$.

Now, no element of R can go to an element of R. For suppose $r_1, r_2 \in R$ and $r_1 \to r_2$. Then we could not reach r_2 from 2 in k or less steps.

Further, at most one element of $L \cup 2^k$, say l_0 can go to $l \in L$.

This can only happen if

(i) $x_1 \in \{2^{k-1}, \dots, 3.2^{k-2} - 1\}$ so that $x_1 \to l_1$ and $x_1 \to l_2$.

$l_1, l_2 \in L \cup 2^k$. Then for some element $l_0 \in L \cup 2^k$ we have $l_0 \to l_1$ or $l_0 \to l_2$;

or

(ii) $x_1 = l_0$.

We will compare the number of elements of L, say N_l and the number of elements of R, say N_r that can be reached from vertex 2^k in k or less steps.

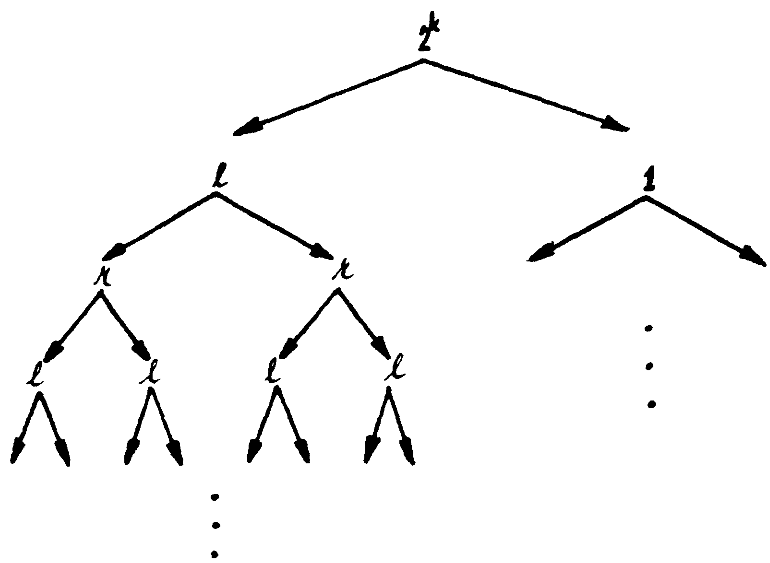
If $G \in G(2^{k-1} - 2, 2, k)$ exists then

$$N_l + N_r = 2^k - 2 \quad \text{and} \quad N_r = N_l.$$

However,we will show that if $k > 2$ then $N_l \neq N_r$ for all the possible cases; this contradiction shows that $G(2^{k-1} - 2, 2, k)$ is empty.

Case 1. If $x \in L$ then $2^k = l_0$ and there cannot be any more elements of L that go to an element of L.

In this case we can only have



where either all $r \in R$ and all $l \in L \cup x_{2^k}$; or all $r \in R \cup x_{2^k}$ and all $l \in L$, where x_{2^k} is the repeat from 2^k.

(a) all $r \in R$ and all $l \in L \cup x_{2^k}$. Then k is obviously odd.

Then

$$N_l = 1 + 4 + \dots 4^{(k-1)/2} - 1 = \sum_{i=0}^{(k-1)/2} 4^i - 1$$

$$N_r = 2^k - 2 - N_l = 2^k - \sum_{i=0}^{(k-1)/2} 4^i - 1$$

(b) all $l \in L$ and all $r \in R \cup x_{2^k}$. Then k is obviously even.

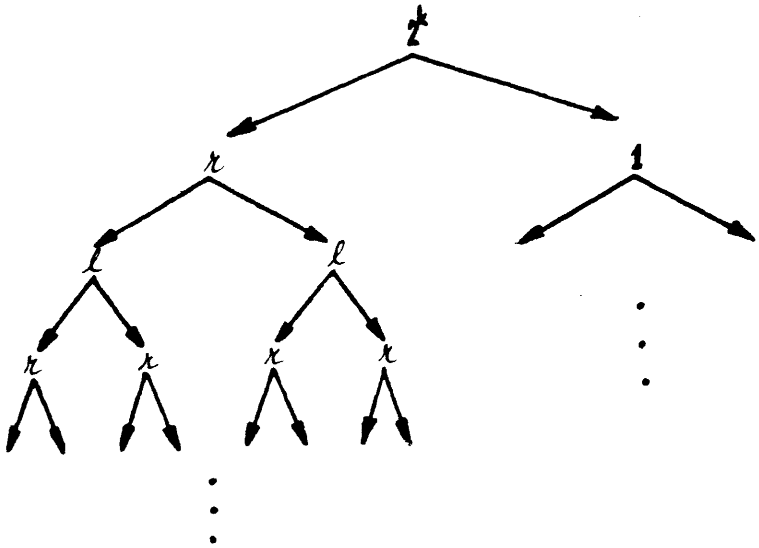
Then

$$N_l = \sum_{i=0}^{(k-2)/2} 4^i$$

$$N_r = 2^k - 2 - N_l = 2^k - \sum_{i=0}^{(k-2)/2} 4^i - 2$$

If $k > 2$ then in both these cases $N_l \neq N_r$.

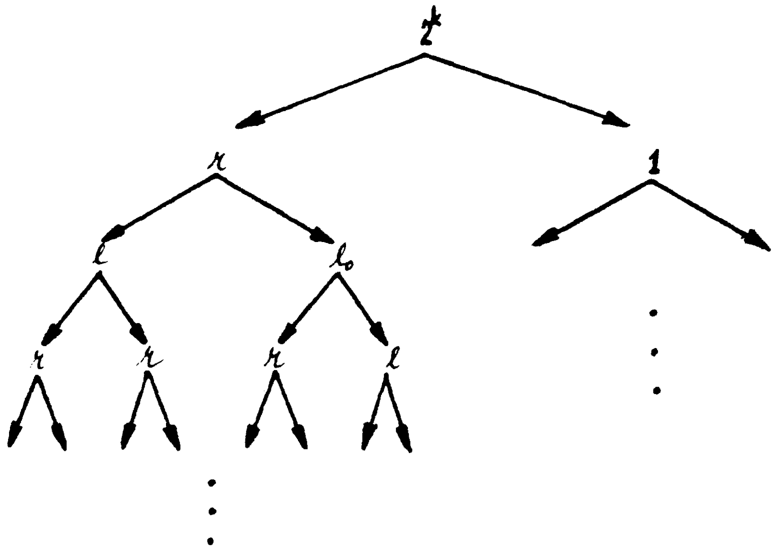
Case 2. If $x \in R$ and either l_0 does not exist or l_0 occurs at k^{th} step from 2^k. Then



where either all $r \in R$ and all $l \in L \cup x_{2^k}$; or all $r \in R \cup x_{2^k}$ and all $l \in L$. It is easy to see that this is the same as case 1 if we interchange N_l and N_r. Thus in this case also $N_l \neq N_r$ for $k > 2$.

Case 3. If $x \in R$ and l_0 occurs at p^{th} step , $p \leq k - 1$. Then obviously p must be even.

The following diagram illustrates this case.



(a) all $r \in R$ and k odd.

Then

$$N_l = 2 \sum_{i=0}^{(k-3)/2} 4^i + \sum_{i=0}^{(k-p-1)/2} 4^i - 2 \sum_{i=0}^{(k-p-3)/2} 4^i - 1$$

$$= 2 \sum_{i=(k-p-1)/2}^{(k-3)/2} 4^i + \sum_{i=0}^{(k-p-1)/2} 4^i - 1$$

$$N_r = 2^k - 2 - N_l = 2^k - 2 \sum_{i=(k-p-1)/2}^{(k-3)/2} 4^i - \sum_{i=0}^{(k-p-1)/2} 4^i - 1$$

(b) all $r \in R$ and k even.

Then

$$N_l = 2 \sum_{i=0}^{(k-2)/2} 4^i + \sum_{i=0}^{(k-p-2)} 4^i - 2 \sum_{i=0}^{(k-p-2)/2} 4^i - 1$$

$$= 2 \sum_{i=(k-p)/2}^{(k-2)/2} 4^i + \sum_{i=0}^{(k-p-2)/2} 4^i - 1$$

$$N_r = 2^k - 2 - N_l = 2^k - 2 \sum_{i=(k-p)/2}^{(k-2)/2} 4^i - \sum_{i=0}^{(k-p-2)/2} 4^i - 1$$

(c) all $l \in L$ and k odd.

Then

$$N_l = 2 \sum_{i=(k-p-1)}^{(k-3)/2} 4^i + \sum_{i=0}^{(k-p-1)/2} 4^i$$

$$N_r = 2^k - 2 - N_l = 2^k - 2 \sum_{i=(k-p-1)/2}^{(k-3)/2} 4^i - \sum_{i=0}^{(k-p-1)/2} 4^i - 2$$

(d) all $l \in L$ and k even.

Then

$$N_l = 2 \sum_{i=(k-p)/2}^{(k-2)/2} 4^i + \sum_{i=0}^{(k-p-2)/2} 4^i$$

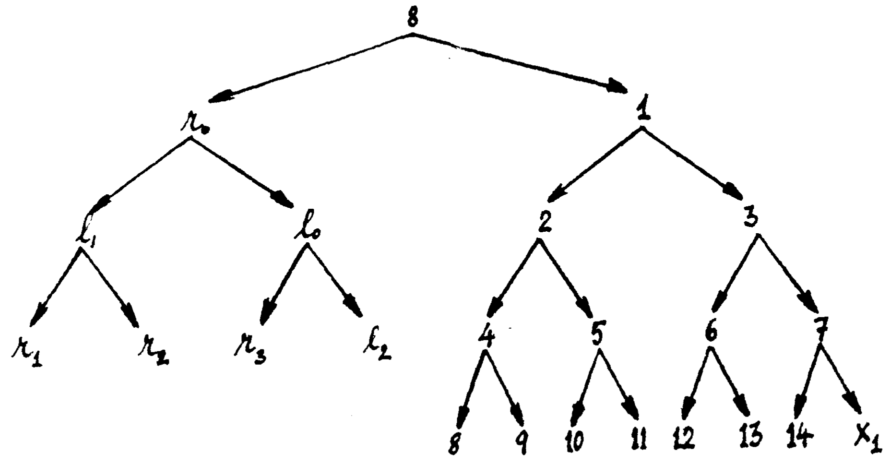
$$N_r = 2^k - 2 - N_l = 2^k - 2 \sum_{i=(k-p)/2}^{(k-2)/2} 4^i - \sum_{i=0}^{(k-p-2)/2} 4^i - 2$$

If $k = 3$ and $p = 2$ then in case (c) $N_l = N_r$;

in all the other cases and for all the possible values of k and p, $N_l \neq N_r$.

Thus it remains to show that $G \in G(14, 2, 3)$ does not exist in case (c).

If G did exist then we could partly draw it as follows.

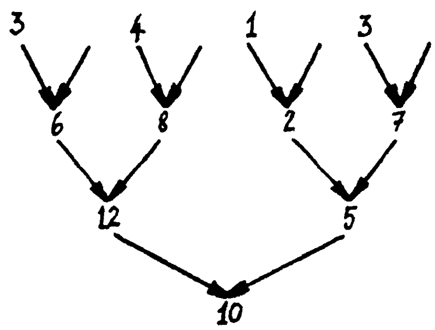


where $l_0, l_1, l_2 \in L = \{9, 10, 11\}$ and all $r_i \in R \cup x_8$, where x_8 is the repeat from 8.

Since $10 \not\to 11$ and $11 \not\to 10$ (else we cannot reach all from 5), one of $\{l_0, l_2\}$ must be 9.

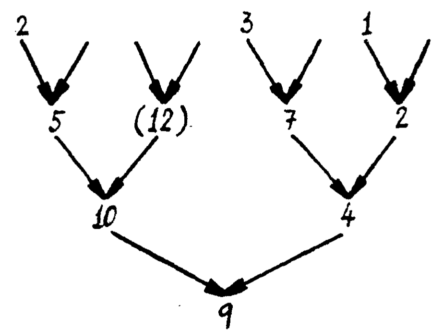
(i) If $l_0 = 9$. Take $l_1 = 10$, $l_2 = 11$. Then $7 \to$ (9 or 5) (to reach 11 from 3). If $7 \to 9$ then we have case 1. Hence take $7 \to 5$ i.e., $x_1 = 5$. To reach 9 from 3, we need $(12, 13$ or $14) \to 9$. But $14 \not\to 9$ so $12 \to 9$ (say). That is, $r_0 = 12$.

Then we have



Thus we need to reach vertex 10 from vertices 9,11,13 and 14. To also reach 2 and 8 from 3 we have either $13 \to 2$ and $14 \to 8$, or $13 \to 8$ and $14 \to 2$. In either case we cannot have $13 \to 4$ or $14 \to 4$ and so we cannot reach 4 from 3.

(ii) If $l_2 = 9$. Take $l_0 = 10$, $l_1 = 11$. Then $x_1 = 4$ (to reach 9 from 3) and we have



where placing (12) is explained below.

To reach 9 from 6, we must have $12 \to 10$ or $13 \to 10$. Take $12 \to 10$, i.e., $r_0 = 12$. To reach 9 from 11, we need $11 \to 7$ (as $11 \not\to 2, 5$ - else cannot reach 2 or 5 from 3). But then we cannot reach 10 from 7.

Hence for all $k > 2$, $G(2^{k+1} - 2, 2, k)$ is empty $\diamond$

Corollary.

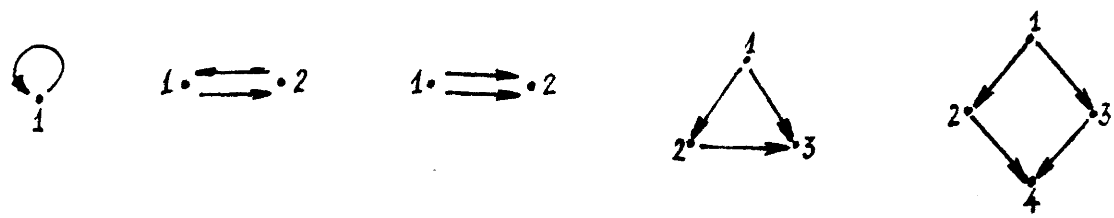
If $k > 2$ then $N(2, k) \leq 2^{k+1} - 3$.

Thus for $k = 3$ we have $12 \leq N(2, 3) \leq 2^4 - 3 = 13$.

In order to show that $N(2, 3) = 12$, we will prove that $G(13, 2, 3)$ is empty. To do this we will use the following lemmas.

• **Lemma 4.**

If $G \in G(13, 2, 3)$ then it does not contain the following subdigraphs.

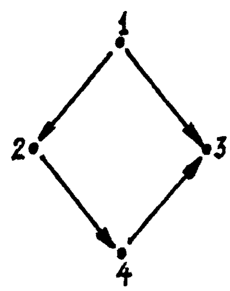


Proof.

Obvious as in each of the above cases we cannot reach all the other 12 vertices from vertex 1 $\diamond$

• **Lemma 5.**

If $G \in G(13, 2, 3)$ then it does not contain the following subdigraph.



Proof.

See Appendix $\diamond$

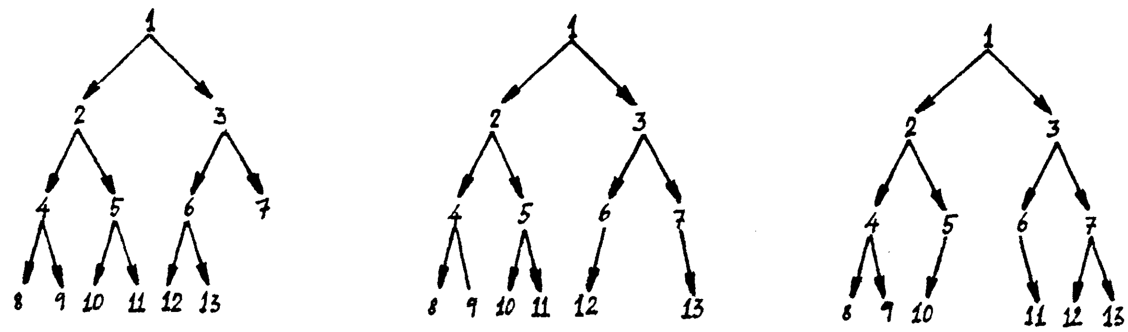
• **Theorem 5.**

$G(13, 2, 3)$ is empty.

Proof.

Suppose $G \in G(13, 2, 3)$.

Labelling the vertices of G by $1, 2, \ldots, 13$, G must contain one of the following subdigraphs.



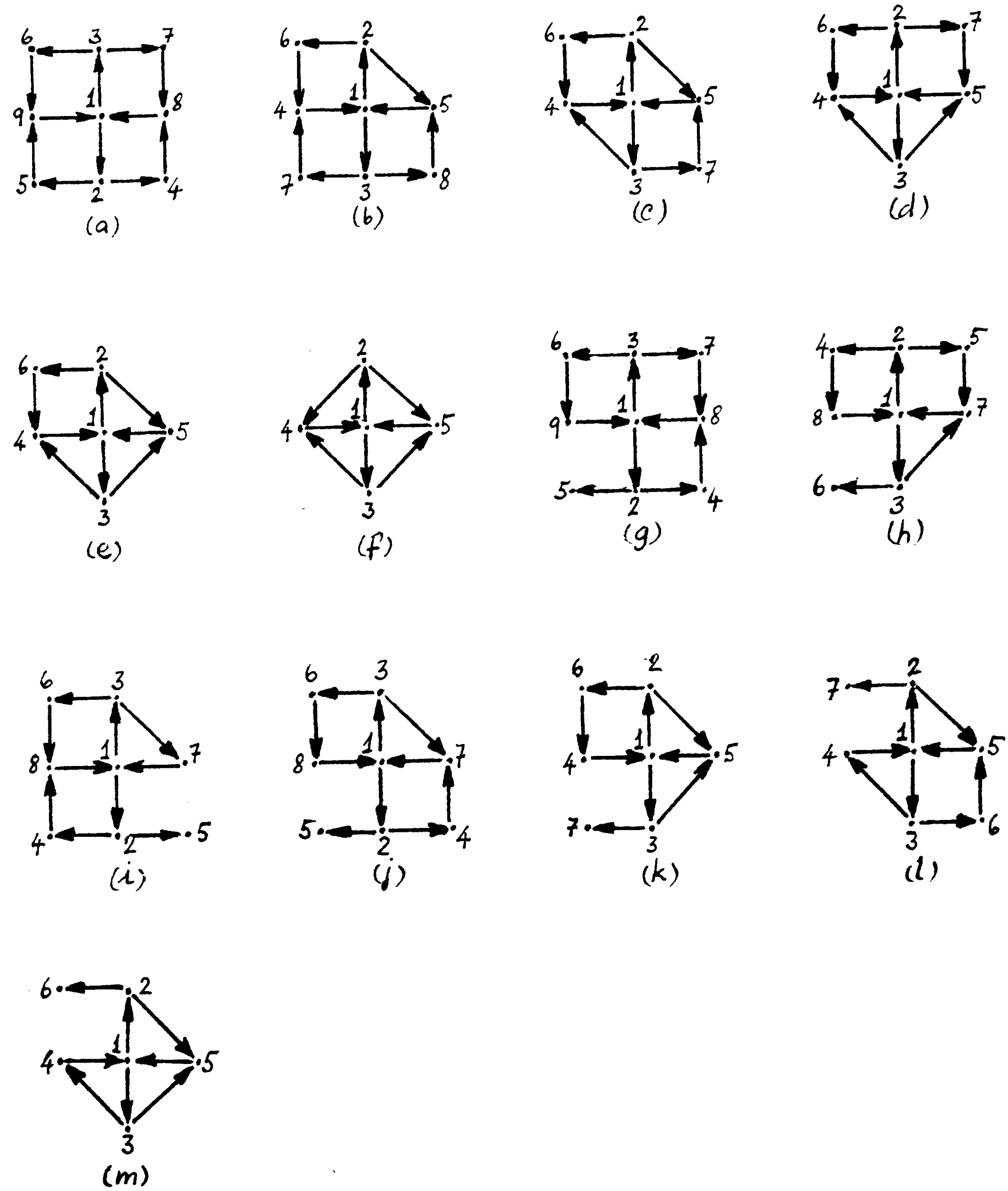
In any case, the vertex 1 must reach itself in 3 or 4 steps in at least two different ways (once through vertex 2 and once through vertex 3 so that vertices 2 and 3 reach 1 in at most 3 steps).

On the other hand, suppose $x \to 1$ and $y \to 1$.

Then obviously $x \neq y$ and the vertex 1 must reach itself in 3 or 4 steps in at least two different ways (once through vertex x and once through vertex y so that vertex 1 reaches vertices x and y in at most 3 steps).

We will show that 1 reaches itself in 3 or 4 steps in exactly two different ways.

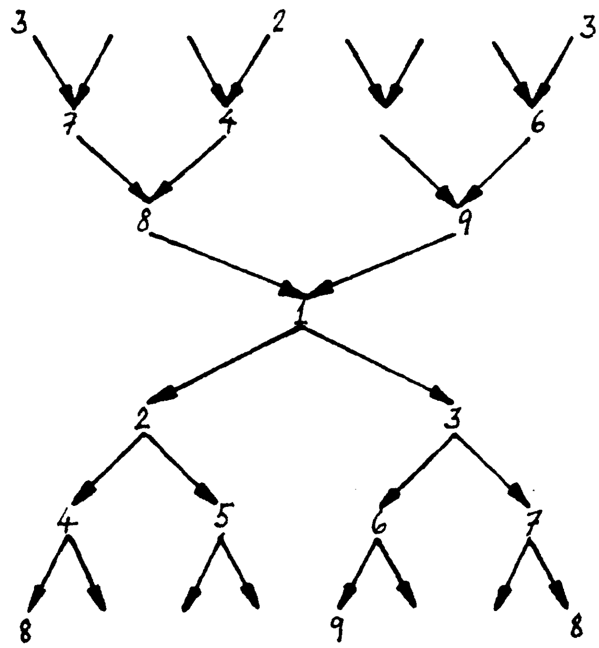
Suppose the contrary. Then one of the following cases must occur.



Cases (b),(c),(d),(e),(f),(k),(l),(m) are obviously impossible as we cannot reach all the other 12 vertices from vertex 1.

Case (a) is a subcase of case (g).

Case (g). We have



To reach all from 1, we need to reach vertices 10,11,12,13 through the vertices 4,5,6,7. That is, ($4 \to 10$ or $5 \to 10$ or $6 \to 10$ or $7 \to 10$) and ($4 \to 11$ or $5 \to 11$ or $6 \to 11$ or $7 \to 11$) and ($4 \to 12$ or $5 \to 12$ or $6 \to 12$ or $7 \to 12$) and ($4 \to 13$ or $5 \to 13$ or $6 \to 13$ or $7 \to 13$). This will be written as $\{4,5,6,7\} \to \{10,11,12,13\}$ for brevity.

(i) Let $4 \to 10, 5 \to 11, 5 \to 12$ and (6 or 7) $\to$ 13.

To reach 1 from 5, we need (11 or 12) $\to$ 9, say $11 \to 9$; to reach 8 from 5, we need (11 or 12) $\to$ 7 (as $11, 12 \not\to 4$).

Then (10 and 13) $\to$ (4 and 6) (to reach 1 from all).

Now $10 \not\to 4$ so $10 \to 6$ and $13 \to 4$; and $12 \not\to 11$ so $12 \to 7$.

But now we cannot reach 3 from 2 as none of 8,10,11,12 can go to 3.

(ii) Let $6 \to 10$, $7 \to 11$, $5 \to 12$ and (4 or 5) $\to$ 13.

–If $5 \to 13$ then to reach 1 from 5, we must have (12 or 13) $\to$ 9, say $12 \to 9$.

To reach 9 from 7, we need $11 \to 12$ (as $8 \not\to 6, 8 \not\to 12, 11 \not\to 6$).

Now $10 \not\to 6$, $10 \not\to 7$ so $10 \to 4$ (to reach 1 from 10).

Since $9 \not\to 2$, $8 \not\to 2$, $10 \not\to 2$ we have $11 \to 2$ (to reach 2 from 3).

To reach 8 from 5, we need (12 or 13) $\to$ 7.

Now $12 \not\to 7$ (else cannot reach 8 from 9) so $13 \to 7$.

To reach 2 from 5, we must have $13 \to 11$ (since $12 \not\to 11$).

But then we cannot reach all from 13.

–If $4 \to 13$ then to reach 8 from 6, we need (9 or 10) $\to$ 4. Now $9 \not\to 4$ so $10 \to 4$.

To reach 2 from 3, we must have $11 \to 2$ (as $8,9,10 \not\to 2$).

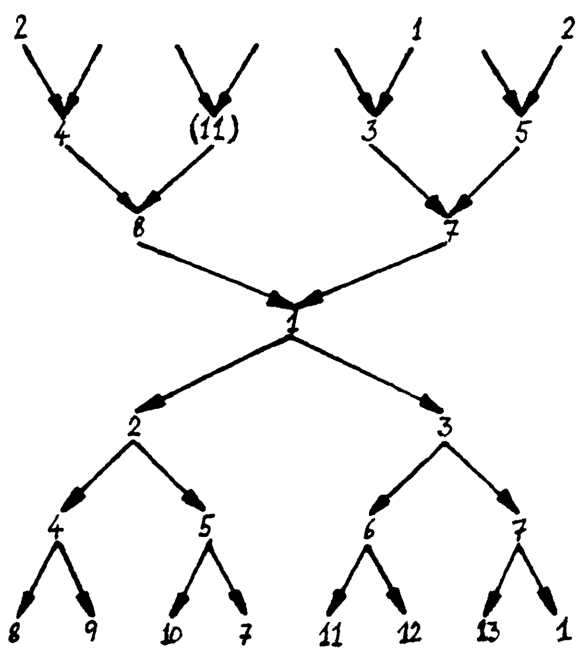
To reach 5 from 3, we need $10 \to 5$ (since $8, 9, 10 \not\to 5$).

To reach 5 from 4, we need $13 \to 10$ (as $8 \not\to 10$).

To reach 10 from 7, we must have $11 \to 13$ (as $8 \not\to \{6, 13\}$, and $11 \not\to 6$).

But now we cannot reach 13 from 6 as $9 \not\to 11$.

Case (h). We have



where placing (11) is explained below and where we have labelled 9,10.11,12,13 arbitrarily.

To reach all from 3, we need $\{11,12,13\} \rightarrow \{4,5,8,9,10\}$.

To reach all from 2, we need $\{8,9,10\} \rightarrow \{3,6,11,12,13\}$.

To reach 1 from 6, we must have 11 or 12 $\rightarrow$ 8, say 11$\rightarrow$8.

To reach 1 from all, we need $\{9,10\} \rightarrow \{3,11\}$ and $\{12,13\} \rightarrow \{4,5\}$.

To reach 7 from 4, we must have 9$\rightarrow$3 (as 8$\not\rightarrow$ (3 or 5); 9$\not\rightarrow$5) and so also 10$\rightarrow$11.

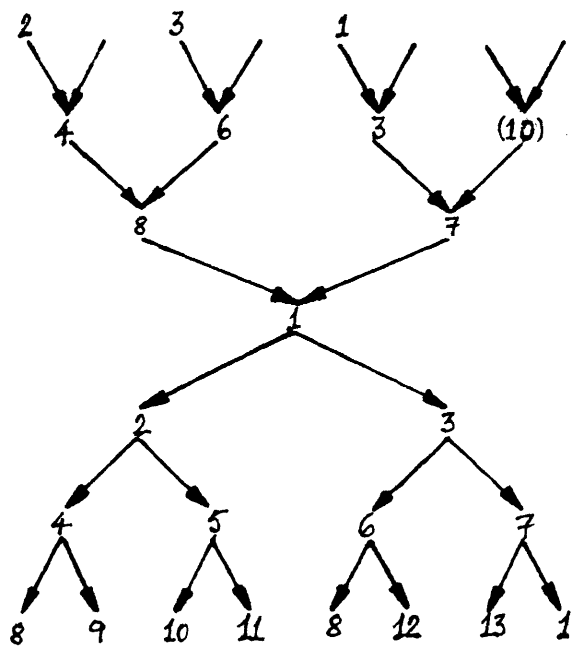
To reach 7 from 6, we must have (11 or 12) $\rightarrow$ 5.

Now 11$\not\rightarrow$5 (else we cannot reach 1 from all) so 12$\rightarrow$5 and then also 13$\rightarrow$4.

To reach 12 from 7, we need 13$\rightarrow$ 10 and 10$\rightarrow$12.

But then we cannot reach 7 from 13.

Case (i). We have



where we have labelled vertices 9,10,11,12,13 arbitrarily.

To reach 1 from 5, we need (10 or 11) $\rightarrow$ 7, say 10$\rightarrow$7.

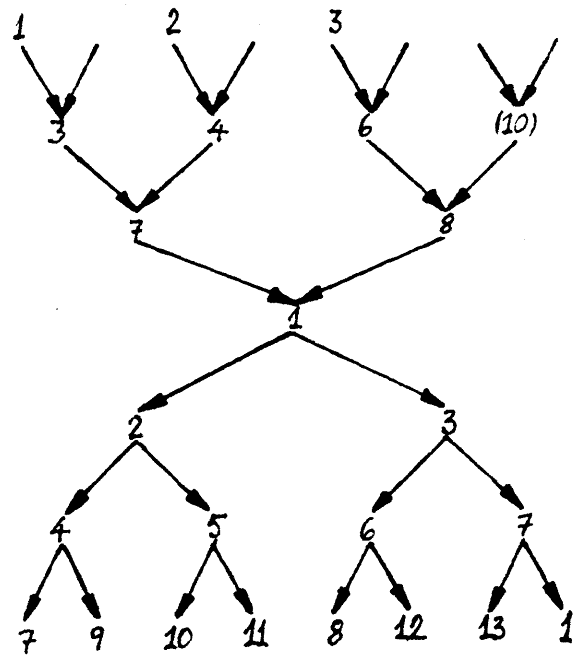
Now 9$\not\rightarrow$10 (else cannot reach 10 from 3) and 9$\not\rightarrow$4 so 9$\rightarrow$(6 or 3) (to reach 1 from 9).

Similarly, $11 \not\to 10$, $11 \not\to 4$ so $11 \to (6 \text{ or } 3)$. Then $\{12, 13\} \to \{4, 10\}$.

Then $13 \not\to 12$ (else we cannot reach 12 from 2), and $13 \not\to \{10, 11\}$ and so we cannot reach all from 7.

If $12 \to 4$ and $13 \to 10$ then we cannot reach 8 from 7.

Case (j). We have



where we have labelled the vertices 9,10,11,12,13 arbitrarily.

To reach 1 from 5, we need $(10 \text{ or } 11) \to 8$, say $10 \to 8$.

To reach 1 from all, we must have $\{9, 11, 12, 13\} \to \{3, 4, 6, 10\}$.

To reach all from 3, we must have $\{8, 12, 13\} \to \{4, 5, 9, 10, 11\}$.

Hence $\{12, 13\} \to \{4, 10\}$ and since $10 \to 8$, $12 \not\to 10$ and so $12 \to 4$, and $13 \to 10$.

To reach 8 from 4, we need $9 \to 6$ (as $7 \not\to 6$).

Then also $11 \to 3$ and $13 \to 11$ (to reach 7 from 13 and since $9, 10 \not\to 11$ - else we cannot reach 11 from 3).

But then we cannot reach 9 from 7 as $10 \not\to 9$ and $11 \not\to 9$.

Hence 1 (and every other vertex of G) reaches itself in at most 4 steps in exactly two different ways, i.e., any vertex lies exactly on either

(a) two 4-circuits

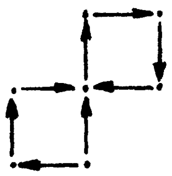
or

(b) one 4-circuit and one 3-circuit

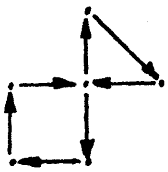
or

(c) two 3-circuits;

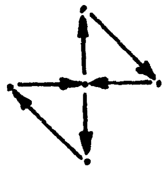
i.e., one of the following three cases must occur



(a)



(b)



(c)

Hence also any two 3- or 4-circuits have at most one point in common. Now, there are $2 \times 13 = 26$ arcs in G. All these arcs lie on the 3- or 4-circuits. If there are x 3-circuits and y 4-circuits in G then we have

$$3x + 4y = 26$$

which gives

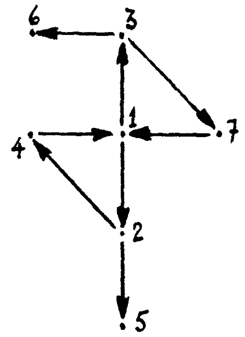
(i) $x = 6$ and $y = 2$

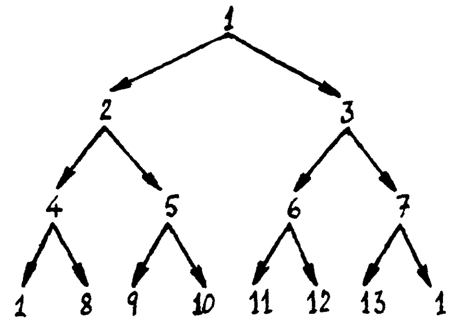
or

(ii) $x = 2$ and $y = 5$.

Case (i).

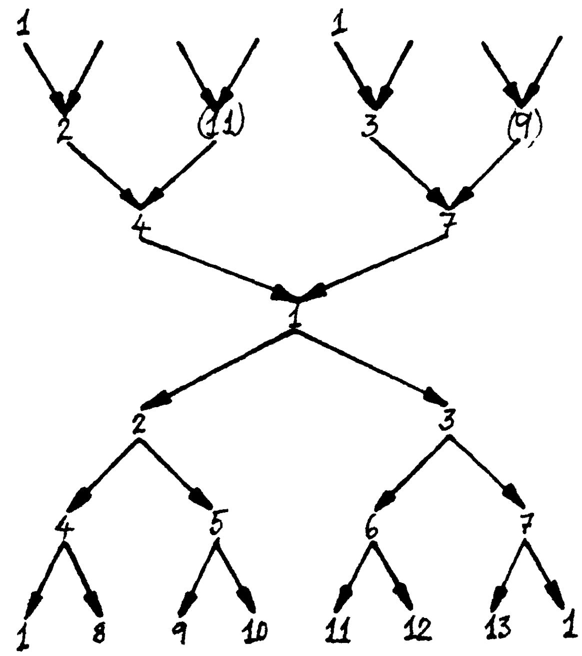
For some vertex, say vertex 1 we have





We will show that it is not possible to have such a subdigraph in G(13,2,3).

We have



To reach 1 from 5, we need (9 or 10) $\to$ 7 (as 9,10 $\not\to$ 4), say 9$\to$7.

To reach 1 from 6, we must have (11 or 12) $\to$ 4, say 11$\to$4.

Now 8$\not\to$(2 or 3) so 8$\to$(9 or 11) (to reach 1 from 8).

Since 8$\not\to$9 (else cannot reach 9 from 3) we have 8$\to$11.

To reach 4 from 5, we must have 10$\to$2 (as 9$\not\to$2 - else we cannot reach all from 9).

Now 9$\not\to$10, 9$\not\to$8 (else cannot reach 8 from 3) so 9$\to$6 (to reach 4 from 9).

Then 13$\to$9 (to reach 1 from 13 and since 13$\not\to$3) and so also 12$\to$3.

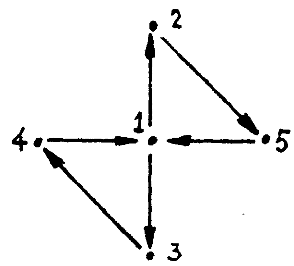
Now $8 \not\to 12$ so $10 \to 12$ (to reach 12 from 2).

To reach 8 from 5 we need $12 \to 8$.

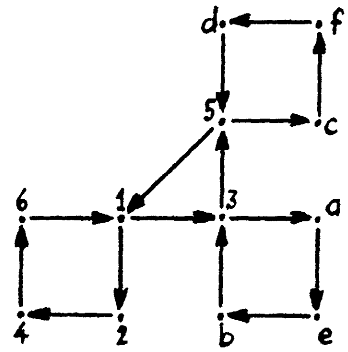
But now we cannot reach 8 from all.

Case (ii).

Since by (i) G cannot contain



there must be in G the following subdigraph



Certainly, the vertices 1,2,3,4,5,6 must all be distinct since the circuits (1,3,5) and (1,2,4,6) already have one point in common.

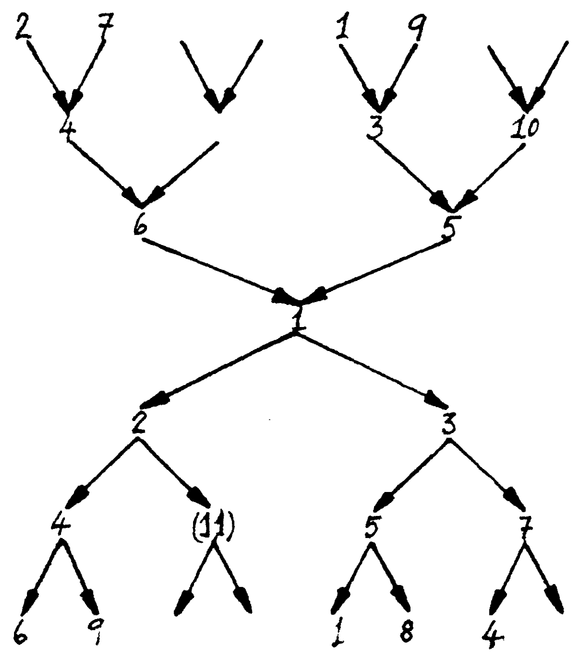
It is obvious that also a and c must be distinct from 1,2,3,4,5,6 and from each other, say $a = 7$, $c = 8$.

It is also obvious that b and d must be distinct from 1,2,3,4,5,6,7,8 and from each other, say $b = 9$, $d = 10$.

Now, e obviously cannot be 1,2,3,5,6,7,8,9,10.

Suppose $e = 4$.

Then we have



If 2$\rightarrow$ x ($x \neq 4$) then $x \neq$1,2,3,4,5,6,7,8,9,10, say x = 11.

Now 7$\not\rightarrow$10 so 11$\rightarrow$10 (to reach 10 from 1) and 11$\rightarrow$12, 7$\rightarrow$13 (say).

To reach 10 from 3, we need (8 or 13) $\rightarrow$ 10; and 8$\not\rightarrow$10 so 13$\rightarrow$10.

To reach 11 from 3, we must have 8$\rightarrow$11 (as 13$\not\rightarrow$11).

But 6 cannot go to (9 and 13); 9 cannot go to (6 and 13); and 13 cannot go to (6 and 9) and so we cannot reach all from 5.

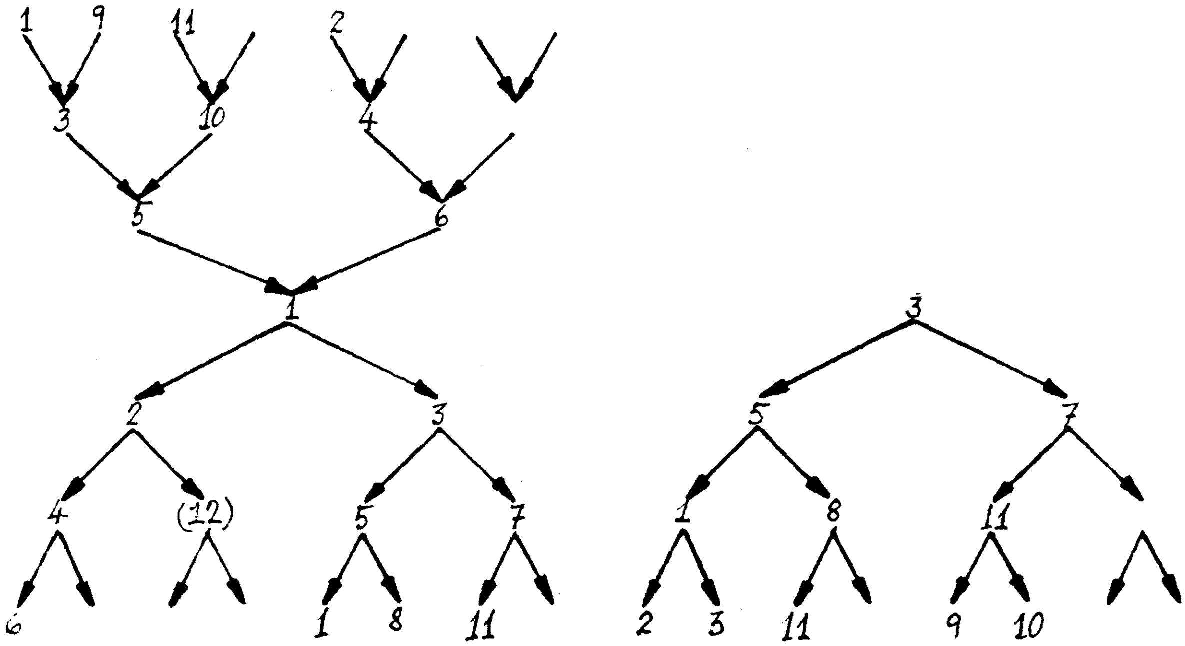
Thus $e \neq 4$, say $e = 11$.

By symmetry, if e cannot be 1,2,3,4,5,6,7,8,9,10 then also f cannot be 1,2,3,4,5,6, 7,8,9,10.

It remains to show that $f \neq e$.

Suppose $f = 11$.

Then we have



If $2 \to x$ $(x \neq 4)$ then $x \neq 1,2,3,4,5,6,7,8,9,10,11$ so $x = 12$ (say).

To reach 3 from 2, we must have (4 or 12) $\to$ 9.

To reach 5 from 2, we need (4 or 12) $\to$10.

-If $4\to9$ and $12\to10$ then to reach 11 from 2, we need $12\to$(7 or 8) so $7\to13$ (to reach 13 from 1).

To reach 1 from 7, we must have $13\to6$.

To reach 11 from 2 we need (6 or 9)$\to$(7 or 8).

To reach 4 from 3 we must have (8 or 13) $\to$ 4.

But $13 \not\to 4$ and $8 \not\to 4$.

-If $4\to10$ and $12\to9$ then to reach 11 from 2, we must have $12\to$(7 or 8) and so $7\to13$ (to reach 13 from 1).

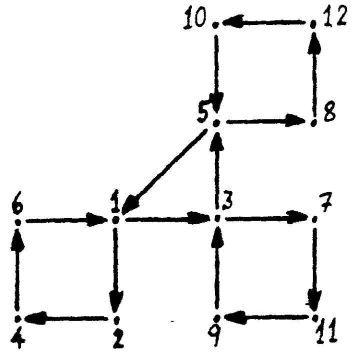
To reach 1 from 7, we need $13\to6$.

But now we cannot reach 5 from 13 (since $13 \not\to 4$).

-If $12 \to 9$ and $12 \to 10$ then we cannot reach 5 from all.

Thus $f = 12$ (say) and all the 12 vertices are distinct. We will show that the second 3-circuit in G must contain the 13^{th} point of G.

We have



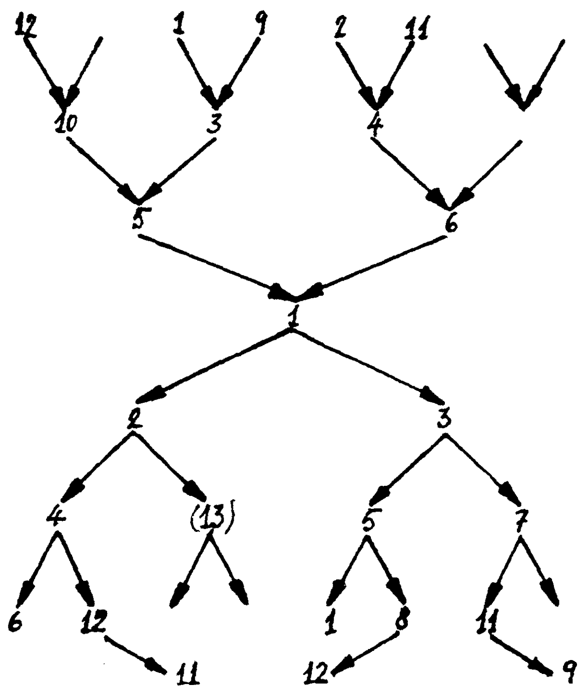
To construct the second 3-circuit using only vertices 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 we have the following possibilities.

(a) $4 \to 12 \to 11 \to 4$ (b) $4 \to 11 \to 12 \to 4$ (c) $6 \to 8 \to 11 \to 6$

(d) $6 \to 11 \to 12 \to 6$ (e) $6 \to 12 \to 11 \to 6$ (f) $2 \to 11 \to 12 \to 2$

(g) $2 \to 12 \to 11 \to 2$

Case (a). We have



If $2 \to x$ then $x \neq 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ so $x = 13$.

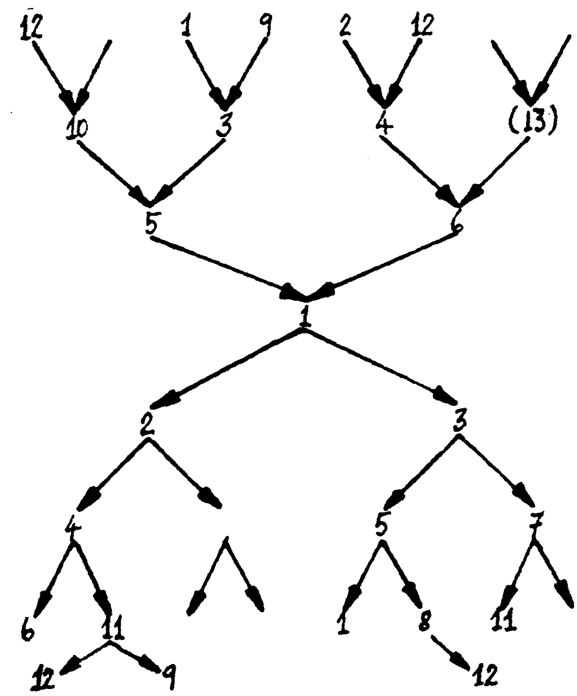
To reach 9 from 1, we need $13 \to 9$ (as $7 \not\to 9$).

To reach 10 from 1, we need $13 \to 10$ (as $7 \not\to 10$).

Now $7 \not\to 8$ and $8 \not\to 7$ so to reach 1 from 7 and 8, we need $7 \to x$, $8 \to x$ and $x \to 6$.

But no such x can exist.

Case (b). We have



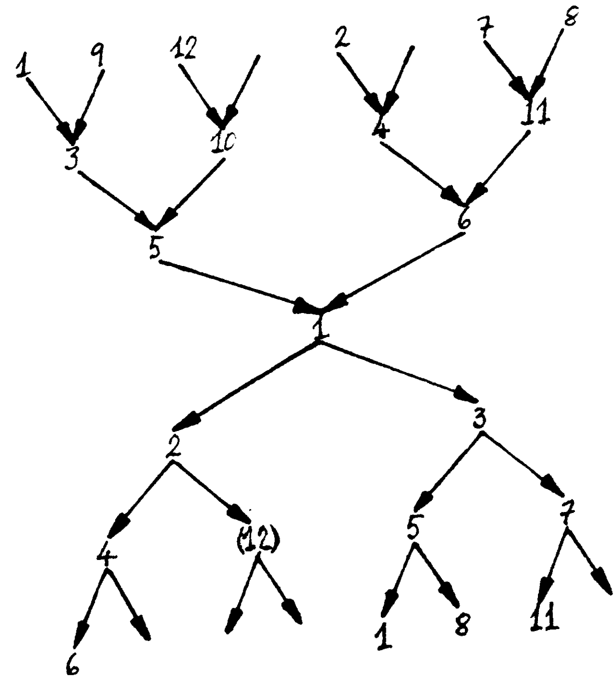
Since $7 \not\to 10$; $7 \not\to 6$ (else we cannot reach all from 3) we need $7 \to x \to 6$ (to reach 1 from 7)

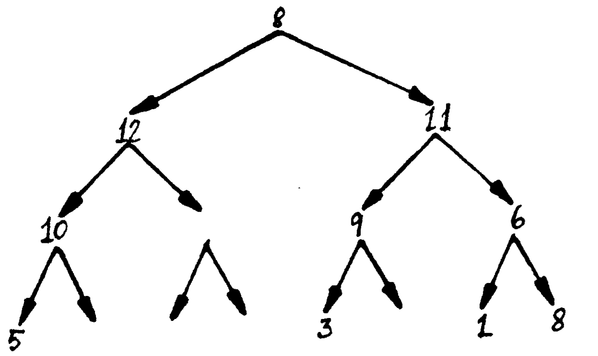
and so $x = 13$.

To reach 1 from 8, we must have $8 \to 13$ (as $8 \not\to 10$).

But then we cannot reach 1 from 11.

Case (c). We have





If $2\rightarrow x (x \neq 4)$ then $x \neq 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$ so $x = 12$ or $x = 13$.

If $7\rightarrow y$ $(y \neq 11)$ then $y \neq 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$ so $y = 12$ or $y = 13$.

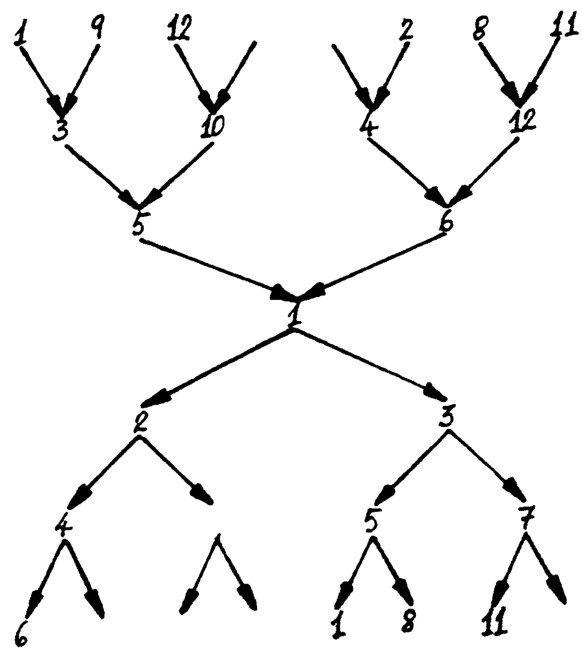
Now, to reach all from 3, we must have $7\rightarrow(4, 10$ or $13)$.

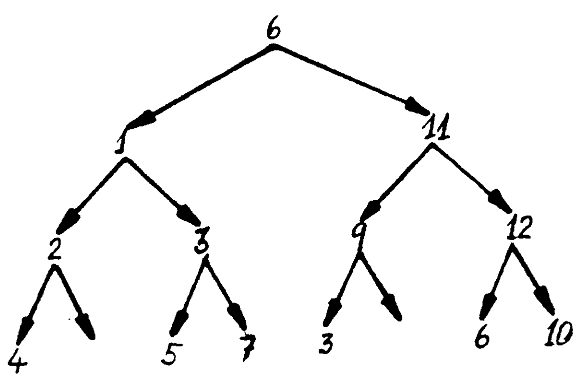
Thus $y = 13$ and $x = 12$.

Then to reach all from 3, we need $13\rightarrow 4$ and $13\rightarrow 10$.

But then since $12\not\rightarrow 13$ we cannot reach 13 from 5.

Case (d). We have



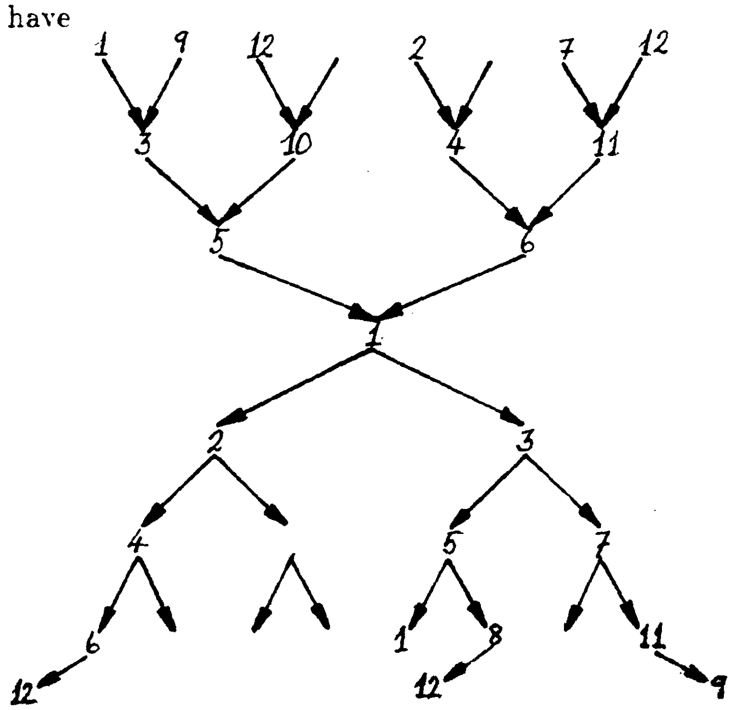


To reach 1 from 7, we must have $7 \to 4$ (as $7 \not\to 10$).

To reach 1 from 13, we need $13 \to 10$.

But then we cannot reach 10 from 3.

Case (e). We have

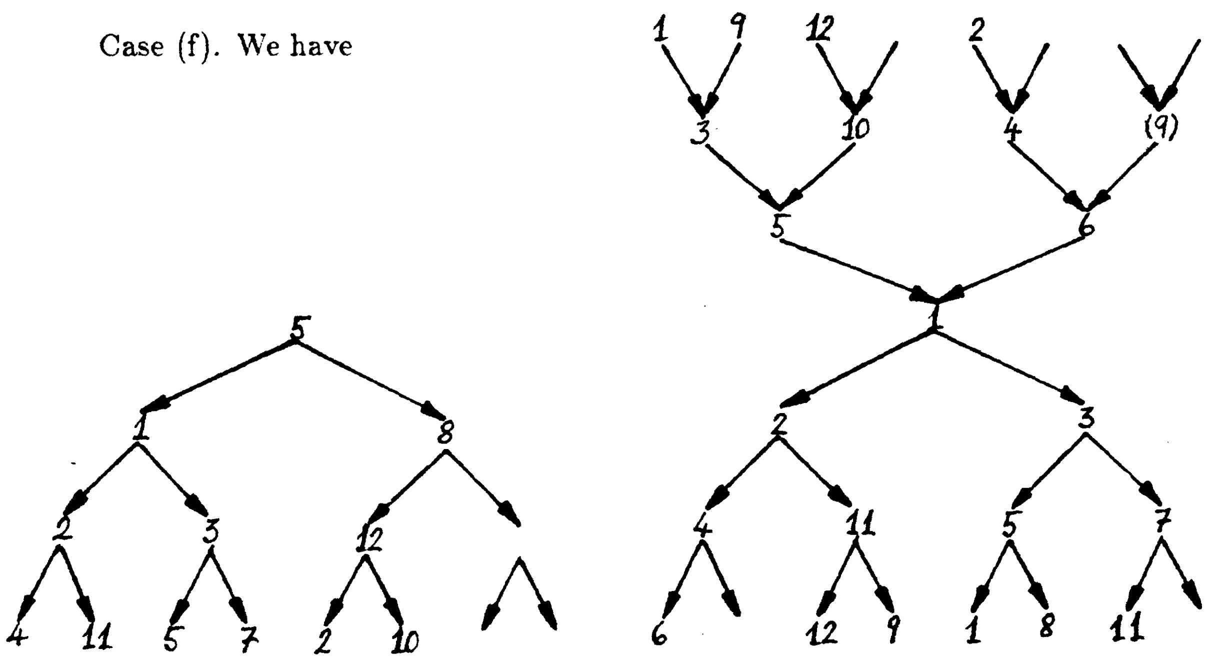


To reach 1 from 8, we need $8 \to 4$ (as $8 \not\to 10$); then also $13 \to 10$ to reach 1 from 13.

To reach 10 and 13 from 3, we must have $7 \to 13$.

But then we cannot reach 12 from 1 as $2 \not\to 6$ and $2 \not\to 8$.

Case (f). We have

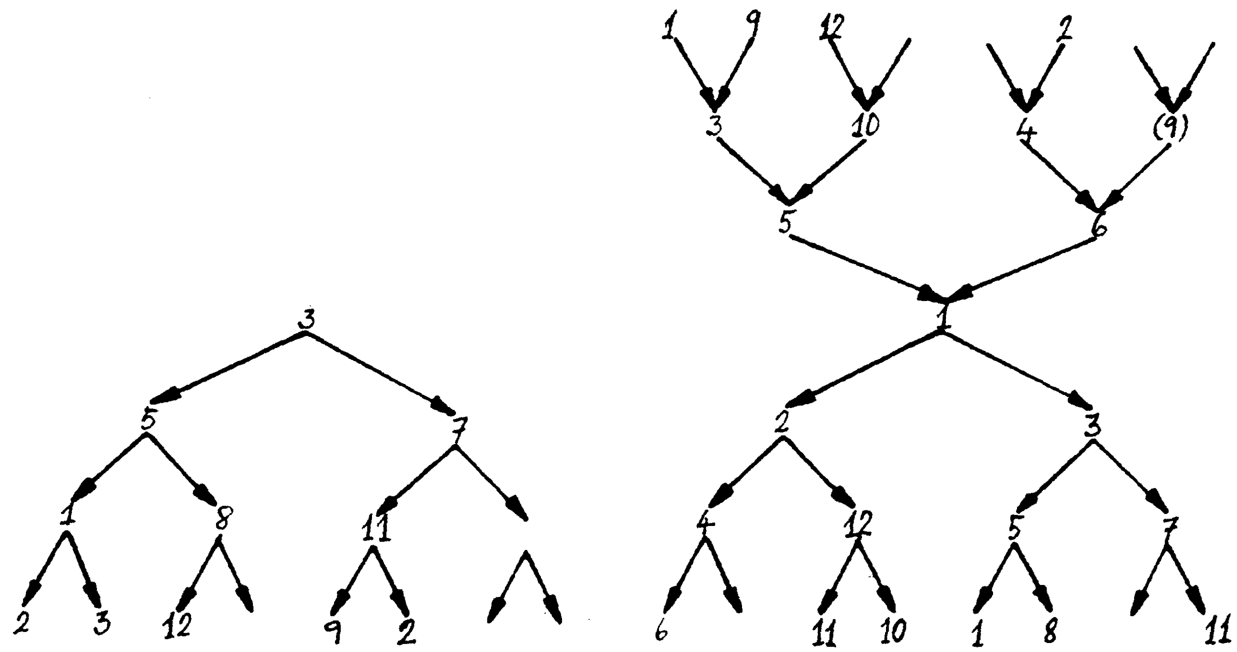


To reach 1 from 11, we need $9 \to 6$.

To reach 1 from 8, we need $8 \to (10 \text{ or } 4 \text{ or } 9)$.

Now , $8 \not\to 10$, $8 \not\to 9$ (else we cannot reach 5 from all) and $8 \not\to 4$ (else we cannot reach all from 5).

Case (g). We have



To reach 1 from 11, we need $9 \to 6$.

To reach all from 1, we need $\{4,7\} \to \{9,13\}$. Now $7 \not\to 9$ so $7 \to 13$ and $4 \to 9$.

But then we cannot reach 1 from 7.

Hence the second 3-circuit must contain the 13^{th} point of G.

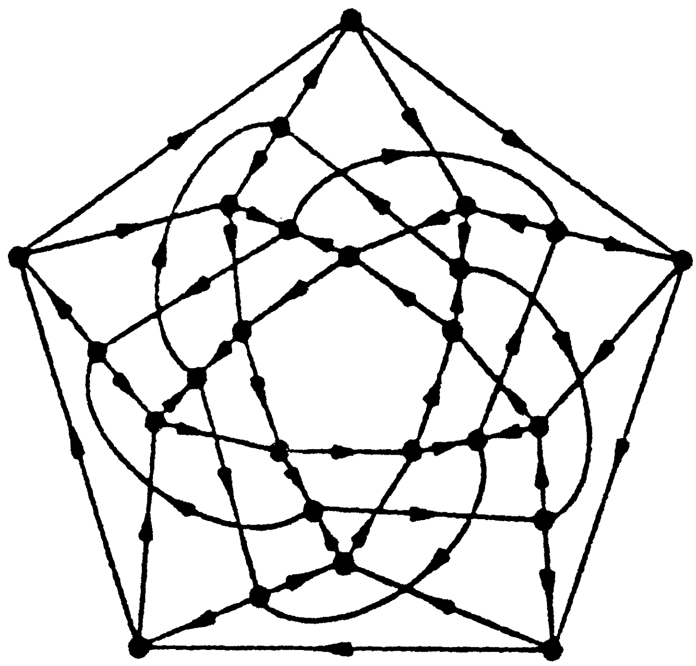
But then we still need two more 4-circuits and since we can use the 13^{th} vertex only once more, and since any two 3 or 4-circuits can have at most one point in common, this is not possible.

Hence $G(13,2,3)$ is empty $\diamond$

Corollary.

$$N(2,3) = 12.$$

Next, for $d = 2, k = 4$ we have $N(2,4) \geq 24$. However. Alegre (1983) constructed a digraph $G \in G(25, 2, 4)$ as follows.



Hence $N(2,4) \geq 25$.

As shown later (Theorem 8) the existence of $G \in G(n, d, k)$ guarantees the existence of $G' \in G(nd, d, k+1)$. thus we have the following table.

k	$N(2,k)$
0	1
1	3
2	6
3	12
4	$25 \leq N(2,4) \leq 29$
5	$50 \leq N(2,5) \leq 61$
6	$100 \leq N(2,6) \leq 125$
7	$200 \leq N(2,7) \leq 253$
8	$400 \leq N(2,8) \leq 509$
9	$800 \leq N(2,9) \leq 1021$
10	$1600 \leq N(2,10) \leq 2045$
$\vdots$	
t	$2^{t-4} \times 25 \leq N(2,t) \leq 2^{t+1} - 3$
$\vdots$	