INTRODUCTION

> Without doubt, geometry is important. It offers us a way to interpret and reflect upon our physical environment ... research that elaborates and extends the van Hiele theory is crucial. This perspective appears to hold much promise for the improvement of both research and instruction, and further elaboration and explication of such notions as "levels of geometric thinking" can help realise this promise ... Perhaps the greatest strengths and weaknesses of the cognitive sciences approach lie in its extreme degree of specification. It provides much needed details on cognitive processes (and thus forces explication on notions too often left vague in other theories, such as "networks of relations").
>
> Clements & Battista (1992, p. 457)

It is evident in the quote above by Clements and Battista (1992) that investigation of students' understandings of geometrical concepts is important and valuable. The extension of knowledge concerning the manner in which students' perceive and develop such concepts has the potential to affect directly the instruction they receive. One area of Geometry that has been the centre of many debates (Burger & Shaughnessy, 1986; Fuys, 1985; Mayberry, 1981; Usiskin, 1982) concerns the difficulties associated with students' understanding of the overview of relationships among figures and properties, and can be described as an understanding of "networks of relations." In particular, an understanding of notions of class inclusion is recognised as a necessary prerequisite for deductive reasoning and also a concept that is a difficult hurdle for many students to overcome (de Villiers, 1993; Pegg & Davey, 1991).

While Clements and Battista (1992, p. 457) stated that cognitive processes are "often left too vague in other theories," two theoretical models, the van Hiele Theory (van Hiele, 1986) and the SOLO model (Biggs & Collis, 1982) explicitly describe the understanding and utilisation of relationships as an essential component of higher-order reasoning. Despite numerous studies exploring aspects of the van Hiele Theory, there has not been a study that has focused specifically upon how an understanding of class inclusion concepts evolve, and, in particular, the manner in which aspects associated with this notion are utilised.

Some studies, not in the field of Geometry, have explored class inclusion and found that it generally provides the learner with a structure that is based on the focus on relations (Greene, 1994; Halford, 1996) rather than requiring specific examples, hence, is regarded as a precursor for higher-order reasoning. While some studies (Lawrence, 1980; Ni, 1998) have acknowledged different levels of classification, lower order classification has been the focus of the majority of research in recent times. In the school setting attempts have been made in secondary education syllabi which acknowledge the difficulties associated with a general lack of understanding of class inclusion notions in Geometry. There remains a need to explore students' perception of the relationships among figures and their properties, and identify aspects affecting students' understanding of relationships.

In the light of findings (Burger & Shaughnessy, 1986; de Villiers, 1993; Mayberry, 1981; Pegg & Davey, 1991; Usiskin, 1982) associated with students' understanding of geometrical concepts, five research themes were employed to guide the present study investigating students' understandings of class inclusion concepts in Geometry. Each of these themes investigates class inclusion in Geometry, the first within the context of triangles, and the second within the context of quadrilaterals. The third theme involves a comparison of the findings across contexts. The fourth theme synthesises the findings in a quantitative manner, while the final theme provides a longitudinal perspective in the form of four case studies.

The theoretical framework providing a base for this study is the van Hiele Theory (van Hiele, 1986) which hypothesises five hierarchical levels of thinking concerning students' understanding of Geometry. The van Hiele level of interest to this study is that of Level 3. This is the level at which van Hiele stated that students focus upon the links that exist between properties and figures, as opposed to a focus upon the properties and figures themselves.

The following discussion provides an outline of the structure of this thesis consisting of eight chapters. Since this study is concerned with the development of ideas leading to an understanding of class inclusion, Chapter 1 provides the background of research directed at the van Hiele Theory, and, in particular, that associated with Level 3 thinking.

In Chapter 2, literature relating to class inclusion in a variety of contexts is reviewed. A more detailed discussion of research directed at class inclusion in Geometry is addressed. This discussion highlights the need for a detailed analysis of this topic, and the need to apply a theoretical framework to assist in the exploration of students' understandings of class inclusion concepts. To meet this need, the SOLO model (Biggs & Collis, 1991) is detailed identifying established links between the van Hiele Theory and the SOLO model (Pegg & Davey, 1998).

As a result of issues arising from previous research in the areas of class inclusion, the van Hiele Theory, and the application of the SOLO model, the design for the present investigation emerged. This study comprises three main studies. Chapter 3 details the design and includes issues relating to context, methodological considerations, data analysis plan, and evaluation of the research methodology.

Chapter 4 concerns Study 1. The first study, within the context of triangles, focused upon two separate areas, these being, relationships among triangle figures and relationships among triangle properties. These involved in-depth interviews incorporating stimulus tasks that were used as a catalyst to initiate student discussion concerning the topics explored. This chapter describes the identified categories of responses to the interview tasks, and the subsequent application of the SOLO model to assist in the interpretation of the response categories.

Study 2, which is detailed in Chapter 5, has a similar focus. The context for this study, namely, quadrilaterals, and utilised the framework already established within the triangle context. As the findings that emerged in Study 1 formed baseline data for Study 2, the SOLO model was applied directly to the response categories. A qualitative synthesis was carried out which identified similarities and differences across contexts resulting in a generic framework.

Chapter 6 utilised additional data, which involved a second intervention of half the original sample two years after the initial intervention. This research is referred too as Study 3. Firstly, utilising all coded responses, the QUEST Analysis Program was administered, utilising the Rasch partial credit model, to provide a quantitative interpretation of the various categories of responses that emerged. This chapter presents a comparative analysis of responses to tasks concerning relationships among figures and those concerning relationships among properties.

A longitudinal perspective is taken in Chapter 7 in the form of four student case studies. The characteristics of student responses over the two-year period are compared concerning relationships among figures, and relationships among properties. In particular, the case studies provided insight into the validity of the frameworks emerging in Study 1 and Study 2, while highlighting individual similarities and differences.

Chapter 8 provides an outline of the possible limitations of the study and a global overview of the research findings as they relate to the five research themes explored. In the light of the findings contained in this summary, the implications to the van Hiele Theory, the SOLO model, and teaching are considered. Finally, future research directions are generated as a consequence of this study.

CHAPTER ONE

REVIEW OF GEOMETRICAL FRAMEWORK

> It is not sufficient to say that a student is not at Level 3 if he/she does not believe a square is a rectangle. Class inclusion is not simply a part of a natural mathematical development. It is linked very closely to a teaching/learning process ... The main feature of Level 3 should not, in my view, be the acceptance of class inclusion but the willingness, ability and the perceived need to discuss the issue.
>
> Pegg (1992a, p. 24)

Introduction

The quote by Pegg (1992a) is written within the context of the levels associated with the van Hiele Theory. This theory hypothesises a five-level framework to describe students' thinking in Geometry. The views concerning behaviour characteristics of Level 3, in particular those relating to class inclusion, may not be accepted by all in the field. For example, some studies which measure levels of thinking via a written test assign Level 3 thinking to a response if the student accepts a single statement involving class inclusion with no means for justification (e.g., Usiskin, 1982).

The resolution of such issues of definition is an important focus of research as it attempts to move theory beyond possible over-simplistic descriptions. Unfortunately, while there has been considerable empirical evidence concerning more holistic aspects of the van Hiele levels, little work has been undertaken which targets underpinning elements of the theory, such as a focused analysis of the understanding and development of class inclusion notions.

This thesis takes up this theme by investigating within the van Hiele Theory the development of ideas associated with class inclusion. To provide a basis for this work, this chapter presents, under four sections, the necessary background. The first section reviews the van Hiele Theory and provides an overview of its three main components. The second section reviews the features of, and research directed at, the characteristics of the van Hiele levels of thinking. The third section includes a detailed discussion of van Hiele's description of Level 3 and past research associated with this level of thinking. The final section titled Conclusion, presents a summary of the key aspects of the literature and clarifies the emerging research themes.

THE VAN HIELE THEORY

The van Hiele Theory was developed in the 1950s when Pierre van Hiele and Dina van Hiele-Geldof undertook companion Ph.Ds while studying at the University of Utrecht. Pierre's doctoral thesis focused on middle-school students' mathematical insights and levels of thinking in Geometry. Dina's doctoral thesis focused on the learning experiences and teacher's role within the classroom during a year-long study involving the documentation of 12-year-old students' progress in terms of van Hiele levels. Her work resulted in the notion of a phase-based approach to lesson planning.

As a consequence of the van Hieles' belief that the content covered in secondary school Geometry was extremely difficult for many students, they linked their work on levels and phases. This integration provided educators with a tool for identifying students' current level of understanding in Geometry, and a teaching sequence to assist the student in moving to the next level of understanding.

Pierre van Hiele (1986, p. 39) identified the difficulties many teachers experience when teaching Geometry, as highlighted through reflection on his own teaching experience. Here "subject matter seemed to be too difficult" and it appeared that he was always "speaking another language" than that of the students he was teaching. Through the consideration of these issues, the van Hieles' work focused on: the importance of insight in learning Geometry; the identification of levels of thinking in Geometry; and, a five-phase approach to instruction which assists students' transition from one level to the next (Fuys, Geddes, & Tischler, 1985, p. 6).

Insight

> Insight is, as it were, the foundation for later thought; success for a great part depends on it.
>
> van Hiele (1986, p. 161)

Central to the work of van Hiele (1986, p. 24) is the notion that learning is evident when a student displays *insight*. Van Hiele defined insight as acting in a *new* situation *adequately* with *intention*. For insight to occur the student must have a sense of ownership of the mathematics needed by being able to apply this knowledge to the questions posed. Students cannot achieve insight by simply reproducing known algorithms or rote-learnt information.

The following statement by van Hiele describes three characteristics of insight:

> Insight can be observed when there has been an adequate action in a new situation. Insight can be ascertained when there has been action on the strength of an established structure from which the answers to new questions can be read. The best examples of insight happen unexpectedly; they are not brought about by planning.
>
> (van Hiele, 1986, p. 154)

The philosophical stance taken by van Hiele in regards to teaching Geometry is embedded within the notion of insight. This opportunity to exhibit and develop insight is described by van Hiele (1986) as the aim of teaching mathematics. Thus, for the promotion of growth in understanding, learners require geometrical tasks that allow them to control their individual problem-solving environment.

In general terms, insight exists when "students understand what they are doing, why they are doing it, and when to do it" (Hoffer, 1983, p. 205) while solving new or non-routine problems (van Hiele, 1986, p. 154). Hoffer went on to describe van Hiele's definition of insight as evident when a person:

> a) is able to perform in a possibly unfamiliar situation;
> b) performs competently (correctly and adequately) the acts required by the situation; and,
> c) performs intentionally (deliberately and consciously) a method that resolves the situation.
>
> Hoffer (1983, p. 205)

Hence, if the opportunity to exhibit insight is to be provided, an environment that encourages students to take on unfamiliar problems is advantageous. Although it may be possible for students to exhibit insight while completing routine questions, it is more observable, and better promoted, through tasks that are unfamiliar to the student (Pegg, 1997a). Such tasks also aim to develop richer mental structures from which students can draw in the future. Thus, the more frequently students experience insight the more sophisticated should their mental structures and mathematical knowledge become.

While van Hiele (1986) described the teacher's role as devising situations where the student is provided with the opportunity to exhibit insight, the indication of a lack of insight also provides valuable information. Providing the task is not targeted at a level well beyond the reach of the student, a perceived lack of insight can indicate those areas that are more difficult for individual students.

In summary, the display of insight indicates a student's understanding of the structure at a particular level. The more opportunities provided for students to display insight, the greater the ownership of their mathematical ideas. When attempting to place students in situations where it is possible to display insight, it is necessary to identify the level of thinking that students are currently operating at so as not to aim the task at a level that is beyond their reach. Through the completion of tasks that require insight, students develop a richer structure, which results in a stronger base from which the next level can evolve.

The van Hiele Levels

The van Hiele Theory hypothesises five levels of thinking in Geometry. The levels are hierarchical and provide a structure for describing student cognitive growth. A description of the five-level framework is provided in Table 1.1. The descriptions are related to 2-D geometrical figures and are adapted from Pegg and Davey (1998).

Table 1.1 Descriptions of the van Hiele levels

Level 1:	Figures are judged by their appearance. A figure is recognised by its form or shape. The properties of a figure play no explicit role in the identification of a figure.
Level 2:	Figures are identified by their properties. The properties, however, are seen to be independent of one another. For example, the properties are not organised in such a way that students realise that a square is a rectangle.
Level 3:	The properties of figures are no longer seen as independent. There is seen to be an ordering of the properties, with one property preceding or following from other properties. Relationships between different figures are also understood.
Level 4	The place of deduction is understood. Necessary and sufficient conditions can be employed. Proofs can be developed, not simply learned by rote. Definitions can be devised.
Level 5	Comparison of various deductive systems can be undertaken. Different geometries can be explored based upon various systems of postulates.

While the broad descriptions in Table 1.1 are those generally accepted by workers in the field, they represent a change from van Hiele's early numbering. Initially, the van Hieles referred to Level 1 (Table 1.1) as the basic level. They then numbered Level 2 (Table 1.1) as Level 1, and so on.

These numbering changes are more than a minor organisational adjustment from one numbering system, basic and 1–4, to another numbering system 1–5. It represents a change in the way van Hiele (van Hiele, 1986, p. 41) perceived the visual level, as the difference in numbering was described as "caused by our not having seen the importance of the visual level (which is now called the first) at that time." The numbering of levels 1–5 is now generally accepted in the literature.

Other suggestions for changes have not been so well received. Van Hiele (1986) described an alternative level structure comprising three levels of understanding in Geometry, and these encompassed the five levels described in Table 1.1. The three alternative levels proposed by van Hiele are:

First level:	the visual level
Second level:	the descriptive level
Third level:	the theoretical level; with logical relations, geometry generated according to Euclid

(van Hiele, 1986, p. 53)

While van Hiele stated that the levels are "characterised in a different manner" and he chose not to use this alternative structure, the terminology used above to describe the framework, such as *visual, descriptive*, and *theoretical levels,* was utilised and incorporated into his writings. As a result, the potential modifications have caused considerable controversy.

The main problem associated with these concerns is the mapping of the original five-level framework onto the proposed alternative three-level model. For example, Fuys, Geddes, and Tischler (1988, cited in Clements & Battista, 1992) claimed to support the alternative characterisation of the levels and link them to the five-level framework in the following manner; visual (Level 1), descriptive (Level 2), and theoretical (Levels 3–5). Clements and Battista (1992, p. 431) mapped the visual level as inclusive of aspects of both Levels 1 and 2, and concluded "the mapping from one level to another is not unambiguous," while Pegg (1992b) linked Level 1 with the visual level, and Levels 2 and 3 with the descriptive level. To avoid confusion, this chapter adopts the five-level framework as described in Table 1.1.

In summary, while the differing descriptions of the levels and numbering systems in van Hiele's writings has caused unnecessary confusion, the original motivation for, and philosophy behind, the van Hiele Theory is maintained and remains the critical issue. The five–level framework devised by van Hiele provides a template which teachers can apply to assist in the identification of an individual student's understanding of geometrical concepts.

The van Hiele Teaching Phases

The work of Dina van Hiele-Geldof offers valuable information regarding instruction that assists the progression from one level to the next. The five teaching phases represent a framework to facilitate the cognitive development of a student through the transition between one level and the next. The phases originate from the idea that "help from other people is necessary for so many learning processes" (van Hiele, 1986, p. 181). This idea stems from the notion that students find it very difficult to move unassisted from one thought level to the next. The van Hiele model acknowledges that this progress is easier for students with careful teacher guidance, the opportunity to discuss relevant issues, and the gradual development of more technical language.

The following description of the phases, as summarised in Table 1.2, has been translated into English by the Geddes Team and was taken from Dina van Hiele-Geldof's final paper.

Table 1.2 Descriptions of the van Hiele teaching phases

1. Information (Inquiry)	Information by means of representative material gathered from the existing substratum of empirical experiences in order to bring the pupils to purposeful action and perception.
2. Directed Orientation	Direct orientation, which is possible when the child demonstrates a disposition towards exploration and is willing to carry out the assigned operations.
3. Explicitation	Explicitation through which subjective experiences are objectified and geometric symbols are formed.
4. Free Orientation	Free orientation, which is the willing activity to choose one's own actions as the object of the study in order to explain the domain of abstract symbols.
5. Integration	Integration which can be recognised as being orientated in the domain, as being able to operate with the figures as a totality of properties.

(van Hiele-Geldof in Fuys, Geddes, & Tischler, 1984, p. 223)

The phases allow a means of defining and aiding progressing from one level to the next. This does not mean that each time the student passes through the five-phase process within concept development that he/she has reached the next level. The phases do, however, provide students with the opportunity to come closer to meeting the need to move to the next level. It is interesting to note that this teaching process is not centred upon one specific form of instruction. The five-phase process lends itself to many teaching styles, and each phase provides a specific and important purpose. The following discussion provides further detail concerning the purpose of each phase, and examples of classroom activities in the content area of quadrilaterals that may comprise each phase.

The purpose of the first phase, named *Information,* is for students to become familiar with the working domain through discussion and exploration. Discussions take place between teacher and students and stress the content to be used. For example, students may be asked to find and discuss as many quadrilaterals as they can in the classroom. This may be followed by a game entitled 'What is my shape?' where students are required to ask up to twenty questions requiring 'yes' or 'no' answers only concerning the identity of a hidden quadrilateral. During this game the teacher probes for as much information as possible from the class.

The second phase, *Directed Orientation,* is designed for students to identify the focus of the topic through a series of single teacher-guided tasks. At this stage, students are given the opportunity to exchange views. Through this discussion there is a gradual implicit introduction of more formal language. A typical activity may involve each student, working in pairs, making and recording twelve different quadrilaterals on geoboards (pin boards) and dot paper.

During the third phase, *Explicitation,* the purpose involves the student becoming conscious of the new ideas and expressing these in accepted mathematical language. The concepts now need to be made explicit using accepted language. Care is taken to develop the technical language with understanding through the exchange of ideas. An example of an activity suitable to this phase is the students cutting out the recorded quadrilaterals. Through discussion with their partner, the students are asked to classify the quadrilaterals into shapes and justify the groups chosen. Class discussion follows concerning two questions: How have you classified the quadrilaterals? and, What could you investigate about each group? Another suitable activity for this phase could involve an investigation and discussion of properties through folding, measuring, and constructing.

The purpose of the fourth phase, named *Free Orientation,* is for students to complete activities in which they are required to find their own way in the network of relations. The students are now familiar with the domain and are ready to explore it. Through their problem solving, the students' language develops further as they begin to identify cues to assist them. For example, the students may be shown a familiar scientific flow-chart of living things. In pairs, students are asked to create a similar classification chart for quadrilaterals. Discussion follows concerning chart design and problems encountered.

The final phase, *Integration,* is when the students build an overview of the material investigated. Summaries concern the new understandings of the concepts involved and incorporate language of the new level. While the purpose of the instruction is now clear to the students, it is still necessary for the teacher to assist during this phase. For example, this phase may involve the constructing of a concept map to show relationships among the different groups of quadrilaterals and their properties.

Overall, the five phases assist in maintaining student ownership of ideas throughout the learning process. During this process, students can seek clarification from each other and from the teacher concerning the language used. In particular, language plays a central role. It is only after students have identified and described concepts using their own language that the more technical language is introduced.

In summary, the five-phase teaching approach provides a structure on which to base a program of instruction. The instructional setting provided assists the learner in moving from one level to the next. As can be seen, the phase approach begins with clear teacher direction involving exploration through simple tasks, and moves to activities that require student initiative in the form of problem solving. The phases are organised in such a way that they acknowledge the assumptions underpinning the van Hiele levels, while providing students with the opportunity to exhibit insight.

Summary

The van Hiele Theory provides teachers with a means to improve teaching practices through the organisation of instruction that considers, within a hierarchy of developmental cognitive growth, students' level of thinking. The Theory is built on the premise that the purpose of instruction should be the development of insight. For a student to pass through the levels of thinking, ownership of the mathematical ideas encountered is necessary which, in turn, provides the opportunity to exhibit insight.

The five–level framework hypothesised by van Hiele provides an identifiable structure from which to view students' thinking in Geometry. While debate has arisen concerning the descriptions and numbering of the levels, the framework has achieved the aim of identifying the difficulties faced by students when exploring geometrical concepts. The hierarchical framework has in turn resulted in five hierarchical levels of thinking to enable identification of individual students' level of thinking in Geometry.

To assist the student in movement between levels, van Hiele-Geldof developed a framework of five teaching phases. The teaching phases provide an opportunity for progression to the next level through a series of carefully guided activities, which promote the gradual shift from teacher-to student-directed tasks, student-initiated language development, and problem–solving situations. The phase approach to instruction provides a mechanism for assisting students to progress to their next level of geometrical thinking.

It is the interrelatedness of the three key aspects of the van Hiele Theory, namely, insight, the five-level framework, and the teaching phases, that has resulted in the perceived usefulness of this theoretical framework from which to view cognitive growth in Geometry. However, of the research directed at the van Hiele Theory, most has been focused on verifying the levels of understanding.

LEVELS: FEATURES AND RESEARCH

Research on the van Hiele levels has focused upon the nature and existence of the levels, as well as their underpinning assumptions. This section addresses previous work in two parts. First, the main features of the levels are summarised, and, second, research investigations into the characteristics of the levels are discussed.

Features of Levels

There are seven key features which can be interpreted as underlying important aspects (or properties) of the levels. These properties are summarised in the following sub-sections.

Hierarchical nature

The five-level framework is hierarchical in nature; that is, a student cannot proceed to a particular level of thinking without an understanding at the previous level (Hoffer, 1981). Thus, it is not possible for a student to skip a level. Van Hiele (1986, p. 51) described the levels as being an "hierarchical arrangement" where "thinking at the second level is not possible without that of the base level (Level 1); thinking at the third level is not possible without thinking at the second level." Although the higher level has evolved from an analysis of the lower level, the focus of the level components change as the student's focus differs. Once the student has made the transformation, the new structure is at the student's disposal.

The progression to the next level is more dependent upon learning experiences than on biological maturation, and is usually achieved after instruction, investigation, and discussion (Clements & Battista, 1992, p. 427). Some students may appear to bypass levels when the teacher has simplified the subject matter and rote learning is adopted. In such cases, the students provide responses that suggest they are at the higher level; however, since real understanding has not been achieved, the higher level is not available to the students.

Different level, different language

Each level has its own language (Clements & Battista, 1992). While it is easy to accept that students are unable to use effectively the language of a level they have not yet achieved, the converse of this is also true. It is very difficult for one to return to the language of a lower level. As stated by van Hiele (1986, p. 126), "one must realise that no argument, no matter how accurately it may be built up, gives the security that the hearer receives it just as the speaker meant it." This highlights the barrier that exists in the communication between people at different levels, as often the same terminology can carry different conceptual meanings in different levels (Burger & Shaughnessy, 1986; Fuys et al., 1988; Mayberry, 1983).

While language hinders communication between levels, it provides a vehicle for the development of a stronger base within a level. The role of language is a critical component to assist in the progression to the higher level as "the transition is not possible without the learning of a new language" (van Hiele, 1986, p. 50). For such a transition to take place "discussion is an indispensable phase" (van Hiele, 1986, p. 263). The effect of the transition period is highlighted in the following statement: "at times one half of the class will speak a language the other half is unable to understand: This is unavoidable." (van Hiele, 1955, cited in van Hiele, 1986, p. 40)

Due to language's unique character at each level, the language used by the student is of assistance in determining the student's level of thinking. For example, in the first level, the language used only acts as a means of communicating the identified name of the identified shape and related objects.

At the second level, the language used describes properties of the shape. This comes about from discussion in the first level, where the invariant features of a shape gradually become part of the discussion. One needs to discuss why, what, and how, and begin defining vocabulary that relates to the new level.

When relations become the focus of discussion, the third level has been entered and the student has the ability to undertake simple proofs. To achieve this level, the student "must have made a study of arguments at the descriptive level (Level 2) and have understood that it is possible to arrange such arguments in an order in which each statement, except those at the beginning, is the outcome of previous statements" (van Hiele, 1986, p. 84).

The notion that each level has its own characteristic language in relation to both the terminology used and the contextual meaning highlights some interesting tensions. While discussion is described as a necessary vehicle for progressing to the next level, the difficulties associated with interaction between different levels is an important issue in the classroom situation.

Crisis of thinking

The movement from one level to the next is not a simple process. When students make the transition within a topic, they need to pass through a 'crisis' of thinking. This involves the reorganisation of mental structures, which were necessary for one level, to take on a different form. For example, when considering the progression from Level 2 to Level 3, a structure that consisted of isolated bits of information is transformed to one where the knowledge is interrelated.

Van Hiele (1986, p. 44) described the learning process between the levels to illustrate how the "necessary crises of thinking can be initiated and how the pupil can be involved not to avoid it, but on the contrary to surmount it." In the learning process leading to a higher level, five phases (discussed earlier) assist teachers in identifying a clear starting point for instruction and the fostering of an environment which is conducive to the transformation required to reach the next level.

The transition that is necessary for the development of students' thinking often needs to take place under the influence of an effective teaching/learning program. The teacher "cannot preclude the crisis; he cannot avoid it, for by this crisis the transition to the higher level will be born" (van Hiele, 1955, pp. 289–290). Van Hiele (1986, p. 44) described one of the teacher's obligations as inducing the student with "appropriate subject matter to a thinking crisis at the right moment" while acknowledging that the "duration of the crisis cannot be legislated." To avoid this crisis of thinking means that the student is likely to continue to work at the same level of thinking. The crisis of thinking is a necessary but difficult hurdle to overcome before progression to the next level.

Level reduction

Level reduction involves the transformation of structures of a higher-level to a lower-level structure. The idea of level reduction is omnipresent in learning mathematics and has both positive and negative aspects. Van Hiele (1986, p. 57) described a positive aspect as evident when students have seen the structure of, say, the third level, have discussed it, and have put the relations of the structure into their own words. When they now come to level reduction, they have made their own contributions to the reductions. If necessary they can find their way back to the deserted level. It needs to be made clear that "Mathematicians also make use of reduced structures when they do routine work. But, when doing so they meet unexpected obstacles, they are usually able to return to the structure of the third level" (van Hiele, 1986, p. 88).

The negative aspect of level reduction occurs when the student is introduced to level-reduced knowledge or procedure prior to reaching the level of thinking required to understand and have ownership of the content being addressed. While these students may appear to be working at the required level of thinking, their work may be based on level-reduced techniques and the students may not be equipped to complete unfamiliar questions at a similar level.

Introduction of level reduction techniques at an inappropriate time in the classroom will usually set students up for failure every time they come across problems relating to the particular concept that require variations of the processes applied. Level reduction only takes on meaning when the student initiates it. As stated by van Hiele (1986, p. 149), "only the flexibility to go back to the higher level guarantees insight."

Progression requires instruction, exploration and reflection

To move students from one level to the next requires the teacher to construct an environment where the students move to the next level and leave behind the structure of the previous level at which they have become comfortable and secure. Van Hiele (1986, p. 177) described the teacher's role as being "principally indirect."

It is necessary for the students to make more and more links to the next level which means providing opportunities that place the students in a situation where it is necessary to make the change. This requires an exploration of the new structure and a gradual exposure to the language necessary to communicate effectively within the structure. Opportunities must exist which allow the student to work within the structure and reflect upon the generalisations formed.

Implicit and explicit understanding

Growth through the levels requires learning experiences which facilitate the analysis of elements of the lower levels. Van Hiele (1986, p. 6) described the attainment of the higher level as evident when "the rules governing the lower structure have been made explicit and studied, thereby themselves becoming a new structure."

Van Hiele used the terms *symbol character* and *signal character* when discussing the thinking processes that occur during the period of transformation between two levels. To illustrate the significance of these terms, consider the nature of Level 1 where the student is able to identify a shape by name or recognise geometrical figures by their shape (van Hiele, 1986, p. 95). Initially, the symbol is comprised of visual content, where the visual image is required to prompt the properties, resulting in Level 2 thinking. Gradually, the visual figure is not required, and the student can verbally state the properties without the need for a visual representation of the figure. Hence, the symbol in Level 1 has become a signal character where the figure is now represented by a group of properties.

The second level is entered when an "internal ordering" of first level objects takes place and figures are considered as "bearers of properties" (van Hiele, 1986, p. 96). The external form is no longer the focus: "it is the properties of the figures that determine their external form." The third level of thinking is characterised by the relationships that exist among the properties of the second level. Through these connections, the student is able to "compare and distinguish figures" based on the internal ordering of the second level. "At each level one is explicitly busy with the internal ordering of the previous level" (p. 96).

Discontinuity

One controversial property of the levels as described by van Hiele (1986, p. 49) is the discontinuity between the levels. This becomes evident when one considers the transition from Level 1 to Level 2 as moving from a "level without a network of relations to a level that has such a network" (van Hiele, 1986, p. 47). The movement from one level to the next is not spontaneous in nature.

It is implied that the movement from one level to the next requires a sudden leap, rather than a gentle progression to the next level (Clements & Battista, 1992). According to van

Hiele (1986), the discontinuity is made evident in students' need to acquire a 'new' language to make the transition to the next level. While van Hiele has not provided empirical evidence that supports this assumption, it raises many issues, such as those concerned with the identification of students in transition, periods between levels, and the teaching phases.

Overview

The five van Hiele levels represent a hierarchical structure in which one can view students' growth in Geometry. Underpinning the five-level framework are many underlying characteristics. These are: the hierarchical nature of levels; the differences in language for each level; crisis of thinking; level reduction; that progression requires instruction, exploration, and reflection; implicit and explicit understanding; and discontinuity.

Each of these features is interconnected. The hierarchical nature of the levels suggests students are unable to skip a level. The evolution, resulting in progression to the next level, involves the change from an implicit understanding of some features of the concept at one level, to an explicit understanding of them at the next. The teacher applying the five phases best achieves this. The movement between the five levels requires direct instruction, exploration and student reflection, and can be promoted if instruction is through a series of five teaching phases.

The individual levels have characteristic language and structure, and students may be on different levels for different concepts. Due to the transformation required to move from one level to the next, the student must pass through a crisis of thinking. It is possible for the structure of a higher level to be reduced to a lower level. This allows questions which typically require, say, Level 3 thinking, to be undertaken at Level 2. If the student does not initiate level reduction, the crisis of thinking is avoided and the consequent ability to reach the higher level when necessary is not made available to the student.

Research Directed At Levels

The previous section established the broad framework of the van Hiele Theory. While the features of the levels described provide depth of understanding to the nature of van Hiele's levels of thinking, research specifically directed at the characteristics of each of the five levels is of particular interest.

The work of van Hiele acted as a catalyst in renewing many researchers' interests in Geometry as it provided a relatively simple framework that had a high degree of intuitive appeal. In the early 1960s the van Hiele Theory was met with enthusiastic response from Russian mathematics educators (Wirszup, 1976). It was Freudenthal (1973), a mentor and at various times a supervisor to both Dina and Pierre van Hiele in their doctoral programs, who was the first to provide details of the van Hieles' work to an English-speaking audience in his book titled *Mathematics As An Educational Task.* Freudenthal presented the van Hieles' findings in a manner which asked mathematics educators to challenge their limited view of Geometry, to move away from teaching Geometry by imposing the deductive system onto students and, instead, acknowledge stages of growth and directed instruction at the students' level using an investigative approach.

The second analysis of the work of the van Hieles for an English-speaking audience was by Wirszup (1976). This investigation reported changes in Soviet Geometry during the 1960s, when the Soviet Geometry curriculum was substantially reworked as a result of the research begun by the van Hieles. Wirszup (1976, p. 96) described the lack of success in Soviet Geometry, prior to 1964, as a result of beginning instruction targeted at Levels 2 and 3, thus leading to a limited understanding which was soon forgotten by the students. Another contributing factor during this prior educational era, as described by Wirszup, was that some students were confronted with Geometry requiring Level 4 thinking when they were in fact entering at Level 1 of their development. This situation comes about when students are provided with a formal, deductive introduction rather than one that is directed at their current level of understanding.

The general view of the inappropriateness of the ordering of geometrical ideas was also pointed out by Coxford (1978, p. 324). He stated "in the case of geometry, curriculum materials are distinguished by their lack of consistency in grade placement and sequence of content" which resulted in teachers skimming over Geometry content due to the teaching difficulties which arose. Coxford (1978) described the van Hiele Theory as a means of working through these issues due to the framework's testability and the manner in which it linked student understanding to instruction. As a result of this dissemination of information, major studies were undertaken in the U.S.

This section focuses on the characteristics of the levels as seen through the work of other researchers. In particular, the investigations of five research groups are explored below in an attempt to bring out major themes in the literature. Of particular interest is the work of Mayberry (1981); the three large American projects, namely, the Chicago Project directed by Zalman Usiskin, the Oregon Project directed by William Burger, the Brooklyn Project directed by Dorothy Geddes; and the work at the University of New England directed by John Pegg often carried out in collaboration with Geoff Davey.

Mayberry's Doctoral Thesis

In 1981, Mayberry completed a Doctor of Education at the University of Georgia titled *An Investigation of the van Hiele Levels of Geometric Thought in Undergraduate Preservice Teachers*. The study was designed to meet the following two purposes: (i) "the production of a valid test" to determine students' levels of thinking, and (ii) to contribute to evidence, either for or against "the existence of van Hiele levels" (p. 14).

In order to explore the hierarchical characteristic of the levels, Mayberry, from van Hiele's descriptions of the levels, wrote behavioural descriptions of each level and designed tasks aimed at typifying thought at each of the levels. Mayberry designed tasks which covered seven common geometrical concepts, these being: the square, right triangle, isosceles triangle, circle, parallel lines, similarity, and congruence. The interview sample consisted of 19 preservice elementary teachers.

Findings of the study supported the hierarchical nature of the levels, and identified that a student could operate on different levels of thinking for different geometric concepts. The questions in her tasks were designed to meet the behavioural description of each level as drawn from the articles of van Hiele. The students' behaviour at each level was described by Mayberry (1981) as follows.

Level 1: At this level a student should
(i) recognise and name figures;
(ii) discriminate a given figure from others which look somewhat the same (p. 47).

Level 2: A student on this level should recognise and name properties of geometric figures (p. 48).

Level 3: On this level a student should
(i) give definitions (since necessary and sufficient conditions are not understood, a definition may include superfluous conditions);
(ii) recognise and name relationships;
(iii) recognise class inclusions and implications (p. 48).

Level 4: A student on this level should

(i) supply reasons for steps in a proof;
(ii) construct a proof (p. 49).

Level 5: On this level a student should
(i) understand the role and necessity of indirect proofs and proofs by contraposition;
(ii) be able to manipulate symbols according to the laws of logic (p. 49).

Using the descriptions, Mayberry identified the following behavioural patterns as indicated by student responses when they attempted questions targeting the different levels. These descriptions arose from the range of attempts made by the sample of students in response to the questions, and do not represent typical behaviour of students' thinking at each level. The patterns observed by Mayberry illustrate similarities and differences among students operating at different levels of understanding.

At Level 1, shape recognition was difficult for some students when in non-standard orientations. Providing examples of concepts was easier for the student than naming the concept when provided with specific examples.

At Level 2, the properties of the figures were often not perceived by the students. Some students still attempted to obtain empirical measurements when asked to describe properties, indicating that a selection of the students had rote-learnt geometrical concepts.

At Level 3, the concepts of class inclusion, relationships, and implications of these, were not understood by many students. The students focused on particular figures rather than responding in a generalised form.

At Level 4, tasks requiring the deduction of relevant facts from given statements appeared to be difficult for the students. When provided with a chain of ordered statements, and asked to explain the reason for the logical argument, most students could not respond in a meaningful manner. Instead, students in the sample demonstrated no perception of a proof as a logical chain which leads from the given information to a conclusion.

At Level 5, students were unable to respond adequately to questions concerned with axioms, indirect proofs, and finite geometry. The questions were not understood by the students and many sought clarifications of concepts and terminology.

An important finding of Mayberry's study arose from 13 percent of response patterns which did not meet the criteria for any of the levels, hence, suggesting that "there is a 'zero' level before figure recognition and discrimination occur" (1983, p. 67).

Through Mayberry's (1981) attempts to produce a valid test based on the level descriptions of van Hiele, brief behavioural descriptions typifying each of the levels were produced which supported the hierarchical nature of the framework. By addressing seven different content areas, Mayberry identified that students could be on different levels for different concepts, and suggested the possible existence of a level below van Hiele's Level 1.

The Chicago Project

This project, titled Cognitive Development and Achievement in Secondary School Geometry (CDASSG), was conceived by Zalman Usiskin and Sharon Senk over a period of three years (1979–1982). The investigation grew from the need to investigate the teaching approach towards secondary-school Geometry in the United States where it is studied in a single year, usually around age 15–16 years. Usiskin (1982) was concerned with the measurement of geometric abilities of students as a function of their van Hiele level. The aim of this study was to test the descriptive and predictive nature of the van Hiele levels.

Usiskin investigated the validity of tests designed to determine students' levels of thinking and the usefulness of the theory in predicting student achievement in the future. The sample in his study consisted of 2900 students in ninth grade Geometry in United States schools. In an attempt to "test the theory by writing items that correspond to the van Hiele descriptions of the levels and measuring the extent to which student responses formed a hierarchy" (Usiskin & Senk, 1990, p. 244) a pre- and post-test design was adopted. The test items designed by Usiskin (1982) required the van Hiele levels to be described with clarity and detail to maintain a rigorous instrument. In an attempt to obtain this accuracy, the project included the examination of nine works of van Hiele for quotes that focused upon the "described behaviours of students at given levels" (p. 9).

Responses to the test items designed by Usiskin highlighted the difficulty in coding students at transition. Usiskin (1982, p. 52) commented that "it is quite likely that many students, having had much exposure to proof during a year's worth of Geometry, would be between Levels 3 and 4." Thus the application of a weaker or stronger criterion would result in the assigning of a higher or lower level. One suggestion conveyed by Usiskin in an attempt to resolve this situation in the future is "to assign each student a mean of the van Hiele levels as calculated using the two criteria."

The testability of Level 5 thinking was also questioned due to the vagueness of van Hiele's Level 5 description. For example, one of the behaviours as described by van Hiele includes "logical thinking itself." Usiskin (1982, p. 79) reached the conclusion that "in the form given by the van Hieles, Level 5 either does not exist or is not testable. All other levels are testable."

The difficulty faced by the researcher was that such a description could be interpreted differently depending upon the situation, subject matter, and researcher. Van Hiele's Level 4 description, "the mind is occupied with the significance of deduction, of the converse of a theorem, of an axiom, of the conditions necessary and sufficient" was described as vague by Usiskin (1982, p. 13). Usiskin raised the issue of the difficulty that exists in designing situations where students are given the opportunity to exhibit behaviour that is described by such terminology as "occupied" and "significant."

In contrast, Usiskin (1982, p. 20) described the descriptions of the earlier levels as "in sufficient quantity and detail to enable testing." The following behaviour was identified as characteristic of the first three levels:

> At level 1, one asks whether a drawing fits one's conception of a member of a class of figures. At level 2, one wonders whether a property is true always, not merely in a single figure. At level 3, one orders properties, needing to know whether one statement always follows from another.
>
> (Usiskin, 1982, p. 20)

There have been criticisms (Wilson, 1990) concerning Usiskin's use of multiple–choice questions, and the limited number of questions per level. Usiskin used only five questions per level and was not concerned about students who were at different levels for different concepts. Wilson was also concerned about individual item characteristics. In response to such criticisms, Usiskin and Senk (1990, p. 242) stated that "concern about the low levels of reliability in our tests matches a concern we had, but we felt that the need for items that matched the theory, and our desire for a brief test were of greater importance than modifying the instrument merely to give it higher reliability."

In general, the formulation of a rigorous written test, which targeted questions to specific levels based on a number of the writings of van Hiele, supported the level descriptions and hierarchical nature of the levels. Usiskin noted the difficulties associated with assigning levels to those students in transition and suggested that this could be overcome by a numerical mean. Usiskin questioned the testability of Level 5 and found tasks which targeted Level 4 difficult to design due to the vagueness of van Hiele's description of this level. This study provided further evidence to support the behavioural characteristics of the van Hiele levels.

The Oregon Project

The investigation known as the Oregon Project, titled "Assessing Children's Intellectual Growth in Geometry," was conducted by a team consisting of William F. Burger (Director), Allan Hoffer, Bruce Mitchell, and Michael Shaughnessy. Under the leadership of Burger and Shaughnessy, the study had three main purposes, namely clarification of: the usefulness of the van Hiele levels in describing students' geometrical thinking; the characterisation of the van Hiele levels operationally; and, the use of interview procedures to reveal the predominant level of thinking utilised by a student on specific tasks (Burger & Shaughnessy, 1986).

The first year of the project focused upon the development of an interview script and analysis pack, which could be readily administered by teachers and researchers. The eight tasks were centred on triangle and quadrilateral concepts and the sample consisted of 48 students ranging from grade one to a university major in Mathematics. In-depth interviews were carried out with 14 of the students.

Unlike Mayberry and Usiskin, Burger and Shaughnessy (1986) designed test items that could be responded to at different levels of understanding. The tasks chosen reflected the van Hiele descriptions of the levels, and student activities as described by Dina van Hiele. The items, which were expected to elicit responses indicating Levels 1 to 3, involved drawing, identifying, and sorting tasks. The formal reasoning tasks included an inference game dealing with a mystery shape, and a series of questions examining the role of proof (p. 34). No attempt was made to collect responses exhibiting Level 5 thinking.

The study involved assigning a predominant van Hiele level to students for each separate task, and confirmed the hierarchical nature of the levels while maintaining that movement is not related to age or grade. Difficulty was experienced by the researchers when attempting to assign a level to students who appeared to be in transition, and, hence, did not support the discreteness of the levels as described by van Hiele. In fact, the levels "appeared dynamic rather than static and of a more continuous nature" (Burger & Shaughnessy, 1986, p. 45). This was particularly evident between Levels 2 and 3, leading the researchers to predict the same transitional pattern between Levels 3 and 4. It was observed that students who "appeared to reason at different levels used different language and different problem solving processes on the tasks" (p. 46).

The assignment of levels was based on data from each task resulting in level indicators. A summary of the indicators follows and assists in operationally characterising the van Hiele levels.

Indicators of Level 1 behaviour are the use of imprecise properties and reference to visual cues when comparing drawings of figures, identifying, characterising, and sorting shapes. Irrelevant attributes, such as orientation of the figure, are included and inconsistency when sorting is evident.

Level 2 behaviour is characterised by the explicit use of properties, usually side properties, when comparing and sorting figures. This is done in a manner that prohibits class inclusion. A list of properties is given as descriptions instead of sufficient properties. The properties are often used when referring to the shapes instead of using the identifying name. Burger and Shaughnessy (1986) concluded that students are at Level 2 when they provide too few or too many properties when attempting to provide the set of properties needed to define a figure.

Level 3 indicators include the "formation of complete definitions" and the ability to modify, accept and use new definitions. There is explicit reference to definition and an "ability to accept equivalent forms of definitions" (Burger & Shaughnessy, 1986, p. 44). Sorting allows for class inclusion concepts and the incorporation of a variety of mathematical attributes. There is explicit use of "if, then" statements and the ability to formulate informal deductive statements is evident. However, confusion exists between the role of axiom and proof.

Level 4 behaviour is characterised by the students' seeking the clarification of ambiguous questions and rephrasing of tasks. The behaviour includes frequent conjecturing and verification, while maintaining a reliance on the deductive proof as the "final authority." There is an understanding of the "rules of the components of mathematical discourse" and an "implicit acceptance of the postulates of Euclidian Geometry" (Burger & Shaughnessy, 1986, p. 45).

The work of Burger and Shaughnessy represents a shift from level specific task design, to a focus on different student responses to the same task. The researchers found that similar tasks could be completed at different levels dependent upon the level of the thinking of the student. This finding resulted in the development of a more detailed characterisation of the first four van Hiele levels based on student behaviour (Burger & Shaughnessy, 1986, pp. 43–45). Through the design of a variety of student tasks, which reflected the writings of both Pierre and Dina van Hiele, and the collection of student response data, a more detailed characterisation of the levels in behavioural terms emerged.

The Brooklyn Project

The Brooklyn Project, which was under the direction of Dorothy Geddes, had four major objectives (Fuys, Geddes, & Tischler, 1985, p. 1). These were:

(i) the development and documentation of a working model of the van Hiele levels based on the translation of van Hiele source materials;

(ii) the characterisation of thinking in Geometry of sixth and ninth grade students in terms of levels;

(iii) to determine if teachers could be trained to utilise the van Hiele levels in the classroom; and,

(iv) the analysis of current school texts in the light of the van Hiele Theory.

As a result of the outcomes of objectives (i) and (ii) above, Fuys et al. (1985, pp. 62–78) described in detail the characterisation of the van Hiele levels in terms of specific behavioural level descriptors. These descriptors provided an operational version of the framework in a similar light to the Oregon Project (Burger & Shaughnessy, 1986). The researchers began with a version of the levels drawn from the level characterisations of Wirszup (1976), and van Hiele and van Hiele-Geldof (1958). From this starting point, later work of Dina van Hiele-Geldof and Pierre van Hiele were analysed, leading to specific behavioural descriptors. The characterisation then underwent a third review drawing upon the expertise of Pierre van Hiele, Alan Hoffer and William Burger, and resulted in the following general descriptors:

Level 1: "Student identifies and operates on shapes (e.g. squares, triangles) and other geometric configurations (e.g. lines, angles, grids) according to their appearance" (Fuys et al., 1985, p. 62).

Level 2: "Student analyses figures in terms of their components and relationships between components, establishes properties of a class of figures empirically, and uses properties to solve problems" (p. 65).

Level 3: "Student formulates and uses definitions, gives informal arguments that order previously discovered properties, and follows and gives deductive arguments" (p. 70).

Level 4: "Student establishes, within a postulational system, theorems and interrelationships between networks of theorems" (p. 76).

Level 5: "Student rigorously establishes theorems in different postulational systems and analyses/compares these systems" (p. 78).

The students characterised as Level 1 thinkers, who stayed at Level 1 throughout the instruction modules and post intervention, showed "a lack of analysis of shapes in terms of their parts, lack of familiarity with basic geometric concepts and terminology, and poor language" (Fuys et al., 1985, p. 185). Some of the students showed instability between Levels 1 and 2 as they were able to formulate properties of familiar classes of shapes, such as squares and rectangles, but had difficulty identifying properties of less familiar figures. Although these students were more verbal then the previous Level 1 thinkers described, they used little mathematical terminology and often relied on concrete materials and visual cues to check properties of figures. Students progressing towards Level 3 expressed themselves more confidently and fluently using Level 2 language that had acquired more accuracy in terms of property descriptions. Some indicated that although they could follow an argument or provide one, "they did not seem sure of the power of their deductive argument" (p. 186).

Level 3 thinkers provided explanations which included sub-class relationships and statements relating properties in a logical order. These students provided simple deductive explanations and were able to "formulate definitions and justify necessary and sufficient conditions in given tasks" (Fuys et al., 1985, p. 187). Only one student in the sample was found to be in transition between Levels 3 and 4. This student provided minimum properties after initial prompt, then subsequent definitions required no prompts. Explanations included sub-class definitions, and deductive proofs were used as justifications. Level 3 thinking characterised some of the students' responses when asked for clarification in some situations.

Since the modules covered three different topic areas—properties of figures, angle sums, and area— it was possible for Fuys et al. (1985, p. 189) to explore whether the students were at the same level of thinking across different concepts. The researchers found that "students frequently lapsed to Level 0 (Level 1) but were quickly able to move to the higher level of thinking that had been reached on a prior concept" (p. 189). Thus, supporting this feature of the van Hiele levels.

The work of Fuys, Geddes, and Tischler (1985) provided further evidence to support the behavioural characterisation of the van Hiele levels. This study resulted in the formulation of single general behavioural statements for each of the five van Hiele levels, based on the observation of level specific characteristics in student responses to tasks.

Projects of New England (Australia) Researchers

While extensive international research has been conducted into the characterisation and the usefulness of the van Hiele levels in describing students' geometrical concept development (Clements & Battista, 1992), the majority of research has focused on defining, in detail, behaviour characteristics of each level. Pegg and his colleagues have adopted an alternative approach by comparing and contrasting basic issues concerning the van Hiele Theory with those of the SOLO model of Biggs and Collis (1982, 1991). While the philosophical stances underlying these two theories are different, Pegg has managed to address these hurdles and has offered new insights into geometrical understanding. From a large body of work, four areas of research stand out: a more detailed characterisation of the development of Level 2 understanding; an exploration of thinking leading to, and at, Level 4; the transition from Level 1 to Level 2 thinking; and, a broadened characterisation of the van Hiele framework.

Firstly, a significant extension of the theory involves the splitting of van Hiele's Level 2 resulting in the following levels known as 2A and 2B:

> Level 2A: Figures are identified in terms of a single property (usually sides).
>
> Level 2B: Figures are identified in terms of properties, which are seen as independent of one another.
>
> (Pegg, 1997b, p. 391)

The above two-part description of van Hiele's Level 2 remains consistent with the original characterisation of Level 2. The difference lies in the description of a progression within Level 2 beginning with the identification of a single property and moving to the identification of more than one independent property (Pegg 1992a; Pegg & Davey, 1989).

Secondly, Pegg and Faithful (1995) and Pegg (1997a) have been able to explore the depth of thinking that indicates the early stages of Level 4, and the pattern of development leading to a deeper understanding at this level (Pegg, 1996). Hence, Pegg has provided evidence of a progression of understanding within Level 4 as reflected in the nature and complexity of student responses.

Thirdly, Pegg and Davey (1989) and Pegg (1992b) have been able to provide detail about the transition between Levels 1 and 2. Here, a series of categories have been identified which appear to describe a path of development of students, from working visually in Geometry to being able to apply a single property (Pegg & Baker, 1999; Whitland & Pegg, 1999). The division of Level 2 allows the transition to be identified: "attaining Level 2A represents a culmination in the thinking process of the development of a single concept or property. As such, it represents an important interface between the visual/intuitive thinking at Level 1 and the identification of several isolated concepts/properties at Level 2B" (Pegg, 1997b, p. 391).

Finally, Pegg (1997b) has broadened the level descriptors while remaining consistent with the van Hiele framework, and allowing more inclusive criteria. The broadened characterisation has allowed for a simpler categorisation of questions that are typical of junior secondary school within the van Hiele framework. For example, problems involving a single concept, such as those involving basic equations with 'real' angle measurements, where the diagram acts as a visual cue, could be characterised as Level 2A. A Level 2B question would involve the application of several concepts and would include a series of steps in which the solution path is not known before beginning the problem (Pegg & Currie, 1998). The extension of the level in general terms has resulted in a more inclusive framework which allows the questions more typical of the junior secondary school to be more accurately described using the van Hiele Theory.

In general, the work of Pegg (1997b) has acknowledged the level descriptions of van Hiele and the detailed characterisations described by other researchers. Although the characterisations appear to have taken on more general terms, the descriptions represent a more applicable and useable framework within the context of typical classroom questions. Most significantly, the SOLO model proved to be a useful tool for extending and exploring the van Hiele framework. A detailed discussion of the SOLO model is provided in Chapter 2.

Overview of research findings

Coming out of the five projects described above are three main themes: research into the existence and nature of the five-level framework which has resulted in more clarity and detail about the characterisations of each level; exploration of underlying assumptions of the van Hiele Theory; and appropriateness of test items.

The testability of Level 5 has been questioned due to van Hiele's description of this level. Mayberry's (1981) findings suggest the need for a level below van Hiele's original base level since students have been identified who have not yet reached the first van Hiele level. This feature was also identified by others researchers, such as Usiskin (1982) and Senk (1989).

Both Burger and Shaughnessy's (1986) and Mayberry's (1983) findings highlight the critical role that language plays in the movement between levels. An implication drawn from these findings is that the concepts underlying the language used by the student may be vastly different from that expected by the teacher. Both studies provide evidence to support Dina van Hiele-Geldof's focus on exploration leading to understanding prior to the introduction of specific mathematical language.

Fuys et al. (1985, p. 234) support the "fixed sequence" aspect of the van Hiele model, particularly for the first three levels, thus reinforcing Mayberry's (1983) findings concerning the hierarchical nature of the levels. While this structure is generally accepted, the notion of discontinuity is not accepted by all due to the identification of students in transition. For example, Burger (1985) and Fuys et al. (1985) identified students in transition between Levels 2 and 3. Although this suggests that the levels appear to be continuous, Fuys et al. (1985, p. 234) stated that "there may be in fact a discontinuity in progress." The notion of discontinuity was neither supported nor rejected due to the process of instruction and the guidance of the interviewer. An interesting implication noted by Fuys (1985, p. 460) is that students in transition between Levels 2 and 3 "need guidance about expectations, and the interviewer–teacher can use a meta-language about thinking to communicate such expectations."

The work of Fuys et al. (1985, p. 234) supports other researchers' findings concerning the notion that each level is characterised by its own "special language" (Burger & Shaughnessy, 1986; Mayberry, 1983). This project found that the level of geometrical language often appears to prevent the progression to the next level, hindering students' ability to express ideas and communicate with others at a different level.

Another important characteristic of the levels, which was explored and supported, is that "at each level what was intrinsic at one level becomes extrinsic at the next level" (Davey & Pegg, 1992, p. 235). In a similar light, Davey & Pegg (1992, p. 239) investigated Levels 1 and 2 in terms of descriptions of 2D and 3D tasks with primary-aged (5–12 years) students. Two findings that were highlighted provide further support to the research described above: "(i) the van Hiele levels were not discrete structures like a series ascending plateau; and, (ii) the way students structured their language was linked to their level on a topic." Development of language coincided with the explicit (extrinsic) description of what was previously implicitly understood (intrinsic).

Through researchers' attempts to derive accurate characterisations of the van Hiele levels, a considerable amount of work has been directed towards assessing the appropriateness of different test items, and interview questions. Usiskin (1982) and Mayberry (1983) designed questions to test specific levels, while Burger and Shaughnessy (1986) and Fuys et al. (1985) designed questions that could be answered at a variety of levels.

Gutiérrez, Jaime, Shaughnessy, and Burger (1991) made a comparison of the pen and paper test and clinical interview as tools for determining the van Hiele level of students' thinking. This study supported the use of open-ended questions followed by short interviews allowing for student explanations. The researchers found that "the traditional assignation of students to a single level is a simplistic view which lost part of the richness of the student's answers, so research should be done aiming to develop new methods of evaluation based on the observation of the ability of students in using four van Hiele levels, as a way for obtaining a more complete picture of the student's thinking" (p. 116).

An alternative method to evaluate students' van Hiele levels of thinking was developed by Gutiérrez, Jaime, and Fortuny (1991). This approach acknowledged that there exists a series of steps leading from no acquisition to complete acquisition of a level. Their study involved students' completing open-ended items and then being assigned a numerical score that is related to a qualitative scale of acquisition. This method took into consideration the notion that one item could be answered at different levels. It was identified "that not all students used a single level of reasoning, but some of them used several levels at the same time, probably depending on the difficulty of the problem" (p. 250).

Pegg and Faithful (1995), while focusing on Level 4 thinking and students' preferred level of functioning, also explored the quality of the response. The student task in this study was adapted from Mayberry (1981) and involved a written deductive proof in which the student had to respond to the question: What have we proved? This approach highlighted the effectiveness of evaluating the quality of a range of responses as opposed to applying a true or false marking scale.

Both Mayberry's and Usiskin's work has included tasks that required an evaluation of the question in terms of the target van Hiele level based on developed criteria depicting each level of thinking. Wilson (1990) questioned criterion-based decisions on the basis that different decisions mean different things for different test items. The coherence of the test items to the designated van Hiele level was also questioned, while the criterion chosen was said not to be sensitive to test items.

The descriptive and predictive nature of the levels has been explored which has led to a more detailed and workable description of the first four levels and assessment items. These items have been explored from two directions. The first approach has concentrated on evaluating the questions/tasks in terms of their target van Hiele level, and the second approach has been to design questions/tasks in which different students can respond to the same task at their own level of thinking. Debate has also taken place about the value of open and closed items, and written and interview tasks for both the assigning of van Hiele levels and exploration of the nature of the levels. The literature indicates that open-ended questions followed by short interviews provide the quality needed to assign and explore the framework. The work of Pegg has highlighted the usefulness of the SOLO model to assist in the characterisation and expansion of the van Hiele levels.

While empirical evidence supports the existence and nature of van Hiele's Level 3, the characterisation of this level is particularly interesting due to the observed difficulty students experience when faced with concepts associated with this level. The following section addresses in detail van Hiele's characterisation of Level 3.

LEVEL THREE

Thinking at Level 3 is focused on the relationships that exist among figures and those that exist among the properties. This section provides a detailed discussion of van Hiele's characterisation of Level 3 as well as other researchers' contributions to those areas of research specific to the formation of networks of relationships among figures and properties. The general characteristics of this level are discussed within two sub-sections titled, Van Hiele's Characterisation of Level Three, and Other Researchers' Characterisation of Level Three.

Van Hiele's Characterisation of Level Three

> Reasoning of the third level deals with the structure of the second level. Conclusions are no longer based on the existence or nonexistence of links in the network of relations of the second level, but on the connection that is supposed to exist between those links.
>
> (van Hiele, 1986, p. 50)

This description of Level 3 thinking stated by van Hiele, raises a key issue that characterises this level of thinking. Relationships that were implicitly recognised at Level 2 are expected to become explicit at Level 3.

> The network of relations of the third level can only come about sensibly after the second level of thinking has been sufficiently built up. When this second network of relations is present in so perfect a form that its structure can, as it were, be read from it, when the pupil is able to speak with others about this structure, then the building blocks are present for the network of the third level.
>
> (van Hiele, 1986, p. 112)

The relationships or links which are explicit in the third level comprise two central networks (van Hiele, 1986). These are: "the network of relations in which the figures are interconnected on the basis of their properties" (p. 95), and the network of relations "between the properties of figures, with the manner in which one property may be deduced from another" (p. 96). For this level of thinking to occur students must be acquainted with the network in such a way that relationships that exist among the figures and within the figures are seen to involve the combination of properties "automatically without any need of pictorial representation" (p. 95).

Van Hiele (1986, p. 110) stated that it is only after the properties of a figure become a "totality" that a logical ordering of these properties becomes possible. Thus, van Hiele (p. 111) suggested that when a student investigates logical ordering, and relationships between properties, the student should first be fully acquainted with the properties involved. "The study of the intrinsic properties of relations leads to the third level of thinking" (p. 169).

If students are to argue at the third level, they must first understand the language of the third level. To understand the language at this level "one must have made a study of the arguments at the descriptive level and to have understood that it is possible to arrange such arguments in an order in which each statement, except those at the beginning, is the outcome of previous statements" (van Hiele, 1986, p. 84). The language of the third level is described by van Hiele (p. 86) as more abstract than that of Level 2. It is no longer based on visual description but expresses logical and causal relationships of the structure. The reasoning required at this level of thinking is not possible without the language characteristic of this level.

As stated earlier, the networks of relations, which are the students' focus when exhibiting Level 3 thinking, can be described as those that deal with the relationships among properties within figures, and relationships among figures. When considering the network of relations within figures, van Hiele (1986, p. 94) made a comparison between students operating at Level 2 and Level 3. A student's thinking at Level 2 is described as recognising an isosceles triangle by its properties. When describing the thinking of the third level within the same concept the properties are no longer the object of the study. The connection between the properties is now the focus. The student now thinks of the two equal sides as implying that the two angles are also equal and that the converse of this is also true.

A comparison between Level 2 and Level 3 thinking also demonstrates the progression to focusing on the network of relations among known figures. An example of a Level 2 statement is described by van Hiele (1986, p. 50) as being "if a quadrilateral is a square, it cannot be a rhombus." This opinion may soon change to "if a quadrilateral is a square it is a rhombus, for a square is a rhombus with some extra properties." While this response provides an example of class inclusion which is one characteristic of Level 3 thinking, van Hiele (p. 50) stressed that such statements alone do not provide the evidence that the student has reached Level 3. This type of statement could in fact be the result of "a learning process," or "a submission to a traditional choice." Van Hiele (p. 42) also described the presentation of problems whose answer suggests that the student is operating at Level 3 but the question has allowed the student to be "helped out of the distress by considering the figure as a totality." Both issues raised, highlight that justification from the student is required to ascertain the classification of Level 3 thinking.

Level 3 thinking is described by van Hiele as a necessary prerequisite for deductive thinking. Van Hiele stated that "without the existence of a network of relations, reasoning is impossible" (van Hiele, 1986, p. 110). Hence, the ability to "operate with known relations of figures known to him" (van Hiele 1955, pp. 290–295 cited in van Hiele, 1986, p. 42), which includes the notion of class inclusion and the implications of properties, is a necessary hurdle to be overcome before Level 4 can be entered. Further examples provided by van Hiele (1986) which illustrate thinking at this level are:

1. To "apply congruence of geometric figures to prove certain properties of a total geometric configuration of which congruent figures are a part" (p. 42).
2. To "deduce the equality of angles from the parallelism of lines" (p. 42).
3. "The square is recognised as being a rectangle because at this level definitions of figures come into play" (1958–59 cited in Usiskin 1982, p. 11).

4. "the child ... [will] recognise the rhombus by means of certain properties, because, e.g. it is a quadrangle whose diagonals bisect each other perpendicularly" (1959, cited in Usiskin 1982, p. 11).

While Level 3 thinking involves a focus upon the connections between geometrical properties, it is not possible to assist a student having difficulties within this level by "showing him a visual geometrical structure. The understanding of the connection is not brought about with the help of visual representation" (van Hiele, 1986, p. 86). Reverting to visual methods is referred to as level reduction, and is viewed by van Hiele as becoming a factor of concern when used by teachers to assist in the progression from Level 2 to Level 3.

The example provided by van Hiele (1986, p. 43) to illustrate this type of level reduction, described the use of the letters Z, U, and F to identify pairs of angles that are alternate, cointerior and corresponding, respectively, when dealing with parallel lines and a transversal. Providing a structure is sometimes viewed by teachers as providing a stimulus for entry to the third level. In reality, "this method can turn out to be harmful if the teacher, in his zeal for quick results has those structures 'learned' by the pupils" (p. 43). In this situation the student is given the means for avoiding the crisis of thinking required before entering the third level. While the issue concerning Level 3 and the danger of inappropriately introducing level reduction techniques is addressed, van Hiele does not exclude methods of level reduction from Level 3 when they are initiated by the learner.

In summary, van Hiele described reasoning at Level 3 as when students focus on the network of relations among known figures, and the relationships that exist among the properties of these figures. Van Hiele (1986, p. 110) implied that Level 3 thinking is not possible until the properties which comprise Level 2 thought become a "totality." Hence, until the properties involved in the relationships are first known separately, it is not possible for the relationships among them to be seen. Issues raised include the importance of obtaining justification for relationships identified by students before characterising their thinking as Level 3. Concepts associated with this level of thinking are viewed as difficult for students to grasp and there appears to be a strong element of rote learning (level reduction) by students in an attempt to address Level 3 issues.

Other Researchers' Characterisation of Level Three

The detail provided by van Hiele has enabled researchers to identify characteristics of Level 3 thinking, and to clarify issues flagged by van Hiele as pertinent to this level. This section outlines research that has included investigation into the characterisation of Level 3 thinking built upon the work of van Hiele. The majority of studies (Burger & Shaughnessy, 1986; Fuys et al. 1985; Mayberry, 1981; Pegg, 1997b; Usiskin, 1982) have focused upon the development of a more detailed and workable characterisation of the five-level framework. Since the studies have a predominantly primary and secondary education context, the findings have centred on the first four levels and, given the nature of the results, the findings are most relevant to Levels 2 and 3. The following discussion of these studies focuses on research directed at the general characteristics of Level 3 thinking.

Through researchers' exploration of the nature of the levels and properties associated with the level framework, a more detailed description of Level 3 thinking has emerged. Mayberry's (1981, p. 1) study which investigated the hierarchical nature of the van Hiele levels described Level 3 as being when a network of relations among the properties is formed and the ability to see how one property leads to another is evident. Mayberry did not set out to investigate the nature of such levels and therefore has drawn no conclusions as to the nature of Level 3. Therefore, Mayberry continued the tradition of van Hiele by describing Level 3 thinking as The Essence of Geometry, thus emphasising the significance of the level.

In Mayberry's attempts to construct a valid instrument comprising questions that targeted particular levels of thinking, Mayberry (1981, p. 48) separated van Hiele's description of Level 3 thinking into three parts. Students at this level were described as being able to:

1. Give definitions (since necessary and sufficient conditions are not understood, a definition may include superfluous conditions);
2. Recognise and name relationships;
3. Recognise class inclusions and implications.

The questions designed by Mayberry (1981) expected students to address these three areas. Examples illustrating Mayberry's questions designed to elicit Level 3 thinking are discussed in the section titled Class Inclusion in the next chapter.

Usiskin (1982) also devised test items targeted at Level 3. To produce valid questions directed at this level, Usiskin (1982, p. 4) began with a general description of Level 3 using the words of Hoffer (1983). Level 3 was described as being evident when "the student can logically order figures and relationships, but does not operate within a mathematical system (simple deduction can be followed, but proof is not understood)." Usiskin then examined the writings of van Hiele to find all descriptions of behaviour at this level. This procedure resulted in the design of five items that required a Level 3 response.

The ability to follow and summarise arguments while being unaware of the power of the deductive proof was also identified by Fuys et al. (1985) as characteristic of Level 3 thinking. Support was later provided by Senk (1989, p. 310) who described knowledge at Level 3 as "derived by short chains of reasoning about properties of a figure that are derived from thinking at the lower levels. At level 3, students can follow a short proof based on properties learned from concrete experiences, but they may not be able to derive such proofs themselves."

Further detail on the characterisation of Level 3 was obtained by Burger and Shaughnessy (1986, p. 44) through the analysis of student responses to structured interview tasks. The study was designed to allow students to respond at different levels to the same tasks, and to allow a comparison of student responses in different tasks and across the same task. The analysis of the interview transcripts resulted in the following indicators of Level 3 thinking:

1. Formation of complete definitions of types of shapes.
2. Ability to modify definitions and immediately accept and use definitions of new concepts.
3. Explicit references to definitions.
4. Ability to accept equivalent forms of definitions.
5. Acceptance of logical partial ordering among types of shapes, including class inclusions.
6. Ability to sort shapes according to a variety of mathematically precise attributes.
7. Explicit use of "if, then" statements.
8. Ability to form correct informal deductive arguments, implicitly using such logical forms as the chain rule (if p implies q and q implies r, then p implies r) and the law of detachment (*modus ponens*).
9. Confusion between the roles of axiom and theorem.

(Burger & Shaughnessy, 1986, p. 44)

This detailed operational characterisation of Level 3 thinking in terms of student behaviours provides details of the types of behaviour exhibited when reasoning at Level

3. The indicators elaborate on the three general areas described by Mayberry (1981) as evident when students focus on the network of relationships. For example, the ability to: (i) give definitions incorporates the indicators 1–5; (ii) recognise and name relationships incorporates all of the nine indicators; and, (iii) recognise class inclusion and implications could also incorporate all of the nine indicators.

Level 3 thinking has been described as when "the student logically interrelates previously discovered properties/rules" (Fuys et al., 1985, p7). This was interpreted within the Geddes Project as indicated through an ability to give and follow informal arguments.

Other researchers, such as Pegg (1997b, p. 395), summarised Level 3 thinking as characteristically concerning "the acceptance and use of relationships between figures." His work was targeted at broadening van Hiele's description of Level 3, using the SOLO model. He associated Level 3 thinking with the ability "to have an overview of relevant elements and to form, on this basis, appropriate generalisations" (p. 395).

Specific behaviour described by Pegg (1997b) includes the identification and monitoring of relevant data, and the ability to work with pronumerals, but requiring the security to replace these with real values if necessary. Such capacities enable students to utilise the relationships that exist between different concepts. Thinking at the third level cannot be done in "separate parts. Only by an overview of all the elements and the structure of the relationships can the pattern be understood" (Pegg, 1997b, p. 396).

In summary, researchers' efforts to gain a deeper characterisation of the Level framework have clarified some issues concerning the nature of Level 3. Similarities exist among the characterisations of Level 3 provided by different researchers. These similarities include the formation, and awareness, of a network of relations among properties and figures; the ability to provide descriptions of figures and properties based upon known relationships; and, the recognition of class inclusion concepts, and the implications of these. Students at this level have also been characterised as having the ability to undertake and follow informal arguments; however, at this level there does not exist an awareness of deductive reasoning.

Summary

Research into the van Hiele level characteristics and properties of the levels has resulted in a more detailed description of Level 3 thinking which builds on the initial ideas of van Hiele. This detail has enabled researchers and teachers to identify Level 3 thinking, and assists in the design of tasks (Crowley, 1987) which provide the opportunity for students to respond using Level 3 reasoning.

Studies have supported the existence of a level where thinking is focused upon the relationships that exist among figures and their properties. Through the exploration of van Hiele's description of Level 3, student thinking has been characterised by behaviour, such as providing definitions (although the use of necessary and sufficient conditions is not applied), recognising and naming relationships between properties and their figures, and classifications involving the notion of class inclusion. The latter concept, namely, class inclusion raises some interesting issues. While the ideas that constitute class inclusion are a necessary prerequisite for deductive thought, the concept is recognised as difficult to grasp.

CONCLUSION

Researchers' attempts to validate the existence and nature of van Hiele's five-level framework of cognitive development have resulted in a large empirical bank of data which provide support for the level characteristics. Two important themes have arisen out of the preceding discussion. They concern (i) class inclusion concepts and (ii) the potential of the SOLO model to provide deeper insights into the van Hiele levels.

It is recognised that Level 3 thinking is a major achievement for students in the compulsory secondary school years and a valuable goal for many of these students. The value of such knowledge by students lies in its ability to strengthen and deepen understandings of figures and their properties. At this level, students can have an overview of the properties of figures, and are able to undertake simple deductions. All these skills provide an important base which appears to assist in the retention and useability of geometrical facts. However, there appears to be no specific work that has explored class inclusion notions in a geometrical sense within the framework offered by the van Hiele Theory.

In particular, it remains unclear how class inclusion concepts evolve and how different aspects of class inclusion, such as relationships between individual properties and individual figures, develop. Hence, an important research undertaking would appear to be a tightly focused investigation directed at class inclusion. The next chapter, in part, takes up this theme by exploring the relevant literature concerning class inclusion in geometrical settings.

In addition, while many studies have supported the nature of the van Hiele levels, some researchers have extended the model to suggest additional levels or sub categories. For example, Mayberry and Usiskin have argued the existence of a Level 0, which characterises those students who have been identified as not yet reaching van Hiele's first level. Some have argued that Level 2 actually comprises two separate aspects, which suggests growth within Level 2. Of particular interest in this context appears to be the work of Pegg who has sought to use the SOLO model to interpret and to provide a deeper interpretation of the meanings of the levels, and to refine further characterisations of the levels. This issue is also addressed in the next chapter by describing in detail the SOLO model and how it might be applied to the van Hiele Theory.

CHAPTER TWO

REVIEW OF CLASS INCLUSION AND THE SOLO MODEL

> Many natural kind categories may be hierarchically arranged. Such hierarchies, most of which are class inclusive, have important implications for issues concerning knowledge representation, cognitive economy, and reasoning. Without class inclusion hierarchies, our ability to draw inferences, which, in turn, allows us to make assumptions, predictions, and generalisations, would be greatly reduced.
>
> (Greene, 1994, p. 72)

Introduction

The quote by Greene, above, supports the importance of class inclusion as a prerequisite for deductive and analytical reasoning. Greene described class inclusion as having implications towards the representation of knowledge and an individual's ability to reason effectively. In addition, Greene indicated that an understanding and appreciation of class inclusion enhances greatly the ability to "make assumptions, predictions, and generalisations," thus, acknowledging class inclusion as an important prerequisite for higher-level reasoning.

Van Hiele saw the importance of class inclusion and, as discussed in the previous chapter, described this aspect as part of his Level 3. Chapter 1 also included a detailed description of van Hiele's characterisation of Level 3 and of subsequent research associated with students' understanding and use of relationships among figures and properties. Due to the nature of the third level, the discussion addressed studies relating to the formation of relationships in the geometrical setting. A consequence was that class inclusion has been shown to be a necessary and important component of van Hiele's Level 3.

This chapter takes up the theme of class inclusion and is organised into four sections. The first two sections are concerned with class inclusion. The first of these sections reviews issues relating to class inclusion concepts in general; the second section includes a discussion of issues specifically concerning geometrical class inclusion. The third section considers the SOLO model, which is an appropriate tool to evaluate student responses. The final section synthesises material in this chapter and the previous chapter, and outlines research themes and questions.

REVIEW OF CLASS INCLUSION CONCEPTS

This section considers the literature concerned with class inclusion. While the amount of research in this area is extensive, this review highlights key aspects. In particular, those aspects considered are Descriptions of Class Inclusion, Benefits of Class Inclusion, Existing Controversies, Methodological Issues, and Theoretical Perspectives.

Descriptions of Class Inclusion

This sub-section illustrates the differences in terminology used by researchers when describing class inclusion. In addition, the discussion indicates that the general concept of class inclusion remains consistent. The examples provided by de Villiers are in the geometrical context and are described in this section. For consistency, this study utilises the term 'class inclusion,' while recognising that this term is synonymous with other terms such as 'inclusion classification' and 'hierarchical classification.'

One of the main concerns in Piaget's studies (Piaget, 1965; 1970) and many subsequent studies in this area (Andrews, 1996; Markman, 1978; Piel, 1987) relates to children's classification competence in terms of the construction of necessary knowledge (Ni, 1998). In this context, necessary knowledge is that knowledge required on a daily basis. Thomas and Horton (1997, p. 1060) described the central problem of the class inclusion task as "to determine whether children understand that the larger of two subclasses is part of a larger subordinate class. For example, if presented with four black and three white horses, does the child understand there are more horses than black horses?"

While research directed at the development of class inclusion has included varying descriptions and degrees of class inclusion concepts, Inhelder and Piaget (1964 cited in Ni, 1998) distinguished two levels of classificatory reasoning, based on membership and inclusion. These were described, respectively, as:

> recognising properties that are common to items in a given class, and that differentiate them from members of other classes
> Inhelder and Piaget (1964 cited in Ni, 1998, p. 281)

> ordering classes, relating the intension (the defining properties constituting the identity of a class) of two classes systematically to their extension. The members of the classes are to be identified by both their negative, inferential properties and by those that are positive and observable
> (p. 282)

De Villiers (1993) used different terminology. He spoke of partition and hierarchical classification. Partition classification is described by de Villiers as being;

> the classification where the various subsets of concepts are considered to be disjoint from one another. For example, squares are not considered to be rectangles or rhombus, nor are rectangles and rhombus considered to be parallelograms.
>
> (de Villiers, 1993, p. 2)

Hierarchical classification is described by de Villiers as being;

> the classification of a set of concepts in such a manner that the more particular concepts form subsets of the more general concepts. For example, rectangles and rhombus are subsets of the parallelograms, with the squares are an intersection between rectangles and rhombus.
>
> (de Villiers, 1993, p. 2)

Thus, similar to Inhelder and Piaget, de Villiers distinguished between two classification systems. Partition classification is similar to membership classification in that it is characterised by disjoint classes of elements. Classification of this type does not allow for an element to be a member of more than one group. The second form of classification, described by de Villiers as hierarchical, groups concepts in a manner where more general groupings include subsets. This form of classification recognises commonalities between elements in different groups, and considers these when forming subsets within and across groupings.

In summary, there is consistency in the definition of class inclusion described above. Class inclusion is compared with, however, does not relate to the lower-order classification known as membership or partition, which considers similarities and differences without considering these in order or involving subsets. Class inclusion and hierarchical classification are considered to be synonymous as both involve ordering of classes based on the perceived properties of the elements within the classes. Thus, there exists a perception of similarities within and across classes resulting in classification involving subsets.

Benefits of Class Inclusion

There are several benefits associated with a sound understanding of class inclusion concepts. This becomes evident when relations are perceived to form the basis of mathematics. It is widely accepted that through the understanding of the relations that underlie concepts or tasks one develops future knowledge. Halford (1996, p. 3) described understanding in Mathematics as depending in "part on representing some of the higher-order relations that link mathematical concepts". This sub-section addresses the significant role class inclusion plays in the representation of higher-order relations.

Historically, it is evident that the explicit representation of relations, such as class inclusion notions, plays an important role in higher cognitive processes. Greene (1991, p. 370) suggested that "if a child notices a structure and organises information from a given domain hierarchically, that, in time, he or she may create a generic structure which could facilitate the organisation of new information from other domains." This highlights the usefulness of students taking specific examples which may then be generalised "to create generic structures that can be used and applied in appropriate contexts" (p. 371). White and Mitchelmore (1997, p. 577) describe the development of abstract concepts generally as "the result of a classification process which starts with the identification of similarities between different experiences."

To appreciate the role of relations, Halford (1996, p. 11) outlined the differences between associations and relations. These differences are:

> *In relations* the type of link can vary and is identified by a symbol. This makes a relation accessible to other cognitive processes, so that a relational instance can be an argument to another relation ...
>
> *Higher order relations*, have relations, or relational instances, as arguments, whereas first order relations have objects as arguments ...
>
> *Associations* can be chained, and can converge or diverge, but the associative link *per se* cannot be an entity in another association, so there is no associative equivalent of higher-order relations, and associations are not recursive.
>
> (Halford, 1996, p. 11)

This system of schemas was used by Halford (1996, p. 12) as "relational schemas have considerable generality and content independence." The relational representations are independent of specific content as the relational links are represented by a symbol, which is explicitly represented. Relational knowledge is described by Halford as "flexible, explicit, and can be organised into complex structures such as lists, trees, and propositional networks" (Halford, 1996, p. 2).

Such organisation of relational information and hierarchical classification is described by de Villiers (1993) as enabling the following important functions:

- it leads to more economical definitions of concepts and formulation of theorems

- it simplifies the deductive systematisation and derivation of the properties of more special concepts
- it often provides a useful conceptual schema during problem-solving
- it sometimes suggests alternative definitions and new propositions
- it provides a useful global perspective

(de Villiers, 1993, p. 5)

The definitions provided by de Villiers (1993, pp. 3–4) illustrate the links between class inclusion and the ability to systematise, generalise, and specialise. He looked at the concept of classification under four distinct headings:

> Descriptive *(a posteriori)* Classification
> the actual classification (as well as the defining of the corresponding concepts) takes place only after the concepts involved have been well-known for some time (i.e. discovered and explored). The main function of such a classification is therefore clearly that of the *systematisation* of existing knowledge.
>
> Constructive *(a priori)* Classification
> the deliberate utilisation of mathematical processes of *generalisation* and *specialisation* to produce new concepts which are immediately placed in either hierarchical or partitional relationship to other existing concepts.
>
> Generalisation
> when a new, more general concept B is constructed from a concept A by *deleting* certain properties (constraints) or *replacing* some of them by more general ones.
>
> Specialisation
> new, more special concept B is constructed from a concept A by demanding additional properties (constraints) from these concepts. Similarly, a new concept C may also be specialised from two or more concepts by demanding that it *combines* all the properties (constraints) of these concepts.
>
> (de Villiers, 1993, pp. 3–4)

While de Villiers focused upon the characterisation of different forms of reasoning, the descriptions provided illustrate the role that class inclusion plays in higher-order cognitive processes. The ability to systematise existing knowledge is described as the main role of classification. De Villiers described the ability to apply hierarchical classification immediately, or class inclusion notions, as necessary to generalise and specialise.

While the formation of relations is fundamental to an understanding of class inclusion, Halford (1996) distinguished between a relationship and a link. Relations are different from associations in that they become accessible to other cognitive processes. Halford (1996) described the difference between relations and associations, as the associate link does not become an entity that is transferable to other conceptual networks. The process of organising information or concepts hierarchically in one domain may result in a generic structure, which can be transferred to other domains (Greene, 1994). The main role of class inclusion is described in terms of the systematisation of knowledge (de Villiers, 1993) where deliberate application of subsets is required for specialisation and generalisation.

In summary, an understanding of class inclusion notions, in any context, provides the learner with a structure, which maintains the focus on the relations involved, as opposed to the need to rely on specific examples. Class inclusion is described as a precursor for higher-order reasoning, such as forming generalisations and systematising existing knowledge.

Existing Controversy

Although consensus has been reached by researchers concerning the importance of class inclusion for cognitive development, there exist three areas of controversy. This subsection considers each of these, namely, when class inclusion develops in the learner, the actual characterisation of class inclusion, and the factors affecting class inclusion.

First, considerable research has focused upon determining what children know about class inclusion concepts. There is, however, still much debate concerning the stage of learning in which the concept can be understood and utilised. Markham (1989) reported that children between the ages of two and three years showed an understanding of class inclusion notions. This finding is in conflict with the theory of Piaget where the criteria are more restrictive and class inclusion is described as requiring mathematical sophistication.

The controversy surrounding when children understand the relations involved in class inclusion has resulted in discrepancies in terms of the various ages in which the class inclusion hierarchies are said to occur. Much of the debate is due to arguments relating to what actually comprises notions of class inclusion. Greene (1994, p. 73) took a moderate position and described researchers as accepting that an "understanding of hierarchical relations is not possible until a child is well into the concrete symbolic" (age 8–9 years old). Ni (1998, p. 281) described Piaget as assuming that "the transition from the pre-operational to the concrete-operational stage entailed attaining a mature understanding of class-inclusion relations."

It is evident that there are numerous examples of tasks that can be described as requiring the application of class inclusion classifications. Due to the variety of contexts and sophistication required for the different tasks, it is not surprising that researchers have not reached a consensus on when class inclusion becomes available as a problem-solving tool.

Piel (1987, p. 8) described class inclusion as "the understanding that an object can be both a part in itself and a member of the whole group simultaneously." The child who does not effectively consider class inclusion will deal with the 3 + 2 = 4 + 1 dilemma by suggesting that the above statement is inaccurate because 4 is greater. On the other hand, the child who is successful in understanding the class inclusion principle will declare, "it doesn't make any difference how you arrange the subsets the result is still the set of five."

Campbell (1991 cited in Greene, 1994, p. 73) described two versions of class inclusion which assist in bridging the gap between the discrepancies: "In the stronger version, termed the exact composition model, Campbell holds that inclusion has extensive mathematical prerequisites, in the weaker version, Campbell suggests that class inclusion has minimal mathematical prerequisites and, in fact, the arithmetic and/or simple algebra involved are the consequences of understanding the logical relations." Through this statement, Campbell acknowledged the wide range of concepts of varying sophistication that involve an understanding of class inclusion. Hence, it is evident that relational processing is inherent in many tasks.

Most investigations into students' understandings of class inclusion are based on results from student responses to standard Piagetian class inclusion problems, and various sorting tasks. For example, Lawrence (1980, p. 383), designed a task which included both number-based and colour-based class inclusion questions. The task required the student to be presented with pictures of two red tomatoes and two green apples. The number-based class inclusion question was: "Are there the same number of tomatoes as things to eat?" The colour-based inclusion question was: "Is the reddest tomato the same as the reddest thing to eat?"

Another typical Piagetian class inclusion question as described by Piel (1987, p. 8) involves the student being provided with an illustration of three daisies and two tulips. The questions asked during the task were:

a) How many daisies do you see?

b) How many tulips do you see?

c) How many flowers do you see?

d) Are there more daisies or are there more flowers?

e) Why?

This task was used to identify those children who responded that "there are more flowers" and, therefore, who determined that there are both daisies and tulips in the set of flowers. Piel (1987, p. 6) stated that this depicted an understanding that "an object can be both a member of a subset and a member of the larger set simultaneously." While Piel (1997, p. 6) identified that this was "a skill many second and third graders have not yet developed," a class-inclusion task from a more difficult content area would result in the notion being evident at an even later stage.

Another study by Ni (1998, p. 283), was also designed to assess directly "the roles of content knowledge and operational structure in classificatory reasoning based on membership and on inclusion." It incorporated five class inclusion questions concerning dinosaur content in the form of the Piagetian class inclusion task. For example, one question was: "If there were five meat-eater dinosaurs and three plant-eater dinosaurs, would there be more dinosaurs or more meat-eater dinosaurs?" Here students were expected to respond by considering the dinosaur class as including a subset of meat-eater dinosaurs.

Studies have revealed an "effect of semantic knowledge on children's performance on the standard class-inclusion tasks, but those studies have also suggested that semantic knowledge alone is not sufficient to lead to the logical understanding of class-inclusion relations (Carson & Abrahamson, 1976; Chapman & McBride, 1992; Lane & Hodkin, 1985). Hence, this conclusion also suggests that it is not possible to reach a global consensus in terms of biological development, or cognitive development to predict the appearance of understanding class inclusion. This study highlighted the role played by semantics and knowledge of the relevant concepts.

Markman's studies (1973, 1978) also provided an interesting demonstration of the effect of semantic knowledge. Markman identified that children were more likely to solve class-inclusion questions involving objects forming collections such as 'family' or 'forest,' than questions that involved objects forming classes. Thus "earlier studies suggest that semantic knowledge plays an important role in the development of classificatory reasoning, but semantic knowledge itself is not sufficient to lead to the understanding of class-inclusion relations" (Ni, 1998, p. 283).

Rather than focusing on semantics, Ni (1998) found that two important factors affecting classificatory reasoning are cognitive structure and content knowledge. He saw these as in interactive, as well as distinctive ways. Ni (1998) concluded that there are different levels of classification, where classification by inclusion is at a higher level than classification by membership.

In summary, when considering the debate concerning the composition of class inclusion and when it becomes available to the learner, a number of issues are raised. It is evident that both composition and timing depend upon factors such as the sophistication of the content, the familiarity of the context, and semantic knowledge. Much research has centred upon this debate and researchers have administered tasks in an attempt to meet the issues that have arisen. Many of these tasks have been Piagetian in style, while others have been less restrictive. This has resulted in two versions of class inclusion, of differing levels of sophistication due to variations in concepts. In the case of Mathematics, the first version requires little prerequisite knowledge, while the second version requires extensive mathematical knowledge. Thus, the latter is described as involving an understanding of the consequences of logical relations. As a consequence of the second version, research highlights the role that understanding of concepts and related semantic knowledge plays in the development of class inclusion notions.

Methodological Issues

Attempts have been made by researchers to address the issues relating to class inclusion described in the previous section. Various methodological issues have also been raised through these attempts. Halford (1996, p. 3) stated that "psychologists have made little attempt to investigate relational knowledge systematically." This section addresses methodological issues raised by researchers involved in the investigation of class inclusion ideas. This methodological debate stands aside the controversies flagged in the previous sub-section.

Rabinowitz, Howe, and Lawrence (1989, p. 379) conducted three experiments to investigate the relationship between reasoning and memory, and the sub-skills used in responding to class inclusion questions. The investigation, which included ten-year-olds, thirteen-year-olds, and college students, reflected the "need for a new conceptualisation of the class inclusion task. Performance seems to be dependent on subjects' abilities to integrate relevant sub-skills, consistent with a resource limited, willed-attention, working memory model." Norman and Shallice (1986, cited in Rabinowitz, 1989, p. 380) hypothesised that class inclusion performance deteriorates as the load on working memory increases. This deterioration reflects primarily the manner in which the subjects interpret the class inclusion question rather than faulty memory.

Greene (1989, 1991), reported in Greene (1994, p. 74), provided passages which could be represented as "four level class inclusion hierarchies" and "provided the children with tree diagrams which accurately depicted the passages." This was a result of a belief that Piagetian class inclusion tasks could detract from obtaining a true picture of children's understandings "by eliciting schemas which might detract from the intended goal" (Greene, 1994, p. 74). The tree diagram was used in conjunction with a question task and construction task to ensure that testing focused upon conceptual understanding rather than measuring memory capacity (Greene, 1994, p. 83).

While Greene (1994, p. 86) described the ages of four–to–seven years as "critical in the development of children's understanding of the relations in multilevel class inclusion hierarchies," he also reported that children at this time "need to learn about the asymmetry relation and about the significance of a branch or a partition." Both drawing and construction tasks help experimenters understand the nature of children's representations. They are also thought to provide converging evidence for the results obtained when employing more traditional question and sorting tasks, and often uncover knowledge that are not normally attributed to young children. Thus, a logical step toward a fuller appreciation of children's abilities in this area is to move into a conceptual focus, rather than a methodological one.

Debate has also surrounded the need to provide justifications for judgements made when solving class inclusion tasks. Case (1985, as cited in Thomas & Horton, 1997, p. 1071) "required a child to provide both a correct judgement and an explanation, which reduces the child's capability of passing a task without possessing the presumed cognitive structure." Case (1985) also was concerned with the central issue of whether children perform at the same cognitive level on different tasks.

Thomas and Horton (1997, p. 1060) stated that the lack of knowledge concerning children's understanding of class inclusion concepts may be "largely rooted in the use of a judgements only criterion." Early studies focusing on class inclusion, such as those by Piaget, tested the child's understanding via clinical methods. This method was rejected by Brainerd (1973) who argued "that only the child's judgements, not the child's justifications or explanations accompanying these judgements, should be used as the basis for assessing the child's understanding. Basing decisions on explanations, he argued, would systematically underestimate the child's cognitive confidence" (Thomas

& Horton, 1997, p. 1060). This opinion was also supported by evidence provided by Siegel (1978 cited in Thomas & Horton, 1973, p. 1060) who considered non-verbal alternatives to Piagetian tasks.

Other researchers, such as Reese and Shack (1974, p. 67), however, did not support this methodology and argued that "limiting one's focus to one response variable would unduly restrict our knowledge at this early date." This raises the issue of eliciting optimum responses to class inclusion tasks. Through an acknowledgment of the processes applied, researchers identified three strategies adopted by children to respond to class inclusion items (Chapman & McBride, 1992; Hodkin, 1987 cited in Thomas & Horton, 1997, p. 1061), namely, "they can reason using sub-class comparison, they can guess, or they can reason with inclusion logic."

The methodology employed makes a difference. As an alternative to the Piagetian account, Chi (1978) reported a proposed knowledge-based account (cited in Ni, 1998, p. 281). Chi argued that classification ability is a function of automatic activation of a content knowledge base. Chi (1978) found that when placing students in a new and novel context, "the expert children's ability to form hierarchical classifications for dinosaurs was argued to exist not because they had available general classification structures, as Piaget assumed, but because their knowledge was already organised so as to permit a retrieval of this structure" (Ni, 1998, p. 281).

Other researchers have carried out a methodological analysis of tasks which attempted to investigate students' understandings of class inclusion. For example, Brainerd and Kaszor (1974) have found that perceptual set factors and question misinterpretation are not significant determinants of class inclusion performance. As a result, class inclusion was understood to be evident relatively late in the concrete-operational stage of acquisition.

De Villiers (1993, p. 2), who studied "the role and function of hierarchical classification of quadrilaterals," stated that the problem seems "to be not so much that of a lack of relational or logical understanding, or even of a proficiency in defining, but one of a lack of functional understanding (i.e. what is the function or value of a hierarchical classification of quadrilaterals)." When investigating class inclusion de Villiers found that the "classification of any set of concepts does not take place independently of the process of defining, but implicitly (or explicitly) involves defining the concepts involved" (p. 2).

De Villiers (1993, p. 3) also stated that partition classification and definitions are not mathematically incorrect, and are "sometimes useful and necessary to clearly distinguish between concepts." While such classifications are described as being a spontaneous and natural strategy, this needs to be taken into consideration when designing tasks to investigate class inclusion. "Since the classification and its corresponding definitions are arbitrary and not absolute, we should acknowledge that the choice between a hierarchical and a partition classification is often a matter of personal choice and convenience" (p. 3). De Villiers' study focused upon the advantages of hierarchical classification (notions of class inclusion) when classifying quadrilaterals, as opposed to partition classification. One important finding of de Villiers (1993, pp. 7–8) was that "several children's difficulty with hierarchical class inclusion (especially older children) does not lie with the logic of the inclusion as such, but with the meaning of the activity, both linguistic and functional: linguistic in the sense of correctly interpreting the language used for class inclusions, and functional in the sense of understanding why it is more useful than a partition classification."

The methodological issues surrounding the investigation of students' understandings of class inclusion notions have been an important focus for many researchers. Such work has highlighted the need to consider factors such as students' justifications as well as judgements. There is also a need to accept that the initial useful natural response to a task may not indicate the full classificatory capabilities of the student, acknowledging students' content base, and investigating for classificatory skills rather than memory.

Theoretical Perspective

The debate concerning the characterisation of class inclusion and its emergence has acknowledged distinct differences in classification based on membership and class inclusion notions. There is also a perceived variance by researchers concerning the time of emergence of the ability to classify hierarchically and utilise such notions dependent upon the complexity of different concepts. While there has been considerable research in the above-mentioned areas, investigations which focus upon the identification of the evolution of class inclusion concepts are limited.

Inhelder and Piaget (1964) described component behaviours found in the preoperational and concrete operational periods, which culminate in class inclusion skills as summarised by Kofsky (1963, 1966). Hooper, Sipple, Goldman and Swinton (1974, p. 3) described the summary provided as "an excellent theoretical discussion of classification skills from the Piagetian orientation." Kofsky summarised the views of Inhelder and Piaget (1964) regarding the culmination of classification skills as follows:

> On the basis of their hypotheses, development appears to proceed in 11 partially ordered steps. They contend that classification begins when the child groups together two objects that are equivalent because they look alike in some way (resemblance sorting). As the child grows he learns to extend the scope of his grouping from two, to more than two (consistent sorting), to all the objects that could be considered equivalent in some respect (exhaustive sorting). The child also learns which are acceptable categories for grouping. Physical proximity becomes a less favoured means of categorising since the resulting groupings are transitory (conservation). Experience in constructing one class at a time prepares the child for more successive and simultaneous classifications and for understanding class inclusion. Slowly the child begins to recognise that objects do not belong exclusively in different categories (multiple class membership), and he actively tries out different groupings of objects, choosing first one and then another single attribute as a focus for grouping (horizontal classification). As his logical abilities develop, his method of choosing criteria becomes more complex. He chooses single attributes to construct successive classes (hierarchical classification). His use of combinational structure (Inhelder & Piaget) enables him to form classes that stand in an inclusion relationship to each other.
>
> (Kofsky, 1966, p. 192, cited in Hooper et.al., 1974, p. 3)

From this, Kofsky developed eleven experimental tasks in an attempt to assess the hypothesised hierarchy of development. The tasks were administered to children aged between four and nine years whose performances could be grouped into the following six levels.

> Level 1: resemblance sorting and consistent sorting, i.e., the ability to match and sort objects on the basis of perceptible attributes;
>
> Level 2: exhaustive sorting, in which all blocks sharing a common attribute were separated from a mixed array, and an understanding of some–all relationships (i.e., after presenting an array of nine blocks which consisted of four blue squares and two blue triangles, and three red triangles, the child was asked a series of questions such as "Are all the triangles red?;
>
> Level 3: a knowledge of multiple class membership in a task setting which included triangular-shaped blocks which varied in two sizes and two color dimensions, and an understanding that the overall number of objects in two sub-classes equals the number in the superordinate class;
>
> Level 4: conservation of classes in which the child had to continue to associate a nonsense syllable label with a specific geometric form in spite of irrelevant transformations, conservation of a class hierarchy (i.e., with an array of two blue and six red square blocks the child is asked, "If I took away all the reds, are there just blues left, just squares left, or both blues and squares?"), and horizontal reclassification in which an array of triangle and square shaped blocks in four colors were sorted and resorted according to the differing potential criteria;

> Level 5: class inclusion skills which were assessed with the same stimulus array as the "some and all" task, and asked the child questions such as "Are there more triangles or blues?"; and,
> Level 6: hierarchical classification skills in which the child had to demonstrate that in an array of four red and three blue triangle-shaped blocks, all the blocks shared one attribute (shape) but that any one of the blocks had an additional attribute (color) shared by only some of the blocks in view.
> (Kofsky, 1966, cited in Hooper et.al., 1974, p. 4)

The six developmental stages described above begin with the ability to match and sort through the recognition of perceived attributes. This stage is followed by the ability to group objects based upon a common identifiable attribute. The third stage concerns the ability to regroup a previously grouped collection of objects and recognise that the groups formed within it together equal the number in the first group formed. The student then develops the ability to classify a group of objects in more than one manner, hence, s/he is able to reclassify and name different classes with differing elements. At this stage, an object can be classified into more than one group depending upon the criteria chosen. The following level involves the ability to respond to questions in the light of 'all' and 'some' based upon the different classifications possible.

Andrews (1996) undertook a project that tested "the proposition that age-related increases in reasoning ability are associated with the ability to represent relations of increasing complexity" (Andrews, 1996, p. 3). The conclusion was that "developmental changes in class inclusion are, at least in part, a consequence of how automatically subjects can perform the necessary sub-skills rather than whether they can perform them at all, given unlimited time" (p. 3). Thus, the ability to initiate and apply hierarchical classification when appropriate indicates an understanding of class inclusion notions. Hence, Andrews suggested growth in the development of class inclusion concepts moving from the ability to move from membership classification to hierarchical classification, to the ability automatically to apply class inclusion to a given task.

Summary

In summary, it is widely accepted that the ability to represent relations explicitly is a necessary prerequisite for higher-order reasoning processes, such as forming generalisations and making predictions. Class inclusion is an essential component of such thought processes. An understanding of class inclusion notions allows one to maintain on overall view based upon a focus on the existing relationships without the need to rely on specific examples. Controversy exists in terms of the biological time in which class inclusion becomes available to the learner; however, this is reconciled by the acceptance that class inclusion is a component of numerous concepts from a range of levels of difficulty (which are encountered at different ages).

A consideration of the methodological issues raised by researchers, most via typical Piagetian class inclusion tasks, shows that there has been an attempt to reach consensus on issues, such as the composition of the concept and biological age when the concept is attained. These investigations have moved researchers' focus to considering different concepts, contexts, and methods of gaining a better grasp of students' classification abilities. There is an identified need to consider students' justifications for their classifications. Attempts to do this adequately include tasks which require the student to represent their classifications through lists and tree diagrams.

GEOMETRICAL CLASS INCLUSION

Research has validated van Hiele's Level 3 description as incorporating the understanding and use of relationships among figures and among properties. This section looks at class inclusion, which is about the interrelationships among figures, and among properties, which combine in a mutually supportive way. To illustrate this point, this section considers several questions and issues raised from individual research projects concerning Level 3 thinking, such as Mayberry (1981), Usiskin (1982), Burger and Shaughnessy (1986), de Villiers (1993), and Pegg and Davey (1989). This section is divided into two sub-sections titled, Geometry Class Inclusion Items, and Issues Identified by Researchers Concerning Class Inclusion.

Geometry Class Inclusion Items

An understanding of class inclusion in the geometrical context can be described as the ability to have an overview of possible relationships that exist among figures and their properties. The following discussion considers a selection of previously designed items, which considered aspects of students' understandings of class inclusion concepts in Geometry. In particular, closed written items and open-response interview items are discussed. Fuys et al. (1985, p. 236) emphasised the need to be aware that some tasks that set out to ascertain a student's understanding and use of class inclusion can be achieved at more than one level. In such cases, the reasoning behind a response is the critical component. The observation described concerns the justification given by students when showing signs of using the notion of class inclusion.

The justifications for the statement, "all squares are parallelograms" have been placed into two different levels. An example of a Level 2 justification involves "listing all properties of a parallelogram and then checking that the square had each of those properties" (Fuys et al., 1985, p. 236). An example of a Level 3 justification is described as "since all squares have opposite sides parallel, then they must be parallelograms, which they defined as quadrilaterals with opposite sides parallel" (p. 236). The difference between Level 2 and Level 3 is described by Fuys et al. (1985, p. 236) as suggesting that the general descriptors of Level 2 thinking could include "subclass inclusion via properties" while retaining "subclass inclusion by deduction" in the Level 3 descriptors. This example illustrates the interrelatedness of the network of relationships among figures and the network of relationships among properties.

Items designed to assess and provide the opportunity to exhibit Level 3 thinking have varied from closed written questions, such as those designed by Mayberry (1981) and Usiskin (1982), to open-response interview tasks, as provided by Burger and Shaughnessy (1986), and Pegg and Davey (1989).

Mayberry's items

Mayberry (1981) developed a number of questions to target relationships among figures and their properties across seven concept areas. The examples below are selected from the right angle concept and are similar to questions developed for the other six content areas.

Of the five questions designed by Mayberry (1981) to target Level 3 thinking, four are concerned with the relationships among figures and their properties. These are Questions 26, 27, 32, and 44.

The first of the following four questions (Q26) covers the right triangle content area. This question aims to assess the ability of the student to provide definitions demonstrating a focus on the network of relations, while not needing to show an understanding of necessary and sufficient conditions.

Q26. Which combination of the following guarantees a figure to be a right triangle?

a) It is a triangle.
b) It has two acute angles.
c) The measures of the angles add up to 180.
d) An altitude is also a side.
e) The measures of two angles add up to 90.
(After response, ask, "Can you use less?")
(Ask again, "Can you use less?")

Question 27 is described by Mayberry as aiming to assess implications of triangle properties.

> Q27. QAB is a triangle.
> a) Suppose angle Q is a right angle. Does that tell you anything about angles A and B? If so, what?
> b) Suppose angle Q is less than 90. Could the triangle be a right triangle? Why?
> c) Suppose angle Q is more than 90. Could the triangle be a right triangle? Why?
> (Implications)

Question 32 is described by Mayberry as requiring the student to recognise class inclusion and its implications, thus outwardly focusing on the relationships among figures but requiring the student to also consider relationships among properties.

> Q32. Which are true? Give reasons.
> a) All isosceles triangles are right triangles.
> b) Some right triangles are isosceles triangles.

The final example of Mayberry's items concerning Level 3 is described as targeting thinking directed at recognising and naming relationships.

> Q44. c) Will figures A and B be congruent
> i) always
> ii) sometimes, or
> iii) never?
> A: a right triangle with a 10cm hypotenuse
> B: a right triangle with a 10 cm hypotenuse

While some of Mayberry's Level 3 items, have been designed with the intention of specifically targeting aspects of the relationships among figures, or among properties, a Level 3 solution requires the student to consider the interrelationship of both networks of relations.

Usiskin's items

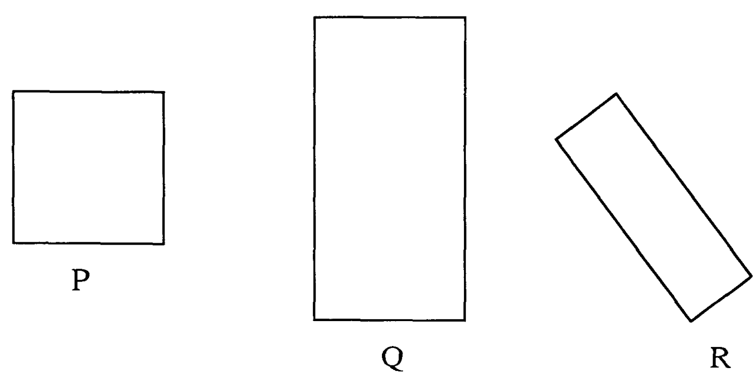
Usiskin (1982, pp. 161–162) included five questions that attempted to target Level 3 thinking. These are Questions 12, 13, 14, and 15.

Question 12 targets the network of relationships among properties, but by incorporating both isosceles and equilateral triangles, the relationships among figures also becomes a focus.

12. Here are two statements.
Statement S: Δ ABC has three sides of the same length.
Statement T: In Δ ABC, ∠B and ∠C have the same measure.
Which is correct?
(A) Statements S and T cannot both be true.
(B) If S is true, then T is true.
(C) If T is true, then S is true.
(D) If S is false, then T is false.
(E) None of (A)–(D) is correct.

Question 13 aims to target the student's understanding, and use, of relationships among figures. When responding to this question at Level 3 the relationships among the quadrilateral properties interrelate when justified adequately.

13. Which of these can be called rectangles?



(A) All can
(B) Q only
(C) R only
(D) P and Q only
(E) Q and R only

Although the obvious focus of questions 14 and 15 is on the relationships among the properties, when responding to these questions using Level 3 reasoning, students find it necessary to work with the interrelationships among the properties and the relationships that exist among the figures.

> 14. Which is true?
>
> (A) All properties of rectangles are properties of all squares.
>
> (B) All properties of squares are properties of all rectangles.
>
> (C) All properties of rectangles are properties of all parallelograms.
>
> (D) All properties of squares are properties of all parallelograms.
>
> (E) None of (A)–(D) is true.

> 15. What do all rectangles have that some parallelograms do not have?
>
> (A) opposite sides equal
>
> (B) diagonals equal
>
> (C) opposite sides parallel
>
> (D) opposite angles equal
>
> (E) none of (A)–(D)

Similarly to Mayberry, Usiskin's Level 3 items also require the consideration of both the relationships among figures and among properties. The knowledge of the interrelationships between the networks is needed to provide an adequate justification for the answer chosen.

Open-response items

Several researchers devised open-response items to explore class inclusion ideas. Open-ended items, also known as free-response items or constructed response items require the respondent to "*create* a response rather than select it from a list" (Collis & Romberg, 1991, p. 84). "Open-ended and free-response questions ... require the student to generate the correct answer, not merely to recognise it. Such assessment items would ... allow for more reliable inferences about the thought processes contributing to the answer" (Alexander & James, 1987, p. 23). Researchers using open-response items include Burger and Shaughnessy (1986), Jaime and Gutiérrez (1990), and Pegg and Davey (1991).

In an attempt to define Level 3 thinking operationally in workable behavioural terms, Burger and Shaughnessy (1986) designed a series of structured interview tasks which did not target a particular level, but instead, enabled the student to respond at their own level of thinking. The drawing task "investigated the properties that students varied to make 'different' figures and explored whether they thought the possible number of triangles was finite or infinite" (p. 54). The identifying and defining task which focused on quadrilaterals "explored the student's definition of class inclusion" (p. 54).

The sorting activity explored students' perceived relationships among triangles. An inference game involving the gradual revealing of a list of clues "elicited formal inference and addressed the role of necessary and sufficient conditions to determine a shape" (Burger and Shaughnessy, 1986, p. 55). The final task involved deductive reasoning. Results obtained from the analysis of the interview transcripts highlighted the interrelatedness of the two networks of relationships.

Jaime and Gutiérrez (1990) used an open-ended written test in an attempt to obtain the similar quantity and quality that can be obtained from interviews where prompts and probes are used to find more information. The task described as a classification question required the students to be given several figures. For each figure, they had to write all the names in a list that are appropriate for the figure. The list of shapes provided included square, rectangle, rhombus, parallelogram and rhomboid. The students were then asked to explain the assignation of names to the figures. Since the written question included the opportunity for explanation, the task had the potential to draw upon students' understandings of both the relationships among figures and relationships among properties.

When Pegg and Davey (1991) investigated Level 3 thinking, one aspect in their study involved a student's ability to provide descriptions using minimum properties, thus targeting the relationships among properties. The task began by asking students to provide a description of a 2D shape. The students were then asked to write a new description using the smallest number of 'things' to allow correct identification of that shape. In addition, students were asked to provide descriptions of shapes, which utilised other known shapes. As with the other items above, this question can be responded to using Level 3 thinking bringing in both the relationships among figures and the relationships among properties. A feature of the design was that care was exercised to ensure that the responses given by students were not provided as a remembered fact.

Overview

In summary, the items described in this section illustrate the manner in which researchers have attempted to design tasks that target Level 3 thinking. It is evident that tasks have been designed with the goal of targeting either the relationships among figures, or the relationships among properties. The result, in most cases, has required the student to focus on the interrelationships among both figures and their properties.

The closed questions described above illustrate the importance of seeking justification and clarification from the student before assigning a label of Level 3 thinking. This accentuates the importance of the underlying structure of the responses to better describe the level of thinking. The results obtained from short answer questions allowing for explanations and interview tasks have enabled researchers to identify in more detail the components of Level 3 thinking. The issues associated with class inclusion as a result of such items are discussed in the following section.

Issues Identified by Researchers Concerning Class Inclusion

The development of class inclusion concepts strengthens and deepens a student's understanding; however, studies have highlighted the difficulty of the task faced by students in achieving and appreciating sub-class relationships (Burger & Shaughnessy, 1986; Fuys et al., 1985; Mayberry, 1981). In particular, issues arising include: the prerequisite nature of class inclusion leading to formal deduction; various elements, such as visual cues and language, that appear to hinder or assist the formation of class inclusion notions; and, the level of dependence upon teaching and curriculum for the development of class inclusion notions.

Mayberry (1981, p. 83) made the following observations concerning the sample: "first, the students answered the questions for particular figures and not for generalised ones. Thus class relationships were not perceived. Second, the role of a definition as a set of minimum conditions was not really understood." Such studies have acknowledged that the class inclusion concept is difficult for students to grasp and appears to require specific instruction rather than maturation. As a result, class inclusion has been described as a hurdle that must be overcome before formal deduction is understood, and as an important step towards the deductive processes in Geometry.

Pegg and Davey (1991) described two contributing factors to the degree of difficulty in developing the notion of class inclusion. These are the requirement that the students have to overcome powerful visual cues, and rely on a relatively high level of logical reasoning. These factors are for example, apparent when considering the visual cues which have to be overcome before a student will acknowledge the square as a subset of the rhombus class of quadrilaterals. It was found by Burger and Shaughnessy (1986) that students thinking at Level 2 prohibit the use of sub-classes in their descriptions or definitions due to the exclusive nature of their classifications. The precluding nature of Level 2 classifications was illustrated by Pegg and Davey (1991) through an example where students described a rectangle as specifically having two sets of opposite sides that are equal where one set is longer than another. Thus, for these students, even entertaining the notion that the square can be classified as a rectangle was not feasible.

Similar behaviour was described by Hoyles and Noss (1988) as the researchers identified students who were capable of providing correct formal definitions that explicitly or implicitly excluded the square from the class of rectangles. The notion of the rhombus being within the class of parallelograms, however, was more often accepted due to the visual "slantiness" of both figures. Davey and Pegg (1989) identified three possible sources which may contribute to this problem as, the initial attention given to squares and right angles in infant classes, the naturally occurring bias to the horizontal and the vertical, and, finally, the strength given to the belief that squares, rectangles, and parallelograms are three distinct classes of figures within our everyday world.

In an attempt to address difficulties associated with the development of Level 3 thinking, such as those described above, Battista and Clements (1992) identified an important aspect concerning a student's ability to apply the notion of class inclusion to classifications of figures. Through the investigation of the use of Logo computer activities to construct concepts of rectangles and squares, and the relationships that exist between them, Battista and Clements (1992, p. 59) found that the use of Logo aided students in moving towards Level 2 thinking, while acting as an "important precursor for hierarchical classifications." Findings illustrated that "the attainment of Level 3 does not automatically result from the ability to follow and make logical deductions; the student must utilise this ability to reorganise her or his knowledge into a new network of relations" (p. 64). The findings in this study suggested that students accepted initially the organisation of the network prior to adopting and utilising the hierarchy of relationships.

An alternative view was offered by de Villiers (1987). He concluded that "contrary to van Hiele's theory, hierarchical class inclusion and deductive thinking develop independently and depend more on teaching strategy than on van Hiele level" (cited in Clements & Battista, 1992, p. 432). Clements and Battista found, however, that when students were asked to explain why a square is a rectangle they would reply "because the teacher told us" (p. 432). Thus, although some students agreed with the statement involving notions of class inclusion, they were unable to justify their response. Although this view was made by de Villiers, the notion of the levels being independent of the curriculum was noted as worthy of more research.

Further research into the characteristics of Level 3 thinking, has supported van Hiele's notion that within this level, students are aware that it is not necessary to provide all known properties of a figure to enable its identification. A series of questions were asked by Mayberry (1981) of the type: What combination of statements from this list, guarantees that the figure is a square? The findings from such questions showed that in the majority of cases, students at Levels 1 and 2 were unable to respond correctly when asked for minimum conditions.

Pegg and Davey (1991) supported this finding, as students giving Level 1 and lower Level 2 responses either repeated the original list contained in the first description, or removed (at random) some properties from the list. The researchers suggested that students need to possess the necessary language skills that would enable them to use expressions such as 'one angle a right angle' and 'a pair of adjacent sides equal.' Hence, these students were unable to provide a meaningful response to the task as "the properties are seen to be independent and whatever properties they know are needed to identify the shape" (p. 12). Level 3 responses included those that could focus on a smaller combination of properties although the end result was not a minimum. Pegg and Davey (1991) stated that a correct minimum description may be classified as indicating Level 4 thinking, but the response would need further probing to ascertain whether the response was rote learnt or whether other minimum descriptions were also able to be provided.

In a study involving the assessment of aspects of students' geometric understanding in the classroom, investigations showed that most students, up to Year 10 (16 year olds), were unable to provide formal definitions requiring the minimum number of properties (Pegg and Davey, 1991). In the past, teachers' expectations were for higher Level 3 and Level 4 responses, when the majority of students were in fact at Levels 1 and 2.

Summary

Many researchers have identified the level of difficulty associated with class inclusion concepts. Class inclusion has been described as a difficult concept to grasp and one that requires specific instruction. While this difficulty is acknowledged, class inclusion is still recognised as a prerequisite for deductive reasoning.

Some interesting findings, such as those of Burger and Shaughnessy (1986) and Pegg (1992a), suggest that at Level 2, students actually preclude classifications involving class inclusion due to their own descriptions of figures. Both visual cues and language use are described as either assisting or inhibiting the development of the notion of class inclusion. These issues imply that class inclusion concepts, which require an understanding of the interrelationships between properties and their figures, is a result of a major transformation and organisation of a student's mental network. While the hurdle exists, it is one that must be overcome before deductive reasoning is possible.

THE SOLO MODEL – SYSTEM TO ANALYSE RESPONSES

In the light of the previous two sections, this section considers the SOLO model as a useful system for analysing students' responses to class inclusion tasks. Previous studies in a variety of key learning areas, such as Geography (Courtney, 1986), Science (Levins, 1992; Stanbridge, 1990; Panizzon, 1999), and general problem-solving in secondary students aged 14–16 years (Bennet, 1987) have identified the SOLO model as a useful framework for interpreting student levels of understanding. In Mathematics specifically, the model has been utilised within the investigation of a wide variety of concepts, such as, fractions (Watson, Campbell & Collis, 1992), multiplication (Watson & Mulligan, 1990), algebra (Coady & Pegg, 1993), statistics (Watson, Collis & Moritz, 1997; Reading & Pegg, 1996), volume of prisms (Campbell, Watson & Collis, 1992), and of special interest to this study, Geometry (Davey & Pegg, 1992; Olive, 1991; Pegg & Woolley, 1994; Pegg & Davey, 1998).

This section provides a brief description of the model, in particular, the five modes of functioning and the levels of thinking within each mode. This is followed by a consideration of the interface between the SOLO model and the van Hiele Theory. The final section provides an overview of the key aspects of the SOLO model in relation to the current study.

Overview

The SOLO model is a categorisation system which evaluates the quality of students' responses. The SOLO (Structure of the Observed Learning Outcome) model first introduced in 1979 by John Biggs and Kevin Collis (Collis & Biggs, 1979), grew from a desire to explore and describe students' understanding in the light of the criticisms of the work of Piaget (Biggs & Collis, 1982). Rather than focus on the level of thinking of the student, the emphasis in the SOLO model is on the structure of students' responses.

Stage theorists such as Piaget hold that the learner passes through the stages in a static manner without the availability to return to a former stage, each stage being characteristically unique. Once a particular stage is reached, the student remains at this stage until maturing to the next stage. Thus, the Piagetian theory is questioned when one considers typical classroom behaviour where a student's thinking is not characteristic of one stage only. The evidence of behaviours of different levels, exhibited from the same person, was described by Piaget as decalage (Biggs & Collis, 1982 p. 20). While this problem was recognised by Piaget, it was not resolved within the Piagetian framework.

Biggs and Collis (1982, p. 22) addressed this issue by simply "shifting the label from the *student* to his *response* to a particular task," and recognising influences such as motivation and prior knowledge of a task. The developers of the SOLO model acknowledged the problems associated with overgeneralising stages of development in terms of ages, and the confusion experienced by teachers when particular students were not performing at the Piagetian stage related to their age.

As a result of the analysis of a large pool of student responses, across a variety of subject areas and learning environments, a structure was identified which remains consistent while still undergoing processes of evolution (Biggs & Collis, 1982, 1991; Coady, 1994; Watson, Collis, Callingham, & Moritz, 1995). The value of the SOLO model lies in its ability to provide a language which can be used to categorise levels of students' responses in a variety of contexts at various stages of development in understanding.

Modes of Functioning

The SOLO model is rooted in the notion that all learning can be described in one of five modes of functioning, or in a combination of these modes. The basis for the theoretical stance taken by Biggs and Collis (Collis & Romberg, 1991, p. 87) was that there are "two phenomena involved in determining the level of an individual's response to an environmental cue: abstraction of the elements utilised, and the complexity of the *structure* of the response within that mode." The five modes identified by Biggs and Collis (1991) assist in the determination of the abstraction used by the individual for a given response. The five modes of functioning are:

Sensori-motor

The response involves a reaction to the physical environment. It is associated with motor activity and can be described as tacit knowledge. Examples include a child learning to walk and an adult playing sport.

Ikonic

The response involves the internalisation of images and linking them to language. Bruner (1964) described the individual as forming internal pictures, images, or 'icons' to aid in thought processes. There is a reliance on images and development of language, and thinking in this mode can be described as intuitive knowledge. Examples include a child developing words for images, and an adult's creation of science fiction images.

Concrete Symbolic

The response involves the application and use of a system of symbols, which can be related to real world experiences. This abstraction enables concepts and operations that are applied to the environment to be manipulated through the medium of symbolic systems, for example, written language and number problems. Responses in this mode can be described as declarative knowledge. Such responses indicate logic and ordering between symbols, and between the elements of the world they represent. Collis (1992, p. 22) claims that "one of the main tasks with which the school is charged is teaching children to operate with the concrete-symbolic systems necessary for successful living within a modern society."

Formal

The response involves the consideration of abstract concepts as there is no longer a need for a real world referent. The formal mode is characterised by a focus upon an abstract system, based upon principles, in which concepts are embedded. Responses in this mode can be described as theoretical knowledge. Such responses include the formation of generalisations and hypotheses concerning perceptions of how elements in the world may be. Indications of thought at the formal level are an expected outcome when studying a discipline at university level.

Post-formal

The response involves the challenge or questioning of the abstract concepts and theoretical perspectives of the formal mode. This involves a further exploration of a discipline to the extent that its knowledge bank expands. While the existence of this mode is debated, it is expected to appear at the postgraduate study level.

While the modes are similar to the stages of Piaget, there are identifiable differences that require consideration. Piaget described four stages of cognitive development, these being: sensori-motor (birth to two years); intuitive/pre-operational (two to six years); concrete operational (seven to fifteen years); and, formal operational (sixteen plus years). While these appear to have similar descriptions to the SOLO modes of thinking, the underlying assumptions are vastly different.

The modes have elements in common with the stages identified by Piaget, however, the SOLO modes represent a growth pattern where the earlier modes remain available to the learner. Unlike the model of Piaget, where one stage subsumes another, the modes of the SOLO model continue to develop with the opportunity to access more than one mode at any one time. Thus, the modes have a supporting influence on each other (Biggs & Collis, 1989). This characteristic is supported by Pegg and Davey (1998, p. 119) who described the differences as "first, a newly developing mode does not subsume or replace earlier modes. Instead, earlier modes continue to evolve and provide support for later acquired modes. Second, these later developed modes can assist further growth in earlier developed modes." In support of this notion, Mulligan and Watson (1998, p. 66) stated that "development of higher modes can also serve to increase sophistication in ikonic functioning."

In addition, in the SOLO model, the concrete symbolic mode goes further than Piaget's concrete operations stage and includes some of what Piaget called formal operations. Hence, the formal mode in SOLO is not seen until around 16 years, whereas in Piaget's framework it is about 12 years. The work of Collis, Watson, and Campbell (1993), which involved the investigation of the solutions to novel mathematical problems by adolescents with high ability, showed that the ikonic mode continued to develop in conjunction with the formal mode within the SOLO model.

Each mode has individual characteristics, which result in their unique identities. The five modes of functioning appear in Figure 2.1 below. The ages given within each mode provide a general indication of when to expect the emergence of the particular mode of thinking. The ages also reflect that the majority of the students at primary and secondary levels of schooling are able to operate in the concrete symbolic mode. While most tasks target this mode, there are still many students who respond in the ikonic mode and those capable of operating in the formal mode.

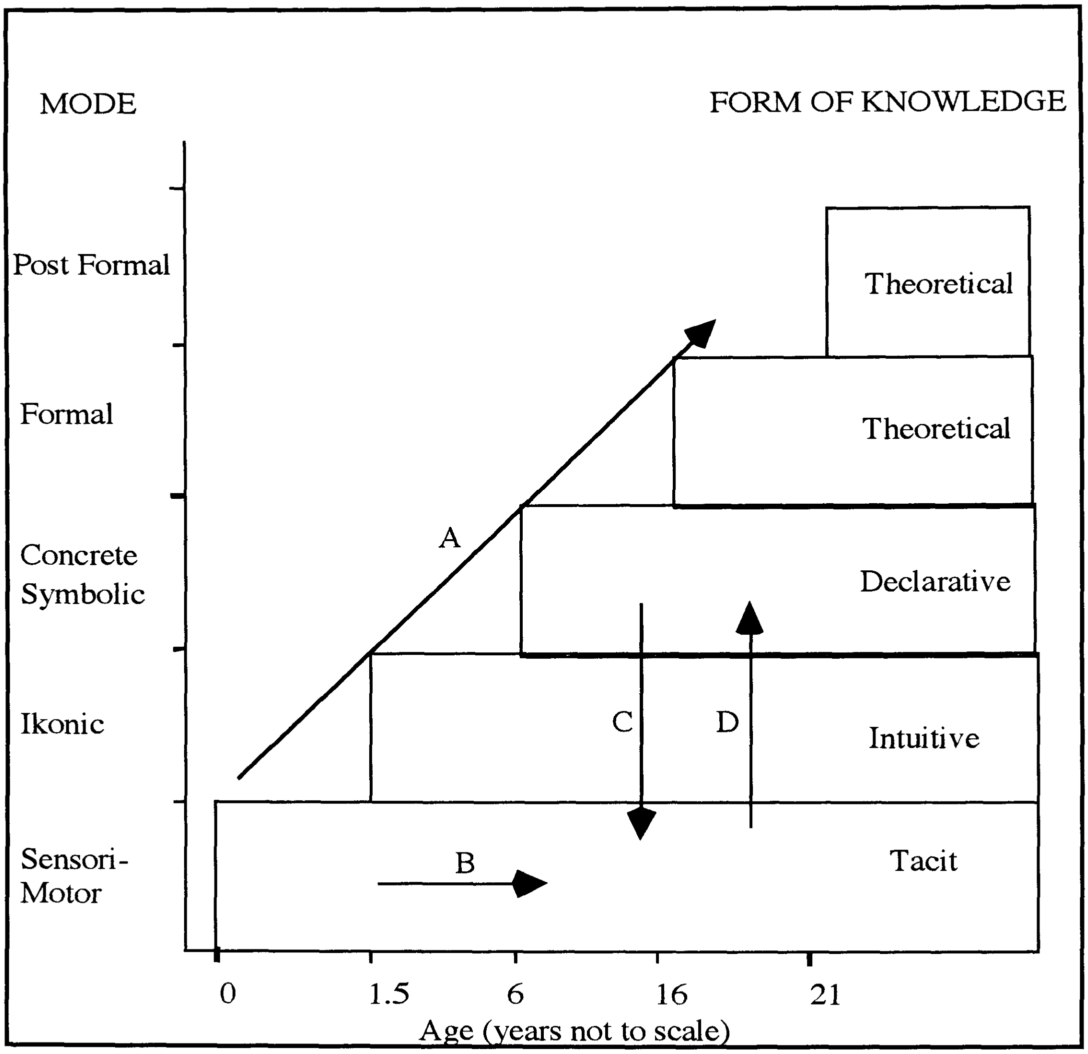


Figure 2.1 The SOLO model: modes and forms of knowledge (Adapted from Biggs & Collis, 1991). A, B, C, and D are explained in the following section.

In summary, an essential element of the SOLO model is represented by the five modes of functioning, these being, sensori-motor, ikonic, concrete symbolic, formal, and post-formal. While they appear to be similar to the developmental stages of Piaget, there are fundamental differences. In the SOLO model the previously acquired modes are not subsumed by later modes and instead assist in the development of newly acquired modes while also playing a supportive role. This extension of the 1982 model, known as multi-modal functioning, is discussed below.

Multi-modal functioning

The SOLO model acknowledges individuals' preferred mode of functioning, and the ability to operate in prior modes, or in more than one mode in particular learning contexts. This ability is known as multi-modal functioning. Figure 2.1 illustrates four different paths of development, all of which are possible within the SOLO model. Firstly, the 'course of optimum development,' widely accepted by stage theorists such as Piaget, is represented by the diagonal arrow (A). This indicates learning characterised by an emerging stage subsuming its predecessor. However, as discussed above, it is more probable for development to involve earlier modes in a supportive capacity which assists the emergence of subsequent modes. Uni-modal learning is represented by the horizontal model (B), thus indicating learning which involves the application of only one mode. The vertical arrows represent 'top-down' (C) and 'bottom-up' (D) development; 'top-down' involving the application of a later acquired mode to enhance learning within an earlier acquired mode, and, 'bottom-up' involving the support of lower modes to enhance performance within a higher mode (Biggs & Collis; 1991).

While a progression occurs from concrete actions to abstract concepts and principles, in the majority of cases, the emergence of one mode does not replace the former mode. Biggs and Collis stated that:

> the modes in fact accrue, the later developing modes existing alongside the earlier modes. The implications of this last statement are twofold:
>
> 1. as the individual matures physiologically, the mode(s) developed earlier continue to develop on the basis of the increasingly mature physical and intellectual background;
> 2. as the model repertoire available increases, multi-modal functioning becomes the norm.
>
> (Collis & Romberg, 1991, p. 87)

Of special interest to Collis & Romberg (1991, p. 93) was "the child's ability to utilise intermodal functioning in solving mathematical problems (e.g., use of the ikonic/intuitive or sensori-motor modes in conjunction with the concrete-symbolic or formal modes." For example, Collis and Romberg described the cognitive characteristics of mathematical tasks in the following manner:

> Let us take for example the area of measurement: the Field A may represent the initial problem of measuring, predicting, or recording a measure of some empirical phenomenon. Consider the typical question that involves finding "How many?" The individual that has to solve the problem has basically two options. One is to use the ikonic mode of functioning and solve the problem by intuition and imaging, perhaps supported by some sensorimotor activity. The other is to translate the relevant aspects to the number field of concrete-symbolic mathematics to operate upon them according to the model which appears appropriate in the field, and then map the result of the calculation back into the empirical field.
>
> (Collis & Romberg, 1991, p. 93)

In summary, the individual operates at a particular preferred mode to solve problems within a certain learning context. While this mode utilises the operations and elements that are available to the individual operating within the mode, the individual always has the option of returning to an earlier acquired mode, or gaining support from one or more earlier acquired modes. Through development, the earlier acquired modes serve as building blocks to subsequent modes, however, the reorganisation to a later acquired mode does not subsume the earlier acquired mode, and, instead, remains available in the form of multi-modal functioning.

Overview

The five SOLO modes represent the level of abstraction of a response. The characterisation of the SOLO modes appears similar to the Piagetian developmental stages, however, there are fundamental differences: the mode of functioning utilised within a response is characterised by the SOLO model, as opposed to characterising the person as at a single developmental mode; newly acquired modes do not subsume previously acquired modes; and, multi-modal functioning is possible. The SOLO modes represent developmental growth where the acquired SOLO modes remain accessible and continue to evolve while supporting other modes.

Levels

Cognitive development is regarded "as a series of hierarchical skill structures that can be grouped into sets of levels" (Collis & Romberg, 1991, p. 86). Within each mode of functioning there occurs development, and development is described in terms of levels. This characteristic of the model provides a vehicle for measuring the level of sophistication of a response to particular tasks within a mode. The existence of levels has been reported by many researchers (Biggs & Collis, 1982, 1989, 1991; Case, 1985; Fischer & Silvern, 1985). The SOLO model does not stand alone in its claims of a series of hierarchical skills that can be grouped into sets (Case, 1985; Fischer, 1980; Fischer & Pipp, 1984; Halford, 1982).

The series of levels, as defined by the SOLO model, are comprised of five different levels based upon the structure of the response. The descriptions below include possible responses at different levels within the concrete symbolic mode. The content area of Geometry is chosen to illustrate the meaning of the levels. The five levels are:

Prestructural

The response is below the target mode. In an attempt to give a response the learner is misled or distracted by irrelevant aspects of the task and responds in a lower mode. A typical response may be "the square is like a box."

Unistructural

The response is characterised by a focus on a single aspect of the problem/task. Since only one relevant piece of information is utilised, the response may be inconsistent. A typical response may be "a square has all sides equal."

Multistructural

The response is characterised by a focus on more than one independent aspect of the problem/task. No relationships are perceived between the components utilised. A lack of integration is evident and some inconsistency is apparent. A typical response may be "a square has all sides equal, four right angles, and all the sides are parallel, and two pairs of opposite sides."

Relational

The response is characterised by a focus on the integration of the components of the problem/task. The relationships between the known aspects are evident with consistency within this system. A typical response may be "a square has four equal sides and a right angle." When probed to provide more information the student explains that there is no need for other properties such as four equal angles or opposite sides parallel, due to the properties already given.

Extended Abstract

The response is taken beyond the domain of the problem/task and into a new mode of reasoning. The response is not singularly reliant on the aspects of the task, but includes generalisations that bring in new and abstract features, e.g., students are able to supply definitions using minimum properties and justify these.

Within each mode there exists cycles of levels. Each cycle comprises of characteristics that are pertinent to that mode of functioning. The prestructural and extended abstract levels are omitted from the diagram as the prestructural response is characterised by irrelevance, or a former developed level in a previously acquired mode, and an extended abstract response is characterised by a level of response in a later acquired mode. (These cycles are shown diagrammatically on Figure 2.2 on the next page.) "There are usually four levels in a complete cycle; the higher levels subsume the lower in the hierarchy and the achievement of the fourth level signals a move to the next mode of functioning" (Collis & Romberg, 1991, p. 91).

The SOLO model grew from the belief that "what is needed is a framework, based upon an understanding of cognitive mechanisms, that could be used both to help in the selection of appropriate open-ended items and to guide the development of valid scaling procedures" (Collis & Romberg, 1991, p. 85). While current assessment items include both 'closed items' and 'open-ended items,' of special interest to Collis and Romberg (1991, p. 102) are the latter. Closed items, which include multiple-choice formats, are described as making clear the type of response required from the context, and providing "little, if any, scope for initiative, investigation, imagination, or cooperation, and there is a unique answer" (p. 102). Open items, on the other hand, "consist basically of items in which the testee has to create a response using whatever resources he/she can bring to bear, hence producing a constructed response" (p. 102).

In an attempt to design assessment items which addressed issues concerning both open and closed questions, Collis (1984, p. 6) devised tasks known as super-items. The tasks contain a stem with four questions specifically targeting the levels described below in increasing order of conceptual difficulty.

Unistructural: "Use of one obvious piece of information coming directly from the stem." (p. 6)

Multistructural: "Use of two or more discrete closures directly related to separate pieces of information contained in the stem."(p. 6)

Relational: "Use of two or more closures directly related to an integrated understanding of the information in the stem." (p. 6)

Extended Abstract: "Use of an abstract general principle or hypothesis, which is derived from or suggested by the information in the stem." (p. 6)

These items are designed in a manner where a response, which correctly achieves Q1, is coded as unistructural. A correct response in both questions 1 and 2 is coded as multistructural, and so on. While each of the questions may be answered independently, it is expected that a correct response to a higher level question would be achieved after the successful completion of the earlier questions.

Cycles of levels

Recent studies (Campbell, Watson, & Collis, 1992; Levins & Pegg, 1993; Panizzon, 1999; Pegg, 1992b) have extended the SOLO model through the suggestion that more than one cycle of unistructural, multistructural, and relational levels exist within each mode. As a result, studies have identified two cycles of levels with the concrete symbolic mode. This is illustrated below in Figure 2.2.

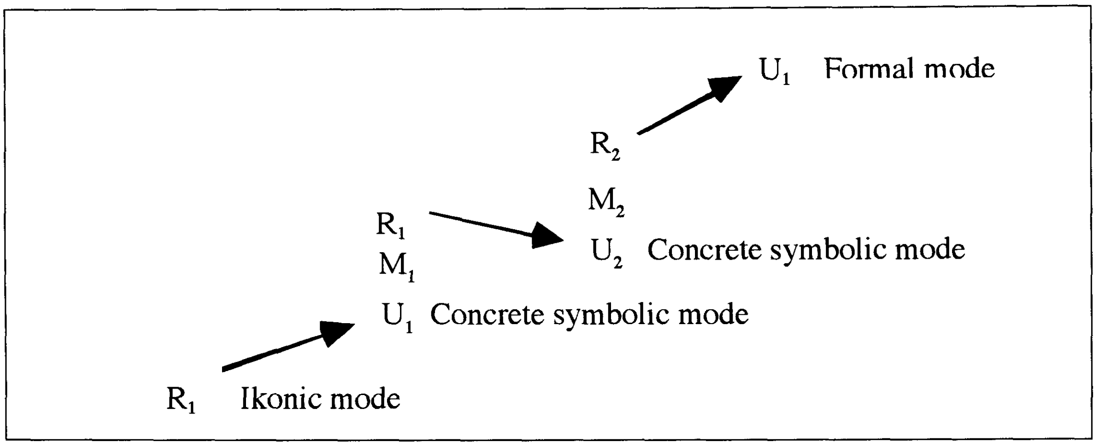


Figure 2.2 Diagrammatic representation of SOLO levels associated with the concrete symbolic mode (Pegg, Guiterrez & Huerta, 1998, p. 284)

To elaborate on Figure 2.2, Pegg et al. (1998, p. 284)) described two cycles within the concrete symbolic mode concerned with students' attempts at describing quadrilaterals. A global description of a shape involving visual images such as pointiness, flatness, and corners would be coded at the relational level (R_2) in the ikonic mode. The student enters the early stages of the concrete symbolic mode by elaborating upon one feature associated with the shape. "The shape has four sides" is an example of a unistructural response of the first cycle (U_1) in the concrete symbolic mode. When the response includes an attempt at quantifying a particular feature, for example, "This shape has a long top and bottom and two short ends," it is an example of a multistructural response (M_1) of the first cycle. A relational response (R_1) of the first cycle in the concrete symbolic mode is characterised by the mention of a single property. For example, "A rectangle has two sides the same length and another two sides the same length."

Pegg et al. (1998, p. 284) described this first cycle of development in the concrete symbolic mode as having "many links to the ikonic mode and represents a transition from an intuitive feeling of space to the development of geometric thinking." A response enters the second cycle (unistructural, U_2) when the property is mentioned succinctly. For example, "A square has all sides equal." Pegg et al. (1998) acknowledged that it is often difficult to distinguish between R_1 and U_2 responses through single statements, however, through student interviews, it is evident that the level of clarity is a distinguishing feature between R_1 and U_2. A typical multistructural response of the second cycle (M_2) may be "a square has all sides equal, four right angles, and all the sides are parallel, and two pairs of opposite sides." A typical relational response of the second cycle (R_2) may be "a square has four equal sides and a right angle." When probed to provide more information the student explains that there is no need for other properties such as four equal angles or opposite sides parallel, due to the properties already given.

Overview

In summary, the five levels describing the complexity of the structure of a response are prestructural, unistructural, multistructural, relational, and extended abstract. The levels can be identified through the observation of the structure of an individual's response to a given task. While unistructural, multistructural, and relational responses appear in each of the modes of functioning, a prestructural response is typified as at a lower level of abstraction required for the task. The extended abstract response goes beyond the requirements of the task. The categorisation of student responses to tasks, rather than the categorisation of the individuals, requires careful construction of assessment items. Both closed and open items are applicable to the SOLO model, however, the most appropriate method for eliciting optimum response is dependent upon the type of investigation. More recently, researchers have identified two cycles of levels within the concrete symbolic mode.

SOLO/Van Hiele Interface

Pegg et al. (1998) identified the following implications when attempting to use the broad categories of the van Hiele theory to characterise in detail learners' understanding over a range of concepts:

(i) learners may be on different van Hiele levels for different concepts
(ii) the van Hiele levels are not particularly suited for fine-grained analyses of a learner's understanding

(Pegg et al., 1998, p. 280)

Through the recognition that there is a need for a system to interpret more clearly students' understanding of geometrical concepts, the SOLO model, while developing independently of the van Hiele theory, has proven to be a useful framework (Pegg & Woolley, 1994; Pegg & Faithful, 1995). Pegg et al. (1998, p. 286) described the following links between the SOLO modes and the van Hiele levels.

Van Hiele Levels	SOLO Modes
Level 1	Ikonic (I)
Level 2 and 3	Concrete symbolic (CS)
Level 4	Formal
Level 5	Post Formal (PF)

Level 1 – SOLO connections

Pegg et al. (1998) hypothesised that there are different levels of acquisition of Level 1, which involves the learner recognising shapes by their whole appearance, which can be interpreted as functioning within the ikonic mode and moving through U_2, M_2, and R_2. Hence, "it would seem that these three SOLO levels may represent a focus on some single aspect of a shape, several unrelated aspects and an overview respectively" (p. 287). It is essential to note that Pegg and Davey (1998) speak of 'aspects.' These are not the same as the properties that develop in the concrete symbolic mode, such as equality of sides and angles, but instead, examples of aspects include pointiness, flatness, and sharpness of shapes. An example of a cycle is: a U_1 level response may describe a rectangle as a flat shape with four sides; an M_1 response may describe a rectangle as having four sides where "the bottom and top are small, but the sides are long"; and, an R_1 response may correctly describe the rectangle as having "two sets of equal sides" (p. 126).

Clements and Battista (1992, p. 429) hypothesised that before reaching van Hiele's visual level, thinking involves a stage where "the objects about which students reason are specific visual or tactile stimuli; the product of this reasoning is a group of figures recognised visually as the same shape." Such work may lead to an interpretation of U_1, M_1, and R_1 (first cycle in ikonic mode) as the debated van Hiele level 0 (as discussed in Chapter 1).

Level 2 – SOLO connections

Since van Hiele's Level 2 requires the learner to recognise figures by properties, this ability can be interpreted as a progression through the U_2 and M_2 in the concrete symbolic mode. This progression is described by Pegg et al. (1998, p. 287) in terms of low, medium and high acquisition, where a U_2 response is based upon the recognition of a single property. An M_2 response can represent a medium to high acquisition depending upon the number of known properties.

In addition there is a transition between the learner moving from a focus on the whole appearance of a figure (ikonic mode), to a focus on a single property (U_2CS). The first cycle of the concrete symbolic mode (U_1, M_1, and R_1) is a useful interpretation of the transitional development between van Hiele Levels 1 and 2.

Level 3 – SOLO connections

Since van Hiele's Level 3 requires the learner to have an understanding of the interrelationships between figures and their properties, there is a strong relationship between this level and R_2 in the concrete symbolic mode. While this connection appears relatively straightforward, Pegg et al. (1998, p. 287) have identified that although deductive reasoning generally is associated with Level 4 thinking, "learners with complete acquisition of Level 3 are able to provide simple formal proofs; however, the main difference between this and Level 4 is that the learners at Level 3 can only use relationships, such as congruency, in relatively straightforward prompted situations and where there is an empirical referent."

Level 4 – SOLO connections

Level 4, described as the ability to apply the interrelationships of figures and their properties and an understanding of the deductive process, can be interpreted within the formal mode. Pegg et al. (1998) described the development within this mode in terms of two of cycles (U_1, M_1, R_1, and U_2, M_2, R_2) in the formal mode. The first cycle "represents a development of the deductive process, where an R_1(F) response describes a learner who can competently apply relationships, such as congruency and similarity, in addition to being able to keep track of most conditions inherent in the question" (p. 287). The U_2 response is characterised by the "successful application of necessary and sufficient conditions in non-prompted situations, and is possibly the best response that capable learners can achieve in the secondary school" (p. 287).

Level 5 – SOLO connections

When matching the characterisation of both van Hiele's Level 5 and the post formal (PF) mode, it is hypothesised that the development of the ability to "challenge old, and adopt new, axiomatic systems" (Pegg et al., 1998, p. 287) is represented by cycles of the PF mode. While research to date, has not reported actual experiences and responses at this level, this level of reasoning is expected to be achieved by professional mathematicians and gifted learners.

Two important features that stand out concerning the SOLO structure when used in conjunction with the van Hiele Theory are the flexibility and versatility offered by the model's characterisation of responses. Pegg et al. (1998, pp. 285–286) identified four issues that become apparent when utilising the SOLO model:

First, since the SOLO categorisations is focused upon the quality of the structure of learners' responses, as opposed to labeling the learner as a particular level, the problems associated with *decalage* (Piaget's and van Hiele's ideas), are avoided. There is not an expectation for the respondent consistently to provide responses that are at a "given level across a number of topic areas or within the same topic on different occasions.

Instead, responses are sensitive to the motivation of the learner, the amount of experience with the material, the learner's interpretation of the stimulus item, and so on" (p. 286);

Second, while the van Hiele model provides five levels of understanding from which to view student responses, the possible categorisations of the SOLO model are extensive. For example, within the current formulation of the SOLO model there are at least six levels within the concrete symbolic mode;

Third, the SOLO model provides a vehicle for exploring the entire structure of the response. This is due to the model's multiple dimensions. Not only is there the "opportunity to describe the quality of responses within a mode but the interaction of other modes allows for an explanation of diversity and individuality in learners' responses that has not previously been available" (p. 286). The opportunity exists not only to consider the optimum response within the target mode, but also to consider the whole problem-solving process that leads to the desired outcome and the support provided by other modes.

Fourth, while the SOLO model does not include a specific teaching model, the model implies important general teaching principles, which assist learning. The SOLO model highlights the need to consider prerequisite skills, both in the target mode and previously acquired modes. The model acknowledges the invaluable ongoing support of former modes to the target mode for the enhancement of learning (p. 286).

Overview

Through the matching of van Hiele levels and the SOLO model, both teachers and researchers are provided with a powerful tool with which to investigate student understandings. The empirical data collected as a result of investigations dealing with van Hiele's Levels 2, 3 and 4, where the SOLO model is used as a means of elaborating on the characterisation of van Hiele levels, assist in generating hypotheses relating to conceptual development.

Summary

It is evident that throughout the evolution of the SOLO model, while new findings have built upon the initial framework, the validity of the initial perceptions has been maintained. The usefulness of the SOLO model lies in its approach where the structure of the observed student's response is analysed. Hence, the focus is not on the person, but instead on the quality of the given response. The model provides a language that can be used to categorise systematically levels of students' responses at various stages of development of understanding. "The SOLO interpretation of responses is more useful than notions of stages of development as such for educators and researchers to describe the level of reasoning on school-related tasks because it points a way to advance and does not merely describe a presumably static state" (Collis, 1984, p. 14).

CONCLUSION

This chapter has presented research studies investigating students' understandings of class inclusion notions. In addition, research directed at van Hiele's Level 3 has also been presented. The results of these studies highlight a number of controversies in regards to the characterisation, development, and factors affecting an understanding of class inclusion. Class inclusion in the context of Geometry has been described as a prerequisite for formal deductive proofs and, hence, is classified as a higher-order skill. Studies have acknowledged that the ability to focus upon sub-class relationships is a difficult cognitive hurdle for students to overcome and a number of contributing factors have been flagged by researchers. Predominantly, the issues involve an understanding of the interrelationships among figures and their properties.

Through the consideration of the issues raised in this, and the preceding chapter, the theoretical framework provided by the SOLO model is used in the present study to explore five themes concerning students' understandings of class inclusion notions in Geometry.

Research Theme 1
To investigate students' understandings of class inclusion concepts concerned with different types of triangles.

1.1 What are the characteristics of students' understandings demonstrated in a classification task of seven different triangles?

1.2 Is there evidence of some developmental pattern in the different responses to a classification task of seven different triangles?

1.3 Does the SOLO model offer a framework to explain the identified categories of responses concerning students' understandings of relationships among different triangles?

1.4 Can students' demonstrated understandings of relationships between triangle properties be categorised into identifiable groups according to similar characteristics?

1.5 Is there evidence of some developmental pattern in the different responses to a task requiring the utilisation of relationships among triangle properties?

1.6 Does the SOLO model offer a framework to explain the identified categories of responses concerning students' understandings of relationships among triangle properties?

Research Theme 2
To investigate students' understandings of class inclusion concepts concerned with different quadrilaterals?

2.1 What are the characteristics, and SOLO classification, of students' understandings demonstrated in a classification task involving six different quadrilaterals?

2.2 What are the characteristics, and SOLO classification, of students' understandings demonstrated in a task concerning relationships among quadrilateral properties?

Research Theme 3
To investigate the comparison of students' understandings of class inclusion concepts concerned with triangles and quadrilaterals.

3.1 What are the similarities and differences when comparing the framework offered by the SOLO model in the context of relationships among triangle figures, and relationships among quadrilateral figures?

3.2 What are the similarities and differences when comparing the framework offered by the SOLO model in the context of relationships among triangle properties, and relationships among quadrilateral properties?

Research Theme 4
Can a quantitative analysis of the results, using a partial credit model, offer insights into students' understandings of class inclusion?

4.1 How do the identified response categories reflect the hierarchical framework of the SOLO model?

4.2 Is there an order of difficulty among the item responses, which can assist in interpreting the complexity of students' understandings of relationships among figures and relationships among properties?

4.3 Which response categories to tasks had relatively larger increase in complexity from the prior response category, and how does this increase reflect upon students' growth in understanding relationships among figures and relationships among properties?

Research Theme 5
To explore the developmental growth in understanding of class inclusion of four different students over a two-year period.

5.1 What are the similarities and differences of students' demonstrated understandings of the relationships among figures over a two-year period?

5.2 What are the similarities and differences of students' demonstrated understandings of the relationships among properties over a two-year period?

(iii) Is there an interrelationship between the developmental pattern of understanding relationships among figures, and understanding relationships among properties?

Consequently, these five themes guide each stage of the following investigation of students' understandings of class inclusion concepts in Geometry, in relation to triangles and quadrilaterals. The following chapter outlines the research design and methodology implemented to explore these research themes.