CHAPTER SEVEN

STUDENT CASE STUDIES: CROSS SECTIONAL AND LONGITUDINAL ANALYSIS

The previous chapter presented the findings, from a quantitative perspective, concerning relationships among figures and relationships among properties. This was achieved through a comparative analysis of the response categories determined by the SOLO classification within two contexts, namely, triangles and quadrilaterals.

The purpose of this chapter is to present the findings of a longitudinal analysis of four students' responses by considering case studies. To assist the investigation, the following research questions are addressed.

Research Theme 5

To explore the developmental growth in understandings of class inclusion of four different students over a two-year period.

5.1 What are the similarities and differences of students' demonstrated understandings of the relationships among figures over a two-year period?

5.2 What are the similarities and differences of students' demonstrated understandings of the relationships among properties over a two-year period?

5.3 Is there an interrelationship between the developmental pattern of understanding relationships among figures, and understanding relationships among properties?

This chapter is divided into five sections. The first four sections are the student case studies. The structure of the four case studies includes the general background of the student, a table summarising the student's SOLO categorisation for Interventions 1 and 2, and longitudinal analysis. The final section, Conclusions, ties together the findings that have emerged from this longitudinal analysis.

The structure of the following four case studies varies to suit the individual characteristics of the student's development over the two-year period. The four students were chosen from the twelve students on the basis of: a large shift from concrete symbolic mode to predominantly formal mode (Narelle); consistent variation in understanding of relationships among figures in different contexts (Brendan); consistent progression over the two-year period in each item while maintaining a mixture of concrete symbolic and formal mode responses (Scott); and finally, the fourth student (Louise) was chosen due to the small change in responses concerning relationships among figures when compared with development in relationships among properties.

STUDENT 1: NARELLE

Narelle was a student in Year 8, Term 4 (aged 13 years and 4 months) when first interviewed. Approximately two years later, when interviewed a second time, Narelle was enrolled in the Year 10 Advanced Course and was then aged 15 years and 7 months.

Results from Interventions 1 and 2

A summary of Narelle's responses to the interview tasks in both Interventions 1 and 2 in terms of SOLO categorisations are presented in Table 7.1.

Table 7.1 Narelle's SOLO codings for Interventions 1 and 2

Concept	**Context**	**SOLO Coding**	
		Int 1	**Int 2**
Relationships Among Figures	Triangles	R_1 (CS)	R_1 (F)
	Quadrilaterals	R_1 (CS)	R_2 (CS)
Relationships Among Properties	Equilateral Triangle	M_2 (CS)	M_1(F)
	Right Isosceles Triangle	M_2 (CS)	U_2 (F)
	Square	R_2 (CS)	M_1 (F)
	Parallelogram	R_2 (CS)	U_1 (F)
	Rhombus	U_2 (CS)	M_1 (F)

Narelle was chosen as one of the four case studies due to the predominance of first cycle responses in the concrete symbolic mode provided in Intervention 1, with a shift to second cycle in the concrete symbolic mode and first cycle in the formal mode during Intervention 2 concerning relationships among figures. Narelle also displayed marked development over time concerning relationships among properties, where she responded at second cycle (CS) in Intervention 1 and responded within the formal mode in Intervention 2. Due to this developmental pattern, the structure of the following sub-sections considers Narelle's Intervention 1 responses in both triangle and quadrilateral contexts, followed by a consideration of Narelle's Intervention 2 responses in both contexts. Each sub-section concludes with a comparison of characteristics across the triangle and quadrilateral contexts. The final sub-section concerns the integration of figure and property relationships.

Relationships Among Figures

Intervention 1

Within the first intervention, Narelle provided the same SOLO level response to both tasks concerning relationships among figures. In both triangle and quadrilateral contexts, Narelle spontaneously identified links based upon the identification of single similar properties or features, such as "at least two sides the same" and "has acute angles."

In the triangle context these included justifications such as:

Int:	*What is that link there for?*
Narelle:	*Because it has got acute angles.*
Int:	*Anything else?*
Narelle:	*Um no.* (pause*) There I have put all the unequal sides together ...*
Narelle:	*They all have three sides and they all have at least one angle that is an acute angle.*
Int:	*What have you done on the next row?*
Narelle:	*Um they all have uneven sides.*
Narelle:	*They all have at least two sides the same ...*

In the quadrilateral context, Narelle described spontaneous links in a similar fashion. It is interesting to note that Narelle did not utilise names of figures in either context; instead she relied upon visual cues from which the properties or features were described. When prompted to utilise quadrilateral names, Narelle did not incorporate class names into her descriptions of links. This is evident in the following excerpt where Narelle described a group of figures as belonging together as "they all have parallel sides," but she did not provide the group of figures with a name:

Int:	*Can you tell me about these links here?*
Narelle:	*They all have at least one set of parallel sides.*
Int:	*So do you think that they all belong together?*
Narelle:	*Um probably not but they all have parallel sides.*
Int:	*Is there a name that you could use to describe that group of shapes?*
Narelle:	*No not really.*
Int:	*And why are these linked?*
Narelle:	*Four equal sides.*

Narelle then proceeded to form other groups of quadrilaterals on the basis of identifiable features that were considered one at a time without consideration of previously formed groups of figures.

Intervention 2

Narelle's response to the task concerning relationships among triangle figures in Intervention 2 was categorised as R_1 in the formal mode. The extract below demonstrates the use of class names, such as scalene, isosceles, and equilateral:

Narelle: *An equilateral and an acute-angled you can link because they have all got under 90 degrees so they have all got acute angles. Um, the scalene triangles.*
Int: *Why can the scalene link?*
Narelle: *Because they are both scalene and they have all got um odd sides. Um, this one because the right-angled and the obtuse-angled scalene have both got an angle of 90 degrees or over. And you could link the acute-angled scalene and the acute-angled isosceles because they have both got all acute angles, you could do the same with the equilateral, and um,* (pause) *that is about it ...*
Narelle: *You can link these two because they have both got right angles and they have both got um a greater hypotenuse on there. These have all got acute angles, three acute angles* (including the equilateral triangle). *These ones* (pause) *are all isosceles triangles. These are all scalene triangles. Those have got obtuse angles, and um nothing else.*

When probed to supply further justification concerning the link between the equilateral triangle and the isosceles class of triangles, Narelle justified the relationship in terms of class inclusion notions:

Int: *Any other reason for linking the equilateral?*
Narelle: *Um* (pause) *it's um the angles are all under, um they are all acute angles, they have got two sides that are the same.*
Int: *So it can go to that group of isosceles?*
Narelle: *Yes, it is an isosceles triangle really.*
Int: *How come?*
Narelle: *Because it has got two equal sides and three acute angles, but then it is in a class of its own as well because it is an equilateral.*
Int: *Does it link to any particular isosceles triangle?*
Narelle: *No, not really.*

In the case of quadrilaterals Narelle's response is not as strong. While encapsulation of properties to form classes of quadrilaterals that are recognised by name is evident, similar to the triangle context, Narelle does not incorporate notions of class inclusion within this context. There is no longer a reliance upon ikonic support and Narelle described links between the square and parallelogram, and the rhombus and the rectangle. The excerpt below illustrates the second cycle relational response (CS) provided by Narelle concerning relationships among quadrilaterals:

Narelle:	*You can link the rectangle and the square because they have both got all 90 degrees angles and opposite sides equal. You can link the rhombus, the parallelogram and trapezium because they have all got at least one pair of parallel sides.* (pause). *Then um, you can link the square and the rectangle again because you can find the area the same way. That is the same for the rhombus.*
Int:	*Why is the square linked to the parallelogram?*
Narelle:	*Because they have all got parallel sides and equal sides and everything. Then the rhombus can go to the square again because they have both got four equal sides. Then* (pause)..
Int:	*Any other links or reasons?*
Narelle:	*I can link the rectangle with the parallelogram. Because, um, you have got parallel sides and you should find the area the same way as well.*

When prompted to incorporate subsets within the parallelogram class of quadrilaterals, Narelle remained focused upon similar property links:

Int:	*You have lots of links there with those four, does that mean anything?*
Narelle:	*Well, they have got 90 degrees and the um diagonals should all be equal. Then um,* (pause) *and the diagonals um, I think that is it.*

Contextual comparison

The language use in Intervention 1 to describe relationships among figures is similar in both contexts. For example, in the triangle context the links were described as "two sides the same," "these are all different," and, "at least two sides the same." In the quadrilateral context similar language-use included "all have parallel sides," and "at least one set of parallel sides." Of particular interest are the uses of "at least," and the fact that property differences are not taken into consideration in either context.

Intervention 2 saw a shift to the inclusion of class names. In the triangle context, Narelle no longer utilised "at least" and confidently described the class of isosceles triangles as having "two equal sides" inclusive of the equilateral triangle. Narelle also utilised the word 'class' in her justification of the equilateral subset by stating "it is in a class of its own as well because it is equilateral." The progression to R_2(CS) in the quadrilateral context also resulted in a shift to the use of class names, and using terms such as "four sides equal" and "opposite sides equal." In this context, Narelle utilised the term "at least" in both interventions, such as "at least one pair of parallel sides" and "has got parallel sides." Narelle did not incorporate subsets within the quadrilateral context.

Relationships Among Properties

Intervention 1

In both contexts, Narelle provided second cycle concrete symbolic responses concerning relationships among properties in the first intervention. While Narelle focused upon more than one property to describe the figures, they were not linked in any manner. The focus remained upon the figure in question, from which the properties were identified.

In the triangle context, as demonstrated below, the justification remained solely upon the right-angled isosceles triangle. While Narelle used "two sides equal" and "two angles equal" when providing minimum combinations, the two properties remained isolated:

MINIMUM COMBINATION 1 (right isosceles triangle)

3 SIDES | HAS RIGHT ANGLE | 2 SIDES EQUAL

Int:	*Can I take out that those two sides are equal?*
Narelle:	*Oh yeah I could take out that one.*
Int:	*Do you need the two sides? I couldn't keep the two angles instead?*
Narelle:	*Oh yeah you could well if you have two sides equal and a right angle it would definitely be it.*

MINIMUM COMBINATION 2 (right isosceles triangle)

3 SIDES | HAS RIGHT ANGLE | 2 ANGLES EQUAL

Narelle: *Yeah I have that. That would work.*

When justifying property combinations in the quadrilateral context a similar response was provided. In the parallelogram context, Narelle was prompted to broaden her focus to other quadrilaterals and their properties, however, she could not justify these notions adequately. Hence, the quadrilateral responses were categorised as second cycle in the concrete symbolic mode:

MINIMUM COMBINATION 1 (parallelogram)

4 SIDES | 4 ANGLES | OPPOSITE SIDES ARE PARALLEL | OPPOSITE ANGLES ARE EQUAL

Int:	*You couldn't make that any simpler?*
Narelle:	*Um not really. I suppose I could take out the opposite angles.*
Int:	*Now what if I looked at that and said um it is a rectangle?*
Narelle:	*Well I don't know I suppose that you could.*
Int:	*Would I be right though?*

Narelle:	*Yep.*
Int:	*How come?*
Narelle:	*Because um it has got that.*
Int:	*If you were asking for parallelogram but I said that would I be right?*
Narelle:	*No.*
Int:	*What could you do to fix that?*
Narelle:	(pause) *I am not sure maybe if I add opposite angles are equal.*
Int:	*What if I said rhombus to that?*
Narelle:	*Well the rhombus might not be equal.*
Int:	*So would I be wrong saying that?*
Narelle:	*Well you won't be wrong but if you knew the other cards then you wouldn't think that.*

Intervention 2

In the second intervention, Narelle provided an $M_1(F)$ response in the equilateral triangle context with regards to property relationships. Narelle focused upon the relationships between equality of sides and equality of angles, equality of angles and equality of sides, and symmetry and equality of sides. While three relationships were the focus of the response, these relationships were not integrated, hence were not described in terms of a network of relationships. In the right isosceles context, Narelle described the interrelationship between equality of sides, and equality of angles, and symmetry. Hence, the relationships were connected and the response given is $U_2(F)$:

MINIMUM COMBINATION 1 (right isosceles triangle)

3 ANGLES	1 RIGHT ANGLE	2 ANGLES EQUAL

MINIMUM COMBINATION 2 (right isosceles triangle)

3 SIDES	1 RIGHT ANGLE	2 SIDES EQUAL

MINIMUM COMBINATION 3 (right isosceles triangle)

3 ANGLES	1 RIGHT ANGLE	1 AXIS OF SYMMETRY

Int:	*As you are moving those cards in and out, have you noticed any that are complementing each other or working together?*
Narelle:	*Well the three angles you can put with three sides, and the two angles equal you put with two sides equal.*
Int:	*Why don't you need both of those as once?*
Narelle:	*Well it would be a bit harder, but if they knew that it had one axis of symmetry, it would have at least two sides equal and two angles equal.*

Narelle provided first cycle responses in the formal mode concerning relationships among quadrilateral properties. A single relationship was focused upon in the parallelogram context, hence, Narelle responded at $U_1(F)$ to the task concerning relationships among parallelogram properties. The relationship identified concerned the link between opposite sides parallel and opposite angles equal. There was no longer a need for a real world referent with the property relationships determining the figure in question:

Int: *Could you make that simpler?*
Narelle: *No probably not.*
Int: *You think that you would need both of those?*
Narelle: *No you probably won't need opposite angles are equal.*
Int: *Why not?*
Narelle: *Um actually, um ... like um if they are parallel, um if both sides are parallel so they will meet at the same angle anyway, so the opposite angles should be equal.*

In the square and rhombus context, it is evident that Narelle focused upon more than one relationship (M_1F). In the rhombus context, Narelle began by focusing upon a relationship between the rhombus and the square. This is evident in the extract concerning relationships among square properties described below:

Int: *How does that one work?*
Narelle: *Okay well there are two pairs, um there are two pairs of equal adjacent sides so that means, two sides have to be equal so the other two have to be equal. All the pairs are equal to each other so there are four axes of symmetry.*

Contextual comparison

It is evident that Narelle's Intervention 1 responses to tasks concerning the relationships among properties, were characterised by a focus upon the figure in question from which the properties were determined. This was particularly evident in the triangle property tasks. The quadrilateral tasks were categorised as second cycle in the concrete symbolic mode, the focus was on a tentative link between the figure in question and other quadrilaterals. In the case of the square and parallelogram the combinations included superfluous properties. In the second intervention it was apparent that Narelle's focus had predominantly shifted to a focus upon the relationships among properties. There was a minor focus upon the relationships among the properties of the figure in question, and properties of other quadrilateral figures. In the tasks concerning relationships among properties, over the two-year period, Narelle moved from a focus upon the figure determining the property, to the relationships among the properties determining the figure.

Integration of Figure and Property Relationships

Since the Intervention 1 responses were predominantly within the concrete symbolic mode, similarities exist across Narelle's responses to the seven tasks. Narelle's first interview comprised responses that spontaneously identified links across figures without reference to generic classes. During this intervention, Narelle also remained focused upon the figure in question when justifying minimum property descriptions. In the second intervention, Narelle focused upon the notion of class inclusion when describing the relationships among triangle properties. During this intervention, Narelle also focused upon the interrelationships among triangle properties of the right isosceles triangle. In the equilateral context, these relationships remained isolated. This also coincided with a shift in perception to the relationships among the properties determining the figure.

STUDENT 2: BRENDAN

At the time of Intervention 1, Brendan was enrolled in Year 8 Term 4 and was at age 13 years and 3 months. After a two-year period, and at the time of the second intervention, Brendan was aged 15 years and 3 months. Brendan was enrolled in the Year 10 Advanced Course.

Results from Interventions 1 and 2

Brendan's responses to the interview tasks in both Interventions 1 and 2 in terms of SOLO categorisations are presented in Table 7.2.

Table 7.2: Brendan's SOLO codings for Interventions 1 and 2

Concept	Context	SOLO Coding	
		Int 1	**Int 2**
Relationships Among Figures	Triangles	M_2(CS)	M_2(CS)
	Quadrilaterals	U_2 (F)	U_2 (F)
Relationships Among Properties	Equilateral Triangle	R_2(CS)	U_2(F)
	Right Isosceles Triangle	M_2(CS)	U_1(F)
	Square	R_2(CS)	U_2(F)
	Parallelogram	M_1(F)	U_2(F)
	Rhombus	M_1 (F)	M_1(F)

Brendan is included in the four case studies, due to the significant variation in understanding of relationships among figures in the different contexts i.e. triangles and quadrilaterals. In tasks concerning the relationships among properties, Brendan also demonstrated growth in understanding in the majority of tasks. Of particular interest are Brendan's formal mode responses within the context of quadrilaterals. In tasks concerning relationships among properties, Brendan progressed to a higher SOLO level in four of the five tasks. The developmental pattern evident in Brendan's response requires a structure, which considers Brendan's responses within each context to illustrate no change in SOLO level concerning relationships among figures, and growth concerning relationships among properties.

Relationships Among Figures

In both Interventions 1 and 2, Brendan had no change in his SOLO categorisation in either context. Despite the two-year period between interventions, Brendan remained at M_2(CS) when responding to tasks concerning the relationships among triangle figures in both interventions. Similarly, both Brendan's responses to tasks concerning relationships among quadrilateral figures remained at U_2 in the formal mode.

Triangle context – Interventions 1 and 2

The extract below illustrates the manner in which Brendan described the three triangle-type classes, constantly referring to their class name. While more than one property was frequently used to describe the particular class of triangles, and angle-type links were made across classes, Brendan did not link the equilateral triangle for any reason:

Int: *Now how come you have got the equilateral triangle on its own?*
Brendan: *Because it doesn't really link to anything.*
Int: *How come?*
Brendan: *Because you can't have a right-angled equilateral triangle. You can't have anything but equilateral...*
Brendan: *Um we have got the isosceles all linked together.*
Int: *How come?*
Brendan: *Because they are all isosceles.*
Int: *Which means what?*
Brendan: *They have got two sides and two angles the same.*
Int: *And how come the isosceles can be linked to the scalene?*
Brendan: *Well that is an obtuse angled triangle and so is that one so in a way that one is linked to that one.*
Int: *Could you put that one going across in? Why is it linked to that one?*
Brendan: *It is an acute angled triangle to an acute angled triangle.*

In the second intervention, Brendan provided a similar response. This was also categorised as M_2(CS). The progression made by Brendan is recognition of angle-type links from the equilateral triangle to the acute isosceles triangle and the obtuse isosceles triangle. When prompted to find another reason to link the equilateral triangle to the isosceles class of triangles, Brendan could not supply another reason. The following excerpt includes Brendan's description of the isosceles class of triangles and his links to the equilateral triangle:

Int: *Why do all the isosceles link together?*
Brendan: *Because they have all got two equal sides and two equal angles.* (pause) *Then there is the equilateral which could possibly be linked to the acute scalene and the acute isosceles because they have all got, um all of them*

have got acute angles, but at the same time it is not a scalene or an isosceles.

Int: *So what do you think that you should do with that?*
Brendan: *I could draw a dotted line.*
Int: *Is there any other way that the equilateral could link in?*
Brendan: *No.*

Quadrilateral context – Interventions 1 and 2

Brendan's response to the quadrilateral tasks made explicit reference to notions of class inclusion. Brendan began the quadrilateral task by applying property relationships to the quadrilateral diagrams as illustrated in the excerpt below:

Int: *Is there a reason why we only need to put one right angle in there?*
Brendan: *Because if they are at right angles it is cointerior there and that is cointerior there so they are all the same.*

Brendan began the justification of links by describing similar properties across classes. These included more difficult links between the rectangle and the rhombus, and the parallelogram and the square:

Int: *Can you tell me about the link from your rectangle to your square?*
Brendan: *They both have four right angles and two sets of parallel sides.*
Int: *And your rectangle to your rhombus?*
Brendan: *Um they both have two sets of parallel sides.*
Int: *And your rectangle to your parallelogram?*
Brendan: *They have two sets of parallel sides and two pairs of equal ones.*
Int: *Your rectangle to your rhombus?*
Brendan: *Um it has two sets of parallel lines.*
Int: *And your rhombus to your square?*
Brendan: *They have got all sides are equal and two sets of parallel lines.*
Int: *Are there any other links that you could make?*
Brendan: *Yes here.*
Int: *How come your square can link to your parallelogram?*
Brendan: *They have both got two sets of parallel lines.*
Int: *Any other reason?*
Brendan: *No.*

Brendan concluded the first intervention task concerning relationships among quadrilaterals, by focusing upon notions of class inclusion to justify the links identified among the figures. He included a justification for the class of parallelograms, and the rectangle class of quadrilaterals, inclusive of the square. Brendan incorrectly described the rhombus as a subset of the square class, and later corrected this by saying "as I say a square is also a rhombus":

Int: *Is there a reason why these three shapes all go to the parallelogram or why these three all go to the rectangle?*
Brendan: *It is to do with the parallel sides and a square is a rectangle but it is special and the same as a rhombus is a square but it is a special one.*
Int: *What about your parallelogram?*

Brendan: *A parallelogram, um a rhombus certainly comes from a parallelogram and as I say a square is also a rhombus.*

During Intervention 2 Brendan began his justification through the spontaneous application of notions of class inclusion. As demonstrated below, Brendan initiated his justification of inclusive classes of quadrilaterals. The progression made by Brendan was his ability to begin his response at the formal level without working from lower levels before providing a formal response, as evident in Intervention 1:

Brendan: *Let's start with the parallelogram because it seems to have the same things as most of the other ones. You could link the other ones to it. And then the rhombus because it has got all the properties of a parallelogram but it is special, it has got four equal sides but not all equal angles. Then, from that there is the square,* (pause) *it is the same as the rhombus only it has four equal angles, and it is also a parallelogram.*
Int: *So why does it link to the parallelogram?*
Brendan: *Because it has got two pairs of sides that are the same as each other. Then from that you could do the rectangle. It has got four equal angles as well, and the rectangle is also a parallelogram because it has got two pairs of equal sides. And,* (pause) *the kite, which is sort of separate from the other ones. It has got two pairs of opposite sides as well. They are the same sides but they are not opposite equal sides. Then the irregular one is just by itself, it is not really linked to anything.*

Contextual comparison

The following points emerged from a consideration of the language-use across both interventions. In the triangle context Brendan consistently referred to class names and the properties associated with each class. Brendan's similar property descriptions in the triangle context utilised phrases such as "two sides and two angles the same." There was a focus upon the class name to encapsulate the known properties. In the quadrilateral context, Brendan made extensive reference to similar properties when justifying the connections among the quadrilaterals. The language use appears to be more developed in the quadrilateral context as Brendan described relationships in terms such as, "two sets of parallel sides" and two pair of equal ones (sides)." When describing class inclusion notions, Brendan's initial attempt utilised phrases such as "a square is a rectangle but it is special" without further justification. In the second intervention, Brendan provided unprompted and unprobed justifications, such as, "the square, (pause) it is the same as the rhombus, only it has four equal angles, and it is also a parallelogram." This is characteristic of a $U_2(F)$ response through the utilisation of an overview of two networks of relationships.

Relationships Among Properties

Triangle context – Interventions 1 and 2

In tasks concerning relationships among properties, Brendan responded in the second cycle (CS) to both triangle tasks in the first intervention. While the characteristics of both the equilateral and the right isosceles triangle property tasks were similar, the equilateral task contained an additional characteristic that lifted the response from M_2(CS) to R_2(CS). The Intervention 1 response to the right isosceles triangle task (M_2CS) demonstrates Brendan's focus upon identified properties or features chosen in the combinations as unique signifiers of the particular triangle. These were "two sides equal" and "two angles equal" as "a feature of the isosceles triangles." Brendan explained that the reason he did not utilise symmetry was "the axis of symmetry is just a feature of it, it is not like a description um a major description":

MINIMUM COMBINATION 1 (right isosceles triangle)

3 ANGLES | HAS RIGHT ANGLE | 2 SIDES EQUAL

Int: *Why you can take out all of those?*
Brendan: *Well if it has got three angles it has most likely got three sides.*
Int: *Is that definite?*
Brendan: *Yes, and the axes of symmetry. You don't have to know that.*
Int: *Why not?*
Brendan: *Because when you work it out from these ones well the axis of symmetry is just a feature of it, it is not like a description um a major description, there is enough information here to do it.*

MINIMUM COMBINATION 2 (right isosceles triangle)

3 SIDES | HAS RIGHT ANGLE | 2 ANGLES EQUAL

Int: *Why can you just have those?*
Brendan: *Because it shows that two of these angles inside are equal is a feature of the isosceles of triangle and it has one right angle.*

In the context of the equilateral triangle reference was made to a link between three sides equal and three angles equal, however, this indicated an ordering between two properties. Brendan provided a combination based upon the link from equality of sides to equality of angles but did not provide the reverse of this link. When applying axes of symmetry, the justification was based on the uniqueness of three axes of symmetry to the equilateral class of triangles:

MINIMUM COMBINATION 1 (equilateral triangle)

3 ANGLES | 3 SIDES EQUAL

Int:	*Now why do we only need three angles and three sides equal?*
Brendan:	*Because it shows that it has three angles and three sides and the three angles would be equal.*
Int:	*Does it mean anything if the three sides are equal?*
Brendan:	*That the three angles are equal.*
Int:	*Anything else?*
Brendan:	*The sides are all the same length.*

MINIMUM COMBINATION 2 (equilateral triangle)

3 SIDES   3 AXES OF SYMMETRY

Int:	*Why do you only need those two?*
Brendan:	*Because there is no other triangle that has three axes of symmetry.*
Int:	*What if I took three sides out?*
Brendan:	*No. Yeah it might.*
Int:	*How come?*
Brendan:	*Because no other shape has three axes of symmetry.*

The second intervention resulted in all possible property combinations of the right isosceles triangle being chosen. The response was coded as $U_1(F)$ as the relationship between equality of sides and equality of angles was utilised, and was described as working with one another. It is interesting to note that while symmetry was incorporated effectively within the property combinations, the justification remained characteristic of a concrete symbolic response:

Int:	*Which ones do you think are working together?*
Brendan:	*Three angles and three sides, two sides equal and two angles equal.*
Int:	*Why do those two compliment each other?*
Brendan:	*If you have got two equal sides in a triangle, then there are just two equal angles.*
Int:	*And vice versa?*
Brendan:	*Yes.*
Int:	*You have the right angle on its own, and one axis of symmetry on its own. Is that right?*
Brendan:	*Yes.*

In the equilateral context, Brendan focused upon the interrelationships among equality of sides, equality of angles, and symmetry. The focus is evident in the transcript below. Brendan provided all possible property combinations for the equilateral triangle:

Int:	*Which ones are working together this time?*
Brendan:	*Three sides and three angles, three angles equal and three sides equal, and three axes of symmetry goes with three angles equal and three sides equal.*
Int:	*How come?*
Brendan:	*Well because you can't have three axes of symmetry in a triangle unless all the angles are equal and all the sides are equal.*

Quadrilateral context – Interventions 1 and 2

In the quadrilateral context Brendan, overall, responded at a higher SOLO classification than in the triangle context. While Brendan's response to the square task is categorised as R_2(CS) in the first intervention, all other responses in both interventions fall into the formal mode. While the square property combinations are correct, there was no reference made to links between other properties or figures; instead, the minimisations were not justified adequately:

MINIMUM COMBINATION 1 (square)

$\boxed{\text{ALL SIDES ARE EQUAL}}$ $\boxed{\text{THERE ARE 4 RIGHT ANGLES}}$

Int: *Now why do we only need those cards when all of these belong?*
Brendan: *Because if it has four right angles and all sides are equal it has to be a square..The other shapes don't have that.*

In the parallelogram context, specific reference was made to relationships between two properties. The minimisations and justifications below illustrate Brendan's focus upon the isolated relationships between two properties. This focus is characteristic of an M_1(F) response. In this case, two relationships are referred to, these being, parallelism and equality of sides, and parallelism and opposite angles equal:

MINIMUM COMBINATION 1 (parallelogram)

$\boxed{\text{4 SIDES}}$ $\boxed{\text{OPPOSITE SIDES ARE PARALLEL}}$ $\boxed{\text{1 AXIS OF SYMMETRY}}$

Int: *How come we can remove opposite sides are equal?*
Brendan: *Because um* (pause) *in a parallelogram if you have um four sides and the opposite sides are parallel then the two parallel sides can't work unless they are equal.*

MINIMUM COMBINATION 2 (parallelogram)

$\boxed{\text{4 ANGLES}}$ $\boxed{\text{OPPOSITE ANGLES ARE EQUAL}}$ $\boxed{\text{1 AXIS OF SYMMETRY}}$

Int: *Now why will that work instead of your parallel lines?*
Brendan: *Because if the opposite angles are equal than those lines must be parallel.*

Brendan incorporated all sides equal and two axes of symmetry separately when providing minimum combination for the rhombus. While Brendan attempted to explain his chosen combination with "but the all sides equal can be swapped with any of these because they can," he is unable to justify the interrelationships:

Int: *So what did change?*
Brendan: *The four sides and four angles, but the all sides are equal could probably be swapped on any of these because they can. I think I could just have four sides and two axes of symmetry.*

In the second intervention, Brendan progresses to $U_2(F)$ in the contexts of both the square and the parallelogram. Brendan provided the following property combinations for the square, and began his justification by making reference to the properties of the square and other quadrilaterals. Brendan's detailed justification that follows the fourth combination, integrated notions of class inclusion and a focus upon the interrelationships of property relationships:

MINIMUM COMBINATION 1 (square)

4 SIDES | ALL SIDES ARE EQUAL | THERE ARE 4 RIGHT ANGLES

MINIMUM COMBINATION 2 (square)

4 ANGLES | ALL SIDES ARE EQUAL | ARE 4 RIGHT ANGLES

MINIMUM COMBINATION 3 (square)

4 ANGLES | THERE ARE 4 RIGHT ANGLES | 4 AXES OF SYMMETRY

Int: *Why do you need those three?*
Brendan: *Four angles means it is a quadrilateral. The four right angles, which narrows it down to a square or a rectangle, and there are four axes of symmetry which make it a square.*

MINIMUM COMBINATION 4 (square)

4 ANGLES | THERE ARE 4 RIGHT ANGLES

DIAGONALS MEET AT RIGHT ANGLES

Int: *How does that work?*
Brendan: *Well you have got it narrowed down to the rectangle and the square for the four right angles, and because the diagonals meet at right angles and that is a property of only squares and rhombuses, so that sort of leaves the square there.*
Int: *Which ones are working together or complementing each other when you do that?*
Brendan: *Four angles and four sides*, (pause, sorts through cards into pairs).
Int: *Why does diagonals are equal go with four right angles?*
Brendan: *Because all the um,* (pause) *um because um whenever it is in a rectangle or a square, in those two shapes, the two four-right-angled shapes the diagonals are equal in both of them.*
Int: *What about opposite sides are parallel and the diagonals bisect?*
Brendan: *The opposite sides are parallel because there are four right angles and it just sort of makes sense.*
Int: *Will the opposite sides always be parallel when the diagonals bisect?*
Brendan: *Um no not really what I meant was that when there are four right angles the diagonals bisect. They are all sort of grouped together. It is sort of ...*
Int: *Which ones goes with opposite sides are parallel?*
Brendan: *Four right angles.*
Int: *And that goes with diagonals bisect?*
Brendan: *All of these sort of go with four right angles.*
Int: *So four right angles goes with opposite angles are equal, diagonals are equal, opposite sides are parallel, and the diagonals bisect.*

The spontaneity of the combinations provided by Brendan in the parallelogram context, and subsequent overview of property relationships are evident in the following excerpt:

MINIMUM COMBINATION 1 (parallelogram)

4 SIDES   OPPOSITE SIDES ARE PARALLEL

MINIMUM COMBINATION 2 (parallelogram)

4 SIDES   OPPOSITE ANGLES ARE EQUAL

MINIMUM COMBINATION 3 (parallelogram)

4 ANGLES   OPPOSITE ANGLES ARE EQUAL

MINIMUM COMBINATION 4 (parallelogram)

4 SIDES   OPPOSITE SIDES ARE PARALLEL

Int: *Which ones are working together this time?*
Brendan: *Um sides and angles, opposite sides are parallel and opposite angles are equal. Opposite sides are equal is by itself, oh no it could probably join with opposite sides are parallel as well, actually it could go in with both of them.*

Brendan's response to the rhombus property task is classified as $M_1(F)$ due to a focus upon isolated property relationships. Brendan's necessity to add a property card "all sides aren't equal" is also characteristic of this level. It is interesting to note that Brendan made a link to other properties and figures, however, he did not transfer the class inclusion notion adopted in the relationships among quadrilateral figures to reconcile the need to distinguish the rhombus from the square:

MINIMUM COMBINATION 1 (rhombus)

4 SIDES   ALL SIDES ARE EQUAL

[plus angles aren't all equal]

Brendan: *I can't sort of just leave them because all of them are properties of a square as well. I don't have anything to define between the square and the rhombus.*
Int: *Is that a problem?*
Brendan: *Yes.*
Int: *Would you like to add a card?*
Brendan: *Yes probably.*
Int: *What would you need?*
Brendan: *One that says the angles aren't all equal.*
Int: *What would you put with that card?*
Brendan: *All sides are equal and all angles are equal.*

Contextual comparison

Overall, Brendan responded at higher SOLO levels in the context of relationships among quadrilateral properties compared with relationships among triangle properties. Of particular interest are Brendan's Intervention 1 responses to the relationships among equilateral triangle properties, relationships among right isosceles triangle properties, and relationships among square properties. Each of the responses was characterised as second cycle concrete symbolic due to Brendan's focus upon the unique properties of the square and equilateral triangle, and his perceived uniqueness of the right isosceles triangle. Hence, when providing and justifying minimum property descriptions, Brendan focused upon the figure determining the properties. In the second intervention, Brendan had shifted to a focus upon the relationships among the properties determining the figure, and hence, each of his responses was characterised as a formal mode response. Brendan focused upon the relationships that exist among the properties in both the triangle and quadrilateral context during Intervention 2.

Integration of Figure and Property Relationships

In both tasks concerning relationships among figures, and tasks concerning relationships among properties, Brendan responded at a higher level in the quadrilateral context, than in the triangle context. In the triangle context, Brendan responded at M_2(CS) concerning relationships among triangle figures. Brendan's responses concerning relationships among triangle properties were in the second cycle of the concrete symbolic mode in the first intervention, and then progressed to the formal mode in the second intervention. While Brendan encapsulated the known properties of classes of triangles and referred to these by name, he did not identify links across triangle classes. It is interesting to note that Brendan was consistent in the triangle property task in Intervention 1 as his focus remained on the properties determining the figure, thus precluding links as workable units between properties and between figures. The response to the equilateral property task was categorised as U_2(F) in the second intervention, however, this did not assist Brendan to proceed above M_2(CS) in relationships among triangle figures.

In the quadrilateral context, Brendan applied notions of class inclusion consistently in both interventions. In the light of his responses to relationships among quadrilateral properties, he predominantly responded within the formal mode. In the first intervention, Brendan responded inconsistently to the relationships among square properties (R_2CS).

Hence, although Brendan had reached the second cycle of the formal mode in respect to relationships among quadrilateral figures, the relationships among square properties were not as developed as the parallelogram and the rhombus. This was not the case in second intervention where Brendan focused upon the relationships among the properties determining the quadrilaterals, and he applied notions of class inclusion to the quadrilateral property task.

STUDENT 3: SCOTT

When initially interviewed, Scott was enrolled in Year 8 and was 14 years and 4 months of age. At the time of the second intervention Scott was enrolled in Year 10 completing the Advanced Mathematics Course and was aged 16 years and 9 months.

Results from Interventions 1 and 2

A summary of Scott's responses, demonstrating progression in terms of SOLO categorisations in response to interview tasks in both Interventions 1 and 2 are presented in Table 7.3.

Table 7.3 Scott's SOLO codings for Interventions 1 and 2

Concept	Context	SOLO Coding	
		Int 1	Int 2
Relationships Among Figures	Triangles	M_2/R_2 (CS)	R_2 (CS)
	Quadrilaterals	M_2 (CS)	R_2 (CS)
Relationships Among Properties	Equilateral Triangle	U_1 (F)	U_2 (F)
	Right Isosceles Triangle	U_1 (F)	M_1 (F)
	Square	U_1 (F)	R_1 (F)
	Parallelogram	R_2 (CS)	U_2 (F)
	Rhombus	U_2 (CS)	U_1 (F)

Scott was chosen as one of the case studies as he consistently responded at a higher level in the second intervention when compared with the first intervention. While there was a consistent change, Scott maintained a mixture of concrete symbolic and formal mode responses within each intervention. Responses to tasks concerning relationships among figures were all in the second cycle of the concrete symbolic mode, ranging from M_2 to R_2. Scott also demonstrated progression in each of the tasks concerning relationships among properties, with movement from a mixture of second cycle responses (CS) and first cycle (F) responses to consistently formal mode responses. Due to this developmental pattern, the following subsections contain a comparison of Intervention 1 and Intervention 2 responses for each task type. These comparisons are divided into relationships among figures, relationships among properties, and integration of figures and properties.

Relationships Among Figures

Triangle context – Interventions 1 and 2

Over the two-year period, Scott responded at a higher SOLO classification concerning relationships among figures in both contexts. Scott responded at a transitional level between M_2(CS) and R_2(CS) in the triangle context which shifted to R_2(CS) in the second intervention. In the quadrilateral context, Scott initially responded at M_2(CS) and in the second intervention progressed to R_2(CS). The transcript below, taken from Intervention 1, illustrates the manner in which Scott made reference to a link between the equilateral triangle and the isosceles class of triangles. When asked to justify this link, Scott was hesitant and withdrew the link due to observed differences:

Scott: *This one has got three sides the same and this one has got two.*
Int: *Well how can they relate together?*
Scott: *Because they have both got two sides the same and one of them has to have the other and it can have two angles the same oh no they can't that can have 60 and all the same.*
Int: *So do you think that they can be linked together for angle reasons?*
Scott: *No they can't.*

In the second intervention, Scott preferred to work with triangle names instead of sketches on the tree diagram. During this intervention, Scott placed in all angle-type links with the addition of a link from the equilateral triangle to the acute isosceles triangle. Scott also linked the equilateral to the isosceles class of triangles as "they have both got two sides that are the same length":

Int: *Why is the acute angled scalene linked to the obtuse angled scalene?*
Scott: *Because neither of them has got angles the same and none of the sides are the same, they are just all different.*
Int: *Why does the isosceles link to the equilateral?*
Scott: *They have both got two sides that are the same length. Like this one has got two and this one has got three, but they are both like equal.*

Later in the intervention Scott moved his focus to angle-type links. Scott acknowledged that the equilateral triangle more appropriately links to the acute isosceles, however, he did not consider the equilateral triangle as a member of the isosceles class of triangles. It is evident that Scott separated links based upon triangle-type and those based upon angle-type links. This is particularly apparent when he described the link between the acute-angled scalene triangle and acute-angled isosceles triangle as "half related":

Int: *Now the equilateral, you have got isosceles written there and then you have got the different types of isosceles, which one does the equilateral relate to?*
Scott: *More to the acute wouldn't it?*
Int: *How come?*
Scott: *Because it doesn't have any obtuse, it doesn't have any large angles on it.*
Int: *Is it related to the other isosceles triangles at all?*

Scott: *Um I am not sure, maybe to the acute.*
Int: *What would your reason be?*
Scott: *Because there are no obtuse angles.*
Int: *Any other reason why it is linked to the acute isosceles triangle?*
Scott: *No.*
Int: *Why do you have the right-angled isosceles to the right-angled scalene?*
Scott: *Because they have both got a right angle which puts them into the same kind of group.*
Int: *What about these?*
Scott: *Well they are scalene, so they are all scalene triangles.*
Int: *What about this here?*
Scott: *Well that is an acute-angled scalene so it has got an acute angle so it is half related to the isosceles, and this one has got an obtuse so it is related to the obtuse.*

Quadrilateral context – Interventions 1 and 2

Similar to Brendan, Scott also incorporated relationships among properties to the markings he placed on the diagrams of quadrilaterals. This is evident in the following transcript:

Int: *Is there a reason why we only need to put one right angle on that square?*
Scott: *Because if you have one right angle, all the rest will be right if all the sides are equal, and if they are parallel and all that.*
Int: *So why will they all be right angled?*
Scott: *Because if they are parallel to one another and they are the same size than the angles will be the same due to the rest like it will be corresponding and alternate and everything else.*

Scott made links between quadrilaterals in the first intervention, based upon more than one property. While Scott was able to make links such as the square to the rectangle, the parallelogram to the rectangle, rhombus to the square, and parallelogram to the rhombus, Scott made no links between the square and parallelogram, or rhombus and rectangle. Scott's response during Intervention 1 is typical of M_2 (CS):

Int: *Do you think that shows how all the quadrilaterals relate together?*
Scott: *Yes these have got to have all the sides the same or two sides the same and parallel lines and corresponding.*
Scott: *Well I can link to the parallelogram here, the rectangle, rhombus and the squares as they have each got two sides the same, they have got more with the parallelogram and the rectangle as they only have two sides the same as each other.*
Int: *Could I link it to the rhombus?*
Scott: *Yes because they got two sides the same.*
Int: *Could it link to your trapezium?*
Scott: *No because it has got no parallel lines.*
Int: *These ones have got parallel lines but it is still linked to that.*
Scott: *They have got the same sort of sides as it.*

During the second intervention, Scott began the interview by making similar links; however, his justifications made on the basis of similar properties are more detailed. It is also interesting to note that Scott linked the kite to the parallelogram in this intervention due to having "two sets of like equal sides." Scott also stated that the trapezium could link to any shape that has "parallel sides." The justifications made by Scott, and additional links to the trapezium and kite are illustrated in the following transcript:

Int: *Why are the rhombus and the square related?*
Scott: *Because they have both got four sides, and they have both got two sets of parallel sides, and they are all equal.*
Int: *What about the rhombus to the parallelogram?*
Scott: *They have got two sets of parallel sides and one, um each has two sets of angles that are equal.*
Int: *Your square to your rectangle?*
Scott: *They have got two sets of even sides, like that has got two sets, they have all got right angles and two sets of parallel lines.*
Int: *Your parallelogram to your rectangle?*
Scott: *They have both got two sets of equal parallel lines.*
Int: *And your parallelogram to your kite?*
Scott: *It has got two sets of like equal sides.*
Int: *Your trapezium to your parallelogram and rectangle?*
Scott: *It has got a set of parallel sides.*
Int: *What about your scalene quadrilateral?*
Scott: *Because it is related to all of them just because it has got four sides.*
Int: *Are there any other links that you could make in there?*
Scott: *I suppose that the trapezium could go to any of them with parallel sides.*

At this stage of the response, Scott had responded at M_2 (CS), however, when probed by the question "Any other links?" Scott spontaneously added the more difficult links that were not provided in the first intervention, these being, the rectangle to the rhombus, and the parallelogram to the square. Each of these links was justified on the basis of similar properties, raising the SOLO level to R_2(CS). Scott described links on the basis of similar properties which reconciled the differing properties accentuated by visual cues:

Int: *Any other links?*
Scott: (pause) *The rectangle to the rhombus.*
Int: *Why is that?*
Scott: *Because it has got two sets of equal angles.*
Int: *Any others?*
Scott: *The square to the parallelogram because it has got four parallel sides.*

Contextual comparison

Overall, when responding to tasks concerning relationships among figures, the structure of Scott's responses developed over the two-year period. In the first intervention, Scott's language-use included terms such as "this one has got three sides the same and this one has got two," and, "it can have two angles the same, oh they can't, they can have 60 and all the same," when providing an M_2/R_2(CS) transitional response in the context of triangles. Similarly, when providing an M_2(CS) response in the quadrilateral context, Scott described the links in terms of "they are parallel to one another and they are the same size."

It is interesting to note that when Scott did link the equilateral with the isosceles class of triangles in the second intervention he justified this using "they are both like equal." In comparison, when Scott provided an R_2(CS) response in the context of quadrilaterals and made links, which were not supported by visual cues, his language had become more precise as he stated "two sets of equal parallel sides" and "two sets of equal angles." Thus the inclusive nature of the description allowed "four equal sides" to link with "opposite sides equal."

Relationships Among Properties

Triangle Context – Interventions 1 and 2

In the first intervention, Scott provided a U_1(F) response to both tasks concerning relationships among triangle properties. When responding a second time after a two-year period, Scott provided U_2(F) and M_1(F) responses to the same tasks. Scott focused upon a single property relationship in the context of the right-angled isosceles triangle and the equilateral triangle. The excerpt below illustrates Scott's focus upon the relationship between equality of sides and equality of angles:

MINIMUM COMBINATION 1 (right isosceles triangle)

3 ANGLES HAS RIGHT ANGLE 2 SIDES EQUAL

Scott:	*I could take one of these away out of the angles and sides. And this one away.*
Int:	*Why can you take that one away when it definitely still has that?*
Scott:	*Because if you leave this one here either two angles equal or two sides equal then there are three angles in there and a right angle then you have enough to figure out what it is. You know it has a right angle and two sides equal.*

MINIMUM COMBINATION 2 (right isosceles triangle)

3 ANGLES HAS RIGHT ANGLE 2 ANGLES EQUAL

Scott:	*I have to leave three angles so that they know it is a triangle, I have to leave that so they know it has two angles. I have to leave something to do with two sides or angles and the right angle.*

In the second intervention, Scott's response was classified as M_1(F) to the task concerning relationships among right isosceles properties as he included multiple property relationships that were not an integrated focus of the response. In addition to the relationship utilised in Intervention 1, Scott also focused upon the relationships between equality of sides and symmetry:

Int: *Which ones do you think are working together this time?*
Scott: *Well these ones always do, you know if it has got three sides it has got three angles, um two sides and two equal angles.*
Int: *Where does the axis of symmetry come in?*
Scott: *If it has got two sides equal it only has one axis of symmetry.*
Int: *Does the acute angle card work with any?*
Scott: *It is just an extra one that is not necessary.*

This analysis was taken a step further in the context of the equilateral triangle as Scott focused upon the interrelationships among equality of sides, equality of angles, and symmetry (U_2, F). Scott utilised the connections among relationships between properties when providing minimum combinations and summed up by saying:

Int: *Which ones are working together this time?*
Scott: *Three sides and three angles, three sides equal and three angles equal and three axes of symmetry work together.*

Quadrilateral context – Interventions 1 and 2

In the quadrilateral context, Scott correctly provided a minimum combination for the square based upon four axes of symmetry at the U_1(F) level in the first intervention. It was evident that the focus of the response was upon the relationships among the properties determining the figure. Scott sometimes provided superfluous properties, depending upon whether the property belonged to the quadrilateral. The R_2(CS) response concerning the parallelogram below included properties that correctly belonged to the quadrilateral and it is evident that Scott was unable to justify his response adequately in terms of links among figures or links among properties:

MINIMUM COMBINATION 1 (parallelogram)

$\boxed{\text{4 SIDES}}$ $\boxed{\text{4 ANGLES}}$ $\boxed{\text{OPPOSITE SIDES ARE PARALLEL}}$
$\boxed{\text{OPPOSITE SIDES ARE EQUAL}}$

Int: *You think that you would need to leave both the opposite sides are equal and the opposite sides are parallel?*
Scott: *Yes they would need to see both.*

MINIMUM COMBINATION 2 (parallelogram)

$\boxed{\text{4 SIDES}}$ $\boxed{\text{4 ANGLES}}$ $\boxed{\text{OPPOSITE SIDES ARE PARALLEL}}$
$\boxed{\text{OPPOSITE ANGLES ARE EQUAL}}$

Scott: *That should do it too. To know it is a parallelogram they would not need the others. This is enough to not say it is one of the others.*

Scott's response in the rhombus context involved the identification of one property that belonged to the rhombus. This property was considered in isolation. When prompted to consider the properties of the square, and justify on the basis of a link to other properties or figures, Scott incorrectly stated that "the diagonals don't meet at right angles on a square":

MINIMUM COMBINATION 1 (rhombus)

4 SIDES | DIAGONALS MEET AT RIGHT ANGLES

Scott: *Just that is enough.*
Int: *It couldn't be any other shape?*
Scott: *No.*
Int: *What if they said a square with one and nine?*
Scott: *They couldn't do it because the diagonals don't meet at right angles on a square.*

The second intervention sees Scott providing comparatively higher level responses in each of the quadrilateral contexts. In the rhombus context, Scott's response shifts from U_2(CS) to U_1(F) as he focused upon the fact that "opposite angles are equal and opposite sides are equal work together." The excerpt below illustrates the manner in which Scott began his response to the task concerning relationships among square properties by making links from the square to the rhombus based upon similar properties. In addition, Scott focused upon relationships between properties, such as "if the opposite sides are equal then the opposite angles will be." Scott made a reference to the interrelationships among properties when he stated "all sides are equal can work with any of them," however, this was not justified and he continued and stated "only a square can have four axes of symmetry so it really works by itself":

MINIMUM COMBINATION 1 (square)

ALL SIDES ARE EQUAL | THERE ARE 4 RIGHT ANGLES

Int: *Why is that enough?*
Scott: *Because all sides are equal, so you know it is either a rhombus or a square, and if it has four right angles, then it has to be a square.*

MINIMUM COMBINATION 2 (square)

ALL SIDES ARE EQUAL | 4 AXES OF SYMMETRY

Int: *Why will that work?*
Scott: *Because it has got all sides are equal so it has to be a rhombus or a square, and if it has got four axes of symmetry and the rhombus doesn't have four axes but the square does.*

MINIMUM COMBINATION 3 (square)

ALL SIDES ARE EQUAL DIAGONALS ARE EQUAL

Int: *Why will that work?*
Scott: *Because the diagonals all meet in the centre and they are all the same length, so all the sides are equal so it will have to be a rhombus or a square.*

MINIMUM COMBINATION 4 (square)

THERE ARE 4 RIGHT ANGLES DIAGONALS ARE EQUAL

Int: *Which ones do you think are working together here?*
Scott: *All sides are equal can work with any of them.*
Int: *What about the four axes of symmetry?*
Scott: *It only works, um like it can work with any of them because it says um that only a square can have the four axes of symmetry so it really works by itself. The right angles are working on its own.*
Int: *Is there another reason why it works rather than just saying it is a square?*
Scott: *I am not sure.*
Int: *What about these others?*
Scott: *Opposite sides are parallel tells you that they are parallel. Um if the opposite angles are equal then the opposite sides will be.*
Int: *What about the diagonals here?*
Scott: *Well if the diagonals are equal they usually bisect.*

In the parallelogram context, it is evident that Scott focused upon the interrelationships among properties as he justified his correct combinations as:

Scott: *Four sides and four angles, and opposite sides are equal and opposite angles are equal, and this one works on its own (opposite sides are parallel) but it could go with these two as well.*

Contextual comparison

The majority of Scott's responses in both contexts fall into the formal mode, hence, the relationships among the properties determine the figure. It is evident that Scott utilised the relationships among known properties as the primary focus when providing minimum descriptions of figures. Scott continually made statements such as "if it has got two sides equal then it has only one axis of symmetry." In some cases Scott was able to integrate relationships and maintain an overview as he described clusters of properties which 'worked together.'

Integration of Figure and Property Relationships

In each of the seven tasks Scott responded at a higher SOLO categorisation in the second intervention when compared with the first intervention while maintaining a mixture of concrete symbolic and formal mode responses. The first intervention was characterised by responses that focused upon the three classes of triangles in which their class names encapsulated the known properties. Scott also focused upon a relationship between two properties when providing minimum combinations. Hence, in both tasks concerning the relationships among triangles and tasks concerning the relationships among triangle properties, Scott did not require specific examples of the triangles to make links or provide minimum combinations, but instead, he focused upon the properties that determined the class.

In the second intervention, Scott's responses in the triangle context were more complex in character as the responses incorporated a link between the equilateral and isosceles classes of triangles, as well as angle-type links across each triangle-type class. When providing responses to relationships among triangle properties Scott no longer needed a real world referent and focused upon the relationship among known properties.

In the context of quadrilaterals, Scott's response in the initial intervention to a task concerning relationships among figures was characterised by links across quadrilateral classes, with the exception of links that were not supported by visual cues. Thus, Scott utilised quadrilateral names to represent a selection of properties belonging to the class of figures, and made links across classes, except for the rectangle to rhombus, and the parallelogram to the square. During this intervention, Scott relied upon the parallelogram and rhombus figures to determine the properties, and hence responded in the second cycle of the concrete symbolic mode (R_2 and U_2 respectively).

In the second intervention, the two links previously not made by Scott when linking quadrilateral figures became a new feature of the response. Scott no longer focused upon the differences in properties between shapes, but reconciled this and acknowledged the similarities observed. Similarly, when responding to a task concerning relationships among quadrilateral properties, Scott perceived the property relationships as determining the figures, and hence focused upon the relationships that existed among the properties of other quadrilaterals or the properties of the figure in question. Of particular interest is that Scott, in the second intervention, focused upon the relationships among the known properties of the rhombus and parallelogram, and was able to make and justify the difficult links concerning relationships among quadrilaterals.

STUDENT 4: LOUISE

The fourth case study to be examined is that of Louise. Louise was in Year 8, aged 14 years and two months at the time of the first intervention. At the time of the second intervention, Louise was aged 16 years and was enrolled in the Year 10 Advanced Mathematics Course.

Results from Interventions 1 and 2

The SOLO categorisations of each of Louise's responses to the interview tasks in both Interventions 1 and 2 in are presented in Table 7.4.

Table 7.4: Louise's SOLO codings for Interventions 1 and 2

Concept	Context	SOLO Coding	
		Int 1	**Int 2**
Relationships Among Figures	Triangles	M_2/R_2 (CS)	M_2/R_2(CS)
	Quadrilaterals	M_2(CS)	R_2 (CS)
Relationships Among Properties	Equilateral Triangle	R_2(CS)	M_1 (F)
	Right Isosceles Triangle	R_2(CS)	M_1 (F)
	Square	U_1(F)	M_1 (F)
	Parallelogram	U_1 (F)	M_1 (F)
	Rhombus	M_1(F)	U_1 (F)

Louise was chosen as one of the four studies, as there was little change evident in her responses to relationships among figures between Intervention 1 and Intervention 2. Louise's responses to relationships among properties did alter SOLO categorisations across interventions. Also, Louise's response to the rhombus task in Intervention 2 (U_1F) was at a lower SOLO level than in Intervention 1 (M_1F). The structure below is divided into the context of tasks where Interventions 1 and 2 are compared across each intervention.

Relationships Among Figures

Triangle context – Intervention 1 and 2

Both Louise's responses to tasks concerning relationships among triangles in Interventions 1 and 2 were categorised as transitional between M_2 and R_2 (CS). While Louise mentioned in the first intervention the possibility of the equilateral triangle as linking to the isosceles class of triangles, the two-year period between interventions did not reconcile the differences observed. As illustrated below, Louise focused upon the three separate classes of triangles, scalene, isosceles, and equilateral, which were described in terms of their known properties. Louise began by linking the equilateral to the isosceles class of triangles, and then hesitantly removed the link:

Louise:	*Well these ones because they are just right angles, these because they are isosceles, these because they are all scalene which means that none of the sides are the same. These being because they are obtuse and then I have the acute together.*
Int:	*Well how does the equilateral fit in?*
Louise:	*Well after you take out that it would come in because it has got two sides the same but it has got another one the same.*
Int:	*Any other reasons why it is linked?*
Louise:	*Well with the acute ones and I think that you could have it linked with the obtuse.*
Int:	*Do you think that you could have an obtuse angled equilateral?*
Louise:	*I think you could, couldn't you, if it went out like that, no I don't think that you could because if it went out like that it would make it into an isosceles. I think that the main reason is that that one would be an acute so it could go with that one.*

In Intervention 2 the same hesitation is held in regards to the equilateral linking to the isosceles class of triangles. After explaining all other angle-type and triangle-type links, Louise stated "I am not quite sure where I can put the equilateral triangle in on that. I might just have to leave that one." When Louise returned to her diagram a second time during the intervention she repeated the links made, as described below; she repeated that "I would leave it" when referring to the equilateral. In the same excerpt, Louise made a similar statement to Scott's expression "half related" when she stated "I think it is a bit of a contradiction":

Louise:	*All right the right angled isosceles is linked to the acute angled isosceles and an obtuse angled isosceles for the fact that they are all isosceles triangles which means they have two sides the same length on the triangle. Um, it is also linked with a right-angled scalene because they both have right angles which is 90 degrees. Um, then the acute angled isosceles is linked with the obtuse angled isosceles because they are both isosceles. Um, the acute angled scalene is linked with the right-angled scalene and the obtuse angled scalene because they are all scalene which means that their sides are not of the same length. Um the obtuse angled scalene is linked with the obtuse angled isosceles because they both have obtuse angles which is over um 90 degrees. The acute angled scalene is linked*

with the acute angled isosceles because they both have acute angles, and I have linked the equilateral triangle with the acute angled isosceles because they have acute angles. I didn't link it with the acute angled scalene because I think it is a bit of a contradiction because that is noted as a scalene, um and it doesn't have any sides the same, but I still put it with the isosceles even though that has two sides the same not all three because I thought it was better fitted there instead of there.

Int: *Would you link the equilateral with any of the other isosceles?*

Louise: (pause*) Um,* (pause*) well I wouldn't link it with an obtuse angled isosceles because an equilateral triangle can't have an obtuse angle. Um, and I wouldn't link it with the right-angled isosceles because an equilateral triangle can't have a 90 degree angle so I would probably leave it.*

Int: *Any other links that you might have missed?*

Louise: (pause*) I don't think so.*

On Louise's final return to her diagram, a link is made from the equilateral to the acute scalene triangle. When prompted to also link the equilateral to the acute isosceles triangle, Louise acknowledged similar properties, differences, and an angle-type link. Finally, Louise decided to make a single link from the equilateral triangle to the acute isosceles triangle due to acute angles:

Int: *How did your equilateral triangle fit into that diagram?*

Louise: *Um, that one I have done, that one I have, um oh actually that one is different, that one I have connected it with the scalene acute. But I have fitted it in there because they all have acute angles so it is really a bit out of it, but I think that is probably where it is best suited.*

Int: *Is that better than to the acute isosceles?*

Louise: *Um, oh not necessarily, I think it would in a sense be good to put two links actually on second thoughts, two links with the equilateral because um it does have equal sides, oh sorry it is that one, because even if it doesn't have all the properties of those triangles it has one property.*

Int: *What is that one property?*

Louise: *The one property is the fact that they have acute angles.*

Quadrilateral context – Interventions 1 and 2

In the quadrilateral context, Louise's response is categorised as M_2(CS) due to her use of class names to encapsulate similar properties and make links between classes that are assisted by visual cues. Louise did not mention a link between the square and parallelogram, or rhombus and rectangle. When prompted to join the rectangle and rhombus, the decision is made that "I don't think they can go together":

Louise: *That one goes to the rhombus because they have the same length sides. That one there would be allied angles.*

Int: *What do you mean by allied angles?*

Louise: *Um something like cointerior.*

I: *What do you mean by that?*

Louise: *Well that one would go with that. Um I think that these ones go together because of the allied angles they both make up the 180 degrees so they go together because they are allied.* (rhombus to the parallelogram*) That and* *that would go together because the sides are the same length. I mean they are equal in length. The angles are the same on each one here.*

Int: *So does the rhombus have equal angles too?*

Louise: *Um no it doesn't does it? I think, no it doesn't have equal angles sorry.*

Int: *What about the kite?*

Louise: *The kite would go in here um* (connected to the rhombus)*. It could go with the rhombus because on both the opposite angles are the same. The trapezium can be added to the um those. I am trying to think of how it comes in. These two go together because both of the sides are the same like these two are equal and these two are equal.*

Int: *Do you think the trapezium fits in anywhere?*

Louise: *I am just trying to think. It would go with these two because these sides are the same yeah they are the same and then um.*

Int: *Have you got all the links in?*

Louise: *I am not sure. I think there might be something else that I am missing.*

Int: *Could I join the rectangle with the rhombus?*

Louise: *Um the angles will be the same because they are all right angles, no I don't think they can go together.*

The second intervention begins in the same manner as Intervention 1, and is typical of M_2(CS). When asked why she did not link the rhombus to the rectangle, Louise spontaneously replied:

Louise: *I probably should have because they both have two pairs of parallel sides.*

Int: *Do you think they can link?*

Louise: *Yes I think they can because they are parallel and they are parallel and that is a property.*

When probed to make further links, Louise linked the square and the parallelogram. The following discussion took place:

Int: *Why did you link the square to the parallelogram?*

Louise: *Because they are parallel as well.*

Int: *Why do you think that you didn't put that in before?*

Louise: *I probably just didn't see it as a um, I think because people tend to relate a square to a rectangle, and a rhombus to a parallelogram and that is probably why I didn't look at it as an important thing.*

Due to the spontaneous reply and justification of the more difficult links, Louise's second intervention response is coded as R_2(CS). At the end of this task, Louise was prompted to form a class of parallelograms, but this was not incorporated into her response:

Int: *Is there any reason why these are linked together here?*

Louise: *Um other than what I have already said?*

Int: *Yes.*

Louise: *Um, these four, um* (pause) *I am not quite sure.*

Contextual comparison

In both contexts, Louise has responded predominantly at $M_2(CS)$ to $R_2(CS)$ concerning relationships among figures. Louise responded at transitional $M_2/R_2(CS)$ during both interventions concerning relationships among triangle figures. Louise utilised phrases such as "two sides the same but it has got another one the same" in both interventions. While Louise acknowledged a similarity between the isosceles class of triangles and the equilateral class of triangles in terms of "sameness" the differences are accentuated, and Louise made the decision not to make the link. In the quadrilateral context in Intervention 1, Louise did not make the difficult links between quadrilateral classes; she accentuated differences, and utilised similar language as in the triangle context. During the second intervention, Louise was content with making links between the parallelogram and square, and rhombus and rectangle, on the basis of similar properties. The property differences were no longer the focus of the response in the context of quadrilaterals.

Relationships Among Properties

Triangle context – Interventions 1 and 2

When responding within the triangle context, Louise responded at $R_2(CS)$ in both property tasks. The second intervention responses to the same task were categorised as $M_1(F)$. It is evident in the transcript below that Louise sometimes provided superfluous properties, however, she made reference to a link between equality of sides and equality of angles. This link has not yet formed a workable unit as Louise did not utilise the link to form the basis of her minimum combination, hence requiring her to include "axis of symmetry." A similar structure and focus was evident in Louise's response to the equilateral task during Intervention 1:

MINIMUM COMBINATION 1 (right isosceles triangle)

3 SIDES	HAS RIGHT ANGLE	AXIS OF SYMMETRY
2 ANGLES EQUAL		

Louise: *Two angles and two sides mean the same thing so I can have either one of those …*

Int: *How come those two can swap?*

Louise: *Because if you have two sides that are equal then the angles are going to be the same too and the other one is three sides because that means three angles.*

The second intervention responses provided by Louise concerning relationships among triangle properties are coded as $M_1(F)$. The focus of her response was upon the relationships that exist between equality of sides and equality of angles, equality of sides and symmetry, equality of angles and symmetry, however, an overview was not evident to allow a focus upon the interrelationships that exist:

Int:	*Why just those two?*
Louise:	*Because three sides tell you it is a triangle, and three sides equal tells you that automatically it is an equilateral.*
Int:	*Is there anything else that three sides equal tells you?*
Louise:	*It also tells you that all the angles are equal.*
Int:	*Anything else?*
Louise:	*It tells you that it also has three axes of symmetry.*

MINIMUM COMBINATION 2 (equilateral triangle)

3 SIDES 3 AXES OF SYMMETRY

MINIMUM COMBINATION 3 (equilateral triangle)

3 ANGLES 3 AXES OF SYMMETRY

MINIMUM COMBINATION 4 (equilateral triangle)

3 ANGLES 3 ANGLES EQUAL

Int:	*Now you are moving these cards in and out, can you tell me which ones are working that way with each other?*
Louise:	*Um so in other words, ones that are similar?*
Int:	*What do you mean by similar?*
Louise:	*When you say that three angles indicate, um like I could say three angles indicates the same sort of thing as three sides because three sides means you have three angles and three angles means you have three sides. Um three angles equal and three sides equal, three axes of symmetry and three angles equal can go together as well.*
Int:	*Is there any other one that three axes of symmetry could go with?*
Louise:	*Three sides equal, I think that is about it.*

When Louise is prompted to discuss the interrelationships that exist among the isolated property relationships between pairs of properties, she is hesitant to group the relationships together:

Int:	*So do you think those three all go together?*
Louise:	*Yeah pretty much, oh um.*
Int:	*Do you think so, or are you a bit worried about that?*
Louise:	*Um* (pause) *not really worried.*
Int:	*What concerns you?*
Louise:	*Um nothing, it is okay like that.*

Quadrilateral context – Interventions 1 and 2

The following three excerpts, taken from Intervention 1 within the quadrilateral context, demonstrate the manner in which Louise's responses are each categorised as within the first cycle of the formal mode. In the first intervention, Louise's combination for the square is correct, and the justification reveals that the focus is upon a single relationship between equality of sides and equality of angles that results in the square. Hence, the relationships among the properties determine the figure (U_1F):

MINIMUM COMBINATION 1 (square)

ALL SIDES ARE EQUAL | THERE ARE 4 RIGHT ANGLES

Int:	*Why can all those go?*
Louise:	*Well when um if you say all sides are equal and you know that the sides are equal and then you have right angles well you know that it can't be a rhombus because all the angles aren't right and from this you know it has to be a square.*

Louise's response within the parallelogram context focused upon the link between the properties of two figures, these being the parallelogram and rectangle. The focus remained upon the properties that determine particular quadrilaterals, however, since the properties of the parallelogram cannot be selected to produce a combination which is not a subset of the properties of other quadrilaterals, such as the square, rectangle, and rhombus, Louise provided additional properties:

MINIMUM COMBINATION 1 (parallelogram)

OPPOSITE SIDES ARE PARALLEL | OPPOSITE SIDES ARE EQUAL | OPPOSITE ANGLES ARE EQUAL

Int:	*I would need the opposite sides are parallel but that could be a rectangle. Um that would make it out like that one the angles. Yeah I think that is right.*
Int:	*Could you make that simpler?*
Louise:	(pause)
Int:	*Do you think that you would need to keep opposite sides are equal?*
Louise:	*I think I would still need it because I have the opposite sides are parallel and the opposite angles are equal. I think I would need to leave that the opposite sides are equal to be sure.*
Int:	*Could you do it another way?*
Louise	(pause*) um I am not sure. No I don't think so.*

Louise identified the same problem in the rhombus task, but she made a combination based upon the link between the properties of the rhombus and the square, and the relationship between "opposite angles are equal" and "all sides are equal." Louise also perceived "one axis of symmetry," as 'only one' axis of symmetry. This response is categorised as $M_1(F)$:

MINIMUM COMBINATION 1 (rhombus)

4 ANGLES | ALL SIDES ARE EQUAL | 1 AXIS OF SYMMETRY

Louise:	*I think those ones. I think that you could probably even take opposite angles are equal out.*
Int:	*How come?*
Louise:	*Well if you know that all the sides are equal and it has got four angles, well that makes it either a rhombus or a square, and if it has only got one axis of symmetry well that means it has got to be a rhombus because a square has four.*

In the second intervention, Louise provided $M_1(F)$ responses for both tasks concerning relationships among square properties, and relationships among parallelogram properties. The following excerpt demonstrates the focus upon the link between the properties of the square and the rectangle, and the following relationships considered in isolation: four axes of symmetry and equality of sides; four axes of symmetry and bisection of diagonals; equality of diagonals and equality of sides; and, diagonals meeting at right angles and four right angles:

MINIMUM COMBINATION 1 (square)

4 SIDES | ALL SIDES ARE EQUAL | THERE ARE 4 RIGHT ANGLES

Int:	*How does that work?*
Louise:	*Um, four sides tells you that it is a quadrilateral. Um four right angles tells you that it is either a square or a rectangle. And the fact that all the sides are equal tells you that it is a square.*

MINIMUM COMBINATION 2 (square)

4 SIDES | 4 AXES OF SYMMETRY

Int:	*Do you think you can just have those two?*
Louise:	*Yes because I think a square is the only regular quadrilateral that has four axes of symmetry.*
Int:	*Is there any other reason why four axes of symmetry works?*
Louise:	*Um, because it has got four equal, um I mean all sides are equal.*
Int:	*Anything else?*
Louise:	*Um because the diagonals bisect.*

MINIMUM COMBINATION 3 (square)

4 SIDES | DIAGONALS ARE EQUAL

DIAGONALS MEET AT RIGHT ANGLES

Int:	*Why will that work?*
Louise:	*Um because four sides tells you that it is a quadrilateral. Diagonals meeting at right angles tells you that um, because if they are meeting at right angle then that means that each of the corner angles is going to be a right angles as well, because if you add up the sum of a triangle. And the diagonals are equal which means that the sides are equal as well, I think. Oh I'm not sure if that necessarily means that the sides are equal but I think that it could be connected to that.*
Int:	*Which ones do you think are indicating each other?*

Louise: *Four sides and four angles, um, (pause) I think that if you put four right angles and four axes of symmetry together with one of those two they work together pretty well. Um,* (pause) *um (*pause*) those two maybe, but I think those two probably.*

Int: *Why only maybe?*

Louise: *Well it says three pairs of equal adjacent sides, and um, so that could indicate that the two pairs are different lengths. But it also could still mean that they are the same. So that is why I think that one is only a maybe.*

Int: *Any others?*

Louise: *Um* (pause*) the diagonals meet at right angles and there are four right angles, because, if um,* (pause*) the diagonals meet at right angles there are going to be four right angles.*

Int: *Do you think that is it?*

Louise: *Yes that is about all that I can work out.*

In the rhombus context, Louise responded at a lower level in the second intervention (U_1F) when compared with the first (M_1F). Louise began the task by questioning the symmetrical property of the rhombus by stating:

Louise: *I am not sure about the axis of symmetry on that.*

Louise included superfluous properties that belong to the rhombus; however, the focus of the response was a single link between the known properties of the square and properties of the rhombus. Louise would like to have rephrased the card "opposite angles equal" to "two pairs of opposite angles equal" in an attempt to achieve a selection that will eliminate the combination as being applicable to the square. It is evident that Louise was not focused upon the relationships that exist among the properties of the rhombus:

MINIMUM COMBINATION 1 (rhombus)

4 SIDES | ALL SIDES ARE EQUAL | OPPOSITE SIDES ARE PARALLEL

1 AXIS OF SYMMETRY

MINIMUM COMBINATION 2 (rhombus)

4 ANGLES | ALL SIDES ARE EQUAL

OPPOSITE ANGLES ARE EQUAL | 1 AXIS OF SYMMETRY

Louise: *I would like a card that said two pairs of opposite angles are equal to add to that. Because I think that that one is a bit, um I mean it could be something else like a square. See if you put it with that one instead with one axis of symmetry it wouldn't be a square.*

Int: *Any others?*

Louise: *um,* (pause) *not really actually, it would basically just be a variation just swapping one card around most probably.*

Int: *Which cards are indicating each other now?*

Louise: *Um, four sides and four angles, um,* (pause*) um I think that is about all.*

Int: *Are any of those others working together?*

Louise: (pause*) Not really, I don't think so.*

Contextual comparison

Of particular interest in Louise's responses to tasks concerning relationships among properties is the 'blurring' effect evident when properties not known by the student are attempted to be incorporated, such as symmetry. This occurred in both the triangle and quadrilateral contexts, specifically in responses to the right isosceles triangle task and the rhombus task. It was also evident that minimum combinations for the square were relatively easier to find than for the rhombus or parallelogram, due to the uniqueness of property combinations to depict the square and eliminate other quadrilaterals. The justification provided, however, depicts the level of the response.

Integration of Figure and Property Relationships

In the triangle context, Louise focused upon the three classes of quadrilaterals while making connections across classes based upon angle-type links. Louise acknowledged that the equilateral triangle does have two sides equal, and after hesitation maintained that a link could not be made due to differences observed. In the same interview, when responding to tasks concerning the relationships among triangle properties, Louise made reference to a link between properties of the particular triangle in question; however, she included superfluous properties that indicate that the link had not formed a workable unit.

Responses to tasks in the quadrilateral context concerning relationships among figures were of a similar structure as Louise's responses within the triangle context. While Louise considered the classes of quadrilaterals separately, and made reference to links across classes when supported by visual cues, she did not make reference to difficult links requiring the reconciliation of visual property differences.

When responding to tasks concerning relationships among properties, Louise focused upon two isolated links between properties and properties of other figures in the case of the rhombus. However, this focus did not extend to the contexts of the parallelogram and the square.

It is interesting to note that in the second intervention, in the context of relationships among quadrilaterals, Louise included links between the square and parallelogram, and rhombus and rectangle. During this intervention, Louise also shifted her focus to more than one isolated relationship among quadrilateral properties and properties of figures in the contexts of the square and parallelogram. This was not the case in the rhombus context where confusion concerning the symmetrical properties of the rhombus occurred.

CONCLUSION

The observed changes indicative of longitudinal development in the responses of the students were the result of a combination of Geometry instruction in accordance with the NSW Mathematics syllabus, and maturation over the two-year period. As the SOLO codings displayed in Appendix I indicate, the majority of students' responses to the seven tasks resulted in an increase in SOLO level at the second intervention. The longitudinal analysis discussed in Chapter 6 included twelve students. It was evident that eleven out of the twelve students indicated developmental growth over the two-year period.

When considering developmental change of the 84 responses provided over the seven tasks, approximately 72.6% of responses increased in SOLO level, 15.5% of responses stayed at the same SOLO level, and 11.9% of responses were at a lower SOLO level. Those students who responded at a lower SOLO level in the second intervention were mainly providing transitional responses which fluctuated between SOLO levels. For example, a M_2/R_2(CS) response was provided in Intervention 1 followed by an M_2(CS) response in Intervention 2 when addressing the same task. Those students who responded at the same SOLO level demonstrated a characteristically identical response in both interventions. Most responses improved in SOLO categorisation over the two-year period. When this improvement occurred the ideas communicated in the original response to tasks concerning relationships among figures were subsumed by new ideas. These new responses followed the same broad framework that emerged through the analysis of Intervention 1 responses. As a result, the longitudinal data supported the hierarchical framework that emerged concerning students' understanding of relationships among figures and relationships among properties.

The following research questions consider the developmental growth evident in the four student case studies.

Research Question 5.1 stated, *What are the similarities and differences of students' demonstrated understandings of the relationships among figures over a two-year period?* The changes observed over the two-year period within the four student case studies followed the general developmental pathway described in Chapter 5 and explored from a quantitative perspective in Chapter 6. The results indicate students may respond at different SOLO levels with reference to relationships among figures, depending upon the familiarity of the context.

The importance of contextual familiarity was particularly evident in Brendan's responses where he responded at M_2(CS) in the triangle context, and at U2(F) in the context of quadrilaterals, in both interventions. The other three case studies demonstrated development between Interventions 1 and 2 indicative of movement along the identified developmental pathway. Observed similarities and differences in response categories of Intervention 2 were characteristic of those identified in Intervention 1. Within each of the four case studies the similarities in language use were particularly evident across the contexts of triangles and quadrilaterals at the same SOLO level.

The SOLO framework provided a means for identifying similarities in the structure of the responses within different contexts which highlighted similarities in degree of difficulty within similar task types. While the overall degree of difficulty was found to be similar across contexts, individual differences were identified within each student case study.

Research Question 5.2 stated, *What are the similarities and differences of students' demonstrated understandings of and the relationships among properties over a two-year period?* The four student case studies indicated an overall increase in SOLO levels at the second intervention when compared with the first. The framework that emerged concerning relationships among properties was supported by the observable changes over the two-year period.

Of particular interest is the progression from the concrete symbolic mode to the formal mode, such as in Narelle's and Scott's cases. This change in SOLO categorisation was a result of moving from a focus upon the figure determining the property, to the relationships among the properties determining the figure. Hence, the progression saw a shift from finding unique properties that belonged to the figure, to a focus upon relationships between properties, or figures. The culmination was a focus upon the interrelationships among properties.

The case studies emphasised that, in the context of the square, particular students were able to provide minimum combinations but this ability did not necessarily mean that the focus was upon relationships among properties. This finding was particularly evident when observing the superfluous properties often included in the parallelogram and rhombus tasks in an attempt to differentiate these figures from the square. Hence illustrating the manner in which the property card manipulation provided a vehicle for eliciting conversation concerning justifications for the decisions made by the student concerning relationships among properties.

Research Question 5.3 stated, *Is there an interrelationship between the developmental pattern of understanding relationships among figures, and understanding relationships among properties?* The student case studies indicate a supportive influence between a student's understanding of the relationships among properties and relationships among figures. The case studies accentuate the need to form classes of figures which are recognised by name and encapsulated by properties to assist in the development of relationships among properties.

There is also an indication that reaching the formal mode and hence, perceiving the relationships among the properties as determining the figures, coincides in most cases with the identification of similar properties across classes of figures. In Narelle's case it appeared necessary for the property relationships to determine the figure before class inclusion notions emerged which were characterised by the recognition and subsequent justification of subsets via properties. In Louise's case, there was evidence of a progression to the formal mode concerning relationships among properties, while maintaining second cycle (CS) responses concerning relationships among figures.

CHAPTER EIGHT

CONCLUSIONS

Introduction

This chapter considers the overall findings of the study in which students' understandings of class inclusion concepts in Geometry were investigated. Initially, limitations imposed by the design of the study are discussed. An overview of the results in the light of the five research themes addressed in Chapters 4 to 7 are presented. This is followed by a consideration of the implications of the findings in relation to the van Hiele Theory, the SOLO model, and for the practice of teaching. Finally, a number of future research directions are generated as a consequence of the findings of the study.

POSSIBLE LIMITATIONS OF THE STUDY

It is necessary for the results described in the preceding chapters to be viewed in the light of possible limitations imposed by features of the research design. This section reviews the strengths and weaknesses of three aspects including: uniformity of the research sample; contextual difficulties associated with the interview tasks; and, the number of students in the sample when considering the Rasch analysis.

The first possible limitation concerns the sample for the study being taken from the top 30% of students from each school. The students interviewed were selected in the first year from the higher ability Year 8 course, and the Advanced Mathematics Course of Years 9 and 10. This group was targeted because of the difficulty, identified in the research literature, of students' understanding class inclusion concepts in Geometry. Initially, this high achieving sample may be perceived as one of little spread. However, when considered in the light of the findings there was considerable diversity within the responses supplied by the students. The purpose of the study was not to obtain population norms, but instead to identify potential developmental pathways in concept development. As a consequence, the diversity evident within the sample of students from similar mathematical backgrounds assisted in the validation of the research findings emanating from this study.

A second possible limitation concerns the student interview tasks that were used as a catalyst for discussion concerning students' understandings of class inclusion concepts in

Geometry. It is possible that the students may have found the tasks unfamiliar, and, hence, this may have caused contextual difficulties. However, both triangle and quadrilateral contexts are explicitly described as compulsory components of the Years 7–8, and Years 9–10 NSW Mathematics syllabi. To alleviate any contextual problems the design enabled students to return to each task on three occasions to provide an environment conducive to the provision of optimum student responses. The 'characteristic cards' associated with the interviews concerning relationships among properties may have contained properties and/or language that were not familiar to some students, however, the design allowed for student questions at any time during the interview for clarification when necessary. The provision was also made for students to add their own property characteristic cards if deemed necessary by the student.

The final issue involves the Rasch analysis. It may have been appropriate to have a larger, possibly broader, sample. When considering the intrinsic nature of the design, and the nature of the interview tasks and questions that entailed repeated exposure to the same stimuli, this was not possible within the context of the study, from both the perspective of the researcher and that of the participating schools. Nonetheless, there was a sufficient number in the sample for the Rasch analysis to be meaningful and to offer an important quantitative perspective to the large amount of qualitative data obtained.

Overall, despite the possible limitations imposed on the study by the nature of the research design and interview tasks, this discussion demonstrates that the effects of these factors were considered carefully during the design phase. While the possible limitations addressed constraints relating to the qualitative emphasis of the design, the study did raise a number of methodological issues concerning the use of interview tasks as a vehicle to promote student discussion concerning their conceptual understanding.

In essence, the design of this study allowed for the collection of detailed qualitative data in regards to students' understandings of class inclusion concepts in Geometry. Aspects of the design accounted for individual conceptual backgrounds of students within the sample and applied a predominantly qualitative approach inclusive of a longitudinal element while incorporating a quantitative perspective. In addition, this study utilised a validated empirical theoretical framework to assist in the interpretation of the response groupings.

OVERVIEW OF RESULTS

The focus of this thesis has been van Hiele's Level 3. This study had a predominantly qualitative perspective, and was designed in recognition of previous studies within class inclusion, exploration of the van Hiele Theory, and the SOLO model. Firstly, the design enabled the student to work within familiar, yet non-routine settings. This proved invaluable. The utilisation of student tasks as a catalyst to initiate discussion regarding students' understandings of relationships among figures and relationships among properties was successful. As a result, several categories of responses were identified and described. The use of these tasks provided the opportunity for the students to respond at a range of levels.

Further, a clearer picture of each student's perception was elicited with the use of prompts and probes by the interviewer. This approach addressed the need for students to identify and justify relationships and class inclusion, as opposed to the acceptance of a judgement only. It was also seen to be particularly beneficial to validate the identified hierarchical frameworks that emerged in Studies 1 and 2, through the utilisation of a longitudinal component. The quantitative perspective also shed light upon the similarities and differences concerning degrees of difficulty across response categories to relationships among figures, and relationships among properties, within two different contexts.

In summary, there are twelve major findings emanating from this study:

1. A generic developmental pathway leading to an understanding of class inclusion notions was identified. This pathway characterises an understanding of relationships among figures, and relationships among properties, which have not been considered previously in such depth.

2. The pathway concerning growth in understanding of relationships among figures is broadly described as the ability to: link individual classes which are supported by visual cues; relate classes on the bases of similar properties; form and utilise sub-class relationships, and, finally, integrate multiple sub-class relationships.

3. The growth in understanding relationships among figures requires the following development in thinking: properties perceived as features of figures which are spontaneously recognised without the formation of classes; recognition of similar properties and the acknowledgment of different properties which place groups of figures into classes; property differences between classes are no longer a primary focus and similarities between classes of figures are accepted; classes of figures develop an encompassing quality based upon the network of property relationships which are characterised by subsets; and, finally, multiple interrelationships among subsets are acknowledged and utilised.

4. The pathway concerning growth in understanding relationships among properties is broadly described as: isolation of properties recognised as unique to individual figures; ordering of properties; utilising relationships between properties; and, focusing upon interrelationships among properties.

5. The growth in understanding relationships among properties requires the following development in thinking: links between properties which require an ordering between two properties; the relationships among properties as determining figures; to, finally, the interrelationships among properties determining figures.

6. In general, the existence of formed property relationships supported the formation of sub-class relationships among figures. Thus the formation of sub-class relationships required the perception of the relationships among properties determining the figure.

7. The formation of property relationships did not emerge according to an identifiable sequence. Instead, the formation of property relationships is dependent upon each student's individual familiarity of properties. While property development did not appear in any particular order, a developmental pattern is evident in terms of language-use. At the lower level, descriptions of properties were exclusive in nature, while at subsequent levels the properties were inclusive.

8. This study validated the notion that thinking at a particular level in one context assists the progression to the same level in other contexts. While the contexts of triangles and quadrilaterals were chosen due to differing levels of complexity, this was not mirrored in the results.

9. It has been established by this study that the previously accepted characterisation of van Hiele's Level 3 highlights some difficulties. Of particular interest is the unpacking of students' understandings of class inclusion notions within the context of Geometry. This investigation provides empirical evidence to explain the perceived hurdle that is associated with an understanding of class inclusion notions in Geometry. In essence, behaviours previously described as requiring Level 3 thinking have been found by this study to include Level 3, Transitional Level 3/4, and Level 4.

10. Through the application of the SOLO model it has been possible to reconceptualise the van Hiele levels of thinking in Geometry. This impacts upon the overall characterisation of Level 2, 3, and 4 when considered in the light of the SOLO model. Previous generalisations of Level 3 have been found to be inappropriate through this present study. This links to previous work associated with the

characterisation of Level 1 and Level 2A where there is a Transitional Level 1/2 (Pegg & Baker, 1998). This is mirrored in the identification of a Transitional Level 3/4 in this study.

11. A transitional level between van Hiele's Level 3 and Level 4 was found. The thinking required in moving from Level 3 to Level 4 provides guidance concerning subsequent teaching implications and strategies. An understanding of class inclusion which involves the utilisation of a global overview of multiple systems of relationships, has been placed into van Hiele's Level 4.

12. This study has found the following SOLO connections to the development of class inclusion notions. These are contained in Table 8.1 below.

Table 8.1 Overview of developmental pathway leading to an understanding of class inclusion notions in Geometry.

Van Hiele levels	SOLO modes and levels
Trans Level 1/2	**R_1(CS):** Properties perceived as 'features.' Spontaneous groups formed upon identification of similar features. All known properties included in minimum descriptions.
Level 2A	**U_2(CS):** Formation of individual classes of triangles based upon one property. One property is utilised as a unique signifier when providing minimum descriptions.
Level 2B	**M_2(CS):** Formation of individual classes of triangles based upon more than one property. Links are made between figures on the basis of visual cues. More than one property utilised as a unique signifier when providing minimum descriptions.
Level 3	**R_2(CS):** Links based upon similar properties between classes that are not supported by visual cues. Link between two properties evident when providing minimum descriptions. This link has not formed a workable unit and is not readily available. An ordering exists between two properties.
Trans Level 3/4	**U_1(F):** Prompted or tentative statements concerning class inclusion. Single relationship has become a workable unit and is the focus when providing minimum descriptions.

	M_1 (F): Accepted notion of class inclusion without adequate justification. More than one relationship is utilised when providing minimum descriptions. **R_1(F):** Class inclusion becomes integrating feature and is justified. Focus upon interrelationships between property relationships but it is not succinct, and may not readily encapsulate all property relationships.
Level 4	**U_2(F):** Further conditions placed upon class inclusion, general overview formed of different relating concepts. Succinct and spontaneous use of interrelationships among property relationships.

While the present study was guided by five research themes, the following discussion addresses these themes globally. The following synthesis is presented within two subsections. The first outlines the characterisation of van Hiele's Level 3 as a consequence of this study, and the second considers developmental issues. Implications for theoretical frameworks and implications for teaching are treated separately later in the chapter.

Characterisation of van Hiele's Level 3 as a Consequence of This Study

> Relationships between previously identified properties of a figure are established. The properties of the figure are seen to have an order. For example, in an isosceles triangle, the fact that the opposite sides are equal is seen to imply that the opposite angles are equal. In addition, relationships between figures are understood. For example, a square is a rectangle because the set of all properties of a rectangle is included in the set of properties of a square.
>
> Pegg (1995, p. 91)

It has been established by this study that the description above concerning van Hiele's Level 3 provided by Pegg (1995) highlights some difficulties associated with the characterisation of this level. This present study articulates the characteristics of an understanding of relationships among figures and relationships among properties which have not been considered in such depth.

This study challenges the suitability of the characterisation of van Hiele's Level 3 inclusive of class inclusion concepts, which necessitates the utilisation of an overview of

multiple networks of relationships in a spontaneous and succinct manner. The developmental pathway identified in this study provides the framework where the more complex notions of class inclusion require the application and interpretation of interrelationships of networks of relationships which is typical of van Hiele's Level 4.
The following discussion outlines the development of this hierarchical structure within the characterisations emanating from this study as summarised in Table 8.1. The outline involves the splitting of van Hiele's Level 2 into 2A and 2B. This structure was validated within this study in regards to early development of links among classes of figures and their properties.

Transitional Level 1/2 – U_1, M_1, and R_1 (concrete symbolic)
The earliest triangle and quadrilateral responses were identified as a transitional group between Level 1 and Level 2. These were classified as first cycle (CS). In the triangle context, this comprised a focus upon a single similar feature to link the triangles into groups. These groups were not recognised by name, and did not form a workable identity. The groupings formed according to the student's focus at the particular time, hence, no dominant relationships were formed. The first cycle (CS) responses were also based on the spontaneous formation of groups based upon the identification of a single feature or property. Again, the groupings changed as frequently as the perceived unifying feature changed.

In the triangle context, the links at R_1(CS) were based on "acute angles," "unequal sides," and "at least two sides equal." In the quadrilateral context, the language-use was very similar. Identifying features included "at least one set of parallel lines," "equal sides," "right angles," and "two sets of equal length sides" and "all have parallel sides." "Parallel sides" was included as an identifying feature in all of the quadrilateral responses. There was also a reliance on ikonic support, which decreased as the identifying feature developed further. The responses to the quadrilateral task incorporated a greater use of ikonic support than in the triangle context. For example, statements were made such as "the rhombus is like a squashed square," and "it is like one but it is longer" in addition to the links described above.

This study identified students at Transitional Level 1/2 (van Hiele), and each of these responses were characterised as R_1(CS) due to the spontaneous use of a single property to link figures. It is also feasible that in a younger sample of students the behaviour at Level 1/2 could be described as the use of a single property to link figures as U_1(CS), the use of multiple properties to link figures M_1(CS), and R_1(CS) being the use of all known properties to form links among different figures. Within this first cycle of the concrete symbolic mode each property is utilised as a recognisable feature that remains the singular focus when describing the link formed.

In both the triangle and quadrilateral contexts there was very little use of class names in descriptions. In the cases of the square, rectangle, and rhombus, however, these words were not used to describe a group of shapes, but instead related only to the specific example provided. Links were based on a single quantifiable feature, and in some instances, the linking feature or property did not belong to all of the figures in the group. This results in inconsistencies. Groups were based upon similarities, which were observed spontaneously, hence, class structure was not understood, instead the groups were static with a single similar feature being the focus of the group.

In the context of relationships among properties, the first cycle (CS) responses were characterised by a strong reliance on visual cues, or tracing of the figure physically from which properties were assigned. It was necessary for the student to check the validity of the property belonging to the figure through visual means.

Responses were driven by the notion that if the property belongs to the figure; it was necessary for it to be included in the minimum combination. A specific visual example was required each time to determine the appropriateness of each property. It was evident that the focus was upon the figure determining the property. While the term 'property' was used in the description characteristic of the response, properties were perceived as features that are determined by the figure.

Level 2A – U_2 (concrete symbolic)

Both the triangle and quadrilateral responses included groups of responses requiring thinking at Level 2A. These responses coincided with a second cycle unistructural response in the concrete symbolic mode. The U_2(CS) triangle responses were characterised by the formation of three distinct classes of triangles. The classes formed upon the identification of a single similar property. These groups formed spontaneously, and were considered workable identities that are characterised by name. The quadrilateral U_2(CS) responses were also characterised by the formation of classes of quadrilaterals that are recognised by a generic name that represents a single property of the class.

In the triangle context, the responses incorporated no links across classes based upon similar properties, as the differences observed prevented these links. Angle-type links were made across classes at varying degrees, culminating in right-angle, obtuse-angle, and acute-angle links across the scalene and isosceles classes only. In the quadrilateral context, the groups incorporated links across classes, such as square to rhombus, square to rectangle, and rectangle to parallelogram; however, the link was supported by strong visual cues.

When these similarities were observed, the differences between the classes were also articulated. Even when prompted, students did not make a link between the square and parallelogram, and the rectangle and the rhombus, as the differences accentuated by visual cues dominated the similar properties and features observed. The class names, scalene, isosceles, and equilateral were incorporated consistently as these names encapsulated the similarities of the group. The groups were seen to be mutually exclusive, hence, the equilateral triangle was described as having "three sides equal," and the isosceles as having "two sides equal."

Responses requiring Level 2A thinking were identified also in both the triangle and quadrilateral contexts concerning relationships among properties. Within the triangle context, a U_2(CS) response is characterised by the recognition of one property as a unique and necessary signifier of a particular triangle type. While the minimisation is effective in the triangle context, the figure remains the point of reference. Similarly to the first cycle (CS) responses, the figure determines the property.

The quadrilateral U_2(CS) response is also characterised by the single reference point being the figure in question. In this context, the minimum combination is not effective due to the increased complexity of the task. The combinations chosen include a single property for each selection with ikonic support to assist in determining the appropriateness of properties to the particular figure. Hence, both contexts have a focus upon an individual property, which is singularly determined by the figure in question.

Level 2B – M_2 (concrete symbolic mode)

When Level 2B is reached the response includes individual classes, which are based upon more than one property. Links are made across classes when supported by visual cues. The quadrilateral M_2(CS) language accentuated the dominance of differences, for example, less sophisticated responses in this category described the link from the square to the rhombus as "same length and parallel sides but right angles."

Visual support was also prevalent, with the addition of statements that accentuated negative instances. This resulted in the preclusion of links. More sophisticated responses incorporated language such as "have equal sides and have equal angles." However, the description of the rhombus and parallelogram as having "two equal obtuse and two equal acute angles" maintained restrictions.

Thinking characteristic of Level 2B was evident in both triangle and quadrilateral contexts concerning relationships among properties. In regards to an understanding of relationships among triangle properties, an M_2(CS) response is characterised by identification of more than one property as a unique signifier of the triangle type. Similarly, in the quadrilateral context, an M_2(CS) response is characterised by the selection of more than one property to depict the quadrilateral with the focus remaining upon the quadrilateral in question. Each of these responses demonstrated an ability to describe more than one property at a time in relation to the figure; however, the focus of the response remained upon the figure in question.

In the triangle context, when compared to U_2(CS) responses, M_2(CS) responses are less reliant on visual cues, and chosen properties are described in terms of necessary indicators of the particular triangle-type. No links exist between the properties of the triangle or the properties of the quadrilaterals, and, hence, the responses are characterised by a series of short closures. Each property is considered in isolation and relates directly back to the figure.

In the quadrilateral context, it becomes evident that minimum is understood to be "less." When considering the rhombus and parallelogram, additional unnecessary properties were also included. Justifications such as "they need it so that they know it is a square" were typical of this category.

Both the triangle and quadrilateral responses included a category coded as transitional between M_2(CS) and R_2(CS). Both of these groups discussed tentative links between the classes of shapes, which previously were described in a manner where differences did not allow a link to be made. In the case of the triangle, a tentative link was made between the isosceles and equilateral classes of triangles. The quadrilateral responses included a tentative link between the square and the parallelogram.

The triangle transitional response indicated awareness that the equilateral triangle did possess two sides equal and/or two angles equal; however, the decision was made to not draw a link between them. In essence, the similarities are noted, however, the differences precluded a link between the equilateral and isosceles classes of triangles.

The quadrilateral transitional response is characterised by the recognition of similar properties between classes of figures where visual appearance accentuates property differences which are described in a more inclusive manner. For example, the parallelogram is described as "these sides are parallel and these sides are parallel and if they are the same then it is a square." Overall, there is a willingness to acknowledge the possibility that the square and the parallelogram link, however, the decision is that parallelism is not important to the square. This suggests the significance of right angles and all sides equal, which maintain the link as tentative. Less restrictive language is used; however, these are still dominated by perceived property differences.

Level 3 – R_2 (concrete symbolic mode)

Level 3 thinking is characterised by R_2(CS) responses in both the triangle and quadrilateral tasks. These responses include links across classes based upon the identification of similar properties. These links are made regardless of previously observed differences, which precluded the formation of such links in prior levels. These links include the equilateral to the isosceles class of triangles, the square to the parallelogram, and the rectangle to the rhombus. While links are formed across these classes, subsets, i.e., the square as a subset of the class of rectangles, are not described in either context.

The formation of a link between the equilateral and isosceles class of triangles is based upon similar properties, such as two equal sides and/or two equal angles. While a relationship exists between the two classes, the equilateral triangle is not described as a subset of the isosceles class of triangles. The quadrilateral R_2(CS) response included the addition of links that were previously precluded due to differences accentuated by appearance, or language used to describe properties. The R_2(CS) responses in both contexts were characterised by links across classes, which were not considered possible at the M_2(CS) level.

Inclusive language is utilised to describe the links between classes of quadrilaterals such as "opposite sides parallel," "opposite angles equal" and "two sets of equal sides." The difficult links are now made, as all sides equal is inclusive of opposite sides equal. While links are observed across classes based on similar properties, these responses do not incorporate subsets. In the triangle context, the student was not prepared to describe the equilateral triangle as within the isosceles class of triangles. Similarly, there is an unwillingness to use the word parallelogram to describe the square, rectangle, and rhombus, despite the formation of links between these parallelograms based upon "parallel sides" and "opposite sides equal."

Both contexts contained a group of responses characterised as R_2(CS) concerning relationships among properties. These responses focused upon a link between two properties which had not formed a workable unit. While it is evident that the response is centred upon a link between two properties, this link is tentative due to the dominating influence of one of the properties involved in the link. For example, a justification may include "if it has got, um, three sides that are equal, it has three angles that are equal." There is a simple ordering of properties but the two-way linking of properties is not a workable unit.

The quadrilateral R_2(CS) response demonstrated an understanding of the notion of minimisation. There is an attempt to link two properties together, however, the justification is verbose or tentative. The link between the two properties is not the focus of the response, but instead, under probing is tentatively discussed. Thus this link is not spontaneous, however, the notion of minimisation is clear. When attempting to incorporate diagonal or symmetrical properties the notion of minimisation cannot be held. Often when attempting to justify the correct minimisation the response was "because it has to be." The single reference remains the figure in question, but when probed, there is mention of the possibility of a link. This group of responses may also include a tentative link between the quadrilateral in question and another quadrilateral, thus incorporating a link between two figures.

Transitional Level 3/4 – U_1, M_1, and R_1 (formal mode)

Transitional Level 3/4 comprised of responses characteristic of first cycle responses in the formal mode. These included U_1(F), M_1(F), and R_1(F). The unistructural response discussed the possibility of the inclusion of a subset within an established class of figures. In the triangle context the student made mention of the isosceles class of triangles inclusive of the equilateral subset, however, this was not fully accepted. The multistructural response accepted the notion of class inclusion, however, there was no means for justification. Clarity of the interrelationships among the classes of triangles was not evident. While there were no responses identified as U_1(F) and M_1(F) in the quadrilateral context, it is envisaged that, similar to the triangle context, these responses would include tentative or unjustified statements concerning the class inclusion notion accepted and justified at R_1(F). This being, the class of parallelograms inclusive of subsets.

Both the triangle and quadrilateral responses included a group characterised as R_1(F). These are characterised by the formation of subsets within a particular class. The triangle task included the equilateral as a subset of the isosceles, while the parallelogram incorporated the square, rectangle, and rhombus. This requires the student to maintain an overview of the relationships among multiple classes, incorporate subsets within a global class, while maintaining the identity of each subset.

The equilateral triangle as the subset of the isosceles class of triangles is described as an important feature of the relationships among triangles and is justified on the basis of similar properties. The dominating notion of class inclusion is evident in the visual interpretation of the relationships provided by the student, and the justification of the relationships. The quadrilateral R_1(F) response acknowledged the class of parallelograms as inclusive of other generic categories. Similar to the triangle responses in this category, the quadrilateral response included a need for the visual representation of the relationships among quadrilateral figures to portray the notion of class inclusion as an important element of the design.

Both contexts utilised inclusive language at this level. This was particularly evident in the quadrilateral context, for example, the class of parallelograms is described as all figures that contain "opposite sides parallel and equal, and opposite angles equal." The generic categories within the class of parallelograms are also described with distinguishing language such as "all sides equal." Hence, there is an acceptance that specific properties have an encompassing quality. Consistency is evident in the description of the parallelogram class enabling subsets to form. The focus of the response remains the network of relationships concerning the class of parallelograms.

For tasks concerning relationships among properties, the U_1(F) groups of responses in both the triangle and quadrilateral context, focus upon a single relationship. In the triangle context, this relationship is between two properties. The combinations are not chosen on the basis of a significant property signifier, but instead, are based upon the fact that two properties work together to form the triangle. Hence, in the case of the equilateral triangle, two minimum combinations may be provided due to the link between "three sides equal" and "three angles equal." The quadrilateral U_1(CS) response is characterised by a focus upon a single relationship between a pair of properties, or a relationship between a pair of figures. When focusing upon the link between the quadrilateral in question and one other, the response contains a method of checking the properties against another figure in an attempt to distinguish it. Through these attempts, many of the responses in this category requested another characteristic card written in terms of negative instances, such as, "opposite sides equal but no right angles." Inconsistency occurred when attempting to apply unfamiliar properties, such as those related to axes of symmetry and diagonals.

When focusing upon a single relationship between two properties, or between two figures, the focus is spontaneous and is central to the justification in both contexts. In the case of quadrilaterals, the U_1(CS) response, contained inconsistencies due to the need to distinguish the quadrilateral from other generic categories known by name. This is indicated through the addition of superfluous properties added in an attempt to differentiate between figures. Essential to both contexts is the notion that the properties determine the figure. The focal point is the link between two properties, or between the properties of two figures.

The utilisation on multiple links is characteristic of M_1(F) responses in both triangle and quadrilateral contexts. The M_1(F) response in the triangle context utilised two or more relationships between two properties. While more than one property relationship exists, these are treated in isolation. The M_1(F) quadrilateral response is also based upon multiple relationships between properties, and/or between multiple pairs of figures. Due to the uniqueness of the square property combinations, which are consistent with a single class name, the students in this category supplied multiple combinations. The rhombus and parallelogram, however, did not necessarily include more than one correct minimum combination due to attempts to distinguish them from other quadrilateral classes.

Hence, inconsistencies still remain in the quadrilateral context when dealing with more difficult properties. Multiple links between pairs of properties are evident, and similarly to the M_1(F) triangle response, these links remain in isolation to one another and are considered separately. There also remains a need to include property descriptions stated in terms of negative instances.

The final progression within Transitional Level 3/4 in regards to students' understanding of property relationships is characterised by responses coded as R_1(F). The response includes a focus upon the connections that exist among known property relationships, however, these connections are not succinctly justified or utilised.

Level 4 – U_2 (formal mode)

Second cycle formal mode responses were identified in both triangle and quadrilateral contexts. These responses to the triangle figure task tied together relationships that exist due to similar properties, angle-type links, and class inclusion. Hence, further conditions were placed upon the equilateral triangle as a subset of the class of isosceles triangles. The quadrilateral U_2(F) response is characterised by the formation of additional subsets within the class of parallelograms. Similar to the triangle context, further conditions are placed upon the class of parallelograms, which incorporates subsets inclusive of generic categories.

The focus element of the formal responses in both contexts is an overview of the system of relationships. In the triangle context this leads to an acknowledgment of the significant link between the equilateral triangle and acute isosceles triangle. The quadrilateral response included a focus upon the class of parallelograms inclusive of the rhombus with the square subset, and/or, the rectangle with the square subset.

Both the triangle and quadrilateral contexts concerning relationships among properties comprised responses coded as $U_2(F)$. These responses were characterised by a focus upon the interrelationships between pairs of properties. In both cases there is consistency due to an understanding of the general overview of relationships. The relationships between pairs of properties are no longer considered in isolation, and instead are perceived as a network.

Developmental Issues

Overall, many key aspects are emphasised through the comparison of developmental growth concerning relationships among figures and relationships among properties. These included the lack of a set pattern concerning the emergence of links or relationships between properties. Instead, the order of properties was dependent upon an individual's familiarity of properties. Hence, Level 3 thinking does not require the student to know the total set of properties as suggested by van Hiele (1986, p. 110).

There was evidence of a supportive role among property relationships to assist class inclusion notions in most cases. It was also evident that some students differed from this pattern, and responded at a lower level to tasks concerning relationships among properties, when compared to their responses to tasks concerning relationships among figures. In regards to relationships among properties, the importance of the perception of the figure determining the property, shifting to the property determining the figure, was a prerequisite to a focus upon the network of relationships. This is closely related to van Hiele's (1986, p. 96) description of Level 2 where "figures [are] considered bearers of properties" which coincides with students' perception of 'figures determining the properties.' In addition, van Hiele (1986, p. 96) described Level 3 as to "compare and distinguish figures" based on internal ordering of properties of the second level. Thus representing the newly defined Level 3 and Transitional Level 3/4.

Burger and Shaughnessy (1986) identified a transitional path between Levels 2 and 3, and predicted a transitional path between Levels 3 and 4. This was validated through the identification of Level 3, Transitional Level 3/4, and Level 4 responses. In addition, the utilisation of an overview of multiple systems of relationships was placed into Level 4. This is inclusive of complex class inclusion notions expected within the secondary school Geometry content.

Fuys, Geddes, and Tischler (1985) found that language utilised at Level 2 was utilised more succinctly at Level 3. This study validates this idea and has identified that students at Level 2 utilise language which often precludes the formation of links among properties and links among figures. At Level 3 the language-use has developed an inclusive nature which permits the acknowledgment of property similarities alongside identified property differences. This finding validates that "transition is not possible without the learning of a new language" (van Hiele, 1986, p. 50) and the crises of thinking evident between levels.

There was a similarity between language used in different tasks within the same cycles and modes. This highlighted the inclusiveness in students' perception of properties, coinciding with students' needs to identify inclusiveness of figures and to move to focusing upon similarities over differences. The longitudinal findings validated and highlighted the developmental growth, and connection between relationships among figures, and relationships among properties.

IMPLICATIONS FOR THEORETICAL FRAMEWORKS

In the present study, two theoretical frameworks underpinned the investigation. These are the van Hiele Theory, and the SOLO model. The implications of the present study to the two frameworks are synthesised below.

The van Hiele Theory

The van Hiele Theory provided the theoretical framework from which this study was grounded. The investigation into students' understandings of class inclusion concepts in Geometry has resulted in a detailed characterisation of Level 3 thinking. In particular, a developmental path leading to an understanding of the overview of relationships among figures and their properties has been identified. The general frameworks of cognitive growth leading to an overview of the network of relationships of figures and their properties articulated in this study concern a series of levels within van Hiele's characterisation of Level 3 thinking. The culmination of these is the ability to utilise class inclusion notions without needing specific examples of the figures.

While the focus of this study concerns van Hiele's Level 3, in addition, various underlying aspects of the Theory were emphasised. Van Hiele (1986) described Level 2 thinking as a consideration of the figure as 'a bearer of properties.' This study has highlighted that at Level 3, in regards to relationships among properties, the property relationships determine the figure, as a means of 'internal ordering' of Level 2 properties.

There was evidence of discontinuity in the responses, however, transitional responses were also apparent. The shift from the identification and utilisation of relationships perceived in isolation required a "crisis of thinking" to shift to focusing upon the network of relationships required for class inclusion notions to be applied. It was found that once this shift was made in one context, the 'boundary' to rise to the same level in another context was not as difficult to pass through.

It was evident that although some students mentioned class inclusion notions, probably as a result of instruction, these students could not utilise or discuss these notions as the network of relationships was not the focus. The ability to place further conditions on aspects of class inclusion and focus upon more than one network of relationships was characterised as van Hiele Level 4.

This study made clear that some language of a prior level, in this case, Level 2A and 2B actually precluded the establishment of relationships necessary for Level 3 thinking. This occurred in tasks concerning relationships among figures and relationships among properties. In both tasks the shift from Level 2B to Level 3 required an acceptance of inclusive properties such as 'two sides equal' also incorporating 'three sides equal.' In addition, visual cues hindered the progression to Level 3 thinking due to the accentuation of differences, rather than a focus upon similarities among classes.

One hurdle for students to overcome was the focus upon a relationship between figures on the basis of properties such as opposite angles equal and opposite sides equal, which were dominated by more visual property differences such as four right angles and all sides equal. For example, this was evident when students attempted to relate the square to the parallelogram, and the rhombus to the rectangle. This was found to be of a similar degree of difficulty as relating the isosceles class of triangles to the equilateral triangle. Once the student resolved this issue, the next stage in growth requires the shift to focusing on aspects of class inclusion.

With the difficulties associated with making the progression along the developmental path, the instruction required to assist in this progression, via the van Hiele teaching phases will target the needs of the learner more adequately. Acknowledging the hierarchical nature of the levels, the detailed characterisation of the cognitive growth leading to an understanding of class inclusion developed in this study, a better view from where to target activities is provided. This assists in addressing the need to maintain activities, which prevent the students from being placed in a position where they need to rely on level-reduction techniques in an attempt to answer questions that are at a higher level then their present understandings may allow.

Overall, the detailed qualitative analysis addressed the issue of past studies where a limited number of questions were provided for each level which did not acknowledge that students may be at a different level in different contexts. The SOLO model provided a means from which to interpret different categories of responses within van Hiele levels 1, 2, 3, and 4 inclusive of transitional responses.

The SOLO Model

The application of the SOLO model to categories of responses concerning students' understandings of class inclusion concepts in Geometry provided deeper insights into van Hiele's Level 3, and, in particular, the development of class inclusion notions. The strengths of the SOLO model was in the deeper interpretation which allowed for the investigation of early responses (Level 1/2) where links were emerging among known figures to Level 4 responses where complex notions of class inclusion were utilised and justified.

The SOLO model provided a means for characterising the manner in which relationships among figures and among properties develop. The manner in which the SOLO model can separate the content of the response to interpret the structure is a powerful tool for carrying out detailed characterisations of developmental growth. For instance, it was possible to identify similarities and differences between responses that required concrete examples from which properties were identified, those that focused upon a single relationship, those that focused upon many isolated relationships, and those that focused upon the network of relationships.

In terms of the interpretation of language-use at each level, the SOLO model proved invaluable. While some students at higher levels did not articulate their responses at a sophisticated level, they did not use language that precluded the formation of relationships among figures and properties. The SOLO categorisations depicted different language-use at different levels, which is an important aspect of the van Hiele Theory. In particular, either inclusiveness, or exclusiveness was evident in the language utilised by the students.

Of interest, is the student's ability to link a number of figures at first cycle (CS) as it is based upon a single spontaneously identified feature or property. At U_2(CS) and M_2(CS) the property descriptions of classes are described in an exclusive nature, which precluded the formation of links across classes. The formation of relationships among classes of figures coincides with inclusive descriptions of properties and a focus upon similarities rather than differences. Within the domain of relationships among properties, the SOLO model assisted in the separation of responses, which provided similar property combinations, however, the justification had a fundamental difference. In the second cycle of the concrete symbolic mode, students focused upon unique property signifiers of the figures, however, in the formal mode there was a shift in focus to the relationships between properties.

This study has highlighted the strengths of the SOLO model when utilised to explore another theoretical framework. The SOLO model has allowed the identification of a generic developmental pathway leading to an understanding of class inclusion concepts in Geometry. The power of this theoretical framework has been demonstrated in an area of Mathematics requiring a detailed exploration due to the difficulties identified by students in their attempt to grasp the content. Through the application of the SOLO model, the cognitive processes undertaken by learners, and hurdles met along that path have been articulated.

IMPLICATIONS FOR TEACHING

As a consequence of this study, a number of important implications for teaching secondary-school Geometry have emerged. The difficulties associated with students' understandings of class inclusion concepts in Geometry have been widely accepted. Much of this concern has centred upon the lack of an articulated framework, which identifies the developmental pathway leading to an understanding of the network of relationships among figures and their properties. In addition, the class inclusion concepts expected within secondary-school Geometry require a spontaneous overview of the interrelationships among figure and property relationships. This study has found that the utilisation and justification of such notions is better interpreted within van Hiele's Level 4. To move forward along this path, it is necessary to consider the characteristics of thinking in terms of SOLO required to respond within the concrete symbolic mode and the formal mode. The following discussion provides suggestions of teaching activities designed to assist students to reach the next level of thinking. The first part deals with relationships among figures and the second part deals with relationships among properties.

In regards to relationships among figures, Transitional Level 1/2 thinking is characterised by the focus upon a single similar feature or property without the use of class names. Level 2A thinking is characterised by the perception of separate class of figures, which are based upon a single property and are recognised by a class name. In order to assist students at Transitional Level 1/2 they require a selection of concrete activities, which explore each of the properties of figures. Based upon the properties investigated, at this level the students require classification activities where the focus is upon the selection of properties which are characteristic of each group, and the identification of class names.

At Level 2B concerning relationships among figures, the focus is upon the collective class of figures, as opposed to specific examples. This begins with complete isolation of classes, and gradually shifts to the recognition of similar properties across classes which are supported by visual cues. To enable this shift, students need to be encouraged to communicate recognised property differences that prevent relationships, and be directed to find similarities across classes of figures.

While it is essential to work within the domains of the students' language use, at Level 2B, students would benefit from the introduction of inclusive property descriptions. Activities which involve materials with moveable parts, students' computer generated diagrams of figures belonging to certain classes are all valuable at this stage. The activities designed to assist the progression to Level 3 should promote an environment conducive to focusing upon relationships that exist between classes that are not supported by visual cues (Level 3).

At Transitional Level 3/4, the shift in focus is upon the network of relationships based upon the inclusive nature of classes of figures. To assist in the progression to Transitional Level 3/4, students would benefit from tasks requiring them to formulate descriptions/definitions of figures incorporating subsets. In addition, tasks requiring the development and discussion of geometrical concept maps and detailed tree diagrams (similar to the interview tasks in this study) target students needs to move through Transitional Level 3/4 and Level 4.

In regards to relationships among properties, the Transitional Level 1/2 responses are characterised by a focus upon the figure determining the property, and hence, there is a need to identify unique signifiers for each particular figure to make the shift to Level 2A. Activities that promote the recognition of multiple property descriptions for figures are useful to assist in reaching Level 2B.

Encouragement to formulate property descriptions via more inclusive language is necessary to enable the development of links between properties at Level 3. At Transitional Level 3/4 the focus is upon the relationships among the properties. Assistance to reach this level would be in the form of drawing and constructing figures initially from one property, and then to satisfying a number of properties. This type of activity promotes the development of property relationships determining the figure. A comparison of figures that result from the same selection of properties would be beneficial to these students. After the focus upon isolated sets of property relationships is established, students would be encouraged to focus upon the network of relationships.

To reach Level 4, students would benefit from practise at initiating and articulating multiple property descriptions to make certain figures, the aim being the development of minimum and sufficient definitions. These students would benefit from activities designed to reflect and formulate succinct justifications based upon the interrelationships that exist among known property relationships.

The discussion above, while brief, has a central theme, namely, the important role the classroom teacher plays in first, identifying levels of understanding, and secondly, devising and facilitating activities to assist in the progression to the next level. The developmental pathway leading to an understanding of class inclusion has a series of 'hard' and 'soft' boundaries. Through the exploration of cognitive processes, teachers are provided with a tool from which to view students' thinking, and a starting point to design activities that assist students in reaching the next level of thinking.

FUTURE RESEARCH DIRECTIONS

A selection of future research directions arises from this study. In particular, six research initiatives stand out as worthy of investigation.

First, as a close extension of this study, a possible area of research relates to the investigation of class inclusion concepts in Geometry within the formal mode. This exploration would extend and provide more data on the developmental pathway that emerged from this study. This would involve a sample derived from senior secondary students, and a selection of students enrolled in Mathematics at the tertiary level.

The second research direction involves the development and trialing of teaching materials based upon the variations of student understanding within the secondary-school system. The aim being to promote higher-order cognitive processing along the developmental pathways identified in this study. As highlighted by tasks concerning the square, an interesting area of research would involve an investigation of instruction that moved the student from ' more general' to 'more specific' figures, e.g., the encountering of quadrilaterals in the order of trapezium, parallelogram, rectangle, and square.

The third future research direction investigates other concepts described as requiring Level 3 thinking. This would involve determining a full appreciation of these concepts and whether they are better considered at Level 3, Transitional Level 3/4, or Level 4.

Fourthly, research involving an investigation of younger students' understandings of the relationships among figures and their properties would shed additional light upon the range of first cycle responses within this concept. This would involve similar qualitative procedures as utilised in this study, however, similar to the pilot study, diagrams of figures would need to be provided.

The fifth area of interest involves an investigation of class inclusion concepts in regards to three-dimensional figures. This would involve a selection of students from Years K–12 (aged 5–18 years) with the aim of identifying a developmental pathway leading to an overview of the relationships of 3D figures and their properties. Of particular interest would be the formation of classes of prisms, pyramids, cones, and spheres.

Finally, the sixth research direction is an additional extension of this study and would involve exploring students' understandings of class inclusion notions via different design procedures involving a large sample of students. This could be achieved through the devising and administering of a written paper based upon the findings of this research to ascertain current population figures.

CONCLUSION

This study investigated students' understandings of class inclusion concepts in Geometry, within the theoretical context of van Hiele's Level 3, utilising the SOLO model as an interpretive tool. The findings that emerged depict a developmental path leading to an understanding of class inclusion concepts in Geometry.

Through this investigation, it has been possible to identify and explore difficulties found by secondary school students when encountering notions of class inclusion. The formulation of a developmental pathway leading to an understanding of the relationships among figures and relationships among properties has resulted in a new characterisation of van Hiele's Level 3.

Emanating from this study is a detailed hierarchical structure, which depicts the manner in which an understanding of the relationships among properties and relationships among figures develop. Acknowledging the widespread view that class inclusion is a difficult concept to be grasped by most secondary students, the investigation validated this view and justified this perception. As a consequence, the accepted characterisation of Level 3 is divided into three hierarchical levels. These being, Level 3, Transitional Level 3/4, and Level 4 behaviours. In addition, this study highlighted early development of relationships among figures and properties as early as Transitional Level 1/2, Levels 2A and 2B thinking.

The characterisation of developmental growth leading to an understanding of class inclusion concepts in Geometry, highlights some various hurdles that must be addressed throughout the learning/teaching process. Hurdles encountered concern the:

- hindering elements of visual cues evident at Levels 2A and 2B;
- exclusive nature of language-use at Levels 2A and 2B as opposed to the inclusive language required to make the shift to Level 3;
- complexities of focusing upon the interrelationships among classes to maintain subsets at Transitional Level 3/4; and,
- shift to the spontaneous utilisation of the interrelationships of multiple networks at Level 4.

In addition, it is necessary to shift from a focus upon the figure determining the property at Level 2A and Level 2B, to relationships among properties determining the figure at Level 3. At Transitional Level 3/4 the property links established at Level 3 become workable units, which can be utilised fully, and with further growth become interrelated.

The major difficulty found by students involved a shift in focus from the real-world reference of the concrete symbolic mode (associated with Transitional Level 1/2, Level 2A, Level 2B, and Level 3) where a linking of classes or ordering of properties takes place, to the more abstract thinking required to utilise and justify class inclusion notions. Level 3/4 and Level 4 thinking required students to form subclass relationships being an expectation of secondary syllabi upon which the students' introduction was based.

In essence, the new characterisation formulated by this study refocusses upon van Hiele's (1986) description of Level 3 behaviour, as a focus upon "the network of relations in which the figures are interconnected on the basis of their properties" (p. 95), and the network of relations "between the properties of figures, and the manner in which one property may be deduced from another" (p. 96). Behaviours previously described as Level 3, such as "recognise class inclusions and implications" (Mayberry, 1981, p. 48) have been characterised by this study as Level 4. In addition, through the identification of hurdles to be encountered when moving along this hierarchical structure, a number of teaching considerations have emerged. This study has highlighted the potential of the SOLO model to further develop the characterisation of the van Hiele levels and to explore associated difficulties encountered by students as they grow in their geometrical understanding.