CHAPTER FIVE

QUADRILATERAL RESULTS

The purpose of this chapter is to present the results of research directed at quadrilaterals. The discussion relates closely to the results of the previous chapter, however, since the focus is on quadrilaterals rather than triangles the complexity is increased. The framework for discussion is achieved by addressing the following two research themes.

Research Theme 2
To investigate students' understandings of class inclusion concepts concerned with different quadrilaterals.

2.1 What are the characteristics, and SOLO classification, of students' understandings demonstrated in a classification task involving six different quadrilaterals?

2.2 What are the characteristics, and SOLO classification, of students' understandings demonstrated in a task concerning relationships among quadrilateral properties?

Research Theme 3
To investigate the comparison of students' understandings of class inclusion concepts concerned with triangles and quadrilaterals.

3.1 What are the similarities and differences when comparing the framework offered by the SOLO model in the context of relationships among triangle figures, and relationships among quadrilateral figures?

3.2 What are the similarities and differences when comparing the framework offered by the SOLO model in the context of relationships among triangle properties, and relationships among quadrilateral properties?

This chapter is divided into three main sections. Section 1 of the quadrilateral component investigates Relationships Among Quadrilateral Figures. Section 2 concerns Relationships Among Quadrilateral Properties. In Section 3, titled Combined Qualitative Framework, two general qualitative frameworks are provided which contain descriptions associated with growth in understanding of the relationships among figures, and relationships among properties on the basis of identified similarities and differences.

RELATIONSHIPS AMONG QUADRILATERAL FIGURES

This section is divided into three sub-sections. These are: Background, which includes interview and analysis procedures; Results, which involves the identification of the characteristics of response categories and their SOLO classification; and, Synthesis, which ties together the characteristics in the categories of coded responses.

Background

A detailed discussion of the methodology of this section of the research appeared previously in Chapter 3. A similar interview structure as that used in the investigation of triangle figures is also used in this section. In summary, the interview began with students' recall and drawing of all possible types of quadrilaterals. Unlike the triangle section, students were not provided with an example of the six quadrilaterals (square, rectangle, kite, rhombus, parallelogram, and trapezium). The interview format enabled the students to work with familiar recalled information, supplemented information, and discussion involving prompts and probes where the stimulus came from the student-designed tree diagrams. The common questions, as asked in each interview, appear in Table 5.1. This structure allowed for the domain to be set with further prompts and probes dependent upon individual student responses.

Table 5.1 Interview format for the task concerning relationships among quadrilaterals

Quadrilateral Relationships	
(i) Int:	I would like you to write a list of all the quadrilateral names you can think of. Draw each quadrilateral.
(ii) Int:	Design a tree diagram which links the different quadrilaterals. Draw a sketch to link each type. (discussion follows concerning the reasons for links and/or lack of links) (the following three points are addressed if required)
(iii) Int:	There are some quadrilaterals that we can add to this list. (provide quadrilaterals not recalled) Draw a sketch of each new quadrilateral.
(iv) Int:	Design a second tree diagram incorporating all the quadrilaterals on the list. (discussion follows concerning the reasons for links and lack of links)
(v) Int:	Return now to your first map. I would like you to add the new quadrilaterals to your original tree. (discussion follows concerning the reasons for links and/or lack of links)

The format provided a means to extract more detailed justifications, and opportunities to add or subtract links via discussion based on more than one tree diagram. Overall, the task analysis, which appears in Appendix G, provides a large number of possible outcomes that could be expected prior to analysis in the student responses. The relationships focused upon and the reasons provided are of particular interest to the characterisation of responses.

The analysis of responses to the task concerning relationships among quadrilaterals is discussed in the following sub-section titled Results. The investigation required analysis of responses to questions (ii), (iv) and (v) that deal with relationships among the six quadrilaterals. A diagrammatical relationship summary was developed which illustrates the relationships drawn, discussed, and justified by the student. The student diagrams in the following discussion represent the quadrilaterals drawn by the students and are not meant to mirror the individual sketches drawn by each student. Each type of response described in the results includes at least one quadrilateral relationship summary, a description of the particular type of response, excerpts from the interview transcript, and SOLO classification. The diagrams have a similar structure for visual comparison with the exception of first cycle (concrete symbolic) responses.

The quadrilateral relationships summaries incorporating diagrammatical data and interview dialogue, were used to divide the responses into types. Although the types were not further divided into categories, some of the types have subtle differences that are described in detail. The types, which were observed, group the responses according to two main focuses. These being, the identification of the relationships formed among six quadrilaterals, and, reasons provided for the existence of these relationships. While the responses in the quadrilateral context supports the categories identified in the triangle context, not all identical categories were identified.

There are seven types of responses identified when applying the SOLO model, and a description of each follows. The justifications for the relationships formed, and not formed, that appear on the diagrams are consistent with the language used by the students within that particular group of responses. In addition, identified categories are considered in the light of the SOLO model, and are classified according to corresponding modes and levels.

Results of Relationships Among Quadrilateral Figures

Analysis of the characteristics of the twenty-four responses places them into seven groups. Of the seven groups, the first group of responses falls into the first cycle of the concrete symbolic mode. The next four groups are characteristic of the second cycle in the concrete symbolic mode. The final two groups are typical of the formal mode.

First cycle (CS)

These responses include quadrilateral relationships based on identification of a single similar property. The relationships exist according to identification of similarities, such as at least one set of parallel lines, equal sides, right angles, two sets of equal sides, and relationships based on appearance only. Since groups of quadrilaterals are based on the identification of only one property at a time, the groupings change as frequently as the perceived unifying property changes.

There were three responses (Frances, Narelle and Tracy) coded as in the first cycle of the concrete symbolic mode. Due to their diversity, each student's quadrilateral relationships summary is included in Figures 5.1, 5.2, and 5.3, respectively. Although each of the three students focuses on parallel sides as a similar property, the students' responses indicate the gradual decrease in ikonic support as the ability to identify different similar properties develops.

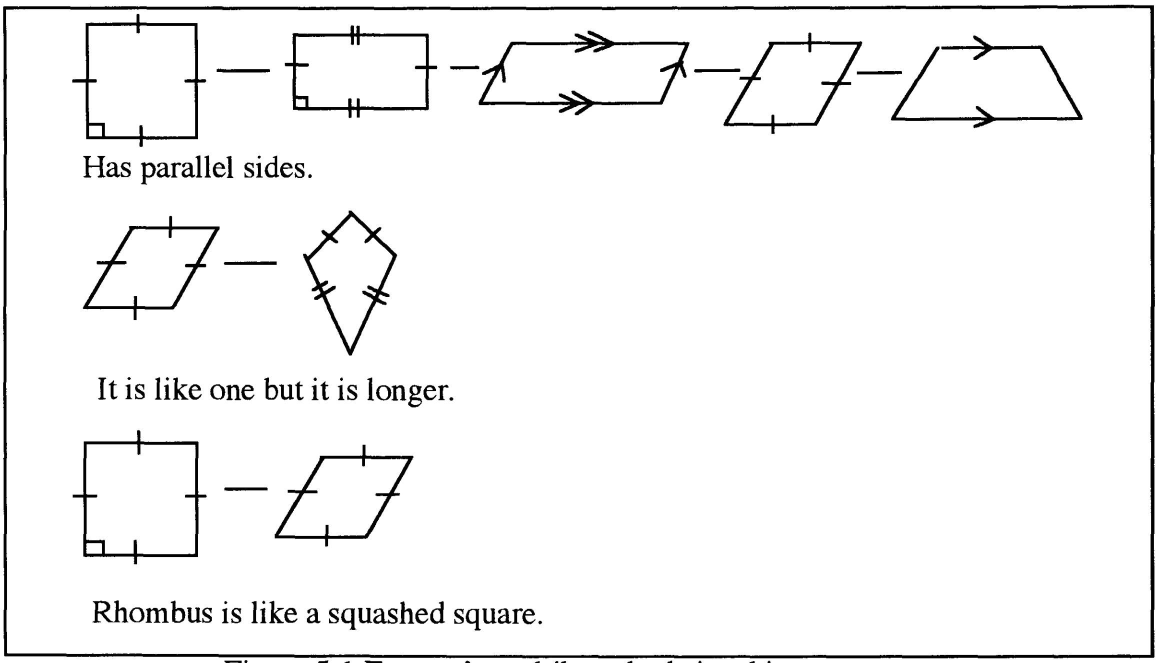


Figure 5.1 Frances' quadrilateral relationships summary

In Figure 5.1 above, it is evident that Frances identified limited links based on separate, individual features or properties. One group formed on the basis of parallel sides, and two groups formed on the basis of ikonic links, such as "it (rhombus) is like a squashed square." Frances' description of the quadrilateral relationships illustrates the spontaneous nature of the links between the different shapes. When grouping according to parallel sides, Frances formed the group as the property was identified, and the other properties of the shape were not taken into consideration. A single focus, which appears to be assisted by visual cues, dominated all other aspects of the shape.

Frances: *They have got parallel lines but I could put that one anywhere.*
Int: *How come?*
Frances: *Because it is kind of like them but it has different lengths, it is sort of the same but it has got parallel sides, so I could put it off all of them.*
Int: *Place in the lines showing me everywhere it goes.*
Frances: *Well um that one could go off any of them too.*
Int: *Show me where.* (trapezium to square)
Int: *How can something with one pair of parallel sides be connected to something that has got two?*
Frances: *Well you could start a different one if you like but it still has got two. They are all like similar but they have little things that make them different.*

When prompted to make other links on the diagram, Frances verbalised the spontaneous nature of the responses in this category. No identified link was more important than another, and the property chosen as an identifying feature had no connection to other properties of the quadrilateral.

Int: *Are there any other links that you could make anywhere?*
Frances: *Well I could link them all really, they don't have to go in a set way, you could find all different similarities with them.*

Narelle's response, as summarised in Figure 5.2, also links the quadrilaterals together on the spontaneous identification of parallel sides, although she believed the kite also has parallel sides. Narelle formed another group based on the similarity of four equal sides and made no ikonic responses.

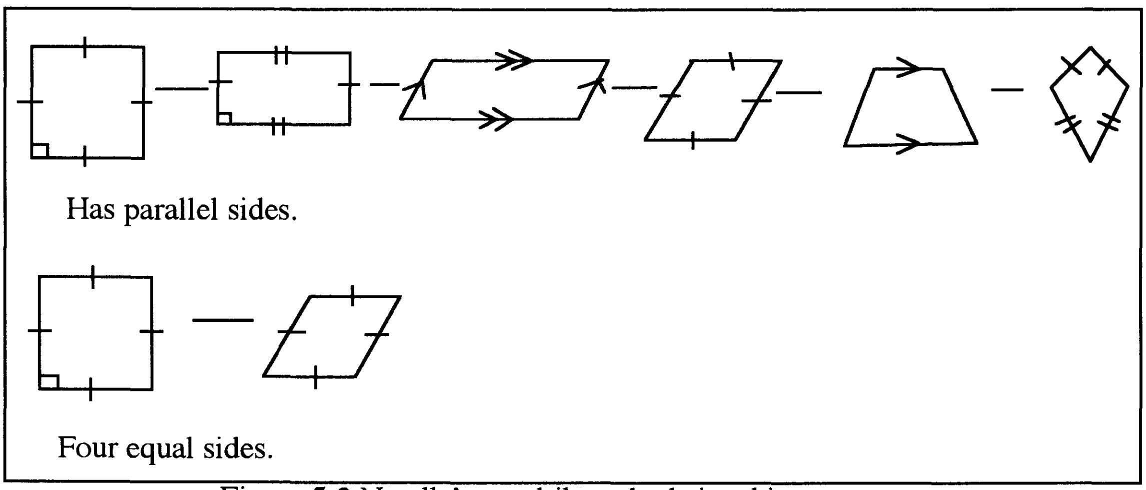


Figure 5.2 Narelle's quadrilateral relationships summary

Narelle's discussion indicates the spontaneous nature of the links formed as the focus feature is identified. The links described have no established structure with only one property the focus of the link. When Narelle was prompted to give another reason as to why the figures in the group containing parallel sides are related, Narelle stated that they probably did not go together but they do all contain parallel sides.

Int: *Can you tell me about these links here?*
Narelle: *They all have at least one set of parallel sides.*
Int : *So do you think that they all belong together?*
Narelle: *Um probably not but they all have parallel sides.*
Int: *Is there a name that you could use to describe that group of shapes?*
Narelle: *No not really.*
Int: *And why are these linked?*
Narelle: *Four equal sides. They have got the parallel sides too.*

Narelle showed confusion concerning appropriate links to the kite, and described it in terms of linking to different shapes depending upon the feature it has. Narelle described a link to the square if the kite has a right angle, or a link to the 'parallel sides group' if the kite contains parallel lines.

Narelle: *Well if it has a right angle and I can just put it on here.*
Int: *Why would that help?*
Narelle: *Because you can put that in there as they all have one oh no they don't.*
Int: *So what could you do with it?*
Narelle: *I suppose I could put it here.*
Int: *Does it have parallel lines?*
Narelle: *Um it could but then it might become parallel. If it has that it can go here.*

Figure 5.3 summarises Tracy's attempt. This is the best of the three first cycle (CS) examples due to the increased ability to identify different unifying features, while still maintaining a single focus at any one time in isolation from all other properties or features of the shape. The links are made as long as the identifying feature is evident. The dialogue to follow illustrates the concentration on a particular feature or property with no mention of the existence of other features, which may make a stronger or weaker link. When prompted to discuss other links, Tracy accepted that they might link for more than one reason, but did not take into consideration the dominating effect of any other property.

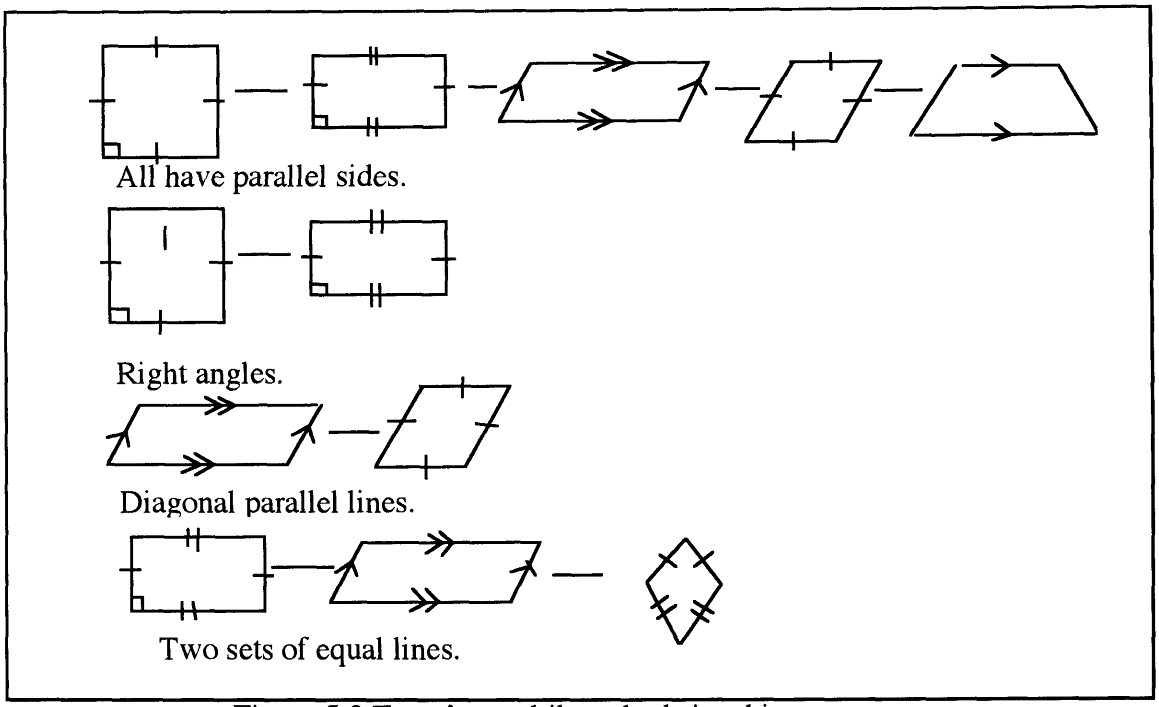


Figure 5.3 Tracy's quadrilateral relationships summary

Int: *How does your kite link to the square?*
Tracy: *Because they both have two sets of equal lines, which mean that those two are the same. This is going to get really messy. I have got all those ones that have at least one set of parallel lines.*
Int: *Can you tell me about these other ones?*
Tracy: *Well this is just connecting the ones that have two sets of equal length sides. That is already linked.*
Int: *Which one do you mean?*
Tracy: *Well that is linked because it has parallel diagonal lines.*
Int: *So that is linked for two reasons?*
Tracy: *Yes so I will put a double line. This one because they have the sides that are perpendicular um at right angles.*

When making a comparison among the three responses described above, within the first cycle (CS) it is evident that there is a definite progression in terms of complexity. While all the responses are typical of first cycle (CS) there appear two possibilities. However, without further responses from students within this cycle to analyse, it is inappropriate to provide a definite answer. For example, it is feasible to code the use of a single property, namely, parallel sides by Frances in Figure 5.1 to form spontaneous links among figures as U_1. The utilisation of more than one property, namely, parallel sides and four equal sides, by Narelle in Figure 5.2, to form spontaneous links among figures could be coded as M_1. An R_1 response category could involve the utilisation of all known properties, as provided by Tracy in Figure 5.3 to form spontaneous links among figures.

Alternatively, the three responses types could be categorised as R_1(CS). In this case, the responses for each student focused upon a single property at any one time to form links among figures without the formation of a class as a workable unit. It is evident in each of the responses that the student focuses upon a single property to form spontaneous links without the consideration of other known properties in terms of similarities or differences. The difference in the three responses lies in the ability to focus on a single property once, or multiple times within the response to the task.

Overall, the responses in this group focus on the spontaneous formation of groups related by the identification of a single unifying feature. The properties of the quadrilaterals are treated in isolation and a relationship to other properties is not evident. Hence, other properties do not affect the relationship when formed on the basis of another focus property. All three responses include parallel sides as one identifying characteristic. Other characteristics include right angles, two sets of equal sides, four equal sides, and ikonic responses. The groupings change as frequently as the identifying feature changes. The students' focus for the spontaneous groupings is based upon a single quantifiable aspect at any one time, which has a strong reliance on visual cues. The relating feature is sometimes not applicable to each member of the group formed, hence, there is a lack of consistency in the chosen approach.

Second cycle (CS)

Unistructural (U_2)

A response coded as unistructural within the second cycle of the concrete symbolic mode included links between classes that were formed upon the identification of a single similar property. There were two responses (Megan and Cameron) coded as U_2(CS). The responses were assisted by visual similarities, and did not include links which are hindered by visual differences. Hence, the students did not form links between the square and parallelogram, or rhombus and rectangle. While the student focused upon similarities and differences of each known class of quadrilaterals, the focus of the links formed was based upon a single property.

Megan's quadrilateral relationships summary in Figure 5.4 is included in this description as it illustrates a U_2(CS) response, which focuses upon a single property for the links described. The response contains links that preclude other properties, such as equal sides, but also includes language, such as two sets of equal sides for the rectangle, parallelogram and kite. As evident in Megan's response, no link is made between the parallelogram and the square, and the rectangle and the rhombus. While these links are not considered feasible within the context of class inclusion notions concerning the rectangle subset, and the rhombus subset, it is an important link to be made in the larger context of the class of parallelograms.

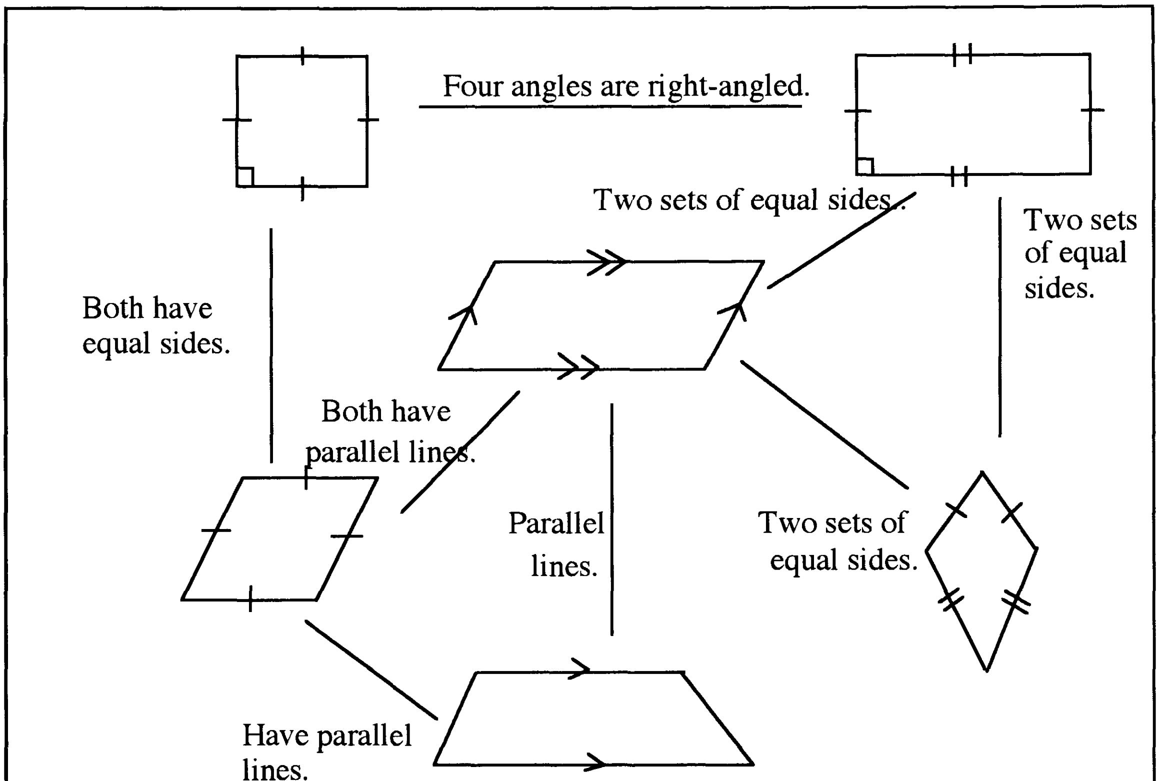


Figure 5.4 Megan's quadrilateral relationships summary

Megan's description of the link between the square and rectangle is restricted to right angles only. The link to the parallelogram is described in terms of two sets of equal sides. This suggests a greater depth to the link and the movement towards inclusive properties. Although this description of the link precludes the parallelogram from linking to the square, and the rectangle linking to the rhombus, Megan's reasons for relating specific quadrilaterals are becoming less dominated by differences.

Megan:	*The square is linked to the rectangle.*
Int:	*How come?*
Megan:	*Because they both have right angles, like all four angles are right angled. That is linked to the parallelogram because it has two sets of equal sides and this would link to the rhombus because um it has parallel lines and the rhombus would link to the square because they both have equal sides, like four sides are equal. The kite would link to the parallelogram and the rectangle because they both have two sets of equal sides. The quadrilateral would link to the rest of them because they all have the same number of sides, it is linked to all of them really.*

It was not until Megan was prompted to identify another reason for the links formed that she added that the "same ones that have equal sides also have equal angles."

Int:	*Is there any other reason these shapes are linked together?*
Megan:	*Well the same ones* (rectangle to parallelogram, rectangle to kite, parallelogram to kite) *that have equal sides also have equal angles.*

When prompted to incorporate the parallelogram class into the diagram, Megan repeated the links already identified in terms of similar properties. The only reason Megan provided for possibly linking the square and the parallelogram is that they are both quadrilaterals.

Int:	*Is there any reason why these four go around together?*
Megan:	*Um it just happened to work out that way I think but these two both have um the square and the rhombus both have equal sides, the rhombus and the parallelogram both have parallel lines um two sets of parallel lines. Oh there is one more and they would connect because they both have parallel lines.*
Int:	*Is it possible to link my square with my parallelogram?*
Megan:	*Only because they are quadrilaterals I think.*
Int:	*Any other reason?*
Megan:	*No.*

In summary, the U_2(CS) responses within the context of quadrilaterals form links among known classes of quadrilaterals on the basis of a single property. Class names are utilised to encapsulate known properties of quadrilaterals. Links formed are supported by visual cues. The language utilised to describe the properties is not inclusive, and hinders links among quadrilaterals that require the student to overcome property differences accentuated by appearance.

Multistructural (M_2)

These responses are characterised by links based on similar properties while maintaining a focus on a selection of properties of which the class or generic category of quadrilaterals is comprised. Although similar properties are identified between certain quadrilaterals, the absence of a certain property will not allow a particular link to be made. For example, links such as square to rectangle, square to rhombus, rectangle to parallelogram, parallelogram to rhombus, parallelogram to trapezium, are made on the basis of more than one similar property, such as right angles, parallel sides, equal sides, and opposite angles equal. When making these links, the student also articulated the differences between the shapes. An important feature of the eight students' responses (Andrew, Arthur, Ellen, Jason, Kathy, Louise, Peter and Scott) coded as multistructural within the second cycle of the concrete symbolic mode is that similar properties were not identified between the parallelogram and the square, or the rhombus and the rectangle, unless the student was prompted to do so.

Although the eight responses provided by students in this category are not divided further, the responses suggest growth within the category. This growth is determined by more detailed descriptions of the relationships between quadrilaterals, hence, stronger relationships are evident; and there is a refining of the language used to describe the properties of the classes. Three responses are discussed in detail below, illustrating the links formed, the reasons for these links, and the type of language used. All links made in this category are supported by visual cues.

Andrew's response, summarised in Figure 5.5, included links typical of a weaker M_2 response. The names of known shapes had developed a generic category, which is a culmination of properties encapsulated by name. The differences between the shapes was expressed as well as the similarities, while the negative instances precluded the forming of the two links described above within the M_2 (CS) response.

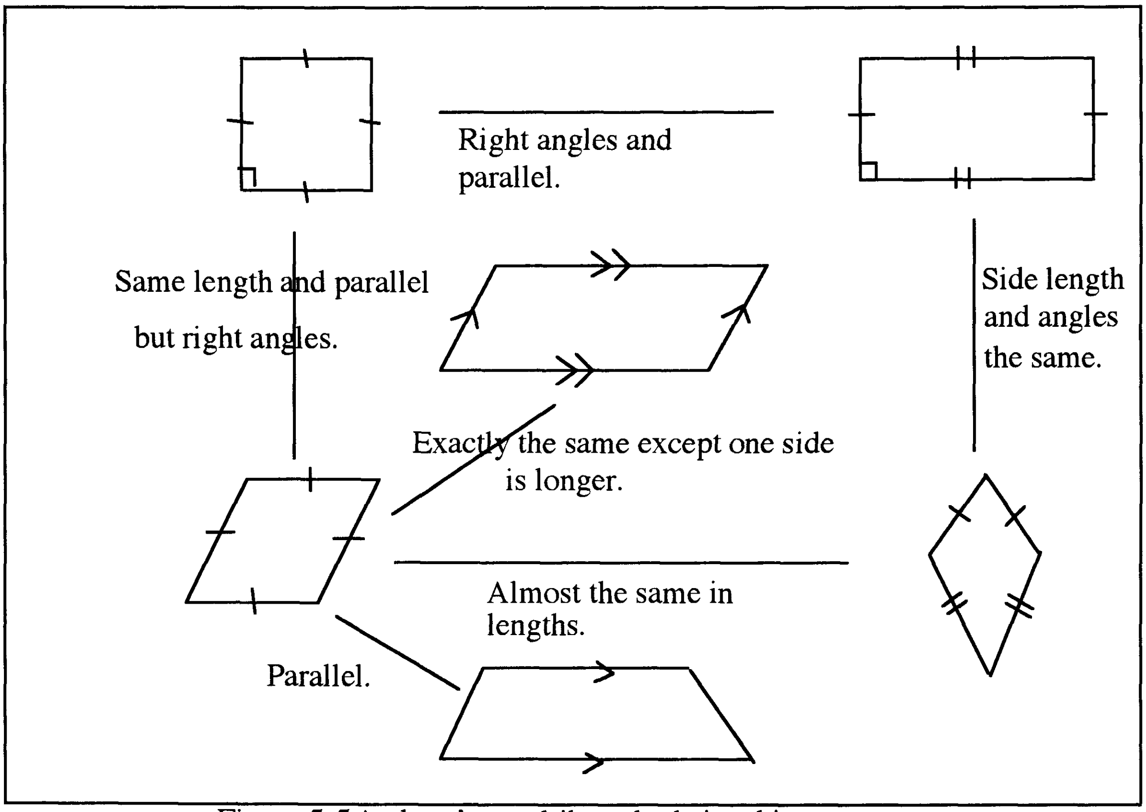


Figure 5.5 Andrew's quadrilateral relationships summary

When discussing the links, Andrew described the properties in restricted terms, such as 'lengths' are described as longer, 'length' and 'angles' are the same, almost the same 'length,' and parallel. The properties of the parallelogram and rectangle were described using language which restricts the inclusion of relationships based on similar properties. For example, the concentration on right angles and the 'same length' was utilised and does not include opposite angles equal and opposite sides equal.

Int: *Do you see the square connecting to the rhombus at all?*

Andrew: *Yes because they are all the same. See they are the same length and they are parallel and they have the same angles. Except that one is 90 degrees and that is about it ...*

Int: *Tell me about your rhombus to your parallelogram.*

Andrew: *Well they are almost exactly the same except one has only one side as longer and that can go to that because that is square but one side is longer so that is just like that.*

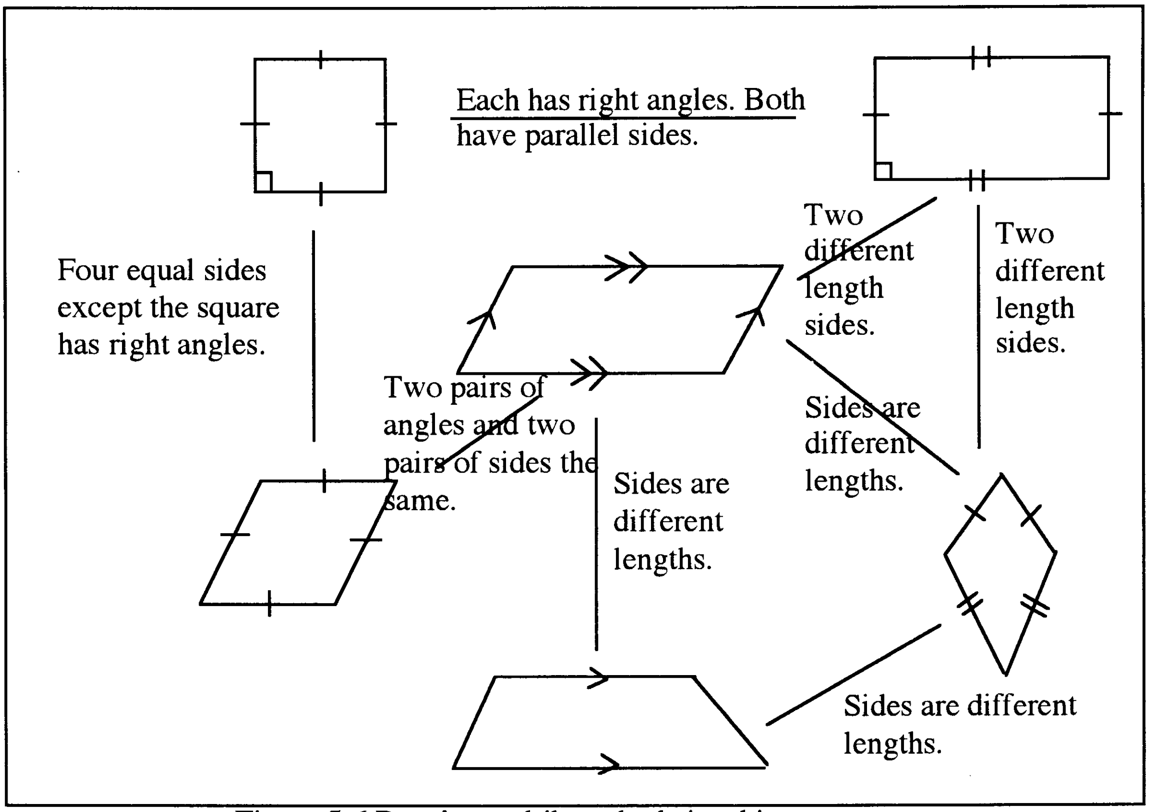


Figure 5.6 Peter's quadrilateral relationships summary

Peter's response illustrated in Figure 5.6 above, has the characteristics of a better M_2(CS) response. The links described between the rectangle and square, square and rhombus, and rhombus and parallelogram are more detailed than other links made. Peter utilised inclusive descriptions of properties when making the link between the rhombus and parallelogram and did not accentuate differences. When describing the link between the square and the rhombus Peter acknowledged both a similarity and difference between the two quadrilateral figures. The following excerpt depicts the elements of the M_2(CS) response:

Peter: *The rhombus relates to the parallelogram because two pairs of angles and two pairs of sides are the same as well.*
Int: *What is the link between the square and the rhombus?*
Peter: *Um the square and the rhombus both have sides the same length except that on this one (square) all the angles are right angles, and on the rhombus they are different.*

When prompted to make a link between the square and the parallelogram Peter noted that this could not be made "except that they have both got four sides." It is interesting to note that Peter utilised the word 'similarity' when justifying his decision to not make a link:

Int: *Do you think that I could make a link between the square and the parallelogram?*
Peter: *Um well probably not except that they have both got four sides that is the only similarity.*

In summary, the multistructural response in the second cycle of the concrete symbolic mode links quadrilaterals on the basis of more than one similar property while acknowledging the overall generic category represented by that shape. Although the links may exist due to certain similar properties, other properties of the quadrilateral are also noted which either reinforce the link, have no role to play in the link, or are negative instances that dominate over other similar properties. For example, the rectangle does not link to the rhombus in these responses as the rectangle is described as having two sets of equal sides where the pairs are of different lengths, and the rhombus has all sides equal. Another dominating difference which does not allow this link to be made concerns the rectangle being described as having four right angles, and the rhombus as having two obtuse and two acute angles. These differences described also dominate and hinder the formation of a link between the square and the parallelogram.

These responses are also characterised by the restrictions placed on the property descriptions. For example, the inability to describe the square as having opposite angles equal, or opposite sides equal precludes a link being formed with the parallelogram. Although this group of responses is characterised by the type of links made, the differences in language-use suggest the gradual formation of stronger links based on more than one similar property, and a refinement of the language used to describe the links while maintaining restrictions. The majority of responses in this category include the addition of ikonic responses, which accentuate the negative instances. This group of responses is characterised by the formation of classes of quadrilaterals, which form identifiable units due to the properties identified. While different classes are linked upon the identification of similar properties, this link relies upon visual cues. Differences are accentuated which exclude the linking of classes where the relationship is not evident visually; hence, inconsistencies are evident.

Multistructural/Relational (M_2/R_2)

This group of responses is classified as transitional (M_2/R_2) as they are characteristic of the M_2(CS) description with the addition of tentative links between the square and the parallelogram. The similar properties are noted and they are described in a more inclusive manner. Although the similarities between the square and parallelogram are of importance, the differences also expressed result in indecision concerning this relationship. There are three students (Alice, Beth, Suzanne) within this category. The summary of Alice's quadrilateral relationships in Figure 5.7 illustrates the conflict evident between inclusive descriptions and the differences identified.

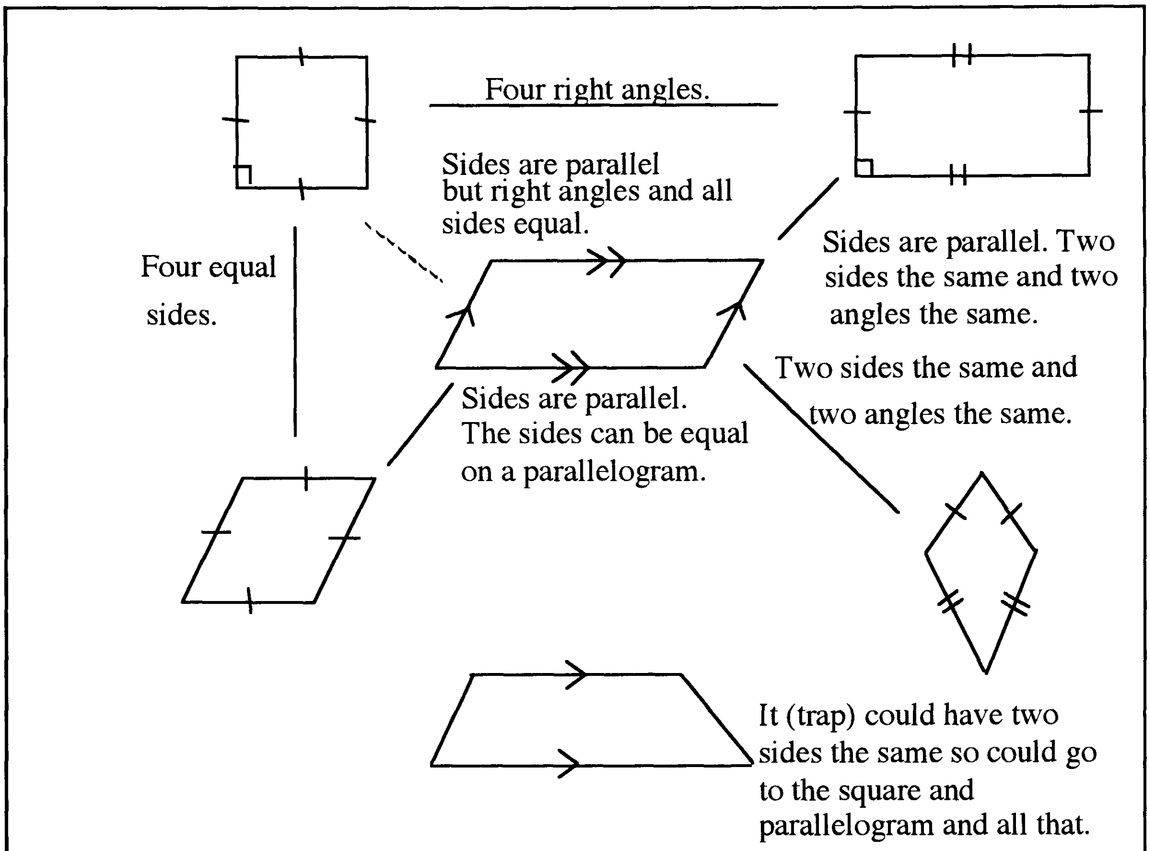


Figure 5.7 Alice's quadrilateral relationships summary

The significant difference between Alice's response and the M_2(CS) responses is Alice's ability to discuss the possibility that the parallelogram can have four right angles as the sides are still parallel. Alice continued by stating that if the sides were the same, the parallelogram would be a square. When prompted to discuss the possibility that the square is a parallelogram, Alice was restricted by other properties of the square and described parallelism as an unimportant characteristic of the square:

Int: *So do you think that they can go together?* (parallelogram and square)
Alice: *Um, no it is not important enough. That one there has got four right angles so it can go to the square. That one can have four right angles as well.*
Int: *The parallelogram?*
Alice: *Yes because all of these sides are parallel and these sides are parallel and if they are all the same then it is a square.*
Int: *So does that mean that the square is a parallelogram?*
Alice: *No because parallel isn't important to a square.*
Int: *Well do you think the square is one?*
Alice: *Yes because the sides are parallel. But yeah oh yeah. They are sort of linked to the rectangle because of the sides.*
Int: *Do you think that the rhombus is linked to the square?*
Alice: *Yeah and um probably to the parallelogram as well.*

Int: *How come?*
Alice: *Because the sides are equal and the sides of the square are always equal. And on the parallelogram they don't have to be equal but they can be.*

The indecisive nature of a link between the square and the parallelogram coincides with the willingness to describe the parallelogram as encompassing right angles and four equal sides. The square, however, is restricted in nature and is dominated by the right angle property; hence, the link remains only a possibility in Alice's response.

In summary, this transitional group of responses (M_2/R_2CS) includes connections based upon similar properties, while taking into consideration all known properties of each generic category. The differences between relationships are not as dominant in nature compared to the M_2(CS) responses allowing a tentative link to be made between the parallelogram and the square. The parallelogram is described in less restricted terms, thus acknowledging the possibility of including right angles and equal sides. These responses can be described as transitional as they are characteristic of the multistructural response with the addition of tentative statements regarding links between classes of quadrilaterals dominated by perceived property differences.

Relational (R_2)

This group of responses is characterised by the addition of relationships which were not included in the M_2(CS) responses, and which were tentatively linked in the transitional response above (M_2/R_2CS). The differences observed, such as right angles and equal sides, do not hinder the relationships that exist between the parallelogram and square, and/or the rhombus and rectangle. These relationships are described in terms of inclusive language. For example, opposite sides parallel, opposite angles equal, and two sets of equal sides. Links exist between all parallelograms in the majority of responses based on similar properties but the classes of parallelograms, rectangles, and rhombuses have not become inclusive of subsets. There are four responses (Allan, Dianne, Jenny and Michael) coded as relational within the second cycle of the concrete symbolic mode. As also evident in the M_2(CS) response, this group of responses includes links based on two or more similar properties. A typical R_2(CS) response is illustrated in Figure 5.8, which is a summary of Jenny's quadrilateral relationships.

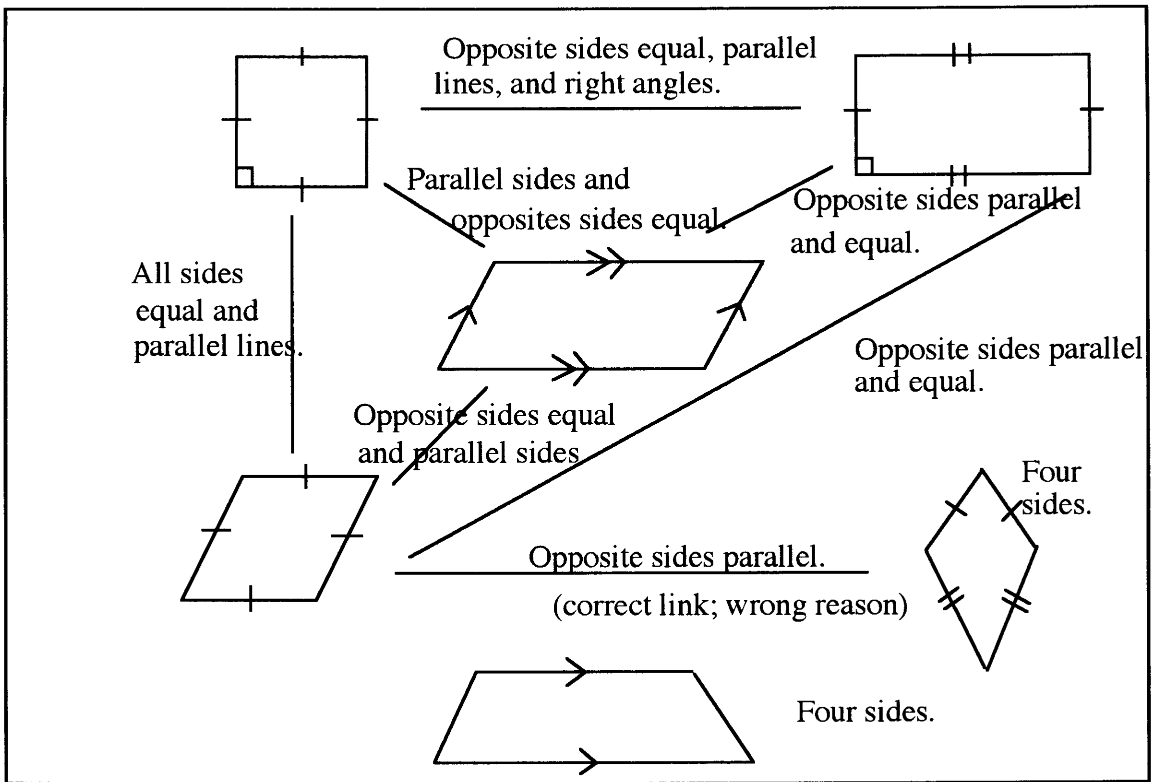


Figure 5.8 Jenny's quadrilateral relationships summary

Jenny's response illustrates the use of language which allows the formation of significant relationships when compared with the M_2(CS) and M_2/R_2(CS) responses. Jenny's response was not focused upon differences between the quadrilaterals, thus similarities described in terms of properties are not dominated by the properties which are overt in appearance. This is also indicated by the lack of ikonic responses. Jenny acknowledged that the square has opposite sides equal and parallel but did not describe the relationship in terms of opposite angles equal:

Int: *Can you tell me why you have the square, the rhombus, the rectangle and the parallelogram all linking up together?*

Jenny: *Well it is more of a progression I suppose as the square has all right angles and so does the rectangle that is why they are linked. And also they have opposite sides equal and parallel lines, the opposite sides are parallel. With the rhombus all the sides are equal and the same with the square and the parallel lines also um these are obviously not right angled as this doesn't have right angles either but the opposite sides are parallel and the opposite sides are equal. These two are linked because everything is the same except this one doesn't have right angles.*

When prompted to include the square, rectangle, and rhombus within the parallelogram class of quadrilaterals, Jenny accepted that they are a group, but was unable to place these shapes within the generic category carried by the word 'parallelogram':

Int: *Do you see these as fitting into a group together?*
Jenny: *Yes.*

Int: *What would you call it?*
Jenny: *I don't know.*

Allan's response, which includes a relationship between the square and parallelogram but does not include the rhombus to the rectangle, focuses on the length of sides for the majority of relationships. Hence, it incorporates the notion that four equal sides is inclusive of two sets of equal sides:

Allan: *Um these can be linked because of the four equal sides.*
Int: *So the rhombus can go to the square.*
Allan: *And this parallelogram with the trapezium because it has two parallel sides which is the same as each other.*
Int: *Does it matter that the parallelogram has two sets?*
Allan: *Well they still have got one pair of parallel sides each. Um the quadrilateral is the one that really doesn't go to the rest because of the four sides. The square and the rectangle have got two equal sides and four equal sides. That is like that because it has two sets of equal lines there and there. I suppose those can go together because they have got 90 degree angles in it.*

Allan's attempt at adding more depth to the relationship descriptions, when prompted, results in the search of a term to describe opposite angles. Allan described these as "diagonal angles" and when provided with the terminology extended this to the rectangle:

Int: *Can you make any more links there?*
Allan: *No. Oh I could link that because some of the diagonal angles are equal.*
Int: *You are talking about your opposite angles there.*
Allan: *I suppose I could do that with the rectangle as well.*
Int: *For the same reason?*
Allan: *Yes.*

When Allan was prompted to include other types of quadrilaterals within the parallelogram class, he repeated the reason for the link as described below. Allan acknowledged the differences in terms of angles, but did not put the link into practice. Allan remained indecisive due to the perceived preclusive nature of the angles:

Int: *Is there a reason why these three here link to your parallelogram?*
Allan: *Um* (pause) *um* (pause) *they are just linked because well your rectangles are linked because they have got two equal sets of sides but they haven't got the same angles. Yeah I am not sure.*

In summary, the relational responses of the second cycle of the concrete symbolic mode are characterised by the addition of links between the parallelogram and square, and/or the rhombus and rectangle. The relationships based on similar properties acknowledge all the known properties of the generic category represented by the quadrilateral name. The relationships are not hindered by the visually dominant properties, and the descriptions encompass a range of property subsets as opposed to the exclusion evident in the M_2(CS) responses. Each of the responses in this category includes none, or few, ikonic responses when justifying relationships. This emphasises reliance upon known properties of different quadrilateral types, and reconciliation between similarities and differences, made possible via the inclusive nature of the language used. Descriptions of quadrilateral classes no longer preclude relationships based upon similar properties, enabling connections across them. Inconsistency lies in the inability to incorporate subsets, such as the square, rectangle, and rhombus as parallelograms.

First cycle (F)

Relational (R_1)

This group of responses includes the class of parallelograms as an important component of the quadrilateral relationships. The class of parallelograms is succinct and is based on similar properties. The parallelogram encompasses other generic categories, such as square, rectangle, and rhombus, with justifications based on more than one similar property. The links made to the trapezium and kite are based on similar properties. There were three responses (Adam, David, and Nathan) coded as within the first cycle of the formal mode. Figure 5.9 contains a summary of David's quadrilateral relationships with reasons for these links.

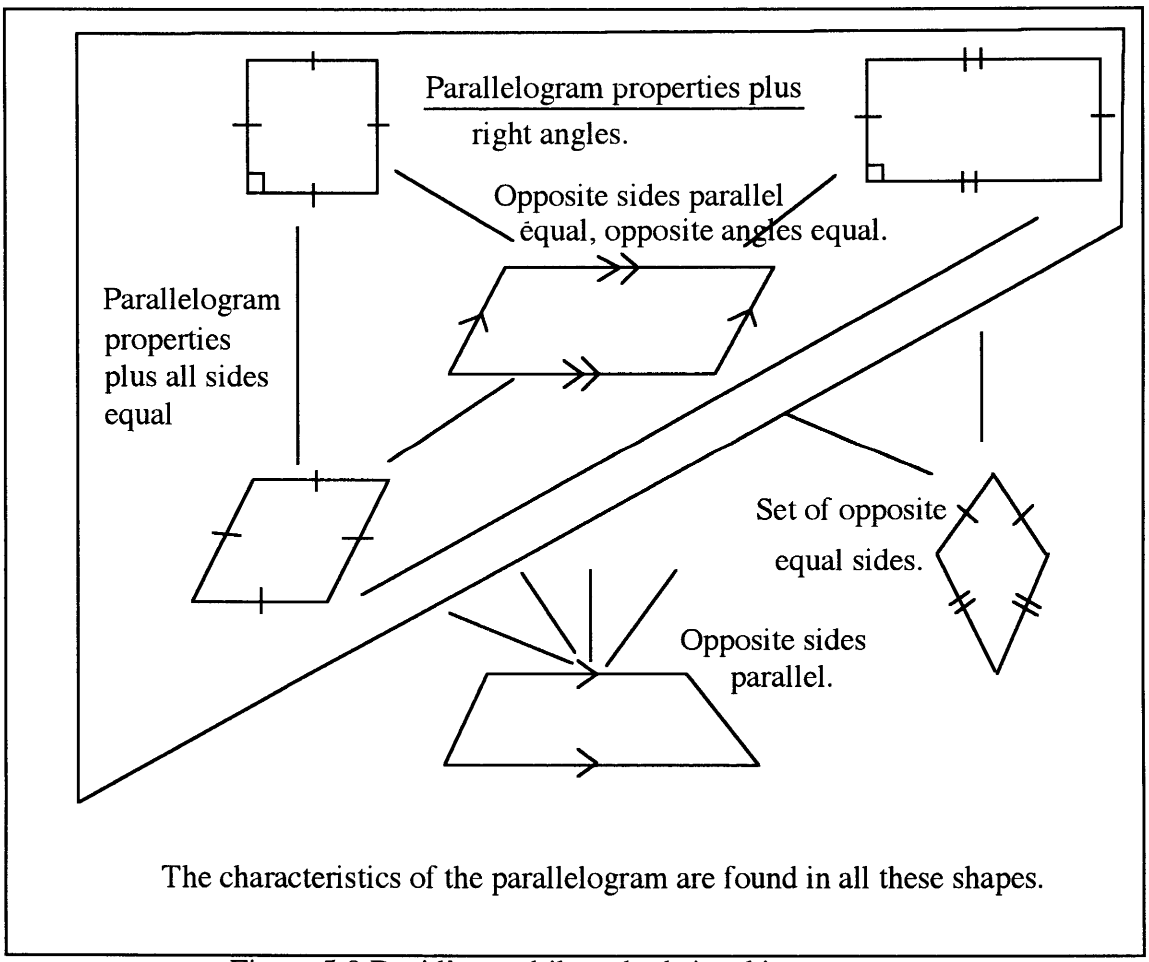


Figure 5.9 David's quadrilateral relationships summary

David's initial comment concerning the design of the tree diagram is centred upon the class of parallelograms including other generic categories. David is not content with his diagram suggesting that the square and rectangle are not part of the parallelogram class and sets out to rectify this problem. Hence, class inclusion has become an integrating feature of the response:

David: *I was going to start with has right angles and doesn't have right angles but a parallelogram would end up over there but a rectangle and a square are parallelograms but I don't think I can differentiate between the two because they are parallelograms because they do have parallel sides. This is a tricky one ...*

The dialogue below illustrates the ability to distinguish between the different quadrilateral types within the class of parallelograms while maintaining the recognition of similarities. Inclusive language is used to describe properties characteristic of the parallelogram class. David described the set of parallelograms as containing opposite sides parallel and equal, and opposite angles equal, while also distinguishing between quadrilateral classes through the use of language such as 'all sides equal':

David: *Well in this case the links are different to the first lot and I have opposite sides are equal and all sides are equal to differentiate between the square.*
Int: *Do you think that shows the relationships between them well?*
David: *Um not really because I have come across the problem of the rectangle and the parallelogram where the angles didn't decide between the two.*
Int: *Do you think that the parallelogram links anywhere else?*
David: *I think it links in with the square.*
Int: *Does your tree show that?*
David: *Well they are on the same branch. I think it would have been even better if I had started this with the opposite sides of equal length because that would have shown that a parallelogram is like a square because the opposite sides are equal and parallel.*

When prompted to address the concept of the class of parallelograms again, David described in detail the justification for this, and stated that the square is a parallelogram but the converse of this is not true:

Int: *Can you tell me why this parallelogram has all these lines coming off everywhere?*
David: *Because it has almost all the characteristics of the other shapes, like it has the parallel sides, the set of equal opposite sides, it can have angles that equal 90 degrees but it can't be um you have to remember that a square is a unique parallelogram as in a parallelogram can't have sides of equal length. As in a parallelogram can be a rectangle but it can't be a square.*
Int: *Can a square be a parallelogram?*
David: *Yes it is a parallelogram.*

David's response accentuates the links between the rectangle and square, and rhombus and square, via the description of the additional similar properties. The concepts of a rhombus class inclusive of the square, and a rectangle class inclusive of the square, are not evident.

Adam's response is similar to David's, but Adam did not include the rhombus in the class of parallelograms. Adam was not familiar with the properties of the rhombus and described it in restricted terms to the square. The other relationships are described using language such as 'opposite sides parallel,' 'opposite sides equal,' and 'one set of parallelograms.' Adam's description of the rhombus relationships is typical of a M_2 response and hence restricts the rhombus from becoming a subset of the parallelogram class:

Adam: *Well they are all right angled so I can link those.* (pause) *They are both parallelograms as well.*
Int: *The square and the rectangle. For what reason are they also parallelograms?*
Adam: *The opposite sides are parallel and the opposite angles are equal.* (pause)
...
Adam: *Well the square and the rhombus because your rhombus is all equal sides and so is your square.*

Nathan's response also includes a class of parallelograms inclusive of other subsets, with the exception of the rhombus. Ikonic support is evident when describing the rhombus, "it is um like a squashed in square but it is not parallel" thus the rhombus is not included within the class of parallelograms. This limited description of the rhombus also inhibits the link between the rectangle and the rhombus while all other relationships are consistent with a first cycle formal response.

In the light of the triangle context, the utilisation and justification of class inclusion notions as an integrating feature of known classes would be characterised as R_1(CS). The quadrilateral data were not available to verify this. The sample of quadrilateral responses did not include tentative statements concerning class inclusion notions (U_1F), or an acceptance of class inclusion notions without justification on the basis of property relationships (M_1F).

In summary, the R_1(F) reponses of the formal mode are characterised by the formation of a parallelogram class inclusive of other quadrilateral figures categorised by a different name. The language used to describe the relationships is inclusive in nature, while the generic categories are distinguished by their differences. The properties as a whole are taken into consideration and have an encompassing quality. The notion of class inclusion is perceived as an integrating feature of the described relationships. There is consistency

Unlike students in the previous category, Brendan began his discussion of the tree diagram by describing in detail the relationships between the quadrilaterals on the basis of similar properties incorporating inclusive language. There was no mention of the classes of parallelogram, rectangle, or rhombus but all three were discussed when he was probed to expand upon his response:

Brendan: *I could link them all here because they have four sides but I haven't worried about that.*
Int: *Can you tell me about the link from your rectangle to your square?*
Brendan: *They both have four right angles and two sets of parallel sides.*
Int: *And your rectangle to your rhombus?*
Brendan: *Um they both have two sets of parallel sides.*
Int: *And your rectangle to your parallelogram?*
Brendan: *They have two sets of parallel sides and two pairs of equal ones.*
Int: *Your rectangle to your rhombus?*
Brendan: *Um it has two sets of parallel lines.*
Int: *And your rhombus to your square?*
Brendan: *They have got all sides are equal and two sets of parallel lines.*
Int: *Are there any other links that you could make?*
Brendan: *Yes here.*
Int: *How come your square can link to your parallelogram?*
Brendan: *They have both got two sets of parallel lines.*
Int: *Any other reason?*
Brendan: *No.*

When Brendan was probed to discuss the parallelogram he began with a description of the rectangle class incorporating the square. The rhombus was described as a special square, which suggests that the rhombus class of quadrilaterals was still evolving. The parallelogram class was described as the basis from which evolve the square, rectangle, and rhombus:

Int: *Is there a reason why these three shapes all go to the parallelogram or why these three all go to the rectangle?*
Brendan: *It is do with the parallel sides and a square is a rectangle but it is special and the same as a rhombus is a square but it is a special one.*
Int: *What about your parallelogram?*
Brendan: *A parallelogram, um a rhombus certainly comes from a parallelogram and as I say a square is also a rhombus.*
Int: *Is a square a parallelogram?*
Brendan: *Yes.*
Int: *What about a rectangle?*
Brendan: *Yes it is but it isn't flat ...*

Although Brendan's description of the classes with subsets were not described with the same detail as David's, Figure 5.9 illustrates the complexity of the relationships formed when compared to a first cycle response in the formal mode. Brendan has integrated relationships based on similar properties to form an overview, which encompasses multiple networks of relationships. The class of parallelograms includes subsets and he also mentioned subsets of the rectangle class, and described the rhombus incorrectly as a subset of the square class.

In summary, the U_2(F) response is categorised by the class of parallelograms including the subsets of square, rectangle, and rhombus; within the parallelogram class of quadrilateral two more classes are formed which incorporate subsets, namely, the rhombus with the square subset and/or the rectangle with the square subset.

Each of these classes can be justified on the basis of similar properties. Broad generalisations are based upon more specific generalisations, hence the focus of the response maintains an overview of interrelationships among quadrilateral classes, and their subsets.

Synthesis

This subsection includes a summary of characteristics of the identified response categories. This summary is considered within the light of their SOLO categorisation, hence assisting with the interpretation of the hierarchical framework that has emerged. The following overview responds to Research Question 2.1, *What are the characteristics, and SOLO classification, of students' understandings demonstrated in a classification task of six different quadrilaterals?*

The students' responses were coded into seven types, when grouped characteristically and considered in terms of their SOLO categorisation. The framework identified placed five of the seven categories into the concrete symbolic mode. The final two groups fell into the first and second cycle of the formal mode as there was evidence of a further abstraction of class inclusion notions. The first group of responses is characteristic of a first cycle response (CS) as it includes links based upon the spontaneous identification of unifying features. The following four groups are characteristic of responses in the second cycle of the concrete symbolic mode, and includes one transitional group. A detailed summary of the response codings, and their SOLO categorisation, is contained in Table 5.2.

Table 5.2 The SOLO model and relationships among quadrilaterals

Level	Description
First Cycle (CS)	One single similar property identified to link the quadrilaterals. The groupings depend upon the unifying feature chosen, and this property remains the single focus of the relationship. The spontaneous groups of quadrilaterals formed depend upon the observed presence of the unifying characteristic. The similar property or feature is the single focus of the link, and other known properties are not taken into consideration. There is a lack of consistency, as the classes have not formed generic categories, and sometimes, the description given to the class is not appropriate to all quadrilaterals in the group.
U_2 (CS)	The formation of quadrilateral classes known by name and categorised by a single property. The classes have no subsets. Links are made between classes of quadrilaterals on the basis of a single property. Links are described between classes when a perceived significant property is evident, and is often ikonically supported. Classes are described using restrictive language. For example, four equal sides does not encompass opposite sides equal. Links do not exist between the square and parallelogram, and rhombus and rectangle as such links are not supported by visual cues.
M_2 (CS)	Similar to the unistructural response above, however, the quadrilateral classes are categorised by a group of properties. The classes have no subsets. M_2 responses incorporate all the known properties, which form the generic category recognised by the quadrilateral name. Each of the quadrilaterals has developed a workable identity, which is encapsulated by their individual names. The observed differences between the shapes play a significant role, and they are accentuated by appearance. Responses in this category do not make links between the square and parallelogram, and between the rectangle and rhombus. Although more than one property is considered when linking the quadrilaterals, the properties do not have an inclusive nature.
TRANS M_2/R_2 (CS)	Characteristic of a multistructural response with a tentative link made between the square and parallelogram. The similarities are noted between the shapes but the differences observed will not allow the link to be made. This is a transition category as the response acknowledges the similarities based on properties between the square and parallelogram. The differences are expressed and indecisive comments are made concerning the appropriateness of this link.

R_2 (CS)	The relationships formed across quadrilateral classes allow for similarities while acknowledging differences. Differences are resolved through the utilisation of inclusive language to describe properties of figures. The link is made between the square and parallelogram, and the rhombus and rectangle, as the properties have developed an encompassing quality. Relationships are not dependent upon ikonic support.
R_1 (Formal)	The first cycle formal responses include the square, rectangle, and rhombus as inclusive of the parallelogram class of quadrilaterals. The square, rectangle and rhombus are considered special parallelograms. The three subsets are justified on the basis of similar properties. A real world referent is not required and relationships are based upon the notion of class inclusion.
U_2 Formal	The class of parallelograms acquires further development within the formal mode. The subsets also contain generic categories of their own, known by name. Within the parallelogram class the rectangle is inclusive of the square, and the rhombus is inclusive of the square. The notion of class inclusion is used to describe the class of parallelograms similar to the first cycle formal responses with the addition of the class of rectangles inclusive of the square, and the class of rhombus inclusive of the square. These responses include a focus upon an overview of the interrelationships among classes and their subsets.

In summary, the SOLO categorisations assist the interpretation of the hierarchical framework, imbedded in the different response types, leading to an understanding of class inclusion notions within the context of quadrilaterals. The first cycle responses of the concrete symbolic mode include links based on similar features or properties without the formation of generic classes. It is not until the second cycle is reached that there is evidence of the emergence of quadrilateral classes based upon the focus of more than one property. The lower level responses in this cycle consistently act upon the perceived significant differences between the square and parallelogram, and the rectangle and rhombus. Progression leads to acceptance of relationships across quadrilateral classes based on similar properties. It is not until the formal mode is reached that the ability to utilise relationships that involve subsets of the class of parallelograms is reached.

Growth through the second level (CS) indicates a shift from the utilisation of descriptions that hinder relationships not cued visually, to descriptions that include relationships, which allow similarities and differences concerning two or more properties. For example, when considered inclusively, the recognition of all sides equal and four equal angles does not hinder relationships to shapes described as having properties of opposite sides equal and opposite angles equal. The first cycle of the formal mode is reached when the focus of the response concerns an overview of the interrelationships among a class of quadrilaterals and their subsets. When the second cycle of the formal mode is reached generalisations are formed upon consideration of more then one class and sub-class relationship. This requires a focus upon the interrelationships that exist between more than one class and sub-set structure. Hence, through the consideration of the structure of responses in the light of the SOLO model, a developmental framework has emerged concerning students' understanding of the relationships among quadrilaterals.

RELATIONSHIPS AMONG QUADRILATERAL PROPERTIES

This section is divided into three sub-sections: Background, Quadrilateral Property Results, and Synthesis. The structure mirrors that adopted in the previous section. The theoretical perspective considered throughout the quadrilateral property results is in the light of the framework that emerged in the triangle property results.

Background

The methodology for this section of Study 2, directed at students' understandings of relationships among quadrilateral properties, incorporates similar procedures to the research directed at students' understandings of the relationships among triangle properties discussed in Chapter 3. Hence, while the nature of the task remains the same, the level of complexity due to the change in context from triangle properties to quadrilateral properties has increased. The quadrilaterals chosen for this task were the square, parallelogram, and rhombus. The interview began by providing the student with a selection of seventeen cards, named characteristic cards.

The student was instructed first to choose all cards that belonged to the square. After selection of all known characteristic cards, the student was placed into the context of leaving clues for a friend to identify the shape accurately. Thus, the student worked with a familiar domain within a context that enabled a minimum description/definition of each quadrilateral depending upon the known property relationships. The interviewer probed for justifications for the chosen combinations, and then asked the student to provide and justify as many combinations as possible. The same procedure was repeated for the parallelogram and rhombus. The characteristic cards are shown in Table 5.3.

Table 5.3 Quadrilateral characteristic cards

4 SIDES
4 ANGLES
ALL SIDES ARE EQUAL
THERE ARE 4 RIGHT ANGLES
OPPOSITE SIDES ARE PARALLEL
OPPOSITE SIDES ARE EQUAL
DIAGONALS ARE EQUAL
DIAGONALS BISECT
DIAGONALS MEET AT RIGHT ANGLES
OPPOSITE ANGLES ARE EQUAL
2 PAIR OF EQUAL ADJACENT SIDES
1 PAIR OF OPPOSITE ANGLES EQUAL
4 AXES OF SYMMETRY
2 AXES OF SYMMETRY
1 AXIS OF SYMMETRY
1 PAIR OF PARALLEL SIDES
1 PAIR OF OPPOSITE SIDES EQUAL

The purpose of this section of the study was to raise the complexity of the property relationships task by placing the students in the context of focusing upon known relationships among quadrilateral properties. A summary of the interview schedule is contained in Table 5.4.

Table 5.4 Interview format for the task concerning relationships among quadrilateral properties

Quadrilateral Property Relationships	
(i) Int:	We are going to look closely at a few quadrilaterals. I have placed some cards in front of you named quadrilateral characteristic cards. I would like you to begin by choosing the cards which belong to the square. (selection made) Look carefully to make sure that you have included all the cards which belong to the square.
(ii) Int:	Suppose you wanted to leave some clues for a friend to identify the shape accurately. Do you think that your friend would need to see all these properties to know that you are thinking about a square? What combination could you leave? (discussion follows concerning reasons for cards included in the combination and those that have been removed) Do you think it could be made simpler? (discussion follows concerning reason for the simplification and inability to make simpler)
(iii) Int:	Let's put all the cards back. I would like you to make a different set of clues for your friend. (point (ii) repeated until student has provided all known combinations).
(iv) Int:	First three steps repeated for parallelogram and rhombus.

Context

Due to the increased number of properties available to the student, a large number of variations occurred in the initial selection. The optimum responses for the selection of characteristic cards for the square, parallelogram, and rhombus appear in Table 5.5 below.

Table 5.5 Optimum selection of characteristic cards

CHARACTERISTIC CARDS	SQUARE	PARA	RHOM
4 SIDES	•	•	•
4 ANGLES	•	•	•
ALL SIDES ARE EQUAL	•		•
THERE ARE 4 RIGHT ANGLES	•		
OPPOSITE SIDES ARE PARALLEL	•	•	•
OPPOSITE SIDES ARE EQUAL	•	•	•
DIAGONALS ARE EQUAL	•		
DIAGONALS BISECT	•	•	•
DIAGONALS MEET AT RIGHT ANGLES	•		•
OPPOSITE ANGLES ARE EQUAL	•	•	•
2 PAIR OF EQUAL ADJACENT SIDES	•		•
1 PAIR OF OPPOSITE ANGLES EQUAL	•	•	•
4 AXES OF SYMMETRY	•		
2 AXES OF SYMMETRY	•		•
1 AXIS OF SYMMETRY	•		•
1 PAIR OF PARALLEL SIDES	•	•	•
1 PAIR OF OPPOSITE SIDES EQUAL	•	•	•

*Key: PARA = Parallelogram
RHOM = Rhombus

• = Cards included in student's initial selection

It is necessary to consider the properties that were known by the sample of 24 students for each of the quadrilaterals. While the selection of characteristic cards varied greatly according to familiarity, it was possible to place them into three groups which are described in general terms below. A table containing the contextual groupings of the 24 students for each of the three tasks appears in Appendix H.

Group 1

All possible characteristic cards for the particular quadrilateral were chosen.

Group 2

The majority of characteristic cards were chosen, excluding some inclusive properties, such as adjacent sides equal when all sides are equal, and one pair of opposite sides parallel when both pairs of opposite sides are parallel. Some of the students did not include a diagonal property card, or included one incorrect diagonal property or symmetrical property.

Group 3

This group of students chose a smaller selection of cards and did not include properties such as opposite sides equal, and opposite angles equal when appropriate. Diagonal properties were not known by this group of students. Overall there was a general lack of understanding of the inclusive nature of properties, and the students' understanding of diagonal and symmetrical properties was limited.

Table 5.6 outlines the number of students within each of the three groups for each quadrilateral.

Table 5.6 Quadrilateral property context groupings

Grouping	Square	Parallelogram	Rhombus
1	6	0	1
2	14	15	13
3	4	9	10

It is interesting to note the variety of the working domain when comparing the three quadrilateral tasks. Although the number of properties of the square is greater in the initial selection, six students chose the complete selection. Fewer students identified the full set of cards for the parallelogram and rhombus. The majority of the students in Groups 2 and 3 displayed a limited understanding of diagonal properties of the parallelogram and rhombus, with a number of students choosing the card 'diagonals are equal.'

Task analysis

As in the triangle context, the investigation concerned students' understandings of the relationships among properties. Within the quadrilateral context, the complexity of the task is increased, thus increasing the variability between the working domains of each student. Considering this, the initial selection of cards provided an avenue for setting the boundaries of an individual's known properties. Of particular interest to this task was each student's discussion concerning the relationships among quadrilateral properties and justifications for the identified relationships. All possible characteristics of a response concern the following relationships:

Set 1: No identified relationships

Set 2:
1. The relationship between equality of sides and equality of angles.
2. The relationship between symmetry and equality of sides.
3. The relationship between symmetry and equality of angles.

4. The relationship between opposite sides equal and opposite angles equal.
5. The relationship between opposite sides parallel and opposite angles equal.
6. The relationship between opposite sides parallel and opposite sides equal.
7. The relationship between equal diagonals and opposite sides equal.
8. The relationship between equal diagonals and opposite angles equal.
9. The relationship between diagonals meeting at right angles and opposite sides equal.
10. The relationship between symmetry and parallelism.

Set 3: Focus upon the interrelationships among properties.

The students' responses have been grouped into seven categories based upon identified links and the students' justifications of each link. The response categories are described below.

Quadrilateral Property Results

Each response to the quadrilateral property tasks was divided into the concrete symbolic or formal mode according to the SOLO model. The discussion is structured in the light of the SOLO framework and is divided into three sections. The first section considers the characterisation of the concrete symbolic mode, while the second section considers those responses typical of the formal mode. The final section titled Synthesis provides an overview of findings concerning the research question directed at relationships among quadrilateral properties.

Table 5.7 below outlines in summary the number of responses within each category. A table appears in Appendix I, which provides the quadrilateral relationship coding for each student over the three quadrilateral property relationships tasks.

Table 5.7 SOLO categorisation of quadrilateral property responses

	Square	Parallelogram	Rhombus	Sub-Total
U_2 (CS)	2	0	2	4
M_2 (CS)	2	3	5	10
R_2 (CS)	6	6	4	16
U_1 (F)	7	5	7	19
M_1 (F)	7	6	4	17
R_1 (F)	0	3	0	3
U_2 (F)	0	1	2	3
Total	24	24	24	72

Overall, three groups of responses were coded into the second cycle of the concrete symbolic mode and are characterised by a single reference to the figure associated to the task. Hence, when attempting to justify combinations relating to the square, specific reference is made to the square only. The unistructural response utilises a single property as a unique signifier of the shape, while the multistructural responses utilise two or more properties as unique signifiers of the shape. Often visual reference is sought by the student from which the properties are generated. The third group of responses falls into the relational level as they are characterised by an attempt to minimise effectively. While this is the driving force of the response, the link between properties or figures required prompting or was verbose in nature. Each of the second cycle concrete symbolic responses is characterised by the perception of the figure determining the properties.

The remaining four groups of responses fall into the formal mode. The first three groups of responses are categorised as first cycle in the formal mode. This cycle sees evidence of a bifurcation concerning those responses that focus upon relationships among properties of the figure, and those that focus upon relationships among classes of figures. The formal mode is entered when a response utilises a single relationship between two properties or between two figures which have formed a workable unit. The multistructural response includes two or more readily available relationships between properties. A relational response focuses upon the interrelationships among property relationships.

The second cycle of the formal mode is entered when students spontaneously and succinctly focus upon the interrelationships among property relationships or class inclusion notions among figures to formulate minimum descriptions of figures. The encompassing feature of the formal mode is the focus upon the property relationships determining the figure. This notion requires an overview of the interrelationships among properties and figures.

Concrete symbolic mode

This sub-section describes the first three groups of responses, which comprise the second cycle of the concrete symbolic mode. The following descriptions of each SOLO category draws from student responses across the square, parallelogram, and rhombus tasks to assist in the depiction of the SOLO coding. These codes are unistructural (U_2) multistructural (M_2), and relational (R_2).

Unistructural (U_2CS)

The first group of responses, coded as unistructural within the second cycle of the concrete symbolic mode, was characterised by the identification of a single property that belongs to the figure while making no reference to other properties or figures. There were four responses in this group. The only reference is the figure, and while the property chosen correctly belongs to the figure, the minimum combination chosen is not effective. Responses in relation to both the square and rhombus included responses within this category. Below is Kathy's response which is typical of U_2.

MINIMUM COMBINATION 1 (square)

4 SIDES THERE ARE 4 RIGHT ANGLES

Int: *Do you think that is enough?*
Kathy: *Yes.*

MINIMUM COMBINATION 2 (square)

4 SIDES DIAGONALS MEET AT RIGHT ANGLES

Int: *That it?*
Kathy: *Um it would go like that, yeah I think that is it.*

Kathy's response illustrates a focus on a single property for each combination. The property chosen was considered in isolation. There was no reference to other figures or properties.

Scott's attempt at providing a suitable minimum combination for the rhombus also depicts the inability to shift the focus from the chosen property. When prompted to connect 'diagonals meet at right angles' to the square, Scott remained focused on the single reference to the rhombus and did not add to his combination.

MINIMUM COMBINATION 1 (rhombus)

4 SIDES DIAGONALS MEET AT RIGHT ANGLES

Scott: *Just that is enough.*
Int: *It couldn't be any other shape?*
Scott: *No.*
Int: *What if they said a square with four sides and diagonals meet at right angles?*

Scott: *They couldn't do it because the diagonals don't meet at right angles on a square.*

MINIMUM COMBINATION 2 (rhombus)

4 ANGLES DIAGONALS MEET AT RIGHT ANGLES

Scott: *I could leave the four angles instead of the sides.*

In summary, the unistructural responses within the concrete symbolic mode are characterised by a focus on an individual property of the particular figure. At no stage do the responses incorporate more than one property within the chosen combination, and the only reference is the figure itself. The responses incorporated ikonic support in the form of tracing and sketching the figure in an attempt to clarify that the property did belong to the figure.

Multistructural (M_2CS)

The second group of responses were coded as multistructural within the second cycle of the concrete symbolic mode. These responses were characterised by the selection of more than one property within their chosen combination to signify the requested figure. There are a total of ten responses in this group. While more than one property of the square is perceived to be significant within combinations, minimum is understood to be 'less,' however, this is not an effective minimisation.

Jenny's selection depicts the manner in which more than one property is selected as necessary to depict the square. While all the properties chosen belong to the square, there are no links between these properties, and the only reference is to the square. When prompted to remove OPPOSITE SIDES ARE PARALLEL, Jenny stated that it is necessary for them to know it is a square, however, the reason for this was not evident.

MINIMUM COMBINATION 1 (square)

ALL SIDES ARE EQUAL THERE ARE 4 RIGHT ANGLES

OPPOSITE SIDES ARE PARALLEL 4 AXES OF SYMMETRY

Int: *Do you think that you could make it simpler than that?*
Jenny: *Well I mightn't need this one?*
Int: *Why not?*
Jenny: *Um you wouldn't need the four angles as well as the sides.*
Int: *Could I take out that the opposite sides are parallel?*
Jenny: *Yes you could but some might not think it is a square.*
Int: *Is there another way that you could do that?*
Jenny: *Not so that they would think it was a square.*

Suzanne's response to the parallelogram task also depicts the manner in which superfluous properties are included, and the inclusion of an incorrect diagonal property, namely, DIAGONALS MEET AT RIGHT ANGLES. When prompted to link 1 PAIR OF

OPPOSITE SIDES EQUAL, Suzanne was unable to explain the reasons for her selection of cards.

MINIMUM COMBINATION 1 (parallelogram)
DIAGONALS MEET AT RIGHT ANGLES
OPPOSITE ANGLES ARE EQUAL   1 PAIR OF OPPOSITE SIDES EQUAL

Int: *Why do you only need one pair of opposite sides are equal?*
Suzanne: *Um because um the sides um you only need one because if they are equal and they are equal.*
Int: *So you would only need to leave the clue that one pair of opposite sides are equal?*
Suzanne: *Because um I really don't know.*
Int: *Could you leave another set of clues?*
Suzanne: *No that is the only way I can think of.*

The multistructural response in the second cycle of the concrete symbolic mode (M_2) is characterised by the selection of more than one property when providing a minimum combination for the figure. These properties are chosen with single reference to the figure in question. The response to the square task included superfluous properties which all belong to the square, while the parallelogram and rhombus included additional properties which do not belong to the figure. While this may appear to be a difference in the context of the square, it needs to be acknowledged that all the characteristic cards provided do belong to the square in the initial selection. In essence, the M_2 response includes a single point of reference determined by the figure, and understanding that 'minimum' means 'less,' as opposed to the selection of a single property.

Relational (R_2CS)

The relational responses of the second cycle of the concrete symbolic mode are characterised by the demonstration of attempts to minimise to reach the smallest possible selection. There are a total of sixteen responses within this category. At this level, appropriate familiar properties are used, and hence, the minimisation is 'correct.' However, it is not uncommon for there to be superfluous properties included in the responses. This was particularly true for the responses concerning the parallelogram and rhombus. The combinations chosen are correct when utilising familiar properties, however, the justification provided for the combination refers to the figure only. Sometimes when probed to provide further combinations involving unfamiliar properties, particularly symmetrical or diagonal properties, the response may include an 'incorrect property.' This category may also include an attempt at linking to either another property or to another figure, however, the response is verbose or tentative.

The response below provided by Megan illustrates that the notion of minimum was utilised and is correct in the case of the square. While the minimisation chosen is effective, Megan was unable to link her properties in any manner. The single reference is the square only, and she is confident that the combination will definitely result in a square. Megan's attempt at providing another combination included the replacing of 4 SIDES with 4 ANGLES.

MINIMUM COMBINATION 1 (square)

4 SIDES | ALL SIDES ARE EQUAL | THERE ARE 4 RIGHT ANGLES

Int: *Now why is that only necessary?*
Megan: *Because if you have got four sides and all the sides are equal and it has got four right angles then it has to be square.*

MINIMUM COMBINATION 2 (square)

4 ANGLES | ALL SIDES ARE EQUAL | THERE ARE 4 RIGHT ANGLES

Megan: *If it has got four angles then it has four sides and all the others are the same.*

The parallelogram response below illustrates Michael's attempt to provide a minimum combination. The properties chosen all belong to the parallelogram, however, the justification provided for the combination refers to the parallelogram only. Although superfluous, Michael's chosen property cards both belong to the parallelogram. When prompted to remove OPPOSITE SIDES ARE EQUAL, Michael removed the card, but was unable to provide a justification for the removal. Michael was also tentative when asked if the other properties, not included in the selection, are still relevant.

MINIMUM COMBINATION 1 (parallelogram)

OPPOSITE SIDES ARE PARALLEL | OPPOSITE SIDES ARE EQUAL

Int: *What if I took opposite sides are equal out?*
Michael: *Yes that would work too.*
Int: *How come?*
Michael: *Because it has to be.*
Int: *Why?*
Michael: *Because it just seems to.*

MINIMUM COMBINATION 2 (parallelogram)

4 SIDES | OPPOSITE ANGLES ARE EQUAL

Int: *So if you use that, what happens to all these other things?*
Michael: (pause)
Int: *Will they all happen still?*
Michael: *Yes possibly they should.*

The relational response in the second cycle of the concrete symbolic mode (R_2) is characterised by the demonstration of attempts to minimise through selection of the 'smallest' combination. There is an attempt to provide an effective minimum combination, however, there remains a single reference to the figure in question. While the familiar properties utilised within the minimisations are correct and effective, the combinations often remain superfluous; however, all properties belong to the figure in question. When prompted, some responses suggested a link between properties; however, this could not be justified effectively.

Formal mode

The following four groups of responses fall into the formal mode when considered in the light of the SOLO model. There is a bifurcation within this cycle that places the responses into two sub-groups. Within the first cycle, two subsets of responses are apparently based upon justification in two forms, namely; relationships between figures and/or relationships between properties. The final group of responses, unistructural in the second cycle of the formal mode, is characterised by a focus upon both figure and property relationships.

Unistructural (U_1F)

The unistructural response in the first cycle of the formal mode is characterised by the selection of minimum combinations, which are based upon a single readily available relationship between a pair of figures, or a pair of properties. There are a total of nineteen responses within this category. While the responses attempt to provide minimum combinations on the basis of a single relationship connecting two familiar properties, superfluous properties remain in some cases. Both groups included some incorrect characteristic cards when probed to provide further combinations and included diagonal or symmetrical properties that were unfamiliar to the student. The following description of the category is divided into justifications based on figure links, and property links.

Focus Upon Figure Relationships

The following response given by Louise is typical of a unistructural response, which focuses upon a readily available relationship between the square and another figure. The combination chosen is accurate, and the justification is based upon a link between the square and the rhombus. Hence, the reference for the response is the relationship between the properties of the square and another figure. Louise's response also indicates inconsistency when incorporating symmetrical properties. The third combination is identical to the first with the addition of the 4 SIDES card.

MINIMUM COMBINATION 1 (square)

ALL SIDES ARE EQUAL THERE ARE 4 RIGHT ANGLES

Int: *Why can all those go?*
Louise: *Well when um if you say all sides are equal and you know that the sides are equal and then you have right angles well you know that it can't be a*

rhombus because all the angles aren't right and from this you know it has to be a square.

MINIMUM COMBINATION 2 (square)

OPPOSITE SIDES ARE PARALLEL   4 AXES OF SYMMETRY

MINIMUM COMBINATION 3 (square)

4 SIDES   ALL SIDES ARE EQUAL   THERE ARE 4 RIGHT ANGLES

Int: *Do you think that you would be right with four sides and all sides are equal?*
Louise: *No I don't actually I think that you would need there are four right angles otherwise it would be a rhombus.*

Megan's response to the rhombus task is typical of a less sophisticated unistructural response in the first cycle of the formal mode. Megan's response illustrates a characteristic of many within this category when attempting to provide a combination for the rhombus or parallelogram, being, the need to provide a card that is unique that distinguishes it from other shapes within the same class. In Megan's case, the focus for the minimum combination is the link between the rhombus and the square. In Megan's attempt to provide a distinguishing property she drew upon the less familiar diagonal properties and incorrectly included DIAGONALS ARE EQUAL. While the property combination included superfluous cards, it demonstrated an attempt to find a distinguishing property between the rhombus and the square, hence the inclusion of OPPOSITE ANGLES ARE EQUAL and ALL SIDES ARE EQUAL. It is also evident in Megan's response that she is unfamiliar with the symmetrical properties of the rhombus:

Megan: *I still have that problem that it* (rhombus) *might be a square.*
Int: *If that is a problem, what can you do about it?*
Megan: *Um I would need to add that it has no axes of symmetry again. Maybe the diagonals bisecting, no that would be a square again. No I don't think I can separate them.*

MINIMUM COMBINATION 1 (rhombus)

4 SIDES   ALL SIDES ARE EQUAL
DIAGONALS ARE EQUAL   OPPOSITE ANGLES ARE EQUAL

Megan: *Um would that work um I would have to um oh no that is still a square.*
Int: *Well what can you do to add that?*
Megan: *Something to do with the axes of symmetry.*
Int: *How many?*
Megan: *None.*

Focus Upon Property Relationships

Suzanne's response below illustrates that the focus on the minimisation is upon a single relationship between two properties, namely four axes of symmetry and equality of angles, that has formed a workable unit. This is evident when Suzanne is asked to justify her combination and she replied "four axes of symmetry tells you that all the angles have to be equal as well."

MINIMUM COMBINATION 1

ALL SIDES ARE EQUAL (square)

4 AXES OF SYMMETRY

Int: *Why is that enough if all these belong?*

Suzanne: *Well four sides are equal um* (pause) *and then four axes of symmetry tells you have all the angles will be equal as well.*

Suzanne then provides a second combination, which utilises the reversibility of the relationship as she utilises THERE ARE FOUR RIGHT ANGLES .

MINIMUM COMBINATION 2 (square)

ALL SIDES ARE EQUAL

THERE ARE 4 RIGHT ANGLES

Suzanne: *Well if you have the right angles they would think either a rectangle or a square but if you add the equal sides it is a square.*

Overall, the unistructural response within the second cycle of the concrete symbolic mode is characterised by a focus upon a single relationship connecting a pair of figures, or a pair of properties. The combinations chosen are correct when utilising familiar properties, however, the combinations chosen sometimes include superfluous properties, in an attempt to find a distinguishing property to isolate the figure from others in the same class. Some of the responses in this category are inconsistent when attempting to incorporate unfamiliar diagonal or symmetrical properties.

Multistructural (M_1F)

The fifth group of responses, which are coded as multistructural in the first cycle of the formal mode, are characterised by a focus upon more than one readily available relationship among multiple pairs of figures, and/or relationships among multiple pairs of properties. There are a total of seventeen responses in this category. The responses included one or more combinations. Similarly to the previous groups of responses, there is inconsistency in unfamiliar cases, such as working with diagonal properties. In such situations, the notion of minimisation is not held. As in the previous category, this group of responses also comprises two subsets, these being, a focus upon figure links, and/or a focus upon property links.

Focus Upon Figure Relationships

Allan's response to the parallelogram task depicts the focus upon more than one pair of figures. Allan has focused his justification upon links between the parallelogram and square, and parallelogram and rhombus. Allan demonstrated a consideration of the comparison of properties, which characterised the figures in an attempt to distinguish the parallelogram from the square and the rhombus. It is interesting to note that Allan mentioned that "a rhombus is a parallelogram." Allan is not able to use this notion effectively and included the card [1 PAIR OF OPPOSITE SIDES EQUAL] in an attempt to eliminate the possibility of naming the figure a rhombus.

MINIMUM COMBINATION 1 (parallelogram)

[4 SIDES] [OPPOSITE SIDES ARE PARALLEL]

[1 PAIR OF OPPOSITE ANGLES EQUAL] [1 PAIR OF OPPOSITE SIDES EQUAL]

Int: *Can you tell me why you only need those cards?*

Allan: *Well you have got the opposite sides are parallel which can either be a square a rhombus or a parallelogram. It has got four sides and that says it is a quadrilateral, and one pair of opposite angles and so if one is equal then on a square they will be all equal and they could say a rhombus but a rhombus is a parallelogram.*

Int: *So it is okay if they say a rhombus?*

Allan: *Yes. Um if you have one pair of opposite sides are equal and then that gets rid of the rhombus.*

MINIMUM COMBINATION 2 (parallelogram)

[4 SIDES] [OPPOSITE SIDES ARE PARALLEL]

[OPPOSITE SIDES ARE EQUAL] [OPPOSITE ANGLES ARE EQUAL]

Allan: *Um no I don't think so. Um if I use... yeah I could use that.*

Andrew's response to the parallelogram depicts the same elements as Allan, with the addition of the need to include a card based on properties described in the negative form, such "four angles are not equal." Through his focus upon the links between the square and parallelogram, and the rectangle and parallelogram, Andrew described that he would need to add the card "four angles are not equal." Andrew's second combination below incorporated the addition of "four sides are not equal" to his original list.

MINIMUM COMBINATION 1 (parallelogram)

[OPPOSITE SIDES ARE PARALLEL] [OPPOSITE SIDES ARE EQUAL]

[OPPOSITE ANGLES ARE EQUAL] [4 AXES OF SYMMETRY]

Andrew: *I think I need more than that because you wouldn't be able to get a parallelogram out of that.*

Int: *What do you think that would get?*

Andrew: *You would get a square or a rectangle or something like that but you wouldn't get a parallelogram.*
Int: *So you don't think you can make that simpler?*
Andrew: *No you would have to add more.*
Int: *So what does it need?*
Andrew: *It needs um four angles are not equal.*
Int: *So you think you would be right if you could add that to your list?*
Andrew: *Yes the fact that they had looked at all these as well.*

MINIMUM COMBINATION 2 (parallelogram)

OPPOSITE SIDES ARE PARALLEL | OPPOSITE SIDES ARE EQUAL
OPPOSITE ANGLES ARE EQUAL | 4 AXES OF SYMMETRY

Andrew: *Well you could take these out and say the four angles are not equal and the four sides are not equal and then you would be right.*

Focus Upon Property Relationships

The response below began as a unistructural response within the first cycle of the formal mode due to the single focus between the square and the rhombus. When Tracy attempted to provide a second minimum combination she incorporated diagonal and symmetrical properties. While the combination is not minimised due to the increase in complexity of the properties chosen, Tracy's justification is based upon links between multiple pairs of properties. Tracy has linked diagonal properties with equality of sides.

MINIMUM COMBINATION 1 (square)

4 SIDES | ALL SIDES ARE EQUAL | THERE ARE 4 RIGHT ANGLES

Tracy: *Yes I would need the right angles otherwise it could be a rhombus.*

MINIMUM COMBINATION 2 (square)

4 SIDES | ALL SIDES ARE EQUAL | DIAGONALS BISECT
DIAGONALS MEET AT RIGHT ANGLES | 4 AXES OF SYMMETRY

Tracy: *Oh they still meet at right angles on the rhombus.*
Int: *So what can you do to separate them?*
Tracy: *Um* (pause) *um*
Int: *Do you think that would do it?*
Tracy: *Yes I think so.*
Int: *Why wouldn't I need that the opposite sides are parallel?*
Tracy: *Um you have all the sides are equal and the diagonals meet at right angles and they bisect as well and that means that that would have to be the same the distance and therefore those sides would have to be the same distance and therefore those sides would have to be the same.*

Brendan's response to the parallelogram task begins with a focus upon the link between "opposite sides are parallel" and "opposite sides are equal." There is a blurring of the response with the addition of 1 AXIS OF SYMMETRY. The second combination provided by Brendan included the same inconsistency with the symmetrical property, however, OPPOSITE ANGLES ARE EQUAL is utilised and OPPOSITE SIDES ARE PARALLEL is removed. When asked to justify the combination, Brendan stated "because if the opposite sides are equal, then those lines must be parallel."

MINIMUM COMBINATION 1 (parallelogram)

4 SIDES OPPOSITE SIDES ARE PARALLEL 1 AXIS OF SYMMETRY

Int: *How come we can remove opposite sides are equal?*
Brendan: *Because um* (pause) *in a parallelogram if you have um four sides and the opposite sides are parallel then the two parallel sides can't work unless they are equal.*

MINIMUM COMBINATION 2 (parallelogram)

4 ANGLES OPPOSITE ANGLES ARE EQUAL 1 AXIS OF SYMMETRY

Int: *Now why will that work instead of your parallel lines?*
Brendan: *Because if the opposite angles are equal then those lines must be parallel.*

In summary, the multistructural responses of the first cycle in the formal mode include one or more minimisations, which are based upon multiple relationships between pairs of figures, and/or pairs of properties. The responses to the three tasks are characterised by inconsistency in terms of minimum combinations when working with unfamiliar properties such as diagonal or symmetrical properties. The parallelogram and rhombus responses were often characterised by a need to include an additional card described in terms of negative instances such as "has no right angles" and "sides are not all equal."

Relational (R_1F)

This group of responses is coded as relational in the first cycle of the formal mode. This category of responses is characterised by the selection of at least one combination based upon the relationships among groups of properties and/or figures. There are a total of three responses within this category which were a response to either the parallelogram or rhombus task. These responses see a focus upon the interrelationships between figures and properties, however this is not succinct and does not readily encapsulate all known property relationships. This overview is not spontaneous and often requires prompting.

Beth's response to the parallelogram selection is interesting as it began typical of a multistructural response within the first cycle of the formal mode. Beth worked through the notion of class inclusion to justify her choice of property cards, and concluded by stating "a square and a rectangle are types of parallelograms."

MINIMUM COMBINATION 1 (parallelogram)

4 SIDES OPPOSITE SIDES ARE PARALLEL

Int:	*Now why is it enough to have four sides and the opposite sides are parallel?*
Beth:	*Um actually it is not.*
Int:	*You don't think so. Why not?*
Beth:	*Well that could be a square or a rectangle.*
Int:	*Is that a problem?*
Beth:	*Oh no because they are parallelograms.*
Int:	*So do you think that is okay?*
Beth:	*Yes because a square and a rectangle are types of parallelograms.*
Int:	*How does that assist you?*
Beth:	*It just does.*

MINIMUM COMBINATION 2 (parallelogram)

4 ANGLES

OPPOSITE SIDES ARE PARALLEL

Int:	*Another way?*
Beth:	*Oh no I don't think so.*

There is a hesitation in Beth's response and inability to expand upon the notion of interrelationships to include more than one combination. Beth is prepared to discuss relationships among groups of quadrilaterals, however, this has not developed into an effective tool which has the capacity to generate spontaneous multiple combinations based upon this notion.

These relational responses of the first cycle of the formal mode indicate the emergence of an overview of the relationships among the properties of quadrilaterals. While the interrelationships are integrated into the response, they are hesitantly utilised and are not spontaneously incorporated.

Unistructural (U_2F)

The responses coded as unistructural in the second cycle of the formal mode spontaneously and succinctly utilise the interrelationships among all known property relationships. The overview of relationships is an integrating focus of the response. There are three responses coded as U_2 in the formal mode.

David provided only one minimum combination, however, his justification was based upon the relationship between a group of properties. David stated that because of the selection chosen, which is inclusive of OPPOSITE SIDES ARE PARALLEL, that all the other properties of the figure "still happen."

MINIMUM COMBINATION 1 (parallelogram)

4 SIDES

OPPOSITE SIDES ARE PARALLEL

Int: *Can you tell me why you can do that?*
David: *Well if it has four sides it has to have four angles, and um if the opposite sides are parallel then they also have to be equal. Because of these all of these still happen.*

Nathan's response to the rhombus task clearly focused on the interrelationships among figures in an inclusive nature. Nathan applied notions of class inclusion when justifying his combinations spontaneously. Nathan's response included "it has got four equal sides so a square can be a rhombus" and the notion that a rhombus is still a parallelogram. Nathan provided three minimum combinations as described below.

Nathan: *A square is a rhombus isn't it ... Oh it could be because it has got four equal sides so a square can be a rhombus so just four sides are equal.*
Int: *So is that enough?*
Nathan: *Yeah it is. Yes um diagonals meet at right angles will be enough also.*

MINIMUM COMBINATION 1 (rhombus)

DIAGONALS MEET AT RIGHT ANGLES

MINIMUM COMBINATION 2 (rhombus)

ALL SIDES ARE EQUAL

Nathan: *Oh actually that could be a parallelogram. No that is all right.*
Int: *What if they said it is a square?*
Nathan: *Well that is still a rhombus.*
Int: *How can it be?*
Nathan: *Because it has got four equal sides.*

In summary, the unistructural response in the second cycle of the formal mode require a generalised overview of the known relationships among figures and properties. The responses in this category provide one or more correct minimum combinations based upon interrelationships among properties and figures in a spontaneous and succinct manner.

Synthesis

This sub-section provides a summary of the SOLO characterisations of student responses concerning the relationships among quadrilateral properties. Due to the variation in context of the three tasks, these being square, parallelogram, and rhombus, it is necessary to investigate the similarities and differences associated with the different quadrilateral property tasks. In the light of the observable differences, a general hierarchical framework emerged with the application of the SOLO model. An overview of the categorisation is addressed via the response to Research Question 2.2, *What are the characteristics, and SOLO classification, of students' understandings demonstrated in a task concerning relationships among quadrilateral properties?*

The student responses were coded into seven categories, three of these falling into the concrete symbolic mode, and the final four falling into the formal mode. The first three groups comprised the second cycle (U_2, M_2, and R_2) of the concrete symbolic mode and were characterised by a focus on the single figure in question only. For example, in the case of the square, the focus of the response was a single unique property signifier of the rectangle at U_2(CS), such as, "opposite sides equal." At M_2(CS) multiple property signifiers were the focus of the response, such as, "opposite sides equal and four right angles because the rectangle is the only shape that has that." A response characterised as R_2(CS) includes a link or ordering between two properties such as "I can just have opposite sides equal and four right angles for the rectangle. The equal sides means it will have axes of symmetry anyway." While the response at R_2(CS) is characterised by an attempt at providing a minimum the focus remains on the figure in question with a single link between two properties.

The first cycle responses within the formal mode (U_1F, M_1F, and R_1F) are characterised by a shift to the property relationships determining the figure. This was evidenced by utilisation of readily available relationships between familiar properties of the figure, and/or relationships between familiar figures. Through this cycle, growth was represented by a single relationship connecting pairs of figures/properties to multiple pairs of figures and/or properties. The relational response involved the utilisation of the overview of relationships among known properties and figures, however this was not succinct. The culmination of the formal mode responses was a unistructural response in the second cycle (U_2) where the focus is on the interrelationships among properties and figures which are succinctly utilised as an integrating feature of the response. A detailed summary of the response groupings appears in Table 5.8.

Table 5.8 The SOLO model and relationships among quadrilaterals

Level	Description
U_2 (CS)	Each of the properties of the quadrilateral is perceived in isolation. The only reference is to the quadrilateral in question from which the single property is derived. There is a strong reliance on ikonic support in the form of sketching and tracing specific examples of the figure. The minimisations are not effective in most circumstances due to the single focus in terms of the property chosen and single focus upon the particular quadrilateral.

M_2 (CS)	While the single reference to the quadrilateral is evident in this category, more than one property is chosen to represent the quadrilateral. The quadrilateral is perceived in isolation, however, more than one property is utilised to determine the figure. Minimisation is understood to be 'less' and includes superfluous properties. Inconsistency is evident with the inclusion of additional and sometimes incorrect properties.
R_2 (CS)	There is evidence of attempts to minimise property combinations, however, the justification provided for the combination refers to the quadrilateral in question only. The response may include an attempt at linking another property or to another figure. Additional unnecessary properties are sometimes included in the minimisations as the student attempts to find unique distinguishing combinations. When prompted to include further combinations, incorrect properties may be included when attempting to use unfamiliar properties, particularly with properties associated with diagonals or symmetry.
U_1 (F)	The focus of the minimisations is a single readily available relationship connecting the properties of two figures, and/or a relationship between two properties of the quadrilateral in question. The property relationship is now perceived to determine the figure. The attempts at minimisations sometimes include additional properties when attempting to provide a selection of distinguishing properties from other classes of quadrilaterals. There is conflict evident in students' desire to reach minimum descriptions while distinguishing the figure from other possible solutions. The relationship between two properties or between two figures has become a workable unit; however, there remains a need to describe the figure in terms that differentiate the class. Incorrect properties are often included when attempting to use unfamiliar properties.
M_1 (F)	The response includes a focus upon multiple readily available relationships connecting pairs of figures, and/or properties. One or more minimum combinations are provided, however, superfluous properties are often included, particularly in the case of the parallelogram and rhombus combinations. There is a blurring in unfamiliar circumstances, such as working with diagonal properties. Notion of minimisation is not maintained in such situations.
R_1 (F)	The response includes a focus upon the interrelationships among known property relationships, however, it is often not succinct, and may not be utilised to its potential. While there is mention of an overview, which integrates the relationships described at M_1, the overview does not lend itself to the formation of multiple combinations with the exception of prompted situations.

U_2 F	The focus of the response is upon the network of relationships among the properties and figures. The parallelogram and rhombus often include a single correct minimum combination, and it is identified based upon relationships among groups of properties and figures. The notion of minimisation can be held in all circumstances.

The detailed descriptions of the SOLO codings above illustrate growth through the levels indicated through the change in focus. In general, second cycle concrete symbolic mode responses focus upon the figure in question, beginning with the inclusion of a single property, to multiple properties, to an understanding of the notion of minimising which remains focused upon the single figure with tentative links. The first cycle of the formal mode is characterised by growth, which encompasses a focus upon relationships among properties and/or figures. The bifurcation that occurs divides the responses into three groups as described by the Venn diagram in Figure 5.11 below.

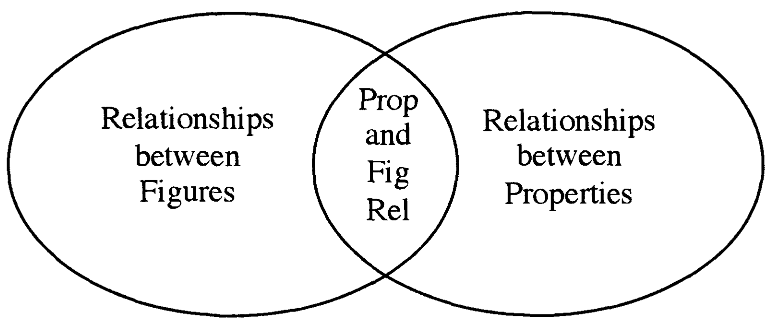


Figure 5.11 Bifurcation evident in first cycle of formal mode

Growth to R_1(F) sees the further development of relationships between figures and properties. At R_1(F) an overview of relationships is utilised within the response, however, it is not succinct. The formal U_2 response is characterised by a focus upon the interrelationships among properties and/or figures. The property links are no longer perceived in isolation and have formed a network from which the figure is derived. Notions of relationships among figures are often utilised to justify the appropriate minimum property combinations in a succinct manner.

The application of the SOLO model, in terms of modes and cycles of levels has assisted in the interpretation of the hierarchical structure evident in groups of responses concerning relationships among quadrilateral properties. Seven groups of responses were identified which were characteristic of the concrete symbolic mode and formal mode. The structure of the first three groups of responses placed them into the three levels of the second cycle of the concrete symbolic mode (U_2CS, M_2CS, and R_2CS) while the final four groups fell into the first cycle of the formal mode (U_1F, M_1F, and R_1F) and second cycle formal mode (U_2F).

COMBINED QUALITATIVE FRAMEWORK

Within this section, a comparison of the identified response categories takes place in relation to the different contexts. This is from a qualitative perspective, which allows a synthesis of the characteristics of the hierarchical structure in different contexts. This section is divided into two sub-sections, namely, Relationships Among Figures, and Relationships Among Properties. Each sub-section includes overall descriptions, which accommodate both sets of data on the basis of observed similarities and differences between the response categories from the triangle and quadrilateral tasks. Thus, a comparison of the support provided by the triangle findings to the context of quadrilaterals takes place.

Relationships Among Figures

This following discussion provides a synthesis of the SOLO response categories concerned with relationships among triangle figures, and relationships among quadrilateral figures. The groupings observed in the responses to the triangle and quadrilateral tasks fell within the same modes and cycles. The discussion below includes first and second cycle responses of the concrete symbolic mode, and those classified as first and second cycle responses in the formal mode. Table 5.9 includes the categorisations of responses when applying the SOLO model to the triangle and quadrilateral response categories.

Table 5.9 Summary of relationships among figures SOLO codings

Triangle Figures		**Quadrilateral Figures**	
First Cycle (CS)	Relational	**First Cycle (CS)**	Responses
Second Cycle (CS)	Unistructural	**Second Cycle (CS)**	Unistructural
	Multistructural		Multistructural
	Transitional		Transitional
	Relational		Relational
First Cycle (F)	Unistructural	**First Cycle (F)**	
	Multistructural		
	Relational		Relational
Second Cycle (F)	Unistructural	**Second Cycle (F)**	Unistructural

The generic descriptions to follow include the eight response categories evident across the triangle and quadrilateral contexts. This does not include transitional groups, which incorporated tentative statements similar to the focus adopted within the following SOLO category. The following general frameworks include three terms requiring clarification. 'Class of figures' refers to the collection of figures that are characterised by similar properties and are recognised by name, e.g., square, parallelogram, equilateral triangle, and isosceles triangle. 'Link' refers to an ordering between two properties or figures. A 'relationship' between two figures or properties refers to a connection that has formed a workable unit, that is, seeing one property or figure related in some way to another property or figure and vice versa, that is readily available.

R_1(CS) response

A single property or feature is identified to link the figures. The focus of the response is upon the identification of an observed single quantifiable aspect, which places figures into spontaneous groups. There is a strong reliance on visual cues.

U_2(CS) response

Classes of figures are known by name and are characterised by a single property. The class represents an identifiable unit. Links do not exist between classes, unless supported by visual cues. Observed differences in traditional shapes, such as right angles, play a significant role.

M_2(CS) response

Similar to the U_2 response above, the M_2 responses incorporate classes of figures, which are known by name. These classes are characterised by more than one property. Links are not made between classes where differences in properties are accentuated by visual differences.

R_2(CS) response

Relationships exist between classes of figures, which are based upon similar properties. Inclusive language is used to describe the classes of figures; hence, property descriptions allow for similarities to be acknowledged.

U_1(F) response

When prompted, tentative statements are made concerning the possibility of subsets within a class of figures. There is no acceptance of this notion, however, it is able to be discussed tentatively.

M_1(F) response

There is an unprompted acceptance of a class of figures containing subsets. While this notion of class inclusion is accepted and utilised, it is not justified adequately.

R_1(F) response

The notion of class inclusion is an integrating feature of the response. A class of figures incorporates subsets, which are inclusive of generic categories identified by other names. Each class maintains a workable identity while the focus is upon the network of relationships based upon the properties of each class.

U_2(F) response

The notion of class inclusion acquires further development. Conditions are placed upon the classes of figures, which acknowledge more than one system of relationships. This requires an overview of the interrelationships among classes and their subsets, which utilises subsets within subsets, and precludes inappropriate examples of figures.

Relationships among Properties

The hierarchical framework evident when applying the SOLO model to the categorisation of responses concerning the relationships among triangle properties comprises eight response groups of varying levels and modes. Tasks within the context of quadrilaterals brought forth seven response categories. Table 5.10 below, depicts the similarities in SOLO categorisation evident in the developmental pathway leading to an understanding of the relationships among properties.

Table 5.10 Summary of SOLO codings concerning relationships among properties

Triangle Properties		Quadrilateral Properties	
First Cycle (CS)	Relational	**First Cycle (CS)**	
Second Cycle (CS)	Unistructural	**Second Cycle (CS)**	Unistructural
	Multistructural		Multistructural
	Relational		Relational
First Cycle (F)	Unistructural	**First Cycle (F)**	Unistructural
	Transitional		
	Multistructural		Multistructural
			Relational
Second Cycle (F)	Unistructural	**Second Cycle (F)**	Unistructural

A comparison of the characteristics of the SOLO framework identified in responses to tasks concerning relationships among properties in triangle and quadrilateral contexts, places both sets of data into the concrete symbolic and the formal modes. The similarities and differences identified are outlined below.

Taking into consideration the similarities and differences observed in the discussions provided in Chapters 4 and 5, a general categorisation is developed to suit both the triangle and quadrilateral contexts concerning understanding of the relationships among properties. The eight categories are:

R_1(CS) response

The focus of the task is upon the figure in question from which all known properties are derived. A specific example of the figure is utilised from which each property is determined. There is a strong reliance on ikonic support. The properties are perceived as features of the figure in question.

U_2(CS) response

The reference for the response is the figure in question. The figure determines a single property. Minimisation is understood to be 'less' and is based upon the uniqueness of a single property to the figure.

M_2(CS) response

The single reference remains the figure in question. The figure determines two or more unique properties, which are utilised to represent the figure. As the figures and their properties remain in isolation there are a series of isolated closures. Minimisation is understood to be 'less.'

R_2(CS) response

The focus of the response is upon a link or ordering between a pair of properties, or a pair of figures within the same context. While there is an attempt to link two properties or two figures, it is characterised by a single dominant property that precludes the utilisation of a relationship that is readily available in both directions.

U_1(F) response

This type of response incorporates a relationship between two properties, or between two figures, and is justified accurately. Two properties are now perceived to work together, and as a result determine the figure. This single relationship has become a workable unit, which forms the basis of the minimisation.

M_1(F) response

The response is based upon the existence of multiple pairs of relationships between properties or relationships between figures. While the focus of the response is on more than one relationship, they are treated in isolation. Minimisations remain in conflict with the need to distinguish a certain figure from other figures within the same global class.

R_1(F) response

The response includes a focus upon the network of relationships among known property and figure relationships. The interrelationships may not incorporate all property relationships.

U_2(F) response

A network of relationships is the focus of the response. There is an understanding of the general overview, which utilises relationships among groups of properties and figures. The notion of minimisation can be held in more than one circumstance spontaneously.

Research Question 3.1 stated, *What are the similarities and differences when comparing the framework offered by the SOLO model in the context of relationships among triangle figures, and relationships among quadrilateral figures?* The comparison of the SOLO categories across the triangle context and the quadrilateral context highlighted some interesting trends. The generalised descriptions above, which take into consideration the similarities and differences observed, provide a framework that is applicable to both contexts concerning growth in understanding relationships among figures. This growth culminates in an understanding of notions of class inclusion. The triangle tasks response characterisations supported the categories identified in the quadrilateral task. While the complexity was increased within the quadrilateral task due to an increased number of classes and subsets, similarities were evident across the SOLO categorisations as described in the general descriptions.

Research Question 3.2 stated, *What are the similarities and differences when comparing the framework offered by the SOLO model in the context of relationships among triangle properties, and relationships among quadrilateral properties?* Through the qualitative comparison of the developmental framework evident in two contexts, concerning the relationships among properties, various similarities and differences were identified. A general description of the SOLO characterisation of levels leading to an understanding of the overview of relationships among properties applicable to both contexts emerged. Of particular interest is the manner in which the concrete symbolic responses in both contexts perceived the figure as determining the properties. The perception of the property relationships determining the figure was not evident until the first cycle of the formal mode where the properties are perceived to determine the figure. This culminates in a focus on the network of relationships that determine the figure as evident when entering the second cycle of the formal mode.

Chapter 5 has established a broad framework for quadrilaterals. The two areas of developmental growth are, relationships among quadrilateral figures, and relationships among quadrilateral properties. Chapter 4 addressed the same issue in the context of triangles. The concluding section of Chapter 5 considered the similarities and differences within the hierarchical frameworks that emerged in Chapters 4 and 5. Through this comparison two general frameworks were derived which are applicable to both the triangle and quadrilateral contexts. These concerned relationships among figures, and relationships among properties. The following chapter addresses the similarities and differences between the two contexts through a quantitative comparative analysis.

CHAPTER SIX

INTEGRATION OF TRIANGLE AND QUADRILATERAL RESULTS

This chapter integrates the frameworks that emerged indicating developmental growth in understanding of relationships among figures, and relationships among properties. This is considered via a quantitative approach, which provides a comparison of the complexity of the different tasks. In addition, this chapter concerns a discussion of the longitudinal growth that highlights the developmental pathways leading to understanding of class inclusion concepts in Geometry.

The research theme and research questions that guide the investigation include:

Research Theme 4

Can a quantitative analysis of the results, using a Rasch analysis model, offer insights into students' understandings of class inclusion?

4.1. How do the identified response categories reflect the hierarchical framework of the SOLO model?

4.2. Is there an order of difficulty among the item responses, which can assist in interpreting the complexity of students' understandings of relationships among figures and relationships among properties?

4.3. Which response categories to tasks had relatively larger increases in complexity from the prior response category, and how does this increase reflect upon students' growth in understanding relationships among figures and relationships among properties?

The results are presented in two sections. The first section, titled Quantitative Synthesis, applies the Rasch partial credit modelling process to assist in the interpretation of response categories, and considers student development over the two-year period. The final section, titled Conclusion, ties the findings together.

QUANTITATIVE SYNTHESIS

In this section, a quantitative analysis of the results is presented. The SOLO categorisations for both figures and properties, in the contexts of triangles and quadrilaterals, have been processed using the QUEST software application (Adams & Khoo, 1993) of the Rasch partial credit modelling process, provided by Masters (1982).

Overview of Data

With the categories for each item being of an ordinal nature, the data assumptions of the Quest application of the Rasch modelling process are consistent with the data of this study. The software (Adams & Khoo, 1993) can be used to provide estimates of item difficulties and case abilities, together with measures of model fit. In addition to the tables of delta scores for item difficulty and case (student) estimates, and the corresponding mean square values, the Quest program presents the information in the form of an item map that explores the unidimensionality of the items.

The analysis of data using the Rasch model allows for the exploration of the existence of developmental structure. Such a hierarchical structure is dependent upon the data to be analysed which in turn rely on the items having construct validity. "The resulting scale is expected to exhibit *segmentation* and *order*" (Wilson, 1990, p. 232). Wilson provided the following definitions of segmentation and order:

> The segmentation occurs when items representing different parts of the learning sequence are contained on separate segments of the scale, with a non-zero distance between segments. Order occurs when the sequence is in the order predicted by the theory.
>
> (Wilson, 1990, p. 232)

Reliability

Reliability of separation produced by the Quest software is described by Adams and Khoo (1993, p. 24), for both items and cases, as the proportion of the observed variance that is considered true. The reliability of separation of the item estimates of this study could not be measured due to the possibility of item overlaps in the relatively small sample required for the in-depth interview approach taken. Due to this factor, the item estimates are to be interpreted conservatively. The reliability of the case estimates could be measured, and the reliability index was 0.78. This demonstrates stable separation between cases. For this study, the internal consistency parameters (Cronbalch's alpha measure) of 0.75 shows an adequate measure of reliability.

Fit statistics

Fit statistics concerning item estimates appear in Table 6.1, and those concerning case estimates are provided in Table 6.2. Fit statistics are the means and standard deviations of the infit (weighted) and outfit (unweighted) fit statistics in the mean square form. When the observed data and estimates are compatible, the expected value of the mean squares is close to 1 with a small standard deviation, and the transformed infit (Infit t) is close to zero. Hence, the items come from the same underlying construct, namely, relationships among figures and relationships among properties.

Table 6.1 Fit statistics for item estimates

Infit Mean Square		Outfit Mean Square	
Mean	1.02	Mean	1.02
SD	0.17	SD	0.18
Infit t		**Outfit t**	
Mean	0.12	Mean	0.13
SD	0.69	SD	0.57

Table 6.2 Fit statistics for case estimates

Infit Mean Square		Outfit Mean Square	
Mean	0.99	Mean	1.02
SD	0.61	SD	0.73
Infit t		**Outfit t**	
Mean	-0.01	Mean	0.09
SD	1.04	SD	0.92

The component infit mean square values are presented in graphical form in Figure 6.1 to assist in interpretation. The infit statistic for each item is the weighted residual based statistic, which indicates quantitatively how appropriately each item fits the model (Fisher, 1993, p. 4). This comparison can be used to confirm the unidimensionality of the items, confirming construct validity of the items.

The figures on the horizontal scale represent the infit mean square scale and the asterisks indicate the magnitude of the fit statistic for each item on the same line. Fit statistics that lie within the two dotted vertical lines are considered acceptable. The well fitting nature of the items to the model indicates that the items represent aspects of a latent trait. The infit mean square map for the seven items, which appear below in Figure 6.1, indicates that six of the seven items are within the acceptable limits. Item 4, which concerns students' understanding of the relationships among properties of the right isosceles triangle, is only slightly to the right-hand side of the acceptable limits. This indicates that for Item 4, there is a small number of reversals. A reversal occurs when a student responds at a higher level on a harder item than his or her response level on an easier item. Overall, the items contribute consistently to the measurement of the underlying construct.

```
INFIT MNSQ
           .63      .71        .83       1.00       1.20       1.40
-----------+--------+---------+---------+---------+---------+-----
  1 item 1               .             *|              .
  2 item 2               .         *    |              .
  3 item 3               .   *          |              .
  4 item 4               .              |              .*
  5 item 5               .              *              .
  6 item 6               .          *   |              .
  7 item 7               .              |    *         .
==================================================================
```

Figure 6.1 Item map

Overall longitudinal performance

While the Intervention 1 response codings (*n*=24) were presented in Chapters 4 and 5, the overall performance of the longitudinal sample (*n*=12) are presented in Tables 6.3, 6.4, 6.5, and 6.6 below. The SOLO response groupings for Intervention 1 and Intervention 2 are characterised using the same criteria. In the following tables, lines indicate the progression of students in each intervention. Due to varying numbers of students following the same development path, different line styles indicate movement of one student (——), two students (═══) and three students (≡≡≡).

Table 6.3 Triangle figures development–comparison of Interventions 1 and 2

SOLO Coding	Group A Year 8 Int 1	Group A Year 10 Int 2	Group B Year 9 Int 1	Group B Year 11 Int 2
R_1 (CS)	1			
U_2			3	
M_2	3	3	3	3
R_2	2	1		3
U_1(F)				
M_1				
R_1		1		
U_2		1		

Table 6.4. Quadrilateral figures development–comparison of Interventions 1 and 2

SOLO Coding	Group A Year 8 Int 1	Group A Year 10 Int 2	Group B Year 9 Int 1	Group B Year 11 Int 2
R_1 (CS)	1			
U_2			1	
M_2	4	1	5	3
R_2		3		3
U_1 (F)				
M_1				
R_1				
U_2	1	2		

Table 6.5 Mean triangle properties development–comparison of Interventions 1 and 2

SOLO Coding	Group A Year 8 Int 1	Group A Year 10 Int 2	Group B Year 9 Int 1	Group B Year 11 Int 2
R_1 (CS)				
U_2			2	
M_2	1			
R_2	3		3	3
U_1 (F)	2		1	1
M_1		1		1
R_1		4		
U_2		1		1

Table 6.6 Mean quadrilateral properties development–comparison of Interventions 1 and 2

SOLO Coding	Group A Year 8 Int 1	Group A Year 10 Int 2	Group B Year 9 Int 1	Group B Year 11 Int 2
R_1 (CS)				
U_2				
M_2	1			1
R_2	1		4	2
U_1 (F)	3	1	2	1
M_1	1	2		2
R_1		3		
U_2				

The tables above illustrate overall growth of SOLO level for tasks concerning relationships among figures and relationships among properties in the second intervention compared with the first intervention. This is discussed from a quantitative perspective in the sub-section titled Longitudinal Development – Intervention 1 to Intervention 2. It is evident that in both contexts the students predominantly responded at a higher SOLO level during the second intervention when compared with the first intervention.

Item difficulty and case estimates

The item difficulty is represented by thresholds (Δ) (Adams & Khoo, 1993, p. 86). The unit of measure for both case and item estimates is called a logit. The threshold represents the transition between an item response category and the following response category. If a case estimate and an item estimate are of the same value on the logit scale the student will have a fifty-percent likelihood of answering correctly. The position of each item along the logit scale indicates the degree of difficulty found by the sample of students associated with each item categorisation.

The information pertinent to item and case estimates is displayed in the variable map in Figure 6.2. This map includes all responses obtained from Intervention 1 and Intervention 2. There are seven tasks in total, and 36 sets of student responses represented. The chart includes a logit scale on the left of the diagram on which both items (*n*=7) and cases (*n*=36) are calibrated. The distribution of students is represented by XXXs on the left-hand side of the chart.

KEY
1. Rel – tri fig
2. Rel – quad fig
3. Rel – equ fig
4. Rel – r-isos prop
5. Rel – squ prop
6. Rel – para prop
7. Rel – rhom prop

LOGIT	STUDENTS (CASES)	ITEMS		
3.0				
	X			
		5.U_2(F)		
2.0				
		5.R_1(F)		
		1.U_2(F)		
		6.U_2(F)	7.U_2(F)	
		2.U_2(F)		
1.0	X	3.U_2(F)		
		1.R_1(F)	3.R_1(F)	4.U_2(F)
	XXX	1.M_1(F)	6.R_1(F)	
		1.U_1(F)	2.R_1(F)	4.M_1(F)
	X	3.M_1(F)		
	XXX	7.M_1(F)		
	XXX			
	XX	2.R_2(CS)		
	XX	5.M_1(F)	6.M_1(F)	
.0	XX	1.R_2(CS)		
	XXX			
	X	3.U_1(F)	4.U_1(F)	
	XX	6.U_1(F)	7.U_1(F)	
	XXXX	3.R_2(CS)	5.U_1(F)	
	XXXX	4.R_2(CS)		
	XX			
		7.R_2(CS)		
		6.R_2(CS)		
		2.M_2(CS)		
-1.0	X	3.M_2(CS)	5.R_2(CS)	
		1.M_2(CS)		
	X		6.M_2(CS)	
		4.M_2(CS)	5.M_2(CS)	
			7.M_2(CS)	
-2.0				

Figure 6.2 Item and case estimates (thresholds)

When considering individual cases, the case estimates derived from the Rasch analysis for eleven students from the sample of twelve, increased in the second intervention when compared with the first intervention. This is depicted in Figure 6.3 below.

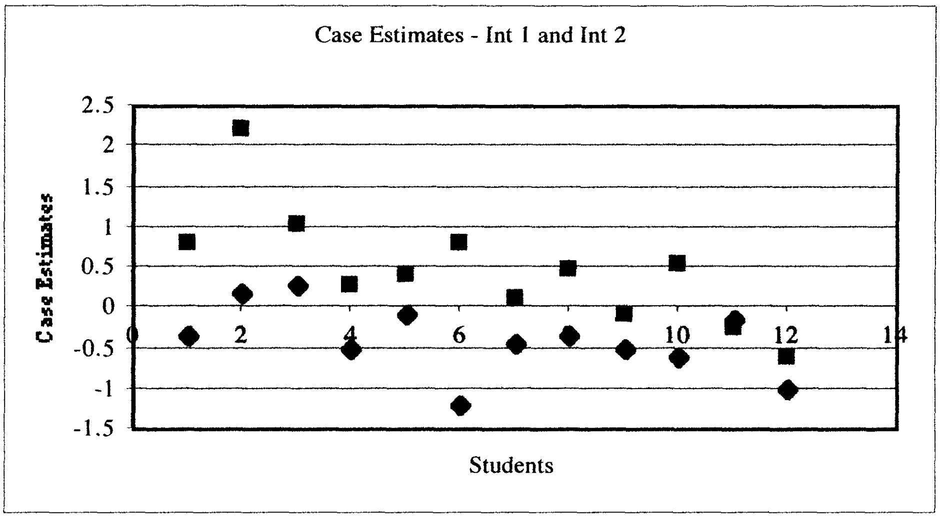


Figure 6.3 Case estimates – Intervention 1 and Intervention 2

A single student's overall performance in Intervention 1 (blue square) compared with Intervention 2 (pink square) is represented by their differing positions along the *y*-axis (case estimates). One student, Megan (Student 11), did not indicate overall developmental growth in terms of the case estimate, however, Megan's response indicated development and maintenance of SOLO level in the majority of tasks. Megan responded at U_2(CS) and M_2(CS) levels in tasks concerning the relationships among figures in Intervention 1, and progressed to R_2(CS) in Intervention 2 for both tasks. For the tasks concerning relationships among properties, Megan remained at U_1(F) for two tasks, R_2(CS) for one task, however, she responded at a lower level for two tasks concerning relationships among properties in Intervention 2.

The Mann Whitney U Test compared the case estimates for Intervention 1 and Intervention 2 to test the null hypothesis, that there is no difference in case estimates between the interventions. This null hypothesis is rejected (z=-3.09, p=0.002). Hence, there is a significant difference between the interventions as the probability is less than the threshold level of 0.05. The conservative Mann Whitney U Test was used as the sample size is small, and, hence, the underpinning assumption of normality required for the t-test could be contravened. Figure 6.4 below graphs Intervention 1 against Intervention 2 and indicates positive correlation.

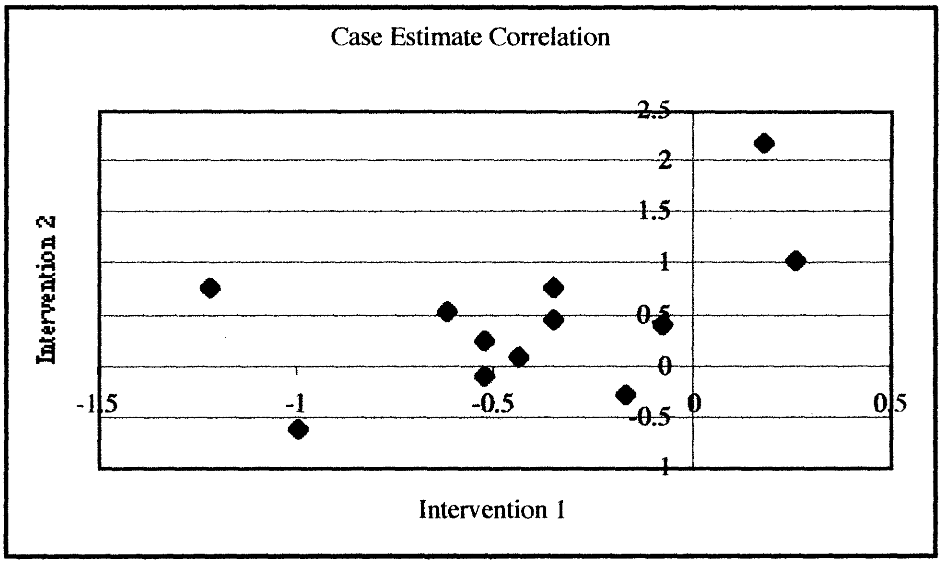


Figure 6.4 Case estimate correlation

The individual item response categories for each item are plotted along the right-hand side of the variable map (Figure 6.2) according to the degree of difficulty. Hence, the higher the item threshold, the greater is the estimated degree of difficulty. Each item has a number, which corresponds to one of the particular tasks. The key for this is located at the upper right-hand side of the chart. Colour coding is used to assist in the identification of SOLO categorisation for each item.

The codings utilised were the general combined structure described in the previous section. These codings are contained in Appendix J. Categories concerning relationships among figures and relationships among properties comprise first cycle responses (R_1) and second cycle responses (U_2, M_2, and R_2) in the concrete symbolic mode (CS), and first cycle (U_1, M_1, and R_1) and second cycle responses (U_2) in the formal mode (F). Hence, the codings from 1 to 8 of Appendix J correlate to these SOLO categories.

An alternative view to the case and item estimates if provided by the overall item difficulty, calculated by the Quest package using the tau statistics option. The overall item difficulty provides an aggregate representative value for each of the items based upon a weighted average of the individual step difficulties. In summary, the overall item difficulty for each of the seven tasks appears in Table 6.7 below. This analysis highlights some interesting trends.

Table 6.7 Overall item difficulty

Relationships Among Figures		Δ
Triangles	Item 1	0.12
Quadrilaterals	Item 2	-0.03
Relationships Among Properties		
Equilateral Triangle	Item 3	0.14
Right Isosceles Triangle	Item 4	0.01
Square	Item 5	0.19
Parallelogram	Item 6	-0.25
Rhombus	Item 7	-0.18

Overall, relationships among figures were found to be the most difficult of the tasks with an overall item difficulty mean of 0.045. This was also evidenced by the supportive nature of relationships among properties in a number of cases, as a prerequisite to a focus upon relationships among figures which are not supported by visual cues. The task concerning the relationships among triangles (0.12) was of a higher degree of difficulty than relationships among quadrilaterals (-0.03). Hence, while the response categories across these two tasks remained at a similar degree of difficulty throughout the varying SOLO categories, the increased number of relationships among classes and properties concerning quadrilaterals did not effect the overall item difficulty.

When considering relationships among properties, of particular interest is the perceived difficulty of relationships among properties of the square (0.19). Although the square was a familiar quadrilateral to the students in the sample, an overview of the relationships of the square was the most difficult for the students to utilise. The students' responses indicate that the visual properties of the square assist in the progression through the second cycle of the concrete symbolic mode, however, the visual dominance of particular square properties hinders the formation of relationships among square properties. Hence, the differences highlighted exceed the similarities perceived.

The relationships among the properties of the parallelogram were found to be the least difficult (-0.25). This was particularly evident in the relative ease with which students identified the relationships among opposite sides parallel, opposite sides equal, and opposite angles equal. The equilateral triangle property task (0.14) was found to be harder than the right isosceles triangle (0.01). This coincides with the unique nature of the equilateral triangle properties, as it is possible for the student to provide minimum descriptions of the figure without justifications based upon the relationships among the properties. Similar to the square, the students found it difficult to shift their focus to one that is not supported by visual cues and no longer requires a real world referent.

Item Analysis

The following discussion addresses the patterns that have emerged concerning item difficulty across item response categories. The features are considered in three areas, namely, relationships among figures, relationships among properties, and, finally, a comparison of item difficulties across figures and property relationships. The comparison involves individual response category item difficulties, which appear in Table 6.8 below.

Table 6.8 Item response category difficulty levels

	Concrete Symbolic			**Formal**			
FIGURES	U_2	M_2	R_2	U_1	M_1	R_1	U_2
Triangles	-2.06	-1.10	0.07	0.74	0.82	0.94	1.42
Quadrilaterals	-1.28	-0.97	0.22			0.70	1.16
PROPERTIES							
Equilateral		-1.08	-0.35	-0.20	0.58	0.91	1.08
Right Isosceles		-1.44	-0.50	-0.12	0.72		0.88
Square		-1.44	-1.03	-0.36	0.19	1.72	2.09
Parallelogram	-1.63	-1.26	-0.77	-0.32	0.16	0.81	1.23
Rhombus		-1.63	-0.74	-0.28	0.48		1.24

The groupings of response categories of similar item difficulty evident in Figure 6.2 and Table 6.8 highlight a number of interesting patterns concerning concept development of the relationships among figures, and relationships among properties in the context of triangles and quadrilaterals. These patterns are discussed below in terms of response category clusters, SOLO categories, and item response category difficulty.

The discussion begins with item response categories found to be less difficult by the sample of students interviewed and ends with the categories found to be more difficult. Similarities and differences in relation to degree of difficulty and characteristics of the responses form the basis of the comparison. This sub-section comprises three parts, namely, relationships among figures, relationships among properties, and a comparison of relationships among figures and relationships among properties.

Relationships among figures

The U_2(CS) response is considered as the lowest level degree of difficulty appearing on the logit scale. The R_1(CS) response category is not assigned a logit value as this value represents the probable transition point between an item response category and the following item response category in terms of difficulty. In the context of relationships among figures, the item difficulty follows the hierarchical structure of modes and levels evident in the SOLO model.

Concrete Symbolic Item Difficulty Concerning Relationships Among Figures

In the second cycle of the concrete symbolic mode, students found it easiest to describe the isosceles, scalene, and equilateral classes of triangles as generic categories known by name and justified on the basis of a single property (U_2CS, -2.06). This is compared with an item difficulty of –1.28 for responses in the quadrilateral context that formed classes of figures on the basis of a single similar property. Both contexts included responses which utilised more than one similar property to justify the formation of classes (M_2CS). These appear at a similar degree of difficulty for both the triangle (-1.10) and quadrilateral (-0.97) contexts.

The predominance of differences was evident in both contexts. In the triangle context differences were noted between the equilateral triangle and the isosceles class of triangles. The quadrilateral context saw the students being unable to link quadrilaterals that were characterised by visual property differences, such as the rhombus to the rectangle, and the square to the parallelogram, and vice versa. While it was possible for the students to make angle-type links between isosceles and scalene classes of triangles, links of a similar nature were made across classes of quadrilaterals when utilising ikonic support. Hence, any property links that could be made were prompted by visual cues.

Due to the increase in complexity of the quadrilateral context, there were many more properties to encapsulate the class of quadrilaterals, and differences to observe. Even so, language use for both tasks at M_2(CS) was of a similar nature. For example, in the triangle context, phrases were used such as "two sides the same only" for the isosceles class of triangles, and "three sides the same" for the equilateral class of triangles.

The quadrilateral task included phrases such as "sides are the same length and parallel but has right angles" to describe the square. The language use for M_2(CS) in both items utilised language which precluded links across classes. The class descriptions provided also accentuated differences in both contexts.

The R_2(CS) responses are positioned close to one another on the logit scale, and hence were found to be of a similar degree of difficulty (0.07 concerning the triangle context, and 0.22 concerning the quadrilateral context). Both R_2(CS) responses in items concerning the relationships among figures required the student to reconcile the differences noted, and focus upon similar properties observed across classes of figures. At this level, students utilised language of an inclusive nature, for example, "three sides equal" is now inclusive of "two sides equal" and "four equal angles" is inclusive of "opposite angles equal." Hence, if the student was able to apply the inclusive nature of one property between two classes not linked by visual cues, this level was entered. As a result, it is evident that the students found the degree of difficulty for linking the equilateral class of triangles to the isosceles class of triangles, at a slightly lower degree of difficulty, than linking the square to the parallelogram, and, the rhombus to the rectangle.

Formal Mode Item Difficulty Concerning Relationships Among Figures

The next group of responses to appear on the item estimates chart were the first cycle responses (U_1, M_1, and R_1) in the formal mode. The R_1(F) response concerning relationships among quadrilaterals was found to be slightly less difficult (0.70) than the U_1 (0.74), M_1 (0.82), and R_1 (0.94) responses concerning relationships among triangles.

The quadrilateral item incorporated six classes of figures: square, rectangle, parallelogram, rhombus, kite, and trapezium. The encapsulation of properties which form a class was also made easier in the triangle context by having three examples of each of the isosceles and scalene classes of triangles: right-angled, acute-angled, and obtuse-angled. In comparison, the quadrilaterals had only one example of each when not utilising notions of class inclusion.

The students found it more difficult to undergo the shift in focus, and perceive the equilateral triangle as a member of the isosceles class of triangles, than to form a class of parallelograms inclusive of subsets categorised by name (such as the square, rectangle, and rhombus). The students found it less difficult to overcome differences associated with the parallelogram class. The class of parallelograms was described as having "opposite sides parallel and equal, and opposite angles equal." In both items, the notion of class inclusion is an integrating influence that remained a central feature of the first cycle formal response.

The U_2 responses in the formal mode were at a similar level in both the triangle (1.42) and quadrilateral (1.16) contexts. The students found placing further conditions on the class of parallelograms to form classes within sub-sets, to be at a lower degree of difficulty than the U_2 formal response in the triangle context. The overview necessary to tie together relationships based upon angle-type links, triangle-type classes, and class inclusion notions was found to be the most difficult considering tasks dealing with relationships among figures. In the triangle context, this required the student to place further restrictions on the subset formed at $R_1(F)$ and acknowledge a more significant relationship from the equilateral to the acute isosceles than from the obtuse isosceles triangle and right isosceles triangle. While the quadrilateral context required further subsets within the class of parallelograms, the class formed at $R_1(F)$ did not require alteration.

Relationships among properties

As in the previous section, this section outlines a comparison of item difficulty in regards to students' responses to tasks concerning relationships among properties in the contexts of triangles and quadrilaterals. Similarly, the discussion follows the clusters of response categories that are illustrated in Figure 6.2.

Concrete Symbolic Item Difficulty Concerning Relationships Among Properties

There is one response category assigned with a degree of item difficulty which is categorised at U_2 to tasks concerning relationships among properties, within the concrete symbolic mode. The students in the sample found recognition of one unique property which signified the parallelogram at an item difficulty of -1.63. This was the lowest level response category concerning relationships among properties to be allocated a threshold.

The second group of responses are the $M_2(CS)$ responses concerning relationships among properties. The five items at this SOLO classification are grouped together, these being the equilateral triangle (–1.08) and right isosceles triangle (-1.44) property tasks, and the square (-1.44), rhombus (-1.63) and parallelogram (-1.26) property tasks. The students found the degree of difficulty to be similar for focusing upon more than one unique property of the right isosceles triangle in isolation, as selecting one or more properties of the square to signify the quadrilateral.

It is interesting to note that the rhombus responses at this SOLO level were at the same degree of difficulty as a $U_2(CS)$ response in the context of the parallelogram. The responses to the rhombus task at $M_2(CS)$ often included a list of properties, which were chosen by referring back to the rhombus each time. The tendency to sketch and trace the figure was more prevalent in the rhombus responses. The $M_2(CS)$ response to the parallelogram task was found to be the lowest degree of difficulty of all property tasks within this cluster of responses.

The next group of responses concerning the relationships among properties are those at the relational level of the second cycle (CS). The three quadrilateral tasks at R_2(CS) are grouped, and hence, are considered to be at a similar level of difficulty. The triangle R_2(CS) responses are above these, with the equilateral triangle property response (-0.35) of a slightly higher degree of difficulty than the right isosceles triangle task (-0.50) at R_2(CS).

In the quadrilateral context, the sample of students found the degree of difficulty for demonstrating an understanding of the notion of minimisation, and attempting to make a link between the figure in question and one other figure, or, attempting to link two properties, similar for the square (-1.03), parallelogram (-0.77), and rhombus (-0.74). The responses at R_2(CS) indicated a confidence in chosen combinations, although sometimes these were superfluous in the cases of the parallelogram and rhombus, but demonstrated difficulty in justifying decisions.

The triangle property tasks at R_2(CS) were characterised by the focus upon a single link, which was not readily available in both directions. While the triangle responses at this level also indicated a confidence that the chosen combination would represent the triangle, the link was dominated by one of the properties of the linking pair. Hence the link had not formed a balanced workable unit. Responses within the triangle context indicate tentative movements towards perceiving the property relationships as determining the figure, however, this remains verbose or not adequately justified.

Formal Mode Responses to Relationships Among Properties

The U_1(F) responses in the context of relationships among properties are grouped closely together, and hence, indicate a similar degree of difficulty. While there is slight variation, the item estimate thresholds chart indicates that a focus upon the properties determining the figures in both triangle and quadrilateral contexts is of a similar degree of difficulty, namely, equilateral triangle (-0.20), right isosceles triangle (-0.12), square (-0.36), parallelogram (-0.32), and rhombus (-0.28).

In the triangle context, the response incorporates a focus upon a single relationship between two properties. The relationship between two properties is described in a balanced manner. This also applies in the quadrilateral context, however, a bifurcation takes place within this group of responses, where the focus is upon a link between two properties, or a link between the figure in question and one other figure. Hence, statements given at R_2(CS) in both contexts are redefined at this new level and are utilised adequately. The links between the properties or figures have become the tool from which the selection of properties is made and justified.

Similarly, the multistructural responses (M_1F) concerning relationships among properties are consistently placed above the $U_1(F)$ responses for each of the tasks. The $R_1(F)$ responses consistently appear at a higher degree of difficulty when compared with the $M_1(F)$ responses. While this consistency is evident and clusters of each level exist, there is a blurring of the first cycle formal mode responses.

In both the triangle and quadrilateral contexts, the responses included a focus upon a single relationship between two known properties, to multiple links between pairs of properties, to a focus upon the interrelationships that exist among the property relationships. The $R_1(F)$ quadrilateral tasks were often characterised by the need for the student to add additional cards described in terms of negative instances. For example, "has no sides equal" or "opposite angles are equal but they are not all the same." In the triangle context, responses were not characterised by the need to add additional information in terms of negative instances. The R(F) responses in the triangle context focused more clearly upon the interrelationships and did not have the blurring effect associated with more difficult and less familiar properties, such as diagonal properties. Within this cluster of response categories, the square $R_1(F)$ responses are at a considerably higher degree of difficulty (1.72) when compared with other $R_1(F)$ property relationship responses, namely, responses to the parallelogram (0.81) and equilateral triangle (0.91).

The final and highest group of responses on the logit scale, in the context of relationships among properties, are those in the second cycle of the formal mode. While the responses to the five tasks are grouped together, there are variations in degree of difficulty worthy of consideration. Each of the $U_2(F)$ responses concerning relationships among properties required a focus upon the interrelationships that exist among the previously identified property relationships in a succinct and spontaneous manner. In the triangle context, the students found it of a slightly higher degree of difficulty to apply this overview in the context of the equilateral triangle (1.08). The students found it easier to relate equality of sides, equality of angles, and symmetry together in the right isosceles triangle context (0.88), than they did in the equilateral context.

The positioning on the logit scale of the quadrilateral tasks concerning property relationships indicates a clear progression in degree of difficulty of $U_2(F)$ responses in this context. The students found a focus upon the overview of property relationships for the parallelogram (1.23) the lowest degree of difficulty followed by the rhombus (1.24) property task, with the square (2.09) property task considered the most difficult by the students interviewed. The U_2(F) responses to the parallelogram and rhombus property tasks were considered the lowest degree of difficulty of all U_2(F) responses in the quadrilateral property context. The parallelogram and rhombus property combinations were based upon the network of relationships surrounding "opposite sides parallel," "opposite angles equal," and "opposite sides equal." When considered in this light, the square required a greater emphasis on the network of relationships in regards to relationships among figures, in particular, notions of class inclusion. A focus upon the network of relationships in the square context was considered most difficult, as it is possible to respond at a lower level and supply correct minimum combinations of properties. The square has unique property signifiers that link to other properties separately. This allows multiple minimum combinations to be devised, such as "sides equal and angles equal," "four axes of symmetry," and "diagonals are equal and bisect at right angles."

Comparison of item difficulty across figures and property relationships

On the basis of item difficulty thresholds appearing in Figure 6.2, this section considers the similarities and differences associated with item difficulty and characteristics of response categories across tasks concerning relationships among figures and relationships among properties. This is considered in clusters of item responses beginning with the lower level SOLO responses which also appear at the lower end of the item estimate threshold. Thus, the following discussion provides an overview while making a comparison across task type and across triangle and quadrilateral contexts.

In the tasks concerning relationships among figures, and those concerning relationships among properties, a hierarchical framework emerged which is evident in the SOLO categorisations and is reinforced by the application of the Rasch analysis. Each of the items followed the SOLO sequence of levels within cycles without exception. Interesting groupings of responses at particular levels also assisted in providing quantitative information concerning the similarities and differences in regard to degrees of difficulty. The following discussion provides a comparison of item estimate thresholds when comparing item difficulty across response categories concerning relationships among figures, and item responses concerning relationships among properties.

The U_2(CS) response category concerning relationships among triangle figures was found by the sample of students to be of the lowest degree of difficulty. This was followed by other groups of U_2(CS) and M_2(CS) responses concerning relationships among figures, and relationships among properties. Hence, the students found the utilisation of the three mutually exclusive classes of triangles, at a similar degree of difficulty to focusing upon unique property signifiers of figures with reference to the figure only. It appears that the progression to finding multiple property signifiers to assist in the formation of minimum combinations assists the encapsulation of multiple properties to form generic categories. While restrictive language which does not facilitate the inclusive nature of properties is utilised in U_2(CS) responses concerning figures and properties, this level is a necessary precursor for developing notions of minimisations.

Next on the logit scale is a cluster of R_2(CS) responses including all five tasks concerning relationships among properties. Hence, the students found ordering between two properties to be at a similar degree of difficulty in both the triangle and quadrilateral contexts. While the U_1(F) responses are grouped together when addressing tasks concerning the relationships among properties, these appear before the R_2(CS) responses in the context of relationships among figures, thus indicating that the students found a focus upon relationships between pairs of properties and/or figures, and making property links across classes, of a similar degree of difficulty in both triangles and quadrilaterals contexts. The U_1(F) response concerning property relationships appears to be a precursor to focusing upon relationships among classes of figures, which are not supported by visual cues. The remaining first cycle formal responses are clustered at a similar degree of difficulty, thus indicating that the utilisation of a single network of relationships among figures, utilising multiple relationships among properties, and an attempt to focus upon the interrelationships among property relationships are at a similar degree of difficulty.

The U_2(F) responses have a greater range in terms of degree of difficulty. This final cluster indicates that the students found the focus upon more than one network of relationships involving notions of class inclusion, and the focus upon the network of relationships to form minimisations, the most difficult groups of responses. In the context of property relationships the students found the right isosceles triangle and parallelogram items the least difficult at this SOLO level. Class inclusion notions requiring the acknowledgment of multiple subsets when relating figures was at a similar degree of difficulty to the utilisation of the network of relationships among properties of the equilateral triangle. This was closely followed by the rhombus task.

It is interesting to note the high degree of difficulty found by the sample of students when forming minimisations of square properties based upon the network of property relationships. This indicates that while the lower SOLO categories indicated a comparatively lower degree of difficulty for the square item compared with other items of the same SOLO level, the shift required to move form $M_1(F)$ to $R_1(F)$ is relatively difficult in the context of the square. The responses indicated that this is due to factors such as visual cues assisting links, and multiple unique properties of the square which assist understanding at lower SOLO levels. In contrast, at the formal mode the student must leave the real world referent behind and focus upon the network of relationships among the properties, as opposed to concrete symbolic justifications.

The degree of difficulties between item response categories, known as step difficulties, further clarifies the similarities and differences among the SOLO categorisations. The step difficulties describe the change in degree of difficulty, found by the sample of students, between one SOLO level and the subsequent SOLO level. These appear in Table 6.9, and also include the mean step difficulty for each SOLO response category within tasks concerning relationships among figures, and relationships among properties.

Table 6.9 Step difficulties

	Concrete Symbolic			**Formal**		
	U_2 to M_2	M_2 to R_2	R_2 to U_1	U_1 to M_1	M_1 to R_1	R_1 to U_2
Item 1	0.96	1.17	0.67	0.08	0.12	0.48
Item 2	0.31	1.19				0.46
FIG MEAN	**0.64**	**1.18**	**0.67**	**0.08**	**0.12**	**0.47**
Item 3		0.73	0.15	0.78	0.33	0.17
Item 4		0.94	0.38	0.84	0.16	0.25
Item 5		0.41	0.67	0.55	1.53	0.37
Item 6	0.37	0.49	0.45	0.48	0.65	0.42
Item 7		0.89	0.46	0.76		
PROP MEAN	**0.37**	**0.70**	**0.42**	**0.68**	**0.67**	**0.30**

Of particular interest, are the higher and lower step difficulties. The step difficulty between a $U_2(CS)$ response and an $M_2(CS)$ response concerning relationships among figures has a mean of 0.64. It was also found to be difficult by the sample of students to respond at $R_2(CS)$ compared with $M_2(CS)$ concerning relationships among figures (mean 1.18). This was similar to the step difficulties concerning relationships among properties, where $M_2(CS)$ to $R_2(CS)$ (0.70) was found to have a comparatively high step difficulty.

In addition, movement through the first cycle of the formal mode is a difficult progression concerning relationships among properties. This is evident by: $U_1(F)$ to $M_1(F)$ (mean 0.68) and $M_1(F)$ to $R_1(F)$ (mean 0.67). It is interesting to note that the highest individual step difficulty concerns the shift from M_1(F) to R_1(F) in regards to relationships among square properties (1.53).

Overall the progression from U_1(F) to M_1(F) concerning relationships among figures has the least step difficulty (0.08). Figures 6.5 and Figure 6.6 illustrates the mean increase in difficulty evident in progressing from one SOLO level to a subsequent level concerning relationships among figures, and relationships among properties.

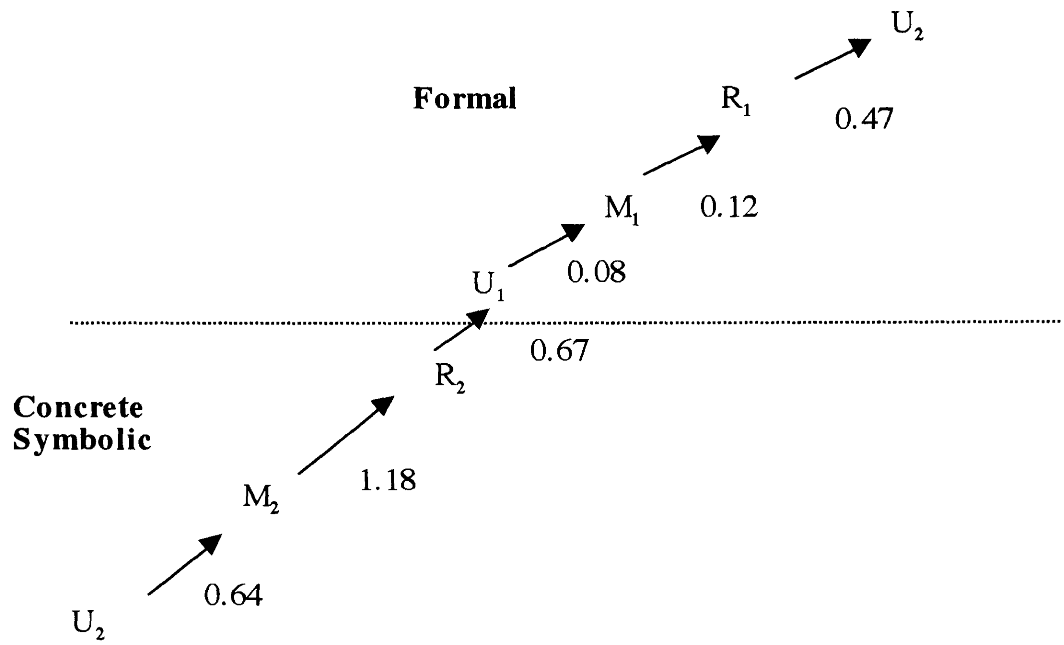


Figure 6.5 Relationships among figures mean step difficulty

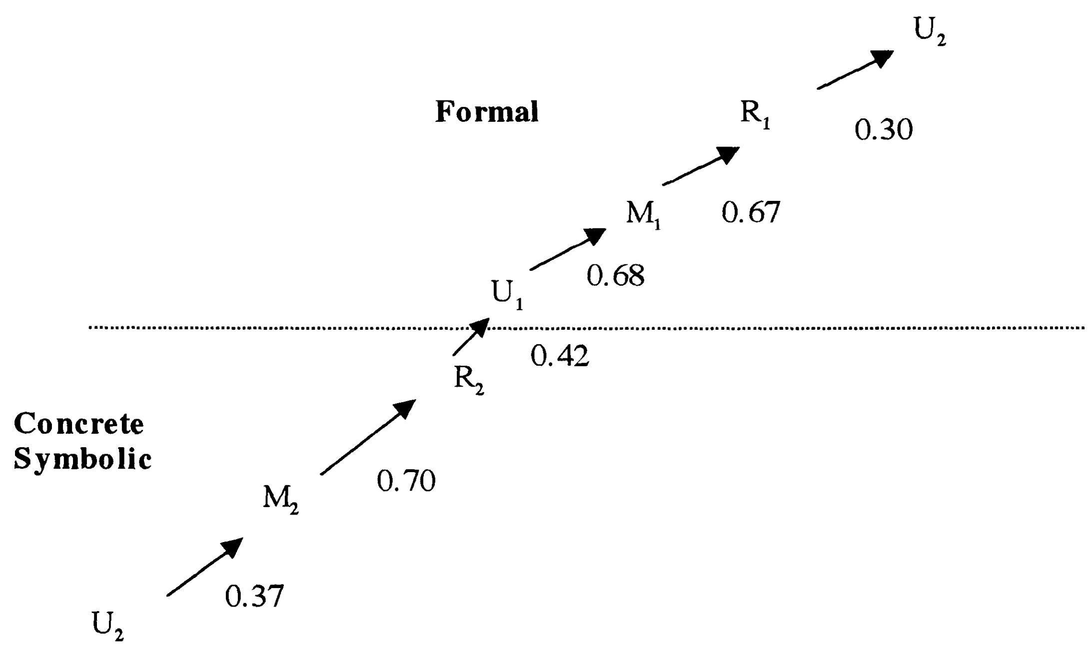
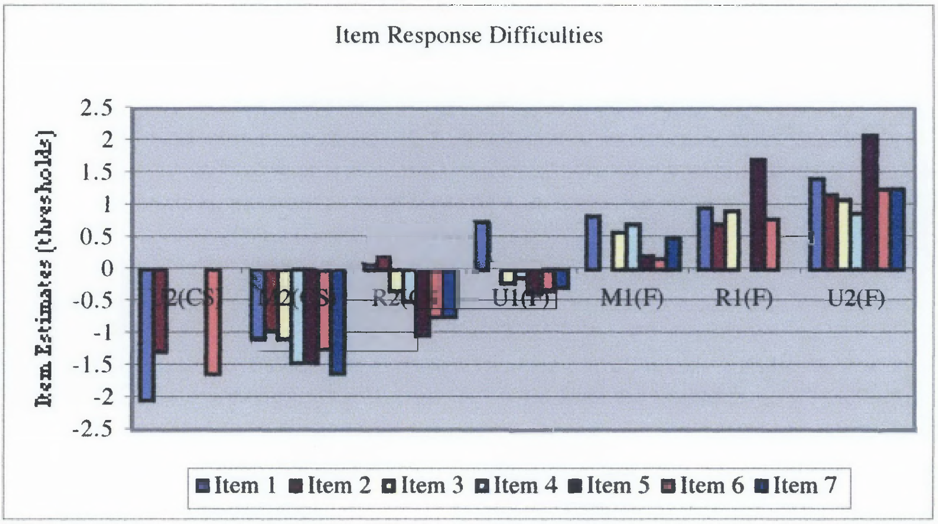


Figure 6.6 Relationships among properties mean step difficulty

An overview of the each item response category degree of difficulty appears in Figure 6.7 below. This graph emphasises visually the hierarchical SOLO frameworks concerning relationships among figures, and relationships among properties evident through the quantitative analysis. While the developmental structure is evident across the seven items, it is also interesting to note the diversities. For example, the $U_1(F)$ response category to Item 1 (relationships among triangle figures) was found to be at a higher degree of difficulty then other $U_1(F)$ responses to other tasks. This category involved tentative statements concerning the equilateral triangle as a subset of the isosceles class of triangles.



Figures 6.7 Item response difficulties

Of particular interest are the predominant clusters which depict the similar item difficulty between the SOLO levels for each of the seven items. Figure 6.7 also illustrates the similarities in difficulty of progression within Item 1 and Item 2 concerning relationships among figures, and the similar hierarchical structure evident within the SOLO categories in Items 3, 4, 5, and 6 concerning relationships among properties. In this graphical representation the difficulty associated with moving through the formal mode in relation to the square property task (Item 5) is particularly evident.

This quantitative analysis has provided a means for ascertaining growth along the developmental pathway leading to an understanding of class inclusion concepts in Geometry. This theme is considered from a qualitative individual perspective in Chapter 7.

CONCLUSION

This chapter employed a quantitative analysis through the use of the Rasch partial credit modelling process provided by Masters (1982). The employment of this technique explored reliability and confirmed the existence of construct validity. In addition, the Quest program plotted the item difficulties for the seven tasks upon a single scale. This analysis offered insights into students' understandings of class inclusion concepts, thus facilitating the comparison of difficulty associated with students' understandings of class inclusion concepts related to two contexts of differing complexity, namely, the triangle and quadrilateral contexts.

Extensive qualitative procedures were utilised in previous chapters, which synthesised the SOLO categorisations through the identification of similarities and differences across the contexts of triangles and quadrilaterals. The quantitative component employed in this chapter was designed to complement and extend the qualitative analysis of results, through a procedure that provided comparative qualitative results across relationships among figures, relationships among properties, and different contexts. Of particular interest was the finding that despite the quadrilateral context being chosen in the study due to an increase in complexity, this was not mirrored by the analysis. The degree of difficulty was found to be similar within the triangle and quadrilateral contexts.

Research Question 4.1 stated, *How do the identified response categories reflect the hierarchical framework of the SOLO model?* The application of the Rasch model supported the developmental sequence that evolved through the SOLO categorisations. The results also highlighted a number of interesting trends. The first of these is the consistency of the groupings evident in the item estimate thresholds when comparing student responses across figure tasks, property tasks, and different contexts. Secondly, the fit statistics and item estimates indicate that the items came from the same underlying construct. This provides confirmation of the appropriateness of the SOLO model.

Research Question 4.2 stated, *Is there an order of difficulty among the item responses, which can assist in interpreting the complexity of students' understandings of relationships among figures and relationships among properties*? The concrete symbolic responses indicate that maintaining a focus upon a single property to encapsulate separate classes of figures is a prerequisite to focusing upon a single unique property of a figure when asked to provide a minimum description of a figure. The M_2(CS) responses indicate that the shift in moving from multiple properties to form individual classes of figures is at the same level as identifying multiple unique property signifiers while maintaining a real world referent. Thus, the figure determines the properties.

The identification of a link between two properties, and the shift to utilising the relationship as a workable unit are necessary precursors to the utilisation of relationships among classes of figures without the need for a real world referent. This progression is a shift into the formal mode in terms of relationships among properties, and is characterised by the property relationships determining the figure in both contexts. When the formal mode is entered, concerning relationships among properties, the degree of difficulty is the same in regards to linking properties or figures despite the bifurcation. The focus upon perceiving the property relationships as determining the figures and utilising inclusive language to describe properties begins at a lower level than focusing upon links across classes of figures. This sequence flows through to a focus upon the network of relationships among figures and properties where there is greater variation in degree of difficulty found by the students across the seven tasks when providing a $U_2(F)$ response.

Research Question 4.3 stated, *Which response categories to tasks had relatively larger increase in complexity from the prior response category, and how does this increase reflect upon students' growth in understanding relationships among figures and relationships among properties*? The higher and lower step difficulties between SOLO response categories assist in the interpretation of the more difficult, and less difficult progressions from one SOLO level to the subsequent SOLO level. The highest increases, or 'hard boundaries,' were found to be in the second cycle of the concrete symbolic mode concerning relationships among figures. These increases concerned the progression from a focus upon single properties to form individual classes of figures, to multiple properties while maintaining mutually exclusive classes, with the hardest boundary being the shift to a focus upon relationships between classes that are not supported by dominant visual differences. Similarly, in the context of property relationships, hard boundaries exist in the shift from a focus upon multiple properties as unique signifiers of a figure, to a focus upon an ordering between two properties.

To a lesser extent, the boundary is relatively difficult when moving from an understanding that the figure determines the property, to a shift into the formal mode where relationships among properties determine the figure. Another hard boundary exists in the progression from a focus upon multiple relationships among properties, to an overview of the network of relationships among properties.

Of particular interest are the supporting influences between relationships among figures, and relationships among properties. These include, the encapsulation of properties to form classes, a shift to perceiving the properties as determining the figure, the dominance of recognised similarities and differences across classes of figures and among properties, and the utilisation of inclusive or exclusive, class or property descriptions.

As a means for exploring these themes that have emerged through the qualitative and quantitative analysis, a longitudinal perspective was taken. The following chapter explores the response patterns that occurred in reference to tasks concerning relationships among figures, and relationships among properties, over a two-year period. This perspective is presented in the form of four student case studies.