INTRODUCTION

The problems associated with children's initial understanding of algebraic concepts have been well documented in the past decade (Booth 1984, 1988; Küchemann 1981; Sleeman 1984; Kieran 1991, 1992). This dialogue has resulted in considerable discussion with regard to alternative methods of developing the initial algebraic understandings. These include the use of number patterns as a context to facilitate the transition from arithmetic thinking to algebraic thinking. This context provides a real referent, understandable to children, for expressing generality.

While such an approach has considerable intellectual appeal, little is known about children's ability to interpret number patterns and to use them in the way indicated by recent syllabus documents (Board of Secondary Education, N.S.W 1988) and curriculum materials (Australian Education Council 1990; Romberg 1989) to develop algebraic notation. These issues provide the central focus of this study which is investigated by addressing four themes. Each theme is developed into a number of specific research questions which are addressed using two complementary research methodologies.

Three of the themes are concerned with how children respond to questions that use number patterns as stimulus items. The first theme involves determining what natural language children use to describe a variety of number patterns. The number patterns vary in both their context and complexity. The second theme considers how natural language responses relate to other response types and to the school year of the respondents. The other responses types include using symbolic notation and extending the number pattern to a large value of the independent variable. The results from these two themes are used to hypothesise the existence of a hierarchy of development in response types that becomes the central issue for the third theme, namely, how do children's descriptions of number patterns, using both natural language and symbolic language, change over time?

To provide a theoretical framework for the study, the SOLO Taxonomy has been chosen. The Taxonomy facilitates an analysis of responses by providing a mechanism for analysing the structure of children's responses and hence provides a means for hypothesising a hierarchy of development in children's pattern description. The efficacy of the SOLO Taxonomy in this regard provides the fourth theme of the study.

The two complementary components of the study are a cross-sectional survey of 1435 children in Years 5 to 8 and a two-year longitudinal component involving 20 children. The two components are reported, using nine chapters.

The first chapter conceptualises algebraic thinking as growing out of arithmetic thinking and identifies some elements of, as well as barriers to, the transition. In particular, some ways of facilitating the cognitive changes necessary for the transition to algebraic thinking are described.

Alternative views of theoretical frameworks are presented in Chapter 2. The position accepted for this study is that theory is a fluid phenomenon which provides a mirror in which to reflect the collected data and to assist in relating the current investigation to the broader context of mathematics education.

In Chapter 3, three of the research themes are developed into a number of research questions and then the research design for the survey component of the study is presented. Considerable detail of procedural matters is provided, together with the instrument design and a data analysis plan. Finally, the constructs of validity and reliability are used as a framework to evaluate the design.

The data collected in the survey is reported and analysed in Chapters 4, 5 and 6. In Chapter 4 the coding procedures and the associated reliability checks are described and the resulting category frequencies reported. The categories of Chapter 4 are analysed in the context of the SOLO Taxonomy in Chapter 5. As a consequence of this analysis a developmental model is described and validated using the Rasch Partial Credit procedure and tests for the homogeneity of effect size. Chapter 6 considers relationships between the various data sources of the survey by identifying a parsimonious model using the technique of loglinear analysis. Four variables are taken into the model and significant interaction effects identified. The application of loglinear modelling used (Norusis 1990) also provides parameter estimates and the associated Z scores to indicate the strength and direction of the association between individual categories of the related variables.

The design for the longitudinal component of the study is presented in Chapter 7. This component responds to a series of issues identified in the survey. These issues are associated with how children's responses to the number pattern stimulus items change over time and are presented as the fourth research theme of the study. The data for this component consists of children's responses to the number pattern stimulus items together with protocols from semi-structured interviews. There were five interventions with each participating child over the two years of the study. The analysis of this data and relevant conclusions are reported in Chapter 8.

The final chapter consists of four sections. The first section presents a brief overview of the findings of the study. The methodological limitations and the limitations of the scope of the study are discussed in the second section prior to a consideration of the implications for further research in the third section. The report concludes with the implications that the findings of the study have for further research.

Chapter 1

ALGEBRAIC THINKING:

A LITERATURE REVIEW

Introduction

> The teaching of algebra in schools is not significantly different today from what it was fifty years ago.... It is time for a major reevaluation of the content of the algebra curriculum and of the instructional strategies that are used in the teaching of algebra.
>
> (Thorpe 1989, p. 11)

Children's difficulties with high school algebra have been a recurring theme in the mathematics education literature since the early 1980s (Booth 1984; Booth 1988;Kieran 1992; Küchemann 1981). While much has been written, little seems to have been done to shift the focus of instruction to overcome these difficulties. This study investigates some issues associated with developing algebraic thought that have been the subject of recent discussion. In particular, expressing generality, which has been included in recommendations for curriculum change, is considered as a vehicle for facilitating children's understanding of early algebraic concepts.

This chapter describes the context for the investigation into children's understanding of expressing generality in three sections. The first section characterises algebraic thinking as growing out of arithmetic thinking and describes some cognitive changes that are needed in that transition, together with some obstacles to those cognitive changes. The second section presents the case for using expressing generality as a context for facilitating the transition from arithmetic thinking to algebraic thinking, together with a review of the recent curriculum documents supporting the approach. The last section

5. based largely upon the operations of counting, adding and combining, and
6. worked almost entirely within the system of whole numbers (and halves).

(p. 37)

Such methods facilitate solutions for a limited range of problems since the implicit nature of the strategies does not lend itself to the formal symbolism of algebra.

The issue of implicit methodology and explicit methodology was taken up by Cortes, Vergnaud and Kavafian (1990) and Vergnaud and Cortes (1986). These writers have described teaching experiments where they have used an arithmetical environment to introduce algebraic methods. To achieve this they introduced algebra as a way to solve arithmetic problems. They justified their approach by arguing that to introduce algebra as a discrete formal mathematics topic is to hide its function as a problem-solving tool. However, while the precise nature of the distinction is not addressed in their work they argued that the shift from arithmetic to algebra involves the shift from using natural language with intuitive tools to having to manipulate chains of symbols with explicit rules. This shift in thinking can be seen as a shift in each of three dimensions, namely,

- implicit to explicit
- natural language to symbolic language
- heuristic to algorithmic.

In this argument the assumption is being made that arithmetic language and operations are part of the natural language of the child and hence are a prerequisite to the development of algebraic concepts. This point is reinforced a little later in this chapter when the possibility of unfamiliar arithmetic calculations being generated by algebraic investigations is discussed.

Usiskin (1988, p. 8) reported MacLane and Birkhoff as saying that

> Algebra starts as the art of manipulating sums, products, and powers of numbers. The rules for these manipulations hold for all numbers, so the manipulations may be carried out with letters standing for the numbers.

He argued that the first sentence describes arithmetic and that the second sentence "is school algebra". While the second sentence might more accurately be seen as the beginning of school algebra, or as a subset of school algebra it remains that algebra is growing out of arithmetic. The transition can be seen to involve some cognitive change. Various attempts to characterise the discontinuity between arithmetic thinking and algebraic thinking. is the focus of the next part.

The Discontinuity

While it has been argued above that algebraic thinking can be seen to grow out of arithmetic thinking, there is evidence of a significant discontinuity between the two modes of thinking, see, for example, Filloy & Rojano (1989), Lee & Wheeler (1989) and Kieran (1989, 1990). These differences have been described using a number of terms that seem to have much commonality in their meaning. Arzarello (1991) and Arzarello, Bazzini and Chiappini (1993) characterised arithmetic thinking as procedural and algebraic thinking as relational. Chiappini and Lemut (1991) developed this distinction in the context of children's responses to the question:

> Draw a rectangle of height "*h*" and base twice the height. Write two formulas which enable you to calculate the perimeter and area of the rectangle respectively.
>
> (p. 201)

Of these responses, those that described a step by step series of operations on the data were classified as procedural. An example of a procedural response is:

$$2h+2h+h+h.$$

A relational response involves "pointing out some relationship between the data" (p. 205). An example of a relational response is:

$$6h.$$

A slightly different dichotomy was used by Kirshner (1992). He used the terms structural and empirical algebra to distinguish between two foci of instruction, not to distinguish between the two modes of thinking. Attempts to teach structural algebra, Kirshner argued, were associated with the "new math" and have been judged to have failed in spite of "an unprecedented investment of resources" (p. 1). Empirical algebra involved developing algebraic meaning in a variety of real world situations, using arithmetic domains and other symbol systems such as graphs and tables. Kirshner's "algebraic structure" was intended to provide the formal underpinnings of school algebra. This was to be "an antidote to the blind symbol manipulation that previously prevailed" (p. 1). Kirshner recognised the importance of empirical algebra but made a plea not to abandon structural algebra as a deductive system.

Kieran (1992) used the distinction between procedural and structural to describe the conceptual differences in the transition from arithmetic thinking to algebraic thinking. She defined procedural as being "arithmetic operations being carried out on number to yield numbers" (p. 392). The substitution of numbers into an algebraic expression and the evaluation of the resultant numerical expression is argued by Kieran to "illustrate a procedural perspective in algebra" (p. 392). The structural perspective refers to a set of operations carried out on algebraic expressions to yield algebraic expressions. The essence of this distinction is that algebra can be perceived as a process closely related to arithmetic or, alternatively, as a structure involving a set of objects. It is the conceptual growth from a process orientation to a structural orientation that Kieran argued is the objective of school algebra. This view is supported by Sfard (1991) who used the term operational in place of procedural:

> There is a deep ontological gap between operational and structural conceptions.... Seeing a mathematical entity as an object means being capable of referring to it as if it was a real thing - a static structure, existing somewhere in space and time. It also means being able to recognise the idea "at a glance" and to manipulate it as a whole, without going into details.... In contrast, interpreting a notion as a process implies regarding it as a potential rather than actual entity, which comes into existence upon request in a sequence of actions. Thus, whereas the structural conception is static, instantaneous, and integrative, the operational is dynamic, sequential and detailed.
>
> (p. 4)

This distinction between the structural conception as integrative and the operational conception as sequential provides a useful distinction and summary of the previous discussion. If this dichotomy of procedural/structural conceptions represents a deep ontological gap, the question needs to be addressed as to how that gap can be bridged. Such a bridge facilitates movement between arithmetic and algebraic thinking. However, there are a number of obstacles in moving from arithmetic thinking to algebraic thinking that are now discussed prior to looking at mechanisms for overcoming them.

Specific Obstacles to Algebraic Thinking

Associated with the need for children to see a set of processes as an object, Kieran (1992) described three adjustments in thinking associated with the shift from procedural to structural conceptualisations that need to be made by the novice algebra student. The first adjustment is to develop the ability to translate problems into algebraic expressions involving the symbolic representation of numerical relationships. Secondly, students need to accept the equal sign as having transitive and symmetric properties rather than merely announcing the answer as in arithmetic thinking. Finally, algebraic thinking requires the modelling of an overview of the data in the form of an equation or other algebraic 'object,' that is, there is a need to 'describe' by presenting the data in an algebraic expression prior to solving. In contrast, arithmetic thinking places emphasis on a linear, sequential approach to a string of operations.

These adjustments are not always supported by instruction procedures. Arzarello, Bazzini and Chiappini (1993) saw traditional teaching practice as an obstacle to algebraic thinking. They believed some teaching strategies provide the opportunity for students to avoid the meaning and use of variables and parameters by allowing students to complete the set tasks without leaving the arithmetic mode of thinking. One is reminded of the Küchemann (1981) classifications of letter not considered, letter as an object, and the like.

Cortes, Vergnaud and Kavafian (1990) also saw algebra concepts arising out of arithmetical experiences. They described a teaching experiment which involved providing early algebraic experiences through solving problems that could be investigated using equations. The aim was to provide "some scaffolding and tutoring" designed to assist the student overcome some conceptual difficulties encountered when first working with algebra. The six conceptual difficulties enumerated were:

1) *The concept of an equation.* This involved making explicit the mathematical relationship implied by the data in the question. This appears to be an example of Kieran's structural 'object' specifically chosen because of the nature of the problems selected for the experiment. They believed students were accustomed to seeing problem solutions as sequences of arithmetic computations and hence equations are thought of as '$x=a+b...$' That is '$x=$' is announcing a set of arithmetical operations and the reflexive property is ignored:

> This tool-like characteristic of an equation ... is not visible when x is incorporated in the analytical expression of the relation $ax + b = c$.
> (Cortes, Vergnaud and Kavafian 1990, p. 28)

Arzarello , Bazzini and Chiappini (1993) extended this difficulty to cover equations and formulas in general. They distinguished between a novice and expert by their ability to choose variables and names for variables from the beginning of the problem-solving process and by showing an understanding of the main relations in the question.

2) *The concept of the unknown.* An unknown is defined as "what is not known in terms of the problem" (Cortes, Vergnaud and Kavafian 1990, p. 28). This is a rather limited definition of the more general concept of a pronumeral which can have other meanings in other contexts. An example is the role of a variable in representing generalisations. However, Cortes, Vergnaud and Kavafian (1990) pointed to the misuse of letters by children when they are used to represent objects or units rather than a number or magnitude.

3) *The meaning of the equal sign.* Once again this conception is similar to that of Kieran's description reported above. There is a transition of meaning from the equal sign "announcing" a result in an arithmetic environment to expressing equivalence and identity relationships in an algebraic environment.

4) *The homogeneity of the equation.* In an arithmetic environment the performing of operations on terms that have the same meaning (units) can be controlled by recording the units in the notation of the algorithm. However, in an algebraic environment this would usually be an inappropriate strategy. Cortes et al. (1990) recommended that the issue be controlled initially by explicitly recording and converting the units to enable operations to be performed. They further argued that pupils should be confronted with the need to control units but have not suggested methods for how this issue might be satisfactorily achieved.

5) *Numbers.* The ability to successfully operate with fractions and directed numbers at the time of introducing algebraic concepts is questioned. Lee and Wheeler (1989) took this issue further and provided evidence that students' arithmetic appeared to be "disturbed" by their algebra. Thus, it is argued that arithmetic skills that are generated by an algebraic environment could be a hindrance to development.

6) *Algebraic calculation - the 'detour' behaviour.* Once the 'object' (in this context the equation) has been identified it needs a series of transformations performed on it in order to produce a problem solution. This conceptual difficulty seems to have three parts. The first is similar to the acceptance of a lack of closure described by Collis (1975). Cortes et al. (1990) implied pupils need to avoid immediate calculations in order to search for an answer. Secondly, students need to accept intermediate equations as valid expressions. This requires a separation from the context of the question. Finally, students need to accept operations on symbols "that may not have an arithmetic meaning" (p. 29). This process is described by Lins (1992) as thinking internally and is described below in the context of a theoretical framework for characterising algebraic thinking.

Characteristics of Algebraic Thinking

Lins' (1992) framework shared several features with the procedural/structural model of Kieran. His model described algebraic thinking as thinking arithmetically, thinking internally and thinking analytically. The arithmetic dimension, he argued, is necessary to characterise the processes and operations involved that are "a fundamental model of our understanding of algebraic operations" (p. 56).

Thinking internally makes explicit the elements that are operated upon:

> By internalism we mean operating exclusively within a Numerical Semantical Field, as opposed, for example, to associating numbers - as measures - to line segments, and from this producing meaning to the manipulation of arithmetico-algebraic relations. The internalism is necessary to allow us to distinguish between algebraic thinking and other models that can be used to produce algebra (eg. geometrical models, ... whole part models, ... or contextualised models).
>
> (p. 56)

This view seems to be consistent with the argument that school algebra arises out of arithmetic experiences. Lins is emphasising the point that the processes and skills need to be applied beyond particular contexts and concrete environments and include the abstract world of numbers and their operations.

Lins' third dimension of algebraic thinking, referred to as thinking analytically, is seen as dealing with relationships involving numbers, arithmetical operations, and equality. These relationships are similar in nature to Kieran's 'objects' of structural algebra. However, Lins did not make explicit the reversibility of the equality sign as a different level of thinking than the use of equality in the arithmetic world. What is of use in his framework is the distinction made between algebra and algebraic thinking:

> A crucial distinction in our framework, is that between algebra and algebraic thinking; in relation to our framework, the production of an algebraic result does not necessarily involve algebraic thinking, nor does the use of algebraic symbolism. It is shown , within our framework, that the development and use of algebraic (literal) symbolism is a possible consequence of, and not an a priori condition for, algebraic thinking. Algebraic symbolism is made possible and adequate by algebraic thinking.
>
> (p. 56)

This distinction has not been recognised by many curriculum writers nor text book writers in the past and has only become the focus of attention relatively recently. These developments are discussed later in this chapter.

This section has presented an argument for perceiving algebraic thinking as growing out of arithmetic thinking and has identified some discontinuities and obstacles to that process. These obstacles are conceptualisations that support the procedural operations of arithmetic that need to adjust to facilitate algebraic thinking. This process of adjustment is not supported by some of the traditional methods of teaching algebra. The next section presents a model for facilitating the transition from arithmetic thinking to algebraic thinking.

FACILITATING THE TRANSITION FROM ARITHMETIC TO ALGEBRA

This section is presented in three parts. The first part develops the case against the Decontextualising of Algebraic Instruction and offers a developmental model explaining the role of a context in concept formation. The second part introduces Expressing Generality as a context for building connections between arithmetic thinking and algebraic thinking while part three reviews the recent curriculum documents that support the role of expressing generality in early algebra instruction.

The Decontextualising of Algebraic Instruction

Kieran (1992) expressed the view that while text book writers often include an introductory chapter that links algebra to arithmetic, this treatment is brief and the emphasis on structural treatments quickly dominates. This focus on the manipulation of algebraic objects leads to what Skemp (1978) would call an instrumental understanding of algebra. Instrumental learning is characterised by learning a fixed set of plans for performing specifically defined tasks. Skemp believed that this type of teaching and learning is often viewed as attractive for a number of reasons. It is easier to understand, the rewards are more immediate and apparent, and hence there appears to be increased efficiency in getting the correct answer more quickly.

Skemp contrasted instrumental learning with relational learning. The key feature of relational learning is the ability to gain an overview of the domain to enable a more flexible approach to problems:

> ... learning relational mathematics consists of building up a conceptual structure (schema) from which its possessor can (in principle) produce an unlimited number of plans for getting from any starting point within his schema to any finishing point.
>
> (p. 14)

One implication of Skemp's dichotomy is that didactics can be a hindrance to the development of algebraic concepts. This view is supported by Kirshner (1992):

> Curriculum innovators of today are again faced with a world of practice in which blind symbol manipulation is the order of the day. Indeed, approaches like those of John Saxon (text book author) (with its single minded focus on repetitive and mixed practice) are particularly effective in minimising students' intellectual engagement.

(p. 1)

Traditional text books have taken the structural view as their focus and presented it in an instrumental way, although they have not explicitly stated this as the case. Thus the primary focus of teaching algebra in schools is to emphasise the structural perspective(Demana & Leitzel 1988; Pozzi 1992; Davis 1989b). That is, the initial algebra experience that children have is associated with the manipulation of symbols. Such a methodology seems to assume an understanding of the meaning of a pronumeral, and ignores the development of a relational understanding (Skemp 1978) and a rationale for the procedures being practised.

Lee and Wheeler (1989) pointed to the lack of connection that children are able to make between arithmetic and algebraic environments following two years of algebraic instruction:

> The pedagogy of algebra (in so far as it exists) seems to have nothing to offer to help students grasp the arithmetic/algebra connection that underlies all these differences between two modes of symbolic behaviour.
> (p. 52)

While Lee and Wheeler did not clearly define what they meant by "the pedagogy of algebra" the reader is left with the view that a traditional emphasis on rules and manipulation is implied. Some of the items that Lee and Wheeler used in their investigations, such as

> The sum of two consecutive numbers is always an odd number. The product of two consecutive numbers is always an even number. Are these two statements true? If they are can you show why?
> (p. 47)

require the student to move from the arithmetic environment of the question to an algebraic environment for the proof. Responses to this question ranged from correct algebraic proof (10%) to the avoidance of all algebraic notation even though the item was presented in the context of an algebra test. To avoid the use of algebraic notation the students used one or two numerical examples to justify their argument, that is, they argued from the particular to the general. In other items, which asked students about the truth or otherwise of an identity, evidence of the separation of the arithmetic world and the algebraic world was reported:

> We see here another illustration of how algebra and arithmetic are two dissociated worlds for these students. They do not spontaneously think of going from algebra into arithmetic and when they are pushed to do so their algebra is not instructed by their arithmetic as one would suppose it ought to be if they perceive algebra as generalised arithmetic.
> (Lee and Wheeler 1989, p. 44)

It seems clear that traditional teaching sequences do not make explicit the connections between arithmetic and algebra, and as a consequence very few students perceive algebra as growing out of arithmetic. The question remains as to how best to begin algebra instruction that might avoid the pitfalls that are so apparent from the focus on the manipulation of symbols. The Mathematical Association, as long ago as 1933, suggested that:

> Historically, algebra grew out of arithmetic, and it ought to grow afresh for each individual
>
> (Mathematical Association 1957, p. 5)

The historical development of algebra reflects three methodologies, namely, rhetorical, syncopated and symbolic. These methodologies might provide some indication of a suitable approach (Kieran 1991; Lins 1990). Human experience with algebra was initially based in a natural language approach, and algebra represented a way of thinking, or a class of problems that were solvable using thought processes and natural language. Parallel with the shift from rhetorical and syncopated algebra to Viete's symbolic algebra, was a shift in the focus or context of algebra. Lins (1990) presented evidence that Arabic mathematicians focused on a series of problems and solved them rhetorically, often relying on geometric strategies. Syncopated algebra involved a shift to operating on the unknown, using all the elements of arithmetic in the same way in which the arithmetic operated on the known. However, both rhetorical and syncopated algebra involved solving certain problems by "means of verbal prescriptions that involved a mixture of natural language and special characters" (Kieran 1991, p. 245). The invention of Viete's symbolism facilitated the process of abstraction in the nineteenth and twentieth centuries in which the focus shifted from a "collection of procedures for solving classes of problems" to a "method that allows us to attack problems in any of those classes" (Lins 1990, p. 95).

These developments represent a decontextualising of algebra, from a method of solving problems to the development of an abstract system. This latter emphasis is reflected in school algebra as the structural/manipulation focus of attention. This issue has been addressed by the considerable discussion in the literature about the need for the initial development of algebra to emerge from a context familiar to the school children who are the focus of the instruction.

This theme was developed further by Thorpe (1989) when he argued that "algebra should not be taught as a collection of tricks, as is common in text books" (p. 12). He believed that children should be presented with new ideas in context and be given the opportunity for reasoning with the elements in order to come to relevant conclusions with confidence and responsibility. Drill and practice are required to reinforce and automatise skills but should only be required of students when they have a clear understanding of why a particular skill is important. Thorpe called this view of curriculum the "Thinking Curriculum ... the correctness of which is self evident" (Thorpe 1989, p. 12).

Context was also an issue for Filloy and Rojano (1989). They believed that the transfer from arithmetic to algebraic thought does not occur spontaneously and that suitable teaching interventions are needed to aid the transition. Instead of starting with the syntactic rules, Filloy and Rojano argued for "modelling in some concrete context" (p. 20). Using this strategy the first elements of algebraic syntax are constructed on the basis of the behaviour of the model. However, while they have observed that "abstractions from the operations" have some standard characteristics in their development, there are also differences between students and differences with the model used. It seems important for teachers to be aware of the range of likely responses from students to their attempts to provide a contextual environment for the development of abstract ideas of algebra.

Linchevski and Sfard (1991) provided a model (see Figure 1.1) for this development of mathematics and the school experience of children by arguing that computational procedures become object-like entities. They use their operational/structural duality mentioned earlier to draw the distinction between these phases of learning. Thus, the computational procedures represent an operational understanding with the object-like entities representing the structural understanding. Once the object-like entities become familiar they in turn become subject to higher level processes. Thus mathematics may be viewed as a "hierarchy, in which what is conceived operationally at one level, should be conceived structurally on a higher level" (p. 318).

Sfard (1991) clarified this model by describing it as a three phase model of conceptual development.

1. *Interiorisation:* operating on familiar mathematical objects.
2. *Condensation*: the operations or processes are squeezed into manageable units.
3. *Reification:* involves seeing something familiar in a new light. A process becomes an object.

Linchevski and Sfard (1991) argued that the process of reification may become too difficult and the child forms a pseudostructural conception in which he or she resorts to symbols as entities which do not represent tangible entities:

> Lacking operational underpinnings, this kind of conception would leave the new knowledge detached from the previously developed system of concepts, and the secondary processes would seem totally arbitrary. The student may still be able to perform these processes, but his understanding will remain instrumental.
>
> (p. 319)

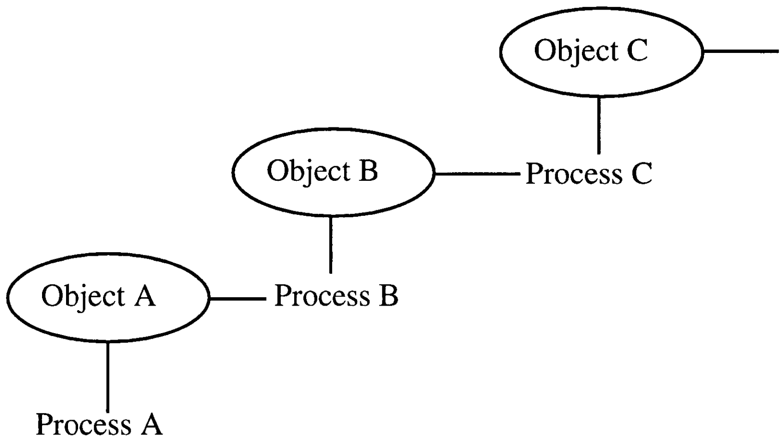


Figure 1.1
Linchevski and Sfard's Model of Development of Mathematical Concepts

This break in the development chain is represented in Figure 1.2 in which one link is missing.

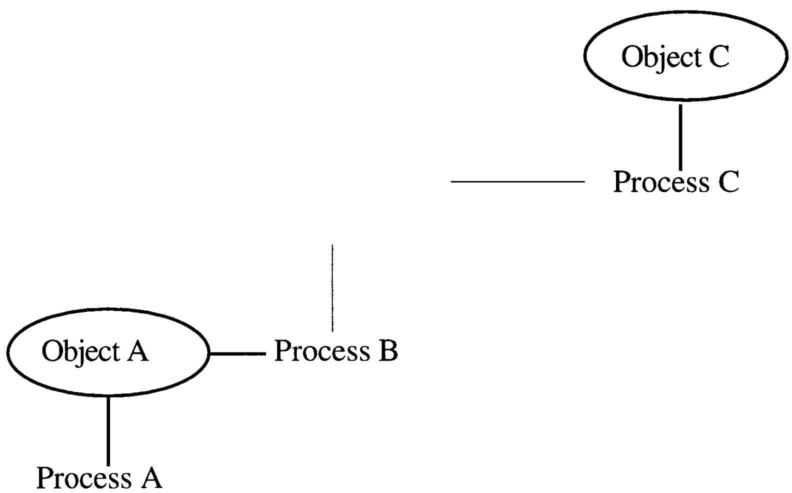


Figure 1.2
Linchevski and Sfard's Model of a Broken Development Chain of Mathematical Concepts

In the context of this discussion of school algebra, it is not the child's inability to reify a process that causes the break in the hierarchical chain of development but the instructional focus of manipulation of symbols that have not grown into an object-like entity via a processing of familiar objects. An approach for overcoming this break in the hierarchical chain of development is the use of expressing generality in the context of number patterns.

Expressing Generality: A Context

If the traditional approach to introductory algebra has resulted in a pseudostructural conception of algebra the question needs to be addressed as to what is a suitable alternative introductory experience. Sawyer (1964) recommended the use of patterns and concrete materials as devices to assist students develop their understanding of algebra. He believed such devices should satisfy three criteria:

> • First, any child should be able to use it with success, so that it builds confidence.
> • Second, the device should call out the child's own powers of discovery and reasoning.
> • Third, the result must be in some way very intriguing, fascinating, remarkable.
>
> (p. 65)

More recently, Mason, Pimm, Graham and Gowar (1985) have developed and popularised these ideas. In their important work, *Routes to, [and] Roots of Algebra*, they presented four views of algebra.

Root 1: Expressing generality. "In order to learn the language of algebra, it is necessary to have something you want to say" (p. 8). A number of settings are provided to facilitate "having something to say". These include "shape sequences that are offered as starting points" (p. 69). Two examples of shape sequences are presented in Figure 1.3. Students were asked to describe the number of elements needed to construct a member of the sequence, thus describing a relationship between a dependent and independent variable.

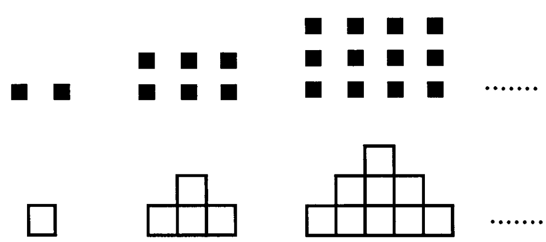


Figure 1.3
Shape Items Suggested by Mason (1985)

Root 2: Rearranging and manipulating. Mason et al. differed from the more traditional assumption of accepting the existence of algebraic objects in introductory algebraic instruction. They implied that the need and appropriateness of rearranging and manipulation should arise "as a result of encountering different expression for the same thing" (p. 29). Further, they believed that students become aware that an expression is an entity in its own right. This development is similar to the model of Linchevski and Sfard described previously.

Root 3: Possibilities and Constraints. This root involves "fostering an awareness of the generality which lies behind symbols" (p.38). Mason et al. argued that there is a subtlety underpinning the meaning of pronumerals that needs to be developed. While arguing that possibilities and constraints are a root of algebra they recognised that:

> Considerable experience of the simple pattern problems suggested in Root 1A: Expressing Generality would be of benefit before tackling the rather more subtle matter of possibilities.
>
> (p. 43)

Root 4: Generalised Arithmetic. This final root involves making explicit the rules of arithmetic. However, as the rules of arithmetic are themselves abstract, this root is an abstraction of an abstract idea. Hence, Mason et al. argued this root can prove difficult. Mason et al.'s separation of the expression of generality and generalised arithmetic provides a clarification of where a beginning point might be for the path to algebra.

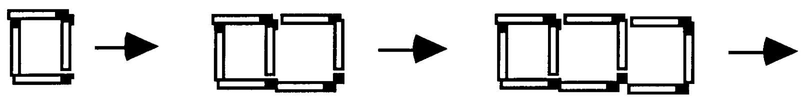
Mason et al. did not present the four roots as a model for a sequential development of algebraic concepts, but a clear sense emerges from their book that the root of expressing generality would provide a suitable starting point. The strategy of expressing generality has considerable appeal as an introduction to algebra. It is a method embedded in the familiar arithmetic context of numbers and their operations and requires the expression of general statements that are themselves expressible in algebraic symbolism. Thus the implementation of expressing generality is achieved via a three-phase model involving seeing, saying and recording a number pattern.

Mason et al. (1985) discussed the need to delay the introduction of algebraic symbolism. This delay is achieved by spending considerable time on "seeing a pattern" (p. 8) and "saying a pattern" (p. 10). What Mason et al. makes explicit is for children to express generality in the natural language prior to developing the symbolic notation of algebra. These ideas were similar to those published in the *New South Wales Mathematics Syllabus Years 7 and 8* (Board of Secondary Education 1988)

In February 1986 the N. S. W. Department of Education authorised a draft syllabus document which required that algebra be introduced via generalisations made of geometric patterns and number patterns. Pre-algebraic instruction included:

- Identifying patterns in geometric and numerical environments.
- Extending a number pattern.
- Tabulating data.
- Describing geometric and number patterns in natural language.
- Encouraging different ways of describing number patterns

The geometric environments were similar to those described above, while the numerical environments included tabulated data that also required the relationship between an independent and dependent variable to be described. Figure 1.4 includes examples of each type taken from the syllabus:



Number of squares	1	2	3	4	
Number of matches	4	7	10	13	

Figure 1.4
Pattern Items from N.S.W. Syllabus

The emphasis in these activities was to develop the language of expressing generality. Indeed, in the final version of the syllabus published in 1989 there were eight objectives specifically relating to using natural language for describing patterns. These natural language objectives followed four objectives associated with building and extending geometric patterns and number patterns. As a third phase of the syllabus the language of algebra emerged as a more succinct way of recording the natural language sentences. This development mirrored closely the seeing, saying and recording trichotomy described by Mason et al. Neither Mason et al. nor the syllabus described how natural language would develop nor the relationship between natural language and symbolic language. However, the ideas expressed did imply a hierarchical development that could be considered in the context of Linchevski and Sfard's model.

Figure 1.5 maps the curriculum ideas onto the model of Linchevski and Sfard. In the model, numbers and operations of the patterns are the initial objects. These are condensed into descriptions of patterns using natural language as the vehicle. These descriptions become the new objects. Once familiarity with these natural language expressions is gained the focus shifts to the use of symbolic language. This results in algebraic expressions becoming the new objects. Of great interest is what happens in the change from one object to the next. What sort of experiences best induce that change, and how difficult is the change? Must the change be sequential or can symbolic notation develop in parallel to the natural language development?

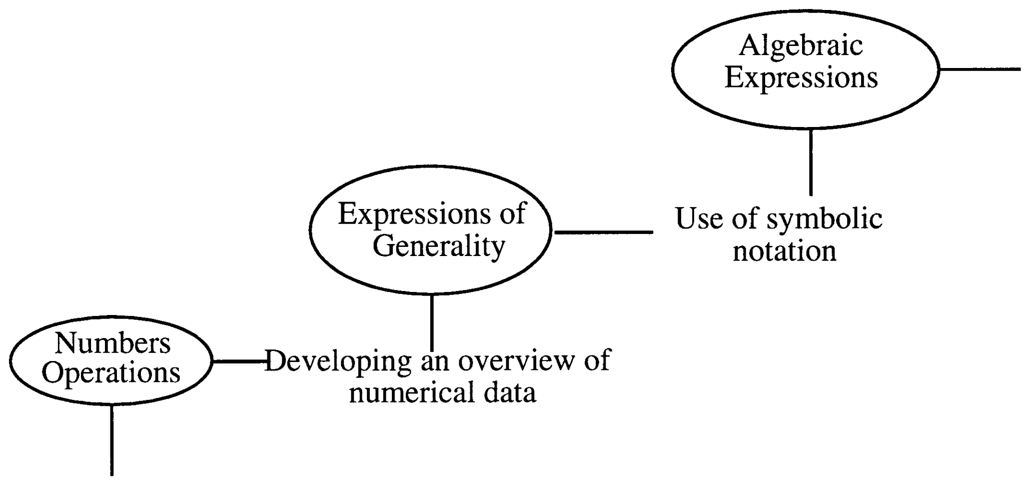


Figure 1.5

Possible Model for Development of Early Algebraic Concepts

Pegg and Redden (1990a) reported on some early work in implementing the ideas of Mason et al. and the N. S. W. syllabus in a classroom environment. They described a three phase model of concrete experiences, language development and algebra emerging. These three phases paralleled those of Mason et al. and Pegg and Redden (1990a) provided considerable detail with regard to their teaching activities, and, while expressing support for the approach, did not provide a detailed account of student progression from their concrete experiences to algebraic expressions. However, they did describe students initial experiments with developing a succinct notation.

> Familiar symbols such as +, -, x and ÷ quickly replaced the words plus, minus, times, and divided by their synonyms. Contracted forms of the expressions "number of matches" and "number of triangles" followed naturally, and abbreviated forms, such as "matches," "mat," and "m," were readily offered by students. Sometimes students used ideograms to abbreviate written sentences.
>
> (p. 389)

There was no attempt to describe the nature of language development although some influences were identified. The need for considerable exposure to number pattern experiences was considered to be important, together with a variety of language activities to make explicit the need for accurate descriptions. These descriptions facilitated applications of the number patterns to activities such as calculating values that are considered to be uncountable. Pegg and Redden (1990a) referred to the value of the 100th term in a pattern sequence as an example of an uncountable example. In another article, Pegg and Redden (1990b) emphasised the importance of language by referring to it as the "missing link" (p. 19) in the development of algebraic skills. They argued that many attempts to use pattern approaches to teach algebra are textbook based and move from numerical experiences with patterns to generalising the pattern in algebraic notation, omitting the component of developing pattern descriptions in natural language. They present some data to suggest that it cannot be automatically assumed that children can describe patterns in terms suitable for expressing in symbolic notation.

The need for extensive experience in pattern work, implied in the work of Mason et al. (1985) and Pegg and Redden (1990a), has since been reflected in a number of influential documents that encourage a focus on patterns and the associated language back into the early school experiences of children. Two documents are reviewed as representative of this encouragement and their content is compared with the relevant sections of the N. S. W. syllabuses

Review of Recent Curriculum Documents

The *Curriculum and Evaluation Standards for School Mathematics* (Romberg 1989) urges teachers to include the study of patterns in the school curriculum from the beginning of formal schooling. The Curriculum Standards for K-4 include:

> • recognise, describe, extend, and create a wide variety of patterns;
> • represent and describe mathematical relationships;
> • explore the use of variables and open sentences to express relationships.
> (p. 60)

They have a broader perspective than the development of algebra. It is claimed that patterns help children see how "mathematics applies to the world in which they live" (p. 60), assist in developing the ability to classify and organise information, and enable children to make connections between mathematical topics. Finally, and importantly in the context of this study,

> The idea of functional relationships can be intuitively developed through observations of regularity and work with generalised patterns.
> (Romberg 1989, p. 60)

The suggested contexts for these developments were both numerical, e.g.,

121 12321 1234321

and geometrical, e.g.,

□ △ ○ □ △ ○ □ △ ○.

The Curriculum Standards for 5-8 (Standard 8) represent a development of the K-4 concepts by encouraging multiple representations of relationships in the form of "tables, graphs and rules" (p. 98) and the analysis of functional relationships "to explain how a change in one quantity results in a change in another" (p. 98). The focus is the informal treatment of functions "relatively unburdened by symbolism" (p. 98). The focus includes verbal descriptions of the relationship between variables.

While in Curriculum Standard 8: Patterns and Functions, symbolism is avoided, it is explicitly presented in Standard 9: Algebra. However, here the use of symbolism is restricted to the description of properties that children discover in a number of settings including pattern generalisations. The development of algebra as an abstract system is delayed until the Curriculum Standards for Grades 9-12.

This brief review of the Curriculum Standard's use of expressing generality/patterns indicates a developmental sequence implied in the experiences suggested for children at each grade level cluster. What is not discussed is how children's responses change as these experiences change. That is, does children's cognitive growth mirror these shifts in focus? Implied in the Curriculum Standards is a significant shift in children's thinking taking place during the Grades 5 to 8. This shift involves moving from merely extending patterns to an informal treatment of functional relationships between variables. As discussed previously, the *N. S. W. Mathematics Years 7 and 8* syllabus (Board of Secondary Studies 1986) required a similar development.

Prior to 1991 the syllabus for years K-6 (New South Wales Department of Education 1967) had no references to patterns or expressing generality. Since 1991, however, a new mathematics K-6 syllabus has been mandatory in N. S. W. This syllabus has several references to the generalisation of pattern, both in a numerical and geometrical environment. Of these pattern references, 13 refer to extending a geometric pattern and two involve developing number facts by investigating patterns. None could be classified as describing relationships between variables.

The second document to encourage expressing generality in early mathematical experiences of children is *A National Statement on Mathematics for Australian Schools*

(Australian Education Council 1990). This document is in general agreement with the views expressed in the curriculum standards and the N. S. W. syllabus for Years 7 and 8, in that it is suggested children be encouraged to recognise and describe regularities and differences in numerical and geometrical (spatial) contexts from the early primary years (Band A) through to the secondary school years (Bands C and D). Within the algebra strand the national statement expresses the view that experiences in developing verbal expressions to describe and summarise spatial or numerical patterns begin in primary school. These experiences should be reported using multiple representations including "verbally, graphically, in writing and physically" (Band AB3 (p. 221)). This is at odds with the N. S. W. syllabus for Years K-6, which does not include the informal treatment of relationships between variables.

The development of algebraic representations is delayed until Band C in the early secondary school years.

> C1: Express a generalisation verbally (orally and in writing) and with standard algebraic conventions.
> C2: Generate elements of a pattern from a verbal or algebraic expression of its rule.
>
> (Australian Education Council 1990, p. 221)

Item types suggested by the National Statement include geometric shapes similar to those described by Mason et al. (see Figure 1.3 above) and the N. S. W. syllabus for Years 7 and 8 (see Figure 1.4 above). Additional item types include:

> From a set of input and output numbers for a 'function machine', write an algebraic expression for the rule that produced the output numbers.
>
> (p. 197)

These suggestions for inclusion in the curriculum are similar to the ideas of Mason et al. (1985) and the two documents reported previously. There appears to be a general agreement on the early introduction of pattern experiences with the activities focusing on recognition, extension and representing patterns prior to describing relationships. This focus on language is a common thread through all these discussions. While language is explicitly mentioned in all these discussions, only the National Statement specifically mentioned the potential for language difficulty implied in the work of Pegg and Redden (1990a, 1990b). In the overview of the algebra strand the National Statement reports:

> Students must first be able to 'see' a pattern. Having identified it, they should continue to develop ways of describing the pattern mathematically. Many students find the verbal expression of generality difficult. The expression of generality often requires quite complex sentence structure and logical connectives but, with help, students will develop the necessary language skills.
>
> (Australian Education Council 1990, p. 194)

This section has criticised standard school textbook approaches to introducing algebra and suggested that the historical development of algebra may provide some insight into alternative developmental sequences. The model of Linchevski and Sfard (1991) was discussed in the context of the transition from arithmetic thinking to algebraic thinking. In particular, the use of expressing generality as a vehicle for the initial stages of this transition was discussed. From this general review the divergent treatment in the various curriculum documents of the role of children's natural language and its relationship to symbolic language was identified.

In the above discussion, while wide support for the use of expressing generality has been identified, very little of this support provides research evidence to justify the claimed benefits of the approach. In the next section the available research evidence is reviewed.

REVIEW OF RESEARCH ON EXPRESSING GENERALITY

Prior to presenting the research findings available in the literature on expressing generality in the second part of this section, a definition of expressing generality is developed in order to provide a focus for the discussion. The third part of this section identifies a number of issues that arise from the review of empirical evidence.

Defining Expressing Generality

This review of research literature concentrates on the concept of expressing generality in the sense implied in the major documents of the previous section, as these represent a suggested vehicle for the transition from arithmetic thinking to algebraic thinking. In all of these discussions it has been suggested that the pattern be of a numerical or geometric (shape) type (see Figure 1.4) for the purpose of expressing generality. A pattern can be seen as a recognisable set of properties that remain invariant over a set of elements. Generalising is used in the sense of identifying and describing the set of properties that are invariant across the set of elements. Definitions such as these can be seen to be general in the extreme and could be applied to children's ability to identify and categorise elements in the simplest of contexts (Davydov 1990). To restrict the discussion to the area of interest in this study it is necessary to restrict the set of elements to be considered. This involves restricting the elements to those describable by counting numbers and the operations that can be performed on those numbers. The rationale behind this restriction is that these pattern types have the potential to be described by children and to be presented in elementary algebraic notation, thus avoiding the need for recurrence relations. The restriction to positive integers is to facilitate searches for relationships using only mental strategies. Also excluded from the discussion are those studies that focus more on generalised arithmetic (Gnepp 1987) rather that the expression of generality. An example of items used by Gnepp is:

> A notebook costs *m* cents and a pencil costs *t* cents. How much would be paid for *d* notebooks and *e* pencils?

Word problems such as this do not fit within the definition of expression of generality presented above since the data is presented to the subject and does not have to be identified from a pattern context.

With the concept of expressing generality appropriately defined the empirical data on children's understanding of the concept can be reviewed.

Research Findings

There have been a number of studies conducted that have used either number patterns or geometrical patterns as stimulus items in an attempt to investigate children's ability to identify, describe or use the invariant features of the pattern. This part reviews the most recent of these investigations prior to identifying a number of issues arising from the research

Using tables of ordered pairs, Ericksen (1988) investigated students' ability to recognise and generalise a pattern. The former was measured by asking children to provide the value of the dependent variable for an independent variable value of 10. Students were then asked to describe how they arrived at their answer prior to being asked to provide an expression for an independent variable of *b* which represented the measure of generalising a pattern. The set of responses from a group of 13 Year 7 children was compared with the responses from 13 Year 9 algebra students. Ericksen found that the Year 9 children were superior at the task to the Year 7 children and that both groups performed better on the less complex task involving one arithmetic operation (e.g., $y=3x$) than on the two operation tasks (e.g., $y=5x-4$). Of some interest was the use of a prompt when students were unable to recognise the pattern. The interviewer sometimes told of the arithmetic operations involved. The students, thus prompted, then appeared to be able to provide algebraic generalisations. Although Ericksen did not make this explicit, it would seem that accurate natural language descriptions of patterns assist in the translation of the pattern to other representational systems. The focus of the study was on finding symbolic representations for data tables, and students were not asked for natural language representations, as suggested by Mason et al. (1985) and Pegg and Redden (1990a). Ericksen does not report a variety of methods used by children in attempting to describe their strategies. The small sample investigated could be a contributing factor in this lack of a detailed analysis.

In an attempt to identify strategies students use in undertaking such tasks, Stacey (1989) used pattern items that could be modelled using linear relationships. The questions included asking for function values of patterns that were at first countable, such as the fourth term in the sequence (f(4)) and then an uncountable value (f(1000)). Initially, Stacey identified a number of strategies by analysing a set of responses from primary school students in Years 4, 5 and 6. The strategies included counting; multiplying by a constant; the whole-object method, which involves identifying the number needed for a countable set and then assuming proportionality; and finally, a linear method ($M(n)=an+b$).

Stacey concluded that:

> The overall impression from the responses of the primary school children was that the children were not reluctant to generalise, but rather that they constructed the generalisation too readily with an eye to simplicity rather than accuracy.
>
> (p. 153)

While she drew conclusions about hypothesised processes that the children used, it is important to note that she did not ask the children to express generality in the sense implied by Mason et al. (1985). If seeing pattern interpretations as an object is to be a vehicle for moving from arithmetic thinking to algebraic thinking then there is a need to consider more than the arithmetic processes. After similar items were presented to secondary school students in Years 7 and 8 they were asked to "explain how you found your answer" (p.157). The majority of responses to this request were in the form of specific arithmetic statements, such as "79x5=395" (p. 157).

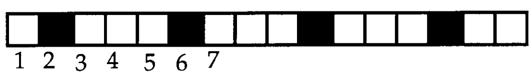
In the same paper Stacey (1989) reported a pre-test/post-test experimental design in which students were provided with some problem-solving instruction that included strategies such as "look for a pattern", "explore simple cases", etc. The pattern items were similar to those referred to above. She reported a considerable shift to using a linear function as the method of solution, but only one child provided a generalised description, namely (sizex3+size-1), when asked for an explanation of methodology used. In spite of these shifts in response categories Stacey stated:

> The qualitative similarity in responses of the primary and secondary students ....
>
> (p. 160)

Hence the reader is left unclear as to the nature of change in children's ability to perform the tasks and under what conditions this change might take place. Of interest is the assertion that children, from as early as Year 4, can interact with the pattern items providing some empirical support for the Curriculum Standards (Romberg 1989) document and the Australian National Statement (Australian Education Council 1990). That is, expressing generality ought to be undertaken in late primary and early secondary school.

O'Brien (1991) reported on the responses of 30 Year 5 children to pattern items in which students were asked to predict the colour of a block placed in a regular sequence such as:

or



Students were asked the colour of box 21, box 40 and box 1047 together with how they did the question. A number of strategies used were similar to those of the students in Stacey's (1989) report. A large number of "Don't Know" responses were recorded, with the proportion increasing as the number of the required term became larger. There were also physical extensions of the pattern together with counting strategies. All the relationships reported by O'Brien involved the specific application of a rule such as "1047 is black, because it's not divisible by 4" (p. 119).

The large number of students who responded in the "Don't Know" category caused O'Brien to express surprise. A similar reaction was caused by the large number of responses in the categories using physical and mental extensions of the pattern. O'Brien identified the need to explore the nature of growth towards what he called "relational thinking" which, although not explicitly defined, appears to have a similar meaning to the expression of generality identified earlier as an important issue in this study. Additionally he suggested that the poor performance of the children on the items which were the focus of the study may be a result of:

a) some developmental limitation,

b) an atypical sample,

c) a mathematics curriculum in which relationships receive little emphasis.

Ursini (1990) was able to categorise children's pattern extensions and attempts at symbolic description of the generalisation but found instability in children's responses. Thus she argued that the categories were "a classification of responses not of children" (p. 155). The item involved presenting children with a sequence of geometric shapes with a sequence of numbers in correspondence. Children were asked to record a rule in symbolic notation and to provide extensions to the number sequence. The response categories Ursini reported were:

> Could not answer any item.
> Could generalise only by drawing.
> Could generalise only by drawing and by numbers.
> Could symbolise algebraically the simplest item.
>
> (p. 152)

This last category seems to refer to Ursini's finding that children had more success with items that involved one arithmetic operation than those with two such operations. While Ursini reported an instability in students' responses, she did not attempt to explain the variability of those responses. The variability due to item complexity, as defined by the number of arithmetic operations, is but one variable. Could it be that the context of the pattern could be a significant influence? Could it be that the children are in a transition phase of development and that the identified instability is associated with an insecurity in the child's perception of the task? That is, they are not seeing the outcome as an 'object' but are focusing on arithmetic processes as distinct from expressing generality as an outcome.

In a follow-up study to the one described above, Ursini (1991) used a LOGO environment to investigate "if and how children" (p. 318) identify patterns and how different representations interact. She related these issues to Mason et al.'s "seeing and saying" and argued that LOGO provided an environment for "saying" other than symbolic algebra, which she defined as "recording" the pattern. It could be argued that the LOGO language is a symbolic system quite different from the natural language referred to by Mason et al. in the context of "saying" the pattern. As part of the investigation children were asked to write a LOGO routine to draw a set of stairs. The children could not record a compact LOGO routine immediately but were able to extend the drawing in direct mode and then recognised a more compact method of proceeding using the REPEAT statement of the LOGO language. The LOGO environment provided the children with a supporting "scaffold" (p. 320) to link their representational systems.

Unlike her 1990 investigation Ursini (1991) reported remarkably stable response patterns both within and between the subjects of this study.

> For almost all of them (subjects) the perceptions of patterns evolved in the same way.
>
> (p. 319)

This may imply that the LOGO environment was presenting a quantitatively easier representational system than the sets of ordered pairs of her previous study. Alternatively, the difference may lie in the nature of the description required. In the earlier work a functional relationship was required which could be classified as an object. It could be argued that in the LOGO environment described by Ursini (1991) the task involved repeating a process such as:

FD 20 RT 90 FD 20 RT 90........

This could be seen as a parallel to some strategies for extending patterns such as repeatedly adding, rather than describing a relationship between a dependent variable and an independent variable.

A different environment again was chosen by Meira (1990) to conduct a micro analysis of how one child perceived a physical device and how he represented those perceptions. The study presented the subject with a mechanical system. An object could be moved to a new position (dependent variable) by turning the handle (independent variable). The relationship was of the type $y=4x+8$. The five hours of interview resulted in a number of different responses that indicated the possibility of a developmental sequence. The various responses could be included in categories reported above and included:

- a successive or iterative response such as "add *3*" (p. 103)
- focusing on one arithmetic operation only such as "$4n=n$" (p. 104)
- an implied function such as "$8+(4n)$" (p. 105).

This investigation, however, revealed more than the replication of categories. It implied a link between the verbal accounts of the model's mechanism and the nature of the symbolism used. Although generalisations cannot be made from a sample of one, there is some evidence in Meira's study to support the view expressed above that there exists a relationship between the natural language descriptions of patterns that children invent and their attempts at symbolic descriptions. This issue is the first of several identified below as issues arising from this review of empirical data.

Issues Arising from the Research Findings

This final part of this section provides an overview of the research findings by identifying five issues arising from the above discussion. The issues are the movement between representational systems, the lack of a systematic categorisation of children's pattern descriptions, factors influencing variability in children's pattern descriptions, how children's pattern descriptions change over time, and, finally, the need for a theoretical framework to facilitate the analysis of the structure of children's responses. These issues are each discussed in turn.

The difficulty students have in being able to translate between representational systems has been immortalised in the 'students and professors problem'. In particular the translation between natural language and algebraic notation has been problematic (Cockcroft 1982; MacGregor 1990; Reeves 1990). In the general discussion about movement between the representational systems, the natural language has been given to the students in the form of a word problem (Clement 1982; Kaput & Sims-Knight 1983; MacGregor 1990).

In the settings of expressing generality, children are being asked to develop their own natural language as a result of studying the pattern data. It might by hypothesised that children's ability to move from one representational system to another would be more successful if they were responsible for generating relationships in both representational systems. That is, if children are asked to describe a pattern in natural language and symbolic language there may be a relationship between the two representational modes. MacGregor (1990) reported that tertiary students were better able to express a word problem in algebraic symbols when first asked to represent the relationship implied by the question in "ordinary English".

The issue of a possible relationship between natural language and symbolic language is one of several that have been identified in this review. It is clear that there is variability in children's perceptions of number patterns. While several descriptions of categories of responses are provided, with considerable agreement between the categorisations, there does not seem to be any attempt to categorise children's natural language descriptions of number patterns. As a consequence there has been no systematic attempt to investigate the relationship between natural language descriptions and the use of symbolic language.

There are some reports of factors influencing children's ability to work with number patterns as defined at the beginning of this section. These include the arithmetic complexity of the relationships and the context within which the pattern was presented. In addition to these there are two other possible influences on children's performance. One is the order in which the data are presented. The pattern extension response to evaluate larger values of the dependent variable might be hindered if the data were not presented in sequential order. Another issue is the amount of data that are presented in the stimulus item. Decisions had to be made as to which of these variables to control and which to vary and to what extent they should vary. These decisions are reported in the research design chapter.

It is clear from the above that there is change in an individual's responses over time. The descriptions of this change ranges from considerable variability (Ursini 1990) to some limited evidence of orderly growth (Ursini 1991; Meira 1990). This change has not been systematically described, nor has it been mapped over an extended period of time. It would seem useful to use a large sample of children's responses to investigate the existence of a hierarchy of development and attempt to validate the hierarchy in a longitudinal study.

This section has reviewed the research literature on children's ability to express generality. It is within this context that the concept of pattern and generalisation are defined. A number of issues have been identified that may affect children's ability to express generality, together with some open questions about how children's pattern representations change over time. Finally, the need for a theoretical framework to facilitate comparisons across items and contexts that assists in mapping changes in children's responses has been identified.

SUMMARY

The initial development of algebraic thinking has been characterised by a set of arithmetic operations being seen as an object. This involves a cognitive shift from a focus on procedural thinking to what Kieran (1992) called structural thinking. While a number of distinctions between arithmetic thinking and algebraic thinking are identified, these distinctions emphasise a dichotomous view of the development required in shifting from one mode of thinking to the other. Arithmetic thinking is seen as an alternative to algebraic thinking. Using the developmental model of Linchevski and Sfard (1991) the issue was raised as to how the transition from arithmetic thinking to algebraic thinking takes place. A vehicle for contributing to this transition that has received wide support is expressing generality, in particular, providing functional descriptions of number patterns. While the curriculum literature has placed considerable emphasis on the role of natural language in assisting the transition, the research literature has investigated this aspect only superficially. Instead, it has focused on symbolic representation of number patterns and the ability to extend the pattern to uncountable examples.

While the research literature has reported considerable variability in children's perception of patterns, no systematic attempt has been made to postulate a developmental sequence to describe the transition from arithmetic thinking to algebraic thinking over time.

Attempts to compare the research studies undertaken have been hampered by the lack of a theoretical framework to facilitate comparison. Such a framework would ideally provide a mechanism for categorising student descriptions and hence enable growth sequences to be identified. To address this and other issues a theoretical framework is now discussed prior to presenting the research questions to be investigated in this study.

Chapter 2

SOLO TAXONOMY:

A THEORETICAL FRAMEWORK

Introduction

Chapter 1 identifies the development of children's responses to expressing generality and the related question of the possibility of describing a hierarchy of growth in these responses as the focus of this study. In this chapter the issue of identifying a suitable theoretical framework is addressed

The first section of this chapter discusses some alternative perspectives on both the role and nature of the theoretical framework. It is argued that theory as an absolute truth is an unsuitable perspective for educational research and that a more fluid, dynamic model is needed than some of the theories that underpin the physical sciences. With this as a background, the development of the SOLO Taxonomy is presented with its antecedents within the Piagetian framework.

THEORY: SOME PERSPECTIVES

These perspectives on theoretical frameworks are presented in three parts. The first part presents some alternative definitions of theory, and the second part considers the role of theory in a number of research traditions. The final part identifies a role and some properties of a theoretical framework considered suitable for this study.

What is Theory

The place of theory in educational research has always been problematical in that few would argue that there has ever been a well developed and rigorous theoretical basis for how children learn and develop (Davis 1989b; Larkin 1989; Wheeler 1989). Much of this lack of confidence is due to a traditional scientific view of theory that is well represented by the definition of Kerlinger (1986):

> A theory is a set of interrelated constructs (concepts), definitions, and propositions that present a systematic view of phenomena by specifying relations among variables, with the purpose of explaining and predicting the phenomena.
>
> (p. 9)

LeCompte and Preissle (1993) referred to this view as "Grand Theory" (p. 135). Grand theory is comprehensive, non probabilistic and explains large complex phenomena. They cited the work of Newton, Einstein, Darwin and Mendel as grand theory. While such a definition imposes severe constraints on what constitutes a theory, others have defined theory in less restrictive terms.

Davis (1989b) argued that there is no "Fundamental Theory" (p. 271) to explain what people do when they think about mathematics. He suggested that development of such a theory should begin with a study of "specific instances" (p. 271) of mathematical behaviour. But in doing this it is necessary to have something less than the grand theory to guide researchers' deliberations. Wiersma (1991) defined theory as:

> a generalisation or series of generalisations by which we attempt to explain some phenomena in a systematic manner.
>
> (p.19)

Central to this definition is the synthesising function of theory, in which theory not only offers the ability to overview the data, but also to link the data being collected and analysed to a broader environment. Wiersma suggested theory is not judged by its truth or falsity, but by the extent to which it enhances the meaning of the data. This view was supported by Larkin (1989) who suggested that truth and reality are not the appropriate criteria for judging theory, as theory is a "construction" (p. 275) of the minds of scientists that is not rigid or prescriptive but rather eclectic and changing. A dynamic view of theory was taken by Mouly (1978) who also suggested that theory is a convenience that assists in understanding phenomena. In addition, he argued that theory is a fluid phenomenon that is constantly being tested and adapted as the result of further investigation. Glesne and Peshkin (1992), in looking at theory from the perspective of a qualitative researcher, supported this dynamic view when they argued that:

> ... theory [is] sometimes referred to as the latest version of what we call truth ...
>
> (p. 21)

This non-static nature of theory carries with it implications for the positioning of theory within the research process. The traditional scientific paradigm involves theory being used to generate hypotheses which are in turn tested by the researcher. Such a view requires that the theory is explicitly stated at the beginning of the research cycle. At the other extreme is the view of Glaser and Strauss (1967) embodied in their grounded theory. Grounded theory is seen as a product of data collection. Glaser and Strauss believed that theory should be embedded in empirical data and hence theory grows out of research rather than precedes it. These two perspectives place the identification of a theory at the opposite ends of the research process. Identifying the role required of theory in a given research environment may help in clarifying its meaning and position within the research cycle. This issue is addressed below.

Role of Theory

Bishop (1992) identified three research traditions of mathematics education, each of which requires a different role of theory. The three traditions are the pedagogic tradition, the empirical-scientific tradition, and the scholastic philosophic tradition.

The pedagogic tradition incorporates names such as Socrates through to Polya, Dienes and Freudenthal. Theory in this tradition exists as the product of reflective teachers who are seen as expert. Intuition and general acceptability validate the theory. Empirical evidence is selective and exemplary. As can be seen from Table 2.1, theory in this tradition is the accumulated and sharable wisdom of expert teachers. It is this type of evidence that has been used to underpin the arguments for using expressing generality as a vehicle to move children from arithmetic thinking to algebraic thinking.

The role of theory in the pedagogic tradition can be contrasted with Bishop's second tradition of empirical science. In this tradition evidence is the "key to knowledge" (p. 712). The research process collects and analyses data, in both a quantitative and qualitative sense, and uses theory to explain the data and then uses the data to test the theory. In this sense the empirical data and the theory interact to provide validity for the data and to test and, if necessary, change the theory. The process is dynamic and interactive.

Table 2.1

Theoretical Traditions (adapted from Bishop (1992, p. 713))

Theory	Goal of Enquires	Role of Evidence	Role of Theory
Pedagogue Tradition	Direct improvement of teaching	Providing selective and exemplary children's behaviour	Accumulated and sharable wisdom of expert teachers
Empirical Scientist Tradition	Explanation of educational reality	Objective data, offering facts to be explained	Explanatory, tested against the data
Scholastic Philosopher Tradition	Establishment of rigorously argued theoretical position	Assumed to be known. Otherwise remains to be developed	Idealised situation to which educational reality should aim

The third tradition described by Bishop is the scholastic philosopher tradition in which models are developed from rational theorising. The real world of children and their classroom environments are seen as imperfect manifestations of the idealised theoretical model. This tradition relies little on empirical evidence, but rather on critical argument and "reflective awareness" (p. 713). Under this tradition, change in education practice takes place by the

> imperfect reality adjusting to a closer approximation of the perfect theory.
>
> (p. 713)

Bishop cited the work of Servais, Steiner and Rosenbloom as characterising this tradition.

Further, he believed that modern research is influenced by all three traditions. Such a position recognises that the three traditions are not mutually exclusive and supports the view of theory as a fluid phenomenon.

Theory: A Fluid Phenomenon

The interaction of Bishop's three traditions points to theory being an evolving concept. Empirical evidence must have influenced the critical argument of scholastic philosophers and reflective teachers, just as the existence of a theoretical position generated by critical argument must influence the choice of theoretical frameworks used and tested by empiricists. This process supports the view of theory taken here. This view sees theory as a dynamic and ever-changing phenomenon whose major role is to provide a mirror on which to reflect the collected data and to assist in placing the context being investigated into the broader framework of mathematics education research.

In Chapter 1 a number of issues were identified that are the major focus of this study. These involved describing children's transition from arithmetic thinking to algebraic thinking using the expressing generality as a vehicle. Bishop would call this "what is" research (p. 714) which he locates in the empirical-scientific tradition, and hence a theoretical framework is required that assists in exploring and ordering the empirical data. In addition, a suitable theoretical framework would facilitate the categorisation of children's responses in a hierarchical model as the shift in cognition takes place. In the light of the above discussion, the desirable framework would not be required to provide a cause-effect explanation of behaviour, but rather a description of questions and responses that transcend the immediate context of the investigation, thus facilitating a comparison in a wider arena.

This section has discussed some alternative definitions of theoretical frameworks and presented a case that these frameworks have played a different role in differing research traditions. The position taken here is to identify an emerging theoretical framework and treat it interactively with the collected data. The following section identifies the Piagetian framework of cognitive growth as being a suitable starting point for discussion. The discussion continues by identifying some shortcoming in the Piagetian model and moves the focus of attention to the more recent work of neo-Piagetian writers.

THE PIAGETIAN FRAMEWORK

The Piagetian framework is presented in three parts. The first part identifies and defines three important features of Piaget's system of cognitive growth. The second part describes the stages of the model. Finally, the model is discussed with particular attention being paid to transition through the stages and the underlying assumptions of the model.

Structure, Function and Content

Piaget's theory of cognitive growth involves the integration of three distinct but related features. The three features are the structure, function and content of developing intelligence.

Structure

Cognitive structures are abstract patterns of organisation of mental operations and can be inferred from a person's behaviour. These structures change with age and it is the developmental aspects of structure that are the major object of Piaget's focus. This developmental model is discussed below as Piagetian Stages.

The patterns of organisation and operations of the cognitive structures result in the formation of schemes. A scheme is a:

> Cognitive structure which has reference to a class of similar action sequences, these sequences of necessity being strong, bounded totalities in which the constituent behavioural elements are tightly interrelated.
> (Flavell 1963, p. 53)

Schemes are defined in terms of overt behaviour (Piaget and Inhelder 1969, p. 4). Examples of sensori-motor behaviour such as grasping, reaching, seeing, tasting and so on are seen as schemes and are said to change as a consequence of cognitive functioning.

Function

While development is seen in terms of cognitive structure, changes in these structures occur as a result of experience and are said to be controlled by two functional invariants, organisation and adaption. Organisation is the cognitive function responsible for the similarities that exist in intellectual behaviour at all levels of development; that is, it is responsible for the continuities in development. Adaption provides the mechanisms responsible for the changes within the cognitive structures and hence is responsible for producing the discontinuities in development. Thus, organisation and adaption complement each other. The former ensures that the reorganisation of cognitive structures produces an ordered whole. The latter ensures that cognitive structures grow internally as elements of the total system.

The mechanisms used by adaption are assimilation and accommodation. Assimilation is the taking in of experience, and its interpretation by existing cognitive structures.

Accommodation is the process of adapting the cognitive structure to facilitate future interpretations of experience:

> An act of intelligence in which assimilation and accommodation are in balance or in equilibrium constitutes an intellectual adaption.
>
> (Flavell 1963, p. 47)

While function remains invariant, content, like structure, changes with age.

Content

Content is "uninterpreted behavioural data" (Flavell 1963, p. 17). It is this behaviour that is used to infer structure and function. Analysis of this observed behaviour facilitates the identification of a series of stages and sub-stages of cognitive growth.

Piagetian Stages

The Piagetian theory of intellectual development involved the application of a stage theory to describe changes in the psychological mode of functioning of children as they move from birth to adulthood. The word "stage" has been used in this discussion for what has variously been referred to as "level" (Piaget 1970) and "period" (Flavell 1963). Piaget believed stages to be real and natural descriptions of cognitive fuctioning rather than "purely arbitrary methods of slicing up development" (Brainerd 1978). Hence, Piaget's stages reflect qualitative changes in cognitive functioning:

> To say that a child is "at" a particular stage of cognitive development is to say that a certain set of sensorimotor or cognitive stuctures is present. To say that a child has entered a new stage of cognitive development is to say that qualitative changes in sensorimotor or cognitive stuctures have occurred.
>
> (Brainerd 1978, p. 31)

The stages and a brief description of each are listed below and are based on the translation of Mays (Piaget 1970) and the reviews of Flavell (1963) and Case (1985) Flavell and Case identified a number of sub-stages that together make up a stage while Piaget in 1970 discussed considerable variation of behaviour in each stage representing growth within the stage in addition to identifying invariant behaviours for each stage.

1. *Stage of Sensorimotor Development* (birth to 2 years). Using up to six sub-stages Piaget used this period to describe children's development from a neo-natal, reflex level to a coherent organisation of their physical environment. During this development physical movements are coordinated and are internalised into rudimentary cognitive

structures. As these structures develop, children's behaviour becomes purposive and goal directed.

2. *Stage of Pre-operational Thought* (2 to 7 years). During this stage children develop a symbolic representation of the world. Flavell argued that Piaget identified three sub-stages of beginning representational thought, simple representations or intuitions, and articulated representations or intuitions:

> Just as the world of objects took on an increasing coherence and stability during the first stage of their lives, so the world that depends on symbolic representations-the world of socially shared meaning, categories, and relations - begins to take on a greater coherence and stability during the second stage of their lives.
>
> (Case 1985, p. 17)

The use of these symbols, mental images and language makes thought possible even though such thought often seems illogical to an adult. It is the development of the ability to reason logically that characterises the concrete operational stage.

3. *Stage of Concrete Operations* (7 to 11 years). Flavell (1963) suggested that children at this stage of development behave as if they have a cognitive system that enables them to present evidence

> of possessing a solid cognitive bedrock, something flexible and plastic and yet consistent and enduring, with which [they] can structure the present in terms of the past without undue strain and dislocation, that is, the ever-present tendency to tumble into the perplexity and contradiction which mark the pre-schooler.
>
> (p. 165)

While children at this stage can think logically, this logical analysis is limited to concretely based information. During this stage children develop ordinal and cardinal number concepts; the ability to argue transitively; the capacity to classify objects according to more than one criterion; the ability to understand spacial transformations; perform class inclusion tasks in logic; and the ability to apply many mathematical concepts to a range of concretely based tasks.

However, the ability to perform operations on even concretely based objects is limited. Children at this stage cannot concurrently carry out the reversibility operations of negation and reciprocation (Flavell 1963, p. 203). The ability to perform these tasks flags the beginning of Piaget's final stage of intellectual development.

4. *Stage of Formal Operations* (12 to 15 years). The new and final reorganisation of cognitive structure facilitates thinking about thinking. The subject of children's thoughts can be thoughts themselves. Children can deal not only with concrete reality but with propositional statements, and are able to generate possibilities and hypotheses whose only immediate referents are prior elements of cognition.

While these stages are seen as discrete, Piaget described how they are related via a transition mechanism. The next part focuses discussion on how transition through these stages is facilitated and some underlying assumptions of the model.

Discussion

Case (1985) attributed to Piaget four factors that influence children's rate of progression through the stages of intellectual development - maturation, physical experience, social experience, and equilibrium. The equilibrium process was seen as common across the stage transitions even though the stages themselves are different. The equilibrium process involved bringing assimilation and accommodation into "balanced co-ordination" (Flavell 1963, p. 239). Further, Flavell reported that maturation, physical experience and social experience contribute in a causal way to the equilibrium process. The process of equilibration implies that higher states of equilibrium incorporate and integrate the elements of the lower states. This relation between adjoining stages serves a unifying function in Piaget's theory, i.e., vertical integration of schemes and horizontal differentiation of the phenomena to which schemes can be applied.

As a component of the social experience, identified above as a causal factor in cognitive development, Piaget ascribed to language a critical role. Representational thought does not begin with or emerge from the verbal signs of the social environment. The first signifiers are private, nonverbal signs that do not arise from language. However, this symbolic function facilitates the development of social signs which includes language (Flavell 1963). Language plays an important role in the development of conceptual thinking:

> Piaget of course admits, in fact, stresses, the enormous role which a codified and socially shared linguistic system plays in the development of conceptual thinking. Language is the vehicle par excellence of symbolisation, without which thought could never become really socialised and thereby logical.
>
> (Flavell 1963, p. 155)

Piaget's theory has had a profound effect on the thinking of educational researchers, as evidenced by the vast number of studies that have attempted to replicate and extend his findings. This attention is due in part to the elegance and parsimony of the theory. Case (1985, p. 23) argued that Piaget's theory had a remarkable breadth in that it explained a great diversity of phenomena and was applicable to both lower-order and higher-order cognitive functioning. These attributes are elsewhere (Biggs and Collis 1982) referred to as assumptions of the Piaget's stage theory of development. They include:

1. Similar understandings are acquired at similar ages across a variety of intellectual domains.
2. There exist sequences of intellectual development in which higher-order understandings are assembled out of lower-order understandings.
3. Certain understandings, which appear obvious to adults, are inaccessible to children until a particular level of logical structure has been constructed.
4. Patterns of development evident in the earliest motor learning are evident in the more complex forms of learning.
5. Children need to achieve a degree of readiness in the form of appropriate logical structures before they can assimilate certain kinds of experiences.

There are two major areas of Piaget's theory that have been subject to considerable attention and criticism. The first is the role of vocabulary in his experimental findings, and the second is the prevalence of horizontal dĕcalage.

Some writers (Flavell 1963, p. 434) argued that Piaget believed that cognitive stage and vocabulary were isomorphic. Critics of Piaget argued that his experiments may represent nothing more than "vocabulary growth" studies (Flavell 1963, p. 434) and do not necessarily produce evidence of change in cognitive structure. Piaget was aware of this issue, but he was not always cautious when inferring cognitive structure from verbal protocols (Flavell 1963, p. 437). Earlier in this section, structure, function and content were identified as three important components of Piaget's work. His experimental method involved making inferences about structure and function based on the evidence of children's observed behaviour (content). It is the closeness of this relationship that is being questioned here. Does the observed behaviour always reflect structure and function?

The second major critisism of Piaget's theory questions the first assumption of stage theory reported above. This assumption states that similar understandings are acquired at similar ages across a variety of intellectual domains. The implication of this is if a child can conserve quantity of matter by recognising that the quantity of an object remains the same irrespective of its shape he/she might reasonably be expected to conserve weight since the concepts are of a similar structure. However, this is not the case and experimental data suggest that one ability occurs before the other. This discontinuity is called a horizontal décalage. A décalage occurs when an individual performs differently on tasks of a similar structure. Thus while it may be useful for an individual to be characterised by a given cognitive structure

> they will not necessarily be able to perform within that structure for all tasks. Task contents do differ in the extent to which they resist and inhibit the application of cognitive structures.
>
> (Flavell 1963, p. 23)

While Piaget recognised the existence of horizontal décalage, extensive research has indicated that décalage is more prevalent than he recognised ((Biggs & Collis 1982; Case 1985; Flavell 1963). This is addressed in a number of theoretical developments of Piaget's work (Kirby & Biggs 1980) that have become known as neo-Piagetian theories (Demetriou 1988). Among these Neo-Piagetian theories is the work of Biggs and Collis that specifically addresses the issues of the relationship between language and cognitive structure and horizontal décalage. Their work is presented in the taxonomy known as the **S**tructure of **O**bserved **L**earning **O**utcomes (SOLO) (Biggs & Collis 1982).

THE SOLO TAXONOMY

This section consists of four parts. The first part traces the development of the SOLO Taxonomy and distinguishes it from the Piagetian model. Part two identifies the components of the model and how the components are related. The factors affecting transition through the modes are considered in part three prior to a discussion of strengths and limitations of the model in the final part of this section.

Development and Overview

The SOLO Taxonomy shifted the focus of attention from the internal construct of the developmental stage of the child to the quality of the learning outcome as evidenced by the children's response to a stimulus item. Biggs and Collis (1982, p.22) distinguished between a "generalised cognitive structure" of the child and the "actual responses" they give to learning tasks. While they accepted the existence of a generalised cognitive structure, they believed it not to be directly measurable and hence referred to a "hypothesised cognitive structure" (HCS). While the HCS may determine the upper limit of functioning, actual responses depend on other factors such as motivation and prior learning experiences.

Biggs and Collis believed that the HCS stages are relatively stable over time and are "independent of instruction," whereas a SOLO level reflects attainment and refers to children's performance on a particular task. The emphasis on a particular task is important as the SOLO Taxonomy assumes that people vary in their performance between tasks that are closely related in terms of underlying logic, thus including the concept of décalage within the model:

> A student can be early formal in mathematics while early concrete in history, or even formal in mathematics one day and concrete the next. Such observations cannot indicate shifts in cognitive development, but rather shifts in more proximal constructs, such as learning, performance, or motivation.
>
> (Biggs & Collis 1991, p. 60)

It is this emphasis on analysing the quality of children's responses that makes the SOLO Taxonomy attractive for this investigation. In the stimulus items used here the focus is not on recording right or wrong answers, but rather on the nature (structure) of children's responses and how they change over time. Thus, by mirroring the coded categories against the SOLO Taxonomy it ought to be possible to infer a hierarchy of responses that reflect change over time and hence to provide a detailed description of the development of children's descriptions of number patterns.

The SOLO Taxonomy has mirrored the evolutionary changes postulated at the beginning of this chapter (Biggs & Collis 1982, 1991; Collis & Biggs 1989; Collis & Biggs 1991). The later versions (Biggs & Collis 1991; Collis & Biggs 1991) retain the concept of levels to describe structural complexity of performance. The earlier construct of stage (Biggs & Collis 1982) has been replaced by the construct of mode (Biggs & Collis 1991) and refers to the degree of abstraction of representations. These modes and levels interact to form an integrated model. A more detailed description of the features of the model, in particular the levels and modes, is presented below.

Levels and Modes

SOLO suggests there are five modes of cognitive functioning rather than the four developmental stages of Piaget. Biggs and Collis have provided a post-formal mode of development to describe shifts in cognitive growth beyond that normally observed among school children. However, one important difference from the views of Piaget is that as new modes become available they do not replace the old mode but develop in parallel to it. That is the "modes accrue from birth to maturity" (Biggs & Collis 1991, p.

61). The latter level represents the upper ceiling to the level of abstraction that the child can perform at, not the level that all performances must conform to. Typically, as more modes become available multi-modal functioning becomes the norm. Before discussing multi-modal functioning in detail a description of the five modes is presented.

1. *Sensorimotor mode*. The focus of attention (or source of elements) is the physical environment. Children develop the ability to coordinate and manage their interaction with the physical environment. The continued development in this mode is exemplified by sporting skills in which a tacit knowledge is acquired.

2. *Ikonic mode*. In this mode symbols and imagery are used to represent the elements of the sensorimotor mode. These signifiers become the elements of the mode which are used for oral communication. Intuition, seen as sporadic and isolated cognitive expressions which do not coalesce, (Flavell 1963, p. 166) is used. Strategies such as guessing, using manipulatives and developing mind pictures are all typical of children operating in the ikonic mode. The sensorimotor and ikonic modes are 'natural' modes for people to operate in. It is the concrete symbolic mode that is the first target of formal schooling.

3. *Concrete symbolic mode.* This mode involves a shift in abstraction from representing the physical world through oral language to using

> written, second order, symbol systems that apply to the experienced world
>
> (Biggs & Collis 1991, p. 63)

The symbol systems have an internal logic and order in addition to facilitating a relationship between the symbol system and the physical environment. Such symbol systems are used extensively in schools in areas such as mathematics, musical notation and written language. Explicit instruction is required to achieve independence within the symbolic system, and hence the concrete symbolic mode distinguishes itself from the earlier modes that can be more naturally accessed by children.

The concrete symbolic mode is the target mode of much school mathematics, since such mathematics attempts to describe and operationalise the children's environment. The pattern approach to introducing algebra, described in Chapter 1, could be said to represent an attempt to locate the instructional activity within the concrete symbolic mode since it involves children describing patterns in their natural language and in developing more concise notational systems for those descriptions. This approach contrasts with the approach of introducing an abstract system of symbolic notation and associated operations that have no real world referent from the child's perspective and result in rote learned responses. Such activity would more appropriately be given to children capable of operating in the formal mode.

4. *Formal mode.* As indicated above, the elements of attention in the formal mode are theoretical constructs without a real world referent. The thinking processes involved are hypothesis formulation and propositional reasoning. Collis and Biggs (1991) believed that this mode represented the target mode for a bachelor's degree student to be able to successfully work within their discipline.

5. *Post-formal mode.* The existence of this mode seems to be hypothesised rather than be supported by empirical evidence. Its essential characteristic is the ability to question the basic tenets of a discipline.

The modes, the approximate age of availability, and the forms of knowledge represented by each mode are presented in Figure 2.1.

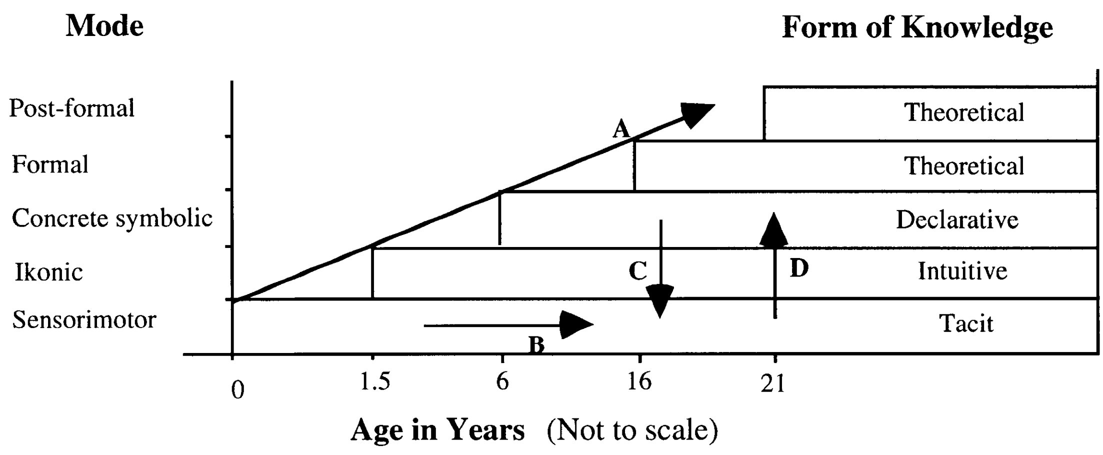


Figure 2.1

Modes and Forms of Knowledge
(Adapted from Biggs and Collis (1991))

With the modes described, the alternative passages of transition through the modes is discussed. This variability is referred to by Biggs and Collis (1991) as multimodal learning. In Figure 2.1 four alternative learning paths are shown by the arrows A, B, C and D. Arrow A is the path assumed by stage theories in which the emerging stage replaces its predecessor. Replacing stages with the concept of modes does not preclude such a path of development. The emerging mode facilitates an added degree of abstraction in the element of analysis. However, the model also allows for the continued development in a mode even if other modes are available to the learner. If the continued development is restricted to one mode (as in arrow B of Figure 2.1) then the development is called unimodal learning.

More typically, to account for the difference between the physical skills of young children and those of elite athletes, more than continued development in the sensorimotor modes is needed. Elite athletes would call on other modes to better understand their performance and hence improve their performance in the target mode. Such modal interaction is called "top-down facilitation of lower order learning" (Biggs and Collis 1991, p. 70) and is represented in Figure 2.1 by arrow C.

In addition to top-down learning, there is the "bottom-up facilitation of higher order learning" (Biggs and Collis 1991, p. 71). In this model, learning activities are located initially in the lower modes and trace a developmental sequence to the target mode. Biggs and Collis (1991, p.71) argued that such a form of learning reflects the work of Dienes and Bruner and can accommodate much of progressive education theories of the last 30 years.

The focus of this part thus far has been the modes and the interaction of the modes in learning. Just as Piaget discriminated between cognitive structures within each stage, so Biggs and Collis identified structural differences of performance within each mode. These differences were called levels and repeated in a cyclical fashion. Within each mode there are three broad levels of structural complexity, namely, unistructural, multistructural and relational. Following the relational level is the extended abstract level that represents the unistructural level in the next mode. A response that fails to engage the question in the target mode is said to be prestructural and may represent a response in a previously developed mode. In all, if the focus is on a particular, or target, mode, there are five broad levels of structural complexity identified within the SOLO Taxonomy with the topmost level of one mode mapping into the bottom (unistructural) level of the next mode. The levels and their characteristics are now described.

1. *Prestructural.* The response indicates an inability to engage the question in a meaningful way. Such a response may involve restating the question, or focusing on some irrelevant data that is incidental to the question. It may reflect that the child is incapable of responding, or does not wish to respond, in the target mode.

2. *Unistructural.* This set of responses uses only one relevant element of data from the stimulus item. A child who responds to the question 3+4+5 with an answer 7 has clearly closed after adding 3+4. Hence only one element of data from the question has been used. A feature of responses at this level is the desire to close quickly and to ignore inconsistencies that may result from the response (Biggs and Collis 1982, p. 20).

3. *Multistructural.* The learner at this level can use multiple data elements, but the elements are not integrated. Hence the response can consist of a number of discrete closures. Typical of these responses would be the following of strict algorithmic procedures that involve a number of steps. However, if a single step was forgotten, or an error made, the respondent would be unable to reconstruct the algorithm. This lack of an overview of the data elements and their relationships makes the response patterns inherently unstable and thus considerable variability may be expected from children responding at this level.

4. *Relational.* In contrast to a multistructural response, a relational response reflects the ability to integrate the elements and operations of the question in a way that enables an overview of the stimulus item. Children using an algorithm at this level would be able to check for errors and inconsistencies, and would be able to reconstruct missing elements of the algorithm. Features of responses at this level include the ability to reverse operations and the set of elements used are internal to the system.

5. *Extended abstract.* The use of data elements external to the system is a feature of an extended abstract response and is the link with the next mode. That is, an extended abstract response at mode N is possibly an unistructural, or higher, response at mode N+1. The generalisation of the elements takes account of new and more abstract features.

A diagramatic representation of the interaction between modes and levels is presented in Figure 2.2.

With the modes and levels described a comparison between the terminology of Piaget and Biggs and Collis can be undertaken. There is a close similarity between the first two stages and the emergence of the third Piagetian stage with the first two modes and emergence of the third mode of the SOLO Taxonomy. The most significant difference lies in the transitions that take place at approximately 11 to 12 years of age. At this age Piaget describes a transition from the stage of concrete operations to formal operations.

The distinguishing characteristic of this transition, described earlier, is the complex operations (concurrent reversibility operations of negation and reciprocation) with elements of concretely based information.

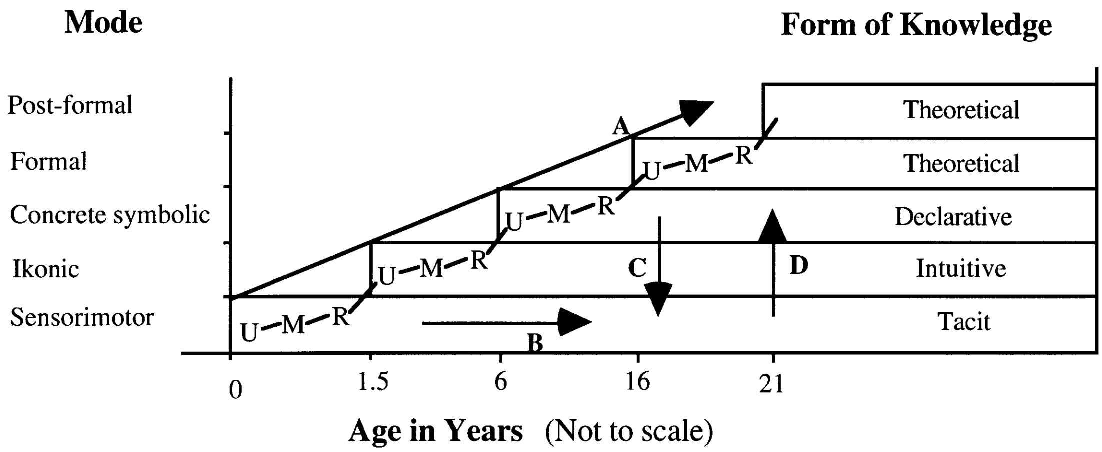


Figure 2.2
Modes, Learning Cycles and Forms of Knowledge
(Adapted from Biggs and Collis (1991))

Biggs and Collis, however, place the nature of the elements being operated upon as the central characteristic of a mode. They have defined the formal mode to emerge when children can operate on elements without a real world referent. This occurs at approximately 16 years of age (Collis 1980). Thus, the complex operations that Piaget characterises as the emergence of formal thinking are included in the concrete symbolic mode of Biggs and Collis. However, as argued in the discussion on levels above, they are included in the relational level of this mode.

The post-formal mode is the second major difference between the stages of Piaget and the modes of SOLO. The post-formal mode is additional to the range of development described by Piaget and as such extends the possible range of cognitive development well past the school age years that were the focus of Piaget.

The next part considers in more detail the issue of relationships between modes and levels and in particular the influences on transitions between them.

Transition Through the Modes

It was noted above that Piaget was not specific about how progression through stages was facilitated. As would be expected Biggs and Collis referred to a variety of causal factors that facilitate these transitions. These included Piaget's factors of physical maturity, the social environment and the physical environment. However, they emphasised a number of other features (Biggs & Collis 1991).

They pointed out that transition across modal boundaries cannot take place without the child being at the relational level in the previous mode. The implication here is that each step in the developmental sequence is a necessary prerequisite to the succeeding step. Piagetian data are invoked to argue this necessity. Such a view is consistent with the model of Linchevski and Sfard reported in Chapter 1.

Biggs and Collis also placed considerable emphasis on the learning experiences of children in facilitating transition. This can be extended to include "confrontation with a problem" (Biggs & Collis 1991, p. 68) as motivation to induce a cognitive reorganisation. Such an influence is similar to the "crisis in thinking" that Van Hiele (1986) argued can induce a shift in the quality of thinking.

While the above factors can be seen as external to the individual, Biggs and Collis also referred to an internal factor influencing transition. Whereas Piaget referred to assimilation and accommodation as mechanisms for internal reorganisation of information, neo-Piagetians refer to working memory capacity and information processing abilities (Case 1985; Halford 1993). The changing use of the working memory facilitates movement through levels and modes. An unsupported argument was presented by Biggs and Collis (1991) to distinguish between three types of shifts. A movement from unistructural to a multistructural response involves changing the ratio of "information to noise" (p. 68). A movement from multistructural to relational responses involves "information becoming better organised" (p. 68) while movement to the next mode "requires a change in the basis of organisation" (p. 68).

Like Piaget, the authors of the SOLO Taxonomy did not provide estimates of the mix of these various factors in inducing cognitive growth. However, Biggs and Collis did argue that the mix may well change for different transitions. The transition from sensorimotor mode to the ikonic mode was seen as a natural transition while transition within the concrete symbolic mode and into the formal mode is more dependent upon external factors such as school-based learned experiences.

Having established the nature of the elements and their relationships within the SOLO Taxonomy, the suitability and shortcomings of the model for the present study is discussed.

Discussion of SOLO Taxonomy

The SOLO Taxonomy was originally developed as an instructional and assessment tool to assist teachers in schools. While the model itself has undergone considerable change since its original publication in 1982, its use as an assessment tool is still a focus of discussion (Collis & Romberg 1991; Pegg 1992). However, it has also become a tool for research in a broad range of curriculum areas. Biggs and Collis (1982) used data from the five curriculum areas of English, mathematics, modern languages, history and geography in the original development of the model. Since then the model has been used in a range of settings including geography (Courtney 1986) and science (Levins 1992; Stanbridge 1990). Within the mathematics education environment SOLO has been used to investigate cognitive growth in a wide variety of topic areas at a wide variety of age groups. These include fractions with children in Years K to 10 (Watson, Collis & Campbell 1992), Multiplication in Years K to 3 (Watson & Mulligan 1990), Geometry in Years 5 to 8 (Davey & Pegg 1992) and in Years 10 to 12 (Pegg & Woolley 1994), Algebra in Years 10 to 12 and tertiary (Coady & Pegg 1994), Statistics in Years 3 to 9 (Watson, Collis & Moritz 1994; Reading & Pegg 1995), problem solving in Years 9 and 10 (Bennett 1987), volume of prisms in Years 2 to 6 (Campbell, Watson & Collis 1992).

The model presents the development in children's responses as an essentially rational phenomena, that can be observed classified and analysed, even if the factors influencing this development cannot be uniquely identified. Others argue that such a structuralist view of human behaviour is an oversimplification. O'Reilly questions the legitimacy of identifying "particular pathways through mathematics" (O'Reilly 1990, p. 77) by questioning the underlying assumptions and the methodology of Hart's (1981) *Concepts in Secondary Mathematics and Science* research project (CSMS). O'Reilly stated that the study began with the assumption that mathematical understanding was hierarchical and the task of the study was to reveal the hierarchies. Such a criticism could be made of the current context. However, in response it is argued that the assumption is explicit and is to be tested within the context of the study. Indeed, this is the role of the theoretical framework and is a reason for choosing the SOLO Taxonomy as such a framework.

O'Reilly also argued that hierarchies are products of contextual, temporal and societal factors, and that small changes in the items used in a study may effect the identification of hierarchies. This should not be seen as an argument against the existence of hierarchies but rather a qualification about the generality of findings. If theoretical frameworks are to be tested, then empirical evidence must be gathered and such evidence must be subject to specific "contextual, temporal and societal" conditions. It is thus necessary for these conditions to be made explicit within the study to facilitate suitable evaluation, extension and replication of the research project.

More specifically, the CSMS project classifies items into hierarchical levels (Hart 1981). This was done using several criteria and not all items initially identified were able to be classified. The strategy to be used in this study shifts the focus from the classification of the item to the classification of the children's response to the item. SOLO facilitates this by providing a structure (mirror) with which to compare the responses. The limitation of such a process is based on the degree of ambiguity of the level criteria presented above. The issue of ambiguity has lead to the identification of two potential problems in using the taxonomy that are already evident from existing research.

The first of these is the concept of a taxonomy itself. Taxonomies are widely used in the biological sciences as a classificatory mechanism for the purpose of bringing accessibility to abundant data. However this parsimony is often seen as artificial. The issue is how to identify groups, and where and how to place boundaries between groups. Gould (1985) saw the need to identify "islands of form" as well as continuity in the biological sciences:

> Islands of form exist to be sure: cats do not flow together in a sea of continuity, but rather come to us as lions, tigers, lynxes, tabbies and so forth. Still, although species may be discrete, they have no immutable essence. Variation is the raw material of evolutionary change.
>
> (p. 160)

Collis and Biggs themselves identified transitional responses that indicated gradations within their groups. The issue here is one of clarity and simplicity in the response categories at the expense of subtlety and nuances in the data. But this is not a criticism restricted to hierarchical models. It is also true of any data reduction process whether it is of the statistical or qualitative paradigm.

The second problem with using the SOLO Taxonomy is for the structure of modes and levels to be over simplistic to accommodate a learning sequence. Some researchers (Campbell, Watson & Collis 1992; Pegg 1992) are hypothesising the existence of repeated unistructural, multistructural, relational cycles (U-M-R) within a mode rather than the single cycle referred to in Figure 2.1. Sometimes these U-M-R cycles are seen to merge (Campbell, Watson & Collis 1992) to form new more complex hierarchies, while at other times they are seen to be sequential (Pegg 1992). The nature and number of cycles seems in part to be a function of the research methodology and the detail of the data collected, as well as being an accurate description of the children's development. It is clear from this discussion that there is a need not only to collect data and to use the SOLO Taxonomy as a tool in the analysis of the data, but also to use the data as a tool to evaluate the Taxonomy.

Some of the above problems occur because a particular set of responses are being considered in an absolute sense. By this it is meant that the responses are being integrated into a broad view of responses and a large slice of cognitive development is being considered simultaneously. Alternatively, the model can be used in a relative sense to consider a restricted set of responses that can be analysed relative to each other rather than to the totality of children's learning experiences. Metaphorically, this is similar to looking at a small piece of the jigsaw in isolation rather than trying to fit this small piece into the total picture. By breaking these links, full benefit of overcoming the problem of horizontal décalage and recognising the effect of learning experience is being used.

The above discussion indicates that the SOLO Taxonomy cannot be seen as a "Grand Theory" in the sense of LeCompte and Preissle (1993) since it has clearly changed since it was originally published in 1982 and is still evolving. However, it has several features that make it attractive as a theoretical framework for this study.

The first of these features is that the focus of analysis is the children's responses, which are more directly accessible and measurable than the more elusive Piagetian focus of cognitive structure. Secondly, the model incorporates learning experiences of the children and hence responds to the issue of horizontal décalage. Thus, it is possible to consider the analysis of curriculum areas separately, while at the same time using SOLO as a common language to facilitate analysis across curriculum areas and with other research within the same curriculum area. Thirdly, the model describes relationships between the categories in the taxonomy and hence it facilitates the analysis of change in children's responses over time, leading to the identification of a hierarchy of development being postulated.

While these are seen as advantages indicating the suitability of the SOLO Taxonomy for this study, it is imperative that it is realised that some reservations with regard to the efficacy of the model have been identified and that the interaction of the model and the data are seen as a two-way process. Hence one research question that needs to be investigated is: Can the children's description of patterns be analysed in the theoretical framework of the SOLO Taxonomy?

SUMMARY

The preceding discussion has three areas of focus. The first was to identify the nature and role of a theoretical framework considered suitable for this study. The traditional scientific view of a grand theory was rejected and replaced with a conception of theory as an emerging phenomenon that assists in linking this study to other research via a common language and shared structures.

The second focus was on Piaget's work as a starting point in the search for a suitable theoretical framework. However, Piaget's focus on the internal concept of cognitive structure as the element of analysis seemed problematic along with the unresolved dilemma of horizontal décalage. Hence the third focus of the chapter was the SOLO Taxonomy that shifted the element of analysis to the more accessible children's responses to stimulus items. While some reservations about the taxonomy were identified, it seemed a suitable framework to use in this context.

Consistent with the concept of theory as a fluid phenomenon, the research is to be theory testing as well as data testing. The next chapter addresses this theme and the issues identified in Chapter 1 and presents some specific research questions together with the research design.

Chapter 3

RESEARCH DESIGN

Introduction

Chapter 1 provided a general overview of the literature relevant to children's understanding of number patterns and identified a number of issues deemed worthy of further investigation. Included in the literature review was a discussion of a series of small studies of children's ability to apply a pattern to an uncountable example or to provide a description of the number pattern in symbolic notation. However, none of the studies attempted to systematically describe children's ability to express generality of number patterns in natural language nor to investigate how these descriptions changed over time. Two other broad issues were identified as worthy of further investigation. The first was the possible existence of relationships between the alternative representations of number patterns, and the second was the development of a common frame of reference and language to facilitate the description of the structure of children's descriptions with other researchers.

These issues form the basis for the description of four research themes that are presented in the first section of this chapter. These themes are in turn developed into a series of research questions to be investigated.

The balance of this chapter describes the research design that has been developed to investigate the research questions outlined in the first section. The research project has two major components, a cross-sectional exploratory survey and a longitudinal developmental study. An overview of the project and some methodological issues are presented in the second section. The design of the survey component of the study is developed in detail and evaluated in the balance of this chapter. Consideration of the design of the longitudinal component of the study is delayed until Chapter 8 as it will be influenced by the findings of the survey component.

RESEARCH QUESTIONS

From the issues identified in the literature review of Chapter 1, and the issues associated with the nature and role of the theoretical framework in Chapter 2, four general research themes have been identified. The themes are identified below prior to presenting specific research questions associated with each of the first three themes. Specific research questions for the fourth theme are developed in Chapter 7.

Theme 1. What natural language do children use to describe patterns associated with numbers in Years 5, 6, 7 and 8?

Theme 2. Can the children's descriptions of number patterns be analysed within the theoretical framework of the SOLO Taxonomy?

Theme 3. Are the natural language descriptions related to the other response types such as use of symbolic notation and calculating an uncountable example, or to the child's school year?

Theme 4. How do children's description of number patterns, using both natural language and symbolic notation, change over time?

In order to more clearly focus on each theme a set of specific research questions is provided below.

Theme 1

What natural language do children use to describe patterns associated with numbers in Years 5, 6, 7 and 8?

> Question 1.1
> Can children's natural language descriptions of number patterns be classified into a discrete set of categories?

Question 1.2
Can children's symbolic language descriptions of number patterns be classified into a discrete set of categories?

Question 1.3
Are the response categories stable across stimulus items that vary in complexity and context?

Theme 2
Can the children's descriptions of number patterns be analysed within the theoretical framework of the SOLO Taxonomy?

Question 2.1
Do the categories of pattern description using natural language exhibit properties that would enable mapping them onto the modes and levels of the SOLO Taxonomy?

Question 2.2
Can a hierarchy of growth be postulated from the response categories of pattern description using natural language?

Question 2.3
Do the stimulus items that vary in complexity and context reveal differences in difficulty?

Are single operation patterns easier than two operation patterns?

Are physical contexts, such as matchstick diagrams, easier than decontextualised number pairs?

Theme 3
Are the natural language descriptions related to the other response types such as use of symbolic notation and calculating an uncountable example, or to the child's school year?

Question 3.1
Is there an association between the categories of pattern description using natural language and the school year of the respondent?

Question 3.2

Is there an association between the categories of pattern description using natural language and the categories of symbolic language used?

Question 3.3
Is there an association between the categories of pattern description using natural language and being able to apply the rule to a large (uncountable) value of the independent variable?

Prior to developing an instrument, implementation procedures and a data analysis plan to investigate these research questions, some general methodological issues are discussed.

METHODOLOGICAL ISSUES

This section contains two parts that provide an overview of the research design. The first part characterises quantitative and qualitative methods as supplementary to each other, rather that mutually exclusive alternatives. The second part identifies the survey as a suitable strategy for collecting data for the initial phase of the study and discusses its advantages and limitations in the context of the study.

Quantitative and Qualitative Data

Many recent reports of research in Mathematics Education (e.g., Hart 1981; Booth 1984) have used as their research strategy the technique of conducting a survey across a large section of the target population with a relatively small number of follow-up interviews of a stratified sample of the cohort. There are at least three justifications for this methodology. The first justification is that the large sample provides a description of the set of possible responses with some potential for generalisability of the results, i.e., external validity. While these data provide a description of the set of responses they do not provide much detail about the processes the respondents used to arrive at their answer. The interviews attempt to provide this additional detail, i.e., they have the potential to enhance internal validity.

A second justification for this methodology is the shift in emphasis from a behavioural approach to an analysis of cognitive processes consistent with the constructivists view of the learning process. The early research that focused solely on classification of error patterns (Sleeman 1984) was of limited value because it failed to provide a reason for the errors and hence had restricted potential in providing directions for future development.

The final justification for using more than one technique for collecting data is that the variety of data gathering strategies provides a balance for their respective strengths and weaknesses. The technique was described by Denzin (1970, p. 472) as "methodological triangulation." By using triangulation the researcher can be more confident that the data generated are not simple artefacts of one specific method of data collection.

Another description that can be applied to this distinction of methodology is that of quantitative versus qualitative research. Quantitative research is characterised by data expressed in numerical form. This contrasts with qualitative research, where data are expressed and described verbally (Crowl 1993, p. 5). Consideration must be given to the application and limitations of both forms of data. The problem of establishing mutual exclusivity of the two approaches was highlighted by Zelditch (1970):

> Quantitative data are often thought of as 'hard,' and qualitative as 'real and deep'; thus if you prefer 'hard' data you are for quantification and if you prefer 'real, deep' data you are for qualitative participant observation. What to do if you prefer data that are real, deep and hard is not immediately apparent.
>
> (p. 217)

Part of the rationale for qualitative methods is based on the notion that objective reality is, at best, a very elusive concept. Partlet and Hamilton (1972) rejected an exclusively quantitative approach because it provides 'tidy' results that can be rarely generalised in an 'untidy reality' (p. 8). This notion has great appeal when trying to monitor children's thought processes. In this study the exclusive use of one method at the expense of another was rejected. Rather, the complementary nature of the two methodologies is emphasised, a view shared by Glaser and Strauss (1967) who said:

> ...There is no fundamental clash between the purposes and capacities of qualitative and quantitative methods of data. We believe that each form of data is useful for both verification and generation of theory, whatever the primacy of emphasis. Primacy depends only on the circumstances of research, on the interests and training of the researcher, and on the kinds of material he needs for his theory. In many instances, both forms of data are necessary: not quantitative used to test qualitative, but both used as supplements, as mutual verification and, most important for us, as different forms of data on the same subject, which, when compared, will each generate theory.
>
> (p.17)

After consideration of these issues it was decided to have two major components of this study. The first is an exploratory survey that would provide both qualitative and quantitative data from a relatively large sample. The second component, while qualitative in nature, combines two important aspects of the study. It provides a deeper understanding of the survey data and will be a longitudinal study thus providing information about student development over time. Figure 3.1 provides a schematic overview of the study.

The design of the longitudinal study is described in detail in Chapter 7, and the next part of this section is devoted to a description of the rationale for implementing the exploratory survey.

Rationale and Discussion of Surveys

From the literature review of Chapter 1 it is clear that little is known about children's ability to describe number patterns or their ability to encode their descriptions in algebraic symbolism. While considerable variability is evident in the studies that have been done, there has been no systematic attempt to describe the range of responses nor to suggest how they might change over time. Hence, this study is best described as an exploratory study that uses the survey as the focus of its research methodology. The research plan is not attempting to confirm or deny a cause-effect relationship in an experimental setting. Its purpose is to investigate the characteristics of a population (Kerlinger 1986). In this sense the survey is being used as a descriptive method of educational research (Cohen & Manion 1989) who argued that surveys collect data at an instance of time with the intention of:

> (a) Describing the nature of existing conditions, or
> (b) Identifying standards against which existing conditions can be compared, or
> (c) Determining the relationships that exist between specific events.
>
> (p. 97)

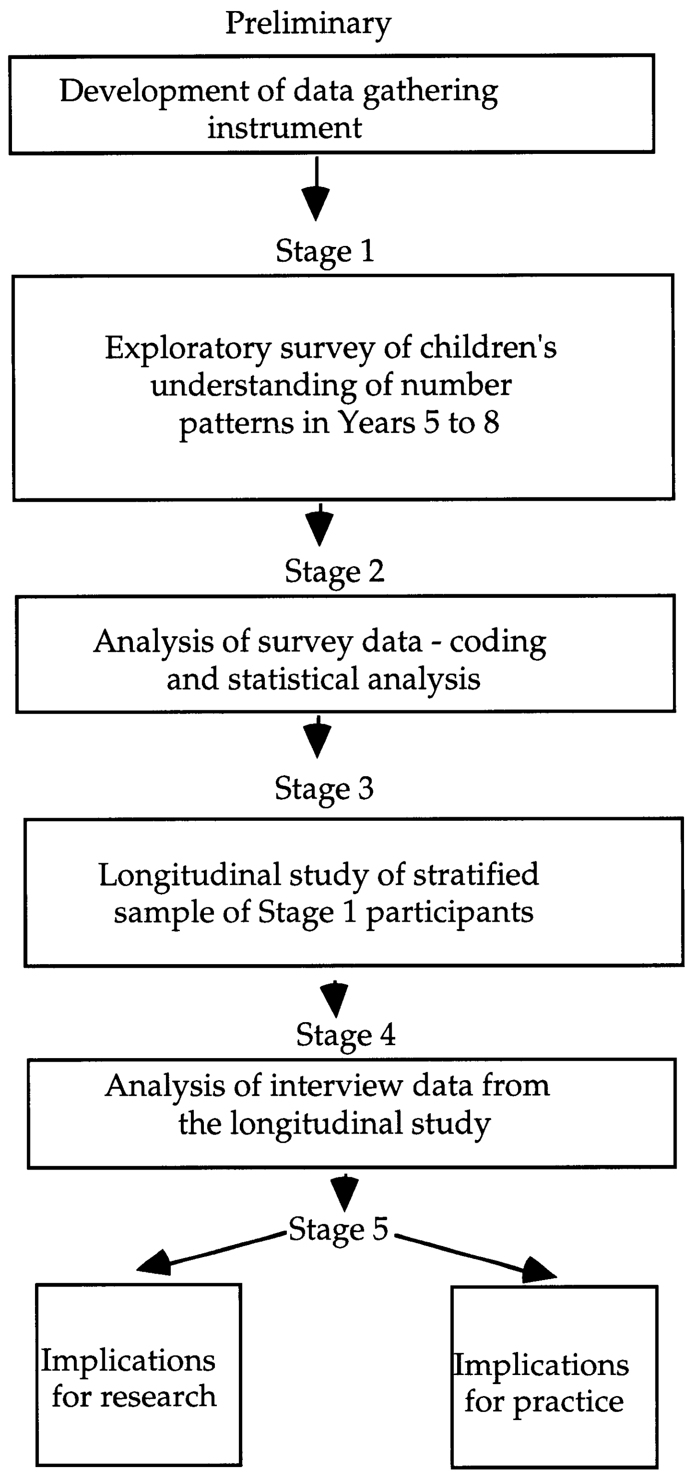


Figure 3.1

Sequence of Components of the Research Study

The research questions presented as the first theme of this study relate closely to the three points of Cohen and Manion in that they focus on determining the nature of children's language (Question 1.1) and use of symbolic notation (Question 1.3). In addition, the research questions require the investigation of relationships between components of the children's responses (Questions 3.1, 3.2 and 3.3). The survey responses can also be used to provide a framework with which to compare the children's responses given during the longitudinal study (Theme 4).

In attempting to describe the nature of existing conditions the survey strategy has the advantage of being able to collect data from a relatively large sample thus adding to the external validity of the study. More discussion of the sample and the consequent influences on external validity is discussed later in this chapter.

While the survey has clear advantages in the context of this study there are some disadvantages. Kerlinger (1986, p. 387) cited five limitations of the survey as a research instrument. These are:

(a) Data collection on such a large scale tends to be superficial and not to "penetrate very deeply below the surface".
(b) Surveys on a large scale are demanding of time and money.
(c) The sampling process is always subject to sampling error.
(d) The act of a survey can remove the respondents from their natural setting and hence reduce the validity of the responses.
(e) Survey results can only be as good as the competence of the researcher and the sophistication of the survey instrument.

Lin (1976, p. 241) added to these disadvantages three others: the response rate, the interaction effect and the lack of dynamics. The response rate is defined as the ratio of returned useable questionnaires to the number of reachable sampled respondents. Obviously the higher the response rate the more representative the survey is of the population. Lin believed that a 50% response rate is considered 'adequate,' and 70% is 'very good'. The interaction effect refers to the interaction between the instrument, the respondent and the interviewer. This interaction can result in a number of outcomes that could be seen as threats to validity. The tendency for respondents "to respond towards neutrality or towards perceived ideals" or for respondents to feel threatened by questions because of the possible use of information by authorities are typical of these outcomes. In addition the tendency to feel threatened if unable to answer questions confidently can induce stress in the respondent, resulting in non-completion of the questionnaire or responses that are not representative of the respondents' true ability.

Lin's third limitation of the survey as a research instrument was its 'lack of dynamics'. By this he meant that it presents a 'frozen slice of reality' without relating the characteristics to the past or the future. While the survey allows the researcher to make some comparisons from within a cohort based on some criteria such as age, sex, school etc., it does not provide a view of how things became what they are nor what they are likely to be in the future. This limitation was recognised earlier when the study was described as an exploratory study.

This section has identified a quantitative/qualitative methodological mix as being an appropriate strategy for maximising external and internal validity. The survey was chosen as the most efficient data collection method for the first exploratory stage of the study. However, a number of threats to the validity of the survey result were enumerated. With these possible limitations and threats to validity in mind, together with the issue of external reliability, the procedures for conducting this study are described in the next section.

PROCEDURES

There are a number of procedural matters that are important for establishing the external reliability of the study. External reliability means the ability of others to replicate the study (Le Compe and Preissle 1993). To assist in achieving external reliability the sample used for the survey and with the administrative details are described prior to discussing some ethical details of the study.

The Sample

While is recognised that as large and diverse a sample as possible tend to enhance external validity, this has to be balanced against practical considerations of time and cost.

The focus of the study is children in Years 5, 6, 7 and 8 (ages 10 to 13 years) of the N.S.W. school system. This group were chosen because they represent the years immediately preceding the introduction of algebra, namely Year 7, and the year following its initial introduction. It is hoped that such a target group might provide some insight into the pattern of development of children in early algebraic thought through the age at which algebra is introduced. However, taking the warning of Lin, outlined above, with regard to the lack of dynamics of the survey, a pattern of development from Year 5 children through to Year 8 children will not be confirmed. At best, the purpose is to propose some hypotheses that could be investigated further in a longitudinal study.

The research was carried out in the northern N.S.W. towns of Armidale and Uralla, with populations of 22000 people and 3000 people, respectively. Uralla is only 20 kilometres away from Armidale and often described as a 'dormitory suburb,' with many people commuting daily between the two towns for both educational purposes and employment. Thus the sample has a rural background. The towns are both influenced by the presence of the University of New England and thus have a more diverse population than many rural communities. This is typified by the depth and breadth of cultural activities in Armidale and by the above average performance of the school students in public examinations.

The sample was drawn from students in the targeted Year 5 to 8 classes at six high schools, one central school and six primary schools. The high schools consisted of two state high schools, a Catholic school and three independent boarding schools. This range of school type, together with the K to 10 central school, enhanced the representativeness of the sample. This diversity was further aided by the independent boarding schools that attracted students from all over northern N.S.W. and southern Queensland. The six primary schools ranged in size from large schools of approximately 600 students to a small two-teacher school with 42 students. The six schools included a Catholic primary school.

The participation rate of the schools was very high. Unfortunately, the smallest of the independent schools was unable to take part in the project, thus reducing the potential sample by approximately 100 students. There were two primary schools not included in the study. One was a special school for severely mentally handicapped children and the other was a small Catholic school that was used for piloting the research instrument. Thus one dimension of the response rate for the survey was very high due to only one school of the targeted 13 schools failing to participate.

The other dimension of the response rate is the number of children absent from their class on the day of administering the instrument. A check was made between the number of children in each class and the number of survey sheets returned. In no case was there any evidence to suggest that the differences were indicative of other than normal school absenteeism. Thus, a response rate of over 90% was achieved. The criterion of Lin (1976), reported above, of 70% being a 'very good' response rate was well exceeded.

While it was felt that the schools chosen were representative of N.S.W. rural schools generally, a decision had to be made as to how many of the 1400 available children to include in the survey. It was decided to use classroom teachers to administer the instrument and to survey all 1400 children. While this decision had the obvious effect of improving external validity by maximising the number in the sample, it had the disadvantage of reducing control over the administration of the survey. The use of the class teachers to administer the survey demanded the preparation of a detailed set of instructions to guide the teachers. A copy of these instructions is included as Appendix 3.1. While this lack of control over the administration can be seen as a disadvantage it does have the compensation of the instrument being administered in a more natural setting. Thus, in addressing Kerlinger's (1986) fourth limitation of surveys relating to the naturalness of the setting, the control over his fifth limitation - associated with the competence of the researcher - was weakened since partial responsibility was handed to the classroom teacher.

Administration of the Survey

The initial approach to the schools was made through the regional offices of the education systems involved. These were the North-West Region of the N.S.W. Department of School Education and the Catholic Education Office of the Armidale Diocese. In addition similar letters were sent to the Principals of the three independent schools. In part, these letters were designed to "establish the legitimacy of the study and the respectability of the researcher" (Tuckman 1988, p. 245). The letters outlined the general intent of the research, its importance and relevance to N.S.W. school children, the nature of the involvement required from each school, the time of year that the cooperation was sought, and the ethical questions of confidentiality to both the individual children and the school were guaranteed. A copy of the letter is included in Appendix 3.2.

Both school systems and two of the independent schools agreed to allow the research to be undertaken in their schools. The Directors of the two school systems placed the proviso that each principal agreed to cooperate. An appointment was then made with each Principal for the purpose requesting his or her cooperation. After canvassing similar issues to those described above they all agreed to allowing the survey to be conducted in their school. The Principal or their nominee acted as a school contact person for the researcher; however, in each school a meeting was arranged with the teachers involved. This meeting was used to explain the nature and purpose of the project and to seek the teachers' cooperation. The instructions for facilitating the administration of the instrument were discussed and agreement reached. These covered the areas of the purpose of the questions, the need to avoid collusion, how to complete the personal details on the top of the first page, how to respond to student enquires and the approximate time the items could be expected to take to complete. The purpose of these was to standardise the administration procedures as far as possible.

Emphasis was placed on the need to have students as relaxed as possible so as to encourage their best efforts. The word 'test' was to be avoided. The most difficult issue to reach a shared understanding on was how much help to give students in the event of questions. Some teachers thought that the items would be very unusual for their students and they anticipated many questions. The instructions (Appendix 3.1) included a description of the structure of the items, which included identifying the number pattern and pointing out that each number pattern had four questions relating to it. Teachers were encouraged to limit their assistance to ensuring that students could understand the question. They were to use synonyms for words that the students had difficulty with; however, they were not to provide assistance in the form of answers or examples.

A class set of questionnaires was left with each set of written instructions. The completed questionnaires were collected from the school one week after the initial contact. The instrument was administered during the month of November in all schools.

The next part of this section canvasses a range of ethical considerations and decisions made in regard to these considerations.

Ethical Considerations

Apart from the obvious moral responsibility of the researcher to conduct research in an ethical manner to protect the rights of the research subjects, there is also the need to protect the good name and reputation of the researcher's institution. Crowl (1993, p. 271) believed that researchers are "accountable to additional ethical and legal constraints that go beyond those that apply to educators generally." The concept of ethical responsibility is operationalised by Crowl by considering a framework adapted from the work of Kimmel for an educational setting (Crowl 1993, p. 271). This framework had six areas for consideration which are discussed in the context of this survey.

1. *Weighing the benefits and costs* or risks of conducting research. Once the decision was made to restrict the survey to the local region the major cost becomes the researcher's time and the teachers' and children's time. While it is almost impossible to objectively compare these costs with outcomes, there is clearly a need to structure the implementation of the survey in such a way as to minimise this cost and not be too intrusive into the normal operations of the classrooms involved. The organisation described above kept these aspects in mind but at the same time attempted to ensure that there was a standardisation of procedures in the many implementation settings.

2. *Informed Consent.* Crowl recommended that consent is always gained from "mentally competent adults" (Crowl 1993, p. 272), but with children he considered that attaining permission from the administrative head of the institution was sufficient. This permission was sought and attained at both the regional level and the school level prior to seeking the voluntary support of the classroom teachers. The children were not offered a choice about participating in the survey; however, while moral suasion was used to gain their cooperation, no coercion was used except in the matter of collusion.

3. *Concealment and Deception.* At no time during the study was there an attempt to conceal information or to deceive the participants about the nature of the research. Teachers were asked to say to their class that:

> We need to know how much students in this school know about number patterns and we have a number of questions we would like you to do your best to answer.
>
> (Appendix 3.1)

Hence it was not felt necessary to use strategies of "dehoaxing and desensitisation" (Crowl 1993, p. 273).

4. *Use of Volunteers.* Teachers and researchers often have a position of authority over students. It is important that this authority is not used in a coercive manner to force students to participate in a study because of fear. This matter was discussed in point (2) above. A further issue addressed by Crowl (1993, p. 275) in this context is the issue of protecting the participants against "mental stress." It was noted in some early trials of the survey items that some students developed a sense of failure as a result of being unable to complete all the items. This trait was particularly evident amongst the younger students when asked to write their answer in "maths symbols." Three

procedures were put in place to help overcome this issue. The first was to emphasis that the questionnaire was not a "test" and that the school was being investigated, not the individual. Secondly, teachers were asked to assist students with difficulties in understanding the question, but if unsuccessful they should instruct the student to "go on to the next question." Finally, the teacher was asked to "debrief" the class after the questionnaires were completed with the aim of identifying particular problems and if necessary to discuss the unusual nature of the questions.

5. *Researcher responsibilities at the end of the study*. These responsibilities include full and honest disclosure of the nature and purpose of the research. The researcher made himself available to each participating school to address a staff meeting on the issues and implications of the research project.

6. *Anonymity and confidentiality*. As noted above, guarantees of anonymity and confidentiality were made in the original contact with authorities and teachers. To facilitate this, each school was given a code number and not named in any reports of the research. Students were asked to write their name at the top of the questionnaire to facilitate follow-up interviews in the second phase of the project. While this would reduce the perceived degree of confidentiality on behalf of the students, the names were only available to the researcher and were not published in the report. The inclusion of names on the questionnaire, while reducing confidentiality, did have the effect of encouraging "best effort" responses from the students.

This section has described a number of procedural aspects of the study including the sample, distribution of the survey and arrangements for its completion. The third part discussed a range of ethical issues and the associated responses to these issues. The next section describes the development of the survey instrument.

INSTRUMENT DESIGN

Prior to presenting the survey instrument a number of issues with regard to the parameters of the stimulus items are discussed in the first part of this section. The remaining two parts deal with the choice of the actual stimulus items and the associated questions or components that require a direct response from the participating children.

Issues of Stimulus Item Design

From the literature review there are a number of issues identified that could be included as variables in this study. In choosing which variables to consider in the initial survey there were three major influences. The first was the constraint of time taken for children to complete the questions and hence the restriction on the number of items that could be included. It seemed desirable for the questionnaires to be comfortably completed in 40 minutes since this was the standard length of a mathematics lesson in the participating schools. The second influence was the desire to include items that were included in the curriculum resource materials commonly available to N.S.W. schools. Finally, the number of variables needed to be restricted to facilitate analysis of the data.

There were five variables that could be used to vary the structure of the stimulus items in the context of this study. The five are enumerated below and the decision as to include them in the study is presented.

1. *Item Complexity.* Item complexity is restricted to the number of arithmetic operations involved in producing the description of the number pattern. Other potential dimensions of item complexity such as the size or form (rational numbers or negative integers) of the numbers have been excluded. Early trialling of items suggested that children were not capable of a satisfactory response when numbers were large (greater than 5) or complex. Item complexity has been included in this study since it has been seen to influence the description of developmental hierarchies in other contexts (Küchemann 1981).

2. *Contextual Embeddedness.* This refers to the context in which the number pattern is presented. Two classes of context have been chosen for the survey study. One involves presenting the data in the form of concrete, real world objects such as tricycles and matchstick models. These items are referred to as "in-context questions." The other "out-of-context questions" involve the data being presented in the form of a table. The issue of context was included in the survey since it represents a major thrust of the N.S.W. Mathematics Syllabus (Board of Secondary Education 1988).

3. *Sequence of data.* Most sets of curriculum materials present the data with the integer independent variable values in ascending order. This policy was pursued in the survey items for two reasons. In item trials, children seemed to be reordering the data into ascending order thus it was expected that there was little chance of a difference being

identified. Secondly, the survey items would not facilitate insight into how children differed in their treatment of in-sequence and out-of-sequence data. This would more appropriately be investigated in an interview situation. As a result of these factors, together with the need to restrict the number of variables, it was decided to only include data in ascending order.

4. *Amount of data*. It is possible that limiting the amount of data may change the nature of the response or create uncertainty in the mind of the child. On the survey study it was decided to use three or four terms of the sequence as the stimulus item for each question. Questions with only one data element were not seen to constitute a pattern in terms of the definition used in this study. Guidance was taken from the number of terms provided in the syllabus documents and standard text book items as these represented the number pattern experiences children were most likely to encounter. A review of these materials revealed that there tended to be three terms presented when the data was in the form of a diagram and four terms were given when the data was presented as a table. These values were accepted for this study and used consistently throughout the study.

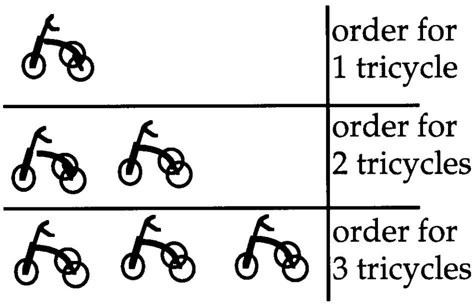
5. *Concrete materials.* As the problems associated with using concrete materials in a survey of over 1400 children seemed insurmountable it was decided to exclude their use from the study.

Of the five variables identified above only two were allowed to change within the study and the other three were not allowed to vary. This decision reflected the content of curriculum support materials for the N.S.W. syllabus. The major focus of all of these items was the in-context, two-operation items. Thus, two of these were included in the survey as it was felt that two items of the same theoretical structure would provide an opportunity to check on the reliability of the coding system that had to be developed. To provide a contrast with these two items, a single-operation, in-context item was included along with an out-of-context, two-operation question. The stimulus items are presented in Figure 3.2 along with the description and order in which they were presented. The order of questions was determined by placing the in-context, single-operation question first since it was hypothesised to be the easiest (this was confirmed in early trialling of the items). The other three questions were randomly allocated but the same order was retained for all children. With hindsight, it might have been preferable to randomly order the three items for each child as there appears to be evidence of attrition in the children's attempts. This would have spread the attrition across the items instead of focusing it in the later items.

Stimulus Item

QUESTION 1

A tricycle manufacturer needs to know how many wheels he needs for different sized orders.



Characteristics

Single Operation

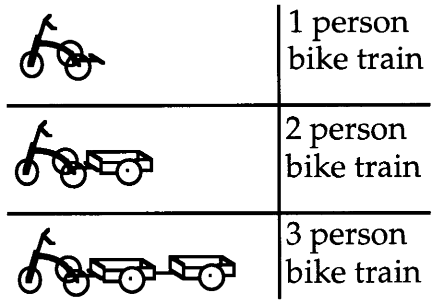
In context

Y=3X

Stimulus Item

QUESTION 2

The same tricycle manufacturer decided to make trailers for his tricycles so that you could make bike trains to carry lots of people.



Characteristics

Two operations

In context

Y=2X+1

Stimulus Item

QUESTION 3

I have a computer that turns the number in the top row into the number in the bottom row.

□	1	2	3	4		
△	2	5	8	11		

Characteristics

Two operations

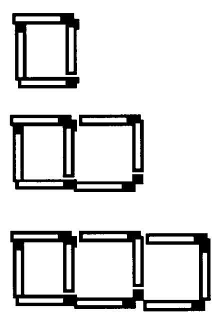
Out of context

Y=3X-1

Stimulus Item

QUESTION 4

Here are some chains of squares built using matches.



Characteristics

Two operations

In context

Y=3X+1

Figure 3.2

Stimulus Items and their Characteristics

Once the stimulus items had been determined it remained to develop the questions to be associated with each of them. In doing so it was felt it would be beneficial to keep in mind the need for a series of follow-up (confirmatory) interviews that were to take place in the longitudinal study.

Question Components

Newman (1977) developed an interview structure that was specifically designed for the analysis of student errors in mathematics. This structure has been further developed and used by others (Booth 1984; Casey 1978; Clements 1980) in a research environment to provide a means of probing children's performance in a number of mathematical contexts. While these studies focused on the "errors" students made, the interview schedule was designed to identify a set of procedures used by the subject in the problem solving process. Booth (1984) added some additional elements to the structure which clarify the child's general understanding of the problem. Hence it was deemed suitable for use in this study where the focus is not specifically on identifying errors, but rather to identify the range of children's responses and to seek from these the nature of children's understanding.

The elements of Newman's (1977, p. 243) structure include:

1. *Reading ability.* Can the students read the question?
2. *Comprehension.* Can they explain the meaning of the question?
3. *Strategy.* Can they select appropriate mathematics for the question?
4. *Process.* Can they perform the mathematical operations necessary for the task?
5. *Encoding ability.* Can they write the answer correctly?
6. *Careless Error.* Was there carelessness involved in arriving at the answer?

To these Booth (1984, p. 100) added:

7. *Consolidation.* Can the student clarify the meaning of the answer?
8. *Verification.* Can the student check to ensure their answer is correct?
9. *Conflict.* Are the answers to components of the question consistent?
10. *Generalisation.* Is there an overview of the strategies involved in the question?

While not all of the issues could be addressed in this survey, the structure provided guidance for the development of questions to be asked of the subjects with regard to each stimulus item. There were four components, each with a specific target. The components are as follows:

Component 1. Write the next term in the sequence.
Component 2. Generalise the pattern using natural language.
Component 3. Apply the generalisation to an uncountable example.
Component 4. Express the generalisation in symbolic notation.

Table 3.1 shows the relationship between these components and the structure developed by Newman (1977) and Booth (1984).

Table 3.1
Relationship Between Newman/Booth Structure and the Stimulus Item and Interview Components

Newman Structure	Survey Component
Reading	
Comprehension	Component A
Strategy	Component B
Process	Component C
Encoding	Component B and D
Careless error	
Consolidation	
Verification	Component C
Conflict	
Generalisation	Component B and D

There was considerable difficulty in developing the appropriate wording for Components B and D. When language was developed for the question components that removed ambiguity many children could not understand it. When children were asked to write a sentence many thought a number sentence was required. The word “generalise” was not understood by many children. While being cognisant of Mason's warning about “any number” being interpreted as any specific number (Mason 1985), a better word than “any” was unable to be found. The wording finally settled on for Component B that seemed to be generally understood by most pupils in early trials asked for a rule relating the dependent and independent variables. Both variables were explicitly mentioned in the question. For Stimulus Item 1, that was set in the context of a manufacturer of tricycles, Component B read as:

> Write an English sentence that tells me a rule for finding how many wheels he will need for a tricycle order of any size.

The intention of Component D was for children to demonstrate their ability to use symbolic notation. To encourage children to use algebraic notation it seemed desirable to ask specifically for algebra. However, while Years 7 and 8 children had all had some experience of algebra, the same was not true of the Year 5 and 6 children. Hence two forms of the questions were developed. Year 7 and 8 children were asked:

> Now write your rule for part (b) in algebraic symbols instead of words.

While Year 5 and 6 children were asked:

> Now write your rule for part (b) in maths symbols instead of words.

The wording for Component A and C was quite straightforward. Component A asked for the next term in the sequence. For stimulus item 1 it read:

> How many wheels will he need for an order for 4 tricycles?

Component C asked for an uncountable example:

> How many wheels would he need for 55 tricycles?

Copies of the survey forms are included as Appendix 3.3.

DATA ANALYSIS PLAN

At the beginning of this chapter the research questions of the study were presented as three themes. While each theme is related to the general issue of children's understanding of number patterns, they each require a different focus of data analysis. For Theme 1 the focus of the data analysis is on data reduction via coding of the children's responses. The analysis for Theme 2 involves discussion of the response patterns in the light of the SOLO Taxonomy and the development of a hierarchy of responses. The Rasch Partial Credit model (Adams & Khoo 1993) is used to sequence the components in order of difficulty. These results are then compared with predictions made using the SOLO Taxonomy. In Theme 3 the focus is on identifying relationships between variables. The technique of analysis for this theme is log-linear analysis. Some general issues associated with each of these three sets of analyses are presented below.

Coding and Frequencies

Coding the children's responses is a method of data reduction that facilitates an overview of the data and borrows from the technique of content analysis. It is suited to the analysis of this data since the focus of the study is to identify the range of responses that children make to the stimulus items, not to provide an interval level measurement of their ability to respond to the pattern items. It is the structure of the children's responses that needs to be identified and described. The coding is done initially by sorting the children's responses into similar groups, identifying the structural features of each group that separate it from other groups and, finally, calculating frequencies for each group.

Before the coding process can begin a number of issues need to be resolved with regard to the procedures used. They include:

- who does the coding,
- whether to use pre-coded categories or post-coded categories,
- what was to be the unit of coding,
- the number of categories to be used,
- how to establish intercoder reliability, and
- how to establish intracoder reliability.

The decisions and methods with regard to all of these issues are described in detail in Chapter 4.

The coded responses were transferred from the coding sheets to a computer environment by a data entry clerk. The description of categories and examples of each response group are reported in Chapter 4. The transfer of the data to a computer environment facilitated the use of SPSS (Norusis 1990) to produce frequency tables for each stimulus item and component. The SPSS platform also allowed the investigation of relationships between components which was the focus of Theme 3 and is discussed in the third part of this section following an outline of the methods used to compare the data with the SOLO Taxonomy.

Comparison of Data with the SOLO Taxonomy

Theme 2 identified a number of issues associated with comparing the coded data with the theoretical framework of the SOLO Taxonomy. The first part of this analysis is done qualitatively and compares the response categories of the natural language component (Component B) with the modes and levels of the SOLO Taxonomy. A model is described that provides for multiple pathways of development as children improve their ability to express generality of number patterns. This model is used to hypothesise an order of difficulty for all 16 components of the survey instrument.

In order to test the model, two quantitative indices for component difficulty are developed and the consistency of these indices is compared both with each other and with the order of difficulty hypothesised from the model. The first index was calculated by allocating an integer value to each of the response categories, which reflected the ordinal nature of the categories. The second index was calculated using the Rasch partial credit model (Andrich 1988). The application of this model is facilitated by the *Quest* software of Adams and Khoo (1993). The details of these methods and resultant analysis for the consideration of the hypotheses of Theme 2 are presented in Chapter 5.

Relationships

Theme 3 identified three relationships of interest in this investigation. In developing a technique for performing this analysis two major issues had to be considered. The first was the level of measurement of the data. The data was not interval or ratio level data and hence techniques for investigating relationships between variables such as ANOVA and MANOVA could not be considered since the data contravened the assumptions underpinning these procedures (Cooksey 1984; Manly 1986).

The second issue was the systematic investigation of the data to identify any possible interaction effects between the variables. In Theme 3 there are four variables identified, namely, natural language (Component B), calculating an uncountable example (Component C), use of symbolic notation (Component D) and the children's school year (class). These variables are to be investigated within three relationships, namely, natural language (Component B) by class, natural language by calculating an uncountable example (Component C) and natural language by use of symbolic notation (Component D). If these three relationships were investigated as a series of two-way tests the identification of possible variance within the two-way test being attributable to either of the other variables would not be facilitated. That is, possible interaction effects between variables would be ignored.

To address both of these issues, log-linear analysis (Norusis 1990; Everitt 1977; Gilbert 1981) was used to systematically identify any significant relationships between the variables. The SPSS (Norusis 1990) computer platform was used to facilitate the modelling. The log-linear analysis had four major advantages.

1. It could be applied to nominal level polychotomous data.
2. It facilitated the identification of a parsimonious model to represent the significant relationships between variables.
3. It provided parameter estimates of relationships between categories of the variables.
4. The relationships could be reported in the form of contingency tables with measures of both strength and direction of association, together with response surfaces, to provide a graphical representation of the data.

A detailed description of the processes of model building and the resultant measures of association and parameter estimates are reported in Chapter 6.

A variety of data analysis strategies have been used in this study. They have borrowed from the traditions of both qualitative and quantitative research. While this eclectic approach adds complexity to analysis, it also adds validity via methodological triangulation.

With this brief outline of the plan for the analysis of the data completed the strengths and limitations of the research design are discussed in the next section.

EVALUATION OF THE DESIGN

As a review of the preceding description of the design it is useful to discuss aspects of the design in terms of both its strengths and weaknesses. This task is undertaken in this section by considering the validity and reliability of the design.

Validity

Validity is most commonly discussed as two separate but related concepts, external validity and internal validity. While this dichotomy is usually applied to an experimental environment, LeCompte and Preissle (1993) have applied it to a wider range of research settings. External validity considers to what extent the research findings are applicable across groups. An extension of this concept is: to what populations do the results generalise? Internal validity in this non-experimental context is concerned with the extent to which there are shared meanings between the participants and the observer. In the discussion that follows, both types of validity are discussed using a number of threats to validity identified by LeCompte and Preissle (1993). These threats are representative of the literature generally and are similar to the work of Campbell and Stanley (1963) and Cook and Campbell (1979). While the threats do not exclusively influence either internal validity or external validity, they do provide a systematic framework for evaluating the research design.

External Validity

The threats to external validity identified by LeCompte and Preissle (1993) are selection effects, setting effects, history effects and construct effects. Each of these is discussed in turn.

Selection Effects. It was argued earlier that the large sample size of approximately 1400 children enhanced the external validity of the research. However, the fact that the sample was not a random sample from a variety of geographical locations is a limitation to the ability to generalise from the findings. The extent of this limitation is not directly measurable, although, it was reported that a wide variety of schools and school systems were included in the study, all of which improves the generalisability of the results.

Setting Effects. "Investigators affect the groups, cultures and settings they study just by studying them" (LeCompte & Preissle 1993, p. 350). The sample was from N.S.W. schools and hence the influences of the N.S.W. curriculum and school system restricts the external validity of the results. Other school systems might well place a different emphasis on the curriculum content area being investigated. Biggs and Collis (1982) argue that learning experiences influence children's progression through levels of the SOLO Taxonomy. Hence, it is possible that children from other systems, not subject to the *setting effects* of the N.S.W. curriculum, could provide a different pattern of responses. However, the common curriculum in N.S.W. provides support for the view that the results should be applicable to a much wider group than the towns of the convenience sample. Additionally, the response rate of over 90% ensures that the targeted group was adequately represented.

History Effects. This is seen as a threat to external validity when groups have different historical experiences. This is considered in two ways in the context of this study. The first is the varying curriculum experiences of school groups that have been discussed above. The second is the variation in responses that is due to the different ages of the sample. However, since this is a variable being specifically addressed in the study it should not be seen as a threat to validity. Thus, no new threats to external validity can be attributed to *history effects.*

Construct Effects. This effect is defined in two ways:

> How effects of observed phenomena are construed ... (or) ... the degree to which instructions for and formats of instruments are mutually intelligible to the instrument designer, to the instrument administrator, and the participants to whom the instrument is applied.
>
> (LeCompte & Preissle 1993, p. 352)

The placing of this item within the category of external validity by LeCompte and Preissle is problematic as both definitions are closely related to the definition of internal validity reported above. It might be seen, more appropriately, to span both internal and external validity. However, wherever it is placed in a taxonomic sense it points to some important issues that should be considered.

When a student's response is coded there seems to be an assumption that the response represents a 'best effort' on behalf of that student. There are at least two threats to this assumption. The first is what is meant by a 'best effort'. Some may interprete this as the simplest response, while others may interpret 'best effort' to mean the most mathematically sophisticated response. Thus, a response may represent less than a child's 'best effort' as judged by the researcher. Secondly, some children and teachers may not take the task as seriously as might be hoped. Hence, flippant answers, or ineffective supervision to prevent copying may have occurred. Closer supervision by the researcher to reduce these effects would have resulted in a smaller sample size due to constraints of time and cost and would have reduced the naturalness of the setting.

The issue of shared meanings of the stimulus items was addressed in two ways. The first was a series of trials of the stimulus items with small groups of children from a primary school that was not included in the main study. While a variety of responses was always provided, the researcher satisfied himself and an independent observer that the children were interpreting the items in the way intended. The second method of considering this matter was in checking the child's response to Component A of each stimulus item. If children were to see a different, but correct, pattern than the one intended by the researcher then this should have been indicated in their response to Component A, and the necessary adjustments to interpreting later responses could be made.

Internal Validity

LeCompte and Preissle (1993) listed five threats to internal validity. They were history and maturation, observer effects, selection and regression, mortality and spurious conclusions. Some of these effects are not at issue due to the circumstances of this study and have considerable overlap with the threats identified in the discussion of external validity above.

History and maturation and *Mortality.* The issue of history effecting external validity was discussed above and referred to variability in the background experiences of groups. In this context LeCompte and Preissle refer to variations in experiences during the conduct of the study. This cannot affect the survey as it represents a 'point in time' rather than a period of time. This is considered further in the discussion of the longitudinal study described in Chapter 7. Mortality refers to attrition of the sample during the study and is also considered in Chapter 7.

Observer effects. The effect is closely related to the naturalness of the setting discussed earlier. The decision to use the class teacher to administer the survey instrument enhances a natural setting in that the children should have felt relaxed and comfortable with their class teacher. On the other hand, using the class teacher reduces the ability to ensure that the instrument was applied consistently across the sample. The net effect of these countervailing pressures on the validity of the design is indeterminate.

Spurious conclusions. This threat has been interpreted to mean the accuracy of the interpretation placed on children's responses by the researcher. In the limitations of survey designs presented early in this chapter the issue of superficial data was identified as a potential problem. This study attempts to address this issue in two ways. The first is to focus on coding the responses in a way that addresses the structure of the responses, rather than the dichotomous right/wrong classification. The details of the coding process are described in Chapter 4. The second, is to conduct interviews with students in the longitudinal study to confirm that the coding categories reflect children's intentions.

Three additional measures are taken to assist in avoiding making spurious conclusions. The first is to compare identified categories with those of other researches using similar items. The second is to compare the findings with the theoretical framework. As the SOLO Taxonomy describes the structure of the responses it should help validate the classifications of the data. Finally, statistical conclusions are based on conservative alpha values. The calculation of the alpha value and the associated family-wise error rate are described in Chapter 6.

This part has discussed the design of the study in terms of internal and external validity. While it is accepted that "perfect validity" is impossible to attain (Wiersma 1991), a number of measures have been described that attempt to improve the validity of this study. Additionally, a number of matters have been identified that are potential threats to validity that remain in the study design. The next part shifts the focus to the related issues of internal and external reliability.

Reliability

Reliability "refers to the extent to which studies can be replicated" (LeCompte & Preissle 1993, p. 332) and can be seen as an elusive concept in research that does not follow "those standardised controls so essential to experimental research" (LeCompte & Preissle 1993, p. 332). Data collected from natural settings, while not being subject to the distortions of manipulation and control of experimental settings, is sometimes considered to be idiosyncratic. It is consideration of issues of reliability that establishes the consistency of:

- the methods used,
- the conditions under which the research was undertaken, and
- the data analysis.

Reliability is a necessary prerequisite for validity. Results that lack reliability cannot be interpreted with confidence or generalised to other settings. Like validity, reliability is often discussed as internal and external reliability (Wiersma 1991; LeCompte and Goetz 1982).

External Reliability

External reliability refers to the ability of independent researchers to replicate the study in the same or similar settings. This ability is enhanced by the detailed description of the research design and procedures used in the research being made explicit. The breadth of the study and the use of classroom teachers to collect data using a standard instrument should have removed idiosyncratic influences from the design. That is, the "researcher status position" (LeCompte & Preissle 1993, p. 334) has been removed from influencing children's responses directly. The researcher's ability to seek the cooperation of the teachers may be a variable influencing possible replication; however, no problems in this regard were apparent.

Internal Reliability

Internal reliability in the context of this study refers to the consistency of data collection and analysis. Two specific measures were taken to monitor the consistency of the coding procedures - intercoder reliability and intracoder reliability.

Intercoder reliability is defined as the ability of two or more coders to agree on the codings of a set of responses. A reliability index was used to establish consistency in both the coding procedures and the clarity of coding definitions. This index expressed the ratio of the number of coding agreements as a percentage of the number of items coded and a criterion of 90% was identified as an acceptable value for the index

Intracoder reliability refers to the ability of a single coder to provide consistent analysis of data over time. Changes in the perception of responses occur as the coder becomes more familiar with the data. This, together with monitoring for careless errors, was the issue addressed using the intracoder reliability index. The index was calculated in a similar manner to the intercoder reliability index. The same criterion of 90% was applied to this index.

Detailed discussion of the processes of calculation of these indices and some specific threats to internal reliability are reported in the next chapter when the issue of coding procedures is discussed in detail.

SUMMARY

This chapter has had three purposes. The first was to present four research themes and the related research questions that were deemed to be of interest and of sufficient importance to warrant further investigation. These themes were derived from the review of literature in Chapter 1 and the identification of a suitable theoretical framework in Chapter 2.

The second purpose of the chapter was to present a research design for investigating Themes 1, 2 and 3. Further consideration of Theme 4 is delayed until Chapter 7. The methods underpinning the study are borrowed from both the qualitative and quantitative traditions of educational research. The major focus of the study is the structure of children's natural language descriptions of number patterns and is firmly placed in the qualitative tradition. Following coding and the calculation of category frequencies, the quantitative technique of log-linear analysis is to be used to build a model showing the relationship between the variables of the survey.

Eight limitations of the survey as a research strategy were identified. These limitations were considered when decisions with regard to the survey procedures were made, in an attempt to minimise their effect. These procedures were reported in considerable detail so as to facilitate replication of the study and thus enhance external reliability.

The final purpose of the chapter was to provide an evaluation of the research design. The concepts of validity and reliability were used as a framework for this evaluation. Validity and reliability have traditionally been applied in an experimental setting and their application to a qualitative environment is a little more problematic. Some practices designed to reduce one threat to validity may increase the effect of another threat to validity. This, together with the limitation of resources, results in it being impossible to produce a perfectly valid and reliable design.

The following three chapters present the data analysis. Chapter 4 provides description of the coding procedures and the resultant category frequencies. Chapter 5 considers the data in the context of the SOLO Taxonomy, while Chapter 6 reports the model building procedures of log-linear analysis .