Chapter 9

Selected Discussion and Comments

9.1 Introduction

The purpose of this Chapter is to present a brief discussion on topics which are related to the experimental design, discussed in Chapter 5, and/or outcomes of this study. Like any other Monte Carlo study, the outcomes of this experiment are influenced by the design of the experiment. That is, the various conclusions that are drawn based on the outcomes of the experiment are subject to the specific form of the model as well as to the various component assumptions made in the experimental design. Thus, before making final conclusions on the outcomes of the study, it is important to examine the restrictiveness of some of the important assumptions made in designing the experiment. For example, as discussed in Chapter 5, the observations on the explanatory variables x_1 and x_2 are kept fixed for all replications in the Monte Carlo experiment. This assumption can be restrictive given that the explanatory variables in a tobit model have a random character. It is perhaps reasonable to raise the question what happens if the explanatory variables were allowed to vary from sample to sample. Similar issues can be raised that are related to the various components of experimental design such as the direction of censoring.

Specifically, this Chapter focusses on the effects of the following on the outcomes of the experiment.

(i) The effects of considering random explanatory variables as compared to fixed explanatory variables.

(ii) The effects of left-hand censoring versus right-hand censoring of observations.

(iii) The importance (effects) of the constant term in the estimation of results associated with the tobit model.

The Sections below present a short discussion on these issues and provide more information in relation to each of the above points. Note that these discussions are not exhaustive. Nevertheless, they provide important information regarding the flexibility of the experimental design as well as the outcomes of the experiment.

9.2 Random Explanatory Variables

Recall that, as discussed in Chapter 5, the experiment in this study is designed so that the observations on the explanatory variables x_1 and x_2 do not vary from sample to sample. This is designed in order to enable one to investigate the effects of violations of the assumptions about the error term with minimum variation due to other factors including the explanatory variables.

However, in practice, this assumption can be restrictive because of the random character of the explanatory variables in the tobit model. For example, in a household expenditure study where the expenditure on durables, y, is expressed as a function of income, x, it is reasonable to treat the x's as random because, in practice, (y, x) is selected at random through something like a random household survey. Thus, given this and other similar examples, it is relevant to raise this issue in our experiment and ask what happens if the observations on the explanatory variables are allowed to change from sample to sample. That is, whether the outcomes from the experiment will be affected significantly by considering random explanatory variables.

One way of assessing this effect is to produce results from an experiment which allows the observations on the explanatory variables to vary from sample to sample and compare these results with those obtained by using fixed explanatory variables (i.e., based on the original design of the experiment). In order to examine this, the experimental design which is discussed in Chapter 5 is modified in such a way that a new set of explanatory variables are selected in each replication of the experiment. Note that all other details of the experimental design remain the same except in this case a new set of explanatory variables are generated for each of the 3000 samples in the Monte Carlo experiment, so that the results can be comparable to those obtained from the original design using fixed explanatory variables. The summary statistics obtained using random explanatory variables are given in Table 9.1 for selected estimators.

Table 9.1 depicts results for H2S, 3SE and MLE using random explanatory variables for the three distributions and a 25% degree of censoring. The interpretation of Table 9.1 is similar to the results reported in Chapter 6 (for example see Table 6.1 in Chapter 6). That is, we obtained the estimated mean (EM), the standard error (SE), the bias (BIAS) and the root mean square error (RMSE) for each coefficient. Furthermore, since the objective is to compare these results with their corresponding estimates obtained based on fixed explanatory variables, the relative root mean square errors (RRMSE) is computed for each coefficient to indicate any changes in reliability of the estimates. The RRMSE is the ratio of the RMSE of a coefficient obtained using random explanatory variables to its corresponding RMSE obtained when the explanatory variables were fixed. That is,

$$RRMSE(\hat{\beta}_k) = \frac{RMSE(\hat{\beta}_k)|\text{random X's}}{RMSE(\hat{\beta}_k)|\text{fixed X's}}, \quad k = 1, 2.$$

As stated above, the RRMSE is used to indicate whether there exists any significance difference between the RMSE obtained with and without allowing the explanatory variables to vary from one sample to the other. For example, as shown in Table 9.1, given a sample size of 100, a 25% degree of censoring and normally distributed errors, the RMSE of $\hat{\beta}_1$ using the H2S estimator is given by 0.206. The corresponding RMSE using fixed explanatory variables is equal to 0.190 (see Table 6.1 in Chapter 6). The RRMSE is therefore the ratio RRMSE=0.206/0.190=1.084. Others are obtained in a similar way.

Given the results tabulated in Table 9.1, one can make the following general points. Under normality, the MLE estimator performs the best followed by the 3SE. The difference between the MLE and 3SE is quite marginal. It is also clear that the MLE is less efficient under the skewed distribution compared to the symmetric distributions. These conclusions are generally consistent with earlier results (see Chapter

Table 9.1: Results for Estimators using Random Explanatory variables, given a 25% Degree of Censoring for the three Distributions.

Sample Size	Estimator		Normal		Students'-t		Chi-Square	
			β_1	β_2	β_1	β_2	β_1	β_2
(1)	(2)		(3)	(4)	(5)	(6)	(7)	(8)
	True values		1.000	1.000	1.000	1.000	1.000	1.000
100	H2S	EM	0.990	0.989	1.141	1.108	1.018	1.010
		SE	0.205	0.258	0.416	0.404	0.192	0.255
		BIAS	-0.010	-0.011	0.141	0.108	0.018	0.010
		RMSE	0.206	0.258	0.437	0.418	0.193	0.255
		RRMSE	1.084	1.102	2.526	1.883	0.869	1.099
	3SE	EM	0.985	0.986	0.957	0.958	0.930	0.934
		SE	0.114	0.197	0.208	0.294	0.123	0.220
		BIAS	-0.015	-0.014	-0.043	-0.042	-0.070	-0.066
		RMSE	0.115	0.198	0.212	0.297	0.141	0.229
		RRMSE	1.008	1.064	1.812	1.650	1.021	1.060
	MLE	EM	1.001	1.001	1.059	1.032	1.044	1.035
		SE	0.097	0.189	0.188	0.247	0.112	0.212
		BIAS	0.001	0.001	0.059	0.032	0.044	0.035
		RMSE	0.097	0.189	0.196	0.249	0.121	0.215
		RRMSE	0.980	1.068	1.921	1.390	1.034	1.086
200	H2S	EM	0.987	0.990	1.035	1.030	1.023	1.006
		SE	0.143	0.181	0.133	0.165	0.133	0.177
		BIAS	-0.013	-0.010	0.035	0.030	0.023	0.006
		RMSE	0.143	0.181	0.137	0.168	0.135	0.177
		RRMSE	0.947	0.978	0.958	0.977	0.978	0.978
	3SE	EM	0.990	0.993	1.014	1.012	0.935	0.933
		SE	0.078	0.141	0.083	0.137	0.084	0.152
		BIAS	-0.010	-0.007	0.014	0.012	-0.065	-0.067
		RMSE	0.079	0.141	0.085	0.137	0.106	0.166
		RRMSE	0.988	0.953	1.000	1.000	0.972	1.044
	MLE	EM	0.999	1.002	1.022	1.025	1.043	1.029
		SE	0.067	0.132	0.072	0.132	0.076	0.146
		BIAS	-0.001	0.002	0.022	0.025	0.043	0.029
		RMSE	0.067	0.132	0.075	0.0132	0.087	0.149
		RRMSE	0.957	1.023	1.000	1.031	1.080	1.000

6). However, the main interest here is that whether there is any significant difference in reliability resulting from the random character of the explanatory variables. As can be seen from the values of the RRMSE, the reliability of the MLE and 3SE estimates remain almost the same as those obtained using fixed explanatory variables in the original design for all samples, provided that the errors are normally distributed. The RRMSEs of the MLE and 3SE estimators under the chi-square distribution also reveal that the outcomes of the experiment are not affected significantly by varying the explanatory variables and provide almost identical results as the sample size increases. However, Table 9.1 also indicates that the reliability of the estimators can deteriorate significantly under the students'-t distribution if the sample size is small and the X's are random. It is, however, interesting to note that as the sample size increases both experiments (i.e., with or without varying the observations on the explanatory variables) provide almost identical results for all estimators and the three distributions (i.e. the RRMSE $\simeq$ 1.000 in all cases). For example, given a sample size of 200, a 25% degree of censoring and the students'-t distribution, the RMSEs of the coefficients for the 3SE using random explanatory variables are identical to those obtained using fixed explanatory variables (i.e., RRMSE=1.000).

In general, the results in Table 9.1 depict that there is enough evidence to conclude that the randomness of the explanatory variables, compared to fixed explanatory variables, has little or no effect on the outcomes of the experiment. That is, the variations in results are mainly attributed to the changes in error structure as well as factors such as the degree of censoring and sample size. In other words, the main conclusions drawn from the original experimental results (i.e., Chapter 6 to 8) would not be altered if one had to allow the explanatory variables to vary from sample to sample instead of using fixed explanatory variables.

9.3 Left Vs Right Hand Censoring

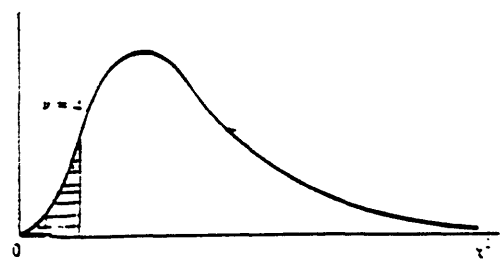
One of the important components in this study is investigating the effects of the degree of censoring on the performance of the various estimators. Recall that the degree of censoring represents the proportion of observations corresponding to the zero values of the dependent variable, i.e., $y_i = 0$ to the total number of observations in the sample. As discussed in Chapters 6 to 8, the quality of the estimators decline in general for higher degrees of censoring. However, the main interest here is not the level (degree) of censoring but its direction.

In this study we considered three distributions for the error term of which two of them are symmetric (i.e., the normal and the students'-t distributions) and a skewed (i.e., the chi-square) distribution. The combined effects of the error distributions and the degrees of censoring on the performance of the estimators depend, among other things, on the type of distribution, the level of censoring and sample size. However, another important aspect which is particularly relevant to the chi-square distribution is the direction (left versus right) of censoring of the observations. It is, perhaps, important to examine the direction of censoring in relation to the skewed distribution (i.e., the chi-square distribution) and its effects on the outcomes of the experiment for the following reasons.

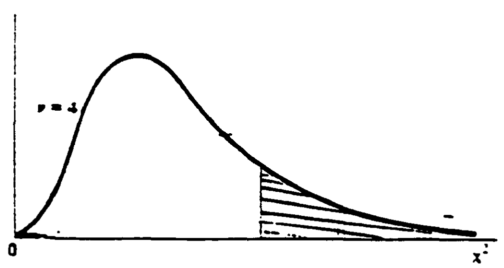
In general, there are two likely situations that exist between the direction of censoring and the error distributions considered in this experiment. If the errors are symmetric, then both left-hand and right-hand censoring are likely to have the same effects on the outcomes of the experiment. However, this may not be true for the skewed distribution. That is, left and right-hand censoring are likely to have different effects on the observations generated from the skewed distribution and hence the outcomes of the experiment. This can be explained intuitively using Figure 9.1 (a) and (b) below. Figure 9.1, (a) depicts a left-hand censoring of the chi-square distribution with four degrees freedom. The shaded part indicates the observations which correspond to the zero values on the dependent variable y_i. In this case, it is clear that the skewness (right-hand tail) of the distribution is less likely to be affected by the censoring of the observations. That is, the observations that contribute more towards the skewness of the distribution are not censored as the censoring is in the opposite tail of the distribution. On the other hand, Figure 9.1 (b) shows when the direction of censoring is to the right, a situation where both censoring and the tail (skewness) of the chi-square distribution are in the same direction. That is, in (b) we have a situation where the observations which contribute most to the skewness of the chi-square distribution are being censored. In other words, as can be seen from Figure 9.1 (b), the chi-square distribution becomes less skewed and perhaps may be approximated by the normal distribution. This implies that, just by comparing the above Figures (a) and (b), the effects of the direction of censoring are likely to be different under the chi-square distribution. Specifically, it can be said that the results from a right-hand censored chi-square distribution (i.e., Figure 9.1, (b)), compared to a left-hand censoring (Figure 9.1, (a)), are likely to be closer to those of the symmetric (or normal) distribution. Thus, it is important to examine whether the results based on a right-hand censoring of observations are significantly different from those obtained using left-hand censoring (i.e., the original design of the experiment). This

Figure 9.1: Left-hand Vs Right-hand Censoring of the Chi-square Distribution.

(a) Left-hand Censored Chi-square curve



(b) Right-hand Censored Chi-square curve



question has important implications in providing more information on whether the skewed distribution is actually represented or not in the experiment and its relationship with the direction of censoring. In other words, it enables one to check whether the data generation process is free of bias against any of the distributions considered in the experiment.

As before, it is possible to examine the effects of using right hand censoring instead of left-hand censoring by making some adjustments to the experimental design. Note that the design of the experiment, discussed in Chapter 5, is based on a left-hand censoring which is similar to Figure 9.1 (a). The main purpose is to modify the experiment in such a way that the censoring and the tail (skewness) of the chi-square observations overlap with each other as in Figure 9.1 (b). A simple way of having such a design is to multiply the chi-square observations by negative one so that the tail of the distribution will lie on the same direction as the censoring. This is the same as Figure 9.1 (b) except in this case the tail of the distribution will be to the left instead of to the right-hand side. In this way one can examine the two cases easily without changing the remaining details of the experimental design.

Following this procedure, we estimated results for selected estimators and the summary statistics are given in Table 9.2. Table 9.2 depicts results for the H2S, 3SE and MLE estimators based on a right hand censoring of the chi-square distribution, given a 25% degree of censoring and for the small and medium sample sizes. It is important to note that the term 'right-hand censoring' is used to indicate the situation in Figure 9.1 (b), where the observations at the right-hand tail of the distribution are censored. The results in Table 9.2 also include two other conditions. As can be seen from the table, Columns (3) and (4) depict results for the estimators when the

Table 9.2: Results for Estimators based on a 25% Left-hand Censoring for the Chi-square Distribution.

Sample Size	Estimator		Fixed X's β_1	Fixed X's β_2	Random X's β_1	Random X's β_2
(1)	(2)		(3)	(4)	(5)	(6)
	True values		1.000	1.000	1.000	1.000
100	H2S	EM	0.955	0.963	0.960	0.969
		SE	0.183	0.248	0.200	0.256
		BIAS	-0.045	-0.037	-0.040	-0.031
		RMSE	0.189	0.253	0.204	0.258
		RRMSE	1.092	1.090	1.179	1.112
	3SE	EM	1.042	1.057	1.042	1.052
		SE	0.101	0.183	0.101	0.179
		BIAS	0.042	0.057	0.042	0.052
		RMSE	0.110	0.192	0.109	0.187
		RRMSE	0.797	0.889	0.790	0.866
	MLE	EM	0.981	0.991	0.979	0.989
		SE	0.079	0.168	0.079	0.161
		BIAS	-0.019	-0.009	-0.021	-0.011
		RMSE	0.081	0.168	0.082	0.162
		RRMSE	0.692	0.848	0.700	0.818
200	H2S	EM	0.962	0.974	0.968	0.979
		SE	0.136	0.174	0.135	0.175
		BIAS	-0.038	-0.026	-0.032	-0.021
		RMSE	0.142	0.176	0.139	0.177
		RRMSE	1.029	0.928	1.001	0.978
	3SE	EM	1.047	1.059	1.051	1.061
		SE	0.069	0.123	0.069	0.127
		BIAS	0.047	0.059	0.051	0.061
		RMSE	0.084	0.136	0.086	0.141
		RRMSE	0.771	0.855	0.789	0.887
	MLE	EM	0.979	0.988	0.981	0.990
		SE	0.054	0.109	0.054	0.114
		BIAS	-0.021	-0.012	-0.019	-0.010
		RMSE	0.058	0.110	0.057	0.114
		RRMSE	0.617	0.738	0.606	0.765

explanatory variables are fixed. That is, the only change on the original design is the direction of censoring of the chi-square errors. The remaining Columns (5) and (6) list results when one uses random explanatory variables as well. Finally, since the interest is to compare these results with those obtained based on the original design of the experiment we have computed the relative root mean square error (RRMSE) for each coefficient. As before, the RRMSE of a coefficient is the ratio of the RMSE obtained with the changes in the direction of censoring to its corresponding RMSE obtained from the original experiment. Specifically, the RRMSE is given by

$$RRMSE(\hat{\beta}_k) = \frac{RMSE(\hat{\beta}_k)|\text{RHC}}{RMSE(\hat{\beta}_k)|\text{LHC}}, \quad k = 1, 2.$$

where RHC and LHC indicate the right-hand (as in Figure (b)) and left-hand (as in Figure (a)) censoring of the chi-square observations, respectively.

For example, as shown in Table 9.2, given a sample of 100 and a 25% degree of censoring, the RMSE of $\hat{\beta}_1$ using the H2S estimator is 0.189. Its corresponding value based on the original design was 0.173 (see Column (7) of Table 6.1 in Chapter 6). The RRMSE is therefore given by the ratio RRMSE=0.190/0.173=1.092. Similarly, the RRMSE for $\hat{\beta}_2$ is given by 1.090. These ratios are relatively larger if one allows the explanatory variables to vary from sample to sample as well (see Table 9.2, Columns (5) and (6)). These results imply that the reliability of the H2S estimator declines if one uses right-hand (instead of a left-hand) censoring of the chi-square distribution. This is contrary to our expectations. However, as the sample increases the H2S estimator yields similar results in both cases (e.g., when N=200 the RRMSE$\simeq$1.000 in all cases).

On the other hand, the 3SE and the MLE estimators reveal more interesting results. In all cases, the RMSE of the coefficients are less than their corresponding RMSE obtained based on the original design; implying an improvement in reliability of the 3SE and MLE estimates under the chi-square distribution, resulting from changes in the direction of censoring. Specifically, when the chi-square distribution is censored towards its tail then the RMSE of the 3SE and MLE estimates can improve up to 20 and 40 percent, respectively, compared to the results obtained from the original design. This is not surprising because, as discussed earlier in this Section, there appears to be little skewness on the chi-square errors and hence improved results. In fact, the results for the 3SE and MLE given in Table 9.2 are almost the same (sometimes even identical) to those obtained under normality of the errors in the original experiment (see Tables 6.1 and 6.9, Chapter 6).

In general, there appears to be enough evidence to conclude that the direction of censoring is important as far as the chi-square distribution is concerned. That is, if the direction of censoring coincides with the tail of the chi-square distribution then the results under the chi-square distribution are likely to the same (or quite close) to those obtained under the normal distribution. In other words, the results may not actually indicate the effects of the skewness of the distribution. This has an important implication for similar other studies which involve censoring (or truncation) and skewed distributions.

9.4 The Effects of the Constant Term

As discussed in Chapter 7, the Monte Carlo results regarding the consistency (inconsistency) of the MLE indicated that in the presence of misspecification there appeared

appeared to be little inconsistency in the coefficients of the model except for the constant term. That is, if one assumes normality when in fact the errors are non-normal (i.e., for the students'-t and the chi-square distributions), then the inconsistency in the constant term, β_0, can be substantial while it is negligible for the remaining coefficients in the model. On the other hand, it should be clear that, unlike the regular regression model, one cannot generally assume that the constant term is not important because the constant term is involved in estimating other results which are associated with the tobit model. In other words, as discussed in Chapter 2, Section 2.10, the constant term of the tobit model plays an important role in deriving useful results that are associated with the tobit model. Thus, it is difficult to make general conclusions regarding the consistency (inconsistency) of the MLE without having further information on the likely effects of the inconsistency in the constant term on these results.

The purpose of this Section is therefore to examine the uses of the constant term and its likely effects if estimated inconsistently. That is, to examine the consequences of inconsistency of the constant term in the estimation of results such as the rate of change in the dependent variable, y_i, for a unit change in x_j, $j = 1, 2$. In order to discuss this further let us reiterate some of the main results that are associated with the tobit model first. These results include (suppressing the observation subscript):

(i) The effects of unit change in x_j on y_i^*, given by

$$\frac{\partial E[y_i^*]}{\partial x_j} = \beta_j, \quad j = 1, 2. \tag{9.1}$$

(ii) The effects of a unit change in x_j on y_i,

$$\frac{\partial E[y_i]}{\partial x_j} = F(x_i'\alpha).\beta_j \tag{9.2}$$

(iii) The effects of a unit change in x_j on the conditional expectation of y_i,

$$\frac{\partial E[y_i|y_i > 0]}{\partial x_j} = \beta_j \, R(x_i'\alpha), \tag{9.3}$$

where

$$R(x_i'\alpha) = \left[1 - (x_i'\alpha).\frac{f(x_i'\alpha)}{F(x_i'\alpha)} - \left(\frac{f(x_i'\alpha)}{F(x_i'\alpha)}\right)^2\right] \quad \text{and} \quad \alpha = \beta/\sigma.$$

Note that the equations (9.1)-(9.3) show that all coefficients including the constant term play an important role in deriving the above derivatives or response equations. More specifically, if one is interested in computing the response of the latent variable, y_i^*, as a result of a unit change in x_j, then the constant term has no relevance. However, the main interest is to examine those equations which involve the constant term, i.e., equations (9.2) and (9.3). For example, if the inconsistency in the constant term has very little effect on the values of the equations (9.2) and (9.3), then one can conclude that inconsistency is not a problem even if the normality assumption of the error term does not hold. Note that, in the discussions below, equations (9.1)-(9.3) will be referred frequently as response equations or simply responses.

It is possible to examine the likely effects of inconsistency of the constant term on (9.2) and (9.3) without requiring Monte Carlo results from the experiment. In other words, one can examine the effects of the constant term based on the true values as illustrated below.

Consider a particular case from the experimental design discussed in Chapter 5, say, a sample size of 200 and a 25% degree of censoring. This implies $\beta_0 = -0.75$, $\beta_1 = \beta_2 = 1$. And let $\bar{x}_1$ and $\bar{x}_2$ be the means of the observations on the explanatory variables x_1 and x_2, respectively. Then, evaluating (9.2) and (9.3) at these true values, we have

$$\frac{\partial E[y_i]}{\partial \bar{x}_j} = F(.).\beta_j = 0.8970 \tag{9.4}$$

and

$$\frac{\partial E[y_i|y_i > 0]}{\partial \bar{x}_j} = R(.).\beta_j = 0.7079 \tag{9.5}$$

It is important to note that the values given above remain the same whether the derivatives are with respect to x_1 or x_2 simply because $\beta_1 = \beta_2 = 1$ in this particular case. Note that the values in (9.4) and (9.5) can be viewed as the changes in the unconditional and conditional expectation of the dependent variable, respectively, per unit change in x_j when no inconsistency exists in the coefficients; and hence they can be used as benchmark values for comparing inconsistency. Since the interest is to examine the effects of inconsistency in β_0 we can then vary the constant term by a certain proportion (upwards or downwards) to account for inconsistency and compare the resulting estimates with those obtained assuming no inconsistency. That is, we can compute new sets of values for equations (9.2) and (9.3) by considering different values for β_0 (but keeping $\beta_1 = \beta_2 = 1$) and these new sets of estimates can be compared with their corresponding true (benchmark) values given by (9.4) and (9.5).

For example, suppose there exists a 10 percent bias in β_0 (i.e., the new value for β_0=1.10×-0.75=-0.825). Then, the resulting new set of estimates for the responses (9.2) and (9.3) will be 0.8829 and 0.6862, respectively. Then, in order to compare these new estimates with their corresponding benchmark values given above, we compute the percent changes in the responses that resulted from the 10 percent inconsistency in β_0 as follows:

The percentage change in $\partial E[y_i]/\partial \bar{x}_j = F(.).\beta_j$ as a result of a 10 percent change in β_0 is given by

$$\%\Delta \,\text{in}\, F(.)\beta_j = [(0.8790 - 0.8829)/0.8790] \times 100 = 1.57\%$$

Similarly, the percentage change in $\partial E[y_i|y_i > 0]/\partial \bar{x}_j = R(.).\beta_j$ as a result of a 10 percent change in β_0 is

$$\%\Delta \,\text{in}\, R(.)\beta_j = [(0.7079 - 0.6862)/0.7079] \times 100 = 3.06\%$$

In other words, a 10 percent inconsistency (bias) in the constant term is likely to cause up to 1.57 and 3.06 percent changes in (9.4) and (9.5), respectively. Similar comparisons can be made by considering varying proportions of inconsistency in β_0.

Recall that, as discussed in Chapter 7, there appeared to be around 20% inconsistency in β_0 if one assumes normality when the errors are generated from the non-normal distributions (i.e., the students'-t and chi-square distributions). In line with this, results for various proportions covering up to 25 percent (upward as well as downward) inconsistencies in β_0 are summarized in Table 9.3 below. Column (1) of the table lists changes in β_0 ranging from -25 to +25 percent. The corresponding values for the response of y_i per unit change in x_j, i.e., $F(.)\beta_j$, are provided in Column (2) and the percentage changes in $F(.)\beta_j$ as a result of the changes in β_0 are given in Column (3). Similarly, Columns (4) and (5) present the value for $R(.)\beta_j$ and the percentage changes in $R(.)\beta_j$, respectively. The negative and positive signs in Columns (3) and (5) indicate whether the effects on the responses are downward or upward effects, respectively. For instance, if β_0 changes by 25% downwards (i.e. the new value for β_0 is 1.25 $\times$-0.75=-0.9375) then the value of the response $F(.)\beta_j$ changes from 0.8790 to 0.8593, implying a 4.20 percent downward effect. Similarly, the value for $R(.)\beta_j$ is given by 0.6533 implying a 7.71 percent decrease in the value

Table 9.3: The likely Effects of Changes in the Constant term (β_0) on Responses associated with the Tobit Model for N=200 and 25% Degree of Censoring

Changes in β_0	$F(.)\beta_j$ (0.8970)*	%Δ in $F(.)\beta_j$	$R(.)\beta_j$ (0.7079)*	%Δ in $R(.)\beta_j$
1.25 × -0.75	0.8593	-4.20	0.6533	-7.71
1.20 ×-0.75	0.8675	-3.29	0.6643	-6.16
1.15 ×-0.75	0.8754	-2.41	0.6753	-4.60
1.10 ×-0.75	0.8829	-1.57	0.6862	-3.06
1.05 ×-0.75	0.8902	-0.77	0.6971	-1.52
0.95 ×-0.75	0.9036	+0.73	0.7186	+1.51
0.90 ×-0.75	0.9098	+1.43	0.7292	+3.01
0.85 ×-0.75	0.9158	+2.09	0.7397	+4.49
0.80 ×-0.75	0.9214	+2.72	0.7500	+5.95
0.75 ×-0.75	0.9268	+3.32	0.7602	+7.39

*Values in brackets are the true (benchmark) values

of the response. These results generally imply that there appears to be little effect on the values of the responses even if the constant term is about 25 percent biased (inconsistent). The evidence also shows that the effects of the constant term on the responses decline further for lower values of inconsistency (bias) in β_0. For example, as can be seen from Table 9.3, about 15 percent inconsistency in β_0 causes less that 5 percent change in the value of the responses. In general, Table 9.3 indicates that inconsistency in β_0 is unlikely to cause any significant problem, given that the degree of censoring is small. The results for the small and large sample size are similar, provided that the degree of censoring remains low.

However, the situation appears to be quite different when the degree of censoring increases. Table 9.4 below depicts the likely effects of the constant term on the responses when the degree of censoring is increased to 50%. Note that the results in Table 9.4 are obtained in the same way as those in Table 9.3 except β_0=-2.000 to adjust the degree of censoring. From Table 9.4, it is evident that inconsistency in β_0 can have serious consequences on the values of the responses. For example, a 10 percent inconsistency in β_0 can have upto 16 percent effect on the value of $F(.)\beta_j$. In short, Table 9.4 depicts that the effects of the constant term on the responses can be generally severe for higher degrees of censoring if β_0 is estimated inconsistently.

Table 9.4: The likely Effects of Changes in the Constant term (β_0) on Responses associated with the Tobit Model for N=200 and 50% Degree of Censoring.

Changes in β_0	$F(.)\beta_j$ (0.5059)*	%Δ in $F(.)\beta_j$	$R(.)\beta_j$ (0.3688)*	%Δ in $R(.)\beta_j$
1.25 ×-2.00	0.3138	-37.97	0.2744	-25.59
1.20 ×-2.00	0.3501	-30.79	0.2911	-21.07
1.15 ×-2.00	0.3878	-23.34	0.3088	-16.27
1.10 ×-2.00	0.4266	-15.67	0.3276	-11.17
1.05 ×-2.00	0.4661	-7.86	0.3477	-5.72
0.95 ×-2.00	0.5457	+7.87	0.3912	+6.07
0.90 ×-2.00	0.5851	+15.65	0.4147	+12.44
0.85 ×-2.00	0.6236	+23.26	0.4393	+27.25
0.80 ×-2.00	0.6609	+30.63	0.4649	+26.05
0.75 ×-2.00	0.6966	+37.69	0.4915	+33.27

*Values in brackets are the true (benchmark) values .

In general, the following points can be noted based on the above discussions. Inconsistency in β_0 can be ignored (or assumed irrelevant) if one is interested in the response of the latent variable, y_i^*. Further, inconsistency in β_0 does not seem to have any significant effects on the responses related to the dependent variable y_i, (i.e., on the values for (9.2) and (9.3)), provided that the degree of censoring is not large. However, the values of the responses (9.2) and (9.3) can be misleading if the degree of censoring is high and β_0 is estimated inconsistently. However, although useful, these conclusions need to be tested in practice using a wide variety of econometric or economic models.

9.5 Summary and Conclusions

In this Chapter we examined three major issues which are related to the design as well as the outcomes of the experiment. Specifically, the main points discussed in this Chapter include the following: (i) The effects of using random explanatory variables as compared to fixed explanatory variables. (ii) The relationships between the direction (left Vs right) censoring and the chi-square (skewed) distribution, and their likely effects on the outcomes of the experiment. (iii) The effects of the constant term in computing the responses (rate of changes) associated with the tobit model.

Note that since any Monte Carlo experiment of this kind is influenced by its design, the first two are aimed to assess the flexibility (or restrictiveness) of the data generation process considered in this experiment. This Chapter provides more information regarding these points and the main conclusions are as follows.

There appears to be enough evidence to suggest that the use of random explanatory variables does not lead to results which are significantly different from those obtained based on fixed explanatory variables in the experiment. That is, the various results and conclusions made throughout the experiment (Chapters 6 to 8) are mainly as result of changes such as the error distribution, sample size and level of censoring.

Another interesting outcome in this Chapter is with regard to the relationships that exist between the direction of censoring and the chi-square (skewed) distribution. The results indicate that if the experiment was designed so that the censoring and the tail (skewness) of the distributions are in the same direction, then the results of the experiment under the chi-square distribution would be as good as those under the normal distribution. In other words, the chi-square distribution is likely to be approximated by the normal distribution unless the experiment is designed in such a way that the censoring and the skewness (tail) of the distribution lie on opposite sides of the distribution. Otherwise, the Monte Carlo results can be misleading as they do not take the skewness of the distribution into account.

The third point discussed in this Chapter, the effects of the constant term, is raised based on the outcomes of the experiment. In particular, when discussing the inconsistency of the MLE (see Chapter 7), the results indicated that inconsistency can be a problem only for the constant term while no substantial problem of inconsistency was observed for other coefficients. Thus, assuming that the constant term can be inconsistent in some cases, we examined its likely consequences on the responses associated with the tobit model. For example, the effects of the constant term on the response of y_i as a result of a unit change, say, in x_j (i.e., $\partial E[y_i]/\partial x_j \ j = 1, 2$).

The results indicated that inconsistency in β_0 does not seem to have any significant consequences if the degree of censoring is small. Further, we also know that the constant term is not required in the estimation of the response of the latent variable y_i^* as a result of a unit change in x_j. However, there appears to be some evidence that the response functions associated with the dependent variable, y_i, can be misleading if the constant term is estimated inconsistently. This happens if the assumption of

Another interesting outcome in this Chapter is with regard to the relationships that exist between the direction of censoring and the the chi-square (skewed) distribution. The results indicate that if the experiment was designed so that the censoring and the tail (skewness) of the distributions are in the same direction, then the results of the experiment under the chi-square distribution would be as good as those under the normal distribution. In other words, the chi-square distribution is likely to be approximated by the normal distribution unless the experiment is designed in such a way that the censoring and the skewness (tail) of the distribution lie on opposite sides of the distribution. Otherwise, the Monte Carlo results can be misleading as they do not take the skewness of the distribution into account.

The third point discussed in this Chapter, the effects of the constant term, is raised based on the outcomes of the experiment. In particular, when discussing the inconsistency of the MLE (see Chapter 7), the results indicated that inconsistency can be a problem only for the constant term while no substantial problem of inconsistency was observed for other coefficients. Thus, assuming that the constant term can be inconsistent in some cases, we examined its likely consequences on the responses associated with the tobit model. For example, the effects of the constant term on the response of y_i as a result of a unit change, say, in x_j (i.e., $\partial E[y_i]/\partial x_j \; j = 1, 2$).

The results indicated that inconsistency in β_0 does not seem to have any significant consequences if the degree of censoring is small. Further, we also know that the constant term is not required in the estimation of the response of the latent variable y_i^* as a result of a unit change in x_j. However, there appears to be some evidence that the response functions associated with the dependent variable, y_i, can be misleading if the constant term is estimated inconsistently. This happens if the assumption of normality of errors does not hold and the degree of censoring is high.

Finally, it is important to note that, although this Chapter provides some highlights on some of the important components related to the experimental design as well as to the outcomes of the experiment, the depth of this investigation is not exhaustive due to time and other constraints. Obviously, it would be interesting to see whether a different data generation mechanism (for example, using real economic or other data) would lead to similar conclusions.

Chapter 10

Summary, Conclusions and Recommendations

Tobit models refer to regression models in which the observations on the dependent variable are observed only within a limited range. The tobit model was first suggested in the pioneering work of Tobin (1958) who analysed the relationship between household expenditure on durables and household income, by realizing the fact that the dependent variable cannot be negative, and takes the value zero for those households who had no expenditure on durables. If traditional least squares techniques are applied to estimate the parameters of the model they provide estimates which are biased and inconsistent and therefore are not appropriate.

In recent years numerous applications of tobit models have appeared covering a wide range of areas in the economics literature. The increase in applications for tobit models has been associated with an increase in survey data for which tobit model analysis is well suited as well as the development of econometric/statistical techniques and computer technology. Theoretical and empirical surveys on tobit models were provided by Amemiya (1981, 1984). The books by Manski and McFadden (1981), Maddala (1983), Amemiya (1985), Judge et al. (1985) and Greene (1991) also provide useful resources. As a result, many types of variations of the tobit have been suggested and various estimation techniques have been proposed to estimate the parameters of the model. However, almost all the theoretical studies have been concerned with the asymptotic properties and/or the computational ease of alternative estimators. Hence, little attention has been paid to the finite sample properties of the various estimators of the model.

> Thus, the purpose of this thesis was to make a contribution towards filling this void. In particular, as there are many generalizations of the tobit model, this study investigates the finite sample properties of the estimators of the standard tobit model (sometimes called the censored regression or simply the tobit model) through a Monte Carlo experiment.
>
> Further, along the lines of the existing Heckman's two-step estimator of the tobit model, this study suggests an alternative three-step estimator for the tobit model and its finite sample properties have been investigated along with other estimators of the model.

Given the above general objectives, the standard tobit model was defined in Chapter 2. Following the definition of the model, we reviewed the various estimators of the model and their properties. These estimators included, the maximum likelihood estimator (MLE), the Heckman's two step (H2S) and its weighted version, the weighted Heckman's two-step estimator (WH2S), the Heckman-type estimators proposed by

Wales and Woodland (1980), and nonlinear estimators of the model. Some highlights on recent developments with regard to semi-parametric, bounded influence and Bayesian estimation of the model were also discussed. Of these estimators, only the MLE and the H2S estimators have been used (and continue to be used) widely in applied research. However, previous studies related to tobit estimators have indicated that these estimators may have serious consequences. Specifically, the MLE can be inconsistent (not only inefficient) if the assumption of normality does not hold, which is usually the case in applied research [Goldberger (1980), Arabmazar and Schmidt (1982)]. This is in contrast to the traditional regression model where estimation by maximum likelihood provides consistent estimates under a wide variety of situations. Further, the H2S estimator which in most cases is preferred for its simplicity, especially for models involving simultaneous equations, usually performs poorly in finite samples [Wales and Woodland (1980), Nelson (1984), Paarsch (1984), Nawata (1994)]. The reason for its poor performance arises mainly because of unavoidable and often strong correlation between the explanatory variables and the estimated inverse of Mill's ratio; a problem which is inherent to the particular form of the model.

In Chapter 3, along the lines of H2S estimator, we proposed an alternative estimator for the tobit model which is referred to as the three-step estimator (3SE). But, unlike the H2S estimator, the 3SE avoids the multicollinearity problem while it preserves the simplicity of the H2S estimator. The 3SE estimator and its asymptotic properties, i.e., consistency and asymptotic distributions have been derived. Moreover, a weighted version of the 3SE estimator, the W3SE, and other generalizations of the three-step estimator, including its potential for extension to other similar models such as the two-limit tobit model have been studied.

Since the design of the experiment is an integral part of a study of this nature, an important consideration was given to its design. Previous studies related to the finite sample properties of the model and other related Monte Carlo/simulation studies were examined carefully in Chapter 4. In Chapter 5, we presented the design of the experiment on which the Monte Carlo experiment in this study was based. The first objective of the experimental design was to define the specific form of the model. This model involved two explanatory variables, three coefficients (including the constant term) and a random disturbance term. The experiment was designed to investigate the following effects on the performance of the estimators of the model: (i) The effects of changes in error distribution, i.e., the effects of violating the assumption about the error term of the model. (ii) The effects of degree of censoring. (iii) The effects of sample size.

To achieve objective (i), we considered three distributions for the error term, namely, the standard normal distribution, the students'-t distribution with three degrees of freedom and the chi-square distribution with four degrees of freedom. The later two distributions represent possible diversions (violations) from the usual normality assumption of the error term. Further, three levels of censoring were considered to investigate the effects of censoring, i.e., 25% (for low), 50 % (for medium) and 75% (for high) levels of censoring. Similarly, the sample sizes of 100 (for small), 200 (for medium) and 400 (for large) were considered to investigate the effects of sample size. Other important details with regard to the data generation process: the generation of the explanatory variables, the determination of the parameters (true values), the generation mechanism of the random variates associated with the three error distributions and other related matters were discussed thoroughly in Chapter 5.

The analysis and discussion of results in this study are presented in Chapters 6 through to 9. In Chapter 6, we examined most of the estimators discussed in the literature. Specifically, the analysis in this Chapter included the comparison of the following estimators.

1. The ordinary least squares estimator using the positive observations on the dependent variable (OLSP).

2. The Heckman's two step estimator (H2S).

3. The weighted Heckman's two-step estimator (WH2S).

4. The three-step estimator (3SE).

5. The weighted three-step estimator (W3SE).

6. The maximum likelihood estimator (MLE).

7. The nonlinear least squares estimator using the positive (non-limit) observations on the dependent variable (NLSP).

8. The ordinary least squares estimator using all observations (OLS).

9. The Heckman's two-step estimator based on the unconditional expectation of the model (H2SU).

10. The weighted Heckman's two-step estimator based on the unconditional expectation of the model (WH2SU).

11. The nonlinear least squares estimator based on the unconditional expectation of the model (NLSU).

These estimators were compared using a wide range of criteria. The main findings and conclusions included the following.

Overall, the MLE and the 3SE estimators provided the best results. The difference between the two estimators appeared to be quite small (marginal) in terms of efficiency. Both estimators yield relatively more efficient estimates under normality and less efficient estimates under the chi-square distribution.

The H2S estimator performed generally less efficiently compared to both the 3SE and MLE estimators. However, its performance deteriorates very rapidly with increases in the degree of censoring. This is in contrast to the 3SE estimator which is much more less sensitive to increases in the degrees of censoring.

The WH2S and the W3SE estimators performed slightly better than their corresponding unweighted versions, i.e., the H2S and the 3SE, respectively, under normality of the errors. However, they are sensitive if the normality assumption does not hold.

The nonlinear least squares estimators provided inefficient estimates compared to the MLE and the 3SE estimates in all cases. More importantly, the nonlinear least squares estimates are highly sensitive to increases in the level of censoring. Computationally, they are very slow and convergence is not always guaranteed.

Not surprisingly, the least squares estimators yield biased estimates; with the bias increasing linearly with the degree of censoring. Similarly, and perhaps surprisingly, the Heckman-type estimators proposed by Wales and Woodland (1980): the H2SU and its weighted version, the WH2SU, perform very poorly in all cases.

Given these and other outcomes as detailed in Chapter 6, several issues and questions were raised while discussing the results. One of the main concerns of this study was to see whether the MLE is inconsistent if one assumes normality when in fact the errors are non-normal, as was indicated by previous related studies. The results in this Chapter did not generally support this claim. That is, the MLE performed quite well in terms of bias under the non-normal distributions; except when the sample size was small (100) and the degree of censoring high (75%).

Given that the MLE is widely used in applied research, the issue of consistency was further considered in Chapter 7; in which, among other things, we analysed the consistency (inconsistency) of the MLE under the three distributions by making use of the normal-based asymptotic results of the MLE estimator. This analysis provided further insights regarding the consistency of the MLE. The results indicated that inconsistency is not generally a serious problem if one assumes normality incorrectly. However, the results also implicated that, if the degree of censoring is not small, inconsistency can be substantial for the constant term of the model under the non-normal distributions, whereas it remains negligible for other coefficients of the model. One of the important implications of this outcome is that, in the presence of misspecification and if the degree of censoring is high then results which involve the constant term can be misleading and need to be treated with caution. Note that, unlike the traditional regression model, the constant term of the tobit model is important in deriving results such as the response (rate of change) of the dependent variable due to a one unit change in an explanatory variable (see Chapters 2 and 9).

Further, an important issue which was analysed in detail is related to the variance-covariance estimation of the coefficients of the model in the ML framework. Note that when estimates are obtained by maximum likelihood procedure, the variance-covariance matrix for the coefficients can be estimated using a number of alternative, but asymptotically equivalent, variance-covariance matrix estimators. These estimators are based on (i) the information matrix (ii) the Hessian matrix (iii) the outer product matrix of the gradient vector and (iv) the robust (White-type) covariance matrix estimator. Significant differences noted between these estimates are usually considered as an indication of misspecification. Given this, we examined the performance of these estimators in estimating the variances of the coefficients under the different distributions, sample sizes and degrees of censoring. Furthermore, the effects of the various variance-covariance matrix estimators for hypotheses testing for the coefficients were examined. The main findings included the following.

> Under normality of the errors, all four estimators provided variances which are quite close (some times even the same) to the true variances of the coefficients as anticipated. The evidence also suggests that the robust (White-type) estimators tend to underestimate the true variances under the correct specification of the model, if the sample is small. Specifically, a sample size of at least 200 is required to obtain reliable results. Consequently, the MLE performed quite well in hypothesis testing regardless of the choice of the variance-covariance matrix estimator. In other words, the choice of the variance-covariance matrix estimator appeared to be neutral, provided that the model has been correctly specified.

However, under the non-normal distributions, the variance-covariance matrix based on the robust (White-type) estimator appeared to be relatively better than others. The variance-covariance matrix based on the outer product matrix performed as the second best. Interestingly, the variance-covariance matrix estimators based on the information and the Hessian matrices yield identical results in almost all cases (i.e., under normal and non-normal errors) but provide confidence intervals which appeared to be slightly narrower (over precise) than they should be.

Note that one of the significant contributions of this study is to suggest the three-step estimator (3SE) for the model, which has been discussed in Chapter 3. As noted earlier, one of the important characteristics of the 3SE estimator is that it avoids the multicollinearity problem whereas the H2S estimator does not. In order to see this, we investigated the finite sample properties of the H2S and the 3SE estimators under various levels of correlation. Specifically, as shown in Chapter 8, we investigated the effects of correlation between the explanatory variables and the estimated inverse of Mill's ratio on the performance of the estimators. The estimators were analysed by considering various levels of correlation ranging from -0.50 (for low) to -0.95 (for high) as well as different sample sizes and degrees of censoring. The main conclusions, among others, included the following.

The 3SE estimator performed better than the H2S estimator in almost all cases. More importantly, the difference between the two estimators becomes substantially large as the correlation level increases. If the correlation between the explanatory variables and the estimated inverse of Mill's ratio becomes high then the H2S estimator can be even worse than the biased ordinary least squares estimator (OLSP) of the tobit model. On the other hand, the 3SE estimator provided results which are marginally close to the MLE estimator for all levels of correlation. The results also indicated that the H2S estimator is likely to provide confidence intervals which are wider than they should be.

In general, one can deduce the following important points, among others, based on the outcome of this research.

Not surprisingly, the MLE performs the best when the model is correctly specified. It is also clear that the MLE performs quite well under the non-normal distributions (i.e., the students'-t and the chi-square distributions) in terms of bias and efficiency, except for the small sample size and high degree of censoring. These results are similar to those of Paarsch (1984) and Moon (1989) in which the MLE performed well under the under the normal as well as the Laplace distribution. However, our results also indicate that in the presence of misspecification there appears to be some inconsistency only for the constant term of the model, a problem which arises if the errors are non-normal and the degree of censoring is high. It is not clear why the constant term behaved differently from others which requires further investigation.

Regarding hypothesis testing (or construction of confidence intervals) for the coefficients of the model, the MLE estimator provides the desired preciseness irrespective of the choice of the variance-covariance matrix, provided that the model is correctly specified and if the sample is not small (100). Otherwise, the robust (White-type) estimator tends to be biased downwards especially for small samples and higher degrees of censoring. In the presence of misspecification (i.e., under the students'-t and the chi-square distribution) the robust (White-type) variance-covariance matrix estimator appears to be better than others. The variance-covariance matrix estimator based on the outer product matrix becomes close second.

The 3SE estimator outperforms the H2S estimator in almost all cases and provides results which are quite close to the MLE. Further, it is simple to use and potentially useful for the estimation of other models such as the two-limit tobit model. It would be more interesting to see if the 3SE estimator could be extended to more general models where estimation by MLE is not attractive, for example Type-II tobit models [see Amemiya (1984,1985), Maddala (1983)]. Specifically, the significance of the 3SE estimator will depend whether it can be extended to more general Tobit models.

Finally, it should be noted that, like other Monte Carlo experiments, this study has its limitations. Thus, the outcomes of this study need to be tested using a wide variety of economic/econometrics models. Further, this study does not include semi-parametric and other estimators of the model. Currently, these estimators are hardly used in applied research mainly because they are computationally cumbersome even for the simplest form of the model [see Paarsch (1984), Moon (1989), Peracchi (1991)]. More research is needed along this line.