Chapter 1

PRELIMINARIES

The purpose of this introductory chapter is to recall some important concepts and theorems from functional analysis which will be used throughout the sequel. §1.1 contains basic definitions, (counter) examples and results which indicate the motivation for our particular study of fixed points for nonexpansive mappings in Banach spaces. In §1.2 we state, without proof, fundamental results, some of which are well-known and can easily be found in standard references, for example, Day [1973] or Diestel [1984]. §1.3 contains proofs of some technical lemmas basic to much of the theory for nonexpansive mappings in Banach spaces.

1.1 Preliminary Definitions

Definition 1.1.1: Let X be a set and let $\mathbb{R}^+$ denote the positive real numbers. We define a distance function $d : X \times X \to \mathbb{R}^+$ to be a *metric* if the following conditions are satisfied:

M1 $\quad d(x,y) \geqq 0 \quad$ for all $x,y \in X$

M2 $\quad d(x,y) = 0 \quad$ if $x = y$

M3 $\quad d(x,y) = d(y,x) \quad$ (symmetry)

M4 $\quad d(x,y) \leqslant d(x,z) + d(z,y) \quad$ for all $x,y,z \subset X$
(triangle inequality)

The set X with metric d is called a *metric space* and is denoted by a pair (X,d). We may denote the space by X alone when the metric d is understood.

Definition 1.1.2: A sequence (x_n) of points of a metric space X is said to *converge* to a point x and we write $x_n \to x$, if corresponding to each $\varepsilon > 0$ there is a positive integer N such that $d(x_n,x) < \varepsilon$ for $n \geq N$. In other words $x_n \to x$ if $\lim_{n\to\infty} d(x_n,x) = 0$.

Definition 1.1.3: A sequence (x_n) of points of a metric space X is said to be a *Cauchy sequence* if for each $\varepsilon > 0$ there is a positive integer N such that $d(x_m,x_n) < \varepsilon$ for all $m,n \geq N$. That is, points in the "tail" of the sequence are arbitrarily close together.

In a metric space all convergent sequences are Cauchy but the converse is not generally true.

Definition 1.1.4: A metric space X is said to be *complete* if every Cauchy sequence of points of X converges in X.

Definition 1.1.5: Given a vector (linear) space X, a *norm* $\|\cdot\|$ on X is a mapping $x \mapsto \|x\|$ from X into the set $\mathbb{R}^+$ of positive real numbers which satisfies the following axioms:

N1 $\|x\| = 0$ if and only if $x = 0$

N2 $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in X$ and $\lambda \in F$ where F is either the field of real numbers or the field of complex numbers

N3 $\|x+y\| \leq \|x\| + \|y\|$ (the triangle inequality).

A vector (or linear) space X on which a norm $\|\cdot\|$ is defined is called a *normed vector space* or a *normed linear space* and is denoted by a pair $(X,\|\cdot\|)$ or X if the norm is understood. The norm function fulfils our intuitive concept of distance from the origin.

Every normed linear space is a metric space with a metric d defined as $d(x,y) = \|x-y\|$.

Definition 1.1.6: A normed linear space X is called a *Banach space* if it is complete as a metric space.

Definition 1.1.7: Let X and Y be normed linear spaces. A linear mapping $T : X \to Y$ is said to be *continuous at a point* $x_0 \in X$ if $\forall \varepsilon > 0 \;\; \exists \delta > 0$ such that

$$\|Tx-Tx_0\| < \varepsilon \quad \text{for all} \quad x \quad \text{satisfying} \quad \|x-x_0\| < \delta \; .$$

Equivalently, T is continuous at $x_0 \in X$ if

$$x_n \to x_0 \quad \text{implies} \quad Tx_n \to Tx_0 \; .$$

T is said to be *continuous* if it is continuous at every point of X. Continuity in other topologies is defined later. Unless otherwise stated, "continuous" means "norm continuous".

Remark 1.1.8 - Dual Spaces: In the sequel, unless otherwise stated, X will denote a Banach space with elements $x,y,\ldots,$ and X^* will denote its first *conjugate (dual) space* with elements $f,g,\ldots$. That is, X^* is the linear space of all continuous linear functionals $f : x \to \mathbb{R}$ (or $\mathbb{C}$, if X is a complex Banach space), endowed with the usual norm:

$$\|f\| = \sup\left\{|f(x)| \; : \; \|x\| = 1\right\} .$$

X^{**} will denote the second conjugate (dual) space of X or the conjugate (dual) space of X^* with elements $F,G,\ldots$.

For any vector x in a Banach space X, the *evaluation functional* $\hat{x}$, mapping X^* into $\mathbb{R}$ (or $\mathbb{C}$) which to every $f \in X^*$ assigns the value $f(x)$ of f at x is a continuous linear functional, that is, an element of the space X^{**}. Moreover, $\|\hat{x}\| = \|x\|$ and the canonical mapping of X into X^{**} defined by $J : X \to X^{**} : x \mapsto \hat{x}$ is linear and one-to-one. We denote the image $J(X)$ of X under J by $\hat{X}$.

Definition 1.1.9: A Banach space X is called *reflexive* if the canonical imbedding $J : X \to X^{**} : x \mapsto \hat{x}$ is onto. That is, if $\hat{X} = X^{**}$.

Example 1.1.10: For $1 < p < \infty$ the Banach spaces ℓ_p and $L_p[a,b]$ are reflexive. This follows immediately from the general form of continuous linear functionals in such spaces. For example, in the space ℓ_p $(1 < p < \infty)$ the general form of a continuous linear functional f is given by the explicit formula:

$$f[(x_1,x_2,\ldots)] = \sum_{i=1}^{\infty} y_i x_i \tag{1.1}$$

with $(y_1,y_2,\ldots) \in \ell_q$ where $\frac{1}{p} + \frac{1}{q} = 1$.
Moreover, $\|f\|$ is equal to the norm of $(y_1,y_2,\ldots)$ in ℓ_q. This allows one to identify the dual space $\ell_p{}^*$ with ℓ_q so that $\ell_p{}^{**} \equiv \ell_p$. Similarly, $(L_p[a,b])^* \equiv L_q[a,b]$ where $\frac{1}{p} + \frac{1}{q} = 1$ so that

$$(L_p[a,b])^{**} \equiv L_p[a,b] \ .$$

The spaces $C[a,b]$, ℓ_1, $L_1[a,b]$, ℓ_∞ and the subspaces c and c_0 of ℓ_∞ are not reflexive.

In the space ℓ_1 every continuous linear functional f is of the form (1.1) with $(y_1,y_2,\ldots) \in \ell_\infty$ and the corresponding norms are equal, so that the space $\ell_1{}^*$ may be identified with the space ℓ_∞.

Similarly, $(L_1[a,b])^* \equiv L_\infty[a,b]$. $c_0^* \equiv \ell_1$ and $\ell_1^* \equiv \ell_\infty$ so, $c_0^{**} \equiv \ell_\infty$.

If a normed linear space X is of finite dimension n, then X* also has dimension n, from which it follows that: every finite dimensional normed linear space is reflexive.

The details of the preceding statements on reflexivity can be found in any standard functional analysis text, for example, Kreyszig [1978].

Definition 1.1.11: Given a vector space X, an *inner product* $\langle , \rangle$ on X is a mapping of $X \times X$ into $\mathbb{R}$ or $\mathbb{C}$; that is, with every pair of vectors x and y there is associated a scalar which is written

$$\langle x,y \rangle$$

and is called the *inner product* of x and y, such that for all vectors x, y, z and scalars α we have

IP1 $\langle x+y,z \rangle = \langle x,z \rangle + \langle y,z \rangle$

IP2 $\langle \alpha x,y \rangle = \alpha \langle x,y \rangle$

IP3 $\langle x,y \rangle = \overline{\langle y,x \rangle}$ (complex conjugation or symmetry if F is real)

IP4 $\langle x,x \rangle \geqslant 0$

$\langle x,x \rangle = 0$ if and only if $x = 0$.

An *inner product space* is a vector space X with an inner product defined on X.

An inner product on X defines a norm on X given by

$$\|x\| = \sqrt{<x,x>}$$

and a metric on X given by

$$d(x,y) = \|x-y\| = \sqrt{<x-y,\ x-y>}\ .$$

Hence inner product spaces are normed linear spaces.

Definition 1.1.12: An inner product space X is called a *Hilbert space* if it is complete in the metric defined by the inner product.

Hence Hilbert spaces are Banach spaces.

It is well known that: every Hilbert space is reflexive.

Definition 1.1.13: Let T be a mapping of a set X into itself (in this case we will refer to T as a *self mapping* of X). A point $x \in X$ is said to be a *fixed point* of T if $Tx = x$. In other words, a point which remains invariant under a mapping is known as a fixed point.

Definition 1.1.14: A topological space X is said to have the *fixed point property* (or X is a fixed point space) if each continuous function $T : X \to X$ has at least one fixed point.

Example 1.1.15: The closed interval $[-1,1]$ has the fixed point property (FPP). For let $T : [-1,1] \to [-1,1]$ be a continuous function.

Define a new function F as $F(x) = T(x) - x$ for each $x \in [-1,1]$. We see that $F(-1) \geq 0$ and $F(1) \leq 0$. Therefore by the Weiestrass Intermediate-Value Theorem, there exists a point $x_0 \in [-1,1]$ such that $F(x_0) = 0$. This gives $T(x_0) = x_0$.

Definition 1.1.16: Let $X = (X,d_1)$ and $Y = (Y,d_2)$ be metric spaces. Let $T : X \to Y$ be a bijection of X into Y. Then T is called an *isometry* if and only if

$$d_2(Tx,Ty) = d_1(x,y) \quad \text{for all } x,y \in X .$$

In particular, if $X = Y$ and the metrics d_1 and d_2 are the same, then $T : X \to X$ is an *isometry* if

(1.2A) $$d(Tx,Ty) = d(x,y) \quad \text{for all } x,y \in X .$$

Definition 1.1.17: A self-mapping $T : X \to X$ of a metric space X is said to be *nonexpansive* if for all $x,y \in X$

(1.2B) $$d(Tx,Ty) \leq d(x,y) .$$

T is said to be *contractive* if for all $x,y \in X$, $x \neq y$,

(1.2C) $$d(Tx,Ty) < d(x,y) .$$

Definition 1.1.18: A self-mapping $T : X \to X$ of a metric space X is said to satisfy a *Lipschitz condition* if there exists a real number $k > 0$ (the Lipschitz constant) such that for all $x,y \in X$

(1.20) $$d(Tx,Ty) \leq k\, d(x,y) .$$

In the special case when $k \in [0,1)$, T is called a *strict contraction.*

Remark 1.1.19: We readily see that the following implications hold, none of which are in general reversible:

$$\begin{array}{ccl} & & \text{strict contraction (1.20)} \\ & & \Downarrow \\ & & \text{contraction (1.2C)} \\ & & \Downarrow \\ \text{isometry (1.2A)} & \Rightarrow & \text{nonexpansive (1.2B)} \\ & & \Downarrow \\ & & \text{Lipschitz (constant } k < 1) \\ & & \Downarrow \\ & & \text{continuous} \end{array}$$

The following example shows that a contractive mapping may fail to have a fixed point.

Example 1.1.20: Let $X = \left\{x \in \mathbb{R} : x \geq 1\right\}$ and set $T : X \to X : x \mapsto x + \frac{1}{x}$. Then it readily follows that T is contractive. Indeed if $x,y \in X$, $x < y$, then

$$T(y) - T(x) = T'(c)(y-x)$$

for some c with $x < c < y$.

Hence

$$|T(y) - T(x)| = (1 - \frac{1}{c^2})\, |y-x| < |y-x| .$$

Clearly T has no fixed point.

However, if a contractive mapping T (and hence a strict contraction) has a fixed point, it will always be unique: for if x_1, x_2 are fixed points of T, that is $Tx_1 = x_1$ and $Tx_2 = x_2$, then

$$d(x_1,x_2) = d(Tx_1,Tx_2) < d(x_1,x_2) .$$

But this is impossible unless $x_1 = x_2$.

□

It is of great importance in the applications to find out if nonexpansive mappings have fixed points.

One of the best known theorems in connection with fixed points of a mapping in a metric space is that given by Banach [1922] and known as the *Banach Contraction Mapping Principle (Theorem)*.

The statement and proof of the theorem is given as follows:

Theorem 1.1.21: *Let (X,d) be a complete metric space and $T : X \to X$ a strict contraction. Then T has a unique fixed point (that is, the equation $Tx = x$ has a unique solution).*

Proof: Let $x_0 \in X$ be arbitrary. Set $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0$, $x_3 = Tx_2 = T^3x_0, \ldots,$ and in general let

$$x_n = Tx_{n-1} = T^n x_0 .$$

We shall show that the sequence (x_n) is a Cauchy sequence. In fact, without loss of generality, taking $n < m$ for $n,m \in \mathbb{N}$, we have

$$\begin{aligned}
d(x_n,x_m) &= d(T^n x_0, T^m x_0) \\
&\leq k\, d(T^{n-1}x_0, T^{m-1}x_0) \quad \text{for some } k \in [0,1) \\
&\leq k^n\, d(x_0, x_{m-n}) \\
&= k^n\{d(x_0,x_1) + d(x_1,x_2) + \ldots + d(x_{m-n-1},x_{m-n})\} \\
&\qquad \text{(by the generalized triangle inequality)} \\
&\leq k^n\, d(x_0,x_1)\{1 + k + k^2 + \ldots + k^{m-n-1}\} \\
&\qquad \text{(by repeated use of } T \text{ as a strict contraction)} \\
&\leq k^n\, d(x_0,x_1)\,\frac{1}{1-k} \qquad \text{(since } k \in [0,1) \text{ ,} \\
&\qquad\qquad 1 + k + k^2 + \ldots + k^{m-n-1} < \sum_{j=0}^{\infty} k^j = \frac{1}{1-k} \text{ .}
\end{aligned}$$

Since $k < 1$, the quantity $k^n\, d(x_0,x_1)\,\frac{1}{1-k}$ is arbitrarily small for sufficiently large n. That is,

(1.3) $$d(x_n,x_m) \leq k^n\, d(x_0,x_1)\,\frac{1}{1-k} \;.$$

$$\to 0 \quad \text{as} \quad n \text{ (and hence } m) \to \infty \;.$$

Thus (x_n) is a Cauchy sequence. Since X is complete, the sequence (x_n) converges in X. Now let

$$\lim_{n\to\infty} x_n = u \;.$$

Now by virtue of the continuity of the mapping T,

$$Tu = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = u \ .$$

Thus the existence of a fixed point of T is proved. Uniqueness of u has been proved already (see remarks after Example 1.1.20). Hence the theorem.

□

Remark 1.1.22: (i) The construction of the sequence (x_n) in the above theorem and the study of its convergence are known as the *method of successive approximations.*

(ii) The theorem has been applied to test existence and uniqueness of solutions to differential and integral equations as has the method of successive approximations.

(iii) The method of successive approximations can be used not only for the proof of existence of unique fixed points u but also for finding an approximate value. Namely, the points x_n are the successive approximations to u. The error of approximations may be estimated by the inequality

$$d(x_n,u) \leq \frac{k^n}{1 - k} d(x_0,x_1)$$

which is obtained by passing to the limit for $m \to \infty$ in the inequality (1.3).

(iv) As the following (simple) examples show, the conclusion of the Banach Contraction Mapping Principle fails to hold if T is more general, for example, a nonexpansive mapping.

Example 1.1.23: (i) The identity mapping $I : X \to X$ on any space X is an isometry (therefore nonexpansive) and every point of X is a fixed point of I. So uniqueness of fixed points of isometries (hence nonexpansive mappings) fails.

(ii) The mapping $T : \mathbb{R} \to \mathbb{R} : x \mapsto x + 1$ is an isometry (therefore nonexpansive). T has no fixed points since $Tx = x + 1 \neq x \ \forall x \in \mathbb{R}$. So existence of fixed points of isometries (hence nonexpansive mappings) fails.

(iii) This example comes from Browder [1965b]. The mapping $T : B[c_0] \to B[c_0] : (x_1, x_2, \ldots) \mapsto (1, x_1, x_2, \ldots)$ is a nonexpansive mapping (in fact an isometry) since $\|Tx-Ty\|_\infty = \|x-y\|_\infty$ for all $x,y \in B[c_0]$. T is fixed-point free, that is, T does not have any fixed point since $(x_1, x_2, \ldots) = (1, x_1, x_2, \ldots)$ would simply mean that $x_1 = x_2 = \ldots = 1$ which is not in $B[c_0]$. Hence T has no fixed points.

Definition 1.1.24: A subset K of a normed linear space X is *convex* if $x,y \in K$ implies that $x + (1-\lambda)y \in K$ for $\lambda \in [0,1]$. That is, given any two points in K, the line segment joining them also lies in K.

$B[X]$ is always convex.

Definition 1.1.25: A metric space X is said to be *compact* if every sequence in X has a convergent subsequence. A subset K of X is said to be *compact* if K is compact considered as a subspace of X, that is, if every sequence in K has a convergent subsequence whose limit is an element of K.

A compact subset of a metric space is closed and bounded. The converse of this result is, in general, false.

However, if X is a finite dimensional normed linear space, then any subset $K \subset X$ is compact if and only if K is closed and bounded.

Unless otherwise stated, compact will mean "norm compact". Compactness in other topologies will be defined in the next section, §1.2.

Having defined the necessary terms, we are now in a position to state a second "classical" fixed point theorem. It is known as the *Schauder Fixed Point Theorem,* and is as follows:

Theorem 1.1.26 (Schauder 1930): *Let K be a compact convex subset of the Banach space X and let $T : K \to K$ be a continuous function. Then T has a fixed point in K.*

(More generally, the norm topology could be replaced by any locally convex topology for X, for example, the weak, or on a dual space, the weak-star topology. These topologies are taken up in the following section, §1.2.)

Remark 1.1.27: The work in this thesis is motivated by a combination of the hypothesis of the two "classical" fixed point theorems we have stated above. Examples 1.1.20 and 1.1.23 suggest that to obtain positive results in the problem of existence of fixed points for nonexpansive mappings T, it is necessary to impose conditions or properties much stronger than completeness on the domain of T. Our purpose will be to investigate additional (geometrical) properties on the Banach space X, for example, uniform convexity (in every direction), normal structure, Opial's condition, near uniform convexity etc., which ensure the existence of fixed points for nonexpansive mappings. We therefore will adopt the approach of Browder [1965a, 1965b], Göhde [1965], Karlovitz [1976a, 1976b, 1976c], Kirk [1965] and others who considered fixed points for nonexpansive mappings in Banach spaces. It is to be noted, that Belluce and Kirk [1969], Cheney and Goldstein [1959], Edelstein [1964], Kirk [1969] and others have studied fixed points for nonexpansive mappings in metric spaces; we will not, however, pursue these generalisations.

1.2 Fundamental and Well-Known Results

In this section we collect together, without proof, standard results which we will frequently mention in the sequel.

"Riesz's Lemma" states: Let Y be a proper closed linear subspace of the normed linear space X and $0 \leq \theta < 1$. Then there is an $x_\theta \in S(X)$ for which $\|x_\theta - y\| > \theta$ for every $y \in Y$.

Our first result is a consequence of Riesz's Lemma:

Theorem 1.2.1: *For each closed bounded subset of the normed linear space X to be compact it is necessary and sufficient that X be finite dimensional.*

That is, in order for each bounded sequence in the normed linear space X to have a (norm) convergent subsequence it is necessary and sufficient that X be finite dimensional.

Remark 1.2.2: If X is infinite dimensional, then $S(X)$ is not compact, although it is closed and bounded.

The next result, due to Grothendieck, conveys the notion that in normed linear spaces (norm) compact sets are small - both algebraically and topologically. But first of all, a definition.

Definition 1.2.3: Let K be a subset of a normed linear space X. The closure of the set

$$\left\{\sum_{i=1}^{k} \lambda_i x_i : \lambda_1, \lambda_2, \ldots, \lambda_k \geq 0, \ \sum_{i=1}^{k} \lambda_i = 1; \ \ x_1, x_2, \ldots, x_k \in K\right\}$$

is called the *closed convex hull* of K and is denoted by $\overline{\text{co}}\, K$. It is easily seen that equivalently $\overline{\text{co}}\, K$ may be defined as the smallest closed convex set in X which contains K. In other words, an element x in X belongs to $\overline{\text{co}}\, K$ is for any $\varepsilon > 0$ there exists a finite sequence of vectors $x_1, x_2, \ldots, x_k \in K$ and a sequence $\lambda_1, \lambda_2, \ldots, \lambda_k$ of non-negative real numbers such that

$$\lambda_1 + \lambda_2 + \ldots + \lambda_k = 1$$

and

$$\|\lambda_1 x_1 + \ldots + \lambda_k x_k - x\| < \varepsilon .$$

Theorem 1.2.4 (see Diestel 1984, for details): If K is a compact subset of the normed linear space X, then there is a sequence (x_n) in X such that $\lim_n \|x_n\| = 0$ and $K \subset \overline{co}\{x_n\}$.

That is, any (norm) compact subset K of a normed linear space is contained in the closed convex hull of some null sequence.

Theorem 1.2.5 (a theorem of Mazur): If K is a (norm) compact subset of a Banach space X then $\overline{co}\, K$ is (norm) compact.

As indicated by theorem 1.2.1, the norm topology is too strong to allow any widely applicable subsequential extraction principles. This fact-of-life leads ut to consider other, weaker topologies on normed linear spaces which are related to the linear structure of the spaces.

The two weaker-than-norm topologies of greatest importance in Banach space theory are the weak (w-) topology and the weak-star (w*-) topology. The first (the weak topology) can be defined in every normed linear space. In order to get any results regarding the existence of convergent or even Cauchy subsequences of an arbitrary bounded sequence in this topology, one must assume additional structural properties of the Banach space. The second (the w*- topology) is defined only in dual spaces.

Definition 1.2.6 - The weak topology: Using the dual space X^* of a normed linear space X, we introduce the weak topology of X in the following way. For a given $\varepsilon > 0$ and a finite number of elements $f_1, f_2, \ldots, f_n \in X^*$, let

$$V(0;\ f_1, f_2, \ldots, f_n;\ \varepsilon) = \left\{x \in X : |f_i(x)| < \varepsilon,\quad i = 1,2,\ldots,n)\right\}.$$

We denote by V the family of all sets $V(0;\ f\ ,f\ ,\ldots,f_n;\ \varepsilon)$ for any choice of ε and any finite sequence $f\ ,f\ ,\ldots,f_n$. It may be easily verified that V satisfies the assumptions for a basis of neighbourhoods of zero in a linear space; translation will carry these neighbourhoods throughout X. Thus, we can make the following definition: A topology defined by the basis V of neighbourhoods of zero in X is called the *weak topology* of X.

Alternatively we could describe the weak topology of X in terms of nets. Take the net (x_α); we say that *(x_α) converges weakly to x_0*, and we write

$$x_\alpha \overset{W}{\to} x_0, \qquad \text{if and only if}$$

$$f(x_\alpha) \to f(x_0) \qquad \text{for every } f \in X^* .$$

Every weakly convergent net (x_α) is necessarily bounded; that is, if (x_α) is a net in X such that $x_\alpha \overset{W}{\to} x_0$, then $\exists\ M > 0$ such that

$$\|x_\alpha\| < M\ \forall\ \alpha .$$

Moreover, the norm of its limit is less than or equal to $\liminf_{\alpha}\|x_\alpha\|$; that is, if (x_α) is a net in X such that $x_\alpha \xrightarrow{w} x_0$, then

$$(1.4) \qquad \|x_0\| \leq \liminf_{\alpha}\|x_\alpha\| ,$$

so the norm is a weak-lower-semicontinuous (w-l-s) functional on X.

The space X endowed with its weak topology, which is obviously coarser than the usual norm topology, is a linear (addition and scalar multiplication are continuous) and Hausdorff (weak limits are unique) topological space. In the sequel by the terms weakly closed set, weakly compact set, weak closure of a set etc. we shall mean closed or compact set, closure of a set etc. in the weak topology. Unqualified topological terms will refer to the norm topology of X, sometimes called the strong topology of X.

The norm topology of a Banach space X and its weak topology are equivalent if and only if X is of finite dimension.

The following theorem states one of the fundamental results in the geometric theory of Banach spaces:

Theorem 1.2.7 (Mazur): *If K is a convex subset of the Banach space X, then the closure of K in the norm topology coincides with the weak closure of K; that is,* $\bar{K}^s \equiv \bar{K}^w$.

A few choice consequences (due to Mazur) follow:

Corollary 1.2.8: If K is a convex set in the Banach space X, then K is norm closed if and only if K is weakly closed.

Corollary 1.2.9: The weak closure of every bounded set K of a Banach space X is contained in its closed convex hull; that is, $\bar{K}^w \subseteq \overline{co}\, K$.

Equivalently, if the sequence (x_n) converges weakly to x, then for every $\varepsilon > 0$ and any positive integer m there is a finite sequence $\lambda_1, \lambda_2, \ldots, \lambda_k$ of nonnegative real numbers such that

$$\|\lambda_1 x_{m+1} + \lambda_2 x_{m+2} + \ldots + \lambda_k x_{m+k} - x\| \leq \varepsilon$$

where $\lambda_1 + \lambda_2 + \ldots + \lambda_k = 1$.

Remark 1.2.10: The weak topology is defined in a projective manner: it is the weakest topology on X that makes each member of X^* continuous. In this case, it is sometimes denoted by $\sigma(X,X^*)$. As a consequence of this and the usual generalities about projective topologies, *if* Ω *is a topological space and* $T : \Omega \to X$ *is a mapping, then*

> *T is weakly continuous if and only if fT is continuous for each $f \in X^*$.*

Let $T : X \to Y$ be a linear map between the normed linear spaces X and Y. Then T is weak-to-weak continuous if and only if for each $g \in Y^*$, gT is a weakly continuous linear functional on X; this, in turn, occurs if and only if gT is a norm continuous functional on X for each $g \in Y^*$.

Now if $T : X \to Y$ is a norm-to-norm continuous linear map it obviously satisfies the last condition of the previous paragraph. On the other hand, if T is not norm-to-norm continuous then $T(B[X])$ is not a bounded subset of Y. Therefore, the Banach-Steinhaus theorem directs us to a $g \in Y^*$ such that $gT(B[X])$ is not bounded, so gT is not a bounded linear functional. Summarizing we get:

Theorem 1.2.11: *A linear map $T : X \to Y$ between the normed linear spaces X and Y is norm-to-norm continuous if and only if T is weak-to-weak continuous.*

Definition 1.2.12 - The weak* topology: In Remark 1.2.10 we defined the weak topology on X, denoted by $\sigma(X,X^*)$, to be the weakest topology on X such that every member f of X^* is continuous. Similarly, $\sigma(X^*,X^{***})$ denotes the weak-topology on the dual space X^*; that is, the weakest topology on X^* such that every member F of X^{**} is continuous. A typical basis set for $\sigma(X^*,X^{**})$ is

$$V(0;\ F_1,F_2,\ldots,F_n;\ \varepsilon) = \{f \in X^* : \|F_i f\| < \varepsilon,\ \ i = 1,2,\ldots,n\} .$$

A more interesting topology on X^* with which we will concern ourselves is the weak* topology. The *w^*-topology* on X^*, denoted by $\sigma(X^*,\hat{X})$, is defined to be the weakest topology such that every evaluation $\hat{x} : f \mapsto f(x)$ is continuous. The family $\mathcal{V}^*$ of sets

$$\begin{aligned} V^*(0;\ \hat{x}_1,\hat{x}_2,\ldots,\hat{x}_n;\ \varepsilon) &= \{f \in X^* : |\hat{x}_i(f)| < \varepsilon,\ \ i = 1,2,\ldots,n\} \\ &= \{f \in X^* : |f(x_i)| < \varepsilon,\ \ i = 1,2,\ldots,n\} \end{aligned}$$

$(\varepsilon > 0;\ x_1, x_2, \ldots, x_n \in X)$ defines a basis of neighbourhoods of zero for the w*-topology $\sigma(x^*, \hat{X})$ on X^*.

As with $\sigma(X, X^*)$ we can describe $\sigma(x^*, \hat{X})$ in terms of nets. We say that the net *(f_α) converges weak* to $f_0 \in X^*$*, and we write

$$f_\alpha \overset{w^*}{\to} f_0$$

if and only if

$$f_\alpha(x) \to f_0(x) \quad \text{for all} \quad x \in X\ .$$

Remark: $F \in X^{**}$ is w*-continuous if and only if $F = \hat{x}$ for some $\hat{x} \in \hat{X}$.

In general, the weak topology in the dual space X^* of a Banach space X is finer than the weak* topology. It is clear, however, that these two topologies coincide if the space X is reflexive.

The most important feature of the weak* topology is contained in the following compactness result.

Theorem 1.2.13 - *Banach-Alaoglu Theorem: For any normed linear space X, $B[X^*]$ is w*-compact.*

The w-topology is a locally convex Hausdorff linear topology and so the separation theorem applies. In this case it allows us to separate points (even w*-compact convex sets) from w*-closed convex sets by means of the w*-continuous linear functionals on X^*.*

Theorem 1.2.14 - The basic separation theorem: If K is a $w(w^*)$-compact convex subset disjoint from the $w(w^*)$-closed convex subset C, then there exists a continuous (w^*-continuous) linear function f such that

$$\sup f(C) < \inf f(K)$$

As seen in the preceding results, regardless of the normed linear space X, *w^*-closed bounded sets in X^* are w^*-compact.* How does a subset K of a Banach space X get to be w-compact? The two are related as shown in the next result. But note the following:

w-compact sets are norm closed and norm bounded.

Let K be a w-compact set in the normed linear space X. If $f \in X^*$ then f is w-continuous, therefore $f(K)$ is a compact set of scalars. It follows that $f(K)$ is bounded for each $f \in X^*$ and so K is bounded. Further, K is w-compact hence w-closed and so norm closed. That is, *K w-compact $\Rightarrow$ K(norm) closed and (norm) bounded.* But the converse is in general false, for example, $B[c_0]$ and $B[\ell_1]$ are closed and bounded but not w-compact.

In fact, the next result, stating the fundamental property of reflexive Banach spaces, provides more examples of sets which are closed and bounded but not w-compact, namely balls in any nonreflexive space.

Theorem 1.2.15 - Bourbabi-Kakutani: A Banach space X is reflexive if and only if its unit ball $B[X]$ is w-compact.

From Theorems 1.2.7 and 1.2.15 it follows immediately that: *in a reflexive space every bounded closed convex set is w-compact.*

Theorem 1.2.16 - *Eberlein-Smulian: A subset K of a Banach space X is w-compact if and only if K is weakly sequentially compact.*

In other words, $K \subset X$ *is w-compact if and only if every sequence* (x_n) *in* K *has a w-convergent subsequence* (x_{n_k}) *with* $x_{n_k} \xrightarrow{w} x \in K$.

Note that $K \subset X^*$ *is w*-compact if and only if every net in* K *has a subnet converging* w^* *to an element of* K.

One final result to be quoted is a consequence of Phillips' Lemma.

Theorem 1.2.17: *In* $\ell_1 (\equiv C_0{}^*)$ *weak and norm convergence are equivalent; that is,* $f_n \xrightarrow{w} f$ *implies* $f_n \xrightarrow{s} f$.

This result is sometimes referred to as the *Schur Property* and we say ℓ_1 is a Schur space.

Before concluding this section, some remarks are called for regarding the following two properties which are the subject of our subsequent investigation.

The weak fixed point property (w-FPP): For every nonempty w-compact convex subset K of a Banach space X and each nonexpansive mapping selfmapping $T : K \to K$, there exists $x_0 \in K$ with $Tx_0 = x_0$; and in the case of a dual space X^*,

The weak-star fixed point property (w*-FPP): For every nonempty w*-compact convex subset K of X^* and each nonexpansive selfmapping $T : K \to K$, there exists $x_0 \in K$ with $Tx_0 = x_0$.

We note that for a dual space X^* *w*-FPP ⇒ w-FPP*. A natural advantage of w*-FPP is the ready supply of w*compact sets guaranteed by the Banach-Alaoglu Theorem 1.2.13. For a reflexive space these two fixed point properties coincide and, by Mazur's Theorem 1.2.8, imply that every nonexpansive mapping of a bounded closed convex set has a fixed point.

The existence of fixed points for nonexpansive self-mappings of weak (weak*) compact convex sets can be seen as a wedding of the Banach and Schauder Fixed Point Theorems.

1.3 Preliminary Lemmas

This section contains standard tools, arguments and lemmas which will be used constantly in what follows to derive fixed point results.

Sufficient conditions for $X(X^*)$ to have the w-FPP (w*-FPP) are obtained by pursuing the following "natural" line of argument:

Lemma 1.3.1 (existence of minimal invariant sets): Suppose K is a nonempty $w(w^*)$-compact convex subset of a Banach space $X(X^*)$ and $T : K \to K$ a nonexpansive selfmapping of K. Then K contains a nonempty $w(w^*)$-compact convex subset K_0 which is minimal for T (that is, $T(K_0) \subseteq K_0$ and no strictly *smaller nonempty $w(w^*)$-compact convex subset of K_0 is invariant under T).*

Proof: Denote by Φ the family of all nonempty $w(w^*)$-compact convex subsets K' of K which are invariant under T (that is, $T(K') \subseteq K'$). The family Φ is nonempty since $K \in \Phi$. In an obvious manner Φ may be ordered (partially) by the relation of set inclusion. It is easy to show that Φ is inductive. To prove this, consider an ordered subfamily Ψ of Φ. The intersection

$$K^* = \bigcap_{K' \in \Psi} K'$$

is a $w(w^*)$-compact convex and invariant subset of K. All sets K' in Ψ are $w(w^*)$-compact convex and the family Ψ has the finite intersection property. By weak compactness of K, it follows that K^* is nonempty so that $K^* \in \Phi$ and is a lower bound for Ψ. Now by Zom's Lemma, there exists in Φ a minimal element, say K_0.

□

Remark 1.3.2: (i) For K a nonempty $w(w^*)$-compact convex set in $X(X^*)$ and $T : K \to K$ nonexpansive, the above lemma establishes the existence of a *minimal invariant subset* K_0 from the class of nonempty $w(w^*)$-compact convex subsets of K which are invariant under T. *When minimal invariant sets are referred to, they must be understood in this sense.*

(ii) $K_0 = \overline{co}\,(T(K_0))$ since $\overline{co}\,(T(K_0))$ is contained in K_0 and is closed convex and invariant under T.

(iii) If K_0 is a singleton, then by the invariance, K_0 is a fixed point of T. Thus we note that: *$X(X^*)$ has the $w(w^*)$-FPP if and only if every minimal element K_0 of the above type contains precisely one point.*

The idea is to find distinguishing properties which such a minimal invariant subset would have were it to contain more than one point, and then to seek "natural" conditions on the space which rule out the occurrence of sets with such properties.

(iv) If K_0 is not a singleton (so $\operatorname{diam} K_0 > 0$), then the next Lemma 1.3.3 states that K_0 contains a sequence (x_n) with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We call such a sequence an *approximate fixed point sequence (a.f.p.s.)* for T.

Lemma 1.3.3 (existence of a.f.p.s.): Let K be a nonempty bounded closed convex subset of a Banach space X, and let $T : K \to K$ be nonexpansive. Then there is a sequence (x_n) in K such that $(x_n - Tx_n) \to 0$ (that is, (x_n) is a.f.p.s.) and furthermore $(x_n - x_{n+1}) \to 0$ as $n \to \infty$.

Proof: Let y_0 be a given point of K. Define T_n, $n = 1,2,\ldots,$ by

$$T_n x = \frac{1}{n} y_0 + (1 - \frac{1}{n})Tx \qquad \text{for each } x \in K .$$

Then $T_n : K \to K$ is a selfmapping of K by the convexity of K. Furthermore, T_n is strictly contractive with

$$\|T_n x - T_n y\| = \|(1-\tfrac{1}{n})\ (Tx - Ty)\| \leq (1-\tfrac{1}{n})\ \|x - y\| \quad \forall x,y \in K\ .$$

Hence by the Banach Contraction Mapping Principle 1.1.21, T_n has a unique fixed point, denoted by x_n. That is,

$$T_n x_n = x_n = \frac{1}{n}\, y_0 + (1-\tfrac{1}{n})Tx_n\ .$$

We have

$$x_n - Tx_n = \tfrac{1}{n}(y_0 - Tx_n) \to 0 \quad \text{as} \quad n \to \infty\ .$$

Also for each $n \in \mathbb{N}$,

$$\begin{aligned}
\|x_n - x_{n+1}\| &= \|\tfrac{1}{n}\, y_0 - (1-\tfrac{1}{n})Tx_n - \tfrac{1}{n+1}\, y_0 + (1-\tfrac{1}{n+1})Tx_{n+1}\| \\
&= \|\tfrac{1}{n(n+1)}\, y_0 - \big(1 - \tfrac{1}{n+1} + \tfrac{1}{n(n+1)}\big)Tx_n + (1-\tfrac{1}{n+1})Tx_{n+1}\| \\
&= \|\tfrac{1}{n(n+1)}\, [y_0 - Tx_n] - (1-\tfrac{1}{n+1})[Tx_n - Tx_{n+1}]\| \\
&\leq \tfrac{1}{n(n+1)}\, \|y_0 - Tx_n\| + (1-\tfrac{1}{n+1})\ \|x_n - x_{n+1}\|\ .
\end{aligned}$$

So $$\frac{1}{(n+1)}\ \|x_n - x_{n+1}\| \leq \frac{1}{n(n+1)}\ \|y\ - Tx_n\|\ .$$

Thus $$\|x_n - x_{n+1}\| \leq \frac{1}{n}\ \|y_0 - Tx_n\| \to 0 \quad \text{as} \quad n \to \infty\ .$$

That is, $$(x_n - x_{n+1}) \to 0 \quad \text{as} \quad n \to \infty\ .$$

□

Remark 1.3.5: (i) Note that the lemma did not require K to be weakly compact, only bounded closed and convex.

(ii) An alternative construction of the required a.f.p.s. would be as follows:

Defining $T_s : K \to K : x \mapsto (1-s)y_0 + sTx$,

where $0 < s < 1$ and $y_0 \in K$ is fixed, yields a strict contraction. Clearly, denoting by x_s the unique fixed point of T_s, it can be seen that $\|Tx_s - x_s\| \to 0$ as $s \to 1$.

A (perhaps the) fundamental property of the minimal invariant subsets, introduced in Lemma 1.3.1, appears in the next Lemma 1.3.7. But first of all a preliminary definition and Lemma 1.3.6:

Definition 1.3.5: We say that a point $x_0 \in K$ is a *diametral point* of K if and only if

$$\sup\{\|x_0 - y\| : y \in K\} = \text{diam } K$$

and that K is *diametral* if all the points of K are diametral points.

Lemma 1.3.6: Let K_0 be a minimal invariant subset of a nonempty w-compact convex set K with $T : K \to K$ nonexpansive. If $\psi : K_0 \to \mathbb{R}$ is a weak-lower-semicontinuous mapping with $\psi(Tx) \leq \psi(x)$ for every $x \in K_0$, then ψ is constant on K_0.

Proof: Let $x_0 \in K_0$ be such that $\psi(x_0) = \inf \psi(K_0)$ and let $E = \{x \in K_0 : \psi(x) = \psi(x_0)\}$. Then E is a nonempty w-compact convex set which is invariant under T and so by minimality $E = K_0$.

□

We now show that nonempty minimal invariant subsets are diametral; basing our proof on Maurey [1980/81]:

Lemma 1.3.7 *(Garkavi 1961, Kirk 1965): If K_0 is a $w(w^*)$-compact convex set which is minimal with respect to invariance under some nonexpansive mapping $T : K_0 \to K_0$, then K_0 is diametral.*

Proof:

$$\psi(x) = \sup \{\|x-y\| : y \in K_0\}$$

$$= \sup \{\|x-Ty\| : y \in K_0\} \quad (\text{as } \overline{co}\, T(K_0) = K_0 \text{ by minimality})$$

satisfies the conditions of Lemma 1.3.6. Thus ψ is constant on K_0 with value

$$\sup_{x \in K_0} \sup_{y \in K_0} \|x-y\| = \text{diam } K_0$$

and so K_0 is diametral.

□

That diametral sets containing more than one point can exist may seem somewhat surprising. However, easy examples are at hand, see below and the next chapter where examples are given in $C[0,1]$ and $(\ell_2, \frac{1}{\beta}\|x\|_2 \vee \|x\|_\infty)$ where $\beta \geq 1$.

Example 1.3.8: In $(C_o, \|\cdot\|_\infty)$, let

$$K = \{x = (x_1, x_2, \ldots, x_n, \ldots) : x_n \geq 0 \ \forall_n \|x\|_\infty \leq 1\}$$

Then K is diametral. To see this, first note that for $x,y \in K$ we have $0 \leqslant x_n, y_n \leqslant 1$ and so $|x_n - y_n| \leqslant 1$ $\forall n$. Thus $\|x-y\|_\infty \leqslant 1$ and diam $K \leqslant 1$. Now for any $x \in K$ and $e_n = (\delta_{in}) \in K$ we have

$$\|x-e_n\|_\infty \geqslant |x_n - 1| \to 1 \qquad \text{as} \quad x_n \to 0.$$

So x is a diametral point. Since x is arbitrary K is diametral.

□

Lemma 1.3.9 (Brodskii-Milman 1948): If K is a closed convex set which is diametral, then for any $\varepsilon \in (0,1)$ there exists a sequence (x_n) in K with

$$\inf_{m \neq n} \|x_m - x_n\| \geqslant \varepsilon \, \mathit{diam} \, K \, .$$

Proof: If daim $K = 0$ the result is trivial, so without loss of generality we may assume diam $K = 1$. Choose points $x_1, x_2 \in K$ such that $\|x_1 - x_2\| \geqslant 1 - \varepsilon$. By convexity $\frac{x_1 + x_2}{2} \in K$ and so there exists $x_3 \in K$ with

$$\left\| x_3 - \frac{x_1 + x_2}{2} \right\| > 1 - \frac{\varepsilon}{2} \, .$$

Continuing in this way we obtain a sequence $x_1, x_2, \ldots, x_m, \ldots$ with

$$\left\| x_{n+1} - \frac{x_1 + x_2 + \ldots + x_n}{n} \right\| > 1 - \frac{\varepsilon}{n} \, .$$

We then have that

$$1 - \frac{\varepsilon}{n} \leqslant \left\| x_{n+1} - \frac{x_1 + x_2 + \dots + x_n}{n} \right\|$$

$$= \frac{1}{n} \| (x_{n+1}-x_1) + (x_{n+1}-x_2) + \dots + (x_{n+1}-x_n) \|$$

$$\leqslant \frac{1}{n} \sum_{k=1}^{n} \| x_{n+1}-x_k \|$$

$$\leqslant 1$$

and so we conclude that $\| x_{n+1}-x_k \| \geqslant 1 - \varepsilon.$

□

In the proof of some fixed point results, a refinement of the above Lemma is more useful, and it is based on the following definition.

Definition 1.3.10: A bounded sequence (x_n) is *diametral* if it is nonconstant and if

$$\mathrm{dist}(x_{n+1}, \overline{\mathrm{co}}\, \{x_1, x_2, \dots, x_n\}) \to \mathrm{diam}\{x_n\}_{n=1}^{\infty} .$$

***Lemma 1.3.11** (Brodskii-Milman 1948): If K is a closed convex set which is diametral, then there exists a sequence (x_n) in K such that*

$$\mathrm{dist}(x_{n+1}, \overline{\mathrm{co}}\, \{x_1, x_2, \dots, x_n\}) > (1 - \tfrac{1}{n})\mathrm{diam}\ K$$

(that is, (x_n) is a diametral sequence where $\mathrm{diam}\, K = \mathrm{diam}\{x_n\}_{n=1}^{\infty}$).

Proof: We wish to construct a diametral sequence in K. Let $0 < \varepsilon < \operatorname{diam} K = d$. As in the previous Lemma, we start with any $x_1 \in K$ and construct inductively x_{n+1} from $x_1, x_2, \ldots, x_n$ such that

$$\left\| \frac{x_1 + x_2 + \ldots + x_n}{n} - x_{n+1} \right\| \geq d - \frac{\varepsilon}{n^2} .$$

Then (x_n) is a nonconstant bounded diametral sequence. For, let $x \in \overline{co}\, \{x_1, x_2, \ldots, x_n\}$ and

$$x = \sum_{i=1}^{n} \alpha_i x_i , \quad \alpha_i \geq 0 \text{ and } \sum_{i=1}^{n} \alpha_i = 1 .$$

Let $\alpha = \sup \{\alpha_1, \alpha_2, \ldots, \alpha_n\} > 0$. Then

$$\frac{x_1 + x_2 + \ldots + x_n}{n} = \frac{1}{n\alpha} x + \sum_{i=1}^{n} \left(\frac{1}{n} - \frac{\alpha_i}{n\alpha}\right) x_i ,$$

the second member being a convex combination of x, x_i. Then by the induction hypothesis,

$$d - \frac{\varepsilon}{n^2} \leq \frac{1}{n\alpha} \| x - x_{n+1} \| + \sum_{i=1}^{n} \left(\frac{1}{n} - \frac{\alpha_i}{n\alpha}\right) \| x_i - x_{n+1} \|$$

$$\leq \frac{1}{n} \| x - x_{n+1} \| + \left(1 - \frac{1}{n\alpha}\right) d .$$

Thus $$d \geq \| x - x_{n+1} \| \geq d - \frac{\varepsilon\alpha}{n} \geq d - \frac{\varepsilon}{n} .$$

Hence $$\operatorname{dist}(x_{n+1}, \overline{co}\, \{x_1, x_2, \ldots, x_n\}) \to d .$$

It follows that $d(= \operatorname{diam} K) = \operatorname{diam} \{x_n\}_{n=1}^{\infty}$ and that the sequence is diametral.

□

For some calculations with minimal invariant subsets, it is convenient to know the sequence (x_n) more explicitly. In this direction we have Lemma 1.3.12 (due to Korlovitz 1976c), which gives an interesting and useful property of nonexpansive mappings, expressed in terms of approximate fixed points. This is then used in conjunction with the geometric properties of some spaces to derive fixed point results.

Lemma 1.3.12 (Korlovitz 1976c): *Let K_0 be a w-compact convex subset of a Banach space X and let $T : K_0 \to K_0$ be nonexpansive. Suppose that K_0 is minimal invariant. If (x_n) is an approximate fixed point sequence for T, then for each $x \in K_0$,*

$$\lim_{n\to\infty} \|x-x_n\| = \operatorname{diam} K_0 \ .$$

Proof: Let (y_n) be any approximate fixed point sequence for T in K, and let

$$\psi(x) = \limsup_n \|x-y_n\| \ .$$

Then ψ satisfies the assumptions of Lemma 1.3.6 and so ψ is constant on K_0 with value D say. Let (y_{n_k}) be a subsequence with $y_{n_k} \overset{w}{\to} y$. Then

$$D \geq \limsup_{x\in K} \|x-y_{n_k}\| \geq \liminf_k \|x-y_{n_k}\| \geq \|x-y_0\|$$

(by inequality 1.4).

Thus $D \geq \sup_{x \in K_0} \|x-y_0\| = \text{diam } K_0$ (by Lemma 1.3.7). Now taking (y_n) to be any subsequence (x_{n_k}) of (x_n) we therefore have

$$\limsup_k \|x-x_{n_k}\| = \text{diam } K_0 \qquad \forall x \in K_0$$

and so $\lim_n \|x-x_n\| = \text{diam } K_0$.

□

Remark 1.3.13: This shows that the sequence (x_n) of Lemmas 1.3.9 and 1.3.11 could be extracted as a subsequence of an approximate fixed point sequence and so would itself be an approximate fixed point sequence.

Chapter 2

NORMAL STRUCTURE

The notion of normal structure was introduced by M.S. Brodskii and D.P. Milman [1948] and has been significant in the development of fixed point theory for nonexpansive mappings in Banach spaces. This notion was used by them to show that every weakly compact convex subset of a Banach space, with normal structure, contains a point, the Brodskii-Milman centre, having the property that all isometries of the set onto itself leave it fixed. In other words, this centre is a common fixed point for all isometries of the set onto itself.

In this chapter we define normal structure, w(w*)-normal structure and asymptotic normal structure as a generalisation of normal structure. We discuss examples and spaces which either satisfy or fail these geometrical conditions. Basic to the chapter is Kirk's [1965] result which states that every weakly compact convex subset K of a Banach space with normal structure has a fixed point for every nonexpansive selfmapping of K.

A recent survey of results on normal structure has been given by Swaminathan [1983].

2.1 Preliminary Definitions

The first class of spaces known to have the w-FPP were obtained by ruling out the existence of bounded closed convex diametral sets containing more than one point. Such spaces are said to have "normal structure". That is;

Definition 2.1.1 (Brodskii-Milman 1948): A set K is *nontrivial* if diam $K > 0$ (that is, if K contains more than one point). A Banach space X is said to have *normal structure* if for each nontrivial bounded closed convex subset K there exists a point $p \in K$ such that

$$\sup\{\|p-x\| : x \in K\} < \text{diam } K .$$

Any point p of K with such a property is called a *nondiametral point* of K.

Remark 2.1.2: We will also need to apply normal structure to sets. In this case, we say that a bounded closed convex set K has *normal structure* if each nontrivial convex subset C of K contains a nondiametral point.

Geometrically, K has normal structure if for every nontrivial convex subset C of K there exists a ball of radius less than diam C centred at a point of C and containing C. That is, if $p \in C$ is a nondiametral point of C then for some r, $0 < r <$ diam C, we have $C \subset B_r(p)$.

Definition 2.1.3 (Lim 1980): $X(X^*)$ is said to have *$w(w^*)$-normal structure* if every nontrivial $w(w^*)$-compact convex set contains a nondiametral point.

Remark 2.1.4: We will see that, in general, normal structure ⇒ w-normal structure ⇒ w-FPP, and for a dual space, normal structure ⇒

w*-normal structure ⇒ w-normal structure, with w*-normal structure ⇒ w*-FPP. For a reflexive space all three notions of normal structure coincide.

Definition 2.1.5: A Banach space is said to have *(weak) uniform normal structure* if there exists a $0 < k < 1$ such that for each non-trivial (weakly compact) bounded closed convex subset K there exists a point $p \in K$ such that

$$\sup\{\|p-x\| : x \in K\} \leqslant k \operatorname{diam} K . \tag{2.1}$$

Definition 2.1.6: Let C be a bounded subset of a Banach space X. Then define, for each $x \in C$

$r_x(C) = \sup_{y\in C} \|x-y\|$: the minimum radius for a ball centred at x to contain C

$r(C) = \inf_{x\in C} r_x(C)$: the smallest possible radius for a ball containing C and centred on a point of C

$r(C)$ is called the self *Chebyshev radius* of C. Call $x_0 \in C$ a *Chebyshev centre* if this infimum is achieved at x_0 and denote the (possibly empty) set of Chebyshev centres by $\zeta(C) = \{x_0 \in C : r_{x_0}(C) = r(C)\}$: the set of possible centres in C for balls of the minimum radius which contain C; that is,

$$x_0 \in \zeta(C) \text{ if and only if } \sup_{y\in C} \|x_0-y\| = \inf_{x\in C} \sup_{y\in C} \|x-y\| .$$

In terms of this notation, inequality (2.1) becomes

$$r(K) \leq k \operatorname{diam} K .$$

We can also give an alternative definition for normal structure, namely,

Definition 2.1.7: X has *normal structure* if and only if for every nontrivial bounded closed convex subset C of X we have $\zeta(C) \subsetneq C$. This is proved in Proposition 2.1.8.

We note that; X does not have normal structure $\Leftrightarrow$ there exists a closed bounded convex set C such that for each $x \in C$, $r_x(C) = r(C)$ $\Leftrightarrow r_x(C) = r(C) = \sup_{x\in C} r_x(C) = \sup_{x,y\in C} \|x-y\| = \operatorname{diam} C \Leftrightarrow \zeta(C) = C$.

Proposition 2.1.8: C has normal structure if and only if $\zeta(C) \subsetneq C$.

Proof: ($\Rightarrow$): Suppose $\zeta(C) = C$. Then $\forall x \in C$, $r_x(C) = r(C)$. This implies that

$$\begin{aligned}
\operatorname{diam} C &= \sup_{x\in C} \sup_{y\in C} \|x-y\| \\
&= \sup_{x\in C} r_x(C) \\
&= \sup_{x\in C} r(C) = r(C) = r_x(C) \quad \forall x \in C.
\end{aligned}$$

Hence C is diametral, contradicting C has normal structure.

($\Leftarrow$): Suppose C fails normal structure. Then $\forall x \in C$, $r_x(C) = \operatorname{diam} C$. This implies that

$$r(C) = \sup_{x\in C} r_x(C)$$

$$= \sup_{x\in C} \sup_{y\in C} \|x-y\|$$

$$= \text{diam } C .$$

So $r_x(C) = r(C)$ $\forall x \in C$. That is, $\zeta(C) = C$, contradicting that $\zeta(C) \subsetneqq C$.

□

The following lemma states some more facts about the set $\zeta(C)$ and conditions under which $\zeta(C)$ must be nonempty.

Lemma 2.1.9: *Let C be a bounded closed convex subset of $X(X^*)$. Then $\zeta(C)$ is closed and convex. Furthermore, if C is $w(w^*)$-compact, then $\zeta(C) \neq \phi$.*

Proof: Assume (c_n) is a sequence in $\zeta(C)$ with $c_n \to c$. For any $\varepsilon > 0$ choose n_0 such that $\|c_{n_0} - c\| < \varepsilon$. Then

$$r(C) = \inf_{x\in C} r_x(C) \leqslant r_c(C) = \sup_{x\in C} \|c-x\|$$

$$= \sup_{x\in C} \|c-c_{n_0}+c_{n_0}-x\|$$

$$\leqslant \varepsilon + \sup_{x\in C} \|c_{n_0}-x\|$$

$$= \varepsilon + r_{c_{n_0}}(C)$$

$$= \varepsilon + r(C)$$

That is, $r(C) \leq r_c(C) \leq \varepsilon + r(C)$. Since ε is arbitrary, $r_c(C) = r(C)$. That is, $c \in \zeta(C)$. Hence $\zeta(C)$ is closed. Since $\zeta(C)$ is closed, it suffices to show that $\zeta(C)$ is midpoint convex. So let $c_1, c_2 \in \zeta(C)$. By convexity $\frac{c_1 + c_2}{2} \in C$. Then for each $x \in C$ we have

$$(2.2) \qquad \left\|\frac{c_1 + c_2}{2} - x\right\| \leq \frac{1}{2}(\|c_1 - x\| + \|c_2 - x\|)$$
$$\leq \frac{1}{2}(r_{c_1}(C) + r_{c_2}(C))$$
$$= r(C) .$$

But then,

$$r(C) \leq r_{\frac{c_1+c_2}{2}}(C) = \sup_{x \in C} \left\|\frac{c_1 + c_2}{2} - x\right\|$$
$$\leq r(C) \qquad \text{by (2.2).}$$

So $r_{\frac{c_1+c_2}{2}}(C) = r(C)$. Thus $\frac{c_1 + c_2}{2} \in \zeta(C)$. That is, $\zeta(C)$ is convex. Finally, let (x_n) be a sequence in C such that

$$r_{x_n}(C) = \sup_{x \in C} \|x_n - x\| \to r(C) = \inf_{y \in C} \sup_{x \in C} \|y - x\| .$$

Then there exists a subsequence (subnet) (x_{n_k}) such that $x_{n_k} \overset{w}{\to} c \in C$. Since the norm is $w(w^*)$-lower-semicontinuous (see inequality (1.4)) we therefore have, for every $x \in C$, $\|c - x\| \leq \lim_k \inf \|x_{n_k} - x\|$.

Thus

$$r(C) \leq r_c(C) = \sup_{x \in C} \|c - x\|$$
$$\leq \sup_{x \in C} \lim_k \inf \|x_{n_k} - x\|$$
$$\leq r(C) .$$

That is, $r_c(C) = r(C)$ which implies that $c \in \zeta(C)$, as required.

□

2.2 Fixed Points for Nonexpansive Mappings

The most fundamental result in the theory of fixed points for nonexpansive mappings involving normal structure is due to W.A. Kirk [1965]:

Theorem 2.2.1 (Kirk 1965): *Let K be a nonempty $w(w^*)$-compact convex subset of $X(X^*)$ and suppose that $X(X^*)$ has $w(w^*)$-normal structure. Then every nonexpansive mapping $T : K \to K$ has a fixed point in K.*

Proof: By Lemma 1.3.1 there exists a nonempty $w(w^*)$-compact convex subset K_0 of K which is minimal with respect to invariance under T. If K_0 is a singleton, then by the invariance, it is necessarily a fixed point of T (cf. Remark 1.3.2(iii)). So suppose that K_0 is not a singleton. Then by Lemma 1.3.3 K_0 contains an approximate fixed point sequence (x_n) for T. That is,

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

(cf. Remark 1.3.2(iv)). Then by Lemma 1.3.12, for each $x \in K_0$,

$$\lim_{n\to\infty} \|x_n - x\| = \operatorname{diam} K_0 \ . \qquad (2.3)$$

This equation (2.3) and/or Lemma 1.3.7 implies that K_0 is diametral, contradicting that K has normal structure. Thus necessarily K_0 is a singleton and hence a fixed point for T.

□

Remark 2.2.2: Before discussing applications of Kirk's result, we need to point out, that since normal structure is such a highly nonintuitive notion, it requires clarification in terms of related and perhaps better known properties. Brodskii and Milman [1948] have restated or characterised normal structure in terms of diametral sequences: *X has normal structure if and only if X does not contain a diametral sequence* (cf. Lemma 1.3.11). Indeed, if X contains a diametral sequence (x_n) then, as can be readily seen (see for example Swaminathan 1983, p. 203), $\overline{co}\{x_n\}$ fails to contain nondiametral points and, conversely, if X fails normal structure, then (by Lemma 1.3.11) it contains a diametral sequence.

The following proposition gives a class of sets with normal structure. It is actually a corollary of Lemma 1 of de Marr [1963] which states:

If K is a nontrivial compact subset of a Banach space, then coK contains a nondiametral point.

Proposition 2.2.3 (Brodskii-Milman 1948): Every compact convex set of a Banach space has normal structure. Consequently, finite dimensional spaces have normal structure, since bounded closed sets are compact.

Proof: Let K be a compact convex subset of X. Suppose that K fails normal structure (of course, we can suppose without loss of generality that diam K = 1, otherwise the result would be trivial). Then by Lemma 1.3.9 (see the construction of the sequence) there exists a sequence (x_n) in K such that for any $\varepsilon \in (0,1)$

$$\|x_{n+1} - x_k\| \geq 1 - \varepsilon \quad \text{for all } k = 1,2,\ldots,n .$$

In this case, the sequence (x_n) has no convergent subsequence, contradicting K compact, and this proves the proposition.

□

There exist spaces, indeed reflexive spaces, which fail normal structure, for example:

Example 2.2.4: (i) In $C[0,1]$ let $K = \{x(t) : 0 \leq x(t) \leq 1, x(0) = 0, x(1) = 1\}$. Clearly K is a bounded convex subset of $C[0,1]$. If $z \in K$ then by continuity given $\varepsilon > 0$ there is a $\delta > 0$ such that $z(t) < \varepsilon$ for $0 < t < \delta$. Clearly there is an $x \in K$ such that $x(t) = 1$ for $t \geq \frac{\delta}{2}$. It follows that $\|z - x\| \geq 1 - \varepsilon$ showing that z is a diametral point of K. Thus $C[0,1]$ fails to have normal structure.

(ii) In ℓ_2, let $\|x\| = \max\{\frac{1}{2}\|x\|_2, \|x\|_\infty\}$ where $\|\cdot\|_2$ is the ℓ_2-norm and $\|\cdot\|_\infty$ is the sup-norm of ℓ_∞-space. Then

$$\frac{1}{2}\|x\|_2 \leq \|x\| \leq \|x\|_2 .$$

Hence the two norms are equivalent and therefore the space $X_2 \equiv (\ell_2, \frac{1}{2}\|x\|_2 \vee \|x\|_\infty)$ is reflexive.

Let $C = \{x = (x_1, x_2, \ldots, x_n, \ldots) \in \ell_2 : \|x\| \leqslant 1, \ x_n \geqslant 0 \ \ \forall n\}$.

Then C is bounded closed convex, hence weakly compact by Theorems 1.2.7 and 1.2.15. In the new norm, diam $C = 1$. Each point of C is diametral. For if $x \in C$ and for any $\varepsilon > 0$ choose n_0 such that $0 \leqslant x_{n_0} < \varepsilon$ and let $y = e_{n_0} \in C$. Then $\|x - y\| > |x_{n_0} - y_{n_0}| > 1 - \varepsilon$. Hence every $x \in C$ is a diametral point. Thus X_2 fails to have normal structure.

This example, due to R.C. James, is of special importance in that it shows that:

weak compactness or even reflexivity of the Banach space are not strong enough properties to imply normal structure.

(iii) We showed in Example 1.3.8 that in the space c_0 the bounded convex set $K = \{x = (x_1, x_2, \ldots, x_n, \ldots) : x_n \geqslant 0 \ \forall n, \|x\|_\infty \leqslant 1\}$ of the unit ball is diametral, hence c_0 fails normal structure. Similar examples show that the spaces ℓ_1 and $L_1[a,b]$ do not have normal structure.

Remark 2.2.5: From the above examples we know that $C[a,b]$, c_0, ℓ_1, $L_1[a,b]$ and $(\ell_2, \frac{1}{2}\|x\|_2 \vee \|x\|_\infty)$ fail to have normal structure. (Indeed, van Dulst [1982] has shown that every Banach space may be equivalently renormed so as to *fail* normal structure.) None-the-less Baillon-Schoneberg [1981] have shown that $(\ell_2, \frac{1}{2}\|x\|_2 \vee \|x\|_\infty)$ has the FPP. Thus, as shown by Karlovitz [1976a] when he proved that the reflexive space $X_{\sqrt{2}} \equiv (\ell_2, \frac{1}{\sqrt{2}}\|x\|_2 \vee \|x\|_\infty)$ has the FPP, that:

normal structure is not necessary for the fixed point property, even in reflexive spaces.

Maurey [1980/81; cf. Elton-Lin-Odell-Szarek 1983] has recently shown that c_0 *has the w-FPP*. Previous to Maurey's result, others were only able to exhibit certain closed convex subsets of c_0 with the FPP. For example, Odel-Sternfeld [1981] showed that the closed convex hull of a weakly convergent sequence (including $\overline{co}\{e_n\}_{n=1}^{\infty}$), which fails normal structure, has the FPP. Also Haydon-Odell-Sternfeld [1981] showed that weakly compact "coordinate-wise star-shaped (c.s.s.)" subsets of c_0 have the FPP. A subset of K of c_0 is said to be c.s.s. if there exists a point $x \in K$ (called the centre of K) such that for all $y \in K$ and $z \in c_0$, if $z(i) \in \overline{co}\{x(i), y(i)\}$ for all i, then $z \in K$.

Maurey [cf. Elton-Lin-Odell-Szarek 1983] also showed that all reflexive subspaces of $L_1[0,1]$ have the w-FPP.

Lim [1980] proved that ℓ_1 in its natural norm has w*-normal structure and hence by Theorem 2.2.1 the w*-FPP. The fact that ℓ_1 has the w*-FPP was previously proved by Karlovitz [1976b]. Note that in ℓ_1, the basis vectors (e_n) form a diametral sequence, hence by Remark 2.2.2, ℓ_1 does not have normal structure. Indeed ℓ_1 fails normal structure, since the closed convex set $K = \{(x_n) : x_n \geq 0 \ \forall n, \sum_{n=1}^{\infty} x_n = 1\}$ is diametral $(\forall x \in K, \ \sup\{\|x - y\| : y \in K\} = \text{diam } K = 2)$. Since weak compactness coincides with compactness in ℓ_1 (see Theorem 1.2.7), it follows that ℓ_1 does have w-normal structure. This justifies some of the Remarks 2.1.4 (normal structure ⇒ w-normal structure but not conversely).

Day-James-Swaminathan [1971] have shown that every separable Banach space can be renormed so as to *have* normal structure and hence the w-FPP.

2.3 Uniformly Convex Banach Spaces

The main result in this section, which is due to Browder [1965b], exhibits a class of spaces with normal structure.

Definition 2.3.1 (Clarkson 1936): A Banach space X is said to be *uniformly convex (u.c.)* (*uniformly rotund*, in the terminology of Day [1973]) if and only if for each $\varepsilon \in (0,2]$ there is a $\delta = \delta(\varepsilon) > 0$ such that for all $x,y \in S(X)$, we have

$$\|x - y\| \geq \varepsilon \Rightarrow \|\frac{x+y}{2}\| \leq 1 - \delta(\varepsilon) .$$

In other words, X is uniformly convex if for any two points x,y on the unit sphere S(X) the midpoint of the segment joining x to y can be close to the sphere only if x and y are sufficiently close to each other. Without loss of generality, S(X) may be substituted by B[X] in our definition of uniform convexity.

In terms of sequences, we have the following equivalent formulation (Clarkson 1936): X is *uniformly convex* if whenever (x_n) and (y_n) are sequences in S(X), then

$$\|\frac{x_n + y_n}{2}\| \to 1 \Rightarrow \|x_n - y_n\| \to 0 \text{ as } n \to \infty .$$

Definition 2.3.2: The *modulus of convexity* of a Banach space X is defined to be the function on $(0,2]$ defined by the relation

$$\delta_X(\varepsilon) = \inf \{1 - \|\tfrac{x + y}{2}\| : \|x\| \leqslant 1, \|y\| \leqslant 1, \|x - y\| \geqslant \varepsilon\} .$$

It is clear that X is uniformly convex if $\delta_X(\varepsilon) > 0$ for every $\varepsilon > 0$.

Example 2.3.3: (i) Any Hilbert space (indeed, an Euclidean space of any dimension) is uniformly convex. To show this, it suffices to recall that in such spaces, the parallelogram identity

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

holds for any pair of vectors x and y. Hence it easily follows that

$$\delta_X(\varepsilon) = \{1 - \tfrac{1}{2}\sqrt{4 - \varepsilon^2}\} > 0 \quad \text{where} \quad \varepsilon \in (0,2] .$$

(ii) It is well known (see Clarkson 1936) that for $1 < p < \infty$ the spaces ℓ_p and $L_p[a,b]$ are uniformly convex. In the space ℓ_1, the midpoint of the segment joining points $(1,0,0,\ldots)$ and $(0,1,0,\ldots)$ of the unit sphere $S(\ell_1)$ also lies on $S(\ell_1)$ so ℓ_1 is not uniformly convex. Similarly, the example of points $(1,1,0,0,\ldots)$ and $(0,1,1,0,\ldots)$ shows that neither the space ℓ_∞ nor c_0 are uniformly convex. Also it may be easily seen that $L_1[a,b]$ and $C[a,b]$ are not uniformly convex.

Now the main result of this section.

Theorem 2.3.4 (Browder 1965b; cf. Edelstein 1963): Every uniformly convex Banach space has normal structure.

Proof: Suppose K is a nontrivial bounded closed convex subset of X of diam K = r. Let p,q ∈ K with $\|p - q\| \geq \frac{r}{2}$. Now for any s ∈ K put $\frac{p - s}{r} = x$, $\frac{q - s}{r} = y$. Then we have

$$\|x\| = \frac{1}{r}\|p - s\| \leq 1$$

(2.4) $$\|y\| = \frac{1}{r}\|q - s\| \leq 1$$

$$\|x - y\| = \frac{1}{r}\|p - q\| \geq \frac{1}{2} .$$

But X is uniformly convex so (2.4) implies that

$$\left\|\frac{x + y}{2}\right\| < 1 - \delta(\tfrac{1}{2})$$

where δ(ε), the "modulus of convexity" is positive for $0 < \varepsilon \leq 2$. Now let $t = \frac{p + q}{2}$. By convexity of K, t ∈ K, and by (2.4) we have

$$\left\|\frac{x + y}{2}\right\| = \frac{1}{r}\left\|\frac{p + q}{2} - s\right\| = \frac{1}{r}\|t - s\| < 1 - \delta(\tfrac{1}{2}) .$$

So for every s ∈ K we have

$$\|t - s\| < r(1 - \delta(\tfrac{1}{2})) < r .$$

That is $\sup\{\|t - s\| : s \in K\} < r$ - diam K. Hence K has a non-diametral point. Since K is arbitrary, X has normal structure.

□

Remark 2.3.5: In the following Corollary 2.3.6, we will need to make use of a well known result of Milman and Pettis which states:

Every uniformly convex Banach space is reflexive.

We will show that all uniformly convex spaces have the FPP for nonexpansive mappings, but more generally, so do all reflexive Banach spaces which have normal structure (see Kirk 1965). It is still an open question as to whether: *every reflexive space has the FPP for nonexpansive mappings?*

It follows from Edelstein [1963] that:

every uniformly convex Banach space has uniform normal structure.

Corollary 2.3.6 *(Browder 1965b; Göhde 1965): If T is a nonexpansive self mapping of a bounded closed convex subset K of a uniformly convex Banach space, then T has a fixed point.*

Proof: Since a uniformly convex Banach space is reflexive (by Milman-Pettis), any bounded closed convex set in it is weakly compact (by Theorems 1.2.7 and 1.2.15). By the preceding Theorem 2.3.4, it has normal structure. Hence the result follows from Kirk's Theorem 2.2.1. That is, U.C. spaces have the w-FPP.

□

2.4 Strictly Convex Banach Spaces

Definition 2.4.1 (Clarkson 1936): A Banach space X is called *strictly convex* (*rotund*, in the terminology of Day 1973) if and only if for all $x,y \in X$, $x \neq y$,

$$\|x\| = \|y\| = 1 \Rightarrow \|\frac{x+y}{2}\| < 1 .$$

Equivalently, X is strictly convex if and only if for $x,y \in S(X)$, we have

$$\|x\| = \|y\| = \|\frac{x+y}{2}\| \Rightarrow x = y .$$

Example 2.4.2: (i) ℓ_1^2 and hence ℓ_1^n, ℓ_1 are not strictly convex. For in ℓ_1^2, the points $x = (-1,0)$, $y = (0,1)$ lie on $S(\ell_1^2)$, but so does their midpoint $\frac{x+y}{2} = (-\frac{1}{2},\frac{1}{2})$.

(ii) ℓ_∞^2 and hence ℓ_∞^n, c_0 and ℓ_∞ are not strictly convex. For in ℓ_∞^2, the points $x = (-1,1)$, $y = (1,1)$ lie on $S(\ell_\infty^2)$, but so does their midpoint $\frac{x+y}{2} = (0,1)$.

(iii) $C[0,1]$ is not strictly convex. For the points $x(t) = 1 - t$, $y(t) = 1 - \frac{1}{2}t$ we have $\|x\|_\infty = \max_{t\in[0,1]} |1 - t| = 1$, $\|y\|_\infty = \max_{t\in[0,1]} |1 - \frac{1}{2}t| = 1$, $\|\frac{x+y}{2}\|_\infty = \max_{t\in[0,1]} |1 - \frac{3}{4}t| = 1$.

As is easily shown, none of the spaces ℓ_1, $L_1[0,1]$, ℓ_∞, c_0 and $C[0,1]$ are strictly convex. However, we have the following observation.

Proposition 2.4.3 (Clarkson 1936): Every uniformly convex Banach space is strictly convex.

Remark 2.4.4: Except for finite dimensional spaces, the converse of the above proposition is false. For example, the space

$$(\ell_2^2 \oplus \ell_3^2 \oplus \cdots \oplus \ell_n^2 \oplus \cdots)_2$$

is strictly convex but not uniformly convex.

The proposition gives an easier means of deciding which spaces fail uniform convexity.

2.5 Banach Spaces Which Are Uniformly Convex (Rotund) in Every Direction (UCED)

In this section we give a generalization of uniform convexity which also implies normal structure and hence the FPP.

Definition 2.5.1 (Garkavi 1964): A Banach space X is said to be *uniformly convex in every direction* (UCED), if given any $\varepsilon > 0$ and $d \in X$ with $\|d\| = 1$, there exists a $\delta = \delta(\varepsilon, d) > 0$ such that for every $x \in X$ with $\|x\| = 1$, we have

$$\|x + \varepsilon d\| = 1 \Rightarrow \|x + \frac{\varepsilon d}{2}\| \leq 1 - \delta .$$

The notion of UCED was first used by Garkavi [1964] to characterize normed linear spaces for which every bounded subset has at most one Chebyshev centre.

Day-James-Swaminathan [1971] gave the following equivalent formulation:

Definition 2.5.2 (Day-James-Swaminathan 1971): A Banach space X is UCED if and only if for any sequences (x_n), (y_n) in X we have

$$\left.\begin{array}{lll} (2.5A) & \|y_n\| \leq 1, \ \|x_n\| \leq 1 \text{ for all } n & \\ (2.5B) & x_n - y_n \to z \quad \text{as } n \to \infty & \\ (2.5C) & \left\|\frac{x_n + y_n}{2}\right\| \to 1 \quad \text{as } n \to \infty & \end{array}\right\} \Rightarrow z = 0 .$$

Remark 2.5.3: The notion of UCED is a generalisation of uniform convexity whose geometric significance is that all chords of the unit ball B[X] that are parallel to a fixed direction and whose lengths are bounded below by a fixed positive number have the property that the mid-point of the chord lies uniformly deep inside B[X].

Proposition 2.5.4: Every uniformly convex Banach space is UCED.

Proof: Suppose X is uniformly convex. Let (x_n) and (y_n) be sequences satisfying conditions (2.5A), (2.5B) and (2.5C) above. By (2.5B), $\|x_n - y_n\| \to \|z\|$. But X is uniformly convex and so by Definition 2.3.1, we have $\|x_n - y_n\| \to 0$ as $n \to \infty$. That is $z = 0$. Hence X is UCED.

□

Remark 2.5.5: The converse of the above result is in general false. For example, let X be the ℓ_2-direct sum of the sequences of spaces $(\ell_n)_{n\geq 2}$; that is, let X be the collection of all sequences $x = (x_n)_{n\geq 2}$ such that $x_n \in \ell_n$ for each n and

$$|x|^2 = \sum_n \|x_n\|_n^2 < \infty .$$

We denote this space by $(\ell_2 \oplus \ell_3 \oplus \ldots)_2$. It is reflexive and UCED (see for example Bynum 1980, p. 432) but (Day-James-Swaminathan 1971; cf. Huff 1980) cannot be equivalently renormed to be UC.

If *X is UCED, then X is strictly convex.* The converse is not true. For example, the space $C[0,1]$ of all real continuous functions on the unit interval with the norm

$$\|f\| = \sup \{|f(t)|\} + \left(\int_0^1 |f(t)|^2 \, dt\right)^{\frac{1}{2}}$$

is strictly convex, but this space is not UCED (see pp. 126-127 of Garkavi 1964).

Proposition 2.5.6 (cf. Corollary 3 of Day-James-Swaminathan 1971): If a Banach space X is UCED, then X has normal structure and hence the w-FPP.

□

Remark 2.5.7: The above proposition follows from the observation: in the proof of Theorem 2.3.4 we only required "uniformity" in the "p - q" direction. Thus, if X is UCED, then every bounded closed convex set in X has a nondiametral point.

Gillespie and Williams [1979] have shown that *a space which is UCED does not necessarily have uniform normal structure.* Hence this shows that the converse of the statement, "uniform normal structure implies normal structure", is not true.

2.6 Locally Uniformly Convex (LUC) Banach Spaces

In §2.3 we introduced the notion of uniform convexity of the norm in a Banach space. Stated in geometrical terms, a norm is UC if whenever the midpoint of a variable chord in the unit sphere of the space approaches the boundary of the sphere, the length of the chord approaches zero. In this section we introduce a weaker type of convexity, which is called *local uniform convexity (rotundity,* LUC). Geometrically this differs from uniform convexity in that it is required that one end point of the chord remain fixed.

We present an example, due to Smith and Turett [1984, preprint], of a reflexive LUC space which does not have normal structure.

Definition 2.6.1 *(Lovaglia 1955):* A Banach space is said to be *locally uniformly convex (rotund)* if and only if given $\varepsilon > 0$ and an element x with $\|x\| = 1$, there exists a $\delta(\varepsilon,x) > 0$ such that

$$\left.\begin{array}{l} \|y\| = 1 \\ \\ \|x - y\| \geq \varepsilon \end{array}\right\} \Rightarrow \left\|\frac{x + y}{2}\right\| \leq 1 - \delta(\varepsilon,x) .$$

Equivalently, a Banach space X is LUC if whenever $x \in X$ and (x_n) is a sequence in X such that $\|x\| = 1$, $\|x_n\| \to 1$, and $\|\frac{x + x_n}{2}\| \to 1$, then $x_n \to x$.

It is clear from the definitions that UC $\Rightarrow$ LUC and LUC $\Rightarrow$ SC.

Example 2.6.2 (Smith-Turett 1984, preprint): Smith [1978] gave an example of a Banach space $(\ell_2, \|\cdot\|_L)$, which is reflexive, LUC but not UCED. We present a proof of Smith and Turett [1984, preprint] that the space $(\ell_2, \|\cdot\|_L)$ does not have normal structure. But first of all an explanation of the norm $\|\cdot\|_L$:

Recall the equivalent norm defined on c_0 by Day [1955]. For u in c_0 enumerate the support of u as (n_k) such that $|u(n_k)| \geq |u(n_{k+1})|$ for $k = 1,2,\ldots$, define Du in ℓ_2 by

$$Du(n) = \begin{cases} \dfrac{u(n_k)}{2^k} & \text{if } n = n_k \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$

and define $|||u||| = \|Du\|_2$, where $\|\cdot\|_2$ is the usual ℓ_2-norm. Then $|||\cdot|||$ is an equivalent strictly convex norm on c_0. For $x = (x^1, x^2, \ldots)$ in ℓ_2, let

$$u = (\tfrac{1}{2}\|x\|_2\,,\ x^1, x^2, x^2, \ldots, \underbrace{x^j, x^j, \ldots, x^j}_{j}, \ldots)$$

be the element of c_0 associated with x and define

$$\|x\|_L = |||u||| .$$

Then $\|\cdot\|_L$ is an equivalent norm on $(\ell_2, \|\cdot\|_2)$.

Now to show that $(\ell_2, \|\cdot\|_L)$ lacks normal structure.

By Remark 2.2.2 (Brodskii-Milman) we just need to show that $(\ell_2, \|\cdot\|_L)$ contains a diametral sequence. In fact, we show that the usual unit vector basis (e_j) in ℓ_2 is a diametral sequence in $(\ell_2, \|\cdot\|_L)$. For $m > n$,

$$\|e_m - e_n\|_L = |||(\tfrac{\sqrt{2}}{2},0,\ldots,0,\underbrace{-1,\ldots,-1}_{n},0,\ldots,0,\underbrace{1,\ldots,1}_{m},0,\ldots)|||$$

$$= \left(\sum_{k=1}^{m+n} 4^{-k} + \frac{4^{-(m+n+1)}}{2}\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{k=1}^{\infty} 4^{-k}\right)^{\frac{1}{2}}$$

and thus $\operatorname{diam}\{e_j\} \leq \frac{1}{\sqrt{3}}$. But, from the computation above, it follows that

$$\lim_{m\to\infty} \|e_m - e_n\|_L = \frac{1}{\sqrt{3}}$$

and hence $\operatorname{diam}\{e_j\} = \frac{1}{\sqrt{3}}$.

Suppose $\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$ where $0 \leq \alpha_i \leq 1$ for $1 \leq i \leq n$. Then, if $p = 1 + (1+2+\ldots+n)$

$$\|e_{n+1} - \sum_{i=1}^{n} \alpha_i e_i\|_L$$

$$= |||(\tfrac{1}{2}\|e_{n+1} - \sum_{i=1}^{n} \alpha_i e_i\|_2, \; -\alpha_1,\ldots,\underbrace{-\alpha_n,\ldots,-\alpha_n}_{n},\underbrace{1,\ldots,1}_{n+1},0,\ldots)|||$$

$$\geq |||(\underbrace{0,\ldots,0}_{p},\underbrace{1,\ldots,1}_{n+1},0,\ldots)|||$$

$$= \left(\sum_{k=1}^{n+1} 4^{-k}\right)^{\frac{1}{2}}$$

and hence

$$\text{dist}(e_{n+1}, \; \text{co}\{e_1,e_2,\ldots,e_n\}) \geq \left(\sum_{k=1}^{n+1} 4^{-k}\right)^{\frac{1}{2}} . \tag{2.6}$$

Also

$$\|e_{n+1} - \sum_{i=1}^{n} \alpha_i e_i\|_L \leq |||(\underbrace{1,\ldots,1}_{p},\underbrace{1,\ldots,1}_{n+1},0,\ldots)|||$$

$$\leq \left(\sum_{k=1}^{\infty} 4^{-k}\right)^{\frac{1}{2}}$$

and hence

$$\text{dist}(e_{n+1}, \; \text{co}\{e_1,e_2,\ldots,e_n\} \leq \frac{1}{\sqrt{3}} .$$

From this last inequality and (2.6), it follows that

$$\text{dist}(e_{n+1}, \; \text{co}\{e_1,e_2,\ldots,e_n\}) \to \frac{1}{\sqrt{3}} = \text{diam}\{e_j\} .$$

That is, (e_j) is a diametral sequence in $(\ell_2, \|\cdot\|_L)$ and hence $(\ell_2, \|\cdot\|_L)$ fails to have normal structure.

□

2.7 Uniformly Smooth Banach Spaces

We show in this section that uniformly smooth Banach spaces have normal structure and hence the FPP.

Definition 2.7.1: The *modulus of smoothness* of a Banach space X is defined, for $\tau > 0$, by

$$\rho_X(\tau) = \sup\left\{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : \|x\| = \|y\| = 1\right\} .$$

A Banach space is said to be *uniformly smooth* if

$$\lim_{\tau\to 0^+} \rho_X(\tau) = 0 .$$

Remark 2.7.2: It is known that: *X is uniformly smooth if and only if X^* is uniformly convex.*

Baillon [1978/79] has shown that: *if X is a reflexive Banach space whose modulus of smoothness ρ_X satisfies*

$$\lim_{\tau\to 0^+} \frac{{}_X(\tau)}{\tau} < \frac{1}{2} ,$$

then X has the FPP; in particular he proved.

***Theorem 2.7.3** (Baillon 1978/79):* *If a Banach space X is uniformly smooth then X has normal structure (and so the FPP for nonexpansive mappings). Also X^* has normal structure (and hence the FPP).*

Remark 2.7.4: By Remark 2.7.2, X^* is uniformly convex if X is uniformly smooth. So by Theorem 2.3.4, X^* has normal structure and hence by Corollary 2.3.6, the FPP for nonexpansive mappings. Thus we need only show that X has normal structure. To this end we present a proof by Turett [1982].

***Proof** (of Theorem 2.7.3 by Turett 1982):* Assume that X fails to have w-normal structure. Then we have:

Step 1: There exists a sequence (x_n) in $B[X]$ such that $x_n \overset{w}{\to} 0$, $\|x_n\| \to 1$, $\operatorname{diam}\{x_1,x_2,\ldots\} = 1$, $x_1 = 0$ and $\operatorname{dist}(x_n, \overline{\operatorname{co}}\{x_1,x_2,\ldots, x_{n-1}\}) > 1 - \frac{1}{n}$.

Proof of Step 1: By Lemma 1.3.11 (Brodskii-Milman 1948 or Gossez-Lami Dozo 1972, p. 567) there exists a sequence $(x_n)_{n=2}^{\infty}$ such that $x_n \overset{w}{\to} 0$, $\|x_n\| \to 1$, $\operatorname{diam}\{x_2,x_3,\ldots\} = 1$ and $\operatorname{dist}(x_n, \overline{\operatorname{co}}\{x_2,x_3,\ldots,x_{n-1}\}) > 1 - \frac{1}{n}$. Defining $x_1 = 0$ yields to desired sequence. The only condition which is not obvious is the last one. Since $x_n \overset{w}{\to} 0$, by Theorem 1.2.7 (Mazur)

$$\operatorname{dist}(0, \overline{\operatorname{co}}\{x_2,x_3,\ldots,x_{n-1}\}) \to 0 .$$

Choose $y_n \in \overline{\operatorname{co}}\{x_2,x_3,\ldots,x_{n-1}\}$ such that $\|y_n\| \to 0$. Then, for $\lambda_k \geq 0$ with $\sum_{k=1}^{n-1} \lambda_k = 1$ we have

$$\begin{aligned}\Big\|x_n - \sum_{k=1}^{n-1} \lambda_k x_k\Big\| &\geq \Big\|x_n - (\lambda_1 y_1 + \sum_{k=2}^{n-1} \lambda_k x_k)\Big\| - \lambda_1 \|y_n\| \\ &\geq \operatorname{dist}(x_n, \overline{\operatorname{co}}\{x_2,x_3,\ldots,x_{n-1}\}) - \|y_n\| \\ &\geq 1 - \frac{1}{n} - \|y_n\| .\end{aligned}$$

Since $\|y_n\| \to 0$, $\operatorname{dist}(x_n, \overline{\operatorname{co}}\{x_1,x_2,\ldots,x_{n-1}\} \to 1$.

□

Step 2: Given $\varepsilon > 0$, there exists norm-one elements $f,g \in X^*$ and $x \in X$ such that

$$\|f - g\|,\ f(x),\ g(x) > 1 - \varepsilon .$$

That is, the dual ball $B[X^*]$ contains arbitrarily "thin" slices with diameter near one (or more). By a slice of thickness ε, we mean the set

$$S(B[X^*],\ x,\ \varepsilon) = \{f \in B[X^*] : f(x) > 1 - \varepsilon\}$$

Proof of Step 2: For each n,

$$B_{1-\frac{1}{n}}(x_{n+1}) \cap \overline{co}\,\{x_k\}_{k=1}^{n} = \phi .$$

Applying the basic separation theorem 1.2.14 to $B_{1-\frac{1}{n}}(x_{n+1})$ and $\overline{co}\,\{x_k\}_{k=1}^{n}$ yields a norm-one functional f_{n+1} such that

$$1 \geq f_{n+1}(x_{n+1} - x_k) > 1 - \frac{1}{n} \text{ for all } k \leq n .$$

In particular, $k = 1$ gives $f_{n+1}(x_{n+1}) > 1 - \frac{1}{n}$. Now choose $j_0 \in \mathbb{N}$ with $1 < j_0 < \frac{2}{\varepsilon}$. Since $x_n \overset{w}{\to} 0$ there exists $n_0 > j_0$ such that $|f_{j_0}(x_{n_0})| < \frac{\varepsilon}{2}$. But then

$$-f_{j_0}\left(\frac{x_{n_0} - x_{j_0}}{\|x_{n_0} - x_{j_0}\|}\right) \geq -f_{j_0}(x_{n_0} - x_{j_0}) > 1 - \varepsilon,$$

$$f_{n_0}\left(\frac{x_{n_0} - x_{j_0}}{\|x_{n_0} - x_{j_0}\|}\right) \geq f_{n_0}(x_{n_0} - x_{j_0}) > 1 - \frac{1}{n_0} > 1 - \varepsilon$$

and $\|f_{n_0} - (-f_{j_0})\| \geq (f_{n_0} + f_{j_0})(x_{n_0}) > 1 - \frac{1}{n_0} - \frac{\varepsilon}{2} > 1 - \varepsilon.$

The result (that is, Step 2) now follows by taking $f = f_{n_0}$, $g = -f_{j_0}$ and $x = \dfrac{x_{n_0} - x_{j_0}}{\|x_{n_0} - x_{j_0}\|}$.

Hence the theorem.

□

Remark 2.7.5: It suffices to have X^* *ε-inquadrate* for some $\varepsilon < 1$, and so, as Turett shows, it is enough to have X satisfy

$$\lim_{\tau\to 0^+} \frac{\rho_X(\tau)}{\tau} < \frac{1}{2} .$$

A space X is ε-inquadrate if $\delta(\varepsilon) > 0$ for some $\varepsilon \in (0,2]$. That is, there exists $\varepsilon \in (0,2]$ and $\delta > 0$ such that $\|x - y\| < \varepsilon$ whenever $x,y \in S(X)$ and $\|\frac{x + y}{2}\| > 1 - \delta$. Clearly, *$X$ is UC if and only if X is ε-inquadrate for every $\varepsilon \in (0,2]$.*

2.8 Inheritance of Normal Structure from a Subspace

In this section we present a lemma which Giles-Sims-Swaminathan [1984, preprint] recently used to give an example of a reflexive Banach space which has normal structure, but which lacks the geometrical properties UCED (and therefore UC by Proposition 2.5.4), LUC and WUKK (see Chapter 4).

Lemma 2.8.1 (Giles-Sims-Swaminathan 1984): If a Banach space X contains a closed subspace M of finite codimension (or a subspace with complement a Schur space, for example, $X = \ell_1 \oplus M$) and with weak uniform normal structure, then X has w-normal structure.

Proof: Assume X fails w-normal structure, then by Lemma 1.3.11 there exists a sequence (x_n) with

$$\mathrm{dist}(x_{n+1}, \overline{\mathrm{co}}\,\{x_1,x_2,\ldots,x_n\}) \to \mathrm{diam}\ \overline{\mathrm{co}}\,\{x_n\}_{n=1}^{\infty} = 1.$$

Since any translate of a subsequence of (x_n) also has this property we may without loss of generality assume that $x_n \overset{w}{\to} 0$. Now, let P be the linear projection from X to M, then $x_n - Px_n \overset{w}{\to} 0$ in the finite dimensional complement of M and so we may choose N_0 so that

$$\|x_n - Px_n\| < \varepsilon < \frac{1-k}{2(1+k)} \quad \text{for } n \geq N_0 .$$

Let $C = \overline{\mathrm{co}}\,\{x_n\}_{n=N_0}^{\infty}$ then $\mathrm{diam}\ C = 1$, C is diametral, indeed $\|c - x_n\| \to 1$ for all $c \in C$, and for each $c \in C$ $\|c - Pc\| \leq \varepsilon$.

[To see this last inequality note that x may be approximated arbitrarily well by an element of the form $\sum_{n=N_0}^{\infty} \lambda_n x_n$ where $\lambda_n > 0$ and $\sum \lambda_n = 1$ and

$$\|\sum \lambda_n(x_n - Px_n)\| \leq \sum \lambda_n \sup \|x_n - Px_n\|$$

$$\leq \varepsilon .]$$

From this it follows that $\mathrm{diam}\ P(C) \leq 1 + 2\varepsilon$ and so by the uniform normal structure of M there exists $m_0 \in P(C)$ with

$$\|m_0 - Pc\| < k(1 + 2\varepsilon) \quad \text{for all} \quad c \in C .$$

But then, choosing $c_0 \in C$ with $Pc_0 = m_0$ we have, for all $c \in C$

$$\begin{aligned}\|c_0 - c\| &\leq \|c_0 - Pc_0\| + \|Pc_0 - Pc\| + \|Pc - c\| \\ &\leq 2\varepsilon + k(1 + 2\varepsilon) \\ &< 1\end{aligned}$$

contradicting the diametrality of C.

□

2.9 Asymptotic Normal Structure

Motivated by a method often employed in proofs of fixed point theorems for nonexpansive mappings (see Lemmas 1.3.1, 1.3.3 and 1.3.12) Baillon and Schöneberg [1981] introduced the following generalization of the concept of normal structure.

***Definition 2.9.1** (Baillon-Schöneberg 1981):* A Banach space X is said to have *asymptotic normal structure*, if for each nontrivial bounded closed convex subset K of X and each sequence (x_n) in K satisfying $x_n - x_{n+1} \to 0$ as $n \to \infty$, there is a point $p \in K$ such that

$$\liminf_{n\to\infty} \|x_n - p\| < \operatorname{diam} K .$$

Evidently normal structure implies asymptotic normal structure, since if $p \in K$ is a nondiametral point and if (x_n) is a sequence in K then

$$\liminf_{n\to\infty} \|x_n - p\| \leqslant \limsup_{n\to\infty} \|x_n - p\|$$

$$< \text{diam } K ,$$

but the converse is not true, as we see below in Theorem 2.9.4.

Using Lemmas 1.3.1, 1.3.3 and 1.3.12, Baillon and Schöneberg proved that:

***Theorem 2.9.2** (Baillon-Schöneberg 1981): Every Banach space with asymptotic normal structure has the w-FPP.*

The interest in Theorem 2.9.2 stems, of course, from the fact that normal structure defines a narrower class of Banach spaces than asymptotic normal structure. This can be seen, for example, from the next result. But first, an explanation of the space used in the result.

Example 2.9.3: The reflexive space $X_2 \equiv (\ell_2, \frac{1}{2}\|x\|_2 \vee \|x\|_\infty)$ was shown in Example 2.2.4(ii) to lack normal structure but was shown by Baillon-Schöneberg [1981] to have the w-FPP.

The reflexive space $X_{\sqrt{2}} \equiv (\ell_2, \frac{1}{\sqrt{2}}\|x\|_2 \vee \|x\|_\infty)$ also lacks normal structure but has the w-FPP, a fact first established by Korlovitz [1976a] using Lemma 1.3.12 and some detailed calculations in the space $(\ell_2, \|\cdot\|)$.

We give a general definition for these spaces.

Let $\beta \geqslant 1$ and let X_β be the real space ℓ_2 renormed by setting, for $x \in \ell_2$,

$$\|x\| = \max\{\frac{1}{\beta}\|x\|_2, \|x\|_\infty\} .$$

Then

$$\frac{1}{\beta}\|x\|_2 \leqslant \max\{\frac{1}{\beta}\|x\|_2, \|x\|_\infty\} \leqslant \|x\|_2$$

and so $\|\cdot\|$ is equivalent to $\|\cdot\|_2$. Hence X_β is reflexive.

Baillon-Schöneberg [1981] proved that:

Theorem 2.9.4 (Baillon-Schöneberg 1981): *(i) X_β has normal structure if and only if $\beta < \sqrt{2}$.*

(ii) X_β has asymptotic normal structure if and only if $\beta < 2$.

Remark 2.9.5: Theorem 2.9.4 is equivalent to

(i)' X_β fails normal structure if and only if $\beta \geqslant \sqrt{2}$.

(ii)' X_β fails asymptotic normal structure if and only if $\beta \geqslant 2$.

From Theorem 2.9.4 we see that Kirk's Theorem 2.2.1 is properly contained in Theorem 2.9.2. On the other hand, Theorem 2.9.4 shows (via Theorem 2.9.2) that for all $\beta < 2$, X_β has the w-FPP. Finally, Theorem 2.9.4 shows that Theorem 2.9.2 cannot be used to decide which of the spaces X_β with $\beta \geqslant 2$ have the w-FPP.

Maurey's C_0 result [1980/81; cf. Elton-Lin-Odell-Szarek 1983] has settled in the affirmative the question of whether X_β has the w-FPP for $\beta \geqslant 2$.

From above we know that X_2 fails asymptotic normal structure but has the w-FPP. Thus asymptotic normal structure is not essential for the fixed point property, even in reflexive spaces. Another example of a reflexive space without asymptotic normal structure is:

Example 2.9.6 (Bynum 1980): Let $\ell_{p,\infty}$ be ℓ_p, $1 < p < \infty$, renormed with

$$|||x||| = \max\{\|x^+\|, \|x^-\|\}$$

where for any $x = (x(n))_{n=1}^{\infty}$ in ℓ_p define

$$x^+ = \max\{x \quad ,0\}$$
$$x^- = (-x)^+ .$$

We note that for each $x \in \ell_p$, x^+ and x^- are in ℓ_p and $|||x||| \leq \|x\| \leq 2^{\frac{1}{p}}|||x|||$ and so $\ell_{p,\infty}$ $(1 < p < \infty)$ is reflexive. Bynum proves that the reflexive space $\ell_{p,\infty}$ lacks asymptotic normal structure. However, $\ell_{p,\infty}$ $(1 < p < \infty)$ has the FPP.

2.10 Examples of Alspach and Lim

We have seen classical results of Browder [1965b], Day-James-Swaminathan [1971], Kirk [1965] and others establish that every UCED space (hence every uniformly convex space) and every space with normal structure has the w-FPP. Unfortunately, not every space has (w) normal structure, and it remained open (see for example, Reich 1976, 1980) as to whether or not every Banach space possessed the w-FPP until Alspach [1981] gave an example of a fixed point free nonexpansive mapping on a weakly compact convex subset of $L_1[0,1]$ (see also Schechtman 1982). Lim settled, also in the negative, the corresponding problem for the w*-FPP in dual spaces.

Example 2.10.1 (Alspach 1981): Let $X = L_1[0,1]$. Let

$$K = \{f \in X : \int_0^1 f = 1, \quad 0 \leqslant f \leqslant 2 \text{ almost everywhere}\} .$$

K is a closed convex subset of the order interval $[0,2]$, hence K is weakly compact.

Let $$T : K \to K : f(t) \mapsto \begin{cases} \min\{2f(2t),2\}, & 0 \leqslant t < \frac{1}{2} \\ \max\{2f(2t-1)-2,0\}, & \frac{1}{2} \leqslant t < 1 . \end{cases}$$

T is an isometry and is fixed point free. That is, $L_1[0,1]$ fails the w-FPP.

□

Example 2.10.2 (Lim 1980): Let $X = C_0^* = \ell_1$ have the equivalent dual norm $\|x\| = \max\{\|x^+\|_1, \|x^-\|_1\}$ where for the sequence $x = (x_i)$ in ℓ_1, $x^+ = (x_i \vee 0)$ and $x^- = (-x)^+$.

Let

$$K = \{x \in \ell_1 : x_i \geqslant 0 \text{ for all } i, \ \|x\|_1 \leqslant 1\} .$$

Then K is a w*-compact convex subset of ℓ_1 and $T : K \to K$ defined by

$$Tx = (1 - \|x\|_1, \quad x_1, x_2, \ldots, x_n, \ldots)$$

is a fixed point free isometry of K in $(\ell_1, \|\cdot\|)$.

□

Remark 2.10.3: These examples still leave open:

(1) Does every reflexive space have the FPP?

The nature of T in both examples also suggests:

(2) If $X(X^*)$ fails to have w-FPP (w^*-FPP) does it necessarily fail for an isometry?

Chapter 3

OPIAL CONDITIONS

In this brief chapter, we give another condition of a more topological type on the norm of a Banach space. This condition was introduced by Z. Opial [1967] and will be referred to in these notes as the *w(w*)-strict Opial condition.* We will see that this condition links the w(w*)-topology to the norm topology and implies the w(w*)-FPP.

An interesting sufficient condition for the strict Opial conditions is given in Theorem 3.2.4.

It is to be noted that a variant of the strict Opial conditions formulated in terms of the Birkhoff-James orthogonality relation was studied by Karlovitz [1976b].

3.1 w(w*)-Strict Opial Condition

Definition 3.1.1 (Opial 1967): A Banach space $X(X^*)$ is said to satisfy the *w(w*)-Opial condition* if and only if for each sequence (x_n) in $X(X^*)$ we have

$$x_n \overset{w(w^*)}{\to} x_\infty \qquad \text{implies}$$

$$\liminf \|x_n - x_\infty\| \leq \liminf \|x_n - x\| \quad \text{for all } x \in X(X^*);$$

and the w(w*)-strict Opial condition if and only if

$$x_n \overset{w(w^*)}{\to} x_\infty \qquad \text{implies}$$

$$(3.1) \qquad \liminf \|x_n - x_\infty\| < \liminf \|x_n - x\| \quad \text{for all } x \neq x_\infty .$$

Example 3.1.2 (Opial 1967; Lemma 1): Every Hilbert space X satisfies the w-strict Opial condition: since every w-convergent sequence is necessarily bounded, both limits in (3.1) are finite. Thus, to prove this inequality (3.1), it suffices to observe that in the equality

$$\|x_n - x\|^2 = \|x_n - x_\infty + x_\infty - x\|^2$$

$$= \|x_n - x_\infty\|^2 + \|x_\infty - x\|^2 + 2\mathrm{Re} \langle x_n - x_\infty, x_\infty - x\rangle$$

the last term tends to zero as $n \to \infty$.

□

Example 3.1.3: It can also be checked that the spaces $\ell_p (1 \leq p < \infty)$ also satisfy the w-strict Opial condition. The space $\ell_1 \equiv C_0^*$ even satisfies the w*-strict Opial condition (see, for example, Karlovitz 1976b). On the other hand, it is shown in Opial [1967] that in the uniformly convex spaces $L_p[0,1]$ $(1 < p < \infty, p \neq 2)$ the w-strict Opial condition fails to hold.

Further examples may be found in the appendix. Recall that the space $(\ell_1, \|x^+\|_1 \vee \|x^-\|_1)$ (example 2.10.2) fails to have the w*-FPP. However, this space has the w*-Opial condition. To show this:

Assume not. Then without loss of generality we may assume there exists a sequence (x_n) with $x_n \overset{w^*}{\to} 0$, $\|x_n\| = 1$, and there exists an x such that $\liminf \|x_n - x\| = 1 - \delta$ for some $\delta > 0$. But then, $\liminf \|(x_n-x)^{\pm}\|_1 \leq 1 - \delta$ while $x_n^{\pm} \overset{w^*}{\to} 0$ and so by the Opial condition for $\|\cdot\|_1$ we must have $\liminf \|x_n^{\pm}\|_1 < 1 - \delta$, contradicting $\|x_n\| = 1$.

□

Thus the w*- and hence w-Opial condition is not sufficient for the w* -FPP. However,

Theorem 3.1.4 (Gossez-Lami Dozo 1972): *If X satisfies the w-strict Opial condition, then X has w-normal structure and hence the w-FPP.*

Proof: Assume that X fails to have w-normal structure, so by Lemma 1.3.11 (or Remark 2.2.2) there is a w-compact convex diametral nontrivial subset K containing a sequence (x_n) with

$$\operatorname{dist}(x_{n+1}, \overline{co}\{x_1,x_2,\ldots,x_n\}) \to \operatorname{diam} K .$$

Since any subsequence also inherits this property, we may assume that

$$x_n \overset{w}{\to} x_\infty \in C_n \quad \text{(by Mazur, that is Corollary 1.2.9)}$$

where

$$C_n = \overline{co}\{x_1,x_2,\ldots,x_n\} .$$

Thus, given any $\varepsilon > 0$, there is a finite sequence $\lambda_1,\lambda_2,\ldots\lambda_{n_0}$ of nonnegative real numbers with

$$\sum_{i=1}^{n_0} \lambda_i = 1$$

and

$$\left\| x_\infty - \sum_{k=1}^{n_0} \lambda_k x_k \right\| \leqslant \frac{\varepsilon}{2} .$$

Also there is an N $(\geqslant n_0 + 1)$ such that for $m \geqslant N$ we have

$$\begin{aligned} \operatorname{dist}(x_m, C_{n_0}) &\geqslant \operatorname{dist}(x_m, C_{m-1}) \\ &\geqslant \operatorname{diam} K - \frac{\varepsilon}{2} . \end{aligned}$$

But then

$$\begin{aligned} \operatorname{diam} K &\geqslant \|x_m - x_\infty\| \\ &\geqslant \left\|x_m - \sum_{k=1}^{n_0} \lambda_k x_k\right\| - \frac{\varepsilon}{2} \\ &\geqslant \operatorname{dist}(x_m, C_{n_0}) - \frac{\varepsilon}{2} \\ &\geqslant \operatorname{diam} K - \varepsilon . \end{aligned}$$

Thus $\|x_m - x_\infty\| \to \operatorname{diam} K$ as $m \to \infty$, and so for any $x \in K$ we have

$$\liminf_m \|x_m - x_\infty\| = \operatorname{diam} K \geqslant \liminf_m \|x_m - x\|$$

contradicting the w-strict Opial condition. Hence X has w-normal structure and so by Theorem 2.2.1 (Kirk) X has the w-FPP.

□

Remark 3.1.5: The use of Mazur's Theorem (Corollary) in the above argument precludes an analogous proof for the w*-case and leaves us with:

Open question: Does the w-strict Opial condition imply w*-normal structure (at least in the dual of a separable space X^* where $B[X^*]$ is w*-sequentially compact)?*

What is known is that the w*-strict Opial condition implies the w*-FPP. This was indirectly proved by Karlovitz [1976b] when he gave a variant of the strict Opial conditions formulated in terms of the Birkoff [1935]-James [1945] orthogonality relation. We present a more direct proof by van Dulst [1982].

Theorem 3.1.6 *(Karlovitz 1976b): If X^* is the dual of a separable Banach space and X^* has the w*-strict Opial condition, then X^* has the w*-FPP.*

Proof: Let K be a w*-compact convex subset of X^* and let $T : K \to K$ be nonexpansive. By Lemma 1.3.3 there exists in K an approximate fixed point sequence (x_n), that is, $\|Tx_n - x_n\| \to 0$. Passing to a subsequence if necessary, we may assume that

$$x_n \overset{w^*}{\to} x_\infty .$$

Then $Tx_\infty = x_\infty$. Indeed,

$$\begin{aligned}\liminf \|Tx_\infty - x_n\| &= \liminf \|Tx_\infty - Tx_n\| \\ &\leqslant \liminf \|x_\infty - x_n\| ,\end{aligned}$$

so $Tx_\infty \neq x_\infty$ would contradict the w*-strict Opial condition.

□

Remark 3.1.7: The above proof applies equally well to the w-case without the need for a separability assumption.

The assumption that X is separable is only required to ensure that the unit ball $B[X^*]$ is w*-sequentially compact and so the above result extends to the dual of any "smoothable" (admits an equivalent smooth norm) space X.

3.2 A Sufficient Condition for the Strict Opial Condition

Definition 3.2.1 (Browder 1966): Let $D : K \to K^*$ be a mapping of a Banach space X into X^*, μ a continuous strictly increasing real-valued function on $\mathbb{R}^+$ with $\mu(0) = 0$ (that is, μ is a *gauge function*). We say that D is a *support mapping* of X into X^* with gauge function μ if both of the following conditions are satisfied:

(i) For every $x \in X$, $D(x)(x) = \|D(x)\| \; \|x\|$;

(ii) For every $x \in X$, $\|D(x)\| = \mu(\|x\|)$.

Browder showed that: for $1 < p < \infty$ $(\frac{1}{p} + \frac{1}{q} = 1)$ the space ℓ_p has a w-continuous duality mapping into $\ell_p^* = \ell_q$, but not the space $L_4([0,2\pi])$.

The duality mapping

$$\mathcal{D} : S(X) \to 2^{X^*} : x \mapsto \mathcal{D}(x) = \left\{ f \in X^* : \|f\| = f(x) = 1 \right\}$$

is *norm to norm upper semi-continuous* if given $\varepsilon > 0$ and $x \in S(X)$, there exists $\delta > 0$ such that for all $y \in S(X)$ and $\|x - y\| < \delta$ we have $\mathcal{D}(y) \subseteq \mathcal{D}(x) + B_\varepsilon[0]$. $\mathcal{D}$ is *norm to norm uniformly upper semi-continuous* if there is a δ for all x.

Remark 3.2.2: Browder [1966] and Opial [1967] considered spaces with a w to w* continuous support mapping (selector for the duality mapping $\mathcal{D}$). Opial showed that uniformly convex spaces with such a mapping have the w-strict Opial condition. Gossez and Lami Dozo showed that the assumption of uniform convexity was unnecessary. By a slight modification of their argument, Sims [1982] arrives at an interesting sufficient condition for the strict Opial conditions. This is presented in Theorem 3.2.4. But first, some more preliminary definitions.

Definition 3.2.3: We say that the extended duality mapping

$$\widetilde{\mathcal{D}} : X \to 2^{X^*} : x \mapsto \widetilde{\mathcal{D}}(x) = \left\{ f \in X^* : f(x) = \|f\| \, \|x\|, \|f\| = \|x\| \right\}$$
$$= \|x\| \, \mathcal{D}\left(\frac{x}{\|x\|}\right)$$
$$(x \neq 0)$$

is *pseudo sequentially continuous w to w** if;

given any w*-neighbourhood N^* of 0, if $x_n \overset{w}{\to} x_\infty$ then eventually

$$\widetilde{\mathcal{D}}(x_n) \cap [\widetilde{\mathcal{D}}(x_\infty) + N^*] \neq \phi$$

(with a similar definition for pseudo sequential continuity w* to w* in the case of a dual space).

$\widetilde{\mathcal{D}}$ is pseudo sequentially continuous w to w* whenever $\widetilde{\mathcal{D}}$ is w to w* upper (or lower) semi-continuous. Since $\widetilde{\mathcal{D}}$ is always n to w* upper-semi-continuous we see that any Schur space has $\widetilde{\mathcal{D}}$ pseudo sequentially continuous w to w*. The same is true of any Hilbert space and the spaces $\ell_p (1 < p < \infty)$. However, by a result of Fixman and Rao

[1982], $\widetilde{\mathcal{D}}$ fails to be pseudo sequentially continuous w to w* (equal to the w to w continuity of the unique support mapping) in $L_p(\Omega,\Sigma,\mu)$ unless every measurable subset of finite positive measure contains an atom. The failure in $L_4[0,1]$ had been previously noted by Browder, and in $L_p[0,1]$ $(1 < p < \infty)$ by Opial.

Now the result.

Theorem 3.2.4 (Sims 1982): If $X(X^*)$ has $\widetilde{\mathcal{D}}$ pseudo sequentially continuous $w(w^*)$ to w^*, then $X(X^*)$ satisfies the $w(w^*)$-strict Opial condition.

Proof: For the *convex* function $\mathbb{R} \to \mathbb{R} : t \mapsto \|x + ty\|$ we have (see, for example, Roberts and Varberg 1973)

$$\|x + y\| = \|x\| + \int_0^1 g^+(x + ty;\ y)dt$$

where

$$g^+(x + ty;\ y) = \lim_{h\to 0^+} \frac{\|x + (t+h)y\| - \|x + ty\|}{h}$$

$$= \max\left\{f(y) : f \in \mathcal{D}\left(\frac{x + ty}{\|x + ty\|}\right)\right\} .$$

Now let $x_n \overset{w(w^*)}{\to} x_\infty$ and $x_0 \in X$, then substituting we obtain

$$\|x_n - x_0\| = \|x_n - x_\infty\| + \int_0^1 g^+((x_n - x_\infty) + t(x_\infty - x_0);\ x_\infty - x_0)dt .$$

So

$$\liminf_n \|x_n - x_0\|$$

$$\geq \liminf_n \|x_n - x_\infty\| + \liminf_n \int_0^1 g^+(x_n - x_\infty) + t(x_\infty - x_0);\ x_\infty - x_0)dt.$$

By Fatou's Lemma, it therefore is sufficient to prove

$$\lim_n \inf g^+((x_n - x_\infty) + t(x_\infty - x_0);\ x_\infty - x_0) > 0$$

for all $t \in (0,1)$ and $x_0 \neq x_\infty$.

Now, for each t let

$$N_t^* = \left\{ g : g(x_\infty - x_0) > -\frac{t}{2} \|x_\infty - x_0\| \right\}$$

since $(x_n - x_\infty) + t(x_\infty - x_0) \overset{w(w^*)}{\to} t(x_\infty - x_0)$, for n sufficiently large there exists

$$\tilde{f}_n \in \tilde{\mathcal{D}}((x_n - x_\infty) + t(x_\infty - x_0))$$

such that

$$\tilde{f}_n = \tilde{f} + g \quad \text{where } \tilde{f} \in \tilde{\mathcal{D}}(t(x_\infty - x_0))$$
$$\text{and } g \in N^* .$$

Thus

$$\tilde{f}_n(x_\infty - x_0) = t\|x_\infty - x_0\|^2 + g(x_\infty - x_0) > \frac{t}{2}\|x_\infty - x_0\|^2$$

and so for n sufficiently large

$$g^+((x_n - x_\infty) + t(x_\infty - x_0);\ x_\infty - x_0) \geq \frac{\tilde{f}_n(x_\infty - x_0)}{\|(x_n - x_\infty) + t(x_\infty - x_0)\|}$$

$$\geq \frac{\|x_\infty - x_0\|}{2 + \frac{2}{t}\|x_n - x_\infty\|}$$

$$> 0$$

as required.

□

Open question: Are the strict Opial conditions characterized by "pseudo continuity" properties of $\widetilde{V}$?

Remark 3.2.5: van Dulst [1982; Theorems 1 and 2] has shown that: *every separable space* X *(separable dual space* X^**) may be given an equivalent (dual) norm* $\|\cdot\|_1$ *which satisfies the w(w*)-strict Opial condition. In particular then,* $(X, \|\cdot\|_1)$ $((X^*, \|\cdot\|_1))$ *has the w(w*)-FPP.*

These observations naturally raise the question of how w(w*)-strict Opial conditions relate to other sufficient conditions for the w(w*)-FPP, for example UCED, and as we see in the next Chapter, w(w*)-shrinking ball properties. We will show by examples that all of these conditions are essentially independent of one another.

Chapter 4

NEAR UNIFORM CONVEXITY AND RELATED PROPERTIES

In this chapter we consider another generalisation of uniform convexity (UC), namely, *near uniform convexity* (NUC; see Huff 1980), and a further weakening of NUC (see van Dulst and Sims 1981) called *the weakly uniformly Kadec-Klee* (WUKK) property. Both properties (NUC and WUKK) imply the w-FPP.

We also see that the WUKK (WUKK*) property is equivalent to the ε;w(w*)-shrinking ball property and these in turn imply w(w*)-normal structure and hence the w(w*)-FPP. This equivalence is due to Sims [1982]. He also shows that the ε;w-shrinking ball property in reflexive spaces is equivalent to Huff's NUC property.

In §4.4 we introduce the concept of w(w*)-asymptotic normal structure (Lim 1974, 1980) which are to be distinguished from the concept of asymptotic normal structure considered in §2.9. We see that w-asymptotic normal structure is equivalent to w-normal structure (and hence the w-FPP) and w*-asymptotic normal structure implies w*-normal structure (and hence the w*-FPP) but that the converse of this last statement still remains an open question. We include a result in relation to the w(w*)-shrinking ball property.

4.1 Preliminary Definitions

Definition 4.1.1 (Day 1973): (The norm of) a Banach space X is said to be *Kadec-Klee* (KK) provided that on the unit sphere sequences converge in norm whenever they converge weakly. Such a space is also said to have *property H*. In particular such a space satisfies:

$$\text{(KK)} \qquad \left.\begin{array}{l} (x_n) \subset B(X) \\ x_n \overset{w}{\to} x_\infty \\ \operatorname{sep}(x_n) > 0 \end{array}\right\} \Rightarrow \|x_\infty\| < 1 \quad .$$

Here, $\operatorname{sep}(x_n) = \inf \{ \|x_n - x_m\| : n \neq m \}$.

Huff [1980] reformulated the (KK) property and introduced two successively stronger notions, namely *uniformly Kadec Klee* (UKK) and *nearly uniformly convex* (NUC). We recall his definitions.

Definition 4.1.2 (Huff 1980): (The norm of) a Banach space X is called UKK is for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\text{(UKK)} \qquad \left.\begin{array}{l} (x_n) \subset B[X] \\ x_n \overset{w}{\to} x_\infty \\ \operatorname{sep}(x_n) \geqslant \varepsilon \end{array}\right\} \Rightarrow \|x_\infty\| \leqslant 1 - \delta \quad .$$

Definition 4.1.3 (Huff 1980): (The norm of) a Banach space X is called NUC if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\text{(NUC)} \qquad \left.\begin{array}{l} (x_n) \subset B[X] \\ \operatorname{sep}(x_n) \geqslant \varepsilon \end{array}\right\} \Rightarrow \operatorname{dist}(0, \operatorname{co} \{x_n\}_{n=1}^{\infty} < 1 - \delta \ .$$

Equivalently, X is NUC if for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ $(\delta < 1)$ such that whenever C is a closed convex subset of $B[X]$ with

$$1 - \delta \leqslant \operatorname{dist}(0,C) \leqslant 1 \quad ,$$

we have that C admits a *finite* ε-*net* (that is, there exists $x_1, x_2, \ldots, x_n \in X$ such that

$$C \subseteq \bigcup_{i=1}^{n} B_\varepsilon[x_i] \ .$$

Remark 4.1.4: Huff [1980] showed that: *X is NUC if and only if X is UKK and reflexive.*

Clearly, NUC is implied by UC so we have

$$UC \Rightarrow NUC \Rightarrow UKK \Rightarrow KK \ .$$

Example 4.1.5: $(\ell_2^2 \oplus \ell_3^3 \oplus \ldots)_2$ is NUC (Huff 1980) but cannot be equivalently renormed to be UC.

Recall (Remark 2.5.5) that the space $(\ell_2 \oplus \ell_3 \oplus \ldots)_2$ is UCED and reflexive but by Huff [1980; Thereom 4] is not UKK for any equivalent renorming.

These examples show that *NUC neither implies nor is implied by UCED.*

Also every Schur space (for example, ℓ_1) is UKK, and since NUC spaces are reflexive, $UKK \not\Rightarrow NUC$.

van Dulst and Sims [1981] gave a weakening of the UKK property, called *weakly uniformly Kadec-Klee* (WUKK), and showed it implies w-normal structure and hence the w-FPP. It is defined as follows:

Definition 4.1.6 *(van Dulst-Sims 1981)*: (The norm of) a Banach space X is called *WUKK* if there exists an $\varepsilon < 1$ and $\delta > 0$ such that

$$\text{(WUKK)} \qquad \left.\begin{array}{l} (x_n) \subset B[X] \\ x_n \overset{w}{\to} x \\ \operatorname{sep}(x_n) \geq \varepsilon \end{array}\right\} \Rightarrow \|x\| \leq 1 - \delta$$

In duals of separable spaces (or more generally, in spaces for which $B[X^*]$ is w*-sequentially compact) WUKK* and UKK* may be reformulated as follows:

Definition 4.1.7 *(van Dulst-Sims 1981)*: If (4.1) denotes the property:

$$(4.1) \qquad \left.\begin{array}{l} A \subset B[X] \\ (x_n) \subset A \\ \operatorname{sep}(x_n) > \varepsilon \end{array}\right\} \Rightarrow \bar{A}^{w^*} \cap B_{1-\delta}(0) \neq \phi$$

then the dual space has WUKK* if (4.1) holds for some $\varepsilon \in (0,1)$ and $\delta > 0$, and has UKK* if for every $\varepsilon \in (0,1)$, (4.1) holds for some $\delta = \delta(\varepsilon) > 0$.

Recall (Karlovitz 1976b, Lim 1980) that ℓ_1 with its natural norm fails normal structure but has w(w*)-normal structure and hence the w(w*)-FPP. In generalizing this result while simultaneously extending the known conditions for w(w*)-normal structure, Sims [1982] examined the *ε;w(w*)-shrinking ball properties* and showed that such spaces have w(w*)-normal structure and hence the w(w*)-FPP. He defines these properties in the following way.

Definition 4.1.8: The *measure of compactness* for a set K, denoted by $\gamma(K)$, is defined to be

$$(4.2) \qquad \gamma(K) = \sup_{(x_n)\subset K} \inf_{m\neq n} \|x_m - x_n\|$$

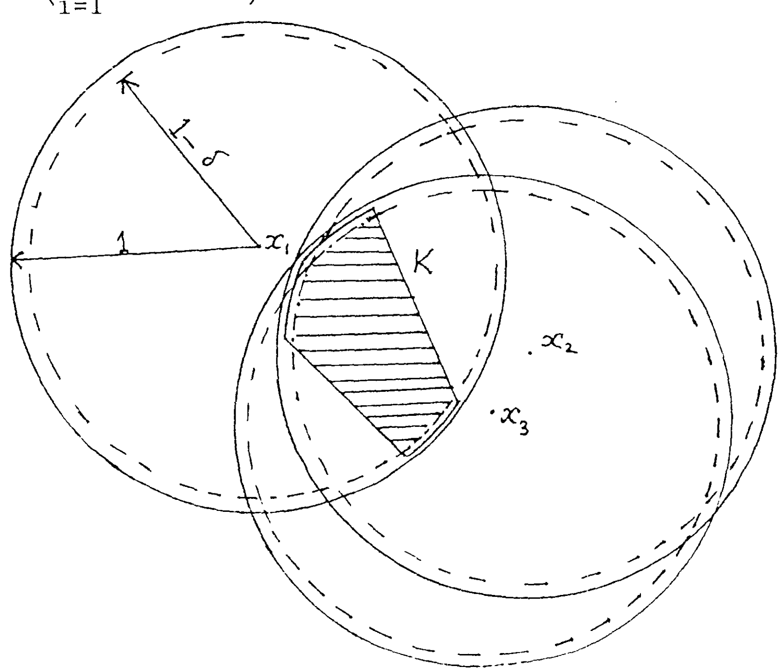
where the supremum is taken over all infinite sequences of points in K.

For $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$ we say that a Banach space $X(X^*)$ has the *ε;w(w*)-shrinking n-ball property* if there exists a $\delta \in (0,1)$ such that whenever K is a nontrivial w(w*)-compact convex subset with

(i) $\gamma(K) > \varepsilon$, and

(ii) $x_1, x_2, \ldots, x_n \in X(X^*)$ with $K \subseteq B_1[x_i]$ $(i = 1,2,\ldots,n)$.

then we have $K \cap \left(\bigcap_{i=1}^{n} B_{1-\delta}[x_i]\right) \neq \phi$.



Remark 4.1.9: Clearly, the ε';w(w*)-shrinking n'-ball property implies the corresponding ε,n property whenever $\varepsilon' \leq \varepsilon$ and $n' \geq n$.

If either or both ε or n are omitted then the property is assumed to hold for all permissible values of that parameter.

In terms of the above definition for γ (4.2), Lemma 1.3.9 may be restated as follows.

Lemma 4.1.10: If K is a closed convex diametral set, then $\gamma(K) = \text{diam } K$.

Remark 4.1.11: We note the following properties of γ:

(4.3A) (a) $\gamma(K) = 0$ if and only if K is (norm) compact;

(4.3B) (b) If $K_1 \subseteq K_2$, then $\gamma(K_1) \leq \gamma(K_2)$;

(4.3C) (c) $\gamma(K_1 \cup K_2) = \max\{\gamma(K_1), \gamma(K_2)\}$;

(4.3D) (d) If $K_1 \supseteq K_2 \supseteq \ldots \supseteq K_n \supseteq \ldots$ is a nested sequence of nonempty sets with $\gamma(K_n) \to 0$, then

$$K = \bigcap_{n=1}^{\infty} K_n \text{ is nonempty and compact.}$$

By property (b), if K admits a finite ε-net, then $\gamma(K) \leq 2\varepsilon$. Now if $\varepsilon_0(K) = \inf\{\varepsilon > 0 : K \text{ admits a finite } \varepsilon\text{-net}\}$, the "usual" measure of compactness, then

$$\varepsilon_0(K) \leq \gamma(K) \leq 2\varepsilon_0(K) \ .$$

(d) follows from this and the analogous result for ε_0 due to Kuratowskii.

4.2 The Properties WUKK(WUKK*) and ε;w(w*)-Shrinking Ball Are Equivalent

Before establishing the equivalence of WUKK(WUKK*) and ε;w(w*)-shrinking ball properties, to render the shrinking ball properties more palatable, we have the following:

Lemma 4.2.1: For any $\varepsilon \in (0,1)$, $X(X^*)$ has the $\varepsilon;w(w^*)$-shrinking ball property if and only if it has the $\varepsilon;w(w^*)$-shrinking 1-ball property.

Proof: Necessity being obvious we only prove sufficiency. Assume the implication fails, then there is a largest n (≥ 1, as the ε;w(w*)-shrinking 1-ball property holds) for which $X(X^*)$ has the ε;w(w*)-shrinking n-ball property. For this n, there must exist a w(w*)-compact convex set K, with $\gamma(K) > \varepsilon$ and points $x_1, x_2, \ldots, x_n$, x_{n+1} such that

(i) $K \subseteq B_1[x_i]$ $(i = 1,2,\ldots,n)$; and

(ii) $A = K \cap \left(\bigcap_{i=1}^{n} B_{1-\delta}[x_i]\right) \neq \phi$; but

(iii) $A \cap B_{1-\delta}[x_{n+1}] = \phi$.

Since A is w(w*)-compact, by Theorem 1.2.14 (the basic separation theorem) applied to (iii), there exists a (w*) continuous linear functional f and $k \in \mathbb{R}$ with

$$\sup f(B_{1-\delta}[x_{n+1}]) < k < \inf f(A) .$$

Let $E_1 = \{x \in K : f(x) \geq k\}$ and

$$E_2 = \{x \in K : f(x) \leq k\} .$$

so that $K = E_1 \cup E_2$.

Then E_1 is a w(w*)-compact convex set with $E_1 \subseteq B_1[x_{n+1}]$ but $E_1 \cap B_{1-\delta}[x_{n+1}] = \phi$, thus by the ε;w(w*)-shrinking 1-ball property we must have $\gamma(E_1) \leq \varepsilon$ and so since $K = E_1 \cup E_2$, by property (4.3C) (see Remark 4.1.11) of γ we have

$$\gamma(E_2) > \varepsilon .$$

But then

$$E_2 \subseteq K \subseteq B_1[x_i] \quad (i = 1,2,\ldots,n)$$

is a w(w*)-compact convex set with

$$\gamma(E_2) > \varepsilon \text{ such that}$$

$$E_2 \cap \left(\bigcap_{i=1}^{n} B_{1-\delta}[x_i] \right) = E_2 \cap A = \phi ,$$

contradicting the choice of n as the largest value for which the implication holds. Hence the result.

□

Now the main result of the section.

Proposition 4.2.2 (Sims 1982): X has the ε;w-shrinking 1-ball property if and only if whenever a sequence $(x_n) \subset B[X]$ has $\text{sep}(x_n) > \varepsilon$ and $x_n \overset{w}{\to} x_\infty$, we have $\|x_\infty\| \leq 1 - \delta$.

Proof: *(⇐):* Assume the implication fails. That is, assume X fails the ε;w-shrinking 1-ball property. Then there exists a w-compact convex set $K \subseteq B_1[0]$ with

$$K \cap B_{1-\delta}[0] = \phi \text{ , but } \gamma(K) > \varepsilon \text{ .}$$

Hence K contains a sequence (x_n) with

$$\|x_m - x_n\| \ (\geq \text{sep}(x_n)) > \varepsilon \quad \text{for } n \neq m \text{ ,}$$

which by compactness, we may assume converges weakly to $x \in K$. But

$$K \cap B_{1-\delta}[0] = \phi$$

so $x \notin B_{1-\delta}[0]$, hence $\|x\| > 1 - \delta$, contradicting the hypothesis that $\|x\| \leq 1 - \delta$.

Now (⇒): Conversely, assume the implication fails. That is, assume that there exists a sequence $(x_n) \subset B[X]$ with

$$\|x_n - x_m\| \geq \text{sep}(x_n)) > \varepsilon \text{ for } n \neq m, \text{ and}$$

$$x_n \overset{w}{\to} x_\infty \text{ but that } \|x\| > 1 - \delta \text{ .}$$

That is,

$$x \notin B_{1-\delta}[0] \text{ .}$$

Hence by Theorem 1.2.14 (basic separation theorem), let $f \in S(X^*)$ such that f strictly separates x from $B_{1-\delta}[0]$ and let n_0 be such that for $n > n_0$

$$f(x_n) \geq \frac{1}{2}(f(x) + 1 - \delta) > 1 - \delta .$$

By Phillips' Lemma (see Theorem 1.2.17)

$$K = \overline{co}\,\{x_n\}_{n=n_0}^{\infty}$$

is a w-compact convex set, with $\gamma(K) > \varepsilon$ such that

$$f(y) \geq \frac{1}{2}(f(x) + 1 - \delta) \quad \text{for all } y \in K .$$

But

$$\frac{1}{2}(f(x) + 1 - \delta) > 1 - \delta$$

so

$$K \cap B_{1-\delta}[0] = \phi ,$$

contradicting the ε;w-shrinking 1-ball property.

□

Remark 4.2.3: The above result essentially shows that *the w-shrinking ball property is equivalent to UKK.* In particular, by Remark 4.1.4, *the w-shrinking ball property plus reflexivity is equivalent to NUC.*

Example 4.2.4: (i) Recall (Example 4.1.5) that the space $(\ell_2 \oplus \ell_3 \oplus \ldots)_2$ is UCED (and reflexive) but is not UKK for any equivalent renorming. This shows that *UCED need not imply the w-shrinking ball property. The reverse implication also fails* since vacuously every finite dimensional space has the w-shrinking ball property.

(ii) The space $(\ell_2 \oplus \ell_3 \oplus \ldots)_2$ also satisfies w-strict Opial condition but is not UKK for any equivalent renorming. On the other hand, the spaces $L_p[0,1]$ $(1 < p < \infty)$ satisfy the w-shrinking ball property but (Opial 1967) fail the w-strict Opial condition. This shows that the properties *w-shrinking ball and w-strict Opial are essentially independent of one another.* To show $(\ell_2 \oplus \ell_3 \oplus \ldots)_2$ has the w-strict Opial condition (recall Theorem 3.2.4):

$$\text{Let } \underset{\sim}{x}_n \overset{w}{\to} \underset{\sim}{x}_0 \neq 0.$$

Then there exists k_0 such that $x_0(k_0) \neq 0$ and for all such k_0 $\underset{\sim}{x}_n(k_0) \overset{w}{\to} \underset{\sim}{x}_0(k_0)$. So $f\underset{\sim}{x}_n(k_0) \to f\underset{\sim}{x}_0(k_0)$. But then $f\underset{\sim}{x}_n \overset{w}{\to} f\underset{\sim}{x}_n$ so unique support (and hence the duality) mapping is w-w-continuous.

□

Further examples may be found in the Appendix.

For the weak* case in a dual space X^* an argument similar to that in Proposition 4.2.2 establishes the equivalence of the ε;w*-shrinking ball property to WUKK*. We also have:

Lemma 4.2.5: *X^* has the ε;w*-shrinking 1-ball property if and only if there exists $k \in (0,1)$ such that for every norm one w*-continuous linear functional f on X^* the slice of the dual unit ball*

$$S[f,k] = \{x \in X^* : \|x\| \leqslant 1 \text{ and } f(x) \geqslant k\}$$

has $\gamma(S[f,k]) \leqslant \varepsilon$.

Proof: *(⇒)* is obvious, since for any $k > 1 - \delta$ where δ is given in the definition of the ε;w*-shrinking 1-ball property, $S[f,k]$ is a w*-closed convex subset of $B_1[0]$ which is disjoint from $B_{1-\delta}[0]$.

(⇐): Let K be a w*-compact convex set and let x be such that $K \subseteq B_1[x]$. Assume $\gamma(K) > \varepsilon$ but that

$$K \cap B_{1-\delta}[x] = \phi$$

where $\delta = 1 - k$.

Then $K' = K - x$ is a w*-compact convex subset of $B_1[0]$ which is disjoint from the closed ball $B_{1-\delta}[0]$, so by the Basic Separation Theorem 1.2.14, there exists a norm one w*-continuous linear functional f with

$$\inf f(K') > \sup f(B_{1-\delta}[0]) = 1 - \delta = k \ .$$

Thus $K' \subseteq S[f,k]$ and so

$$\gamma(S[f,k]) \geqslant \gamma(K') = \gamma(K) > \varepsilon,$$

contradicting our hypthesis on K, hence the result.

□

Example 4.2.6: ℓ_1 satisfies the conditions of above Lemma for any $\varepsilon \in (0,2)$ - take $k = 1 - \frac{\varepsilon}{2}$.

4.3 The Importance of NUC and Its Related Properties for Our Purposes Lies in the Following:

Theorem 4.3.1 (Sims 1982): If $X(X^*)$ has the $\varepsilon;w(w^*)$-shrinking ball property for some $\varepsilon \in (0,1)$, then $X(X^*)$ has $w(w^*)$-normal structure and hence the $w(w^*)$-FPP.

Proof: Assume K is a diametral nontrivial $w(w^*)$-compact convex set in $X(X^*)$. Without loss of generality suppose $\operatorname{diam} K = 1$. Then by Lemma 4.1.10 $\gamma(K) = 1 > \varepsilon$ and for each $x \in K$, $K \subseteq B_1[x]$.

Let

$$E_x = K \cap B_{1-\delta}[x] .$$

Then E_x is a $w(w^*)$-closed subset of K which is nonempty by the $\varepsilon;w(w^*)$-shrinking 1-ball property. Further, by the full $\varepsilon;w(w^*)$-shrinking ball property, the family $\{E_x : x \in K\}$ has the finite intersection property and by the $w(w^*)$-compactness of K, there exists an $x_0 \in \bigcap_{x \in K} E_x$. But then, for any $x \in K$ we have

$$x_0 \in E_x \subseteq B_{1-\delta}[x] ,$$

so $\|x_0 - x\| \leq 1 - \delta$, and so x_0 is a nondiametral point of K, contradicting the assumption on K.

□

***Corollary 4.3.2** (van Dulst-Sims 1983): A Banach space $X(X^*)$ satisfying WUKK (WUKK*) has $w(w^*)$-normal structure and hence the $w(w^*)$-FPP.*

Proof: By Proposition 4.2.2 the WUKK (WUKK*) property is equivalent to $\varepsilon;w(w^*)$-shrinking ball property. By the above theorem, such spaces have $w(w^*)$-normal structure and hence the $w(w^*)$-FPP by Kirk's Theorem 2.2.1.

□

Remark 4.3.3: (1) Since NUC ⇒ UKK ⇒ WUKK spaces which are NUC and UKK also have the w-FPP. Similarly, UKK* ⇒ w^*-FPP.

(2) A direct proof that WUKK ⇒ w-FPP is also possible namely (see van Dulst-Sims 1983):

It suffices to show that every nontrivial w-compact convex subset K of X contains a nondiametral point. Suppose not. Then, by Lemma 1.3.11 there exists a sequence (x_n) in K satisfying

$$\lim_n \operatorname{dist}(x_{n+1}, \overline{co}\,\{x_1, x_2, \ldots, x_n\}) = \operatorname{diam} K. \tag{4.4}$$

Any subsequence of (x_n) again satisfies (4.4), so we may, by weak compactness, assume that $x_n \xrightarrow{w} x$. By applying first a translation and then a multiplication, we may further simplify the situation and assume that $x_n \xrightarrow{w} 0$ and $\operatorname{diam} K = 1$. Since the weak and the norm closure of $\overline{co}\{x_n\}$ coincide, (4.4) implies in particular that $\lim_n \|x_n\| = 1$. Now let $\varepsilon < 1$ and $\delta > 0$ be as in the definition of WUKK. Choose $n_0 \in \mathbb{N}$ such that $\|x_{n_0}\| > 1 - \delta$ and such that $\text{dist}(x_{n+1}, \overline{co}\{x_1, x_2, \ldots, x_n\}) > \varepsilon$ whenever $n \geq n_0$. Consider now the sequence $(x_{n_0} - x_n)_{n=n_0+1}^{\infty}$. Clearly

$$\|x_{n_0} - x_n\| \leq 1 \quad (n = n_0+1, n_0+2, \ldots) ,$$

$$\text{sep}(x_{n_0} - x_n) \geq \varepsilon \quad \text{and}$$

$$x_{n_0} - x_{n_0} \xrightarrow{w} x_{n_0} .$$

This contradicts WUKK since $\|x_{n_0}\| > 1 - \delta$.

□

4.4 Lim's w(w*)-Asymptotic Normal Structure

In Example 4.2.6 we saw that ℓ_1 has ε;w*-shrinking ball property for any $\varepsilon \in (0,2)$, and so we conclude Lim's [1980] result: *ℓ_1 has w*-normal structure and hence the w*-FPP.* But Lim actually proved more.

Definition 4.4.1 (Lim 1974): For a nonempty bounded convex subset K of a Banach space X and $\{A_\alpha : \alpha \in \Lambda\}$ a decreasing net of bounded nonempty subsets of X, we define the following numbers.

For each $x \in K$ and $\alpha \in \Lambda$,

$$r_x(A_\alpha) = \sup\left\{\|x-y\| : y \in A_\alpha\right\}$$

$$r(x) = \inf\left\{r_x(A_\alpha) : \alpha \in \Lambda\right\}$$

$$= \lim_\alpha r_x(A_\alpha)$$

$$r = \inf\left\{r(x) : x \in K\right\} .$$

Then the set $AC = \left\{x \in K : r(x) = r\right\}$ (the number r) will be called the *asymptotic centre (asymptotic radius)* of $\{A_\alpha : \alpha \in \Lambda\}$ with respect to K.

In particular, if we take $A_\alpha = B$ for each $\alpha \in \Lambda$, where B is a nonempty bounded set, then the asymptotic centre (asymptotic radius) is just the Chebyshev centre (Chebyshev radius) of B with respect to K. It is denoted by

$$C(B,K) = \left\{x \in K : r_x(B) = \inf_{y \in K} r_y(B)\right\} .$$

Definition 4.4.2 (Lim 1980): We say $X(X^*)$ has *w(w*)-asymptotic normal structure* if for every nontrivial w(w*)-compact convex subset K of $X(X^*)$, the asymptotic centre of any decreasing net of nonempty subsets of K with respect to K is a proper subset of K.

Remark 4.4.3: By Definition 2.1.6 and Proposition 2.1.8, K is diametral if and only if $C(K,K) = C(K) = K$. So we see that: *w(w*)-asymptotic normal structure implies w(w*)-normal structure and hence the w(w*)-FPP.*

Lim [1974; Theorem 1 and Corollary 1] showed that *w-asymptotic normal structure is equivalent to w-normal structure.* However, *no such equivalence seems to be known in the w* case*: that is, it is not known whether w*-normal structure implies w*-asymptotic normal structure.

None-the-less Lim [1980] proved that: *ℓ_1 has w^*-asymptotic normal structure* and so deduced a common fixed point result for certain semigroups of nonexpansive selfmappings of K. In fact, what Lim [Theorem 3] proved is that: *for any nonempty w^*-closed convex subset K of ℓ_1, the asymptotic centre of a decreasing net $\{A_\alpha : \alpha \in \Lambda\}$ of bounded nonempty subsets of K with respect to K is a nonempty (norm) compact convex subset of K.*

This leads naturally to the following open question.

Question 4.4.4: Does the w^*-shrinking ball property for X^* imply that for any nonempty w^*-compact convex subset K of X^* the asymptotic centre of a decreasing net of nonempty subsets of K with respect to K is nonempty and (norm) compact?

Sims [1982] provides a partial answer to this question.

Lemma 4.4.5 (Sims 1982): *If $X(X^*)$ has the $w(w^*)$-shrinking ball property, then for any nonempty $w(w^*)$-compact convex subset K and any bounded subset B of $X(X^*)$, the Chebyshev centre $C(B,K)$ is nonempty compact and convex.*

Proof: By Lemma 2.1.9 $C(B,K)$ is closed convex and nonempty. We only need to prove compactness. Thus, assume $C(B,K)$ is not compact and without loss of generality take the Chebyshev radius of B with respect to K, $r = \inf\{r_y(B) : y \in K\}$, to be one. Then $C(B,K)$ contains a sequence (x_n) with $\|x_n - x_m\| > \varepsilon$ for al $m \neq n$ and some $\varepsilon > 0$, and for each $x \in B$, $C(B,K) \subseteq B_1[x]$. Thus by the $w(w^*)$-shrinking ball property

$$E_x = C(B,K) \cap B_{1-\delta}[x]$$

is a nonempty w(w*)-compact subset of K. The argument now proceeds along similar lines to the last part of Theorem 4.3.1.

□

CONCLUSION

We saw in Chapter 2 that Banach spaces with normal structure or more generally asymptotic normal structure have the w-FPP. Unfortunately, as we found out, not every space has normal structure or asymptotic normal structure. In fact, we saw examples of spaces, for example, $X_{\sqrt{2}}$ and $\ell_{p,\infty}$, which respectively failed these properties but still possessed the w-FPP. Examples by Alspach [1981] and Lim [1980] show that some restriction on the Banach space is necessary. What about spaces which are strictly convex and reflexive? What about locally uniformly convex spaces - we do know that LUC spaces fail normal structure? The nature of the nonexpansive mappings on these examples also suggests: If $X(X^*)$ fails to have $w(w^*)$-FPP does it necessarily fail for an isometry?

In Chapter 3 we found that spaces which have the w-strict Opial condition have w-normal structure and hence the w-FPP. But it is still not known whether the w*-strict Opial condition implies w*-normal structure. What is known is that the w*-strict Opial condition implies the w*-FPP.

Chapter 4 contains another unsolved problem: Does w*-normal structure imply w*-asymptotic normal structure? The reverse implication is trivially true. Equivalence of the w-case was proved by Lim [1974].

These and similar questions seem worthy of further investigation.

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APPENDIX

We summarize most of the geometrical conditions, considered in the thesis, which imply the w(w*)-FPP in terms of the following diagram, where

$\Rightarrow$ denotes "implies"

$\not\Rightarrow$ denotes "does not imply or possess"

$\dashrightarrow$ denotes "implication unsolved"

without an accompanying arrow $\Rightarrow$ or $\dashrightarrow$ means $\rightleftarrows$

A.N.S. denotes "asymptotic normal structure"

S.B.P. denotes "shrinking ball property"

S.O.C. denotes "strict Opial condition"

and all the other symbols are obvious from the text.

$$
\begin{array}{ccccccccccc}
 & & \text{N.S.} & & & & & & & & \\
 & & \not\Uparrow & & & & & & & & \\
 & & \text{LUC} & \Rightarrow & \text{SC} & & \text{US} & & \text{w-A.N.S.} & & \text{A.N.S.} \\
 & & \Uparrow & & \Uparrow & & \Downarrow & & \Updownarrow & & \Downarrow \\
\text{Hilbert} & \Rightarrow & \text{UC} & \Rightarrow & \text{UCED} & \Rightarrow & \text{N.S.} & \Rightarrow & \text{w-N.S.} & \Rightarrow & \text{w-FPP} \\
 & & & & & & & \nearrow & & & \\
 & & \Downarrow & & & & \text{w-S.O.C.} & & \Uparrow & & \\
 & & \text{NUC} & \Rightarrow & \text{UKK} & \Rightarrow & \text{WUKK} & \Leftrightarrow & \varepsilon;\text{w-S.B.P.} & &
\end{array}
$$

$$
\begin{array}{ccccccc}
\text{w*-A.N.S.} & \Rightarrow & \text{w*-FPP} & \Leftarrow & \text{w*-S.O.C.} & \dashrightarrow & \text{w*-N.S.} \\
\Downarrow\ \overset{\wedge}{\vdots} & \nearrow & & & & & \\
\text{w*-N.S.} & \Rightarrow & \text{w-N.S.} & \Rightarrow & \text{w-FPP} & & \\
\Uparrow & & & & & & \\
\text{WUKK*} & \Leftrightarrow & \varepsilon;\text{w*-S.B.P.} & & & &
\end{array}
$$

We also tabulate below some of the common and important (counter) examples used to illustrate independence or (non) implication for some of the properties above. By following the arrows above (forwards and backwards) one can deduce further properties satisfied or failed by the given space:

Space	Properties satisfied (⇒) or failed (⇏)	Reference
$\ell_p (1 < p < \infty)$	⇏ UC, w(w*)-S.O.C.	2.3.3, 3.1.3
ℓ_1	⇒ w(w*)-S.O.C., w*-N.S.; w(w*)-S.B.P.; ⇏ N.S.	2.2.4, 3.1.3
$L_p (1 < p < \infty)$	⇒ UC; ⇏ w(w*)-S.O.C., ε;w(w*)-S.B.P.	3.1.3
$L_1[0,1]$	⇏ w-FPP	2.10.1
$(\ell_1, \|\|x^+\|\|_1 \vee \|\|x^-\|\|_1)$	⇏ w*-FPP, N.S.; ⇒ w*-O.C.	2.10.2, 3.1.3
c_0	⇏ N.S.; ⇒ w-FPP	2.2.4, 2.2.5
$C[0,1]$	⇏ N.S.	2.2.4
ℓ_∞	⇏ w(w*)-FPP	
$(\ell_2 \oplus \ell_3 \oplus \ldots)_2$	⇒ UCED: ⇏ ε;w(w*)-S.B.P.	2.5.5, 4.1.5
$(\ell_2^2 \oplus \ell_3 \oplus \ldots)_2$	⇒ NUC; ⇏ UC	4.1.5
$X_\beta (\beta \geq 1)$	⇒ w-FPP	2.9.5
$X_\beta (\beta < \sqrt{2})$	⇒ N.S.	
$X_\beta (\beta < 2)$	⇒ A.N.S.	2.9.4
$X_\beta (\beta \geq \sqrt{2})$	⇏ N.S.	2.2.4, 2.9.4
$X_\beta (\beta \geq 2)$	⇏ A.N.S.	2.9.4