FIXED POINTS OF NONEXPANSIVE MAPPINGS ON WEAK AND WEAK-STAR COMPACT CONVEX SETS IN BANACH SPACES

by

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TABLE OF CONTENTS

NOTATION

Throughout this thesis the following symbols will remain unchanged, maintaining the meaning for which they are defined, namely:

$\mathbb{N}$	the set of all natural numbers
$\mathbb{R}$	the set of all real numbers
$\mathbb{C}$	the set of all complex numbers
w-	weak
w*-	weak-star
FPP	fixed point property for nonexpansive mappings
w-FPP	weak fixed point property for nonexpansive mappings
w*-FPP	weak-star fixed point property for nonexpansive mappings
X	a Banach space with elements $x,y,\ldots$
X^*	the dual space of X with elements $f,g,\ldots$
X^{**}	the dual space of X^* with elements $F,G,\ldots$
$\hat{X}$	the image of X in X^{**} under the canonical embedding $J : X \to X^{**} : x \mapsto \hat{x}$
$B(X)$	$\{x \in X : \|x\| < 1\}$: the open unit ball of X
$B[X]$	$\{x \in X : \|x\| \leq 1\}$: the closed unit ball of X
$S(X)$	$\{x \in X : \|x\| = 1\}$: the unit sphere of X
$B_r(x)$	$\{y \in X : \|x-y\| < r\}$: open ball in X with centre x and radius $r > 0$; in particular, $B_1(0) = B(X)$
$B_r[x]$	$\{y \in X : \|x-y\| \leq r\}$: closed ball in X with centre x and radius $r > 0$; in particular, $B_1[0] = B[X]$
$\zeta(K)$	the set of Chebyshev centres of a nonempty set K
$\mathrm{co}K$	the convex hull of K
$\overline{\mathrm{co}}K$	the closed convex hull of K

Notation (continued)

$\bar{K}^s \equiv \bar{K}$	the strong or norm closure of K
$\bar{K}^w$	the weak closure of K
$\bar{K}^{w*}$	the weak-star closure of K
diam K	$\sup\{\lVert x-y\rVert : x,y \in K\}$: the diameter of K
dist(x,K)	$\inf\{\lVert x-y\rVert : y \in K\}$: the distance from a point x to K
sup K	supremum (least upper bound) of K
inf K	infimum (greatest lower bound) of K
$\lim \equiv \lim_n$	limit (as $n \to \infty$)
$\text{sep}(x_n)$	$\inf\{\lVert x_m-x_n\rVert : m \neq n\}$: the separation constant of the sequence (x_n)
$\Leftrightarrow$	if and only if
$\overset{s}{\to}$	strong or norm convergence (the s may be omitted)
$\overset{w}{\to}$	weak convergence
$\overset{w*}{\to}$	weak-star convergence
C[a,b]	the Banach space of all continuous real-valued functions x on the closed interval [a,b] with norm defined by $\lVert x\rVert_\infty = \max\{\lvert x(t)\rvert : t \in [a,b]\}$
$L_p[a,b]$ $(p \geq 1)$	the Banach space which is the completion of the normed linear space of all continuous real-valued functions x on [a,b] with norm defined by $\lVert x\rVert_p = \left(\int_a^b \lvert x(t)\rvert^p \, dt\right)^{1/p}$

We will also refer to the following Banach spaces which are infinite dimensional subspaces of $\mathbb{R}^\infty$:

ℓ_∞	the space of all bounded sequences $x = (x_n)$ with norm defined by $\lVert x\rVert_\infty = \sup\{\lvert x_n\rvert : n = 1,2,\ldots\}$

Notation (continued)

c	the subspace of ℓ_∞ consisting of all convergent sequences of scalars
c_0	the subspace of ℓ_∞ consisting of all sequences of scalars convergent to zero
ℓ_1	the space of all absolutely summable sequences $x = (x_n)$ (that is, $\sum_{n=1}^{\infty} \lvert x_n \rvert < \infty$) with norm defined by $\lVert x \rVert_1 = \sum_{n=1}^{\infty} \lvert x_n \rvert$
$\ell_p (1 < p < \infty)$	the space of all p-summable sequences $x = (x_n)$ (that is, $\sum_{n=1}^{\infty} \lvert x_n \rvert^p < \infty$) with norm defined by $\lVert x \rVert_p = \left(\sum_{n=1}^{\infty} \lvert x_n \rvert^p \right)^{1/p}$

ℓ_∞^n, ℓ_1^n and ℓ_p^n $(1 < p < \infty)$ are the n-dimensional analogues of ℓ_∞, ℓ_1 and ℓ_p $(1 < p < \infty)$

INTRODUCTION

Let X be a Banach space, K a nonempty bounded closed convex subset of X and T a nonexpansive selfmapping of K. The purpose of this thesis is to investigate the following question:

What further assumptions (of a geometrical nature) on K (or X) can we make to ensure that there exists a point x in K for which $Tx = x$?

Such points in K are called *fixed points* of T in K. X is then said to have the *fixed point property (FPP)* if for every nonempty bounded closed convex subset K of X, each nonexpansive selfmapping T of K has a fixed point. This, together with the following two properties, will be the subject of our subsequent investigation.

The weak fixed point property (w-FPP): For every nonempty weak compact convex subset K of X and each nonexpansive selfmapping $T : K \to K$, there exists $x \in K$ with $Tx = x$; and in the case of a dual space X^*,

The weak-star fixed point property (w*-FPP): For every nonempty weak-star compact convex subset K of X^* and each nonexpansive selfmapping $T : K \to K$, there exists $x \in K$ with $Tx = x$.

The existence of fixed points for nonexpansive selfmappings of weak (weak-star) compact (convex) sets can be viewed as a wedding of the "classical" Banach [1922] and Schauder [1930] Fixed Point Theorems. These theorems are included in §1.1 There is a strange feature in this combination: the condition on K usually concerns the weak or weak-star topology while that on T concerns the norm topology. The seeming lack of connection between these conditions is what makes the problem interesting and challenging.

The study of our problem has its origins in four papers which appeared in 1965:

(1) Browder [1965a] showed that evert Hilbert space has the FPP;

(2) subsequently he [1965b] extended this result to the much wider class of all uniformly convex spaces;

(3) this latter result was independently obtained by Göhde [1965]; while

(4) Kirk [1965] obtained the same result for an even wider class of spaces, namely, he showed that every Banach space with w(w*)-normal structure has the w(w*)-FPP. These fundamental results are included in Chapter 2 as part of our discussion on normal structure.

Certain generalizations of uniform convexity also imply normal structure and hence the w-FPP. For example,

(1) see §2.5: spaces which are uniformly convex in every direction (Garkavi 1961, 1964; cf. Day-James-Swaminathan 1971);

(2) see Chapter 3: spaces which satisfy the weak-strict Opial condition (Opial 1967; cf. Gossez-Lami Dozo 1972).

Although w*-strict Opial condition implies w*-FPP [Karlovitz 1976b], it is not known whether this condition implies w*-normal structure;

(3) see Chapter 4: spaces which are weakly uniformly Kadec-Klee [van Dulst-Sims 1981]. Uniformly Kadec-Klee and Near Uniform Convexity are successive strengthenings of weakly uniformly Kadec-Klee property and so they imply w-normal structure.

Despite these results, the assumption of normal structure is not essential for positive nonexpansive fixed point results. For example,

(1) see §2.2: it is known [Maurey 1980/81; cf. Elton-Lin-Odell-Szarek 1983] that c_0 has the w-FPP but fails normal structure. An earlier example was given by James/Karlovitz (see below). Indeed, van Dulst [1982] has shown that every Banach space may be equivalently renormed so as to fail normal structure. In §2.6 we discuss an example, due to Smith-Turett [1984], of a Banach space which is reflexive and locally uniformly convex, but fails to have normal structure. This answers negatively the question, posed by Swaminathan [1983, p. 212], as to whether every locally uniformly convex space has normal structure.

(2) see §2.9: Maurey also shows that the following reflexive (in fact superreflexive) space X_β $(\beta \geq 1)$ has the w-FPP:

X_β ($\beta \geq 1$) $\equiv \ell_2$ renormed by setting, for $x \in \ell_2$,

$\|x\| = \max\{\frac{1}{\beta}\|x\|_2, \|x\|_\infty\}$

where $\|\cdot\|_2$ ($\|\cdot\|_\infty$) is the usual ℓ_2 (ℓ_∞) norm.
Karlovitz [1976a] showed that $X_{\sqrt{2}}$ has the w-FPP although as observed by R.C. James [see Belluce-Kirk-Steiner 1968] for $\beta \geq \sqrt{2}$, X_β fails normal structure. Baillon-Schöneberg [1981] introduced a class of spaces which have, what they called, asymptotic normal structure, and which properly contains the class having normal structure, while still retaining the w-FPP. They showed that X_β has asymptotic normal structure if and only if $\beta < 2$. They also showed that X_2 has the w-FPP. It is also known [see Bynum 1980] that asymptotic normal structure is not essential for the FPP. Indeed X_β has w-FPP for all β.

Other sufficient conditions for $X(X^*)$ to have the $w(w^*)$-FPP are also discussed. These include uniformly smooth spaces (see §2.7 and Baillon 1978/79 and Turett 1982), and Lim's [1974, 1980] $w(w^*)$-asymptotic normal structure (see §4.4) which is distinct from the concept of asymptotic normal structure discussed above.

Recently Giles-Sims-Swaminathan [1984] gave an example of a reflexive Banach space which has normal structure, but which lacks most of the geometrical conditions considered in this thesis. These conditions include uniform convexity, local uniform convexity, uniform convexity in every direction and the weakly uniformly Kadec-Klee property. In §2.8, we give their lemma, based on the idea of uniform normal structure.

The long standing question [see Reich 1976, 1980] of whether every Banach space possessed the w-FPP was settled in the negative by Alspach [1981] when he gave an example of a fixed point free isometry on a nonempty w-compact convex subset of $L_1[0,1]$ (see § 2.10 and Schectman 1982). The corresponding question for dual Banach spaces was also settled in the negative by Lim [1980] when he exhibited a fixed point free isometry on a nonempty w*-compact convex subset of ℓ_1 with the equivalent dual norm $\|x\| = \max\{\|x^+\|_1, \|x^-\|_1\}$ (see §2.10).

The Appendix summarises most of the geometrical properties considered in the thesis which imply positive nonexpansive fixed point results. It also includes some examples of spaces which satisfy or fail these properties. In the conclusion we collect together a few open problems related to the thesis and which lend themselves for further investigation. Chapter 1 is largely introductory. It introduces some important concepts and results from functional analysis which will be used throughout the thesis. The lemmas in §1.3 are basic to much of the theory for nonexpansive mappings in Banach spaces.