Chapter 5

Double Density Filter Banks and Framelets

In this chapter we look at the design of oversampled filter banks and the resulting framelets. The *undecimated wavelet transform* is known for its shift invariant properties and has applications in areas such as denoising [29]. The framelets we will design in this chapter will have improved shift invariant properties over *decimated wavelet transform.* Shift invariance has applications in many areas particularly denoising [22, 40, 20] and coding and compression [56].

We will look at a special class of framelets from a filter bank perspective, in that we will design double density filter banks (DDFB's) as shown in Figure 5.1. Using the basic multirate identities we obtain the following expression for $Y(z)$.

$$
\begin{aligned}
Y(z) = \frac{1}{2}\left(H_0(z)\tilde{H}_0(z) + H_1(z)\tilde{H}_1(z) + H_2(z)\tilde{H}_2(z)\right) X(z) + \\
\frac{1}{2}\left(H_0(z)\tilde{H}_0(-z) + H_1(z)\tilde{H}_1(-z) + H_2(z)\tilde{H}_2(-z)\right) X(-z).
\end{aligned}
\tag{5.1}
$$

Now, for the perfect reconstruction, i.e. $Y(z) = X(z)$, it must be necessary that

$$H_0(z)\tilde{H}_0(z) + H_1(z)\tilde{H}_1(z) + H_2(z)\tilde{H}_2(z) = 2, \tag{5.2}$$

$$H_0(z)\tilde{H}_0(-z) + H_1(z)\tilde{H}_1(-z) + H_2(z)\tilde{H}_2(-z) = 0. \tag{5.3}$$

Alternatively we can write the above perfect reconstruction conditions in the polyphase domain. Given the following polyphase matrices:

$$\tilde{H}(z) = \begin{bmatrix} \tilde{H}_{00}(z) & \tilde{H}_{01}(z) \\ \tilde{H}_{10}(z) & \tilde{H}_{11}(z) \\ \tilde{H}_{20}(z) & \tilde{H}_{21}(z) \end{bmatrix} \text{ and } H(z) = \begin{bmatrix} H_{00}(z) & H_{01}(z) \\ H_{10}(z) & H_{11}(z) \\ H_{20}(z) & H_{21}(z) \end{bmatrix} \tag{5.4}$$

where $\tilde{H}(z)$ is the type 1 analysis polyphase matrix, and $H(z)$ is the type 2 synthesis polyphase matrix, we can write the perfect reconstruction conditions as

$$[H(z)]^T \tilde{H}(z) = I. \tag{5.5}$$

5.1 Constraints on The Scaling Filter

We first formalize the low pass nature of scaling filters for the double density filter banks.

Definition 7 *A filter $A(z)$ is said to be a possible scaling filter if $A^f(0) = \sqrt{2}$.*

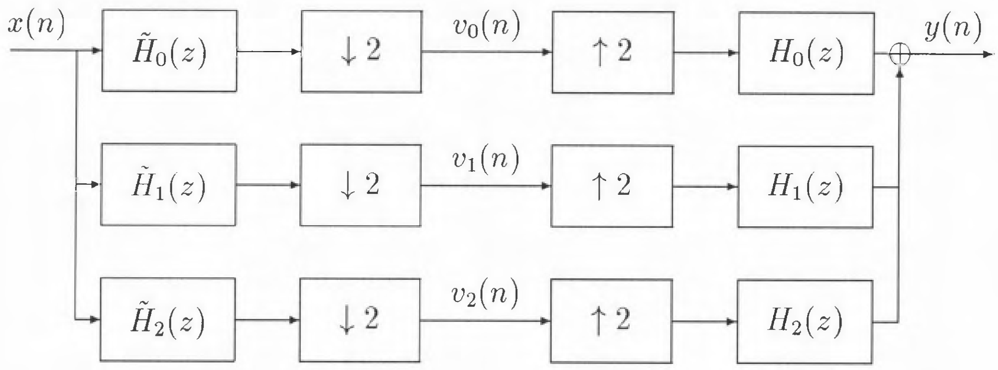

Figure 5.1: The double density filter bank

The following lemma provides the relationship between the zeros of the scaling filters at the aliasing frequency π and the first vanishing moments of wavelet filters.

Lemma 8 *Let* $H_0^f(0) = \sqrt{2}$, $\tilde{H}_0^f(0) = \sqrt{2}$. *Then the following are true.*

- *If at least one of* $H_1^f(0)$ *and* $H_2^f(0)$ *are nonzero,* $H_0^f(\pi) = 0$, *and* $\tilde{H}_0^f(\pi) = 0$ *then* $\tilde{H}_1^f(0) = 0$ *and* $\tilde{H}_2^f(0) = 0$.
- *If at least one of* $\tilde{H}_1^f(0)$ *and* $\tilde{H}_2^f(0)$ *are nonzero,* $H_0^f(\pi) = 0$, *and* $\tilde{H}_0^f(\pi) = 0$ *then* $H_1^f(0) = 0$ *and* $H_2^f(0) = 0$.
- *If* $H_1^f(0) = 0, H_2^f(0) = 0, \tilde{H}_1^f(0) = 0$, *and* $\tilde{H}_2^f(0) = 0$ *then* $H_0^f(\pi) = 0$, *and* $\tilde{H}_0^f(\pi) = 0$.

Proof: First, we prove the first case and the second follows similarly. We can write equation 5.3 in frequency domain as

$$H_0^f(\omega)\tilde{H}_0^f(\omega+\pi) + H_1^f(\omega)\tilde{H}_1^f(\omega+\pi) + H_2^f(\omega)\tilde{H}_2^f(\omega+\pi) = 0 \qquad (5.6)$$

from which we can deduce the following after substituting zero and π for ω respectively.

$$H_1^f(0)\tilde{H}_1^f(\pi) + H_2^f(0)\tilde{H}_2^f(\pi) = 0 \qquad (5.7)$$

$$H_1^f(\pi)\tilde{H}_1^f(0) + H_2^f(\pi)\tilde{H}_2^f(0) = 0 \tag{5.8}$$

The equation 5.2 can be written in frequency domain as

$$H_0^f(\omega)\tilde{H}_0^f(\omega) + H_1^f(\omega)\tilde{H}_1^f(\omega) + H_2^f(\omega)\tilde{H}_2^f(\omega) = 2 \tag{5.9}$$

from which we deduce that

$$H_1^f(0)\tilde{H}_1^f(0) + H_2^f(0)\tilde{H}_2^f(0) = 0. \tag{5.10}$$

Since at least one of $H_1^f(0)$ and $H_2^f(0)$ are nonzero, the coefficient matrix of the equations 5.10 and 5.7 must be singular. Thus,

$$\tilde{H}_1^f(0)\tilde{H}_2^f(\pi) - \tilde{H}_2^f(0)\tilde{H}_1^f(\pi) = 0. \tag{5.11}$$

Now we have the following system of equations in variables $\tilde{H}_1^f(0)$ and $\tilde{H}_2^f(0)$.

$$\begin{aligned} \tilde{H}_1^f(0)\tilde{H}_2^f(\pi) - \tilde{H}_2^f(0)\tilde{H}_1^f(\pi) = 0 \\ \tilde{H}_1^f(0)H_1^f(\pi) + \tilde{H}_2^f(0)H_2^f(\pi) = 0 \end{aligned} \tag{5.12}$$

The coefficient matrix of the system has full rank since

$$H_1^f(\pi)\tilde{H}_1^f(\pi) + H_2^f(\pi)\tilde{H}_2^f(\pi) = 2. \tag{5.13}$$

To see this, we assume

$$\begin{bmatrix} H_1^f(\pi) \\ \tilde{H}_2^f(\pi) \end{bmatrix} = k \begin{bmatrix} H_2^f(\pi) \\ -\tilde{H}_1^f(\pi) \end{bmatrix} \tag{5.14}$$

and find a contradiction. Substitute $H_1^f(\pi) = kH_2^f(\pi)$ and $\tilde{H}_1^f(\pi) = \frac{-\tilde{H}_2^f(\pi)}{k}$ in equation 5.13 to see the contradiction. Since coefficient matrix has full rank, we must have $\tilde{H}_1^f(0) = 0$ and $\tilde{H}_2^f(0) = 0$.

The third case is easier. Substituting $\omega = \pi$ in equation 5.6 we get $H_0^f(\pi)\tilde{H}_0^f(0) = 0$, which implies $H_0^f(\pi) = 0$. Similarly we get $\tilde{H}_0^f(\pi) = 0$ after substituting $\omega = 0$ in equation 5.6. ∇

Thus unlike in the wavelets which corresponds to the maximally decimated filter banks, vanishing moments of the high pass filters are not guaranteed simply by enforcing zeros at the aliasing frequency for the scaling filters in general. But vanishing moments of the high pass filters are still guaranteed for the orthogonal like double density filter banks.

Corollary 1 *Let H_0 be the scaling filter of a orthogonal like double density filter bank and H_1 and H_2 are the wavelet filters. Then $H_1^f(0) = 0$ and $H_2^f(0) = 0$ if and only if $H_0^f(\pi) = 0$.*

Proof: It follows immediately from Lemma 8 since vanishing moments for the analysis wavelet filter implies vanishing moments for the synthesis wavelet filter and vice versa. ∇

Now, the issue of design of scaling filter is driven by three main constraints.

- Necessary conditions for perfect reconstruction.
- Vanishing moments for wavelet filters.
- Smoothness for scaling functions.

5.1.1 Necessary conditions for perfect reconstruction

In the two-band maximally decimated filter banks, for perfect reconstruction it is necessary that the scaling filters, $H_0(z)$ and $\tilde{H}_0(z)$, satisfy the bi-orthogonality constraint, $H_0(z)\tilde{H}_0(z)[\downarrow 2] = 1$. Thus most of the designs were dominated to ensure this bi-orthogonality constraint [11, 1, 16]. In the design of double density filter banks we no more have the bi-orthogonality constraint. Thus strictly speaking we do not have bi-orthogonal or orthogonal double density filter banks. But we will design *bi-orthogonal-like* and *orthogonal-like* double density filter banks. We use the term orthogonal-like when the analysis filters and synthesis filters are related via, $\tilde{H}_i(z) = H_i(z^{-1})$ for $i \in \{0, 1, 2\}$. When the filters are not orthogonal-like we say the double density filter bank is bi-orthogonal-like. Thus we can select scaling filters as we like and complete the filter bank such that it satisfy the perfect reconstruction. But vanishing moments for the wavelet filters do impose constraints on the design of scaling filters.

5.1.2 Vanishing moments for wavelet filters

First we will look at necessary conditions which must be satisfied by the scaling filters such that wavelet filters has a given number of vanishing moments. The following lemma gives the minimal number of higher derivatives of scaling filters which vanish at the aliasing frequency.

Theorem 10 *Let the analysis wavelet filters has at least $\tilde{K}_w$ vanishing moments and the synthesis filters has at least K_w vanishing moments. Then $H_0^{f^{(i)}}(\pi) = 0$ and $\tilde{H}_0^{f^{(i)}}(\pi) = 0$ for $i \in \{0..K_m\}$ where $K_m = min(\tilde{K}_w, K_w)$.*

Proof: We prove the Theorem by mathematical induction. The case for $i = 0$ is covered by the Lemma 8. Now we assume that the Lemma is true for $i = l - 1$, i.e. $H_0^{f^{(i)}}(\pi) = 0$ and $\tilde{H}_0^{f^{(i)}}(\pi) = 0$ for $i \in \{0..l-1\}$. By taking the l^{th} derivative of equation 5.6 and substituting $\omega = \pi$ we get

$$H_0^{f^{(l)}}(\pi)\tilde{H}_0^f(0) = 0 \tag{5.15}$$

which implies $H_0^{f^{(l)}}(\pi) = 0$. Similarly, by taking the l^{th} derivative of equation 5.6 and substituting $\omega = 0$ we prove that $\tilde{H}_0^{f^{(l)}}(\pi) = 0$. ∇

The vanishing moments of wavelet filters imposes further constraints on the scaling filter. The following Lemma indicates that the corresponding product filter $P_0(z) = H_0(z)\tilde{H}_0(z)$ of the scaling filters must have the *coiflet* like property, i.e. zeros of higher derivatives at zero.

Lemma 9 *Let analysis wavelet filters $\tilde{H}_1$ and $\tilde{H}_2$ has $\tilde{K}_1$ and $\tilde{K}_2$ vanishing moments and the synthesis wavelet filters H_1 and H_2 has K_1 and K_2 vanishing moments. Then*

$$P_0^{f^{(i)}}(0) = 0 \qquad 1 \le i \le min(K_1 + \tilde{K}_1, K_2 + \tilde{K}_2). \tag{5.16}$$

Proof: Let $P_1(z) = H_1(z)\tilde{H}_1(z)$ and $P_2(z) = H_2(z)\tilde{H}_2(z)$. Then the equation 5.2 can be written as

$$P_0(z) + P_1(z) + P_2(z) = 2. \tag{5.17}$$

Now the result follows after taking the derivatives of the above equation in the frequency domain and then substituting $\omega = 0$. ∇

The above results are useful in the design of bi-orthogonal-like scaling filters. Now we ask the reverse question which is useful in the design of orthogonal-like scaling filters and wavelet filters. How would the zeros at aliasing frequency at the derivative of scaling filters automatically transfer to the vanishing moments of wavelet filters?

Theorem 11 *Consider an orthogonal-like double density filter bank with H_0 as the scaling filter and H_1 and H_2 as the wavelet filters. Let $H_0^{f^{(i)}}(\pi) = 0$ for $i \in \{0...K_0\}$ and that $P_0^{f^{(i)}}(0) = 0$ for $i \in \{1...2K_s\}$ such that $K_s \le K_0$. Then for $0 \le i \le K_s$ $H_1^{f^{(i)}}(0) = 0$ and $H_2^{f^{(i)}}(0) = 0$.*

Proof: We prove by mathematical induction on i. For $i = 0$, the hypothesis is true by Corollary 1. Assume the hypothesis is true for $0 < i \le l - 1$.

We will prove the hypothesis for $i = l$ by contradiction. Assume that at least one of $H_1^{f^{(l)}}(0)$ and $H_2^{f^{(l)}}(0)$ are nonzero. Taking the l^{th} derivative of the equation 5.6 and substituting $\omega = 0$ and $\omega = \pi$, we get

$$H_1^{f^{(l)}}(0)\tilde{H}_1^f(\pi) + H_2^{f^{(l)}}(0)\tilde{H}_2^f(\pi) = 0, \tag{5.18}$$

$$H_1^f(\pi)\tilde{H}_1^{f^{(l)}}(0) + H_2^f(\pi)\tilde{H}_2^{f^{(l)}}(0) = 0. \tag{5.19}$$

Taking $2l^{th}$ derivative of the equation 5.2 we get

$$P_0^{f^{(2l)}}(\omega) + \sum_{r=1}^{2}\sum_{n=0}^{2l}\binom{2l}{n}\frac{d^n}{d\omega^n}H_r^f(\omega)\frac{d^{2l-n}}{d\omega^{2l-n}}\tilde{H}_r^f(\omega) = 0. \tag{5.20}$$

Substituting $\omega = 0$ we get

$$H_1^{f^{(l)}}(0)\tilde{H}_1^{f^{(l)}}(0) + H_2^{f^{(l)}}(0)\tilde{H}_2^{f^{(l)}}(0) = 0. \tag{5.21}$$

By the assumption, the coefficient matrix of the equations 5.21 and 5.18 must be singular. Thus we get

$$\tilde{H}_2^f(\pi)\tilde{H}_1^{f^{(l)}}(0) - \tilde{H}_1^f(\pi)\tilde{H}_2^{f^{(l)}}(0) = 0. \tag{5.22}$$

Consider the following system of equations:

$$\begin{aligned} H_1^f(\pi)\tilde{H}_1^{f^{(l)}}(0) + H_2^f(\pi)\tilde{H}_2^{f^{(l)}}(0) = 0 \\ \tilde{H}_2(\pi)\tilde{H}_1^{(l)}(0) - \tilde{H}_1(\pi)\tilde{H}_2^{(l)}(0) = 0 \end{aligned} \tag{5.23}$$

Using a similar argument as in Lemma 8 it can be shown that the coefficient matrix of the system has full rank and hence $\tilde{H}_1^{f^{(l)}}(0) = 0$ and $\tilde{H}_2^{f^{(l)}}(0) = 0$. Thus in the orthogonal-like setting we get $H_1^{f^{(l)}}(0) = 0$ and $H_2^{f^{(l)}}(0) = 0$, which is a contradiction. ∇

5.1.3 Smoothness of the scaling filter

The most popular way of achieving smoothness for the scaling filter is to enforce polynomial interpolation property for the scaling filter. The conditions for polynomial interpolation are well known as we have seen in the chapter 4. We will enforce a factor $(1 + z^{-1})^{K_0}$ for the scaling filter $H_0(z)$, which in turn results in $H_0^{f^{(i)}}(\pi) = 0$ for $i \in \{0..K_0 - 1\}$.

5.2 Design of the Scaling Filter

We will discuss a technique due to [18]. In that, Selesnick describes a maximally flat symmetric FIR filter which was originally described by Herrman [32]. Let $P_0(z)$ be the product filter which must satisfy the constraints $P_0^f(0) = 2$, $P_0^{f^{(i)}}(0) = 0$ for $i \in \{1...2K_c\}$, $P_0^{f^{(i)}}(\pi) = 0$ for $i \in \{0...2K_0\}$, and $P_0(z)$ is symmetric, then the product filter is given by

$$P_0(z) = 2\left(\frac{z + 2 + z^{-1}}{4}\right)^{K_0} \sum_{n=0}^{K_c-1} \binom{K_0 + n - 1}{n} \left(\frac{-z + 2 - z^{-1}}{4}\right)^n . \quad (5.24)$$

Now orthogonal-like scaling filters can be obtained by spectral factorization of the product filter while bi-orthogonal-like scaling filters can be obtained by polynomial factorization and appropriate regrouping of the factors.

5.3 Example Designs: Bi-orthogonal-Like Ad Hoc designs

We assume that both scaling filters have regularity K_l and wavelet filters have regularity K_h such that

$$\begin{aligned}
H_0(z) &= \left(\frac{1+z^{-1}}{2}\right)^{K_l} A_0(z), \\
\tilde{H}_0(z) &= \left(\frac{1+z}{2}\right)^{K_l} \tilde{A}_0(z), \\
H_1(z) &= \left(\frac{1-z^{-1}}{2}\right)^{K_h} A_1(z), \\
\tilde{H}_1(z) &= \left(\frac{1-z}{2}\right)^{K_h} \tilde{A}_1(z), \\
H_2(z) &= \left(\frac{1-z^{-1}}{2}\right)^{K_h} A_2(z), \\
\tilde{H}_2(z) &= \left(\frac{1-z}{2}\right)^{K_h} \tilde{A}_2(z).
\end{aligned} \tag{5.25}$$

Now the perfect reconstruction conditions can be written as follows.

$$\begin{aligned}
&\left(\frac{1-z^{-1}}{2}\right)^{K_h}\left(\frac{1-z}{2}\right)^{K_h}(A_1(z)\tilde{A}_1(z)+A_2(z)\tilde{A}_2(z)) \\
&\qquad = 2-\left(\frac{1+z^{-1}}{2}\right)^{K_l}\left(\frac{1+z}{2}\right)^{K_l} A_0(z)\tilde{A}_0(z), \\
&\left(\frac{1-z^{-1}}{2}\right)^{K_h}\left(\frac{1+z}{2}\right)^{K_h}(A_1(z)\tilde{A}_1(-z)+A_2(z)\tilde{A}_2(-z)) \\
&\qquad = -\left(\frac{1+z^{-1}}{2}\right)^{K_l}\left(\frac{1-z}{2}\right)^{K_l} A_0(z)\tilde{A}_0(-z).
\end{aligned} \tag{5.26}$$

When the scaling filters are appropriately designed we can divide RHS of each of above equations by $\left(\frac{1-z^{-1}}{2}\right)^{K_h}\left(\frac{1-z}{2}\right)^{K_h}$ and $\left(\frac{1-z^{-1}}{2}\right)^{K_h}\left(\frac{1+z}{2}\right)^{K_h}$ respectively. Then we get the following system of equations.

$$\begin{aligned} A_1(z)\tilde{A}_1(z) + A_2(z)\tilde{A}_2(z) &= B(z) \\ A_1(z)\tilde{A}_1(-z) + A_2(z)\tilde{A}_2(-z) &= C(z) \end{aligned} \tag{5.27}$$

The determinant of the system is given by

$$\Delta(z) = \tilde{A}_1(z)\tilde{A}_2(-z) - \tilde{A}_1(-z)\tilde{A}_2(z). \tag{5.28}$$

Note that $\Delta(z) = -\Delta(-z)$ and hence $\Delta(z)[\downarrow 2] = 0$. The solutions of the system can be written as

$$\begin{aligned} A_1(z) &= \frac{1}{\Delta(z)}(\tilde{A}_2(-z)B(z) - \tilde{A}_2(z)C(z)), \\ A_2(z) &= \frac{1}{\Delta(z)}(-\tilde{A}_1(-z)B(z) + \tilde{A}_1(z)C(z)). \end{aligned} \tag{5.29}$$

Example 2 *We design a double density system with* $K_l = 2$ *and* $K_h = 1$. *The smallest low pass product filter is given by* $H_0(z)\tilde{H}_0(z) = 2\left(\frac{z+2+z^{-1}}{4}\right)^2$ *such that* $H_0(z) = \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^2$ *and* $\tilde{H}_0(z) = \sqrt{2}\left(\frac{1+z}{2}\right)^2$. *We set one of the analysis high pass filters as the Haar filter such that* $\tilde{A}_1(z) = \sqrt{2}$. *We set* $\Delta(z) = -4z$. *This gives us the other minimum length analysis wavelet filter as the unit time advanced of the Haar filter such that* $\tilde{A}_2(z) = \sqrt{2}z$. *The following table summarizes the filter system. Both analysis and synthesis*

Analysis filter	Synthesis filter
$\sqrt{2}\left(\frac{1+z}{2}\right)^2$	$\sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^2$
$\sqrt{2}\left(\frac{1-z}{2}\right)$	$\frac{\sqrt{2}}{4}\left(\frac{1-z^{-1}}{2}\right)(z^{-1}+3)$
$\sqrt{2}z\left(\frac{1-z}{2}\right)$	$\frac{\sqrt{2}}{4}\left(\frac{1-z^{-1}}{2}\right)(3z^{-1}+1)$

scaling filters are second order splines while analysis wavelet filters are Haar filters. All the filters are symmetric except synthesis wavelet filters.

Example 3 *We design a double density system with* $K_l = 4$ *and* $K_h = 2$. *The smallest low pass product filter is given by*

$$H_0(z)\tilde{H}_0(z) = 2\left(\frac{z+2+z^{-1}}{4}\right)^4(-z+3-z^{-1})$$

such that $H_0(z) = \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^4(-z+3-z^{-1})$ *and* $\tilde{H}_0(z) = \sqrt{2}\left(\frac{1+z}{2}\right)^4$. *We set one of the analysis high pass filters as the second order high pass spline such that* $\tilde{H}_1(z) = \sqrt{2}\left(\frac{1-z}{2}\right)^2$. *We set* $\tilde{A}_1(z) = \sqrt{2}$ *and* $\Delta(z) = -4z$. *This gives us the other minimum length analysis wavelet filter as the unit time advanced of the second order high pass spline filter such that* $\tilde{A}_2(z) = \sqrt{2}z$. *The following table summarizes the filter system.*

Analysis filter	Synthesis filter
$\sqrt{2}\left(\frac{1+z}{2}\right)^4$	$\sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^4(-z+3-z^{-1})$
$\sqrt{2}\left(\frac{1-z}{2}\right)^2$	$\frac{\sqrt{2}}{16}\left(\frac{1-z^{-1}}{2}\right)^2(3z^2+18z+38+18z^{-1}+3z^{-2})$
$\sqrt{2}z\left(\frac{1-z}{2}\right)^2$	$\frac{\sqrt{2}}{16}\left(\frac{1-z^{-1}}{2}\right)^2(z^2+6z+17+32z^{-1}+17z^{-2}+6z^{-3}+z^{-4})$

The example designs construct a class of double density filters where the analysis filters are delays of each other. Thus the wavelet coefficients can be produced by filtering by a single filter and with no down-sampling. When analysis filters are produced by splines shifted by π in frequency domain as in the above examples the resulting filter system consists of symmetric filters when $K_l - K_h$ is even as in example 3. To see this, when $K_l - K_h$ is even both $B(z)$ and $C(z)$ are symmetric, which results in symmetric synthesis filters.

5.4 A Polyphase based Design Approach

As with many problems in signal processing, polyphase constructions substantially simplify constraints. It is the same with double density filter banks. Let the analysis and synthesis wavelet filters are of the form

$$\begin{aligned} \tilde{H}_1(z) &= (1-z)^{\tilde{K}_h}\tilde{A}_1(z), \\ \tilde{H}_2(z) &= (1-z)^{\tilde{K}_h}\tilde{A}_2(z), \\ H_1(z) &= (1-z^{-1})^{K_h}A_1(z), \\ H_2(z) &= (1-z^{-1})^{K_h}A_2(z). \end{aligned} \tag{5.30}$$

Let the type 1 polyphase matrix of analysis wavelet filters and type 2 polyphase matrix of synthesis wavelet filters are given by

$$\tilde{H}_h(z) = \begin{bmatrix} \tilde{H}_{10}(z) & \tilde{H}_{11}(z) \\ \tilde{H}_{20}(z) & \tilde{H}_{21}(z) \end{bmatrix} \text{ and } H_h(z) = \begin{bmatrix} H_{10}(z) & H_{11}(z) \\ H_{20}(z) & H_{21}(z) \end{bmatrix}.$$

Then from equation 5.5 it is easy to see that

$$H_h^T\tilde{H}_h = I - \begin{bmatrix} H_{00}(z) \\ H_{01}(z) \end{bmatrix} \begin{bmatrix} \tilde{H}_{00}(z) & \tilde{H}_{01}(z) \end{bmatrix}. \tag{5.31}$$

We can obtain a simplified expression for both $\tilde{H}_h(z)$ and $H_h(z)$ as in the proof of Lemma 7. First consider the type 1 polyphase vector $\begin{bmatrix} \tilde{H}_{10}(z) \\ \tilde{H}_{11}(z) \end{bmatrix}$. With a simmilar argument as in the proof of Lemma 7 it can be shown that

$$\begin{bmatrix} \tilde{H}_{10}(z) \\ \tilde{H}_{11}(z) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -z & 1 \end{bmatrix}^{\tilde{K}_h} \tilde{B}_1(z).$$

Simmilarly

$$\begin{bmatrix} \tilde{H}_{20}(z) \\ \tilde{H}_{21}(z) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -z & 1 \end{bmatrix}^{\tilde{K}_h} \tilde{B}_2(z).$$

Thus

$$\tilde{H}_h(z) = \tilde{A}(z)\begin{bmatrix} 1 & -z \\ -1 & 1 \end{bmatrix}^{\tilde{K}_h} \tag{5.32}$$

where $\tilde{A}(z) = \begin{bmatrix} \tilde{B}_1^T(z) \\ \tilde{B}_2^T(z) \end{bmatrix}$. Simmilarly

$$H_h(z) = A(z)\begin{bmatrix} 1 & -z^{-1} \\ -1 & 1 \end{bmatrix}^{K_h}. \tag{5.33}$$

Let

$$P(z) = \begin{bmatrix} 1 & 1 \\ z^{-1} & 1 \end{bmatrix}^{K_h} \left[I - \begin{bmatrix} H_{00}(z) \\ H_{01}(z) \end{bmatrix} \begin{bmatrix} \tilde{H}_{00}(z) & \tilde{H}_{01}(z) \end{bmatrix} \right] \begin{bmatrix} 1 & z \\ 1 & 1 \end{bmatrix}^{\tilde{K}_h} \tag{5.34}$$

and $Q(z) = \frac{1}{(1-z^{-1})^{K_h}(1-z)^{\tilde{K}_h}} P(z)$. Since

$$\begin{gathered} \begin{bmatrix} 1 & -z \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & z \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1-z & 0 \\ 0 & 1-z \end{bmatrix}, \\ \begin{bmatrix} 1 & 1 \\ z^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -z^{-1} & 1 \end{bmatrix} = \begin{bmatrix} 1-z^{-1} & 0 \\ 0 & 1-z^{-1} \end{bmatrix} \end{gathered} \tag{5.35}$$

we get

$$A^T(z)\tilde{A}(z) = Q(z). \tag{5.36}$$

Many factorization methods of Laurent polynomial matrices are crucially dependent on the determinant of the matrix. Fortunately, for a good subclass of double density filter banks, we have a simplified result for the determinant of $Q(z)$. We assume that the number of vanishing moments of analysis high pass filters and synthesis high pass filters are equal, i.e. $K_h = \tilde{K}_h = M$. It can be seen that the determinant of $H_h^T \tilde{H}_h$ is given by $D(z) = 1 - H_{00}(z)\tilde{H}_{00}(z) -$

$H_{01}(z)\tilde{H}_{01}(z)$. Since $Q(z)$ is FIR and both $\begin{bmatrix} 1 & -z \\ -1 & 1 \end{bmatrix}^M$ and $\begin{bmatrix} 1 & -z^{-1} \\ -1 & 1 \end{bmatrix}^M$ are factors of $H_h^T\tilde{H}_h$, it must be that $(1-z)^M(1-z^{-1})^M$ is a factor of $D(z)$. We have the following Lemma.

Lemma 10 *Let the high pass filters has M number of vanishing moments each and the low pass filters, H_0 and $\tilde{H}_0$, have the equal regularity, K, and given by*

$$H_0(z)\tilde{H}_0(z) = 2\left(\frac{z+2+z^{-1}}{4}\right)^K \sum_{n=0}^{M-1} \binom{K+n-1}{n} \left(\frac{-z+2-z^{-1}}{4}\right)^n. \tag{5.37}$$

Then the determinant of $Q(z)$ is a real number when,

$$M = \left\lfloor \frac{K+M-1}{2} \right\rfloor. \tag{5.38}$$

Proof: We have

$$\begin{aligned} \det(H_h^T(z)\tilde{H}_h(z)) &= (1-z)^M(1-z^{-1})^M \det(Q(z)) \\ &= 1 - H_{00}(z)\tilde{H}_{00}(z) - H_{01}(z)\tilde{H}_{01}(z). \end{aligned} \tag{5.39}$$

Also note that, $H_{00}(z)\tilde{H}_{00}(z) + H_{01}(z)\tilde{H}_{01}(z) = H_0(z)\tilde{H}_0(z)[\downarrow 2]$. Now the largest power of z in $\det(H_h^T(z)\tilde{H}_h(z))$ is $\lfloor\frac{K+M-1}{2}\rfloor$ and hence the largest power of z in $\det(Q(z))$ is $\lfloor\frac{K+M-1}{2}\rfloor - M$. By symmetry, the smallest power of z in $\det(Q(z))$ is $-\lfloor\frac{K+M-1}{2}\rfloor + M$. ∇

The above lemma covers some useful number of double density filter banks irrespective whether they are bi-orthogonal-like or orthogonal-like such as $(K, M) = (2,1), (3,1), (3,2), (4,2), (4,3), (5,3), (5,4), (6,4), (6,5)$ etc.

5.4.1 Factorizations for Orthogonal-like DDFB's

Note that the polyphase matrix of the orthogonal-like DDFB is a 2×3 lossless system [36]. Selesnick [18] designs the high pass filters indirectly by first designing a 3×3 lossless system and then extracting only the first two rows to form a 2×3 lossless system. We provide a more direct method for a class of DDFB's by factorizing $Q(z)$ when $\det(Q(z))$ is a real number (as in Lemma 10).

Lemma 11 *Let*

$$Q_0(z) = P_{0,k}^T z^k + P_{0,k-1}^T z^{k-1} + \ldots + P_{0,0} + \ldots + P_{0,k-1} z^{-(k-1)} + P_{0,k} z^{-k}$$

and assume that $det(Q_0(z))$ *is a nonzero real number. Let*

$$Q_0^{-1}(z) = R_{0,\tilde{k}}^T z^{\tilde{k}} + R_{0,\tilde{k}-1}^T z^{\tilde{k}-1} + \ldots + R_{0,0} + \ldots + R_{0,\tilde{k}-1} z^{-(\tilde{k}-1)} + R_{0,\tilde{k}} z^{-\tilde{k}}$$

and $A_0(z) = R_{0,\tilde{k}} + zR_{0,\tilde{k}}^T$ *and* $A_0(z)$ *is nonsingular. Then for some* $Q_1(z)$,

$$Q_0(z) = [A_0(z^{-1})]^{-T} Q_1(z) [A_0(z)]^{-1} \tag{5.40}$$

where

$$Q_1(z) = P_{1,k-1}^T z^{k-1} + P_{1,k-2}^T z^{k-2} + \ldots + P_{1,0} + \ldots + P_{1,k-2} z^{-(k-2)} + P_{1,k-1} z^{-(k-1)}.$$

Proof: Since $Q_0(z)Q_0^{-1}(z) = I$, we have

$$P_{0,k} R_{0,\tilde{k}} = 0 \tag{5.41}$$

$$P_{0,k}^T R_{0,\tilde{k}}^T = 0 \tag{5.42}$$

and

$$P_{0,k-1}R_{0,\tilde{k}} + P_{0,k}R_{0,\tilde{k}-1} = 0. \qquad (5.43)$$

From equations 5.41 and 5.42 it is clear that

$$Q_0(z)A_0(z) = (P_{0,k}^T R_{0,\tilde{k}} + P_{0,k-1}^T R_{0,\tilde{k}}^T)z^k + ... + (P_{0,k}R_{0,\tilde{k}}^T + P_{0,k-1}R_{0,\tilde{k}})z^{-(k-1)}.$$

Now

$$A_0^T(z^{-1})Q_0(z)A_0(z) = (R_{0,\tilde{k}}^T P_{0,k}^T R_{0,\tilde{k}} + R_{0,\tilde{k}}^T P_{0,k-1}^T R_{0,\tilde{k}}^T)z^k + \\ ... + (R_{0,\tilde{k}}P_{0,k}R_{0,\tilde{k}}^T + R_{0,\tilde{k}}P_{0,k-1}R_{0,\tilde{k}})z^{-k}.$$

From equations 5.43 and 5.42, we get

$$R_{0,\tilde{k}}P_{0,k-1}R_{0,\tilde{k}} = 0. \qquad (5.44)$$

Thus it is clear that the coefficients of z^{-k} and z^k in $A_0^T(z^{-1})Q_0(z)A_0(z)$ are zero. ∇

Remark 1 *Note since $A_0^T(z^{-1}) = z^{-1}A_0(z)$,*

$$Q_0(z) = [A_0(z)]^{-1}Q_1(z)[A_0(z^{-1})]^{-T} \qquad (5.45)$$

is also a possible factorization but it does not give us new filters!

Assuming each degree reduction step is invertible, repeating the process given in Lemma 11, we could write

$$A_k^T(z^{-1})...A_0^T(z^{-1})Q_0(z)A_0(z)...A_k(z) = Q_{k+1} \qquad (5.46)$$

where Q_{k+1} is a constant symmetric matrix (i.e. $Q_{k+1} = Q_{k+1}^T$). Then Q_{k+1} is orthogonally diagonalizable and let $Q_{k+1} = A_{k+1}^T A_{k+1}$. However we require the eigenvalues of Q_{k+1} be positive. Then we could write

$$A^T(z) = A_0^{-T}(z^{-1})...A_k^{-T}(z^{-1})A_{k+1}^T. \qquad (5.47)$$

We have one degree of freedom in the factorization as given by

$$A^T(z) = A_0^{-T}(z^{-1})...A_k^{-T}(z^{-1})A_{k+1}^T \begin{bmatrix} cos\ \theta & -sin\ \theta \\ sin\ \theta & cos\ \theta \end{bmatrix}. \quad (5.48)$$

Such parameterization is used by Selesnick [18] to achieve near shift invariance among the wavelet functions.

Example 4 *Consider the (2,1) orthogonal-like DDFB with* $H_0(z) = \sqrt{2}\left(\frac{1+z}{2}\right)^2$ *and* $\tilde{H}_0(z) = \sqrt{2}\left(\frac{1+z^{-1}}{2}\right)^2$. *We get* $Q(z) = \begin{bmatrix} \frac{3}{8} & \frac{z}{8} \\ \frac{z^{-1}}{8} & \frac{3}{8} \end{bmatrix}$ *which leads to*

$$A^T(z) = \left[\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} + z^{-1}\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\right]^{-T} \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2} & \frac{-1}{2\sqrt{2}} \end{bmatrix}.$$

The synthesis high pass filters are given by $H_h^T(z) = \begin{bmatrix} 1 & -1 \\ -z^{-1} & 1 \end{bmatrix} A^T(z)$. *This gives* $H_1(z) = \frac{1}{2} - \frac{1}{2}z^2$ *and* $H_2(z) = \frac{2\sqrt{2}}{4} - \frac{2\sqrt{2}}{2}z + \frac{2\sqrt{2}}{4}z^2$.

Table 5.1 shows parametric orthogonal-like filters for smaller values of K and M. Clearly Lemma 11 is not always applicable since there is no guarantee that the degree reduction step is invertible. But so far in all the examples I have computed, I have not run into this problem. Whether it is that we have just been lucky or that we haven't uncovered some of the hidden results, is not yet clear to me. But in the next section when we generalize into multiple density case, we will realize that we have been lucky in the double density case.

Alternate ways of factorizations are also available. In the following, I will adopt a factorization technique used for M-band bi-orthogonal wavelet filter banks [25].

K	M	Orthogonal-like double density filter
2	1	$H_0(z) = \frac{1}{2\sqrt{2}}(1 + 2z + z^2)$
		$H_1(z) = \frac{1}{2}cos\theta + \frac{1}{2\sqrt{2}}sin\theta - \frac{z}{\sqrt{2}}sin\theta + z^2(\frac{1}{2\sqrt{2}}sin\theta - \frac{1}{2}cos\theta)$
		$H_2(z) = \frac{1}{2\sqrt{2}}cos\theta - \frac{1}{2}sin\theta - \frac{z}{\sqrt{2}}cos\theta + z^2(\frac{1}{2}sin\theta + \frac{1}{\sqrt{2}}cos\theta)$
3	1	$H_0(z) = \frac{1}{4\sqrt{2}}(1 + 3z + 3z^2 + z^3)$
		$H_1(z) = \frac{\sqrt{6}}{4}cos\theta + \frac{\sqrt{2}}{8}sin\theta + z(-\frac{\sqrt{6}}{4}cos\theta + \frac{3\sqrt{2}}{8}sin\theta) - \frac{3\sqrt{2}z^2}{8}sin\theta - \frac{\sqrt{2}z^3}{8}sin\theta$
		$H_2(z) = -\frac{\sqrt{6}}{4}sin\theta + \frac{\sqrt{2}}{8}cos\theta + z(\frac{\sqrt{6}}{4}sin\theta + \frac{3\sqrt{2}}{8}cos\theta) - \frac{3\sqrt{2}z^2}{8}cos\theta - \frac{\sqrt{2}z^3}{8}cos\theta$
3	2	$H_0(z) = \frac{1}{8\sqrt{2}}(1 + z)^3(3z - 1)$ maximum phase
		$H_1(z) = z^{-2}(-\frac{1}{8}\sqrt{\frac{3}{10}}cos\theta + \frac{1}{2\sqrt{5}}sin\theta) - \frac{1}{4}\sqrt{\frac{3}{10}}cos\theta - \frac{3}{2\sqrt{5}}sin\theta$ $+z(-\sqrt{\frac{3}{10}}cos\theta + \frac{1}{\sqrt{5}}sin\theta) - \frac{5z^2}{8}\sqrt{\frac{3}{10}}cos\theta$
		$H_2(z) = z^{-2}(\frac{1}{8}\sqrt{\frac{3}{10}}sin\theta + \frac{1}{2\sqrt{5}}cos\theta) + \frac{1}{4}\sqrt{\frac{3}{10}}sin\theta - \frac{3}{2\sqrt{5}}cos\theta$ $+z(\sqrt{\frac{3}{10}}sin\theta + \frac{1}{\sqrt{5}}cos\theta) + \frac{5z^2}{8}\sqrt{\frac{3}{10}}sin\theta$
3	2	$H_0(z) = \frac{1}{8\sqrt{2}}(1 + z)^3(z - 3)$ minimum phase
		$H_1(z) = z^{-2}(-\frac{3}{14\sqrt{5}}cos\theta - \frac{3}{112}\sqrt{\frac{6}{5}}sin\theta) - z^{-1}(\frac{4}{7\sqrt{5}}cos\theta + \frac{1}{14}\sqrt{\frac{6}{5}}sin\theta) + \frac{15}{14\sqrt{5}}cos\theta$ $+\frac{25}{56}\sqrt{\frac{6}{5}}sin\theta + z(\frac{3}{7\sqrt{5}}cos\theta - \frac{4}{7}\sqrt{\frac{6}{5}}sin\theta) + z^2(-\frac{5}{7\sqrt{5}}cos\theta + \frac{25}{112}\sqrt{\frac{6}{5}}sin\theta)$
		$H_2(z) = z^{-2}(\frac{3}{14\sqrt{5}}sin\theta - \frac{3}{112}\sqrt{\frac{6}{5}}cos\theta) - z^{-1}(-\frac{4}{7\sqrt{5}}sin\theta + \frac{1}{14}\sqrt{\frac{6}{5}}cos\theta) - \frac{15}{14\sqrt{5}}sin\theta$ $+\frac{25}{56}\sqrt{\frac{6}{5}}cos\theta + z(-\frac{3}{7\sqrt{5}}sin\theta - \frac{4}{7}\sqrt{\frac{6}{5}}cos\theta) + z^2(\frac{5}{7\sqrt{5}}sin\theta + \frac{25}{112}\sqrt{\frac{6}{5}}cos\theta)$

Table 5.1: Parametric Double Density Filter Banks.

Lemma 12 *Let*

$$Q_0(z) = \sum_{i=1}^{k} P_{0,i}^T z^i + P_{0,0} + \sum_{i=1}^{k} P_{0,i} z^{-i},$$

$$Q_0^{-1}(z) = \sum_{i=1}^{\tilde{k}} R_{0,i}^T z^i + R_{0,0} + \sum_{i=1}^{\tilde{k}} R_{0,i} z^{-i}$$

and the determinant of $Q_0(z)$ is a real number. If both $R_{0,\tilde{k}-1}$ and $P_{0,k-1}$ are invertible matrices, then there exists a nilpotent matrix N, with $N^2 = 0$, such that

$$Q_0(z) = [A_0(z^{-1})]^{-T} Q_1(z) [A_0(z)]^{-1} \tag{5.49}$$

where

$$A_0(z) = I + N - Nz^{-1}, \quad Q_1(z) = \sum_{i=1}^{k-1} P_{0,i}^T z^i + P_{0,0} + \sum_{i=1}^{k-1} P_{0,i} z^{-i}.$$

Proof: Since $Q_0(z)Q_0^{-1}(z) = I$, we have

$$P_{0,k} R_{0,\tilde{k}} = 0, \quad P_{0,k} R_{0,\tilde{k}-1} + P_{0,k-1} R_{0,\tilde{k}} = 0.$$

Now set

$$N = P_{0,k-1}^{-1} P_{0,k} = -R_{0,\tilde{k}} R_{0,\tilde{k}-1}^{-1}.$$

Then we have

$$N^2 = 0, \quad P_{0,k} N = 0, \quad P_{0,k} - P_{0,k-1} N = 0,$$

and hence the result follows. ∇

Remark 2 *The condition that both $R_{0,\tilde{k}-1}$ and $P_{0,k-1}$ are invertible matrices, is a very strong condition for the degree reduction process. All we need is a nilpotent matrix N which satisfy*

$$N^2 = 0, \quad P_{0,k} N = 0, \quad P_{0,k} - P_{0,k-1} N = 0. \tag{5.50}$$

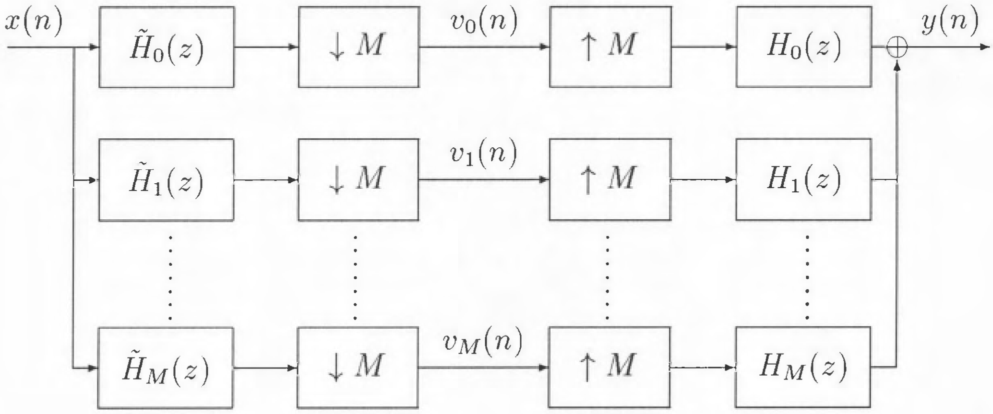

Figure 5.2: The M-band multiple density filter bank

5.5 Generalizations: The M-band Multiple Density Filter Banks

In this section we look at how what we have discussed so far generalizes for M-band multiple density filter banks (MDFB's) as shown in figure 5.2. As with double density filter banks, multiple density filter bank has one extra redundant subband while sub-sampling factor in each subband is $M(\geq 2)$.

The generalization do not have obvious generalization of our discussion about the vanishing moments but our polyphase based design approach do generalize for the multiple density filter banks. First we look at how we can design the scaling filter and then we give a factorization approach for a subclass of multiple density filter banks. Also our discussion is restricted to orthogonal-like multiple density filter banks.

5.5.1 Design of the Scaling Filter

Same as with the double density filter banks, our motivation is to have shorter filters while reducing the number of vanishing moments of wavelet filters.

The design approach is similar to the critically sampled M-band scaling filters [38]. In that we first design the product filter subject to the vanishing moment constraints of the wavelet subbands and regularity constraints of the scaling subband. Other than the equations which set aliasing terms to zero after down-sampling, we require the following equation for the perfect reconstruction.

$$P_0^f(\omega) + P_1^f(\omega) + ... + P_M^f(\omega) = M \tag{5.51}$$

where $P_i^f(\omega)$ is the product filter $\tilde{H}_i^f(\omega)H_i^f(\omega)$ corresponding to the i^{th} subband. If each high pass filter has at least L vanishing moments we must have

$$P_i^{f^k}(0) = 0 \quad i \in \{1, ..., M\} \text{ and } k \in \{0, ..., L-1\}. \tag{5.52}$$

The above equation leads to the following maximum flatness condition on P_0:

$$P_0^{f^k}(0) = M\delta(j) \quad k \in \{0, ..., L-1\}. \tag{5.53}$$

Using the same notation as in the section 4.2, a product filter of the scaling subband with regularity K and subject to the equation 5.53 is given by

$$\mathcal{P}(x) = \mathcal{E}^K(x)\mathcal{R}(x) \tag{5.54}$$

where

$$\mathcal{R}(x) = M \sum_{n=0}^{L-1} \left[\frac{1}{n!} \left(\frac{d}{dx} \right)^n \mathcal{E}^{-K}(x) \right]_{x=1} (x-1)^n \tag{5.55}$$

and $\mathcal{E}^K(x)$ is as given in the section 4.2.

5.5.2 Design of Wavelet Filters

In this section we look at how our polyphase based design approach generalize to the multiple density case. Let the type 1 polyphase matrix of analysis wavelet filters and type 2 polyphase matrix of synthesis wavelet filters are given by

$$\tilde{H}_h(z) = \begin{bmatrix} \tilde{H}_{1,0}(z) & \ldots & \tilde{H}_{1,M-1}(z) \\ \vdots & \vdots & \vdots \\ \tilde{H}_{M,0}(z) & \ldots & \tilde{H}_{M,M-1}(z) \end{bmatrix}$$

and

$$H_h(z) = \begin{bmatrix} H_{1,0}(z) & \ldots & H_{1,M-1}(z) \\ \vdots & \vdots & \vdots \\ H_{M,0}(z) & \ldots & H_{M,M-1}(z) \end{bmatrix}.$$

Then as with the double density case it can be seen that

$$H_h^T \tilde{H}_h = I - \begin{bmatrix} H_{0,0}(z) \\ \vdots \\ H_{0,M-1}(z) \end{bmatrix} \begin{bmatrix} \tilde{H}_{0,0}(z) & \ldots & \tilde{H}_{0,M-1}(z) \end{bmatrix}. \tag{5.56}$$

The determinant of $H_h^T \tilde{H}_h$ is not as obvious as with the double density case but it is similar.

Lemma 13 *The determinant of* $H_h^T \tilde{H}_h$ *is* $1 - \sum_{i=0}^{M-1} H_{0,i}(z)\tilde{H}_{0,i}(z)$.

Proof: It can be easily seen that

$$\det(H_h^T \tilde{H}_h) = \det(I) - \sum_{i=0}^{M-1} \det(A_i)$$

where A_i has ones in the diagonal except the i^{th} location as given by

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & \dots & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & & & & & & \vdots \\ -H_{0,i}(z)\tilde{H}_{0,0}(z) & \dots & \dots & \dots & -H_{0,i}(z)\tilde{H}_{0,i}(z) & \dots & \dots & \dots & -H_{0,i}(z)\tilde{H}_{0,M-1}(z) \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

Clearly the determinant of A_i is $-H_{0,i}(z)\tilde{H}_{0,i}(z)$ and hence the result follows. ∇

Since each high pass filters has at least L vanishing moments, we can write the polyphase matrices of the high pass filters as

$$H_h(z) = A(z)[R(z)]^L \text{ and } \tilde{H}_h(z) = \tilde{A}(z)[R(z^{-1})]^L \tag{5.57}$$

where

$$R(z) = \begin{bmatrix} 1 & 0 & 0 & \dots & -z^{-1} \\ -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

Now we give the equivalent of Lemma 10 for the multiple density filter banks.

Lemma 14 *Let the high pass filters has L number of vanishing moments each and the low pass filters, H_0 and $\tilde{H}_0$, have the regularity, K, and the product filter of the scaling filter is given by 5.54. Then the determinant of $Q(z) = A^T(z)\tilde{A}(z)$ is a real number when*

$$L = \left\lfloor \frac{(M-1)K + L - 1}{M} \right\rfloor. \tag{5.58}$$

Proof: We have

$$\begin{aligned}\det(H_h^T(z)\tilde{H}_h(z)) &= (1-z)^L(1-z^{-1})^L\det(Q(z))\\ &= 1-\sum_{i=0}^{M-1} H_{0,i}(z)\tilde{H}_{0,i}(z).\end{aligned} \tag{5.59}$$

Also note that, $\sum_{i=0}^{M-1} H_{0,i}(z)\tilde{H}_{0,i}(z) = H_0(z)\tilde{H}_0(z)[\downarrow M]$. Now the largest power of z in $\det(H_h^T(z)\tilde{H}_h(z))$ is $\left\lfloor \frac{(M-1)K+L-1}{K} \right\rfloor$ and hence the largest power of z in $\det(Q(z))$ is $\left\lfloor \frac{(M-1)K+L-1}{M} \right\rfloor - L$. By symmetry, the smallest power of z in $\det(Q(z))$ is $-\left\lfloor \frac{(M-1)K+L-1}{M} \right\rfloor + L$. ∇

Note that the Lemma 11 and Lemma 12 are independent of whether the filter bank is double density or multiple density and hence we can possibly use them whenever Lemma 14 is applicable. For $M = 3$, Lemma 14 is applicable when $(K, L) = (2,1), (3,2), (4,3), (5,4)$ etc. Now we will look at an example where both Lemma 11 and Lemma 12 are not applicable.

Example 5 *Consider* $(K, L) = (3, 2)$ *for* $M = 3$. *We get*

$$Q(z) = \begin{bmatrix} \frac{1}{27}(3z+17+3z^{-1}) & \frac{1}{81}(15z+43+2z^{-1}) & \frac{1}{81}(3z^2+32z+25) \\ \frac{1}{81}(2z+43+15z^{-1}) & \frac{1}{243}(10z+187+10z^{-1}) & \frac{1}{243}(2z^2+53z+125) \\ \frac{1}{81}(25+32z^{-1}+3z^{-2}) & \frac{1}{243}(125+53z^{-1}+2z^{-2}) & \frac{1}{243}(25z+157+25z^{-1}) \end{bmatrix}.$$

Lemma 11 is not useful since the determinant of $A_0(z)$ *is not a monomial and hence degree reduction process cannot be repeated. The determinant of* $A_0(z)$ *is a monomial when* $B = R_{0,\tilde{k}}P_{0,k}^T + R_{0,\tilde{k}}^T P_{0,k}$ *is nonsingular since then we can construct FIR inverse of* $A_0(z)$ *via* $[P_{0,k}^T + z^{-1}P_{0,k}]B^{-1}$. *In this example it can be seen that* B *is singular. Lemma 12 is also not applicable since* $P_{0,k-1}$ *is not invertible.*

5.6 Conclusion and Further Research

We have developed a factorization approach to obtain double density wavelet filters for a special case where the determinant of transfer polyphase matrix is a real number. We have generalized the approach to M-band multiple density setting. We have analytically obtained number of example filters for with small number of filter coefficients. However, optimization of such filters was left undone since it can be done similarly as in Selesnick's [18].

Chapter 6

Towards Hexagonal Filter Banks

We will consider the special case of hexagonal sampling where

$$\mathbf{V} = \begin{bmatrix} \sqrt{3}/2 & 0 \\ 1/2 & 1 \end{bmatrix}$$

due to its special properties of directional decomposition of images. The rectangular lattice consists of two main directions represented by the two vectors $[1 \quad 0]^T$ and $[0 \quad 1]^T$. A single point shifted by an integer linear combinations of these direction vectors result in the complete lattice. In the hexagonal lattice we have three main directions, $\mathbf{d}_1^T = [1 \quad 0], \mathbf{d}_2^T = [-1/2 \quad \sqrt{3}/2]$, and $\mathbf{d}_3^T = [-1/2 \quad -\sqrt{3}/2]$. With the use of these directions we can alternatively define the hexagonal lattice as

$$LAT(V) = \{n_1\mathbf{d}_1 + n_2\mathbf{d}_2 + n_3\mathbf{d}_3 \mid (n_1, n_2, n_3) \in \mathcal{N}^3\}. \tag{6.1}$$

Thus we have a three directional representation of $\mathbf{x} \in \mathcal{R}^2$ given by

$$\mathbf{x} = <\mathbf{x} \mid \mathbf{d}_1 > \mathbf{d}_1 + <\mathbf{x} \mid \mathbf{d}_2 > \mathbf{d}_2 + <\mathbf{x} \mid \mathbf{d}_3 > \mathbf{d}_3. \tag{6.2}$$

The directional representation is not unique. We can make the directional representation unique by imposing the directional components to be positive. We can also represent the Fourier and Z transforms in the three directional representation. Let $(\omega_1, \omega_2, \omega_3)$ be the Fourier domain in the directional representation with $\omega_1 + \omega_2 + \omega_3 = 0$ and $\mathbf{n} = [n_1, n_2, n_3]^T$ be the directional representation of a point $\mathbf{n} \in LAT(V)$ then the the three directional Fourier transform is defined as

$$X(\omega_1, \omega_2, \omega_3) = \sum_{\mathbf{n} \in LAT(\mathbf{V})} x(\mathbf{n}) e^{-j(n_1\omega_1 + n_2\omega_2 + n_3\omega_3)} \tag{6.3}$$

and its Z-transform is defined as

$$X(z_1, z_2, z_3) = \sum_{\mathbf{n} \in LAT(\mathbf{V})} x(\mathbf{n}) z_1^{-n_1} z_2^{-n_2} z_3^{-n_3}. \tag{6.4}$$

Unless otherwise specified, when we refer to three directional representation we refer to the unique representation. In this chapter, we omit the superscript notation for the Fourier transform for simplicity.

6.1 Hexagonal Filter Banks Generated by a Pair of Scaling Filters

In the bi-orthogonal setting these filters are known as conjugate mirror filters. We have the following theorem.

Theorem 12 *Consider the hexagonal lattice with*

$$\mathbf{V} = \begin{bmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix}$$

and $\tilde{\mathbf{k}}_1 = (-\pi, -\pi/\sqrt{3}), \tilde{\mathbf{k}}_2 = (\pi, -\pi/\sqrt{3})$, *and* $\tilde{\mathbf{k}}_3 = (0, 2\pi/\sqrt{3})$. *Let*

$$\frac{\tilde{H}_0(\omega + \tilde{\mathbf{k}}_1)}{\tilde{H}_0(\omega + \tilde{\mathbf{k}}_2)} = \frac{H_0(\omega + \tilde{\mathbf{k}}_1)}{H_0(\omega + \tilde{\mathbf{k}}_2)} \tag{6.5}$$

$$\frac{\tilde{H}_0(\omega + \tilde{\mathbf{k}}_2)}{\tilde{H}_0(\omega + \tilde{\mathbf{k}}_3)} = \frac{H_0(\omega + \tilde{\mathbf{k}}_2)}{H_0(\omega + \tilde{\mathbf{k}}_3)} \tag{6.6}$$

Then we can find delays $\mathbf{d}_1, ..., \mathbf{d}_3$ *such that*

$$H_i(\omega) = e^{-j\omega^T.\mathbf{d}_i}\tilde{H}_0(\omega + \tilde{\mathbf{k}}_i) \quad (i = 1, ..., 3), \tag{6.7}$$

$$\tilde{H}_i(\omega) = e^{-j\omega^T.\mathbf{d}_i}H_0(\omega + \tilde{\mathbf{k}}_i) \quad (i = 1, ..., 3). \tag{6.8}$$

Proof: The delays are $\mathbf{d}_1 = (1, 0), \mathbf{d}_2 = (-1/2, \sqrt{3}/2)$, and $\mathbf{d}_3 = (-1/2, -\sqrt{3}/2)$.

Conversely we have the following theorem.

Theorem 13 *Let* $\mathbf{d}_1 = (1, 0), \mathbf{d}_2 = (-1/2, \sqrt{3}/2)$, *and* $\mathbf{d}_3 = (-1/2, -\sqrt{3}/2)$ *and*

$$H_i(\omega) = e^{-j\omega^T.\mathbf{d}_i}\tilde{H}_0(\omega + \tilde{\mathbf{k}}_i) \quad (i = 1, ..., 3), \tag{6.9}$$

$$\tilde{H}_i(\omega) = e^{-j\omega^T.\mathbf{d}_i}H_0(\omega + \tilde{\mathbf{k}}_i) \quad (i = 1, ..., 3). \tag{6.10}$$

Then

$$\frac{\tilde{H}_0(\omega + \tilde{\mathbf{k}}_1)}{\tilde{H}_0(\omega + \tilde{\mathbf{k}}_2)} = \frac{H_0(\omega + \tilde{\mathbf{k}}_1)}{H_0(\omega + \tilde{\mathbf{k}}_2)}, \tag{6.11}$$

$$\frac{\tilde{H}_0(\omega + \tilde{\mathbf{k}}_2)}{\tilde{H}_0(\omega + \tilde{\mathbf{k}}_3)} = \frac{H_0(\omega + \tilde{\mathbf{k}}_2)}{H_0(\omega + \tilde{\mathbf{k}}_3)}. \tag{6.12}$$

These are the two dimensional multiple channel extensions of *conjugate mirror filters* of two channel one dimensional case. Under these substitutions for high pass filters, equation 2.34 becomes the Nyquist property of the product filter $\tilde{H}_0(\omega)H_0(\omega)$. It is also equivalent to the bi-orthogonal condition between $\tilde{H}_0(\omega)$ and $H_0(\omega)$.

When H_0 and $\tilde{H}_0$ are hexagonally symmetric, one of 6.5 and 6.6 can be dropped. When H_0 and $\tilde{H}_0$ are complex conjugate of each other (i.e. orthogonal) and symmetric, both 6.5 and 6.6 are satisfied and we have the following Corollary which was used to design filters in [33].

Corollary 2 *Let H_0 be cetrosymmetric orthogonal low pass filter (i.e. $H_0 = H_0^*$) in a four channel hexagonal perfect reconstruction filter bank. Then high pass filters can be obtained by*

$$H_i(\omega) = e^{-j\omega^T \cdot \mathbf{d}_i} H_0^*(\omega + \tilde{\mathbf{k}}_i) \qquad (i = 1, ..., 3). \tag{6.13}$$

6.2 Symmetric Hexagonal Filter Banks Generated by High Pass Filters

We will look at Hexagonally symmetric filter banks generated by a pair of high pass filters. One such pair generates the scaling filters while the other generates the wavelet filters. Such a filter bank family was developed by [2]. In this family, the high pass filters are generated by the $2\pi/3$ rotation of a single filter. First, we define the following,

Definition 8 *We say a filter $H(\omega_1, \omega_2, \omega_3)$ is hexagonally symmetric if*

$$H(\omega_1, \omega_2, \omega_3) = H(\omega_2, \omega_3, \omega_1) = H(\omega_3, \omega_1, \omega_2). \tag{6.14}$$

Definition 9 *We say a triplet of filters (H_1, H_2, H_3) is globally symmetric by $2\pi/3$ if*

$$H_2(\omega_1, \omega_2, \omega_3) = H_1(\omega_2, \omega_3, \omega_1), \tag{6.15}$$

$$H_3(\omega_1, \omega_2, \omega_3) = H_1(\omega_3, \omega_1, \omega_2). \tag{6.16}$$

We can gain much more deeper results of the above using polyphase analysis. We first start with some definitions. We say the vectors $\mathbf{d}_1, \mathbf{d}_2$, and $\mathbf{d}_3$ are globally rotational symmetric by $2\pi/3$ if $R_\theta(\mathbf{d}_1) = \mathbf{d}_2, R_\theta(\mathbf{d}_2) = \mathbf{d}_3$, and $R_\theta(\mathbf{d}_3) = \mathbf{d}_1$ for $\theta = 2\pi/3$. We can see that $\mathbf{d}_1, \mathbf{d}_2$ and $\mathbf{d}_3$ are globally rotational symmetric polyphase shift vectors. Then the polyphase decomposition of the hexagonal filter $H(\omega)$ is defined to be

$$H(\omega) = H_0(\mathbf{K}\omega) + \sum_{i=1}^{3} e^{-j<\mathbf{d}_i|\omega>} H_i(\mathbf{K}\omega). \tag{6.17}$$

We define the polyphase vector of $H(\omega)$ as $[H_0(\omega), ..., H_3(\omega)]^T$. We say the polyphase vector of a filter is hexagonally symmetric if the filter is hexagonally symmetric. We have the following lemma.

Lemma 15 *Let $H^p(\omega) = [H_0(\omega), ..., H_3(\omega)]^T$ be the polyphase vector of the filter $H(\omega)$. Then $H(\omega)$ is hexagonally symmetric if and only if*

$$H^p(\omega) = PH^p(R_\theta^T \omega) \tag{6.18}$$

where

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Proof: By hexagonal symmetry we have

$$[1 e^{-j<\mathbf{d}_1|\omega>} e^{-j<\mathbf{d}_2|\omega>} e^{-j<\mathbf{d}_3|\omega>}] H^p(\omega) = [1 e^{-j<\mathbf{d}_2|\omega>} e^{-j<\mathbf{d}_3|\omega>} e^{-j<\mathbf{d}_1|\omega>}] H^p(R_\theta^T \omega).$$

Now the result follows by substituting

$$[1 e^{-j<\mathbf{d}_2|\omega>} e^{-j<\mathbf{d}_3|\omega>} e^{-j<\mathbf{d}_1|\omega>}] = P[1 e^{-j<\mathbf{d}_1|\omega>} e^{-j<\mathbf{d}_2|\omega>} e^{-j<\mathbf{d}_3|\omega>}]$$

$\bigtriangledown$

We also have the following observation regarding the system 2.35.

Theorem 14 *Let $\tilde{H}_0$ and H_0 are hexagonally symmetric and $(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3)$ and (H_1, H_2, H_3) are globally symmetric by $2\pi/3$. Then the system of equations 2.33 reduces to a single equation.*

Proof: We rewrite equations 2.33 in three direction notation. Firstly note that $\tilde{\mathbf{k}}_1 = (-\pi, 0, \pi), \tilde{\mathbf{k}}_2 = (\pi, -\pi, 0)$, and $\tilde{\mathbf{k}}_3 = (0, \pi, -\pi)$. Then equations 2.33 equivalent to

$$\begin{aligned} &H_0(\omega_1, \omega_2, \omega_3)\tilde{H}_0(\omega_1 + \pi, \omega_2, \omega_3 + \pi)+ \\ &H_1(\omega_1, \omega_2, \omega_3)\tilde{H}_1(\omega_1 + \pi, \omega_2, \omega_3 + \pi)+ \\ &H_1(\omega_2, \omega_3, \omega_1)\tilde{H}_1(\omega_2, \omega_3 + \pi, \omega_1 + \pi)+ \\ &H_1(\omega_3, \omega_1, \omega_2)\tilde{H}_1(\omega_3 + \pi, \omega_1 + \pi, \omega_2) = 0, \end{aligned} \tag{6.19}$$

$$\begin{aligned} &H_0(\omega_1, \omega_2, \omega_3)\tilde{H}_0(\omega_1 + \pi, \omega_2 + \pi, \omega_3)+ \\ &H_1(\omega_1, \omega_2, \omega_3)\tilde{H}_1(\omega_1 + \pi, \omega_2 + \pi, \omega_3)+ \\ &H_1(\omega_2, \omega_3, \omega_1)\tilde{H}_1(\omega_2 + \pi, \omega_3, \omega_1 + \pi)+ \\ &H_1(\omega_3, \omega_1, \omega_2)\tilde{H}_1(\omega_3, \omega_1 + \pi, \omega_2 + \pi) = 0, \end{aligned} \tag{6.20}$$

$$\begin{aligned} &H_0(\omega_1, \omega_2, \omega_3)\tilde{H}_0(\omega_1, \omega_2 + \pi, \omega_3 + \pi)+ \\ &H_1(\omega_1, \omega_2, \omega_3)\tilde{H}_1(\omega_1, \omega_2 + \pi, \omega_3 + \pi)+ \\ &H_1(\omega_2, \omega_3, \omega_1)\tilde{H}_1(\omega_2 + \pi, \omega_3 + \pi, \omega_1)+ \\ &H_1(\omega_3, \omega_1, \omega_2)\tilde{H}_1(\omega_3 + \pi, \omega_1, \omega_2 + \pi) = 0. \end{aligned} \tag{6.21}$$

Firstly lets see $\tilde{H}_0(\omega_1 + \pi, \omega_2 + \pi, \omega_3)$ is a $2\pi/3$ rotated version of $\tilde{H}_0(\omega_1 + \pi, \omega_2, \omega_3 + \pi)$. Since $\tilde{H}_0$ is hexagonally symmetric

$$\tilde{H}_0(\omega_1, \omega_2, \omega_3) = \tilde{H}_0(\omega_3, \omega_1, \omega_2).$$

Now $\tilde{H}_0(\omega + \tilde{\mathbf{k}}_1) = \tilde{H}_0(\omega_3 + \pi, \omega_1 + \pi, \omega_2)$. It can be seen that $\tilde{H}_0(\omega_1 + \pi, \omega_2 + \pi, \omega_3)$ is a rotated version of $\tilde{H}_0(\omega_3 + \pi, \omega_1 + \pi, \omega_2)$. Now note that equation 6.20 is a $2\pi/3$ rotated version of equation 6.19. Similarly equation 6.21 is a $2\pi/3$ rotated version of equation 6.20. Hence equations 6.20 and 6.21 are redundant. ∇

6.3 Regularity Issues

In one dimensional two band setting, the regularity linearly increases with the power of the factor $(1 + e^{-j\omega})$. We would like similar factor for the hexagonal case as well. Courant interpolating function $C(\omega)$ has been used for this purpose [2]. Courant interpolating function (CIF) is generated by $C(\omega) = (1 + e^{-j\omega_1})(1 + e^{-j\omega_2})(1 + e^{-j\omega_3})$.

Definition 10 *We say that a scaling filter is CIF-Regular of order r if the filter has a factor $C^r(\omega)$. We also say that a wavelet filter H_i is CIF-Regular of order r if the filter has a factor $C^r(\omega + \tilde{\mathbf{k}}_i)$.*

6.3.1 Existence for Filter Banks Generated by A Pair of Scaling Filters

As with one-dimensional two band setting, (and its separable extension to two dimensional setting), we would like to find a filter bank generated by a pair of CIF-Regular scaling filters. In particular consider that the analysis scaling filter is given by $C^k(\omega)$, and that the synthesis scaling filter has a factor $C^l(\omega)$. With the further requirement that the scaling filters are hexagonally symmetric, it is found that a filter bank cannot be generated by the scaling filters as given by the following theorem.

Theorem 15 *Let* $\mathbf{d}_1 = (1,0), \mathbf{d}_2 = (-1/2, \sqrt{3}/2)$, *and* $\mathbf{d}_3 = (-1/2, -\sqrt{3}/2)$ *and*

$$H_i(\omega) = e^{-j\omega^T.\mathbf{d}_i}\tilde{H}_0(\omega + \tilde{\mathbf{k}}_i) \quad (i = 1, ..., 3), \tag{6.22}$$

$$\tilde{H}_i(\omega) = e^{-j\omega^T.\mathbf{d}_i}H_0(\omega + \tilde{\mathbf{k}}_i) \quad (i = 1, ..., 3). \tag{6.23}$$

Then there do not exist hexagonally symmetric scaling filter pair $\tilde{H}_0(\omega) = C^k(\omega)$ *and* $H_0(\omega)$ *such that* $H_0(\omega)$ *has a factor* $C^l(\omega)$.

Proof: Case $l \geq k$: Let $H_0(\omega) = C^k(\omega)H_0'(\omega)$. Then we get

$$H_0'(\omega + \tilde{\mathbf{k}}_1) = H_0'(\omega + \tilde{\mathbf{k}}_2) = H_0'(\omega + \tilde{\mathbf{k}}_3). \tag{6.24}$$

Thus $H_0(\omega)$ has a factor $H_0'(\mathbf{K}\omega)$. Looking at the bi-orthogonal equation 2.45, this is not possible.

Case $l < k$: We prove this by contradiction. w.l.o.g assume

$$\frac{C(\omega + \tilde{\mathbf{k}}_1)}{C(\omega + \tilde{\mathbf{k}}_2)} = \frac{H_0'(\omega + \tilde{\mathbf{k}}_1)}{H_0'(\omega + \tilde{\mathbf{k}}_2)} \tag{6.25}$$

where $H_0'(\omega)$ do not have a factor $C(\omega)$. Then we get

$$(1 - e^{j\omega_1})(1 + e^{j\omega_3})H_0'(\omega + \tilde{\mathbf{k}}_2) = (1 + e^{j\omega_1})(1 - e^{j\omega_3})H_0'(\omega + \tilde{\mathbf{k}}_1) \tag{6.26}$$

This implies $H_0'(\omega)$ has a factor $C(\omega)$, which is a contradiction. ∇

6.3.2 CIF-Regularity of Filter Banks Generated by a Pair of High Pass Filters

Alternatively we may design analysis high pass filters, $\tilde{H}_i$ $(i \in \{1,2,3\}$, first. By designing CIF-Regular analysis high pass filters, we automatically get CIF-Regular synthesis scaling filter.

Theorem 16 *Let $\tilde{H}_1, \tilde{H}_2$, and $\tilde{H}_3$ be globally symmetric by $2\pi/3$ and $\tilde{H}_1(\omega + \tilde{\mathbf{k}}_1)$ has a factor l^{th} order CIF, $C^l(\omega)$. Then the low pass filter produced by equation 6.38 has a factor $C^l(\omega)$.*

Proof: The result can be seen trivially by expanding the determinant in the equation 6.38. ∇

6.4 Design of the Dual Scaling Filter

In this section we will discuss parametric solutions for the dual filters G_i of a given spline analysis filter H_i. We will only discuss the solutions for the scaling filter. The solutions for the high pass filters may be obtained by shifting the scaling filters by aliasing frequencies $\tilde{\mathbf{k}}_i$ in the Fourier domain. Fortunately, we can follow similar techniques discussed in [31, 1]. We will drop the subscripts for simplicity. Let

$$H(\omega_1, \omega_2, \omega_3) = (1 + e^{-j\omega_1})^r (1 + e^{-j\omega_2})^r (1 + e^{-j\omega_3})^r, \tag{6.27}$$

$$G(\omega_1, \omega_2, \omega_3) = (1 + e^{-j\omega_1})^r (1 + e^{-j\omega_2})^r (1 + e^{-j\omega_3})^r G'(\omega_1, \omega_2, \omega_3). \tag{6.28}$$

We need to find $G'(\omega_1, \omega_2, \omega_3)$ such that equation 2.45 is satisfied. Let the minimum length solution is $G'_m(\omega_1, \omega_2, \omega_3)$ such that

$$(1 + e^{-j\omega_1})^{2r} (1 + e^{-j\omega_2})^{2r} (1 + e^{-j\omega_3})^{2r} G'_m(\omega_1, \omega_2, \omega_3) \downarrow_K = 1. \tag{6.29}$$

Note that

$$C^{2r}(\omega_1,\omega_2,\omega_3) = \frac{(1-e^{-2j\omega_1})^{2r}(1-e^{-2j\omega_2})^{2r}(1-e^{-2j\omega_3})^{2r}}{(1-e^{-j\omega_1})^{2r}(1-e^{-j\omega_2})^{2r}(1-e^{-j\omega_3})^{2r}} \tag{6.30}$$

such that

$$\begin{aligned}(1-e^{-j\omega_1})^{2r}(1-e^{-j\omega_2})^{2r}(1-e^{-j\omega_3})^{2r}C^{2r}(\omega_1,\omega_2,\omega_3) = \\ (1-e^{-2j\omega_1})^{2r}(1-e^{-2j\omega_2})^{2r}(1-e^{-2j\omega_3})^{2r}.\end{aligned} \tag{6.31}$$

Thus $(1-e^{-j\omega_1})^{2r}(1-e^{-j\omega_2})^{2r}(1-e^{-j\omega_3})^{2r}C^{2r}(\omega_1,\omega_2,\omega_3)$ has terms only in the zeroth coset. So it must be that

$$C^{2r}(\omega_1,\omega_2,\omega_3)(1-e^{-j\omega_1})^{2r}(1-e^{-j\omega_2})^{2r}(1-e^{-j\omega_3})^{2r}G_a'(\omega_1,\omega_2,\omega_3)\downarrow_K = 0 \tag{6.32}$$

where $G_a'(\omega_1,\omega_2,\omega_3)$ is centro-symmetric and hexagonally symmetric filter which do not have terms in the zeroth coset. Thus

$$\begin{aligned}G(\omega_1,\omega_2,\omega_3) = C^{r}(\omega_1,\omega_2,\omega_3)(G_m'(\omega_1,\omega_2,\omega_3) + \\ C^{2r}(\omega_1+\pi,\omega_2+\pi,\omega_3+\pi)G_a'(\omega_1,\omega_2,\omega_3)).\end{aligned} \tag{6.33}$$

Note that

$$\begin{aligned}G(\omega_1,\omega_2,\omega_3) = C^{r}(\omega_1,\omega_2,\omega_3)G_m'(\omega_1,\omega_2,\omega_3) + \\ C^{r}(\omega_1+\pi,\omega_2+\pi,\omega_3+\pi)G_a'(\omega_1,\omega_2,\omega_3)\end{aligned} \tag{6.34}$$

is also a dual filter.

6.4.1 CIF-Regular Parametric Dual Filters: The General Case

We will state a theorem which can be used to find parametric bi-orthogonal filter for any filter.

Theorem 17 *Let* $\tilde{\mathbf{k}}_1 = (-\pi, 0, \pi), \tilde{\mathbf{k}}_2 = (\pi, -\pi, 0)$, *and* $\tilde{\mathbf{k}}_3 = (0, \pi, -\pi)$ *be the aliasing frequencies. Let* G *is the minimal length bi-orthogonal filter to* H. *Then any other bi-orthogonal filter* G^{new} *can be written as*

$$G^{new}(\omega) = G(\omega) + S^{1,3}(\omega)H(\omega + \tilde{\mathbf{k}}_1) + S^{1,2}(\omega)H(\omega + \tilde{\mathbf{k}}_2) + S^{2,3}(\omega)H(\omega + \tilde{\mathbf{k}}_2) \tag{6.35}$$

where $S^{i,j}(\omega)$ *is a filter with coefficients in only* i^{th} *and* j^{th} *cosets.*

Proof: We will need to show that

$$H(\omega)(S^{1,3}(\omega)H(\omega + \tilde{\mathbf{k}}_1) + S^{1,2}(\omega)H(\omega + \tilde{\mathbf{k}}_2) + S^{2,3}(\omega)H(\omega + \tilde{\mathbf{k}}_2)) \downarrow_K = 0. \tag{6.36}$$

We will show only $H(\omega)S^{1,3}(\omega)H(\omega + \tilde{\mathbf{k}}_1) \downarrow_K = 0$. Similar arguments hold for the other terms. Firstly, assume $S^{1,3}(\omega) = e^{-j\omega_1}S(2\omega)$ has terms only in the 1^{st} coset. In the polyphase domain we have

$$\begin{aligned} H(\omega)S(\omega)H(\omega + \tilde{\mathbf{k}}_1) &= e^{-j\omega_1}S(2\omega)(H_0(2\omega) + e^{-j\omega_1}H_1(2\omega) + e^{-j\omega_2}H_2(2\omega) \\ &+e^{-j\omega_3}H_3(2\omega))(H_0(2\omega) - e^{-j\omega_1}H_1(2\omega) + e^{-j\omega_2}H_2(2\omega) - e^{-j\omega_3}H_3(2\omega)). \end{aligned} \tag{6.37}$$

The 0^{th} coset of the above is

$$\begin{aligned} -e^{-2j\omega_1}S(2\omega)H_0(2\omega)H_1(2\omega) + e^{-2j\omega_1}S(2\omega)H_1(2\omega)H_0(2\omega) - \\ S(2\omega)H_2(2\omega)H_3(2\omega) + S(2\omega)H_3(2\omega)H_2(2\omega) = 0 \end{aligned}$$

which completes the proof. ∇

Similar results were obtained for 2-band one-dimensional setting in [49, 30] and 2-band multidimensional setting in [49].

6.5 Example Designs

The following lemma was proved in [2] which can be used to find the synthesis scaling filter from the analysis wavelet filters.

Lemma 16 *If $\tilde{H}_0, ..., \tilde{H}_3$ and $H_0, ..., H_3$ has perfect reconstruction and $\tilde{H}_0$ and H_0 are hexagonally symmetric, then it satisfies*

$$H_0 = \frac{1}{C} \begin{vmatrix} \tilde{H}_1(\omega + \tilde{\mathbf{k}}_1) & \tilde{H}_2(\omega + \tilde{\mathbf{k}}_1) & \tilde{H}_3(\omega + \tilde{\mathbf{k}}_1) \\ \tilde{H}_1(\omega + \tilde{\mathbf{k}}_2) & \tilde{H}_2(\omega + \tilde{\mathbf{k}}_2) & \tilde{H}_3(\omega + \tilde{\mathbf{k}}_2) \\ \tilde{H}_1(\omega + \tilde{\mathbf{k}}_3) & \tilde{H}_2(\omega + \tilde{\mathbf{k}}_3) & \tilde{H}_3(\omega + \tilde{\mathbf{k}}_3) \end{vmatrix} \tag{6.38}$$

where C is a nonzero constant and equal to the determinant of the system 2.35:

$$\begin{bmatrix} \tilde{H}_0(\omega) & \tilde{H}_1(\omega) & \tilde{H}_2(\omega) & \tilde{H}_3(\omega) \\ \tilde{H}_0(\omega + \tilde{\mathbf{k}}_1) & \tilde{H}_1(\omega + \tilde{\mathbf{k}}_1) & \tilde{H}_2(\omega + \tilde{\mathbf{k}}_1) & \tilde{H}_3(\omega + \tilde{\mathbf{k}}_1) \\ \tilde{H}_0(\omega + \tilde{\mathbf{k}}_2) & \tilde{H}_1(\omega + \tilde{\mathbf{k}}_2) & \tilde{H}_2(\omega + \tilde{\mathbf{k}}_2) & \tilde{H}_3(\omega + \tilde{\mathbf{k}}_2) \\ \tilde{H}_0(\omega + \tilde{\mathbf{k}}_3) & \tilde{H}_1(\omega + \tilde{\mathbf{k}}_3) & \tilde{H}_2(\omega + \tilde{\mathbf{k}}_3) & \tilde{H}_3(\omega + \tilde{\mathbf{k}}_3) \end{bmatrix}. \tag{6.39}$$

We will look at number of filter banks mainly obtained using the Lemma 16 for the construction of synthesis scaling filters. The analysis wavelet filters were obtained using ad hoc methods such that they leads to useful synthesis scaling filters via Lemma 16.

Example 6 *Consider the following lifting [12, 51, 52, 53] like analysis polyphase matrix for the hexagonal filter bank.*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -a(z_1, z_2, z_3) & 1 & 0 & 0 \\ -a(z_2, z_3, z_1) & 0 & 1 & 0 \\ -a(z_3, z_1, z_2) & 0 & 0 & 1 \end{bmatrix}. \tag{6.40}$$

Its synthesis polyphase matrix is given by

$$\begin{bmatrix} 1 & a(z_1, z_2, z_3) & a(z_2, z_3, z_1) & a(z_3, z_1, z_2) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6.41}$$

The system does not have a useful analysis low pass filter. Thus we can design a new analysis low pass filter while retaining the analysis high pass filters and the synthesis low pass filter. Note the three analysis high pass filters are globally rotational symmetric by $2\pi/3$ *while synthesis low pass filter is rotationally symmetric by* $2\pi/3$. *When* $a(z_1, z_2, z_3) = 1$, *each analysis high pass filter is the one-dimensional Haar filter along the corresponding directions. This specific set of analysis Haar filters are given by* $-1+z_1^{-1}, -1+z_1^{-1}, -1+z_1^{-1}$ *and the synthesis low pass filter is given by* $1+z_1+z_2+z_3$.

Example 7 *Select* $a(z_1, z_2, z_3) = \frac{1+z_1^{-1}}{2}$ *in the equation 6.40. Now the analysis high pass filters are second order one dimensional high pass splines along each directions,* $-\frac{1}{2}+z_1^{-1}-\frac{1}{2}z_1^{-2}$, $-\frac{1}{2}+z_2^{-1}-\frac{1}{2}z_2^{-2}$, $-\frac{1}{2}+z_3^{-1}-\frac{1}{2}z_3^{-2}$ *and the synthesis low pass filter is the Caurant Interpolating Function (CIF)* $\frac{1}{2}(1+z_1)(1+z_2)(1+z_3)$.

In [2], they have proposed the following algorithm to design the high pass analysis filters, and then synthesis low pass filters, analysis low pass filters, and synthesis high pass filters respectively.

Algorithm 1

- *Design* $\tilde{H}_1$ *and select* $\tilde{H}_2$ *and* $\tilde{H}_3$ *such that* $(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3)$ *is globally symmetric by* $2\pi/3$.
- *Select* H_0 *using 6.38.*
- *Design* $\tilde{H}_0$ *using 2.45.*
- *Design* H_1, H_2 *and* H_3 *such that* (H_1, H_2, H_3) *is globally symmetric by* $2\pi/3$ *subject to the system 2.35.*

The algorithm is essentially the adaptation of Theorem 3 for the hexagonal setting and the computations are done in the transform domain rather than polyphase domain. It can also be adapted for computations done in the polyphase domain as well. When $(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3)$ is globally symmetric by $2\pi/3$ and the resulting H_0 is hexagonally symmetric we can design $\tilde{H}_0$ to be hexagonally symmetric as well. Then we can design globally symmetric (H_1, H_2, H_3) subject to the two equations given by Theorem 14.

6.6 Factorization Techniques for Hexagonal Filter banks

Since Daubechies [16] technique of wavelet design in 1-D do not have a straightforward generalization to more than 1-D we are lead to explore sub classes of filter banks which have simple and manageable form. Along this line, factorable filter banks and wavelets were originally discussed by [13] and further explored in 2-D by [9, 10].

We restrict ourselves to polyphase matrices which can be factored as a product of factors in one variable. In factorial representation in [9, 10], each factor corresponds to one of the main directions of the rectangular lattice. In the hexagonal lattice we have three main directions corresponding to z_1, z_2, and z_3. Since each of these directions has the same prominence we define the factorial polyphase matrices on hexagonal lattice in three variables . Thus factorial polyphase matrices of order (α, β, γ) in z_1, z_2, and z_3 is defined as

$$H^p(z_1, z_2, z_3) = \prod_{i=1}^{\alpha+\beta+\gamma} \left(I + (z_{r_i}^{-1} - 1)P_i\right) H_0 \qquad (6.42)$$

where the product contains α factors of z_1, β factors of z_2 and γ factors of z_3. H_0 is the characteristic matrix of $H^p(z_1, z_2, z_3)$ [39].

Imposing Hexagonal Symmetry

In order to achieve successful factorization, we need to find the form of invertible polyphase matrix factors, such that upon multiplication with the original polyphase vector of a hexagonally symmetric filter, results in a new hexagonally symmetric filter. The following lemmas provide the form of such matrices.

Lemma 17 *Let $H^p(\omega)$ be the polyphase vector of a hexagonally symmetric filter $H(\omega)$. Then $L(\omega)H^p(\omega)$ is a polyphase vector of some hexagonally symmetric filter if and only if*

$$L(\omega) = PL(R_\theta^T \omega)P^T. \tag{6.43}$$

Proof: We have

$$L(\omega)H^p(\omega) = PL(R_\theta^T \omega)H^p(R_\theta^T \omega)$$

and

$$H^p(\omega) = PH^p(R_\theta^T \omega).$$

Combining the above two equations we get

$$L(\omega)P = PL(R_\theta^T \omega).$$

$\triangledown$

Unlike linear phase conditions, hexagonal symmetry cannot be imposed on a single degree one factor alone. We will impose hexagonal symmetry on product terms of the form $(I + (z_1^{-1} - 1)P_1)(I + (z_2^{-1} - 1)P_2)(I + (z_3^{-1} - 1)P_3)$. We have the following theorem.

Theorem 18 *Let $H_p(z_1, z_2, z_3)$ be hexagonally symmetric polyphase vector and*

$$P_2 = PP_1P^T, \tag{6.44}$$

$$P_3 = PP_2P^T \tag{6.45}$$

then $(I + (z_1^{-1} - 1)P_1)(I + (z_2^{-1} - 1)P_2)(I + (z_3^{-1} - 1)P_3)H_p(z_1, z_2, z_3)$ is also a hexagonally symmetric polyphase vector.

6.7 Conclusion

We have investigated whether we can design conjugate mirror filters on the hexagonal lattice. We have obtained constraints under which such filters exist. We also investigated the hexagonal symmetry of scaling filters and global symmetry by $2\pi/3$ and their consequence to the perfect reconstruction constraints. We also investigated the existence of conjugate mirror filters with CIF-regularity, parametric solution to the CIF-regular scaling filters, and factorization methods for certain classes of polyphase matrices of hexagonal filter banks. However we were unable to make a major breakthrough in the completion of hexagonal filter banks for some given scaling filters.