Chapter 3

Wavelet Theory

An oscillating function of time and/or space is usually referred to as a wave. Fourier analysis is wave analysis where it expand signals or functions in terms of sinusoids. A wavelet is a small wave, which has its energy concentrated in time. Wavelets allow both time-domain analysis and frequency-domain analysis simultaneously due to its energy localization both in time and in frequency. Thus wavelets are very suitable for the analysis of transient, time-varying, or non-stationary phenomena.

A signal $f(t)$ is often represented as a linear decomposition given by

$$f(t) = \sum_l a_l \psi_l(t) \tag{3.1}$$

where the sum may be finite or infinite. When coefficients can be calculated by *inner products*:

$$a_l = \langle f(t), \psi_l(t) \rangle = \int f(t)\psi_l(t)\, \mathrm{d}x. \tag{3.2}$$

we say the collection $\{\psi_l(t)\}_{l\in\mathbb{Z}}$ is a *frame*, and the individual functions $\psi_l(t)$ as *atoms*. When the representation is unique, we refer to the frame as a *basis*. An important question is that, given frame coefficients of a signal, can we recover the signal? Such questions are answered by *frame theory*. It also addresses such issues as completeness, stability and redundancy of linear signal representations.

Whether we can recover a signal from its frame coefficients, is the equivalent of perfect reconstruction in filter bank theory. Such issues have led to a more precise definition of a frame.

Definition 1 *(Frame) The collection $\{\psi_l(t)\}_{l\in\mathbb{Z}}$ is a frame of a Hilbert space $\mathcal{H}$ if there exist two constants $A > 0$ and $B > 0$ such that for any $f \in \mathcal{H}$,*

$$A\|f\|^2 \leq \sum |\langle f, \psi_l\rangle|^2 \leq B\|f\|^2. \tag{3.3}$$

When $A = B$ the frame is said to be tight.

A frame defines a complete and stable signal representation, which may also be redundant. When the atoms $\{\psi_l(t)\}$ are normalized such that $\|\psi_l\| = 1$, this redundancy is measured by the frame bounds A and B. If $A > 1$ then the frame is redundant and A can be interpreted as a minimum redundancy factor. The frame is an *orthonormal basis* if and only if $A = B = 1$. When the collection $\{\psi_l(t)\}_{l\in\mathbb{Z}}$ linearly independent, the frame is said to be a *Riesz basis.*

The reconstruction of f from its frame coefficients is done with a dual frame $\{\tilde{\psi}_l(t)\}_{l\in\mathbb{Z}}$. We can construct the dual frame such that, Mallat [48],

$$\frac{1}{A}\|f\|^2 \leq \sum |\langle f, \psi_l\rangle|^2 \leq \frac{1}{B}\|f\|^2 \tag{3.4}$$

and

$$f = \sum \langle f, \psi_l\rangle \tilde{\psi}_l = \sum \langle f, \tilde{\psi}_l\rangle \psi_l \tag{3.5}$$

where A and B are the frame bounds of $\{\psi_l(t)\}_{l\in\mathbb{Z}}$.

Wavelets are two-parameter atoms, $\psi_{j,k}(t)$, and $\{\psi_{j,k}(t)\}_{(j,k)\in\mathbb{Z}^2}$ is a frame. Thus wavelets come from frame theory, and in particular wavelets constitute Riesz bases. Let $\{\tilde{\psi}_{j,k}(t)\}_{(j,k)\in\mathbb{Z}^2}$ be the dual frame of $\{\psi_{j,k}(t)\}_{(j,k)\in\mathbb{Z}^2}$. The atoms, $\tilde{\psi}_{j,k}(t)$, of the dual frame are usually referred to as *bi-orthogonal wavelets.* Due to explosive development in the wavelet theory the term wavelet is rather abused, but we use the term wavelets when they form Riesz bases, and we use the term *framelets* when they form more general frames (possibly redundant).

One of the index of wavelets run through the scale space while the other run through the time. Thus wavelet representation of a signal is a multi-resolution representation. Most of the earliest wavelets constructed were dyadic wavelet representations. An attractive property in these systems is that each wavelet basis function, $\psi_{j,k}(t)$, is constructed by translation and dilation of a single basis function $\psi(t)$ known as the *mother wavelet*:

$$\psi_{j,k}(t) = \frac{1}{\sqrt{2^j}}\psi\left(\frac{t-2^jk}{2^j}\right). \tag{3.6}$$

We also expect the dual wavelet $\tilde{\psi}_{j,k}(t)$ to satisfy

$$\tilde{\psi}_{j,k}(t) = \frac{1}{\sqrt{2^j}}\tilde{\psi}\left(\frac{t-2^jk}{2^j}\right). \tag{3.7}$$

But in general such duals do not exist, Chui [23, Pages 13-14]. When we refer to wavelets we refer to wavelet bases where the wavelet base and its dual base satisfy equations 3.6 and 3.7 respectively. Under certain conditions, Cohen, Daubechies and Feauveau [1], $f(t) \in \mathrm{L}^2(\mathbb{R})$ can be decomposed in the wavelet basis or its dual basis:

$$f(t) = \sum_{j,k\in\mathbb{Z}} \langle f(t), \tilde{\psi}_{j,k}(t)\rangle \psi_{j,k}(t) = \sum_{j,k\in\mathbb{Z}} \langle f(t), \psi_{j,k}(t)\rangle \tilde{\psi}_{j,k}(t). \tag{3.8}$$

Now the *wavelet transform* of f relative to the dual $\tilde{\psi}$ of ψ at (j,k) is defined to be

$$\langle f(t), \tilde{\psi}_{j,k}(t)\rangle = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2^j}} f(t)\tilde{\psi}\left(\frac{t-2^j k}{2^j}\right) \mathrm{d}t. \tag{3.9}$$

3.1 Multi-resolution Analysis

As shown by Stephane Mallat the atoms which form a wavelet frame can be structured to span a sequence of subspaces which constitute a multi-resolution approximation.

Definition 2 *(Multi-resolutions) A sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $\mathrm{L}^2(\mathbb{R})$ is a multi-resolution approximation if the following 6 properties are satisfied:*

$$\forall (j,k) \in \mathbb{Z}^2, f(t) \in V_j \Leftrightarrow f(t-2^j k) \in V_j, \tag{3.10}$$

$$\forall j \in \mathbb{Z}, V_{j+1} \subset V_j, \tag{3.11}$$

$$\forall j \in \mathbb{Z}, f(t) \in V_j \Leftrightarrow f\left(\frac{t}{2}\right) \in V_{j+1}, \tag{3.12}$$

$$\lim_{j\to+\infty} V_j = \bigcap_{j\to-\infty}^{+\infty} V_j = \{0\}, \tag{3.13}$$

$$\lim_{j\to+\infty} V_j = \text{Closure}\left(\bigcup_{j\to-\infty}^{+\infty} V_j\right) = \mathrm{L}^2(\mathbb{R}). \tag{3.14}$$

There exists ϕ such that $\{\phi(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis of V_0.

The function $\phi(t)$ is known as the *scaling function.* If we define

$$\phi_{j,k}(t) = 2^{-j/2}\phi(2^{-j}t - k) \tag{3.15}$$

then it follows that, for every j, $\{\phi_{j,k}(t)\}_{k\in\mathbb{Z}}$ constitute a Riesz basis for V_j. Given $f(t) \in L^2(\mathbb{R})$, its approximation, $f_j(t)$, in V_j is given by

$$f_j(t) = \sum_k \langle f, \tilde{\phi}_{j,k}\rangle \phi_{j,k} \tag{3.16}$$

where $\{\tilde{\phi}_{j,k}(t)\}_{k\in\mathbb{Z}}$ is the dual frame of $\{\phi_{j,k}(t)\}_{k\in\mathbb{Z}}$. The dual basis $\{\tilde{\phi}_{j,k}(t)\}_{k\in\mathbb{Z}}$ constitute a Riesz basis for the dual space $\tilde{V}_j$. Thus we have a dual multi-resolution analysis and $\tilde{\phi}(t)$ is known as the *dual scaling function* or *bi-orthogonal scaling function* where $\tilde{\phi}(t)$ is a Riesz basis for $\tilde{V}_0$ and

$$\tilde{\phi}_{j,k}(t) = 2^{-j/2}\tilde{\phi}(2^{-j}t - k). \tag{3.17}$$

Now the wavelets $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ and the dual wavelets $\{\tilde{\psi}_{j,k}\}_{k\in\mathbb{Z}}$ forms Riesz bases for the detail spaces W_j and $\tilde{W}_j$ such that

$$V_j \oplus W_j = V_{j-1} \text{ and } \tilde{V}_j \oplus \tilde{W}_j = \tilde{V}_{j-1}. \tag{3.18}$$

The bi-orthogonality implies that W_j is orthogonal to $\tilde{V}_j$ and $\tilde{W}_j$ is orthogonal to V_j. When $\phi(t)$ is orthogonal we have only one multi-resolution hierarchy $\{V_j\}_{j\in\mathbb{Z}}$ and W_j is orthogonal to V_j.

Since $\phi\left(\frac{t}{2}\right) \in V_0$, it can be decomposed in $\{\phi(t-n)\}_{n\in\mathbb{Z}}$ such that

$$\frac{1}{\sqrt{2}}\phi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} h[n]\phi(t-n) \tag{3.19}$$

with

$$h[n] = \frac{1}{\sqrt{2}}\left\langle \phi\left(\frac{t}{2}\right), \tilde{\phi}(t-n)\right\rangle. \tag{3.20}$$

Similarly since $\psi\left(\frac{t}{2}\right) \in V_0$, it can be decomposed in $\{\phi(t-n)\}_{n\in\mathbb{Z}}$ such that

$$\frac{1}{\sqrt{2}}\psi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} g[n]\phi(t-n) \tag{3.21}$$

with

$$g[n] = \frac{1}{\sqrt{2}}\left\langle \psi\left(\frac{t}{2}\right), \tilde{\phi}(t-n)\right\rangle. \tag{3.22}$$

The Equations 3.19 and 3.21 are known as the *refinement equations*. Similar results can be obtained for the bi-orthogonal scaling function and the wavelet function. It is the sequences $h[n]$ and $g[n]$ (and the bi-orthogonal sequences $\tilde{h}[n]$ and $\tilde{g}[n]$) , which connect wavelet theory to the filter bank theory.

3.1.1 Fast Wavelet Transform and Filter Banks

The multi-resolution analysis has been used to develop the fast wavelet transform by Mallat [45, 46, 48]. Let

$$d_j[n] = \langle f, \tilde{\psi}_{j,n}\rangle \quad a_j[n] = \langle f, \tilde{\phi}_{j,n}\rangle. \tag{3.23}$$

Then the transform values at resolution $2^{-(j+1)}$ is given by [48, pages 254,267]

$$a_{j+1}(z) = \tilde{H}(z)a_j(z)\,[\downarrow 2] \quad d_{j+1}(z) = \tilde{G}(z)a_j(z)\,[\downarrow 2] \tag{3.24}$$

where $\tilde{H}(z)$, $\tilde{G}(z)$, $a_j(z)$, $a_{j+1}(z)$ and $d_{j+1}(z)$ are z-transforms of $\tilde{h}[n]$, $\tilde{g}[n]$, $a_j[n]$, $a_{j+1}[n]$, and $d_{j+1}[n]$ respectively. We can recover a_j from a_{j+1}, and d_{j+1} via [48, pages 254,267]

$$a_j(z) = H(z)a_{j+1}(z^2) + G(z)d_{j+1}(z^2) \tag{3.25}$$

where $H(z)$ and $G(z)$ are z-transforms of $h[n]$ and $g[n]$ respectively. Now it is clear that $\tilde{H}(z)$,and $\tilde{G}(z)$ are analysis filters, and $H(z)$ and $G(z)$ are synthesis filters of a perfect reconstruction filter bank.

3.1.2 Filter banks and Wavelet Bases

In practice, we first design h and $\tilde{h}$ and see whether it leads to stable bi-orthogonal wavelet bases. In the two band filter bank setting, once we design h and $\tilde{h}$ there will not be any degrees of freedom left for the design of g and $\tilde{g}$. We can obtain the basis functions ϕ and $\tilde{\phi}$ from the infinite products:

$$\phi(\omega) = \prod_{p=1}^{+\infty} \frac{h(2^{-p}\omega)}{\sqrt{2}} \quad \text{and} \quad \tilde{\phi}(\omega) = \prod_{p=1}^{+\infty} \frac{\tilde{h}(2^{-p}\omega)}{\sqrt{2}}. \tag{3.26}$$

The convergence and stability of the above infinite products is not always guaranteed and such issues are studied in [1, 17].

3.2 Generalizations of the Dyadic System

General wavelet systems were designed by generalizing the refinement equation. One way of generalizing the refinement equation is to generalize the scaling parameter from dyadic to a more general rational number. In particular we can generalize the scaling parameter to any integer $M \geq 2$. Such systems are known as M-band wavelet systems. M-band wavelet systems were mainly motivated from M-band filter banks. The refinement equation for the M-band scaling function ψ_0 has the form

$$\frac{1}{\sqrt{M}}\psi_0\left(\frac{t}{M}\right) = \sum_{n=-\infty}^{+\infty} h_0[n]\psi_0(t-n). \tag{3.27}$$

Now the each subspace V_j is spanned by $\psi_{0,j,k}$ $(k \in \mathbb{Z})$ which is defined as

$$\psi_{0,j,k}(t) = M^{-j/2}\psi_0(M^{-j}t - k). \tag{3.28}$$

The detail space W_j is spanned by $M - 1$ wavelet functions, ψ_i $(i \in \{1..M-1\})$, which has the refinement equation:

$$\frac{1}{\sqrt{M}}\psi_i\left(\frac{t}{M}\right) = \sum_{n=-\infty}^{+\infty} h_i[n]\psi_0(t-n) \quad (i \in \{1..M-1\}). \tag{3.29}$$

Unlike in 2-band setting, M-band systems offer extra degrees of freedom in the design of wavelet filters h_i $(i \in \{1..M-1\})$, Steffen, Heller, Gopinath and Burrus [38]. Like in 2-band systems, $\tilde{h}_i$ $(i \in \{0..M-1\})$ and h_i $(i \in \{0..M-1\})$ form analysis and synthesis filters of a M-band perfect reconstruction filter bank.

Chapter 4

M-band Bi-orthogonal Filter Banks and Wavelet Bases

Central to 2-band wavelets of Daubechies and M-band wavelets of [38] is the low pass *scaling filter* with a specified order of regularity. One approach to constructing M-band wavelets would be to start with a multi-resolution analysis (MRA) as in the 2-band case [16, 17, 1, 30, 45, 46] with a scaling factor of M. In this approach one first constructs the scaling filter and then the wavelet filters and wavelets. This approach was followed in Steffen, Heller, Gopinath and Burrus [38], but was restricted to orthogonal M-band wavelets. In our work, we discuss more general M-band bi-orthogonal wavelet bases. In this case, we first design bi-orthogonal pair of scaling filters. The wavelet filters and wavelets follow from this bi-orthogonal scaling filters.

Definition 3 *(Bi-orthogonal pair of scaling filters) A bi-orthogonal pair of scaling filters* $(\tilde{h}_0, h_0)$ *consists of two sequences* $\tilde{h}_0(n)$ *and* $h_0(n)$ *that satisfies the following linear and quadratic constraints:*

$$\sum_k \tilde{h}_0(k) h_0(Ml - k) = \delta(l), \tag{4.1}$$

$$\sum_k \tilde{h}_0(k) = \sqrt(M), \quad \sum_k h_0(k) = \sqrt(M). \tag{4.2}$$

The quadratic condition does not arise in practice, since one usually design a single scaling filter h_0 and then find a dual scaling filter $\tilde{h}_0$ by solving the linear constraints. It is also the same condition satisfied by the low pass filters in a bi-orthogonal filter bank, i.e.

$$[\downarrow M]\tilde{H}_0(z)H_0(z) = \sum_k \tilde{H}_{0,k}(z)H_{0,k}(z) = 1. \tag{4.3}$$

The linear condition arises from the elegant wavelet analysis which will be discussed later. We will define bi-orthogonal wavelet filters in relation to bi-orthogonal scaling filters and filter bank theory.

Definition 4 *(Bi-orthogonal wavelet filters) Given a pair of scaling filters* $(\tilde{h}_0, h_0)$, *we define bi-orthogonal wavelet filters* $\tilde{h}_i, h_i, i \in \{1, ..., M-1\}$ *such that the bi-orthogonal scaling filter and bi-orthogonal wavelet filters together are filters of a bi-orthogonal filter bank.*

When conditions of the filter bank theory were satisfied as above wavelet filters also satisfy the quadratic bi-orthogonality constraint:

$$[\downarrow M]\tilde{H}_i(z)H_i(z) = \sum_k \tilde{H}_{i,k}(z)H_{i,k}(z) = 1 \quad i \in \{1, ..., M-1\}. \tag{4.4}$$

It is also evident that each filter is orthogonal to all other filters on the opposite side except its dual,i.e.

$$[\downarrow M]\tilde{H}_i(z)H_j(z) = \sum_k \tilde{H}_{i,k}(z)H_{j,k}(z) = 0 \quad i \neq j. \tag{4.5}$$

Combining 4.3, 4.4 and 4.5, we can write

$$\sum_k \tilde{h}_i(k)h_j(Ml-k) = \delta(i-j)\delta(l). \tag{4.6}$$

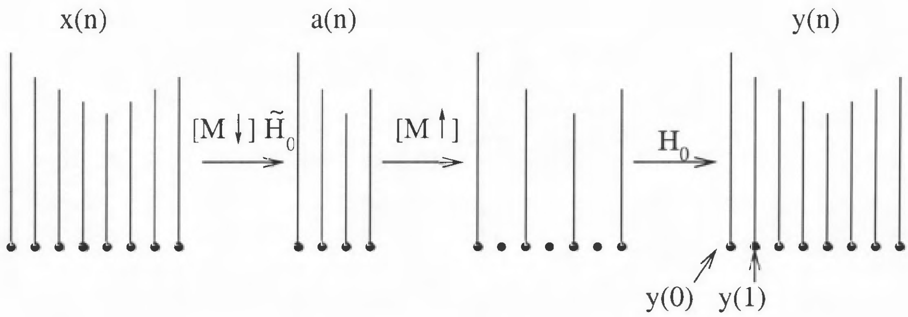

Figure 4.1: The Scaling band for M=2.

4.1 Characterization of Regularity

Classical wavelet transforms are designed to preserve Polynomials (of degree $K-1$) at the scaling subband, i.e. when $x(n)$ is a polynomial sequence of degree $K-1$, $a(n)$ is a M-band subsampled $K-1$ degree polynomial sequence, and $y(n)$ is equal to $x(n)$ subject to some time shift. They are also designed to vanish polynomials at the wavelet subbands.

4.1.1 Discrete Polynomial Preservation

An example of the scaling band is shown in figure 4.1. Suppose the scaling band preserves discrete monomials (say up to degree K-1). When discrete monomials up to degree $K-1$ are preserved, any polynomial sequence up to degree $K-1$ is also preserved. Given that polynomial sequence spaces are shift invariant, it is sufficient to consider output coefficient values at locations $i, 0 \leq i \leq M-1$, to show the equality of the *partial moments* of the synthesis scaling filter H_0. Let

$$a(n) = (Mn)^k.$$

We require that

$$\mu(i,k) = \sum_n (-Mn)^k h_0(Mn+i) = c(i+j)^k \tag{4.7}$$

where c is a signal independent constant, j is a signal independent time shift and $0 \leq i \leq M-1$. Note that $y(i) = \mu(i,k)$ for $0 \leq i \leq M-1$. The partial moments of the filter are defined as

$$m(i,k) = \sum_n (Mn+i)^k h_0(Mn+i) = c(i+j)^k. \quad (4.8)$$

Then

$$m(i,k) = \sum_{p=0}^{k} \binom{k}{p} (i)^{k-p}(-1)^p \mu(p,i) = c(-j)^k. \quad (4.9)$$

But

$$m(0,k) = (-1)^k \mu(0,k) = c(-j)^k. \quad (4.10)$$

Thus

$$m(i,k) = m(0,k). \quad (4.11)$$

When partial moments are equal the $l^{th}(l < K)$ derivatives vanish at the M^{th} roots of unity. To see this,

$$\frac{d^l H_0(e^{\frac{-j2\pi n}{M}})}{d\omega^l} = (-j)^l \left(\sum_{k=0}^{M-1} e^{\frac{-j2\pi k}{M}} \right) m(l) = 0 \quad (4.12)$$

where $m(l)$ is the l^{th} partial moments of H_0 and $0 \leq n \leq M-1$.

The minimal degree Courant polynomial for which the $l^{th}(l < K)$ derivatives vanish at the M^{th} roots of unity is $\left(\frac{1-z^M}{1-z}\right)^K$. A scaling filter with such a factor is termed as *K-Regular* by [38].

Definition 5 *A M-band scaling filter is said to be K-regular or regular of order K if it has a factor* $\left(\frac{1+z^{-1}+\ldots+z^{-(M-1)}}{M}\right)^K$.

In [38], it is shown that, among other equivalent characterizations, polynomials of degree $K-1$ are reproduced by K-regular scaling filter. Thus $K-1$ degree polynomials are preserved by the scaling band if and only if the scaling filters are K-Regular.

The M^{th} roots of unity are aliasing frequencies of the M-band filter bank. Thus K-regularity implies that the derivatives of the scaling filter vanish at the aliasing frequencies. As we will discuss later, zeros at aliasing frequencies are useful in designing multidimensional wavelets.

Alternatively, K-Regularity implies that the magnitude squared frequency response of the scaling filter is flat of order $2K$ at $\omega = 0$. This Flatness property has been used to design orthogonal wavelets [38, 36], and other filters arising in the classical signal processing applications [32].

4.1.2 Discrete Polynomial Annihilation

A filter when convolved with a polynomial sequence up to degree $(K-1)$ results in a zero sequence if and only if it has a zero of order K at the zero frequency. When a scaling filter H_0 is K-regular,i.e. polynomial sequences up to degree $K-1$ are preserved, we would expect the wavelet filters $\tilde{H}_i, i \in \{1, ..., M-1\}$ on the opposite side to vanish on polynomial sequences up to degree $(K-1)$. We will show that it is actually the case as a direct consequence of equation 4.5.

Theorem 1 *(Regularity of wavelet filters from scaling filter) Let H_0 is a K-regular synthesis scaling filter in a M-band filter bank and $\frac{d^l}{d\omega^l} H_0^j(0) \neq 0$ for $l = 0, ..., K-1$. Then moments up to order $(K-1)$ of the analysis wavelet filters $\tilde{H}_i, i \in \{1, ..., M-1\}$ vanish.*

Proof: We will prove the theorem by mathematical induction. We show that the theorem is true for the zeroth moments of the wavelet filters. Equation 4.5 in Fourier domain can be expressed as

$$\sum_{i=0}^{M-1} \tilde{H}_k^f\left(\omega - \frac{2\pi i}{M}\right) H_l^f\left(\omega - \frac{2\pi i}{M}\right) = 0 \quad k \neq l. \tag{4.13}$$

In particular, wavelet filters are orthogonal to the dual scaling filter.

$$\sum_{i=0}^{M-1} \tilde{H}_l^f\left(\omega - \frac{2\pi i}{M}\right) H_0^f\left(\omega - \frac{2\pi i}{M}\right) = 0 \quad l \neq 0. \tag{4.14}$$

Setting ω to zero we get $\tilde{H}_i^f(0)H_0^f(0) = 0$. Since $H_0^f(0) \neq 0$ it must be that $\tilde{H}_i^f(0) = 0$. Now assume that $l-1$ moments of wavelet filters vanish. Using Leibnitz's rule for the derivative of a product (e.g., see p. 147 of [27]), we obtain

$$\sum_{k=0}^{M-1}\sum_{n=0}^{l} \begin{pmatrix} l \\ n \end{pmatrix} \frac{d^n}{d\omega^n}\tilde{H}_i^f\left(\omega - \frac{2\pi k}{M}\right) \frac{d^{l-n}}{d\omega^{l-n}} H_0^f\left(\omega - \frac{2\pi k}{M}\right) = 0.$$

$$l = 0, ..., K-1 \tag{4.15}$$

Setting ω to zero we get

$$\sum_{n=0}^{l} \begin{pmatrix} l \\ n \end{pmatrix} \frac{d^n}{d\omega^n}\tilde{H}_i^f(0)\frac{d^{l-n}}{d\omega^{l-n}} H_0^f(0) = 0.$$

By our assumption that $\frac{d^n}{d\omega^n}\tilde{H}_i^f(0) = 0$ for $n = 0, ..., l-1$ and since $\frac{d^l}{d\omega^l}H_0^f(0) \neq 0$, it must be that $\frac{d^l}{d\omega^l}\tilde{H}_i^f(0) = 0$. ∇

The converse of the above theorem is also true, i.e. vanishing moments of order 0 up to $K-1$ of analysis wavelet filters implies K-regularity of the synthesis scaling filter.

Theorem 2 *(Regularity of scaling filter from wavelet filter) Let* $\tilde{H}_i$ $(i \in \{1, ..., M-1\})$ *be analysis wavelet filters for which moments of order 0 up to* $K-1$ *vanish. Then the synthesis scaling filter is K-regular.*

Proof: We will prove the theorem by mathematical induction. First we prove it for $l = 0$, i.e. if the zeroth moments of analysis wavelet filters vanish, synthesis scaling filter vanish at aliasing frequencies. By setting $\omega = 0$ in equation 4.14, we get

$$\sum_{k=1}^{M-1} \tilde{H}_i^f\left(-\frac{2\pi k}{M}\right) H_0^f\left(-\frac{2\pi k}{M}\right) = 0 \quad i \neq 0. \tag{4.16}$$

Lemma 1 implies (also see equation 2.37) that the transpose of AC matrix

$$\begin{bmatrix} \tilde{H}_0^f(\omega) & \tilde{H}_0^f\left(\omega - \frac{2\pi}{M}\right) & \dots & \tilde{H}_0^f\left(\omega - \frac{2\pi(M-1)}{M}\right) \\ \tilde{H}_1^f(\omega) & \tilde{H}_1^f\left(\omega - \frac{2\pi}{M}\right) & \dots & \tilde{H}_1^f\left(\omega - \frac{2\pi(M-1)}{M}\right) \\ \vdots & \vdots & & \vdots \\ \tilde{H}_{M-1}^f(\omega) & \tilde{H}_{M-1}^f\left(\omega - \frac{2\pi}{M}\right) & \dots & \tilde{H}_{M-1}^f\left(\omega - \frac{2\pi(M-1)}{M}\right) \end{bmatrix}$$

has maximal rank for all ω. Setting $\omega = 0$ in the above matrix, we get

$$\begin{bmatrix} \tilde{H}_0^f(0) & \tilde{H}_0^f\left(-\frac{2\pi}{M}\right) & \dots & \tilde{H}_0^f\left(-\frac{2\pi(M-1)}{M}\right) \\ 0 & \tilde{H}_1^f\left(-\frac{2\pi}{M}\right) & \dots & \tilde{H}_1^f\left(-\frac{2\pi(M-1)}{M}\right) \\ \vdots & \vdots & & \vdots \\ 0 & \tilde{H}_{M-1}^f\left(-\frac{2\pi}{M}\right) & \dots & \tilde{H}_{M-1}^f\left(-\frac{2\pi(M-1)}{M}\right) \end{bmatrix}$$

which has maximal rank if and only if

$$\begin{bmatrix} \tilde{H}_1^f\left(-\frac{2\pi}{M}\right) & \dots & \tilde{H}_1^f\left(-\frac{2\pi(M-1)}{M}\right) \\ \vdots & & \vdots \\ \tilde{H}_{M-1}^f\left(-\frac{2\pi}{M}\right) & \dots & \tilde{H}_{M-1}^f\left(-\frac{2\pi(M-1)}{M}\right) \end{bmatrix}$$

has maximal rank. Thus the coefficient matrix of 4.16 has full rank and hence $H_0^f\left(-\frac{2\pi k}{M}\right) = 0$ $(k \in \{1, ..., M-1\})$.

Now we assume that the derivatives of H_0^f of order upto $l-1$ vanish at aliasing frequencies, i.e. $\frac{d^n}{d\omega^n}\tilde{H}_i^f\left(-\frac{2\pi k}{M}\right) = 0$ $(n \in \{0, ..., l-1\})$. By setting $\omega = 0$ in equation 4.15 we get

$$\sum_{k=1}^{M-1} \tilde{H}_i^f\left(-\frac{2\pi k}{M}\right)\frac{d^l}{d\omega^l}H_0^f\left(-\frac{2\pi k}{M}\right) = 0 \quad i \neq 0. \tag{4.17}$$

Since the coefficient matrix of equation 4.17 has maximal rank, $\frac{d^l}{d\omega^l}H_0^f\left(-\frac{2\pi k}{M}\right) = 0$ $(k \in \{1, ..., M-1\})$. ∇

4.2 Existence and Design of the Bi-orthogonal Scaling Filters

Given a synthesis filter H_0 (or an analysis filter), it is useful to ascertain the existence of an analysis filter (or a synthesis filter) subject to the bi-orthogonality constraint equation 4.3, which in Fourier domain

$$\frac{1}{M}\sum_{k=0}^{M-1} \tilde{H}_0^f\left(\omega - \frac{2\pi k}{M}\right) H_0^f\left(\omega - \frac{2\pi k}{M}\right) = 1. \tag{4.18}$$

In the two band setting, the existence of the dual filter originates from Bezout's theorem [1, 17].

Fact 1 *If p_1, p_2 are two polynomials of degree n_1, n_2 respectively, and if p_1, p_2 have no common zeros, then there exist unique polynomials q_1, q_2 of degree at most $n_2 - 1, n_1 - 1$ respectively, so that*

$$p_1(x)q_1(x) + p_2(x)q_2(x) = 1. \tag{4.19}$$

The above result has been elegantly used by [1] to design bi-orthogonal wavelet bases in the two band setting. They have solved for $q_1(x)$ and $q_2(x)$

by either expanding equation 4.19 using Taylor series around every zero of either $p_1(x)$ or $p_2(x)$, or by means of Euclid's algorithm on polynomials. To see how equation 4.18 in the two band setting leads to Bezout's identity, 4.18 can be written as

$$\left(\frac{1+z}{2}\right)^{\tilde{N}}\left(\frac{1+z^{-1}}{2}\right)^{N}\tilde{Q}_0(z)Q_0(z)+ \left(\frac{1-z}{2}\right)^{\tilde{N}}\left(\frac{1-z^{-1}}{2}\right)^{N}\tilde{Q}_0(-z)Q_0(-z)=2. \tag{4.20}$$

where N is the K-regularity of synthesis scaling filter $H_0(z)$ and $\tilde{N}$ is the K-regularity of analysis scaling filter $\tilde{H}_0(z)$ and $H_0(z)=\left(\frac{1+z^{-1}}{2}\right)^{N}Q_0(z)$ and $\tilde{H}_0(z)=\left(\frac{1+z}{2}\right)^{\tilde{N}}\tilde{Q}_0(z)$. It is the product filter $P(z)=\tilde{Q}_0(z)Q_0(z)$, which is solved using Taylor series expansion or the Euclid's algorithm. Under linear phase conditions, explicit formulas for $P(z)$ were derived analytically in [1]. Even though there is no unique solution to $P(z)$, it is the unique minimal length solution we are usually interested in. Finally the $H_0(z)$ and $\tilde{H}_0(z)$ are obtained by factorizing $P(z)$ and moving those factors among $H_0(z)$ and $\tilde{H}_0(z)$ appropriately.

Fact 2 *Moving factors from $H_0(z)$ to $\tilde{H}_0(z)$ and vice versa will not violate the equation 4.18.*

In the M-band setting, fortunately the existence of the dual filter is still guaranteed by a special case of Hilbert's Nullstellensatz [8, 50].

Fact 3 *If $p_1, ..., p_s$ are polynomials with no common zeros, then there exist polynomials $q_1, ..., q_s$ so that*

$$p_1(x)q_1(x)+...+p_s(x)q_s(x)=1. \tag{4.21}$$

Even though Hilbert's Nullstellensatz proves existence, it is not directly used in solving 4.18 in the general M-band setting. Let

$$\begin{aligned} E(\omega) &= \left| \frac{1+e^{-i\omega}+\ldots+e^{-i(M-1)\omega}}{M} \right|^2 \\ &= \frac{1}{M^2}\left[M + \sum_{k=1}^{M-1} 2(M-k)\cos\ k\omega\right] \\ &= \frac{1}{M^2}\left[M + \sum_{k=1}^{M-1} 2(M-k)T_k(x)\right] \\ &= \mathcal{E}(x) \end{aligned} \tag{4.22}$$

where $T_k(x)$ is the kth Chebyschev polynomial defined by

$$T_k(x) = \begin{cases} 2xT_{k-1}(x) - T_{k-2}(x) & k \geq 2 \\ x & k = 1 \\ 1 & k = 0 \end{cases} \tag{4.23}$$

Now the product filter $P(z) = \tilde{H}_0(z)H_0(z)$ can be written as

$$\mathcal{P}(x) = \mathcal{E}^K(x)\mathcal{R}(x). \tag{4.24}$$

Now $\mathcal{R}(x)$ could be found using techniques in [38, 31]. Using maximum flatness condition at zero frequency and expanding $\mathcal{R}(x)$ in a Taylor series about $x = 1$, it is found in [38] that

$$\mathcal{R}(x) = M\sum_{n=0}^{K-1}\left[\frac{1}{n!}\left(\frac{d}{dx}\right)^n \mathcal{E}^{-K}(x)\right]_{x=1}(x-1)^n. \tag{4.25}$$

Table 4.1 provides some examples of spline scaling filters in the three band case.

Now, we can write $\mathcal{R}(x)$ as a product of real first and second order polynomials.

$$\mathcal{R}(x) = A\prod_{j=1}^{j_1}(x - x_j)\prod_{j'=1}^{j_2}(x^2 - 2\mathrm{Re}z_{j'}x + \mid z_{j'} \mid^2). \tag{4.26}$$

Regrouping of these factors leads to all the possible H_0 and $\tilde{H}_0$.

$\tilde{K}$	K	Analysis filter	Synthesis filter
2	2	$\frac{1}{9\sqrt{3}}(-4z^{-1}+3+6z+17z^2+6z^3+3z^4-4z^5)$	$\frac{\sqrt{3}}{9}(1+2z^{-1}+3z^{-2}+2z^{-3}+z^{-4})$
3	3	$\frac{1}{27\sqrt{3}}(7z^{-2}-z^{-1}+9-6z+27z^2+$ $9z^3+27z^4-6z^5+9z^6-z^7+7z^8)$	$\frac{\sqrt{3}}{27}(1+3z+6z^2+7z^3+6z^4+3z^5+z^6)$
4	4	$\frac{1}{243\sqrt{3}}(112z^{-3}-168z^{-2}+88z^{-1}-487+636z$ $-358z^2+888z^3-693z^4+888z^5-358z^6$ $+636z^7-487z^8+88z^9-168z^{10}+112z^{11})$	$\frac{\sqrt{3}}{81}(1+4z^{-1}+10z^{-2}+16z^{-3}+19z^{-4}$ $+16z^{-5}+10z^{-6}+4z^{-7}+z^{-8})$

Table 4.1: 3-band spline scaling filters

4.3 Filter Bank Completion

In the two-band setting the wavelet filters are directly obtained from scaling filters via [48, 47, 16, 1]

$$\begin{aligned} \tilde{H}_1(z) &= zH_0(-z), \\ H_1(z) &= z^{-1}\tilde{H}_0(-z). \end{aligned} \tag{4.27}$$

So there are no degrees of freedom in the choice of wavelet filters in two-band setting. There will be degrees of freedom left in the choice of wavelet filters in the more general M-band ($M > 2$) setting. In [38], a comprehensive analysis of these degrees of freedom was given together with number of techniques of finding such wavelet filters. But the methods of [38] apply only to orthogonal M-band wavelet filters.

The synthesis scaling filter can be fully obtained from analysis wavelet filters by obtaining the cofactors, which correspond to synthesis scaling filter, of the analysis polyphase matrix. Alternatively we could design analysis wavelet filters such that the cofactors which correspond to synthesis scaling filter are given by polyphase components of the synthesis scaling filter. The determinant of the analysis polyphase matrix is set by the bi-orthogonality constraint between scaling filters, $[\downarrow M]H_0(z)\tilde{H}_0(z) = 1$. Finally the synthesis wavelet filters are obtained by inverting the analysis polyphase matrix.

We have proved the following result.

Theorem 3 *Let the following be satisfied.*

- *The analysis wavelet filters* $\tilde{H}_i(z)$ $(i \in \{1, ..., M-1\})$ *satisfy*

$$H_{0,i}(z) = C_{1,i}(z)$$

  where $C_{1,i}(z)$ *is the cofactor of analysis polyphase matrix corresponding to synthesis scaling polyphase component* $H_{0,i}(z)$.

- *Design* $H_0(z)$ *such that* $[\downarrow M]H_0(z)\tilde{H}_0(z) = 1$.

- *The synthesis wavelet filters are obtained by inverting the analysis polyphase matrix.*

Then the filter bank satisfies the perfect reconstruction.

4.3.1 Design of Spline Wavelets

Theorems 1 and 2 say that K-regularity of synthesis scaling filter implies vanishing moments of analysis wavelet filters of order 0 up to $K-1$ and vice versa. We will construct filter banks where the synthesis scaling filters are K-regular splines. We start with the following result which indicates the form of analysis wavelet filters.

Theorem 4 *Let* $H_0(z) = \sqrt{M}\left(\frac{1+z^{-1}+...+z^{-(M-1)}}{M}\right)^K$ *be the synthesis scaling filter. Then analysis wavelet filters must be of the form*

$$\tilde{H}_i(z) = (1-z^{-1})^K(z^{-1}P_{i,1}(z^M) + z^{-2}P_{i,2}(z^M) + ... + z^{-(M-1)}P_{i,M-1}(z^M)). \tag{4.28}$$

Proof: The analysis wavelet filters must satisfy that $[\downarrow M]H_0(z)\tilde{H}_i(z) = 0$. Note that $(1-z^{-1})^K H_0(z) = \sqrt{M}\left(\frac{1-z^{-M}}{M}\right)^K$ and hence $(1-z^{-1})^K H_0(z)$ has terms only in the zeroth coset. Thus if $P_i(z)$ has terms on cosets other than the zeroth coset, $P_i(z)(1-z^{-1})^K H_0(z)$ has terms on cosets other than the zeroth coset. The result follows. ∇

The smallest possible analysis wavelet filter is given by $(1-z^{-1})^K z^{-i}$ $i \in \{1, ..., M-1\}$. In fact we will show that the choice $\tilde{H}_i(z) = (1-z^{-1})^K z^{-i}$ for the analysis wavelet filters leads to the synthesis spline scaling filter $H_0(z) = \sqrt{M}\left(\frac{1+z^{-1}+...+z^{-(M-1)}}{M}\right)^K$.

Theorem 5 *Let* $\tilde{H}_i(z) = cz^{-i}(1-z^{-1})^K$ *for* $i = 1, ..., M-1$ *and some constant* c *and* $H_0(z) = \sqrt{M}\left(\frac{1+z^{-1}+...+z^{-(M-1)}}{M}\right)^K$. *Then an M-band perfect reconstruction filter bank can be designed with the above filters as analysis wavelet filters and synthesis scaling filter respectively. The constant* c *is given by*

$$c^{M-1} = \frac{\sqrt{M}}{det(\tilde{H}_p(z))}\left(\frac{1}{M}\right)^K. \tag{4.29}$$

Before we prove Theorem 5 observe that

$$(1-z^{-1})^K H_0(z) = \sqrt{M}\left(\frac{1-z^{-M}}{M}\right)^K.$$

Let $(1-z^{-1})^K = F_0(z^M) + z^{-1}F_1(z^M) + ... + z^{-(M-1)}F_{M-1}(z^M)$ (the type 1 polypase representation) and $H_0(z) = H_{0,0}(z^M)+zH_{0,1}(z^M)+...+z^{(M-1)}H_{0,M-1}(z^M)$

(the type 2 polyphase representation). Let

$$F_p(z) = \begin{bmatrix} F_0(z) & F_1(z) & F_2(z) & \dots & F_{M-1}(z) \\ z^{-1}F_{M-1}(z) & F_0(z) & F_1(z) & \dots & F_{M-2}(z) \\ z^{-1}F_{M-2}(z) & z^{-1}F_{M-1}(z) & F_0(z) & \dots & F_{M-3}(z) \\ \vdots & \vdots & \vdots & & \vdots \\ z^{-1}F_1(z) & z^{-1}F_2(z) & z^{-1}F_3(z) & \dots & F_0(z) \end{bmatrix}. \tag{4.30}$$

Then the type 2 polyphase vector of $(1 - z^{-1})^K H_0(z)$ is given by

$$F_p(z) \begin{bmatrix} H_{0,0}(z) \\ H_{0,1}(z) \\ H_{0,2}(z) \\ \vdots \\ H_{0,M-1}(z) \end{bmatrix} = \begin{bmatrix} \sqrt{M}\left(\frac{1-z^{-1}}{M}\right)^K \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{4.31}$$

We will need the determinant of $F_p(z)$ to prove Theorem 5.

Lemma 7 *The determinant of* $F_p(z)$ *is* $(1 - z^{-1})^K$.

Proof: Let $T(z) = (1 - z^{-1})S(z)$ for some $S(z)$. Then type 1 polyphase representation of $T(z)$ is given by $T_p(z) = R_p(z)S_p(z)$ where $T_p(z)$ and $S_p(z)$ are the type 1 polyphase representations of $T(z)$ and $S(z)$ and

$$R_p(z) = \begin{bmatrix} 1 & 0 & 0 & \dots & -z^{-1} \\ -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}. \tag{4.32}$$

Then it must be that the type 1 polyphase vector of $T^k(z) = (1 - z^{-1})^K S(z)$ can be written as

$$T_p^k(z) = [R_p(z)]^K S_p(z) \tag{4.33}$$

where $T_p^k(z)$ is the type 1 polyphase vector of $T^k(z)$. Now it is clear that the type 1 polyphase vector of $(1 - z^{-1})^K z^{-i}$ is $[R_p(z)]^K e_i$ where e_i is the vector with i^{th} component equal to one and zero elsewhere (Note: $e_0 = [1\ 0\ 0\ ...\ 0]^T$). Then the type 1 polyphase matrix of the system $(1 - z^{-1})^K z^{-i} \quad i \in \{0, 1, ..., M-1\}$ is $[R_p(z)]^K$. This system must be unique and equal to $F_p(z)$. Thus

$$F_p(z) = [R_p(z)]^K. \tag{4.34}$$

The determinant of $R_p(z)$ is $(1 - z^{-1})$ and hence the determinant of $[R_p(z)]^K$ is $(1 - z^{-1})^K$. ∇

Now we are ready to prove Theorem 5.

Proof of Theorem 5 : From equation 4.31 we get

$$H_{0,p}(z) = \frac{1}{\det(F_p(z))} \operatorname{adj}(F_p(z)) \begin{bmatrix} \sqrt{M}\left(\frac{1-z^{-1}}{M}\right)^K \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{4.35}$$

Thus using Lemma 7

$$H_{0,p}(z) = \operatorname{adj}(F_p(z)) \begin{bmatrix} \sqrt{M}\left(\frac{1}{M}\right)^K \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{4.36}$$

The type 1 analysis polyphase matrix is given by

$$\tilde{H}_p(z) = \begin{bmatrix} \tilde{H}_{0,0}(z) & \tilde{H}_{0,1}(z) & \tilde{H}_{0,2}(z) & \dots & \tilde{H}_{0,M-1}(z) \\ cz^{-1}F_{M-1}(z) & cF_0(z) & cF_1(z) & \dots & cF_{M-2}(z) \\ cz^{-1}F_{M-2}(z) & cz^{-1}F_{M-1}(z) & cF_0(z) & \dots & cF_{M-3}(z) \\ \vdots & \vdots & \vdots & & \vdots \\ cz^{-1}F_1(z) & cz^{-1}F_2(z) & cz^{-1}F_3(z) & \dots & cF_0(z) \end{bmatrix}. \tag{4.37}$$

Now $H_{0,p}(z)$ is given by the first column of $\frac{\mathrm{adj}(\tilde{H}_p(z))}{\det(\tilde{H}_p(z))}$. It must be that the first column of $\mathrm{adj}(\tilde{H}_p(z))$ is c^{M-1} times that of $F_p(z)$. Now it is clear from equation 4.36 that the synthesis scaling filter obtained from $\tilde{H}_p(z)$ is in fact $\sqrt{M}\left(1 + z^{-1} + \dots + z^{-(M-1)}\right)^K$ and the constant c must be given by [1]

$$\frac{c^{M-1}}{\det(\tilde{H}_p(z))} = \sqrt{M}\left(\frac{1}{M}\right)^K. \tag{4.38}$$

∇

Theorem 5 provides the simplest possible analysis wavelet filters which lead to K-regular synthesis spline filter. Table 4.2 gives an example with smallest analysis wavelet filters (subject to a delay). The problem with such designs is that the corresponding synthesis wavelet filters tend to be large and not symmetric.

We now provide the condition which can be used to construct all possible analysis wavelet filters. Note we already have the form of analysis filters in Theorem 4.

[1]The determinant of the filter bank can be any monomial but our representation of analysis wavelet filters and the synthesis scaling filter enforces that the determinant is a constant.

Analysis filter	Synthesis filter
$\frac{1}{9\sqrt{3}}(-4z^{-1}+3+6z+17z^2+6z^3+3z^4-4z^5)$	$\frac{\sqrt{3}}{9}(1+2z^{-1}+3z^{-2}+2z^{-3}+z^{-4})$
$cz^{-1}(1-z^{-1})^2$	$\frac{1}{81c}(5z^{-1}+10+15z-40z^2+$ $-14z^3+12z^4+8z^5+4z^6)$
$cz^{-2}(1-z^{-1})^2$	$\frac{1}{81c}(4z^{-1}+8+12z-14z^2$ $-40z^3+15z^4+10z^5+5z^6)$

Table 4.2: (2,2) 3-band spline filter bank.

Theorem 6 *Let the analysis wavelet filters be*

$$\tilde{H}_i(z) = (1-z^{-1})^K(z^{-1}P_{i,1}(z^M) + z^{-2}P_{i,2}(z^M) + \ldots + z^{-(M-1)}P_{i,M-1}(z^M)).$$

This $\tilde{H}_i(z)$ *leads to the synthesis scaling filter*

$$H_0(z) = \sqrt{M}\left(\frac{1+z^{-1}+\ldots+z^{-(M-1)}}{M}\right)^K$$

of a perfect reconstruction filter bank when

$$det(P(z)) = \frac{\sqrt{M}}{M^K}det(\tilde{H}_p(z)) \tag{4.39}$$

where

$$P(z) = \begin{bmatrix} P_{1,1}(z) & P_{1,2}(z) & \ldots & P_{1,M-1}(z) \\ P_{2,1}(z) & P_{2,2}(z) & \ldots & P_{2,M-1}(z) \\ \vdots & \vdots & & \vdots \\ P_{M-1,1}(z) & P_{M-1,2}(z) & \ldots & P_{M-1,M-1}(z) \end{bmatrix}$$

and $\tilde{H}_p(z)$ *is the analysis polyphase matrix.*

Proof: Let the type 1 polyphase vector of $(1-z^{-1})^K z^{-i}$ is denoted by $A_i(z)$. Then the type 1 polyphase vector of $\tilde{H}_i(z)$ is given by

$$\tilde{H}_{ip}(z) = \sum_{l=1}^{M-1} P_{i,l}(z)A_i(z). \tag{4.40}$$

Let

$$F^p_{1:M-1}(z) = \begin{bmatrix} z^{-1}F_{M-1}(z) & F_0(z) & F_1(z) & \dots & F_{M-2}(z) \\ z^{-1}F_{M-2}(z) & z^{-1}F_{M-1}(z) & F_0(z) & \dots & F_{M-3}(z) \\ \vdots & \vdots & \vdots & & \vdots \\ z^{-1}F_1(z) & z^{-1}F_2(z) & z^{-1}F_3(z) & \dots & F_0(z) \end{bmatrix}.$$

Then analysis polyphase wavelet matrix is given by $P(z)F^p_{1:M-1}(z)$. Thus the analysis polyphase matrix is given by

$$\tilde{H}_p(z) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{c}P(z) \end{bmatrix} \tilde{H}^{old}_p(z) \tag{4.41}$$

where $\tilde{H}^{old}_p(z)$ is the analysis polyphase matrix of the filter bank given in Theorem 5 and the synthesis polyphase matrix is given by

$$H_p(z) = \begin{bmatrix} 1 & 0 \\ 0 & cP^{-T}(z) \end{bmatrix} [\tilde{H}^{old}_p(z)]^{-T}. \tag{4.42}$$

Now the determinant of the analysis polyphase matrix is given by

$$\left(\frac{1}{c}\right)^{M-1} \det(P(z))\det(\tilde{H}^{old}_p(z)) = \det(\tilde{H}_p(z)).$$

Now using equation 4.29 we get the required valve for $\det(P(z))$. ∇

The following proposition is useful to construct analysis high pass filters which leads to more general synthesis low pass filters. It is also useful to provide an alternative proof to the Theorem 5.

Proposition 1 *Let $H_{p,0}(z) = A(z)e_0$ be the synthesis scaling filter where det($A(z)$) is not necessarily a monomial. Let $A^{-1}(z) = \frac{B(z)}{m(z)}$ where $B(z)$ and $m(z)$ are not rational. Then last M-1 rows of $B(z)$ as the analysis wavelet filters leads to the scaling filter $H_{p,0}(z)$.*

Proof: We can write

$$\mathrm{diag}(1, m(z), ..., m(z))\frac{B(z)}{m(z)}A(z)\mathrm{diag}(1, m^{-1}(z), ..., m^{-1}(z)) = I. \quad (4.43)$$

Thus $\mathrm{diag}(1, m(z), ..., m(z))\frac{B(z)}{m(z)}$ as analysis polyphase matrix and

$$[A(z)\mathrm{diag}(1, m^{-1}(z), ..., m^{-1}(z))]^T$$

as synthesis polyphase matrix leads to a perfect reconstruction filter bank. By changing analysis scaling filter while keeping the same analysis wavelet filters and synthesis scaling filter, we can construct a FIR perfect reconstruction filter bank. ∇

It can be seen that the proposition 1 construct analysis filters of the form $z^{-i}b(z), i = 1, ..., M-1$ when $A(z)$ is a pseudo-circulant matrix since adjoint of a pseudo-circulant matrix is also a pseudo-circulant matrix. An interesting property to observe is that $b(z)H_{p,0}(z) = c(z^M)$ for some $c(z)$. Thus filters parameterized by $b(z)d(z)$ where $d(z)$ do not have terms in zeroth coset, are all orthogonal to $H_{p,0}(z)$. But whether any filter orthogonal to $H_{p,0}(z)$ can be written in the form of $b(z)d(z)$ is not true in general. As an example let $H_{p,0}(z) = 1 + z^{-1} + z^{-2} - z^{-3} + z^{-4}$. Then z^{-1} is orthogonal to $H_{p,0}(z)$ but cannot be written in the form of $b(z)d(z)$.

Alternate proof of Theorem 5: We can write

$$H_{p,0}(z) = [P(z)]^K e_0 \quad (4.44)$$

where

$$P(z) = \begin{bmatrix} 1 & z & z & \dots & z \\ 1 & 1 & z & \dots & z \\ 1 & 1 & 1 & \dots & z \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}. \quad (4.45)$$

It can be shown that $P^{-1}(z) = \frac{R_p(z)}{(1-z^{-1})}$. Now

$$P^{-K}(z) = \frac{[R_p(z)]^K}{(1-z^{-1})^K} = \frac{F_p(z)}{(1-z^{-1})^K}$$

and the result follows by proposition 1. ∇

Now we will show how to construct analysis wavelet filters which lead to more general scaling synthesis filters.

Theorem 7 *Let* $H_0(z) = (1 + z^{-1} + ... + z^{-(M-1)})^K Q(z)$ *and write the polyphase components of* $H_0(z)$ *as* $H_{p,0}(z) = [P(z)]^K Q_c(z) e_0$ *where* $Q_c(z)$ *(the pseudo-circulant matrix corresponds to type 2 polyphase representation of* $Q(z)$*) is*

$$\begin{bmatrix} Q_0 & zQ_{M-1} & \dots & zQ_1 \\ Q_1 & Q_0 & \dots & zQ_2 \\ \vdots & \vdots & & \vdots \\ Q_{M-1} & Q_{M-2} & \dots & Q_0 \end{bmatrix}.$$

Then the last $M-1$ *rows of* $adj(Q_c(z))F_p(z)$ *as analysis wavelet filters leads to the synthesis scaling filter* $H_0(z)$.

Proof:

$$\left[[P(z)]^K Q_c(z)\right]^{-1} = \frac{\operatorname{adj}(Q_c(z))F_p(z)}{\det(Q_c(z))(1-z^{-1})^K}.$$

Then by Proposition 1 the result follows. ∇

4.4 Design of Symmetric Wavelet Filters

The symmetry we consider here is the linear phase (LP) symmetry. It is known that in some applications, particularly image coding, it is crucial to have linear phase in both analysis and synthesis filters. Furthermore, LP filters allow us to employ simple symmetric extension methods to effectively handle the boundaries of finite length signals, see Chen, Nguyen and Chan [24]. Symmetric extension eliminates the annoying energy leakage due to discontinuities at the borders when circular convolution and periodic extension are used to implement non LP filter banks. We will look for symmetric wavelet filters for a given scaling filter pair(i.e. analysis and synthesis scaling filters). We will provide solutions for some special cases. To the knowledge of the author, the full solution remains an open problem.

A filter $H(z)$ is said to have a linear phase symmetry if and only if $H(z) = z^{Mr+k}H(z^{-1})$. In the polyphase domain, linear phase symmetry is given by

$$H_p(z) = z^{r+1} \begin{bmatrix} J_{(k+1)\times(k+1)} & 0 \\ 0 & zJ_{(M-k-1)\times(M-k-1)} \end{bmatrix} H_p(z^{-1}) \tag{4.46}$$

where $H_p(z)$ is the type 1 polyphase vector of $H(z)$ and J is the anti-diagonal identity matrix. Now let the analysis filters are symmetric such that $\tilde{H}_i(z) = z^{Mr_i+k}\tilde{H}_i(z^{-1})$. It can be seen that the analysis polyphase matrix $\tilde{H}_p(z)$ must satisfy

$$\tilde{H}_p(z) = DZ(z)\tilde{H}_p(z^{-1})J(z) \tag{4.47}$$

where D is a diagonal matrix whose entry is $+1$ when the corresponding filter is symmetric and -1 when the corresponding filter is anti-symmetric, and $Z(z)$ is the diagonal matrix $\text{diag}(z^{r_0+1}, z^{r_1+1}, ..., z^{r_{M-1}+1})$, and

$$J(z) = \begin{bmatrix} J_{(k+1)\times(k+1)} & 0 \\ 0 & zJ_{(M-k-1)\times(M-k-1)} \end{bmatrix}.$$

When analysis filters satisfy the equation 4.47, the synthesis filters are either symmetric or anti-symmetric since

Analysis filter	Synthesis filter
$\frac{1}{9\sqrt{3}}(-4z^{-1}+3+6z+17z^2+6z^3+3z^4-4z^5)$	$\frac{\sqrt{3}}{9}(1+2z^{-1}+3z^{-2}+2z^{-3}+z^{-4})$
$a(z^{-1}-z^{-2})(1-z^{-1})^2$	$\frac{1}{162a}(z^{-1}+2+3z-26z^2+$ $+26z^3-3z^4-2z^5-z^6)$
$b(z^{-1}+z^{-2})(1-z^{-1})^2$	$\frac{1}{18b}(z^{-1}+2+3z-6z^2$ $-6z^3+3z^4+2z^5+z^6)$

Table 4.3: (2,2) 3-band symmetric spline filter bank.

$$\tilde{H}_p^{-T}(z) = DZ(z^{-1})\tilde{H}_p^{-T}(z^{-1})J(z^{-1}). \tag{4.48}$$

Example 1 *We use the equation 4.47 and the theorem 6 to construct a symmetric 3-band filter bank with (2,2) vanishing moments with the synthesis scaling filter is the second order 3-band spline filter. From the theorem 6 the two analysis high pass filters can be of the form* $a(1-z^{-1})^2(z^{-1}-z^{-2})$ *and* $b(1-z^{-1})^2(z^{-1}+z^{-2})$. *We can see that equation 4.47 is applicable and hence both analysis and synthesis filters are symmetric or anti-symmetric as given in Table 4.3. Also notice that the one of the analysis wavelet filters has higher regularity than the other one even though we did not enforce it.*

The filter bank constructed in the example 1 is not merely a coincidence but is a result due to the Theorem 9. Before we establish the theorem we define *linear phase symmetric matrices.*

Definition 6 *A linear phase symmetric matrix is an invertible matrix such that coefficients of each row forms a linear phase filter.*

Linear phase symmetric matrices can be parameterized as in Theorem 8.

Theorem 8 *Let A be a linear phase symmetric matrix with m rows and J be an antidiagonal matrix. If m is even, it can be parameterized as*

$$\begin{bmatrix} D & DJ \\ E & -EJ \end{bmatrix}$$

up to a matrix multiplication by a permutation matrix where D is an invertible matrix with $m/2$ rows, and E is an invertible matrix with $m/2$ rows. If m is odd, it can be parameterized as

$$\begin{bmatrix} D & f & DJ \\ E & 0 & -EJ \end{bmatrix}$$

up to a matrix multiplication by a permutation matrix where $[D \quad f]$ is an invertible matrix with $(m+1)/2$ rows and E is an invertible matrix with $(m-1)/2$ rows.

Proof: Consider the vector space, $\mathbb{R}^m$, spanned by the rows of a linear phase symmetric matrix. Since arbitrary symmetric vector is orthogonal to arbitrary antisymmetric vector, $\mathbb{R}^m$ consists of two orthogonal subspaces, one formed by symmetric vectors and the other formed by antisymmetric vectors. Thus number of rows of D must correspond to the dimension of the subspace spanned by the symmetric vectors while number of rows of E must correspond to the dimension of the subspace spanned by the antisymmetric vectors. Let e_j denote the j^{th} row of the identity matrix. Since $e_j + e_{m+1-j}, \ \ j = 1, ..., m/2$ (or $(m+1)/2$ for m odd), D has $m/2$ rows if m is even and $(m+1)/2$ rows if m is odd. Also since $e_j - e_{m+1-j}, \ \ j = 1, ..., m/2$ (or $(m-1)/2$ for m odd), E has $m/2$ rows if m is even and $(m-1)/2$ rows if m is odd. It can easily be seen that both D and E must be invertible for the linear phase matrix to be invertible for m is even while both $[D \quad f]$ and E must be invertible for the linear phase matrix to be invertible for m is odd. ∇

Note that the above proof was motivated and similar to the Theorem 2.2 in Turcajova [41].

Theorem 9 *Let the synthesis scaling filter is* $\sqrt{M}\left(\frac{1+z^{-1}+...+z^{-(M-1)}}{M}\right)^L$ *and the analysis scaling filter is as constructed by the equation 4.25. Let analysis wavelet filters are given by* $a_i z^{-1}(1-z^{-1})^L p_i(z), i = 1, ..., M-1$, *where*

$$p_i(z) = p_i[1\ z^{-1}\ ...\ z^{-(M-2)}]^T$$

and p_i *is the* i^{th} *row of a linear phase symmetric* $(M-1)\times(M-1)$ *bi-orthogonal matrix. Then the filter bank is symmetric, FIR, and satisfy perfect reconstruction.*

Proof: It is easy to see that the filter bank is FIR and satisfy perfect reconstruction due to the Theorem 6. Now we only need to show that the analysis filters and synthesis filters are symmetric(or anti-symmetric). We prove the symmetry by showing that the equation 4.47 is applicable. All the analysis wavelet filters are of same size and satisfy

$$\tilde{H}_i(z) = \pm z^{-(L+M)}\tilde{H}_i(z^{-1}) \quad (i = \{1, ..., M-1\}). \tag{4.49}$$

Now using equation 4.25, we deduce that

$$\tilde{H}_0(z) = z^{L(M-1)}\tilde{H}_0(z^{-1}). \tag{4.50}$$

Since $-L-M \equiv L(M-1) \pmod{M}$ equation 4.47 is applicable. ∇

Table 4.4 gives more examples of bi-orthogonal 3-band spline filter banks.

4.5 Conclusion and Further Research

We were successful in obtaining shortest analysis wavelet filters of equal size and the number of vanishing moments equal to the degree of K-regularity

$\tilde{K}$	K	Analysis filter	Synthesis filter
4	2	$az^{-2}(1+z+z^2)^4(7z^{-2}-34z^{-1}$ $+57-34z+7z^2)$	$\frac{1}{729a}(1+z^{-1}+z^{-2})^2$
		$b(z^{-1}-z^{-2})(1-z^{-1})^2$	$\frac{-1}{486b}(1-z)^4(5z^{-4}+30z^{-3}+105z^{-2}+238z^{-1}$ $+378+378z+238z^2+105z^3+30z^4+5z^5)$
		$c(z^{-1}+z^{-2})(1-z^{-1})^2$	$\frac{1}{1458c}(1-z)^4(-z^{-4}-6z^{-3}-21z^{-2}-42z^{-1}$ $-42+42z+42z^2+21z^3+6z^4+z^5)$
6	2	$az^{-4}(1+z+z^2)^6(-40z^{-3}+276z^{-2}$ $-768z^{-1}+1073-768z+276z^2-40z^3)$	$\frac{1}{19683a}(1+z^{-1}+z^{-2})^2$
		$b(z^{-1}-z^{-2})(1-z^{-1})^2$	$\frac{1}{131220b}(1-z)^6(28z^{-7}+224z^{-6}+1008z^{-5}$ $+3085z^{-4}+7040z^{-3}+12276z^{-2}+16434z^{-1}$ $+16434+12276z+7040z^2+3085z^3+1008z^4$ $+224z^5+28z^6)$
		$c(z^{-1}+z^{-2})(1-z^{-1})^2$	$\frac{1}{39366c}(1-z)^6(4z^{-7}+32z^{-6}+144z^{-5}$ $+425z^{-4}+880z^{-3}+1188z^{-2}+726z^{-1}$ $-726-1188z-880z^2-425z^3-144z^4$ $-32z^5-4z^6)$
1	3	$az^2(1+z+z^2)(-4z^{-1}+11-4z)$	$\frac{1}{81a}(1+z^{-1}+z^{-2})^3$
		$b(z^{-1}-z^{-2})(1-z^{-1})^3$	$\frac{1}{162b}(1-z)(-1-4z-10z^2+10z^3+4z^4+z^5)$
		$c(z^{-1}+z^{-2})(1-z^{-1})^3$	$\frac{-1}{18b}(1-z)(1+4z+10z^2+10z^3+4z^4+z^5)$
3	3	$a(1+z+z^2)^3(7z^{-2}-34z^{-1}+57-34z+7z^2)$	$\frac{1}{729a}(1+z^{-1}+z^{-2})^3$
		$b(z^{-1}-z^{-2})(1-z^{-1})^3$	$\frac{1}{1458b}(1-z)^3(z^{-3}+6z^{-2}+21z^{-1}$ $+42+42z-42z^2-42z^3-21z^4-6z^5-z^6)$
		$c(z^{-1}+z^{-2})(1-z^{-1})^3$	$\frac{1}{486c}(1-z)^3(5z^{-3}+30z^{-2}+105z^{-1}+238$ $+378z+378z^2+238z^3+105z^4+30z^5+5z^6)$
5	3	$az^{-2}(1+z+z^2)^5(-40z^{-3}+276z^{-2}$ $-768z^{-1}+1073-768z+276z^2-40z^3)$	$\frac{1}{19683a}(1+z^{-1}+z^{-2})^3$
		$b(z^{-1}-z^{-2})(1-z^{-1})^3$	$\frac{1}{393660b}(1-z)^5(-4z^{-6}-32z^{-5}-144z^{-4}$ $-425z^{-3}-880z^{-2}-1188z^{-1}-726$ $+726z+1188z^2+880z^3+425z^4+144z^5$ $+32z^6+4z^7)$
		$c(z^{-1}+z^{-2})(1-z^{-1})^3$	$\frac{-1}{13122c}(1-z)^5(28z^{-6}+224z^{-5}+1008z^{-4}$ $+3085z^{-3}+7040z^{-2}+12276z^{-1}+16434$ $+16434z+12276z^2+7040z^3+3085z^4+1008z^5$ $+224z^6+28z^7)$

Table 4.4: 3-band symmetric spline filter banks.

of synthesis spline scaling filter. The filter banks we have developed also have the linear phase symmetry. We are currently investigating two further problems.

1. How can we achieve balance in filter lengths between analysis and synthesis wavelet filters for a given synthesis spline scaling filter.

2. The more general problem of filter bank completion for a given arbitrary synthesis scaling filter with K-regularity.