Chapter 1

SUBSETS CHARACTERIZING THE NUMERICAL RANGE

1.1 Introduction

This chapter is largely expository. In it we consider the numerical range of an operator on a Hilbert space as a convex subset of the complex plane. We also study the behaviour of certain sets of vectors associated with different points of the numerical range.

Convexity of the numerical range is well-known, however, because we shall later use the technique given in the proof by Raghavendran (1969) and also because of its simplicity we include that proof in section 1.2.

In section 1.3, following Embry (1970), we associate a set of vectors from the Hilbert space to each point of the numerical range and show that linearity of the set forces the point to be an extreme point of the numerical range. Stampfli (1966) proved the converse of this result. We also include the results of Embry (1970) for the case when the point is a non-extreme boundary point or an interior point of the numerical range which show that although the corresponding set is no longer linear, we can always associate a subspace with it.

In the last section we state the Cauchy-Schwartz type inequalities proved by Embry (1975) for the vectors from these particular sets and provide different or modified proofs for them. Similar proofs will be later applied when we extend these results to cover the case of unattained boundary points of the numerical range.

1.2 The Numerical Range

The numerical range $W(T)$ for an operator (that is, a bounded linear transformation) T over a finite dimensional inner product space was first defined by Toeplitz in 1918. If H is a Hilbert space and $T \in B(H)$, we have the following definition.

Definition 1.1 For a Hilbert space H and an operator T on H, the *numerical range of T* is the set

$$W(T) = \left\{ \langle Tx, x \rangle : \|x\| = 1, \quad x \in H \right\},$$

that is, $W(T)$ is the image of the unit sphere of H under the quadratic form associated with T.

It is well-known that the numerical range is convex. There are many proofs of this theorem. We give below a modification of Raghavendran's (1969) proof which is simple and interesting. We shall later make use of the technique given in his proof.

Theorem 1.2 *(Toeplitz-Hausdorff) The numerical range $W(T)$ of an operator T is a convex subset of the complex plane.*

Proof Let

$$\xi = \langle Tf, f\rangle, \qquad \eta = \langle Tg, g\rangle$$

with

$$\|f\| = \|g\| = 1, \qquad f, g \in H .$$

Let

$$A = \alpha T + \beta I$$

where

$$\alpha = \frac{1}{\xi - \eta} ,$$

$$\beta = \frac{-\eta}{\xi - \eta} .$$

Hence

$$\langle Af, f\rangle = \alpha\langle Tf, f\rangle + \beta\langle f, f\rangle = \alpha\xi + \beta = 1$$

and

$$\langle Ag, g\rangle = \alpha\langle Tg, g\rangle + \beta\langle g, g\rangle = \alpha\eta + \beta = 0 .$$

We will first show that

$$t\xi + (1 - t)\eta \in W(T)$$

if and only if

$$t \in W(A) .$$

Let

$$t\xi + (1 - t)\eta \in W(T) .$$

So there exists $h \in H$, $\|h\| = 1$ such that

$$\langle Th,h\rangle = t\xi + (1 - t)\eta ,$$

or,

$$\langle Ah,h\rangle = \frac{1}{\xi - \eta} [t\xi + (1 - t)\eta - \eta] = t .$$

So $t \in W(A)$.

Conversely if $t \in W(A)$ so that $\langle Ah,h\rangle = t$, $\|h\| = 1$, then

$$t = \langle Ah,h\rangle = \frac{1}{\xi - \eta} \langle Th,h\rangle - \frac{\eta}{\xi - \eta} .$$

So

$$\langle Th,h\rangle = t\xi + (1 - t)\eta \in W(T) .$$

The proof is completed by showing $[0,1] \subset W(A)$. In fact we show that for any $t \in (0,1)$, it is always possible to get a complex scalar $z = x + iy$ such that

$$\frac{\langle A(f + zg), f + zg\rangle}{\langle f + zg, f + zg\rangle} = t .$$

This is equivalent to

$$\frac{\langle Af,f\rangle + |z|^2 \langle Ag,g\rangle + z\langle Ag,f\rangle + \bar{z} \langle Af,g\rangle}{\langle f,f\rangle + |z|^2 \langle g,g\rangle + z\langle g,f\rangle + \bar{z}\langle f,g\rangle} = t$$

or

$$\frac{1 + z\langle Ag,f\rangle + \bar{z}\langle Af,g\rangle}{1 + |z|^2 + z\langle g,f\rangle + \bar{z}\langle f,g\rangle} = t$$

or

$$|z|^2 t + t + 2t \operatorname{Re}(\bar{z}\langle f,g\rangle) = 1 + z\langle Ag,f\rangle + \bar{z}\langle Af,g\rangle .$$

Separating the real and imaginary parts we get an expression of the form

$$x^2 + y^2 + ax + by + \frac{t - 1}{t} = 0 \qquad \ldots(1.1)$$

and

$$cx + dy = 0 \qquad \ldots(1.2)$$

where a, b, c and d are some real numbers independent of x, y.

Now

$$\frac{t-1}{t} < 0, \quad \text{since} \quad 0 < t < 1 .$$

Hence equation (1.1) is a circle enclosing the origin and $cx + dy = 0$ is a line through the origin so that we shall always get a real pair (x,y) satisfying equations (1.1) and (1.2).

This proves the existence of $z = x + iy$. Hence $[0,1] \subset W(A)$ and consequently the numerical range is convex. □

We shall use the following terminology for the convex set $W(T)$.

Definition 1.3 An *extreme point of $W(T)$* is an element of $W(T)$ which is not contained in the interior of any line segment lying in $W(T)$.

Definition 1.4 Two extreme points of $W(T)$ are said to be *adjacent extreme points* if the line segment joining them lies in the boundary of $W(T)$.

Definition 1.5 A line L is a *line of support for $W(T)$* if $W(T)$ lies in one of the closed half planes determined by L and L contains at least one element of the closure of $W(T)$.

Definition 1.6 An extreme point z of $W(T)$ is a *corner of $W(T)$* if there exist more than one line of support for $W(T)$, passing through z.

1.3 Characterization of the Numerical Range

Embry (1970) associated certain subsets of the Hilbert space H with different points of the convex set $W(T)$. The definitions of these subsets are given below.

Definition 1.7 The set $M_z(T)$ corresponding to each point z in $W(T)$ is given by

$$M_z(T) = \left\{x \in H : \langle Tx,x\rangle - z\|x\|^2 = 0\right\} .$$

$\gamma M_z(T)$ is the linear span of $M_z(T)$.

The set $M(T)$ corresponding to a line of support L of $W(T)$ is defined by

$$M(T) = \left\{x \in H : \langle Tx,x\rangle - z\|x\|^2 = 0,\ z \in L \cap W(T)\right\} .$$

NOTE: Since $M_z(T)$ is homogeneous,

$$\begin{aligned}\gamma M_z(T) &= M_z(T) + M_z(T)\\ &= \left\{x + y : x,\ y \in M_z(T)\right\} .\end{aligned}$$

Also

$$M(T) = \bigcup_{z \in L} \left\{ M_z(T) \right\}.$$

Both $M_z(T)$ and $M(T)$ are closed.

The question arises of when $M_z(T)$ is linear and hence a subspace. Another question is how we can relate a subspace of H to $M_z(T)$ when it fails to be linear. Lemma 2 of Stampfli (1966) and theorem 1 of Embry (1970) answer these questions. Before giving their proofs we develop some necessary lemmas.

The following standard lemma gives a special property of positive operators which we shall use frequently in this chapter. Its extension to bounded sequences of vectors will be important in subsequent chapters.

Recall an operator S is *positive* if for all x in H, $\langle Sx,x \rangle \geqslant 0$.

Lemma 1.8 *For a positive operator S and x in H, $\langle Sx,x \rangle = 0$ if and only if $Sx = 0$.*

Proof If $Sx = 0$, obviously $\langle Sx,x \rangle = 0$. For the converse, let $\sqrt{S}$ be the positive square root of S. Then

$$\langle Sx,x \rangle = 0 \quad \text{implies} \quad \|\sqrt{S}\, x\| = 0$$

and hence

$$Sx = \sqrt{S}\ \sqrt{S}\ x = 0 .$$

□

Lemma 1.9 *Let L be a line of support of $W(T)$ and*

$$M(T) = \left\{ x \in H : \langle Tx, x\rangle - z\|x\|^2 = 0, \text{ some } z \in L \cap W(T) \right\}$$

Let $\theta = 0$ if L is parallel to the real axis, otherwise let θ be the acute angle between L and the real axis. Then

(i) $M(T) = \left\{ x \in H : e^{-i\theta}(T - z)x - e^{+i\theta}(T^* - \bar{z})x = 0 \right\}$,

(ii) *$M(T)$ is a closed subspace of H, and*

(iii) *$M(T) = H$ if and only if $W(T) \subset L$.*

Proof (i) Since $W(\alpha T + \beta I) = \alpha W(T) + \beta$ for any complex scalars α, β, by carrying out the standard reduction $T \to e^{i\theta}(T - zI)$ we can assume, without loss of generality, that

$$L \text{ is the imaginary axis, and}$$

$$\operatorname{Re} W(T) \geq 0 .$$

Then
$$\begin{aligned} M(T) &= \left\{ x \in H : \operatorname{Re} \langle Tx, x\rangle = 0 \right\} \\ &= \left\{ x \in H : \langle \operatorname{Re} Tx, x\rangle = 0 \right\} \quad (\text{where } \operatorname{Re} T = \tfrac{1}{2}(T+T^*)) \\ &= \left\{ x \in H : \operatorname{Re} Tx = 0 \right\} \end{aligned}$$

by lemma 1. as Re $W(T) \geqslant 0$ implies Re T is positive.

This proves part (i) of the lemma. (ii) and (iii) follow immediately from (i).

□

The above proof is a modified version of that given by Embry. For the next lemma instead of giving Embry's proof, we shall use an argument similar to that given in the proof of theorem 1.2.

Lemma 1.10 *Let $a, b \in W(T)$ and z be an interior point of the line segment with end points a and b. If $x \in M_a(T)$, $y \in M_b(T)$, $||x|| = ||y|| = 1$, then*

$$x + \lambda y \in M_z(T)$$

for two distinct complex values of λ. Consequently,

$$M_a(T) \subset \gamma M_z(T) .$$

Proof As shown in the proof of theorem 1.2, without loss of generality, we may assume $a = 1$, $b = 0$. The same proof shows that for $z \in (0,1)$, we have a non-trivial circle enclosing the origin and a line passing through the origin so that we shall always have two distinct complex values of λ, say λ_1, λ_2 such that

$$x + \lambda_i y \in M_z(T), \quad i = 1,2 .$$

This, together with the homogeneity of $M_z(T)$, gives

$$(\lambda_2 - \lambda_1)x \in M_z(T) + M_z(T) ,$$

that is,

$$x \in M_z(T) + M_z(T) = \gamma M_z(T) .$$

Hence

$$M_a(T) \subset \gamma M_z(T) .$$

□

Now we are ready to prove the main theorem.

Theorem 1.11 *Every point z in $W(T)$ can be characterized as follows:*

i) z is an extreme point of $W(T)$ if and only if $M_z(T)$ is a subspace, where

$$M_z(T) = \left\{x \in H : \langle Tx,x\rangle - z\|x\|^2 = 0\right\} .$$

ii) If z is a nonextreme boundary point of $W(T)$, then $\gamma M_z(T)$, the linear span of $M_z(T)$ is a closed subspace of H and

$$\gamma M_z(T) = M(T)$$

where

$$M(T) = \left\{x \in H : \langle Tx,x\rangle - z\|x\|^2 = 0,\ z \in L \cap W(T)\right\} ,$$

L being a line of support of $W(T)$ passing through z. In this case $W(T) \subset L$ if and only if $\gamma M_z(T) = H$.

iii) If $W(T)$ is not a line segment, then z is an interior point of $W(T)$ if and only if $\gamma M_z(T) = H$.

Proof i) Suppose z is an extreme point of $W(T)$. Without loss of generality we may take $z = 0$ and $\operatorname{Re} W(T) \geq 0$.

For $x,y \in M_z(T)$ and $\lambda = \pm 1$, we have

$$\langle T(x + \lambda y), x + \lambda y\rangle$$

$$= \langle Tx,x\rangle + |\lambda|^2 \langle Ty,y\rangle + \lambda\langle Tx,y\rangle + \lambda\langle Ty,x\rangle$$

$$= \lambda\langle Tx,y\rangle + \lambda\langle y,T^*x\rangle$$

$$= \lambda\langle Tx,y\rangle - \lambda\langle y,Tx\rangle$$ (since by lemma 1.8, $\operatorname{Re} W(T) \geq 0$ implies $\operatorname{Re} T = 0$)

$$= 2i\lambda \operatorname{Im}\langle Tx,y\rangle .$$

If $\operatorname{Im} \langle Tx,y\rangle \neq 0$, with $\lambda = \pm 1$, we have two nonzero points of $W(T)$ on the positive and negative imaginary axes contradicting that 0 is an extreme point of $W(T)$.

Thus $\operatorname{Im} \langle Tx,y\rangle = 0$ and hence homogeneity being obvious, $M_z(T)$ is a subspace.

For the converse, if z is a nonextreme point of $W(T)$, z is in the interior of a line segment with end points a and b in $W(T)$ and lemma 1.10 gives

$$M_a(T) \subset \gamma M_z(T) .$$

But $a \neq z$. Hence

$$M_a(T) \cap M_z(T) = \{0\} .$$

This shows

$$M_z(T) \neq \gamma M_z(T) ;$$

that is, $M_z(T)$ is not a subspace.

(ii) Let z be a nonextreme boundary point of $W(T)$. Then lemma 1.10 implies

$$M_a(T) \subset \gamma M_z(T) \quad \text{for all} \quad a \in W(T).$$

Consequently,

$$M(T) = \bigcup_{a \in L} \left\{ M_a(T) \right\} \subset \gamma M_z(T)$$

But $M(T)$ is a subspace by lemma 1.9 (ii). Hence

$$\gamma M_z(T) \subset M(T)$$

as $\gamma M_z(T)$ is the smallest subspace containing $M_z(T)$. Thus

$$\gamma M_z(T) = M(T) .$$

Hence, by lemma 1.9 (iii),

$$W(T) \subset L \text{ if and only if } \gamma M_z(T) = H .$$

(iii) If $W(T)$ is not a line segment and z is an interior point of $W(T)$, lemma 1.10 gives

$$M_a(T) \subset \gamma M_z(T) \quad \text{for each} \quad a \in W(T) .$$

Thus

$$H = \bigcup_{a \in W(T)} \left\{ M_a(T) \right\} \subset \gamma M_z(T) .$$

Hence

$$\gamma M_z(T) = H .$$

On the other hand, if z is a boundary point of $W(T)$,

$$\gamma M_z(T) = \begin{cases} M_z(T) & \text{when } z \text{ is extreme,} \\ M(T) & \text{when } z \text{ is nonextreme,} \end{cases}$$

and thus lemma 1.9 (iii) gives

$$\gamma M_z(T) \neq H . \qquad \square$$

1.4 A Cauchy-Schwartz Inequality

Embry (1975) deduced a Cauchy-Schwartz inequality for the elements of

$$M(T) = \left\{ x \in H : \langle Tx,x \rangle - z\|x\|^2 = 0, \quad z \in L \cap W(T) \right\}$$

where L is a line of support of $W(T)$. We give the inequality in the next theorem with a proof different from that given by Embry.

Theorem 1.12 *Let L be a line of support for $W(T)$ and*

$$M(T) = \left\{ x \in H : \langle Tx,x\rangle - z\|x\|^2 = 0, \quad z \in L \cap W(T) \right\} .$$

Let b be an element of L such that either b is an extreme point of $W(T)$ or $b \notin W(T)$. Then for all x and y in $M(T)$,

$$|\langle (T - b)x,y\rangle|^2 \leq \langle (T - b)x,x\rangle \langle y,(T - b)y\rangle .$$

Proof First note that by virtue of lemma 1.9 (i), the right hand side of the inequality is real. Without loss of generality we may take

$$b = 0,$$
$$W(T) \cap L \subset R^+$$
$$\text{and} \quad \operatorname{Im} W(T) \geq 0 \quad (\text{or} \leq 0) .$$

Let us exclude the obvious case when $x = 0$ or $y = 0$.

Let $t_1, t_2 \in R^+$ be such that

$$\frac{\langle Tx,x\rangle}{\|x\|^2} = t_1 \qquad \text{and} \qquad \frac{\langle Ty,y\rangle}{\|y\|^2} = t_2 .$$

Consider points of $W(T)$ of the form

$$g(\lambda) = \frac{\langle T(x + \lambda y), x + \lambda y\rangle}{\|x + \lambda y\|^2}$$

where λ is any complex scalar.

We have assumed $x + \lambda y \neq 0$ because if $x + \lambda y = 0$, the inequality is trivially true.

Since $x \in M(T)$ and L is the real axis,

$$\text{Im} \langle Tx,x\rangle = 0 .$$

Thus since $\text{Im } W(T) \geqslant 0$, lemma 1.8 gives $\text{Im } Tx = 0$ where

$$\text{Im } T = \frac{1}{2i} (T - T^*) .$$

So $Tx = T^*x$ and hence

$$g(\lambda) = \frac{\langle Tx,x\rangle + |\lambda|^2 \langle Ty,y\rangle + \bar{\lambda} \langle Tx,y\rangle + \lambda \langle Ty,x\rangle}{\|x + \lambda y\|^2}$$

$$= \frac{t_1 \|x\|^2 + t_2 |\lambda|^2 \|y\|^2 + 2\text{Re } (\bar{\lambda} \langle Tx,y\rangle)}{\|x + \lambda y\|^2} .$$

This shows $g(\lambda)$ is real and hence positive, since

$$g(\lambda) \in L \cap W(T) .$$

So we have

$$t_1 ||x||^2 + t_2 |\lambda|^2 ||y||^2 + 2\mathrm{Re}(\bar{\lambda}\langle Tx,y\rangle) \geq 0. \qquad \ldots(1.3)$$

Choose λ such that

$$\mathrm{Re}(\bar{\lambda}\langle Tx,y\rangle) = \pm|\langle Tx,y\rangle| .$$

Then the condition that $|\lambda|$ satisfies (1.3) gives

$$4|\langle Tx,y\rangle|^2 - 4t_2 ||y||^2 t_1 ||x||^2 \leq 0 .$$

Hence

$$|\langle Tx,y\rangle|^2 - \langle Tx,x\rangle \langle Ty,y\rangle \leq 0 ,$$

and so

$$|\langle Tx,y\rangle|^2 - \langle Tx,x\rangle \langle y,Ty\rangle \leq 0 .$$

□

As given in theorem 1.11 (i), Stampfli (1966) proved that $M_z(T)$ is a subspace if z is an extreme point of $W(T)$.

This result can also be deduced from theorem 1.12.

Corollary 1.13 *If b is an extreme point of $W(T)$, then $M_b(T) = \left\{x \in H : \langle Tx,x\rangle - b\|x\|^2 = 0\right\}$ is a subspace.*

Proof Homogeneity being obvious we only have to prove the linearity.

Let

$$x_1, x_2 \in M_b(T) .$$

Thus $x_1, x_2 \in M(T)$ as $M_b(T) \subset M(T)$.
But $M(T)$ is a subspace by lemma 1.9 (ii).

So

$$x_1 + x_2 \in M(T) .$$

Now since $x_1, x_2 \in M_b(T)$ and $x_1 + x_2 \in M(T)$, theorem 1.12 gives

$$\langle (T-b)\ (x_1+x_2),\ x_1 + x_2\rangle = 0 .$$

So, $x_1 + x_2 \in M_b(T)$.

□

Corollary 1.14 *If b is an extreme point of $W(T)$, then*

$$<(T-b)x,y> = 0 \quad \text{and} \quad <(T^*-\bar{b})x,y> = 0$$

for all $x \in M_b(T)$ and $y \in M(T)$ where

$$M_b(T) = \left\{x \in H : <Tx,x> - b\|x\|^2 = 0\right\}$$

and

$$M(T) = \left\{x \in H : <Tx,x> - z\|x\|^2 = 0, \; z \in L \cap W(T)\right\},$$

L being a line of support for $W(T)$ passing through b.

Proof Obvious from theorem 1.12 and lemma 1.9 (i). □

Corollary 1.15 *With the same notations as in corollary 1.14, if $x \in M_b(T)$ and $Tx \in M(T)$, then*

$$Tx = bx \quad \text{and} \quad T^*x = \bar{b}x .$$

Proof Since by lemma 1.9 (ii), $M(T)$ is a subspace, $Tx \in M(T)$, $x \in M_b(T) \subset M(T)$ together imply

$$Tx - bx \in M(T) .$$

But by corollary 1.14,

$$<Tx - bx, y> = 0 \quad \text{whenever} \quad y \in M(T) .$$

Taking $y = Tx - bx$, we have $\|Tx - bx\|^2 = 0$. Consequently $Tx = bx$ and so by lemma 1.9 (i), $T^*x = \bar{b}x$.

□

All the above corollaries are due to Embry. We give below another inequality given by her (with modified proof), from which orthogonality of subspaces associated with adjacent extreme points of $W(T)$ can be deduced.

Theorem 1.16 *Let b and c be adjacent extreme points of $W(T)$ and let*

$$d = tb + (1 - t)c, \quad 0 \leq t \leq 1 .$$

If $x \in M_b(T)$ and $y \in M_d(T)$, then

$$|\langle x,y\rangle| \leq \sqrt{t}\, \|x\|\, \|y\| .$$

In particular, $\langle x,y\rangle = 0$ whenever $x \in M_b(T)$ and $y \in M_c(T)$.

Proof Without loss of generality, we may take

$$b = 0,$$
$$c = 1,$$
$$\operatorname{Im} W(T) \geq 0 \ (\text{or} \leq 0)$$
$$\text{and } L \cap W(T) \subset R^+ .$$

For any complex scalar λ, if $x + \lambda y = 0$, we have

$$|\langle x,y\rangle| - \sqrt{t}\,\|x\|\,\|y\| = 0.$$

So let us assume that $x + \lambda y \neq 0$.

Consider elements of $W(T)$ of the form

$$g(\lambda) = \frac{\langle T(x + \lambda y),\ x + \lambda y\rangle}{\|x + \lambda y\|^2}$$

$$= \frac{\langle Tx,x\rangle + |\lambda|^2 \langle Ty,y\rangle + \bar{\lambda} \langle Tx,y\rangle + \lambda \langle Ty,x\rangle}{\|x + \lambda y\|^2}$$

$$= \frac{|\lambda|^2 (1 - t) \|y\|^2}{\|x + \lambda y\|^2}$$

since lemma 1.8, with our assumptions, gives $Tx = T^*x$ and by corollary 1.14, $\langle Tx,y\rangle = 0$.

Thus $g(\lambda)$ is real and hence must belong to $[0,1]$. So we have

$$|\lambda|^2 (1 - t) \|y\|^2 \leqslant \|x + \lambda y\|^2 ,$$

or

$$|\lambda|^2 (1 - t) \|y\|^2 \leqslant \|x\|^2 + |\lambda|^2 \|y\|^2 + 2\mathrm{Re}\,(\bar{\lambda} \langle x,y\rangle).$$

Choose any λ so that

$$\mathrm{Re}\ (\bar{\lambda}\ \langle x,y\rangle) = \pm\ |\lambda|\ |\langle x,y\rangle|\ .$$

Hence

$$t|\lambda|^2\ \|y\|^2 \pm 2|\lambda|\ |\langle x,y\rangle| + \|x\|^2 \geqslant 0\ . \qquad \ldots(1.4)$$

Then the condition that $|\lambda|$ satisfies (1.4) gives

$$4|\langle x,y\rangle|^2 - 4t\ \|y\|^2\ \|x\|^2 \leqslant 0\ ,$$

that is,

$$|\langle x,y\rangle| \leqslant \sqrt{t}\ \|x\|\ \|y\|\ .$$

□

The following theorem of Embry (1975) considers two lines of support of $W(T)$ and relates the subsets associated with them to each other.

Theorem 1.17 *Let L_1 and L_2 be two non-parallel lines of support intersecting at the point c. Let*

$$M_j(T) = \left\{x \in H : \langle Tx,x\rangle - z\ \|x\|^2 = 0,\ z \in L_j\right\},\quad j = 1,2.$$

Then

$$\langle (T - c)x_1,x_2\rangle = 0 \quad \text{whenever} \quad x_j \in M_j(T),\ j = 1,2.$$

Proof Let θ_j be the acute angle between L_j and the real axis. Let

$$x_j \in M_j(T), \quad j = 1,2.$$

Then by lemma 1.9 (i),

$$e^{i\theta_j}(T - c)x_j - e^{-i\theta_j}(T^* - \bar{c})x_j = 0, \quad j = 1,2.$$

Thus

$$\begin{aligned} & e^{2i\theta_1} <(T - c)x_1, x_2> \\ &= <(T^* - \bar{c})x_1, x_2> \\ &= <x_1, (T - c)x_2> \\ &= <x_1, e^{-2i\theta_2} (T^* - \bar{c})x_2> \\ &= e^{2i\theta_2} <(T - c)x_1, x_2> . \end{aligned}$$

Since L_1 and L_2 are non-parallel, $e^{2i\theta_1} \neq e^{2i\theta_2}$ and hence $<(T - c)x_1, x_2> = 0$.

□

In this chapter we have dealt with the numerical range as a convex set and defined subsets $M_z(T)$, $M(T)$ associated with its different points and lines of support.

In section 1.3, conditions for linearity of these subsets have been examined. We also saw how the argument given in the proof of convexity of the numerical range from section 1.2 can be conveniently applied to the proof of the main lemma required for characterization of the numerical range by these subsets.

In section 1.4, we gave two inequalities for the vectors from these subsets and saw how a result from the previous section, namely, linearity of $M_z(T)$ when z is an extreme point of $W(T)$, can be deduced as a corollary of one of these inequalities.

Note that all these theorems are inapplicable to the unattained boundary points of the numerical range. So a need for extension of these results to all points in the closure of the numerical range is realized. In our next chapter we attempt to supply such an extension.

Chapter 2

SUBSETS CHARACTERIZING THE CLOSURE OF THE NUMERICAL RANGE

2.1 Introduction

In this chapter we attempt to generalize all the results of Embry given in the previous chapter. We define certain subsets associated with each point of the closure of the numerical range. As we see in section 2.2, these sets are very similar in properties to those defined in Chapter 1. But they consist of bounded sequences of vectors from the Hilbert space.

Let $W(T)^-$ denote the closure of $W(T)$. Since $W(T)$ is convex, so is $W(T)^-$. But an extreme point of $W(T)$ need not be an extreme point of $W(T)^-$ and vice versa. Also a non-extreme boundary point of $W(T)^-$ can be an extreme point of $W(T)$ or may not belong to $W(T)$ at all.

In sections 2.3, 2.6 and 2.9 we show that the subset associated with an extreme point of the closure of the numerical range is in fact a subspace and if the subset associated with a point of $W(T)^-$ is linear, then the point has to be extreme. We then consider the case when the point is a nonextreme boundary point or an interior point of $W(T)^-$ and achieve results of the same type, but not exactly similar to those given by Embry for corresponding points of the numerical range.

To prove some of these results a modification of a technique given by Berberian (1962) and Berberian and Orland (1967) proves very useful, though the results can be obtained without the use of this technique as well. For example, Das and Craven proved the linearity of the subset associated with an extreme point of W(T) by a direct method. This has been illustrated in section 2.3. However, since our technique has many applications we shall use it frequently throughout our dissertation.

By using this technique we extend the Hilbert space to another Hilbert space and consider a faithful *-representation of our operator on this new space. The numerical ranges of these two operators are related; in fact the numerical range of the new operator is the closure of the numerical range of the original one. This was first shown by Berberian and Orland (1967). However, we shall prove this result without a Banach algebra approach. This enables us to use known results on numerical ranges for this new space and operator. Often this involves some calculations. Thus we obtain results for the closure of the numerical range. Sections 2.4 and 2.5 of this chapter explain this technique in detail.

In section 2.4 we develop a technical lemma to show the existence of a normalized positive linear functional which strictly separates any non-null sequence of positive numbers from the set of real null sequences. This functional has all the properties of a Banach-Mazur generalized limit except translation invariance. We modify Berberian's technique in that we use this new functional instead of the Banach-Mazur generalized limit to define a pseudo-inner product on the space of bounded sequences of vectors from our Hilbert space. Positivity of this functional is essential to our proofs.

In sections 2.8 and 2.9 we generalize the Cauchy-Schwartz type inequalities given in the first chapter to sequences of vectors. To do this we first use Berberian's technique and then use a direct method by which stronger inequalities can be obtained. From one of these inequalities we see that the results of Das and Craven can be deduced as a corollary.

2.2 Certain Subsets and Their Properties

Let $\ell_\infty(H)$ be the set of all bounded sequences of vectors from H. We associate certain subsets of $\ell_\infty(H)$ with different points of the convex set $W(T)^-$. The definitions of these subsets are given below.

Definition 2.1 The set $N_z(T)$ corresponding to each point z in $W(T)^-$ is given by

$$N_z(T) = \left\{ (x_n) \in \ell_\infty(H) : \langle Tx_n, x_n \rangle - z\|x_n\|^2 \to 0 \right\} .$$

$\gamma N_z(T)$ is the linear span of $N_z(T)$. The sets $N(T)$ and $N_L(T)$ corresponding to a line of support L of $W(T)$ are defined by

$$N(T) = \left\{ (x_n) \in \ell_\infty(H) \ : \ \langle Tx_n, x_n \rangle - z\|x_n\|^2 \to 0, \ z \in L \cap W(T)^- \right\}$$

and

$$N_L(T) = \left\{ (x_n) \in \ell_\infty(H) \ : \ \inf_{z \in L} |\langle Tx_n, x_n \rangle - z\|x_n\|^2| \to 0 \right\} .$$

NOTE: i) $N_z(T)$ is closed and homogeneous.

ii) Since $N_z(T)$ is homogeneous,

$$\begin{aligned} \gamma N_z(T) &= N_z(T) + N_z(T) \\ &= \left\{ (x_n + y_n) \ : \ (x_n), \ (y_n) \in N_z(T) \right\} . \end{aligned}$$

iii) $N(T) = \bigcup_{z \in L} \left\{ N_z(T) \right\}$.

iv) If we look upon H as embedded in $\ell_\infty(H)$ with the correspondence $x \to (x, x, \ldots)$, then $M_z(T)$ (defined in the last chapter) is embedded as subset of $N_z(T)$ whenever $z \in W(T)$. For unattained boundary points of $W(T)$, $M_z(T)$ will consist of the zero vector only, while $N_z(T)$ will be a nontrivial set of sequences. Similar relations hold for $M(T)$ and $N(T)$.

v) If L is a line of support of $W(T)$ and $z \in L \cap W(T)^-$, then

$$N_z(T) \subset N(T) \subset N_L(T) \ .$$

A question likely to be asked is whether $N(T)$ and $N_L(T)$ are closed subspaces. The author is unable to prove the linearity of $N(T)$, though lemma 2.3 will show that $N(T)$ is closed.

The following standard theorem from Real Analysis is needed in the proof of lemma 2.3.

Theorem 2.2 *(Iterated Limit Theorem)* *Let (a_{mn}) be a double sequence in R^p. Suppose that the single limits $b_m = \lim_n (a_{mn})$, $c_n = \lim_m (a_{mn})$ exist for all natural numbers m and n, and that the convergence of one of these collections is uniform. Then both interated limits $b = \lim_m (b_m)$ and $c = \lim_n (c_n)$ exist and are equal.*

Lemma 2.3 *Let L be a line of support of $W(T)$ and*

$$N(T) = \left\{ (x_n) \in \ell_\infty(H) : \langle Tx_n, x_n \rangle - z \|x_n\|^2 \to 0, \quad z \in L \cap W(T)^- \right\} .$$

Then $N(T)$ is closed in the norm topology of $\ell_\infty(H)$.

Proof If $L \cap W(T)^-$ consists of only one point z, then $N(T) = N_z(T)$ and without loss of generality we may take $z = 0$.

Let $x^{(m)} \to x^{(\circ)}$ in $\ell_\infty(H)$ as $m \to \infty$ where

$$x^{(m)} = (x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}, \ldots)$$

and

$$x^{(\circ)} = (x_1^{(\circ)}, x_2^{(\circ)}, \ldots, x_n^{(\circ)}, \ldots) .$$

Thus

$$\|x^{(m)} - x^{(\circ)}\| \to 0$$

and hence $(x^{(m)})$ converges uniformly to $x^{(\circ)}$.

Let $x^{(m)} \in N_\circ(T)$ for each m, that is, for each m,

$$\langle Tx_n^{(m)}, x_n^{(m)}\rangle \to 0 \quad \text{as} \quad n \to \infty .$$

We have to show $\langle Tx_n^{(\circ)}, x_n^{(\circ)}\rangle \to 0$. Obviously as

$$\|x^{(m)} - x^{(\circ)}\| = \sup_n \|x_n^{(m)} - x_n^{(\circ)}\| \to 0 ,$$

we have for each n,

$$x_n^{(m)} \to x_n^{(\circ)} \quad \text{as} \quad m \to \infty .$$

Thus for each n,

$$\langle Tx_n^{(m)}, x_n^{(m)}\rangle \to \langle Tx_n^{(\circ)}, x_n^{(\circ)}\rangle \text{ as } m \to \infty .$$

Hence $\lim_n \langle Tx_n^{(m)}, x_n^{(m)}\rangle$ and $\lim_m \langle Tx_n^{(m)}, x_n^{(m)}\rangle$ both exist for all natural numbers m and n. Also the convergence of $(\langle Tx_n^{(m)}, x_n^{(m)}\rangle)$ as $m \to \infty$ is uniform.

Thus considering a complex sequence as a sequence in R^2, we can apply theorem 2.2 to the double sequence $\langle Tx_n^{(m)}, x_n^{(m)}\rangle$ and so conclude that both iterated limits are equal, that is,

$$\lim_n \lim_m \langle Tx_n^{(m)}, x_n^{(m)}\rangle = \lim_m \lim_n \langle Tx_n^{(m)}, x_n^{(m)}\rangle .$$

But the left hand side limit is nothing but $\lim_n \langle Tx_n^{(\circ)}, x_n^{(\circ)}\rangle$ and since for each m, $\lim_n \langle Tx_n^{(m)}, x_n^{(m)}\rangle = 0$, the right hand side limit is zero.

So, $\lim_n \langle Tx_n^{(\circ)}, x_n^{(\circ)}\rangle = 0$. Hence $N(T)$ is closed. If $L \cap W(T)^-$ is not a single point, then by a suitable translation and rotation, without loss of generality we may take

$$L \cap W(T)^- = [0,1]$$

and

$$\operatorname{Im} W(T) \geq 0 .$$

In this case if $x^{(m)} \in N(T)$, we have for all m, $\langle Tx_n^{(m)}, x_n^{(m)}\rangle - z^{(m)} \|x_n^{(m)}\|^2 \to 0$ where $z^{(m)} \in [0,1]$.

We have to show

$$\langle Tx_n^{(\circ)}, x_n^{(\circ)}\rangle - z\|x_n^{(\circ)}\|^2 \to 0 \quad \text{for some} \quad z \in [0,1].$$

As before, the convergences

$$\langle Tx_n^{(m)}, x_n^{(m)}\rangle \to \langle Tx_n^{(\circ)}, x_n^{(\circ)}\rangle \quad \text{as} \quad m \to \infty$$

and

$$\|x_n^{(m)}\| \to \|x_n^{(\circ)}\| \quad \text{as} \quad m \to \infty$$

are uniform.

If $z^{(m)}$ does not converge, there exists a subsequence $z^{(m_k)}$ such that

$$z^{(m_k)} \to z \in [0,1], \text{ as } z^{(m_k)} \in [0,1] \ .$$

Thus $\langle Tx_n^{(m_k)}, x_n^{(m_k)}\rangle - z^{(m_k)}\|x_n^{(m_k)}\|^2$ converges uniformly to $\langle Tx_n^{(\circ)}, x_n^{(\circ)}\rangle - z\|x_n^{(\circ)}\|^2$ as $m \to \infty$.

Also,

$$\lim_n \ [\langle Tx_n^{(m)}, x_n^{(m)}\rangle - z^{(m)}\|x_n^{(m)}\|^2] = 0 \ .$$

Hence $\langle Tx_n^{(m_k)}, x_n^{(m_k)}\rangle - z^{(m_k)} \|x_n^{(m_k)}\|^2$ must tend to zero as $n \to \infty$.

Now application of theorem 2.2 gives the two iterated limits are equal, that is,

$$\lim_n [\langle Tx_n^{(\circ)}, x_n^{(\circ)}\rangle - z\|x_n^{(\circ)}\|^2] = 0 .$$

□

We shall need the following lemma to show that $N_L(T)$ is a closed subspace.

Lemma 2.4 *For a positive operator S and (x_n) in $\ell_\alpha(H)$,*

$$\langle Sx_n, x_n\rangle \to 0 \text{ if and only if } Sx_n \to 0 .$$

Proof If $Sx_n \to 0$, obviously $\langle Sx_n, x_n\rangle \to 0$. For the converse, let $\sqrt{S}$ be the positive square root of S. Then

$$\langle Sx_n, x_n\rangle \to 0 \text{ implies } \|\sqrt{S}\, x_n\| \to 0$$

and hence

$$Sx_n = \sqrt{S}\,\sqrt{S}\, x_n \to 0 .$$

□

Lemma 2.5 *Let* L *be a line of support of* $W(T)$ *and*

$$N_L(T) = \left\{(x_n) \in \ell_\infty(H) : \inf_{z \in L} |\langle Tx_n, x_n\rangle - z\|x_n\|^2| \to 0\right\} .$$

Let $\theta = 0$ *if* L *is parallel to the imaginary axis, otherwise let* θ *be the acute angle between* L *and the real axis. Then for any* $z \in L$ *we have*

i) $N_L(T) = \left\{(x_n) \in \ell_\infty(H) : e^{i\theta}(T - z)x_n - e^{-i\theta}(T^* - \bar{z})x_n \to 0\right\}$ *, and*

ii) $N_L(T)$ *is a closed subspace of* $\ell_\infty(H)$.

Proof By carrying out the standard reduction $T \to e^{i\theta}(T - zI)$, we may, without loss of generality, assume that

$$L \text{ is the imaginary axis}$$

and

$$\text{Re } W(T) \geq 0 .$$

Then

$$\begin{aligned} N_L(T) &= \left\{(x_n) \in \ell_\infty(H) : \text{Re } \langle Tx_n, x_n\rangle \to 0\right\} \\ &= \left\{(x_n) \in \ell_\infty(H) : \langle \text{Re } Tx_n, x_n\rangle \to 0\right\} \\ &= \left\{(x_n) \in \ell_\infty(H) : \text{Re } Tx_n \to 0\right\} \end{aligned}$$

by lemma 2.4 as $\text{Re } W(T) \geq 0$ implies $\text{Re } T$ is positive.

Also,

$$\left\{(x_n) \in \ell_\infty(H) : e^{i\theta}(T - z)x_n - e^{-i\theta}(T^* - \bar{z})x_n \to 0\right\}$$

$$= \left\{(x_n) \in \ell_\infty(H) : (T - ib)x_n + (T^* + ib)x_n \to 0\right\}$$

[by the choice of θ and z]

$$= \left\{(x_n) \in \ell_\infty(H) : \text{Re } Tx_n \to 0\right\} .$$

This proves part (i) of the lemma. Part (ii) follows immediately.

□

2.3 Linearity on the Boundary of the Numerical Range

Das and Craven first generalized theorem 1.11 (i) for extreme points of $W(T)^-$. We shall here give their proof (modified) of this generalized theorem and then use a technique given by Berberian to give an alternative proof which is more conceptual and less computational in the next section.

Theorem 2.6 *For any point z in $W(T)^-$, let*

$$N_z(T) = \left\{(x_n) \in \ell_\infty(H) : \langle Tx_n, x_n\rangle - z\|x_n\|^2 \to 0\right\} .$$

Then $N_z(T)$ is a subspace of $\ell_\infty(H)$ if and only if z is an extreme point of $W(T)^-$.

Proof Without loss of generality we may assume that

$$z = 0 \quad \text{and} \quad \text{Re } W(T) \geq 0 .$$

Suppose z is an extreme point of $W(T)^-$. Homogeneity being obvious we only have to prove linearity of $N_z(T)$.

Let (x_n), $(y_n) \in N_z(T)$. Since $<\text{Re}\, Tx_n, x_n> \to 0$, lemma 2.4 gives $\text{Re}\, Tx_n \to 0$. Thus

$$<T(x_n+y_n), x_n+y_n> - [<Tx_n,x_n> + <Ty_n,y_n> + 2i\text{Im}<Tx_n,y_n>] \to 0.$$

Since $<Tx_n, x_n>$ and $<Ty_n, y_n>$ both tend to zero, we only have to show $\text{Im} <Tx_n, y_n> \to 0$. If $\text{Im} <Tx_n, y_n>$ does not tend to zero, we will get a contradiction as shown below.

Case 1 $||x_n + y_n||$ and $||x_n - y_n||$ are bounded away from zero for all n.

Passing on to subsequences if necessary, we may, without loss of generality, assume

$$\frac{\text{Im} <Tx_n, y_n>}{||x_n + y_n||^2} \to a$$

and

$$\frac{||x_n + y_n||^2}{||x_n - y_n||^2} \to b$$

where a, b are nonzero real numbers.

Thus

$$\frac{\langle T(x_n + y_n),\ x_n + y_n\rangle}{\|x_n + y_n\|^2} \to 2ia$$

and

$$\frac{\langle T(x_n - y_n),\ x_n - y_n\rangle}{\|x_n - y_n\|^2} \to -2ib$$

Since $2ia$ and $-2ib$ belong to $W(T)^-$ and $b > 0$, this contradicts that 0 is an extreme point of $W(T)^-$.

Case 2 $\|x_n + y_n\|\ \|x_n - y_n\|$ is not bounded away from zero.

Consider the disjoint partition of the sequence (n) of all natural numbers such that

$$(n) = (n') \cup (n'')$$

and

$$\min\left\{ \|x_{n'} + y_{n'}\|,\ \|x_{n'} - y_{n'}\| \right\} < \frac{\varepsilon\|T\|}{2M}$$

where M is an upper bound for $\|x_n\|$.

Since

$$|\langle Tx_{n'}, y_{n'}\rangle| \leq |\langle Tx_{n'}, x_{n'}\rangle| + |\langle Tx_{n'}, x_{n'} \pm y_{n'}\rangle| ,$$

we have

$$|\langle Tx_{n'}, y_{n'}\rangle| \leq |\langle Tx_{n'}, x_{n'}\rangle| + \frac{\varepsilon}{2} .$$

Thus, since $\langle Tx_{n'}, x_{n'}\rangle \to 0$, $|\langle Tx_{n'}, y_{n'}\rangle|$ can be made less than ε by choosing n' sufficiently large. For the sequence (n''), we can apply case 1. Hence $N_z(T)$ is linear.

For the converse, if z is not an extreme point of $W(T)^-$, then either z is an interior point of $W(T)$ and theorem 1.11 (i) shows that $M_z(T)$ and hence $N_z(T)$ is not linear; or z is a nonextreme boundary point of $W(T)^-$, that is, there exist two sequences of unit vectors (x_n), (y_n) such that

$$\langle Tx_n, x_n\rangle \to ia \quad \text{and} \quad \langle Ty_n, y_n\rangle \to -ia \text{ (say)}.$$

Let $\lambda = x + iy$. Then

$$\langle T(x_n + \lambda y_n), x_n + \lambda y_n\rangle - ia(1 - |\lambda|^2) - 2i\,\mathrm{Im}(\bar{\lambda}\langle Tx_n, y_n\rangle) \to 0.$$

Passing on to a subsequence if necessary, we may assume

$$\mathrm{Im}\ (\bar{\lambda}\langle Tx_n,\ y_n\rangle) \to b + ic\ .$$

Thus

$$\langle T(x_n + \lambda y_n),\ x_n + \lambda y_n\rangle \to ia(1 - |\lambda|^2) + 2i(cx - by)\ .$$

Hence

$$(x_n + \lambda y_n) \in N_0\,(T)$$

for at least two distinct values of λ satisfying the equation of the circle

$$x^2 + y^2 + \frac{2(by - cx)}{a} - 1 = 0\ .$$

This shows $N_z(T)$ is not linear.

□

The following interesting example given by Das and Craven shows that though $N_z(T)$ is linear whenever z is an extreme point of $W(T)^-$, the set

$$N_z'(T) = \left\{(x_n) \in \ell_\infty(H) : \langle Tx_n,\ x_n\rangle/||x_n||^2 \to z\right\}$$

which is quite similar to $N_z(T)$ is not necessarily linear.

Suppose (e_n) and (e'_n) are two disjoint sets of orthonormal elements of H. Define a linear operator V such that

$$Ve_n = e_n$$

and

$$Ve'_n = \frac{1}{n} e'_n \ .$$

It is easy to verify that V is selfadjoint.

Let

$$x_n = \frac{e_n + ne'_n}{\sqrt{1 + n^2}} \quad \text{and} \quad y_n = \frac{e_n - ne'_n}{\sqrt{1 + n^2}} \ .$$

Thus $\|x_n\| = \|y_n\| = 1$ and $Vx_n = \dfrac{e_n + e'_n}{\sqrt{1 + n^2}} \to 0.$

Similarly $Vy_n \to 0.$

If we define $T = V^2$, then 0 is an extreme point of $W(T)^-$ and we note that though both $\dfrac{\langle Tx_n, x_n\rangle}{\|x_n\|^2}$ and $\dfrac{\langle Ty_n, y_n\rangle}{\|y_n\|^2}$ tend to zero,

$$\frac{\langle T(x_n + y_n), x_n + y_n\rangle}{\|x_n + y_n\|^2} = 1 \quad \text{for all } n.$$

This shows that $N'_o(T)$ is not linear.

In the next few sections we construct an alternative approach to establish the result given in theorem 2.6. We employ a technique of S.K. Berberian (1962) and S.K. Berberian and G.H. Orland (1967). This approach appears to be more conceptual in that it enables us to deduce theorem 2.6 from theorem 1.11 (i). It also allows us to deduce sufficiency in the same theorem as a corollary from a Cauchy-Schwartz type inequality.

Using the same technique other results may also be generalized to unattained boundary points of the numerical range. This is illustrated in section 2.8 where we extend results of Embry (1975).

The results in sections 2.4-2.7 (except theorem 2.8, corollaries 2.11 and 2.12 and lemma 2.13) have been included in a joint paper by S. Majumdar and Brailey Sims.

2.4 A Technical Lemma

Let ℓ_∞, ℓ_∞^+, c and c_0 be the sets of real bounded, bounded nonnegative, convergent and null sequences respectively. Let $x = (x_1, x_2, \ldots, x_n, \ldots) \in \ell_\infty$ and ℓ_∞^* be the dual of ℓ_∞.

We prove a simple lemma which will be used in the following sections to achieve our main results.

Lemma 2.7 *For any* $y \in \ell_\infty^+ \backslash c_0$, *there exists* $f \in \ell_\infty^*$ *such that*

i) $f(y) \gneqq 0$,

ii) f *is positive, that is,* $f(x) \geqslant 0$ *for all* $x \in \ell_\infty^+$,

iii) $f(e) = 1$ *where* $e = (1,1,\dots)$ *and so* $\|f\| = 1$,

iv) $f|_{c_0} = 0$, *and*

v) *for all* $x \in \ell_\infty$, $\liminf x_n \leqslant f(x) \leqslant \limsup x_n$; *in particular, for* $x \in c$, $f(x) = \lim x_n$.

In other words, y *may be strictly separated from* c_0 *by a 'normalized positive linear functional'.*

Proof Let $A = \left\{x \in \ell_\infty : \limsup x_n \leqslant 0\right\}$. We shall show that $A = c_0 - \ell_\infty^+$.

Let $x = s - t$ where $s \in c_0$, $t \in \ell_\infty^+$ and suppose $\limsup x_n \gneqq 0$.

Take $0 < \varepsilon < \frac{1}{2} \limsup x_n$,

then there exist but a finite number of terms of s greater than ε and hence only a finite number of terms of x greater than ε. This contradicts that the limit superior of x is strictly positive. So $x \in A$.

Conversely, let $x \in A$. Write $x_n = s_n + t_n$ where

$$s_n = \begin{cases} x_n & \text{if } x_n \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $(s_n) \in c_0$ and $(t_n) \in \ell_\infty^-$.

So $x \in c_0 - \ell_\infty^+$.

To prove that A is closed, let x be a limit point of A, that is,

$$\|x - x^{(m)}\| = \sup_n |x_n - x_n^{(m)}| \to 0 \quad \text{as} \quad m \to \infty, \text{ where } x^{(m)} \in A.$$

Therefore, for given $\varepsilon > 0$,

$$|x_n - x_n^{(m)}| < \varepsilon$$

for sufficiently large m and all n.

Assume $a = \lim\sup x_n > 0$.

So $(a - \varepsilon, a + \varepsilon)$ where $\varepsilon = \frac{a}{4}$ must contain an infinite number of x_n and consequently an infinite number of $x_n^{(m)}$ for sufficiently large m. This contradicts that $x^{(m)} \in A$.

Convexity being obvious, we conclude that A is a closed convex subset of ℓ_∞.

Obviously, $y \notin A$ since $y \in \ell_\infty^+ \backslash c_0$. Hence by the separation theorem, there exists $g \in \ell_\infty^*$ with

$$g(y) > 0 = \sup g(A) .$$

If $x \in c_0$, then $x, -x \in c_0 \subset A$.

So $g(-x) \leqslant 0$, or, $g(x) \geqslant 0$;
that is, g is positive on ℓ_∞^+.

Further, $\|y\|e - y \in \ell_\infty^+$.

So $g(\|y\|e - y) \geqslant 0$ and we get $g(e) \underset{\neq}{>} 0$.

Write $f = g/g(e)$.

So $f(e) = 1$ and f is positive.

Thus $f(\|x\|e - x) \geqslant 0$,

or $f(x) \leqslant \|x\|$.

Thus $\|f\| \leqslant 1$,

but since $f(e) = 1$, this gives $\|f\| = 1$.

Again $f((\lim\sup x_n)e - x) \geqslant 0$,

that is, $\lim\sup x_n \geqslant f(x)$.

Similarly, $\lim\inf x_n \leqslant f(x)$.

Hence f satisfies all the properties required in lemma 2.7.

□

2.5 A Modification of Berberian's Technique

S.K. Berberian (1962) used the existence of a Banach-Mazur generalized limit, glim, for bounded sequences of real numbers to introduce a pseudo-inner product on $\ell_\infty(H)$ and thereby obtained a Hilbert space extension K of H. In fact glim was only required to be an element of ℓ_∞^+ satisfying the properties (ii) to (v) of section 2.4. Thus for every f of the type described by lemma 2.7 we have the following construction.

An extension K of H

Suppose $s = (x_n)$ and $t = (y_n)$ belong to $\ell_\infty(H)$. Since $|<x_n, y_n>| \leq \|x_n\| \|y_n\|$, it is permissible to define

$$\phi(s,t) = f((\text{Re} <x_n, y_n>)) + if((\text{Im} <x_n, y_n>)) .$$

Evidently, $\phi(s,t)$ is a pseudo-inner product on $\ell_\infty(H)$ and so satisfies the Cauchy-Schwartz inequality, hence

$$\begin{aligned} N &= \left\{ s \in \ell_\infty(H) : \phi(s,s) = 0 \right\} \\ &= \left\{ s \in \ell_\infty(H) : \phi(s,t) = 0 \quad \text{for all} \quad t \in \ell_\infty(H) \right\} \end{aligned}$$

is a closed (can be easily verified from the properties of f) subspace of $\ell_\infty(H)$.

We write s' for the coset $s + N$ and define the quotient inner product space

$$K = \ell_\infty(H)/N$$

with inner product

$$<s', t'> = \phi(s,t).$$

If x is in H, we write (x) for the sequence all of whose terms are x and x' for the coset $(x) + \mathcal{N}$. Hence $\langle x', y' \rangle = \langle x, y \rangle$ and $x \to x'$ is an isometric linear map of H onto a closed subspace H' of K.

A representation of B(H)

Every operator T in H determines an operator T° in K as follows.

Since $\|Tx_n\| \leq \|T\| \; \|x_n\|$,

if $(x_n) \in \ell_{\infty}(H)$, so is (Tx_n).
Define the linear map $T_{\circ} : \ell_{\infty}(H) \to \ell_{\infty}(H)$ by $T_{\circ}s = (Tx_n)$.
Hence, by positivity of ϕ we have

$$\phi(T_{\circ}s, T_{\circ}s) \leq \|T\|^2 \phi(s,s) \; . \qquad \ldots(2.1)$$

This shows that if $s \in \mathcal{N}$, that is $\phi(s,s) = 0$, then

$$\phi(T_{\circ}s, T_{\circ}s) = 0$$

and hence

$$T_{\circ}s \in \mathcal{N} \; .$$

Thus the linear map $T^{\circ} : K \to K$ defined by $T^{\circ}s' = (T_{\circ}s)'$ is well defined and since from (2.1),

$$\langle T^{\circ}s', T^{\circ}s' \rangle \leq \|T\|^2 \langle s', s' \rangle \quad \text{for all} \quad s' \in K \; ,$$

T° is continuous and $\|T^\circ\| \leq \|T\|$.
But $T^\circ x' = (Tx)'$ for all $x \in H$ and hence $\|T^\circ\| \geq \|T\|$.
Thus we have $\|T^\circ\| = \|T\|$.

It can be easily verified that the mapping $T \to T^\circ$ is a faithful *-representation of $B(H)$ into $B(K)$, that is for $S,T \in B(H)$,

i) $(S + T)^\circ = S^\circ + T^\circ$

ii) $(\lambda T)^\circ = \lambda T^\circ$

iii) $(ST)^\circ = S^\circ T^\circ$

iv) $(T^*)^\circ = (T^\circ)^*$

v) $I^\circ = I$, and

vi) $\|T^\circ\| = \|T\|$.

Also it is easily seen that T is positive if and only if T° is positive.

Berberian and Orland (1967) have shown in the proposition of section 3 of their paper that $W(T^\circ) = W(T)^-$. This fact is basic to our proofs. We give below a simple proof of this result, which, unlike the proof given by Berberian and Orland, needs no reference to Banach algebra; and instead makes use of a normalized positive linear functional f with the properties given in lemma 2.7. This proof was suggested to the author by B. Sims.

Theorem 2.8 *For any operator T in H, $W(T^\circ)$ is closed; indeed, $W(T^\circ) = W(T)^-$.*

Proof The inclusion $W(T)^- \subset W(T^\circ)$ can be shown as follows.

Let $\lambda = \lim\langle Tx_n, x_n\rangle$ where $(x_n) \in \ell_\infty(H)$, $\|x_n\| = 1$. Writing $s = (x_n)$ and $s' = s + N$ as before, we have $\|s'\| = 1$ and

$$\langle T^\circ s', s'\rangle = f((\mathrm{Re}\langle Tx_n, x_n\rangle)) + if((\mathrm{Im}\langle Tx_n, x_n\rangle))$$

(where f is as described in lemma 2.7)

$= \lim\langle Tx_n, x_n\rangle = \lambda$.

For the converse we show $\lambda \notin W(T)^-$ implies $\lambda \notin W(T^\circ)$.

If $\lambda \notin W(T)^-$, there exists a half-plane U such that $\lambda \notin U$ and $W(T)^- \subset U$. Thus by carrying out the standard transformation $T \to \alpha T + \beta$ with suitably chosen complex α, β, without loss of generality we may assume $\lambda = 0$ and $\mathrm{Re}\, W(T)^- < 0$. It will be sufficient to show that

$$\sup \mathrm{Re}\, W(T^\circ) \leqslant \sup \mathrm{Re}\, W(T) .$$

Let $\mu \in W(T^\circ)$.

Then

$$\text{Re } \mu = f((\text{Re}\langle Tx_n, x_n\rangle)) \quad \text{for some } (x_n) \text{ with } f((\|x_n\|^2)) = 1,$$

$$= f((\text{Re } \mu_n \ \|x_n\|^2))$$

where $\mu_n \in W(T)$.

(If $x_n = 0$ for some n, we put μ_n equal to any point of $W(T)$.)

Thus

$$\text{Re } \mu \leq f((\|x_n\|^2 \sup \text{Re } W(T)))$$

by positivity of f,

or, $\quad \text{Re } \mu \leq \sup \text{Re } W(T)$.

□

2.6 Linearity of $N_z(T)$

We are now ready to give an alternative proof of

Theorem 2.6 *For any point z in $W(T)^-$, let*

$$N_z(T) = \left\{(x_n) \in \ell_\infty(H) : \langle Tx_n, x_n\rangle - z\|x_n\|^2 \to 0\right\}.$$

Then $N_z(T)$ is a subspace of $\ell_\infty(H)$ if and only if z is an extreme point of $W(T)^-$.

Proof By carrying out the standard reduction $T \to e^{i\theta}(T - zI)$ where θ is a suitably chosen real number, we can assume without loss of generality that $z = 0$ and $\text{Re } W(T) \geqslant 0$.

We first prove sufficiency. Homogeneity being clear, we need prove only linearity of $N_z(T)$.

By the construction of section 2.5, for each f of the type described in lemma 2.7 we have

$$W(T^\circ) = W(T)^- .$$

Indeed if $\langle Tx_n, x_n\rangle \to 0$, then $0 = \langle T^\circ s', s'\rangle$ where $s' = s + \mathcal{N}$, $s = (x_n)$.

Now let (x_n), (y_n) be such that both $\langle Tx_n, x_n\rangle$ and $\langle Ty_n, y_n\rangle$ tend to zero where 0 is an extreme point of $W(T)^-$.
Then $\langle T^\circ s', s'\rangle = \langle T^\circ t', t'\rangle = 0$ is an extreme point of $W(T^\circ)$. So by theorem 1.11 (i),

$$\langle T^\circ(s' + t'), s' + t'\rangle = 0,$$

or

$$\langle T(x_n + y_n)', (x_n + y_n)'\rangle = 0 .$$

Thus by the form of the inner product in K, for <u>every</u> possible choice of f we have

$$f((\operatorname{Re} \langle T(x_n + y_n), x_n + y_n\rangle)) = 0 \qquad \ldots(2.2)$$

and

$$f((\operatorname{Im} \langle T(x_n + y_n), x_n + y_n\rangle)) = 0 \ . \qquad \ldots(2.3)$$

Now $\alpha = (\alpha_n) = (\operatorname{Re} \langle T(x_n + y_n), x_n + y_n\rangle) \in \ell^+_\infty$ and so by (2.2) and lemma 2.7, $\alpha \in c_o$, that is, $\alpha_n \to 0$.

To show $\beta = (\beta_n) = (\operatorname{Im} \langle T(x_n + y_n), x_n + y_n\rangle) \in c_o$ requires a little more work.

First note that

$$\liminf \beta_n \leq f(\beta) \leq \limsup \beta_n \ .$$

Also, by (2.3), $f(\beta) = 0$.

Assume $a = \limsup \beta_n \gneq 0$, then there exists a subsequence (n_k) such that

$$\operatorname{Im} \langle T(x_{n_k} + y_{n_k}), x_{n_k} + y_{n_k}\rangle \to a \ .$$

Passing on to a further subsequence we may assume

$$\|x_{n_k} + y_{n_k}\| \to L \neq 0 .$$

(If $L = 0$, then $\beta_{n_k} \to 0$ contradicting $a > 0$.)

Thus

$$\operatorname{Im}\left\langle \frac{T(x_{n_k} + y_{n_k})}{\|x_{n_k} + y_{n_k}\|} , \frac{x_{n_k} + y_{n_k}}{\|x_{n_k} + y_{n_k}\|} \right\rangle \to \frac{a}{L^2}$$

while

$$\operatorname{Re}\left\langle \frac{T(x_{n_k} + y_{n_k})}{\|x_{n_k} + y_{n_k}\|} , \frac{x_{n_k} + y_{n_k}}{\|x_{n_k} + y_{n_k}\|} \right\rangle \to 0 .$$

So $ia/L^2 \in W(T)^-$.

If also $b = \liminf \beta_n \lneq 0$, we would similarly have $ib/\ell^2 \in W(T)^-$ where $\ell \neq 0$ is the limit of the norm of a suitable subsequence of $(x_n + y_n)$.

Thus $b/\ell^2 \lneq 0 \lneq a/L^2$

contradicting that 0 is an extreme point of $W(T)^-$.

Thus at least one of a and b is zero. Now β can be decomposed as

$$\beta = \beta^\circ + (\beta - \beta^\circ)$$

where

$$\beta_n^{\circ} = \begin{cases} \beta_n - a, & \text{if } \beta_n \geq a; \\ 0 & \text{otherwise.} \end{cases}$$

So $\beta^{\circ} \in c_o$ and $ae - (\beta - \beta^{\circ}) \in \ell_\infty^+$

If $a = 0$, $\beta - \beta^{\circ} \in \ell_\infty^-$

and similarly

if $b = 0$, $\beta - \beta^{\circ} \in \ell_\infty^+$.

But then for all f satisfying the conditions of lemma 2.7 we have

$$0 = f(\beta) = f(\beta - \beta^{\circ})$$

and so $\beta - \beta^{\circ} \in c_o$.

Thus $\beta \in c_o$ and consequently $N_o(T)$ is linear.

To prove the converse, if 0 is not an extreme point of $W(T)^-$, then either 0 is an interior point of $W(T)$ and theorem 1.1 (i) shows that $M_o(T)$ and hence $N_o(T)$ is not linear; or 0 is a nonextreme boundary point of $W(T)^-$ in which case we may assume that 0 lies on the join of ia and $-ib$ where ia and $-ib$ belong to $W(T)^-$, $a, b > 0$. We will show that $N_o(T)$ is not linear.

Let $s = (x_n)$ and $t = (y_n)$ be two sequences of unit vectors such that

$$<Tx_n, x_n> \to ia \quad \text{and} \quad <Ty_n, y_n> \to -ib \ .$$

Then since $<(T + T^*)x_n, x_n> \to 0$, an extreme point of $W(T + T^*)^-$ and so an approximate eigenvalue of the Hermitian operator $T + T^*$, we have $Tx_n + T^*x_n \to 0$.

Further by passing on to subsequences if necessary, we may assume that for any λ, $\text{Im}(\bar{\lambda}<Tx_n, y_n>)$ is convergent and hence it follows that $(<T(x_n + \lambda y_n), x_n + \lambda y_n>)$ is convergent.

Now, given any f satisfying the conditions of lemma 2.7, we have

$$<T^\circ s', s'> = ia \quad \text{and} \quad <T^\circ t', t'> = -ib$$

and so by lemma 1.10, we have

$$<T^\circ (x_n + \lambda y_n)', (x_n + \lambda y_n)'> = 0$$

for two distinct values of λ.

By (v) in lemma 2.7 and the construction of K, $T°$, we therefore have for both these values of λ that

$$\lim \langle T(x_n + \lambda y_n), x_n + \lambda y_n \rangle$$

$$= f((\mathrm{Re} \langle T(x_n + \lambda y_n), x_n + \lambda y_n \rangle)) + if((\mathrm{Im} \langle T(x_n + \lambda y_n), x_n + \lambda y_n \rangle))$$

$$= 0 ,$$

that is, $(x_n + \lambda y_n) \in N_o(T)$ for two distinct values of λ.

Hence $N_c(T)$ is not linear.

□

2.7 Generalization of a Cauchy-Schwartz Inequality

In theorem 1.12 we have seen a version of the Cauchy-Schwartz inequality for the vectors associated with points of $L \cap W(T)$, where L is a line of support for $W(T)$. We translate this into a statement about sequences of vectors associated with points of $L \cap W(T)^-$. We then illustrate how other results may be extended to unattained boundary points of $W(T)$ by deriving generalizations for some of the consequences given in section 1.4 of Chapter 1. In particular, the results of Das and Craven can be deduced as a corollary to a generalization of a Cauchy-Schwartz inequality.

Throughout let L be a line of support for W(T) and let

$$N_L(T) = \left\{ (x_n) \in \ell_\infty(H) \;:\; \inf_{z \in L} |\langle Tx_n, x_n\rangle - z\|x_n\|^2| \to 0 \right\} .$$

Lemma 2.5 (ii) shows that $N_L(T)$ is a subspace of $\ell_\infty(H)$.

Let f satisfy the conditions of lemma 2.7. For any complex sequence (λ_n), define $f((\lambda_n))$ by

$$f((\lambda_n)) = f((\text{Re } \lambda_n)) + if((\text{Im } \lambda_n)) .$$

We have the following lemma.

Lemma 2.9 *Let f be as above and z be a point of L such that either z is an extreme point of* $W(T)^-$ *or* $z \notin W(T)^-$. *Then for all* $(x_n), (y_n) \in N_L(T)$,

$$|f((\langle (T-z)x_n, y_n\rangle))|^2 \leq f((\langle (T-z)x_n, x_n\rangle))f((\langle y_n, (T-z)y_n\rangle)) .$$

Proof By a suitable translation and rotation we may assume that L is the imaginary axis, z = 0 and Re W(T) $\geq$ 0.

For the given f, let K and T° be as in section 2.5 and let $s = (x_n)$, $t = (y_n)$, then

$$\text{Re } \langle T^\circ s', s'\rangle = f((\text{Re } \langle Tx_n, x_n\rangle)) = 0 \text{ as Re } Tx_n \to 0 .$$

Similarly $\text{Re}\ \langle T^\circ t', t'\rangle = 0$.

Theorem 1.12 therefore applies to give

$$|\langle T^\circ s', t'\rangle|^2 \leq \langle T^\circ s', s'\rangle \langle t', T^\circ t'\rangle ,$$

or, using the definition of inner product in K, that

$$|f((\langle Tx_n, y_n\rangle))|^2 \leq f((\langle Tx_n, x_n\rangle))\ f((\langle y_n, Ty_n\rangle))$$

as required.

□

Corollary 2.10 *If z is an extreme point of $W(T)^-$ and L is a line of support for $W(T)$ passing through z, then*

$$\lim \langle (T - z)x_n, y_n\rangle = 0$$

and

$$\lim \langle (T^* - \bar{z})x_n, y_n\rangle = 0$$

for all $(x_n) \in N_z(T)$ and $(y_n) \in N_L(T)$.

Proof Without loss of generality assume $z = 0$, L is the imaginary axis and $\text{Re}\ W(T) \geq 0$.

Assume $\langle Tx_n, y_n\rangle$ does not converge to 0, then there exist subsequences (x_{n_k}), (Y_{n_k}) such that either the real or imaginary parts of $(\langle Tx_{n_k}, y_{n_k}\rangle)$ form a subsequence in $\ell_\infty^+ \backslash c_0$ (or $\ell_\infty^- \backslash c_0$).

By lemma 2.7, there is an f with the stated properties such that $f((\langle Tx_{n_k}, y_{n_k}\rangle)) \neq 0$.

To derive a contradiction we note that $(x_{n_k}) \in N_z(T)$.

So $\langle Tx_{n_k}, x_{n_k}\rangle \to 0$

and

$$(y_{n_k}) \in N_L(T) .$$

Thus by lemma 2.9,

$$f((\langle Tx_{n_k}, y_{n_k}\rangle)) = 0.$$

So we have

$$\lim \langle (T - z)x_n, y_n\rangle = 0$$

whenever z is an extreme point of $W(T)^-$ and by lemma 2.5 (i) we also have

$$\lim \langle (T^* - \bar{z})x_n, y_n\rangle = 0.$$

□

Corollary 2.11 *Let z and L be as in corollary 2.10. If $(x_n) \in N_z(T)$ and $(Tx_n) \in N_L(T)$, then*

$$\lim (T - z)x_n = \lim (T^* - \bar{z})x_n = 0.$$

Proof Again assume $z = 0$, L is the imaginary axis and $\mathrm{Re}\, W(T) \geq 0$.

Since $(x_n) \in N_z(T)$, by definition $\langle Tx_n, x_n \rangle \to 0$ and so by lemma 2.9,

$$f((\langle Tx_n, y_n \rangle)) = 0 \quad \text{for all} \quad (y_n) \in N_L(T) .$$

In particular, taking $y_n = Tx_n$ we have

$$f((\|Tx_n\|^2)) = 0 .$$

Now $(\|Tx_n\|^2)$ is in ℓ_∞^+, so by lemma 2.7 we conclude that $Tx_n \to 0$ and since $\mathrm{Re}\, Tx_n \to 0$, that $T^*x_n \to 0$.

□

Corollary 2.12 (Das and Craven) If z is an extreme point of $W(T)^-$, then $N_z(T)$ is a subspace of $\ell_\infty(H)$.

Proof Homogeneity being obvious we only have to prove linearity.

Let $(x_n^{(1)}), (x_n^{(2)}) \in N_z(T)$.
Thus $(x_n^{(1)}), (x_n^{(2)}) \in N_L(T)$ where L is a line of support for $W(T)$ passing through z.

But $N_L(T)$ is a subspace by lemma 2.5 (ii). So $(x_n^{(1)} + x_n^{(2)}) \in N_L(T)$.

Now since $(x_n^{(i)}) \in N_z(T)$, $i = 1,2$ and $(x_n^{(1)} + x_n^{(2)}) \in N_L(T)$, by corollary 2.10,

$$\lim <(T - z)x_n^{(i)}, x_n^{(1)} + x_n^{(2)}> = 0 \text{ for } i = 1,2.$$

Hence

$$\lim <(T-z)(x_n^{(1)} + x_n^{(2)}), x_n^{(1)} + x_n^{(2)}> = 0.$$

So $(x_n^{(1)} + x_n^{(2)}) \in N_z(T)$.

□

Let f be any linear functional satisfying the conditions of lemma 2.7. As before, for any complex sequence (λ_n), define $f((\lambda_n))$ by

$$f((\lambda_n)) = f((\text{Re } \lambda_n)) + i\, f((\text{Im } \lambda_n)) .$$

We have the following lemma.

Lemma 2.13 *Let b and c be adjacent extreme points of $W(T)^-$ and let $a = tb + (1-t)c$, $0 \leq t \leq 1$. If $(x_n) \in N_b(T)$ and $(y_n) \in N_a(T)$, then for all f of the type described above,*

$$|f((\langle x_n, y_n \rangle))|^2 \leq t\, f((\|x_n\|^2))\, f((\|y_n\|^2)).$$

In particular, if $(x_n) \in N_b(T)$ and $(y_n) \in N_c(T)$, then $\lim \langle x_n, y_n \rangle = 0$.

Proof For the given f, let K and $T°$ be as in section 2.5 and let $s = (x_n)$, $t = (y_n)$, then an easy application of theorem 1.16 gives

$$|\langle s', t' \rangle| \leq \sqrt{t}\, \|s'\|\, \|t'\| \quad ,$$

or, in terms of f,

$$|f((\langle x_n, y_n \rangle))|^2 \leq t\, f((\|x_n\|^2))\, f((\|y_n\|^2)).$$

In particular, if $(y_n) \in N_c(T)$, then $t = 0$ and hence $f((\langle x_n, y_n \rangle)) = 0$.

If $\langle x_n, y_n \rangle$ does not converge to zero, then there exist subsequences (x_{n_k}), (y_{n_k}) such that either the real or imaginary parts of $(\langle x_{n_k}, y_{n_k} \rangle)$ form a subsequence in $\ell_\infty^+ \setminus c_0$ (or $\ell_\infty^- \setminus c_0$).

By lemma 2.7, there is an f with the stated properties such that

$$f((\langle x_{n_k}, y_{n_k} \rangle)) \neq 0 .$$

But $(x_{n_k}) \in N_b(T)$, $(y_{n_k}) \in N_c(T)$.

Hence

$$f((\langle x_{n_k}, y_{n_k} \rangle)) = 0$$

and we get a contradiction.

Therefore $\langle x_n, y_n \rangle \to 0$.

□

In lemmas 2.9 and 2.13 we have obtained inequalities in terms of f. As we shall see in Chapter 4, these inequalities are sufficient to enable us to deduce as a corollary, results of Garske (1979) and Das and Craven on weak convergence on the boundary of the numerical range. However, in the next section we use a direct method to get inequalities for the elements of $N(T)$ in terms of limit supremum. Property (v) of lemma 2.7 shows that these inequalities are sharper than those obtained in this section. The contents of the next section cover part of a joint paper by Das, Majumdar and Sims (1).

2.8 Inequalities for N(T) in Terms of Limit Supremum

In theorem 1.12 we have seen a sharper version of the Cauchy-Schwartz inequality for the vectors associated with the points of $L \cap W(T)$. In the last section, using a modification of Berberian's technique which involves a change of Hilbert space and operator via a construction based on normalized positive linear functionals in ℓ_∞^*, we have extended theorems 1.12 and 1.16 to the case of vectors associated with the points of $L \cap W(T)^-$. Here we shall not use this technique; instead we exploit the notions of limit supremum and limit infimum to obtain somewhat sharper inequalities.

We prove the following theorem.

Theorem 2.14 *Let L be a line of support for $W(T)$ and*

$$N(T) = \left\{ (x_n) \in \ell_\infty(H) : \langle Tx_n, x_n \rangle - z\|x_n\|^2 \to 0, \quad z \in L \cap W(T)^- \right\} .$$

Let z be an element of L such that either z is an extreme point of $W(T)^-$ or $z \notin W(T)^-$. Then for all (x_n), $(y_n) \in N(T)$,

$$\limsup [|\langle (T-z)x_n, y_n \rangle|^2 - |\langle (T-z)x_n, x_n \rangle| \; |\langle (T-z)y_n, y_n \rangle|] \leq 0 .$$

Proof Let either z be an extreme point of $W(T)^-$ or $z \notin W(T)^-$.

Without loss of generality we can take $z = 0$, $W(T)^- \cap L$ on the positive real axis and $\operatorname{Im} W(T) \geq 0$ (or ≤ 0). We may assume $\|x_n\|$ and $\|y_n\|$ are nonzero for all n, because if zero, they will not alter the inequality.

Let t_1, t_2 be two positive real numbers such that

$$\langle Tx_n, x_n\rangle - t_1 \|x_n\|^2 \to 0$$

and

$$\langle Ty_n, y_n\rangle - t_2 \|y_n\|^2 \to 0.$$

Consider points of $W(T)$ of the form

$$g_n(\lambda_n) = \frac{\langle T(x_n + \lambda_n y_n), x_n + \lambda_n y_n\rangle}{\|x_n + \lambda_n y_n\|^2}$$

where λ_n's are complex scalars such that $|\lambda_n| = n$ for all n.

We have assumed $x_n + \lambda_n y_n \neq 0$ for all n, because if $x_n + \lambda_n y_n = 0$ for some n, it will not change the inequality.

Since $\operatorname{Im} W(T) \geq 0$, we have $Tx_n - T^* x_n \to 0$. So

$$g_n(\lambda_n) - h_n(\lambda_n) \to 0$$

where

$$h_n(\lambda_n) = \frac{t_1\|x_n\|^2 + t_2 n^2\|y_n\|^2 + 2\mathrm{Re}(\bar{\lambda}_n \langle Tx_n, y_n\rangle)}{\|x_n + \lambda_n y_n\|^2}$$

Hence

$$\mathrm{Im}\ g_n(\lambda_n) \to 0$$

and

$$\mathrm{Re}\ g_n(\lambda_n) - h_n(\lambda_n) \to 0 .$$

Thus for any $\varepsilon > 0$,

$$-\varepsilon + \mathrm{Re}\ g_n(\lambda_n) \leq h_n(\lambda_n) \leq \varepsilon + \mathrm{Re}\ g_n(\lambda_n)$$

for sufficiently large n, or,

$$-\varepsilon + \liminf \mathrm{Re}\ g_n(\lambda_n) \leq \liminf h_n(\lambda_n) \leq \varepsilon + \liminf \mathrm{Re}\ g_n(\lambda_n) .$$

If $\liminf \mathrm{Re}\ g_n(\lambda_n) = a < 0$, then there exists (n_k) such that

$$\mathrm{Re}\ g_{n_k}(\lambda_{n_k}) \to a$$

and hence

$$g_{n_k}(\lambda_{n_k}) \to a \quad \text{as} \quad \operatorname{Im} g_{n_k}(\lambda_{n_k}) \to 0 \ .$$

So $a \in W(T)^-$ and thus $a \geq 0$ since $L \cap W(T)^- \subset R^+$. So,

$$-\varepsilon + a \leq \liminf h_n(\lambda_n) \leq \varepsilon + a$$

where $a \geq 0$.

This shows $\liminf h_n(\lambda_n) \geq 0$.

Moreover, since $(x_n + \lambda_n y_n) \in \mathcal{L}_\alpha(H)$, we must have

$$\liminf [t_1 \|x_n\|^2 + t_2 \eta^2 \|y_n\|^2 + 2\operatorname{Re}(\bar{\lambda}_n \langle Tx_n, y_n\rangle)] \geq 0 \ .$$

Choose λ_n such that

$$\operatorname{Re}(\bar{\lambda}_n \langle Tx_n, y_n\rangle) = \pm\eta |\langle Tx_n, y_n\rangle| \ .$$

Given $\varepsilon > 0$, we have for sufficiently large n,

$$t_1 \|x_n\|^2 + t_2 \eta^2 \|y_n\|^2 \pm 2\eta |\langle Tx_n, y_n\rangle| \geq \varepsilon \ .$$

So, by the condition for the above inequality to have solutions,

$$4|\langle Tx_n, y_n\rangle|^2 - 4t_2 \|y_n\|^2 (t_1 \|x_n\|^2 + \varepsilon) \leq 0.$$

Hence

$$\limsup [\,|\langle Tx_n, y_n\rangle|^2 - |\langle Tx_n, x_n\rangle|\,|\langle Ty_n, y_n\rangle|\,] \leq 0 .$$

□

A somewhat similar argument [see K.C. Das S. Majumdar and Brailey Sims (1)] yields the corresponding result for $N_L(T)$, from which the result of Das and Craven for an extreme point of $W(T)^-$ can be deduced as a corollary.

Corollary 2.15 *If $z \in L$ is an extreme point of $W(T)^-$, then*

$$\lim \langle (T - z)x_n, y_n\rangle = 0$$

and

$$\lim \langle (T^* - \bar{z})x_n, y_n\rangle = 0$$

where $(x_n) \in N_z(T)$ and $(y_n) \in N(T)$.

Proof By theorem 2.14, obviously $\langle (T - z)x_n, y_n\rangle \to 0$ and since $(y_n) \in N(T) \subset N_L(T)$, lemma 2.5 (i) gives $\langle (T^* - \bar{z})x_n, y_n\rangle \to 0$.

□

Theorem 2.16 *Let b and c be adjacent extreme points of $W(T)^-$ and let $a = tb + (1 - t)c$, $0 \leq t \leq 1$. If $(x_n) \in N_b(T)$ and $(y_n) \in N_a(T)$, then*

$$\limsup [\,|\langle x_n, y_n\rangle| - \sqrt{t}\,\|x_n\|\,\|y_n\|\,] \leq 0.$$

Proof Without loss of generality we may take $b = 0$, $c = 1$ and $W(T)^{-} \cap L$ on positive real axis.

Let λ_n's be complex scalars such that $|\lambda_n| = n$ for all n.

If $x_n + \lambda_n y_n = 0$ for some n, we have

$$|\langle x_n, y_n \rangle| - \sqrt{t}\,\|x_n\|\;\|y_n\| = 0 \;.$$

So let us assume that $x_n + \lambda_n y_n \neq 0$ for any n. Consider

$$g_n(\lambda_n) = \frac{\langle T(x_n + \lambda_n y_n),\; x_n + \lambda_n y_n \rangle}{\|x_n + \lambda_n y_n\|^2} \;.$$

Since $\langle Tx_n, y_n \rangle \to 0$ by corollary 2.15, we have

$$g_n(\lambda_n) - h_n(\lambda_n) \to 0$$

where

$$h_n(\lambda_n) = \frac{n^2(1 - t)\,\|y_n\|^2}{\|x_n + \lambda_n y_n\|^2} \;.$$

Hence, by our assumption, $\limsup h_n(\lambda_n) \leq 1$.

Thus for any $\varepsilon > 0$ and large n,

$$\eta^2(1-t)\|y_n\|^2 \leqslant (1+\varepsilon)[\|x_n\|^2 + \eta^2\|y_n\|^2 + 2\mathrm{Re}(\bar{\lambda}_n\langle x_n,y_n\rangle)] \ .$$

λ_n can be so chosen that

$$\mathrm{Re}(\bar{\lambda}_n \langle x_n,y_n\rangle) = \pm\eta|\langle x_n, y_n\rangle| \ .$$

Hence

$$t\eta^2\|y_n\|^2 \pm 2\eta|\langle x_n,y_n\rangle| + \|x_n\|^2 \geqslant -\varepsilon\|x_n + \lambda_n y_n\|^2 \geqslant -M\varepsilon$$

where $\sqrt{M}$ is an upper bound for $\|x_n\| + \eta\|y_n\|$. So

$$|\langle x_n,y_n\rangle|^2 - t\|x_n\|^2\|y_n\|^2 \leqslant M\varepsilon t\|y_n\|^2 \leqslant Mmt\varepsilon$$

where m is an upper bound for $\|y_n\|^2$. Hence

$$\limsup[|\langle x_n,y_n\rangle|^2 - t\|x_n\|^2\|y_n\|^2] \leqslant 0 \ .$$

□

It is worth noting that since for each $z \in L$, a line of support for $W(T)$,

$$N_L(T) = \left\{ (x_n) \in \ell_\infty(H) : e^{i\theta}(T-z)x_n - e^{-i\theta}(T^*-\bar{z})x_n \to 0 \right\}$$

where θ is the acute angle between L and the imaginary axis, $(Tx_n) \in N_L(T)$ if and only if $(T^*x_n) \in N_L(T)$ for any operator T. Furthermore, if (x_n) is a non-null sequence of $N_L(T)$ and $Tx_n - zx_n \to 0$, then necessarily $z \in L$ and $T^*x_n - \bar{z}x_n \to 0$.

Thus if (x_n) is a bounded sequence of approximate eigenvectors associated with the boundary of $W(T)^-$ and (y_n) is a bounded sequence of approximate eigenvectors for some other approximate eigenvalue, then $\langle x_n, y_n \rangle \to 0$. This may be compared with the similar results for eigenvalues (see, for example, Embry (1975)).

For convexoid operators, that is, the operators for which $W(T)^-$ is the convex hull of the spectrum, any extreme point of $W(T)^-$ is an approximate eigenvalue, and so this will hold for all extreme points of $W(T)^-$. Theorem 2.16 shows that $\langle x_n, y_n \rangle \to 0$ whenever $(x_n) \in N_b(T)$, $(y_n) \in N_c(T)$ where b and c are adjacent extreme points of $W(T)^-$.

The following generalization of theorem 1.17 is true for two non-parallel lines of support of $W(T)$.

Theorem 2.17 *Let L_1 and L_2 be two non-parallel lines of support of $W(T)$, $L_1 \cap L_2 = \{c\}$ and*

$$N_j(T) = \left\{ (x_n) \in \ell_\infty(H) : \langle Tx_n, x_n\rangle - z\|x_n\|^2 \to 0, \quad z \in L_j \cap W(T)^- \right\},$$
$$j = 1,2 .$$

Then $\langle (T-c)x_n^{(1)}, x_n^{(2)}\rangle \to 0$ whenever $(x_n^{(j)}) \in N_j(T)$.

Proof Let θ_j be the acute angle between L_j and the imaginary axis.

Let $(x_n^{(j)}) \in N_j(T)$, $j = 1,2$.

Then since $N_j(T) \subseteq N_{L_j}(T)$, $j = 1,2$ and by lemma 2.5 (i),

$$N_{L_j}(T) = \left\{ (x_n) \in \ell_\infty(H) : e^{i\theta_j}(T-c)x_n - e^{-i\theta_j}(T^*-\bar{c})x_n \to 0 \right\},$$
$$j = 1,2 ,$$

we have

$$e^{i\theta_j}(T-c)x_n^{(j)} - e^{-i\theta_j}(T^*-\bar{c})x_n^{(j)} \to 0, \; j = 1,2 .$$

A simple manipulation shows that

$$e^{2i\theta_1}\langle (T-c)x_n^{(1)}, x_n^{(2)}\rangle - e^{2i\theta_2}\langle (T-c)x_n^{(1)}, x_n^{(2)}\rangle \to 0 .$$

Since L_1, L_2 are non-parallel, $e^{2i\theta_1} \neq e^{2i\theta_2}$ and hence

$$<(T-c)x_n^{(1)}, x_n^{(2)}> \to 0 .$$

□

In this section we have seen how the orthogonal tendency of vectors can be derived from Cauchy-Schwartz type inequalities. We have also mentioned that the result of Das and Craven can be deduced as a corollary to a similar inequality for elements of $N_L(T)$. This result is based on the case when z is an extreme point of $W(T)^-$. The cases when z is a non-extreme boundary point or an interior point of $W(T)^-$ will be discussed in the next section. The contents of the next section have been used in a joint paper by Das, Majumdar and Sims (2).

2.9 Characterization of $W(T)^-$

Theorem 1.11 of Chapter 1 characterizes every point of $W(T)$ as either an extreme point or a nonextreme boundary point or an interior point in terms of the subset $M_z(T)$ and its linear span $\gamma M_z(T)$ where

$$M_z(T) = \left\{x \in H : <Tx, x> - z\|x\|^2 = 0\right\} .$$

This theorem, though very interesting, cannot characterize the unattained boundary points of the numerical range.

In this section we attempt to fill this gap by achieving a generalization of these results which can be applied to every point of $W(T)^-$. In section 2.3 we have seen that the corresponding result to theorem 1.11 (i) holds for $N_z(T)$ when z is an extreme point of $W(T)^-$. In section 2.6 we proved the same result from another approach involving Berberian's technique. The cases when z is a nonextreme boundary or an interior point of $W(T)^-$ are yet to be considered. We begin by proving the following preliminary lemma.

Lemma 2.18 *Let z be in the interior of a line segment with endpoints a and b in $W(T)^-$. Then the set $N_a'(T) \subset \gamma N_z(T)$ where*

$$N_a'(T) = \left\{ (x_n) \in \ell_\infty(H) \;:\; \langle Tx_n, x_n\rangle / \|x_n\|^2 \to a \right\} .$$

Proof Let $(x_n) \in \ell_\infty(H)$ be such that

$$\langle Tx_n, x_n\rangle / \|x_n\|^2 \to a .$$

Without loss of generality we may take $a = 1$, $b = 0$ and $\|x_n\| = 1$.

Let $(y_n) \in N_o(T)$, $\|y_n\| = 1$.

By separately rotating each y_n we may, without loss of generality, assume

$$\mathrm{Re}\, \langle \mathrm{Im}\, Tx_n, y_n\rangle = 0 .$$

Let $h_n = r_n x_n + y_n$, $r_n \in R$. Then

$$<\text{Im}\, Th_n, h_n> = r_n^2 <\text{Im}\, Tx_n, x_n> + <\text{Im}\, Ty_n, y_n> + 2r_n \text{Re} <\text{Im}\, Tx_n, y_n> \to 0$$

with our assumptions.

So $\text{Im} <Th_n, h_n> \to 0$.

For large n and any fixed $z \in (0,1)$, consider the equation

$$<\text{Re}\ Th_n, h_n> - z \|h_n\|^2 = 0 . \qquad \ldots(2.4)$$

We want to show the existence of two distinct real values of r_n such that (2.4) holds. (2.4) is equivalent to

$$r_n^2 (<\text{Re}\, Tx_n, x_n> - z) + 2r_n \text{Re} <(\text{Re}\, T - z) x_n, y_n>$$
$$+ (<\text{Re}\ Ty_n, y_n> - z) = 0 .$$

Let $\varepsilon_n = <\text{Re}\ Tx_n, x_n> - 1$ and $\varepsilon_n' = <\text{Re}\ Ty_n, y_n>$. Then ε_n, ε_n' both tend to zero as $n \to \infty$. Hence (2.4) is equivalent to

$$r_n^2 (1 - z + \varepsilon_n) + 2r_n \text{Re} <(\text{Re}\, T - z) x_n, y_n> + (\varepsilon_n' - z) = 0 .$$

This is of the form

$$A_n r_n^2 + B_n r_n + C_n = 0 .$$

Now

$$B_n^2 - 4A_n C_n$$

$$= 4[\mathrm{Re} <(\mathrm{Re}\, T - z)x_n, y_n>]^2 - 4(1 - z + \varepsilon_n)(\varepsilon_n' - z)$$

$$= 4[\mathrm{Re} <(\mathrm{Re}\, T - z)x_n, y_n>]^2 + 4z(1-z) + \delta_n(\varepsilon_n, \varepsilon_n')$$

where $\delta_n(\varepsilon_n, \varepsilon_n')$ is the sum of terms containing ε_n and ε_n'. Thus since z is a fixed constant in $(0,1)$, $\delta_n(\varepsilon_n, \varepsilon_n')$ can be made sufficiently small for large n so that $B_n^2 - 4A_nC_n > 0$.

So there exist two distinct values of r_n, say, $r_n^{(1)}$, $r_n^{(2)}$ such that

$$r_n^{(1)} - r_n^{(2)} = \frac{\sqrt{B_n^2 - 4A_nC_n}}{A_n}$$

But for sufficiently large n, $\dfrac{\sqrt{B_n^2 - 4A_nC_n}}{A_n}$ is uniformly bounded away from zero. So we have

$$(r_n^{(1)} x_n + y_n) \in N_z(T)$$

and

$$(r_n^{(2)} x_n + y_n) \in N_z(T) \quad ,$$

that is,

$$((r_n^{(1)} - r_n^{(2)})x_n) \in N_z(T) + N_z(T) = \gamma N_z(T) \;,$$

or, $(x_n) \in \gamma N_z(T)$ since $r_n^{(1)} - r_n^{(2)}$ is uniformly bounded away from zero.

□

Remark The above lemma shows an easy way to prove the convexity of $W(T)$ (theorem 1.2) as follows.

Let z lie in the interior of a line segment with endpoints $a, b \in W(T)$. Let $\langle Tx, x\rangle = a$, $\langle Ty, y\rangle = b$, $||x|| = ||y|| = 1$. We want to show there exists an $h \in H$ such that $\langle Th, h\rangle / ||h||^2 = z$.

Without loss of generality we may take $a = 1$, $b = 0$, $z \in (0,1)$ and $\text{Re}\langle \text{Im } Tx, y\rangle = 0$.

Since $a = 1$, $b = 0$, x and y are linearly independent. Let $h = x + ry$, $r \in R$.
Thus $||h|| \neq 0$ and

$$\langle \text{Im } Th, h\rangle = \langle \text{Im } Tx, x\rangle + r^2\langle \text{Im } Ty, y\rangle + 2r\,\text{Re}\langle \text{Im } Tx, y\rangle$$

$$= 0 \text{ with our assumptions.}$$

Thus $$\frac{\langle Th,\ h\rangle}{||h||^2} = \frac{\langle \text{Re } Th, h\rangle}{||h||^2} \;.$$

Consider the equation

$$\frac{\langle \mathrm{Re}\, T(x + ry),\ x + ry\rangle}{\|x + ry\|^2} = z \quad \text{where} \quad z \in (0,1), \qquad \ldots (2.5)$$

or

$$\frac{\langle \mathrm{Re}\, Tx,\ x\rangle + r^2\langle \mathrm{Re}\, Ty,\ y\rangle + 2r\ \mathrm{Re}\langle \mathrm{Re}\, Tx,\ y\rangle}{1 + r^2 + 2r\ \mathrm{Re}\langle x,\ y\rangle} = z\ ,$$

or,

$$r^2 z + 2r\ \mathrm{Re}\langle (zI - \mathrm{Re}\, T)x,\ y\rangle + z - 1 = 0\ .$$

Now since

$$[\mathrm{Re}\ \langle (zI - \mathrm{Re}\, T)x,\ y\rangle]^2 - z(z - 1) > 0\ ,$$

there exist two distinct values of r which satisfy equation (2.5) and thus prove the existence of h as required. Note that in contrast with the proof of convexity given by Halmos (1967), this method gives two values of r explicitly.

Now we are ready to prove the main theorem of this section.

Theorem 2.19 *Every element z of $W(T)^-$ can be characterized as follows.*

i) z is an extreme point of $W(T)^-$ if and only if $N_z(T)$ is a subspace.

ii) If z is a nonextreme boundary point of $W(T)^-$ and L the line of support for $W(T)^-$ passing through z, then

a) $\gamma N_z(T) = N(T) + N_z(T)$.

b) $N_L(T) = \ell_\infty(H)$ if and only if $W(T)^- \subset L$.

iii) If $W(T)^-$ is not a straight line segment, then z is an interior point of $W(T)^-$ if and only if $N'(T) \subset \gamma N_z(T)$ where

$$N'(T) = \left\{(x_n) \in \ell_\infty(H) : \langle Tx_n, x_n\rangle/\|x_n\|^2 \to a,\ a \in W(T)^-\right\}.$$

Proof i) Already proved in section 2.6.

ii)(a) We first show that $N_a(T) \subset \gamma N_z(T)$ whenever $a \in W(T)^- \cap L$.

Without loss of generality we may take L as the real axis and $\operatorname{Im} W(T) \geqslant 0$.

Let $(x_n) \in N_a(T)$ and $(y_n) \in N_b(T)$, $\|y_n\| = 1$. By multiplying (y_n) with α_n, $|\alpha_n| = 1$, if necessary, we may take $\operatorname{Re}\langle y_n, x_n\rangle = 0$.

Thus corollary 2.10 gives

$$\mathrm{Re} \langle Ty_n, x_n\rangle \to 0.$$

For each choice let table

$$r_n = \pm\sqrt{\frac{a - z}{z - b}}\ \|x_n\|$$

Since $\mathrm{Im}\langle Ty_n, y_n\rangle \to 0$ and $\mathrm{Im}\ W(T) \geqslant 0$, we have $Ty_n - T^*y_n \to 0$ and thus

$$\langle Tx_n, y_n\rangle + \langle Ty_n, x_n\rangle - 2\mathrm{Re}\langle Ty_n, x_n\rangle \to 0 .$$

Hence

$$\langle T(x_n+r_ny_n), x_n + r_ny_n\rangle - z\|x_n + r_ny_n\|^2$$

$$- [\langle Tx_n, x_n\rangle - z\|x_n\|^2 + r_n^2\langle Ty_n, y_n\rangle - zr_n^2] \to 0$$

so that we have

$$\langle T(x_n + r_ny_n), x_n + r_ny_n\rangle - z\|x_n + r_ny_n\|^2 \to 0$$

with the chosen values of r_n. This shows

$$(x_n \pm\sqrt{\frac{a - z}{z - b}}\ \|x_n\|\ y_n) \in N_z(T) ,$$

or,
$$(x_n) \in \gamma N_z(T) .$$

Thus $N_a(T) \subset \gamma N_z(T)$ for all $a \in L \cap W(T)^-$, that is, $N(T) \subset \gamma N_z(T)$. So we have

$$N_z(T) \subset N(T) \subset \gamma N_z(T) \ ,$$

or,

$$\gamma N_z(T) = N_z(T) + N_z(T) \subset N(T) + N_z(T) \subset \gamma N_z(T) + N_z(T)$$

which gives

$$N(T) + N_z(T) = \gamma N_z(T)$$

since

$$N_z(T) \subset \gamma N_z(T) \ .$$

(b) Without loss of generality we may take L as the imaginary axis. So

$$N_L(T) = \left\{ (x_n) \in \ell_\infty(H) : \text{Re}\langle Tx_n, x_n\rangle \to 0 \right\} \ .$$

Now if $W(T)^- \subset L$,

$$(x_n) \in \ell_\infty(H) \quad \text{implies} \quad \text{Re}\langle Tx_n, x_n\rangle / \|x_n\|^2 = 0$$

for all nonzero x_n.

Also if $x_n = 0$ for some n, $\langle Tx_n, x_n \rangle = 0$.

Thus $(x_n) \in N_L(T)$,

that is, $N_L(T) = \ell_\infty(H)$.

Again if $W(T)^- \not\subset L$, there exists $(x_n) \in \ell_\infty(H)$, $\|x_n\| = 1$, such that $\text{Re}\langle Tx_n, x_n \rangle \neq 0$,

or, equivalently, $(x_n) \notin N_L(T)$.

Hence $N_L(T) = \ell_\infty(H)$.

iii) If z is an interior point of $W(T)^-$, by lemma 2.18,

$$N'(T) = \left\{ (x_n) \in \ell_\infty(H) : \langle Tx_n, x_n \rangle / \|x_n\|^2 \to a,\ a \in W(T)^- \right\}$$

$$\subset \gamma N_z(T) \quad .$$

On the other hand, if z is a boundary point of $W(T)^-$, then $\gamma N_z(T) \subset N_L(T)$ since $N_L(T)$ is a subspace. But $N'(T)$ is not a subset of $N_L(T)$ as $W(T)^- \not\subset L$. Thus $N'(T)$ is not contained in $\gamma N_z(T)$.

□

In this chapter we defined the subsets $N_z(T)$ and $\gamma N_z(T)$; and the subsets $N(T)$ and $N_L(T)$ (for a line of support L of $W(T)$) associated with points of $W(T)^-$. We saw that though $N_L(T)$ and $\gamma N_z(T)$ are subspaces, $N_z(T)$ is so if and only if z is an extreme point of $W(T)^-$. Linearity of $N(T)$ we were unable to prove.

Then we gave a characterization of $W(T)^-$ in terms of these subsets and developed a modification of a useful technique given by Berberian, which enabled us not only to prove the linearity of $N_z(T)$ when z is an extreme point of $W(T)^-$, but also to achieve generalizations of Cauchy-Schwartz type inequalities given by Embry (1975). The use of limit supremum and limit infimum helped us to sharpen these inequalities for the elements of $N(T)$. Many corollaries follow from these two versions of these inequalities, for example, the existence of limits of certain sequences of vectors and the orthogonal tendency of vectors from $N_z(T)$ and $N_L(T)$.

Our next chapter will be on the numerical range of different operators. We first discuss various results that hold for points of the numerical range and then extend these results to points of $W(T)^-$. These extensions cover a part of the paper by Das, Majumdar and Sims (1).

Chapter 3

RESULTS ON NUMERICAL RANGE OF SPECIAL OPERATORS

3.1 Introduction

In this chapter we obtain various results for normal, seminormal, convexoid and other particular types of operators in terms of their numerical range.

Embry (1971) has shown that it is possible to classify some of these special operators by means of subsets associated with their numerical range. We have seen the definitions of these subsets in Chapter 1. In section 3.2 we give these theorems of Embry and then extend the results to points of the closure of the numerical range. For this we use subsets associated with the closure of the numerical range as defined in Chapter 2.

In section 3.3, as given by Stampfli (1966) and de Barra (1981), we see that if the sets associated with the numerical range are subspaces then possibly subject to some additional conditions, they are reducing for the operator. For example, in one of the theorems we need the operator to be seminormal. We also prove a theorem generalizing Lin (1975) to obtain some necessary and sufficient conditions for an extreme point of the closure of the numerical range of a convexoid operator to be an eigenvalue.

All the results in section 3.3 are then extended in section 3.4 to cover the case of unattained boundary points of W(T). Berberian's technique of Chapter 2 is again used to give a simple proof for one of these results. The same technique is used again to provide an alternative proof of the known result that a seminormal operator is convexoid.

3.2 Classification of Operators by $M_z(T)$ and $N_z(T)$

In Chapter 1 we have defined various subsets associated with different points of the numerical range. In Chapter 2, following a similar line we have defined subsets associated with different points of the closure of the numerical range and noticed that properties of these two types of subsets are very similar. It seems natural to ask whether these subsets behave in a particular fashion if the operator T has special characteristics or vice-versa. In this section, as shown by Embry (1971), we prove that in many cases the type of operator and behaviour of $M_z(T)$ are related. We then extend these results for elements of $N_z(T)$.

We begin with the following definitions.

Definition 3.1 The operator T is *normal* if $TT^* = T^*T$ and *hyponormal* if $T^*T - TT^*$ is positive. T is *seminormal* if either T or T^* is hyponormal. Also following Embry, T is called an *isometry* if $T^*T = I$ and *unitary* if $T^*T = TT^* = I$.

Let ker T denote the kernel or null space of T. The results and proofs of this section are essentially due to Embry (1971).

Lemma 3.2 *If f, g, h and k are bilinear functionals on H, then the condition*

$$f(x,x)\ g(x,x) = h(x,x)\ k(x,x) \quad \text{for all } x \in H \qquad \ldots(3.1)$$

is equivalent to

$$f(x,y)\ g(x,y) = h(x,y)\ k(x,y) \quad \text{for all } x \text{ and } y \text{ in } H. \qquad \ldots(3.2)$$

Proof *(outline)* Let $x, y \in H$ and λ be an arbitrary complex scalar. Substitute $x + \lambda y$ for x in equation (3.1) and equate coefficients of $\bar{\lambda}^2$ to obtain equation (3.2). The converse is obvious.

□

Theorem 3.3 *T is a scalar multiple of an isometry if and only if for each complex z,*

$$\left\{Tx : x \in M_z(T)\right\} \subset M_z(T) .$$

Proof Equivalently we need prove that for all $x \in H$,

$$<T^2x,\ Tx>||x||^2 = <Tx,\ x>||Tx||^2 \qquad \ldots(3.3)$$

whenever T is a scalar multiple of an isometry and vice-versa.

Suppose equation (3.3) is true for all $x \in H$. Thus by lemma 3.2,

$$\langle T^2x, Ty\rangle\langle x, y\rangle = \langle Tx, y\rangle\langle Tx, Ty\rangle \quad \text{for all } x, y \in H. \qquad \ldots(3.4)$$

Thus

$$\{x\}^{\perp} \subset \{Tx\}^{\perp} \cup \{T^*Tx\}^{\perp}$$

and interchanging x and y in (3.4), we have

$$\{x\}^{\perp} \subset \{T^*x\}^{\perp} \cup \{T^*Tx\}^{\perp} .$$

Since $\{y\}^{\perp}$ is a subspace, we get

$$\{x\}^{\perp} \subset \{T^*Tx\}^{\perp} \quad \text{or} \quad \{x\}^{\perp} \subset \{Tx\}^{\perp} \cap \{T^*x\}^{\perp} .$$

Both cases show the existence of a scalar r_x such that

$$T^*Tx = r_x x .$$

It now follows by standard arguments that T is a scalar multiple of an isometry (see, for example, the proof of lemma 3.6 where a similar argument is detailed). The converse is obviously true.

□

Theorem 3.4 *T^* is a scalar multiple of an isometry if and only if for each complex z,*

$$\left\{T^*x : x \in M_z(T)\right\} \subset M_z(T) .$$

Proof Follows from applying theorem 3.3 to T^* and noting that $M_z(T^*) = M_{\bar{z}}(T)$ for each complex z. □

Theorem 3.5 *T is a nonzero scalar multiple of a unitary operator if and only if for each complex z,*

$$\left\{Tx : x \in M_z(T)\right\} = M_z(T) .$$

Proof Combine theorems 3.3 and 3.4 to give T is a scalar multiple of a unitary operator if and only if for each complex z both

$$\left\{Tx : x \in M_z(T)\right\} \subset M_z(T)$$

and

$$\left\{T^*x : x \in M_z(T)\right\} \subset M_z(T) .$$

Thus if T is nonzero, this is equivalent to

$$\left\{Tx : x \in M_z(T)\right\} \subset M_z(T) \subset \left\{Tx : x \in M_z(T)\right\}$$

proving the result.

□

For the next theorem, the following lemma is required.

Lemma 3.6 *If* T *and* A *are operators on* H *such that*

$$\ker T \subset \ker A$$

and for each $x \in H$ *either*

i) $\|Tx\| = \|Ax\|$, *or*

ii) there exists a real number r_x *such that*

$$T^*Tx = r_x\, A^*Ax \ ,$$

then T^*T *is a scalar multiple of* A^*A .

Proof For $x,y \in H$, let $z = tx + (1 - t)y$ where $0 < t < 1$.

Suppose A^*Ax and A^*Ay are linearly independent and condition (ii) holds, that is, there exist real numbers r_x and r_y such that

$$T^*Tx = r_x\, A^*Ax, \quad T^*Ty = r_y\, A^*Ay \ .$$

Hence either there exists a real number r_z such that

$$T^*Tz = r_z\, A^*Az \quad \text{or} \quad \|Tz\| = \|Az\| \ .$$

But if $T^*Tz = r_z\, A^*Az$,

since $0 < t < 1$, linear independence of A^*Ax and A^*Ay gives

$$r_x = r_y = r_z \;.$$

Now suppose $r_x \neq r_y$, then we must have $||Tz|| = ||Az||$ where $z = tx + (1-t)y$, $0 < t < 1$.

Letting t approach 1 and 0, we have

$$||Tx|| = ||Ax|| \quad \text{and} \quad ||Ty|| = ||Ay|| \;.$$

Since A^*Ax and A^*Ay are nonzero, this gives

$$r_x = r_y = 1 \;.$$

Thus in all cases if A^*Ax and A^*Ay are linearly independent, we have

$$r_x = r_y = r \quad \text{(say)}.$$

Now suppose A^*Az and A^*Ay are linearly dependent and $T^*Tx = r_x\, A^*Ax$ and $T^*Ty = r_y\, A^*Ay$. In this case, since $\ker T \subset \ker A$, we can choose $r_x = r_y = r$.

So for all $x \in H$, we have

either

$$||Tx|| = ||Ax|| \quad \text{or} \quad T^*Tx = r\, A^*Ax \;.$$

This gives

$$\|Tx\| \leqslant \|Ax\| \quad \text{for all} \quad x \in H, \text{ or, } \|Tx\| \geqslant \|Ax\| \quad \text{for all } x \in H \ .$$

In either case the set

$$\left\{ x \in H : \|Tx\| = \|Ax\| \right\}$$

is linear by theorem 1.11 (i) and so

$$H = \left\{ x \in H : T^*Tx = r\ A^*Ax \right\} \cup \left\{ x \in H : \|Tx\| = \|Ax\| \right\}$$

which shows either $T^*T = r\ A^*A$ or $T^*T = A^*A$.

□

Theorem 3.7 *T is normal if and only if for each complex z,*

$$\left\{ x : Tx \in M_z(T) \right\} = \left\{ x : T^*x \in M_z(T) \right\} .$$

Proof Suppose the above two sets are equal, then

$$\langle T^2x, Tx\rangle \|T^*x\|^2 = \langle TT^*x, T^*x\rangle \|Tx\|^2 . \qquad \ldots(3.5)$$

Also we note that the following are equivalent:

i) $Tx = 0$,

ii) $Tx \in M_z(T)$ for all complex z,

iii) $T^*x \in M_z(T)$ for all complex z,

iv) $T^*x = 0$,

and hence $$\ker T = \ker T^* . \qquad \ldots(3.6)$$

Using the same techniques as in theorem 3.3, we can show that if $x \in H$,
either there exists $b \in R$ such that $TT^*x = bT^*Tx$,
or there exist $c, d \in R$ such that

$$TT^{*2}x = cTT^*x \quad \text{and} \quad T^*T^2x = dT^*Tx .$$

These last two equations together with (3.5) and (3.6) give either

$$Tx = T^*x = 0 \qquad \ldots(3.7)$$

or

$$c = \bar{d} .$$

They also imply that

$$T^{*2}x = cT^*x \quad \text{and} \quad T^2x = dTx \ .$$

Now (3.6) gives

$$TT^*x = cTx \quad \text{and} \quad T^*Tx = dT^*x \ .$$

Thus if (3.7) does not hold we have

$$||T^*x||^2 = c\langle Tx,x\rangle = \bar{d}\langle x,T^*x\rangle = ||Tx||^2 \ .$$

Hence we see that both T and T^* satisfy the conditions of lemma 3.6 and thus there exists a real number r such that

$$TT^* = r\ T^*T \ .$$

Thus $r = \pm 1$.

But if $r = -1$, choosing an $x \in \ker T$ we arrive at the contradiction

$$||Tx||^2 = ||T^*x||^2 = 0 \ .$$

Hence $r = 1$ and so T is normal.

□

Corollary 3.8 *Let T be an invertible operator on H. Then the following are equivalent:*

i) *T is normal;*

ii) $\left\{T^{-1}x : x \in M_z(T)\right\} = \left\{T^{*-1}x : x \in M_z(T)\right\}$ *for each complex z;*

iii) $\left\{T^{-1}x : x \in M_z(T^*T^{-1})\right\} = \left\{T^{*-1}x : x \in M_z(T^*T^{-1})\right\}$ *for each complex z.*

Proof If T is invertible, theorem 3.7 shows the equivalence of (i) and (ii). Again, application of theorem 3.5 to the operator T^*T^{-1} gives the equivalence of (i) and (iii).

□

If we look upon H as embedded in $\ell_\infty(H)$ with the correspondence $x \to (x,x,\ldots)$, then it is obvious that

$$\left\{(Tx_n) : (x_n) \in N_z(T)\right\} \subset N_z(T)$$

implies

$$\left\{Tx : x \in M_z(T)\right\} \subset M_z(T)$$

for all complex numbers of z.

This enables us to generalize all the above results as follows.

Theorem 3.9 *T is a scalar multiple of an isometry if and only if for each complex z,*

$$\left\{(Tx_n) : (x_n) \in N_z(T)\right\} \subset N_z(T) .$$

Theorem 3.10 *T^* is a scalar multiple of an isometry if and only if for each complex z,*

$$\left\{(T^*x_n) : (x_n) \in N_z(T)\right\} \subset N_z(T) .$$

Theorem 3.11 *T is a nonzero scalar multiple of a unitary operator if and only if for each complex z,*

$$\left\{(Tx_n) : (x_n) \in N_z(T)\right\} = N_z(T) .$$

Theorem 3.12 *T is normal if and only if for each complex z,*

$$\left\{(x_n) \in \ell_\infty(H) : (Tx_n) \in N_z(T)\right\}$$

$$= \left\{(x_n) \in \ell_\infty(H) : (T^*x_n) \in N_z(T)\right\} .$$

Corollary 3.13 *Let T be an invertible operator. Then the following are equivalent:*

i) *T is normal;*

ii) $\left\{(T^{-1}x_n) : (x_n) \in N_z(T)\right\} = \left\{(T^{*-1}x_n) : (x_n) \in N_z(T)\right\}$ *for each complex z;*

iii) $\left\{(T^{-1}x_n) : (x_n) \in N_z(T^*T^{-1})\right\} = \left\{(T^{*-1}x_n) : (x_n) \in N_z(T^*T^{-1})\right\}$ *for each complex z.*

The proof for each of the above 'if and only if' theorems consists of easy verification for one side and use of the corresponding theorem of Embry for the converse.

3.3 Results on Attained Points of $\partial W(T)$ for Special Operators

In this section we deal with results on the attained boundary points of the numerical range for convexoid and semi-normal operators. As proved by Lin (1975), we obtain some necessary and sufficient conditions for an extreme point of the numerical range of a convexoid operator to belong to the point spectrum. In the next section this result will be extended to unattained boundary points of $W(T)$.

Stampfli (1966) has shown that if T is hyponormal and z is an extreme point of $W(T)$, then $M_z(T)$ is a reducing subspace of T. de Barra (1981) has shown that for such T, $M(T)$ is a reducing subspace and $T|_{M(T)}$ is normal. We first give these theorems for seminormal operators borrowing proofs from Stampfli and using a modification of the proof given by de Barra. We then show in the next section that similar properties hold for $N_z(T)$ (with z an extreme point of $W(T)^-$) and $N_L(T)$.

First we recall some definitions.

Definition 3.14 If $z \in \partial W(T)$ and z_o is the centre of a closed disc D such that $z \in \partial D$ and $\partial W(T) \cap D = \{z\}$, then z_o is said to be an *outer centre point with respect to* z. (In general take Z_O $W(T)$).

Definition 3.15 If $z \in \partial W(T)$ and there exists a closed disc D such that $z \in \partial D$ and $W(T)^- \subset D$, then z is said to be a *bare point of* $W(T)^-$.

Definition 3.16 If z is a bare point of $W(T)^-$ and z_o is the centre of a closed disc D such that $z \in \partial D$ and $W(T) \subset D$, then z_o is said to be an *inner centre point with respect to* z.

Definition 3.17 The *numerical radius* of the operator T is defined by

$$w(T) = \sup_{\lambda \in W(T)} |\lambda|$$

and the *spectral radius* by

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$$

where $\sigma(T)$ is the spectrum of T.

Let $d(z_o, W(T))$ denote the distance of z_o from $W(T)$; $E(T)$ and $B(T)$ respectively the sets of extreme and bare points of $W(T)^-$ and $\sigma_p(T)$ and $\sigma_{ap}(T)$ respectively the point and approximate point spectra of T.

Theorem 3.18 *Let* T *be a convexoid operator and* z *an extreme point of* $W(T)^{-}$ *such that* $z = \langle Tx,x\rangle$, $\|x\| = 1$. *Let* z_o *be an outer centre point with respect to* z. *Then the following are equivalent:*

i) $Tx = zx$;

ii) $\|Tx - z_o x\|^{-1} = r((T - z_o)^{-1})$;

iii) $\|Tx - z_o x\|^{-1} = \|(T - z_o)^{-1}\|$.

Proof Since T is convexoid, $E(T) \subset \sigma(T)$. Thus $z \in \sigma(T) \cap \partial W(T)$.

Also since

$$\|(T-z_o)^{-1}\| \leq [d(z_o,W(T))]^{-1} = [d(z_o,\sigma(T))]^{-1}$$

$$= r((T-z_o)^{-1}) \leq \|(T-z_o)^{-1}\| ,$$

we have

$$r((T-z_o)^{-1}) = \|(T-z_o)^{-1}\| = [d(z_o,W(T))]^{-1} .$$

Thus

(i) implies (ii) since

$$\|(T-z_o)x\|^{-1} = |z-z_o|^{-1} = [d(z_o,W(T))]^{-1} = r((T-z_o)^{-1}),$$

(ii) implies (iii) since

$$r((T-z_o)^{-1}) = \|(T-z_o)^{-1}\|$$

and (iii) implies (i) for

$$||Tx - z_0 x|| = d(z_0, W(T)) = |z - z_0| = |\langle (T - z_0)x, x\rangle|$$

and hence by the condition for equality in Cauchy Schwartz, $(T-z_0)x = \lambda x$ for some complex λ. Since $\langle Tx,x\rangle = z$, this gives $Tx = zx$. □

If $z = \langle Tx,x\rangle \in \partial W(T)$, $||x|| = 1$, by lemma 1.9 (i), $Tx = zx$ if and only if $T^*x = \bar{z}x$. The following corollary given by Lin can be easily verified from this fact and the proof of the above theorem.

Corollary 3.19 *For any operator T,*

1) *If $\langle Tx,x\rangle = z \in \partial W(T)$, $||x|| = 1$ and z_0 is an outer centre point with respect to z, then the following are equivalent:*

   i) $||Tx - z_0 x|| = d(z_0, W(T))$;

   ii) $Tx = zx$;

   iii) $T^*x = \bar{z}x$

   and

2) *If $\langle Tx,x\rangle = z \in B(T)$, $||x|| = 1$ and z_0 is an inner centre point with respect to z, then the following are equivalent:*

   i) $||Tx - z_0 x|| = w(T - z_0)$;

   ii) $Tx = zx$;

   iii) $T^*x = \bar{z}x$.

The next theorem gives conditions for $M(T)$ to be a reducing subspace for T and for the restriction of T to $M(T)$ to be normal.

Theorem 3.20 *Let T be a seminormal operator and*

$$M(T) = \left\{x \in H : \langle Tx, x\rangle - z\|x\|^2 = 0, \quad z \in L \cap W(T)\right\}$$

where L is a line of support for $W(T)$. Then $M(T)$ is a reducing subspace for T and $T|_{M(T)}$ is normal.

Proof By lemma 1.9 (ii), $M(T)$ is a subspace. For any $z \in L$, by carrying out the standard reduction $T \to e^{i\theta}(T-zI)$, without loss of generality we may assume that L is the imaginary axis and $\mathrm{Re}\, W(T) \geqslant 0$.

Thus as in lemma 1.9 (i),

$$M(T) = \left\{x \in H : Tx + T^*x = 0\right\} .$$

Hence

$$\langle T^*Tx - TT^*x, x\rangle = 0 .$$

Thus by lemma 1.8,

$$T^*Tx = TT^*x \quad \text{as} \quad T^*T - TT^* \leqslant 0 \quad \text{or} \quad \geqslant 0 .$$

Now

$$(T+T^*)Tx = T^2x + T^*Tx = T^2x + TT^*x = T(Tx+T^*x) = 0 .$$

Similarly

$$(T + T^*)T^*x = 0 \ .$$

Hence $M(T)$ is reducing and since $T^*Tx = TT^*x$ for all x in $M(T)$, we have $T|_{M(T)}$ is normal.

□

The following theorem proves the reducing property of $M_z(T)$ with z an extreme point of $W(T)$ and T semi-normal.

Theorem 3.21 *Let T be seminormal and z be an extreme point of $W(T)$.*

Let $M_z(T) = \left\{x \in H : \langle Tx,x\rangle - z\|x\|^2 = 0\right\}$.

Then $M_z(T)$ is a reducing subspace of T.

Proof Without loss of generality we may assume $z = 0$ and $\mathrm{Re}\, W(T) \geqslant 0$.

$M_0(T)$ is a subspace by theorem 1.11 (i) and $M_0(T) \subset M(T)$. But T is normal on $M(T)$ and therefore since $\mathrm{Re}\, W(T) \geqslant 0$, the condition

$$\langle Tx,x\rangle = 0 \ \text{ implies that } \ Tx = 0 \ .$$

Hence obviously $M_0(T)$ is reducing for T.

□

In the next section we shall use Berberian's technique to achieve a generalization of theorem 3.21. We shall also generalize by direct calculations the other two theorems in this section. Note that since a seminormal operator is convexoid, theorem 3.18 is valid for seminormal operators. Using Berberian's technique an alternative proof of this known result, that a seminormal operator is convexoid, will also be given.

3.4 Generalized Results on $\partial W(T)$ for Special Operators

The following theorems deal with sequences of vectors from H rather than H itself and thus the results are in terms of limits.

Theorem 3.22 *Let T be a convexoid operator and z an extreme point of $W(T)^-$. Let (x_n) be a sequence of unit vectors such that $\langle Tx_n, x_n\rangle \to z$. Let z_o be an outer centre point with respect to z. Then the following are equivalent:*

i) $\lim \|Tx_n - zx_n\| = 0$;

ii) $\lim \|Tx_n - z_o x_n\|^{-1} = r((T-z_o)^{-1})$;

iii) $\lim \|Tx_n - z_o x_n\|^{-1} = \|(T-z_o)^{-1}\|$.

Proof Similar to the proof of theorem 3.18. Only note that in (i) $\Rightarrow$ (ii) we use the fact that if $\|Tx_n - zx_n\| \to 0$, then $\|Tx_n - z_o x_n\| \to |z - z_o|$.

This is so because

$$\|Tx_n - z_\circ x_n\|^2 = \|(T-z)x_n + (z-z_\circ)x_n\|^2$$

$$\rightarrow |z-z_\circ|^2 \quad \text{since} \quad (T-z)x_n \rightarrow 0$$

and hence

$$(\|(T-z_\circ)x_n\| + |z-z_\circ|)(\|(T-z_\circ)x_n\| - |z-z_\circ|) \rightarrow 0,$$

that is,

$$\|Tx_n - z_\circ x_n\| \rightarrow |z-z_\circ|$$

as $\|(T-z_\circ)x_n\| + |z-z_\circ|$ is bounded away from zero.

Also in (iii) $\Rightarrow$ (i) ,

$$\|Tx_n - zx_n\|^2 = \|(T-z_\circ)x_n - (z-z_\circ)x_n\|^2$$

$$= \|(T-z_\circ)x_n\|^2 + |z-z_\circ|^2 - 2\,\mathrm{Re}((\overline{z-z_\circ})<(T-z_\circ)x_n,x_n>) \rightarrow 0$$

as $\|(T-z_\circ)x_n\| \rightarrow |z-z_\circ|$.

Thus we get

$$\|Tx_n - zx_n\| \rightarrow 0.$$

□

The following corollary is readily verified from the proof above and lemma 2.5 (i).

Corollary 3.23 *Let T be an arbitrary operator.*

1) *If (x_n) is a sequence of unit vectors such that $\langle Tx_n, x_n\rangle \to z \in \delta W(T)$ and $z_\circ$ is an outer centre point with respect to z, then the following are equivalent:*

   i) $\|(T-z_\circ)x_n\| \to d(z_\circ, W(T))$;

   ii) $Tx_n - zx_n \to 0$;

   iii) $T^*x_n - \bar{z}x_n \to 0$

   and

2) *If (x_n) is a sequence of unit vectors such that $\langle Tx_n, x_n\rangle \to z \in B(T)$ and $z_\circ$ is an inner centre point with respect to z, then the following are equivalent:*

   i) $\|(T-z_\circ)x_n\| \to w(T-z_\circ)$;

   ii) $Tx_n - zx_n \to 0$;

   iii) $T^*x_n - \bar{z}x_n \to 0$.

The next theorem is a generalization of theorem 3.20 for elements of the subspace $N_L(T)$. We give below a proof by direct method. The result can also be proved using Berberian's technique.

Theorem 3.24 *Let T be a seminormal operator and*

$$N_L(T) = \left\{(x_n) \in \ell_\infty(H) : \inf_{z\in L} |\langle Tx_n, x_n\rangle - z\|x_n\|^2| \to 0\right\}$$

where L is a line of support for $W(T)$. Then for each $(x_n) \in N_L(T)$, $(Tx_n) \in N_L(T)$ and $(T^*x_n) \in N_L(T)$. Also T approximates normal behaviour on sequences in $N_L(T)$ in the sense that if $(x_n) \in N_L(T)$, then $(T^*T-TT^*)x_n \to 0$.

Proof By lemma 2.5 (ii), $N_L(T)$ is a subspace. Without loss of generality we may take L as the imaginary axis and $\operatorname{Re} W(T) \geqslant 0$ in which case as we have seen

$$N_L(T) = \left\{ (x_n) \in \ell_\infty(H) : Tx_n + T^*x_n \to 0 \right\} .$$

Let $(x_n) \in N_L(T)$.
Hence $Tx_n + T^*x_n \to 0$,
or, $||Tx_n||^2 - ||T^*x_n||^2 \to 0$,
or, $(T^*T - TT^*)x_n \to 0$
by lemma 2.4, since either $T^*T - TT^*$ or $TT^* - T^*T$ is positive.

Also continuity of T gives $T^2x_n + TT^*x_n \to 0$,
or $(T^2x_n + T^*Tx_n) - (T^*Tx_n - TT^*x_n) \to 0$.
Thus $T^2x_n + T^*Tx_n \to 0$.
Hence $(Tx_n) \in N_L(T)$.
In a similar way it can be proved that

$$(T^*x_n) \in N_L(T).$$

□

By $T \to T°$ we will denote the faithful *-representation constructed by Berberian as explained in section 2.5. The following simple lemma is used in the proofs of later theorems.

Lemma 3.25 *T^o is seminormal if and only if T is seminormal.*

Proof By the properties of T^o as given in section 2.5,

$$(T^o)^*T^o - T^o(T^o)^* = (T^*)^oT^o - T^o(T^*)^o = (T^*T-TT^*)^o.$$

Thus, since T^o preserves positivity,

$$(T^o)^*T^o - T^o(T^o)^* \geq 0 \quad \text{or} \quad \leq 0$$

if and only if

$$T^*T - TT^* \geq 0 \quad \text{or} \quad \leq 0 \quad \text{respectively.}$$

□

Theorem 3.26 *Let T be seminormal and z be an extreme point of $W(T)^-$. Let*

$$N_z(T) = \left\{(x_n) \in \ell_\infty(H) : \langle Tx_n, x_n\rangle - z\|x_n\|^2 \to 0\right\},$$

Then for each $(x_n) \in N_z(T)$,

$$(Tx_n) \in N_z(T) \quad \text{and} \quad (T^*x_n) \in N_z(T).$$

[NOTE: This theorem can be deduced as a corollary of theorem 3.24, if z is not an endpoint of a straight line segment on $\partial W(T)$.]

Proof Since by theorem 2.6, $N_z(T)$ is a subspace, $(Tx_n) \in N_z(T)$ if and only if $e^{i\theta}(Tx_n - zx_n) \in N_z(T)$. Thus by the standard transformation $T \to e^{i\theta}(T-zI)$, without loss of generality we may assume $z = 0$ and $\text{Re } W(T) \geq 0$.

Now as in the proof (using Berberian's technique) of theorem 2.6, with the same notations,

$$(x_n) \in N_o(T) \quad \text{implies} \quad s' = (x_n)' \in M_o(T)$$

where

$$M_o(T^\circ) = \left\{ s' \in K : \langle T^\circ s', s' \rangle = 0 \right\} .$$

Since by lemma 3.25, T° is seminormal, theorem 3.21 gives $T^\circ s' \in M_o(T^\circ)$, in fact the proof of that theorem shows $T^\circ s' = 0$.

$$\text{Thus} \quad f((\|Tx_n\|^2)) = 0$$

for all f where f is any linear functional with the properties given in lemma 2.7.

Since $(\|Tx_n\|^2) \in \ell_\infty^+$, by lemma 2.7 we conclude that $Tx_n \to 0$.

Again since $\langle \text{Re } Tx_n, x_n \rangle \to 0$, lemma 2.4 gives $\text{Re } Tx_n \to 0$ and hence we have $T^*x_n \to 0$.

Thus $(Tx_n) \in N_z(T)$ and $(T^*x_n) \in N_z(T)$.

□

Putnam (1965) and Stampfli (1965) have shown independently that a seminormal operator is convexoid. We give below an alternative proof using Berberian's technique. Let co denote the convex hull. We need the following lemma given by Berberian (1962).

Lemma 3.27 *For any operator T,*

$$\sigma_{ap}(T^\circ) = \sigma_{ap}(T) .$$

Proof A complex number μ does not belong to $\sigma_{ap}(T)$ if and only if there exists $\varepsilon > 0$ such that $(T-\mu I)^*(T-\mu I) \geq \varepsilon I$ which is equivalent to $(T^\circ - I)^*(T^\circ-\mu I) \geq \varepsilon I$ by the properties of T° given in section 2.5. □

Theorem 3.28 *For a seminormal operator T,*

$$W(T)^- = co\ \sigma(T) .$$

Proof By lemma 3.25, T° is seminormal. An application of theorem 3.21 to T° gives

$$E(T^\circ) \cap W(T^\circ) \subset \sigma_p(T^\circ) .$$

But since by theorem 2.8, $W(T^\circ) = W(T)^-$, we have

$$E(T) = E(T^\circ)$$

and hence

$$E(T) \subset \sigma_p(T^\circ) \subset \sigma_{ap}(T^\circ) = \sigma_{ap}(T)$$

by lemma 3.27.

Thus $\text{co } E(T) \subset \text{co } \sigma(T)$.

But $W(T)^- = \text{co } E(T)$

and $\text{co } \sigma(T) \subset W(T)^-$.

So we must have $W(T)^- = \text{co } \sigma(T)$.

□

In this chapter we looked at different operators with special characteristics in terms of $M_z(T)$ and the action of the operator T on them. Then we extended the results to $W(T)^-$ and saw that the same type of set inclusions still holds for elements of $N_z(T)$.

In section 3.3 we provided two equivalent conditions in terms of spectral radius and operator norm for an extreme point of a convexoid operator to belong to the point spectrum. We showed that for a seminormal operator T, $M(T)$ is a reducing subspace of T. Moreover T on $M(T)$ behaves as a normal operator. Also for seminormal T, if z is an extreme point of $W(T)$, $M_z(T)$ has the same reducing property. This was shown in theorem 3.21.

In the final section we obtained generalizations to the results of section 3.3. In some cases it was convenient to use Berberian's technique concerning change of operators and Hilbert space. By the same technique we gave an alternative proof of the essentially known result that a seminormal operator is convexoid.

In the following concluding chapter we will consider weak convergent sequences of unit vectors which generate sequences of points in the numerical range converging to the boundary of the numerical range. We shall also discuss the question of convexity for a newly defined restricted numerical range. The convexity of $W(T)$ and Stampfli's numerical range $W_{\delta}(T)$ will follow as corollaries.

Chapter 4

CONVEXITY OF DIFFERENT NUMERICAL RANGES AND WEAK CONVERGENCE ON $\partial W(T)$

4.1 Introduction

In this chapter we define a restricted numerical range in terms of appropriate subsets of S of the unit sphere and investigate conditions on S which will ensure the restricted numerical range is convex. The convexity of Stampfli's numerical range follows as a corollary. Kyle (1977) used a different technique to prove this result. We include his method in section 4.3.

In section 4.2 we consider the weak convergence of a sequence of unit vectors corresponding to a sequence of points in the numerical range with its limit on the boundary of $W(T)$. de Barra *et al.* (1972), Sims (1974), Das (1973, 1974, 1977) and Garske (1979) investigated which boundary points of $W(T)$ are attained. Das and Craven gave a bound for the norm of the weak limit of vectors when the corresponding boundary point is not attained, but lies on the straight line segment on the boundary. We use the method of proof for these results given by Garske and Das and Craven. We then demonstrate how all these results can be obtained as a simple corollary to one of the inequalities obtained in Chapter 2.

4.2 Weak Convergence on $\partial W(T)$

In the previous chapters we obtained results for those boundary points of the numerical range which are attained by the operator T and then extended these results to unattained boundary points of $W(T)$. The question arises as to which boundary points of $W(T)$ are in fact attained.

Garske (1979) showed that if λ is an extreme point of $W(T)^-$, then the following statement is true.

(A) Let (x_n) be a sequence of unit vectors in H with weak limit $x \in H$ and $\langle Tx_n, x_n\rangle \to \lambda \in \partial W(T)$. Then either

i) $x = 0$,

or ii) $\langle Tx, x\rangle / \|x\|^2 = \lambda$.

Weak compactness of the unit sphere in H ensures the existence of such a sequence.

The following example given by Garske (1979) shows that (A) need not hold for all boundary points of $W(T)^-$.

Example 4.1 Let $T : L^2[-1,1] \to L^2[-1,1]$ be the self-adjoint multiplication operator defined by

$$(Tf)(t) = tf(t)$$

for $f \in L^2[-1,1]$, $t \in [-1,1]$.

It follows that $W(T) = (-1,1)$ and so $0 \in \partial W(T)$ is not an extreme point of $W(T)^-$.

$$\text{Let} \quad f_n(t) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } -1 \leqslant t < 0, \\ \cos \pi nt & \text{if } 0 \leqslant t \leqslant 1. \end{cases}$$

and
$$f(t) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } -1 \leqslant t < 0, \\ 0 & \text{if } 0 \leqslant t < 1. \end{cases}$$

Then $||f_n|| = 1$, $||f|| = \frac{1}{\sqrt{2}}$ and f_n converges to f weakly.

But
$$<Tf_n, f_n> = \int_{-1}^{0} \frac{1}{2} t \, dt + \int_{0}^{1} t(\frac{1}{2} + \frac{1}{2} \cos 2\pi nt) dt$$
$$= \int_{-1}^{1} \frac{1}{2} t \, dt + \frac{1}{2} \int_{0}^{1} t \cos 2\pi nt \, dt \to 0$$

whereas $<Tf, f> = \int_{-1}^{0} \frac{1}{2} t \, dt = \frac{1}{4}$.

Thus $<Tf_n, f_n> \not\to \frac{<Tf, f>}{||f||^2}$.

Das and Craven considered points on a line segment on the boundary of the numerical range and gave a bound for the norm of the weak limit for such points. We shall later state the results of Garske and Das and Craven in a single theorem and give their method of proof; but first we begin with a shortened proof of the following lemma due to Das and Craven.

Lemma 4.2 *Let $\lambda \in L \cap W(T)^-$ where L is a line of support for $W(T)$. Let $x_n \rightharpoonup x$ be a weakly convergent sequence of vectors such that $<Tx_n, x_n> \to \lambda$. Then either $x = 0$ or $<Tx, x>/||x||^2 \in L$.*

Proof By a suitable translation and rotation, we may, without loss of generality, assume that L is the imaginary axis and $\mathrm{Re}\, W(T) \geq 0$.

Hence $\langle \mathrm{Re}\, Tx_n, x_n \rangle \to 0$
and thus by lemma 2.4, $\mathrm{Re}\, Tx_n \to 0$.

But $\mathrm{Re}\, Tx_n \rightharpoonup \mathrm{Re}\, Tx$
and hence the uniqueness of the weak limit gives $\mathrm{Re}\, Tx = 0$.

So $\mathrm{Re}\langle Tx, x\rangle = 0$ and hence
either $x = 0$ or $\langle Tx, x\rangle / \|x\|^2 \in L$.

□

The following theorem is a combination of results of Garske and Das and Craven. We first give their method of proof.

Theorem 4.3 *Let $x_n \rightharpoonup x$ be a weakly convergent sequence of unit vectors such that $\langle Tx_n, x_n\rangle \to \lambda \in \partial W(T)$. Thus either*

i) $x = 0$, or,

ii) $\langle Tx, x\rangle / \|x\|^2 = \lambda$, or

iii) λ is not an extreme point of $W(T)^-$ and $x \neq 0$, in which case λ and $\langle Tx, x\rangle / \|x\|^2$ lie in a line segment on the boundary of $W(T)$ and $\|x\|^2 \leq \frac{p}{q}$ where p and q are respectively the distances from λ and $\langle Tx, x\rangle / \|x\|^2$ to the extreme point of $W(T)^-$ collinear with λ and

$\langle Tx,x\rangle/\|x\|^2$ and on the opposite side of λ from $\langle Tx,x\rangle/\|x\|^2$.

Proof First consider the case when λ is an extreme point of $W(T)^-$.

Let $\langle Tx_n,x_n\rangle \to \lambda$, $\|x_n\| = 1$.
This gives $\|x\| \leqslant 1$ and so if $y_n = x_n - x$, then $\|y_n\| \leqslant 2$.
Thus passing on to a subsequence we may assume

$$\|y_n\| \to \varepsilon \in R^+ .$$

(We can exclude the trivial case when $\varepsilon = 0$.)

So we have

$$1 = \|x_n\|^2 = \|y_n\|^2 + 2\ \mathrm{Re}\langle x_n,x\rangle + \|x\|^2 ,$$

or, since $x_n \rightharpoonup x$, this gives

$$\varepsilon^2 + \|x\|^2 = 1 .$$

Again

$$\langle Tx_n,x_n\rangle = \langle Ty_n,y_n\rangle + \langle y_n,T^*x\rangle + \langle Tx,y_n\rangle + \langle Tx,x\rangle \to \lambda ,$$

or,

$$\langle Ty_n,y_n\rangle \to \lambda - \langle Tx,x\rangle .$$

If we call $\mu = \langle Tx,x\rangle/||x||^2$ (assuming $x \neq 0$), this gives

$$\alpha = \frac{\lambda - ||x||^2\mu}{\varepsilon^2}$$

where

$$\alpha = \lim \frac{\langle Ty_n, y_n\rangle}{||y_n||^2}$$

Thus $\lambda = \varepsilon^2\alpha + ||x||^2\mu$.

Since $\varepsilon^2 + ||x||^2 = 1$, λ lies on the line segment from μ to α and thus as λ is an extreme point of $W(T)^-$, either $\lambda = \mu$ or $\lambda = \alpha$.

If $\lambda = \alpha$, we have

$$\mu||x||^2 = (1 - \varepsilon^2)\alpha = \alpha||x||^2$$

and hence again $\lambda = \mu$.

Now consider the case when $\lambda \neq \mu$ and λ is not an extreme point of $W(T)^-$. If $||x|| = 1$, then $x_n \to x$ and thus $\lambda = \mu$. So if $x \neq 0$, we may assume $0 < ||x||^2 < 1$.

Consider

$$\frac{\langle T(x+tx_n),\ x + tx_n\rangle}{||x + tx_n||^2} \qquad (t \in R)$$

which, under the assumption $x_n \to x$, is equal to

$$\lambda = \frac{(\mu-\lambda)(2t+1)||x||^2}{t^2 + (2t+1)||x||^2} .$$

Let

$$u = \frac{(2t+1)\,||x||^2}{t^2 + (2t+1)\,||x||^2} .$$

It can be easily verified that

$$\frac{||x||^2}{||x||^2 - 1} \leqslant u \leqslant 1 .$$

By lemma 4.1, μ and $\lambda + \dfrac{(\mu-\lambda(||x||^2}{||x||^2 - 1}$ are collinear with λ and clearly lie on the opposite sides of λ.

Hence

$$\frac{|\mu-\lambda|\,||x||^2}{1 - ||x||^2} \leqslant p ,$$

or,

$$||x||^2 \leqslant \frac{p}{|\mu-\lambda| + a} = \frac{p}{q} .$$

□

The above theorem can be deduced as a simple corollary of theorem 2.14 as given below.

Corollary 4.4 *Let $x_n \rightharpoonup x$ be a weakly convergent sequence of unit vectors such that $\langle Tx_n, x_n\rangle \to \lambda \in \partial W(T)$. Let L be a line of support of $W(T)$ passing through λ. Then either*

a) λ is an extreme point of $W(T)^-$ in which case one of the following holds:

i) $x = 0$;

ii) $\langle Tx,x\rangle/\|x\|^2 = \lambda$; or

b) λ is a nonextreme boundary point of $W(T)^-$ in which case one of the following holds:

i) $x = 0$;

ii) $\langle Tx,x\rangle/\|x\|^2 = a$ where $a \in L$ is an extreme point of $W(T)^-$.

In this case $\|x\| \leq \sqrt{\dfrac{\lambda - a}{a - b}}$ where b is the other extreme point of $W(T)^- \cap L$.

iii) $\|x\| \leq \sqrt{\dfrac{\lambda - a}{\mu - a}}$ where $\mu = \langle Tx,x\rangle/\|x\|^2$ and $a \in L$ is an extreme point of $W(T)^-$.

Proof By lemma 4.2, $\lambda, \mu, a, b \in L$. If we consider the sequence $(x,x,\ldots)$, it is obvious that $(x,x,\ldots) \in N(T)$ where $N(T)$ is as given in theorem 2.14. Also $(x_n) \in N(T)$. Thus an application of theorem 2.14 with λ as an extreme point gives

$$\lim|\langle (T-\lambda)x_n,\ x\rangle| = 0 \quad \text{since} \quad \langle (T-\lambda)x_n,\ x_n\rangle \to 0$$

and hence

$$<Tx,x> = \lambda ||x||^2 ,$$

that is, either $x = 0$ or $<Tx,x>/||x||^2 = \lambda$.

If λ is a nonextreme boundary point of $W(T)^-$, another application of theorem 2.14 with a as an extreme point of $W(T)^-$ gives

$$\lim|<(T-a)x_n,x>|^2 \leq \lim|<(T-a)x_n,x_n> \lim|<(T-a)x,x>| .$$

Thus if $x \neq 0$,

$$|<(T-z)x,x>|^2 \leq |\lambda-a||\mu-a| ||x||^2 ,$$

or

$$|\mu-a|^2 ||x||^2 \leq |\lambda-a||\mu-a| ,$$

that is, $\mu = a$ or $||x||^2 \leq \frac{|\lambda-a|}{|\mu-a|} = \frac{\lambda-a}{\mu-a}$ since λ, μ and a are collinear and a is an extreme point of $W(T)^-$.

If $\mu = a$, application of the same theorem with b as the other extreme point gives

$$||x||^2 < \frac{\lambda-a}{a-b} .$$

□

Note that the inequality given in theorem 4.3 (iii) is equivalent to the combined two inequalities given in corollary 4.4 (b).

We also note that the above result could also be obtained as a corollary to lemma 2.9.

Throughout our dissertation we used the fact that $W(T)$ is a convex set. In our next section we define a new restricted numerical range and investigate under what condition convexity holds for this set. As a corollary we obtain a result given by Kyle (1977). These results are contained in a paper by Das, Majumdar and Sims (3).

4.3 Restricted Numerical Range and Convexity of $W_\delta(T)$

Stampfli (1970) introduced the concept of $W_\delta(T)$, a modification of $W(T)$ and asked if $W_\delta(T)$ is convex. He defined

$$W_\delta(T) = \text{closure}\left\{\langle Tf,f\rangle : \|f\| = 1, \|Tf\| \geqslant \delta,\ f \in H\right\} .$$

Kyle (1977) settled this question in the affirmative using ideas which are improvements on basic ideas of Dekker (1969).

In the next section we define a restricted numerical range by

$$W_S(T) = \left\{\langle Tf,f\rangle : \|f\| = 1,\ \ f \in S \subset H\right\} .$$

We obtain conditions on S which ensure that $W_S(T)$ is convex. Our results are more general than those of Kyle, convexity of both $W(T)$ and $W_\delta(T)$ following as corollaries.

We begin with some generalizations and modifications of results originally used by Kyle to obtain the convexity of $W_\delta(T)$.

Lemma 4.5 *Let A and B be self-adjoint operators and*

$$M = \left\{f \in H : \|f\| = 1,\ \langle Af,f\rangle \geq \delta \text{ and } \langle Bf,f\rangle = 0\right\} .$$

Then M is path connected.

Proof Suppose $f, g \in M$. If f, g are linearly dependent, they both lie on an arc of

$$\left\{e^{i\theta} f : 0 \leq \theta \leq 2\pi\right\}$$

which lies in M whenever $f \in M$.

If f, g are linearly independent, since f and $e^{i\theta}f$ with suitably chosen real values of θ are path connected and g and $(-1)^n g$, $n = 1,2$ are path connected, without loss of generality we may assume

$$\text{Re}\langle Bf,g\rangle = 0 \quad \text{and} \quad \text{Re}\langle (A-\delta)f,g\rangle \geq 0 .$$

Let $$f(t) = \frac{tf + (1-t)g}{||tf + (1-t)g||} .$$

Then

$$<Bf(t), f(t)>$$

$$= \frac{t^2<Bf,f> + (1-t)^2 <Bg,g> + 2t(1-t) \text{ Re}<Bf,g>}{||tf + (1-t)g||^2}$$

$= 0$ with our assumptions.

Also,

$$<Af(t), f(t)>$$

$$= \frac{t^2<Af,f> + (1-t)^2<Ag,g> + 2t(1-t) \text{ Re}<Af,g>}{t^2 + (1-t)^2 + 2t(1-t) \text{ Re}<f,g>}$$

$$\geq \frac{t^2\delta + (1-t)^2\delta + 2\delta t(1-t) \text{ Re}<f,g> + 2t(1-t) \text{ Re}<(A-\delta)f,g>}{t^2 + (1-t)^2 + 2t(1-t) \text{ Re}<f,g>}$$

$$= \delta + \frac{2t(1-t) \text{ Re}<(A-\delta) f,g>}{||tf + (1-t)g||^2}$$

$\geq \delta$ since $\text{Re}<(A-\delta)f,g> \geq 0$.

Thus $t \to f(t)$ is a path connecting f to g in M as required.

□

Lemma 4.6 *Let T_1, T_2 and A be self-adjoint operators and*

$$V = \left\{(<T_1f,f>, <T_2f,f>) : ||f|| = 1, <Af,f> \geq \delta, f \in H\right\} .$$

Then V is a convex subset of R^2.

Proof Let L be any straight line in R^2 given by

$$ax + by + c = 0 .$$

It is sufficient to show $V \cap L$ is connected.

Let $B = aT_1 + bT_2 + c$.

Then the mapping π given by $\pi(f) = (\langle T_1 f,f\rangle, \langle T_2 f,f\rangle)$ is continuous and the set

$$\left\{f : ||f|| = 1, \langle Af,f\rangle \geq \delta \text{ and } \pi(f) \in L\right\}$$

$= M$ where M is as given in lemma 4.5.

Thus $V \cap L = \pi(M)$ is connected.

□

Theorem 4.7 *Let T be any operator and let A be a self-adjoint operator. Then the set*

$$W = \left\{\langle Tf,f\rangle : ||f|| = 1 \text{ and } \langle Af,f\rangle \geq \delta\right\}$$

is convex.

Proof Suppose $T = T_1 + iT_2$ where T_1 and T_2 are both self-adjoint. Then

$$W = \left\{x + iy : (x,y) \in V\right\} .$$

where V is as given in lemma 4.6.

Hence W is convex.

□

Corollary 4.8 *For any two operators T and A, the set*

$$\left\{\langle Tf,f\rangle \;:\; \|f\| = 1 \;\text{ and }\; \|Af\| \geq \delta\right\}$$

is convex.

Proof Obvious from theorem 4.7 by replacing A by A^*A and noting that

$$\langle A^*Af,f\rangle \geq \delta^2 \;\text{ if and only if }\; \|Af\| \geq \delta \;.$$

□

Corollary 4.9 *$W_\delta(T)$ is convex.*

Proof Take $A = T$ in corollary 4.8. Thus the set

$$\left\{\langle Tf,f\rangle \;:\; \|f\| = 1 \;\text{ and }\; \|Tf\| \geq \delta\right\}$$

is convex.

$W_\delta(T)$ is the closure of the above set and hence $W_\delta(T)$ is convex.

□

At the beginning of this section we have defined the restricted numerical range $W_S(T)$. We now impose certain properties on S so that $W_S(T)$ becomes convex.

Let $S \subset H$ satisfy the following two properties:

Property (i) $f \in S$ implies $\alpha f \in S$, $|\alpha| = 1$.

Property (ii) $f, g \in S$ implies for all real positive r, either $\dfrac{f + rg}{||f + rg||} \in S$ or $\dfrac{f - rg}{||f - rg||} \in S$.

We give below some examples of such S.

Example 4.10 H itself or any subspace of H, for example, the range or null space of any operator A trivially satisfies properties (i) and (ii).

Example 4.11 A useful example of such a set is

$$S = \left\{ f \in H : ||f|| = 1, \ \langle Af, f\rangle \geq \delta, \ A = A^* \right\} .$$

That S satisfies properties (i) and (ii) can be verified as follows.

If r is real and $f, g \in S$,

$$\frac{\langle A(f+rg), f + rg\rangle}{||f + rg||^2}$$

$$= \frac{\langle Af, f\rangle + r^2\langle Ag, g\rangle + 2r \operatorname{Re}\langle Af, g\rangle}{1 + r^2 + 2r \operatorname{Re}\langle f, g\rangle}$$

$$\geq \delta + \frac{2r \operatorname{Re}\langle (A-\delta) f, g\rangle}{||f + rg||^2} .$$

It is obvious therefore that either $\dfrac{f + rg}{||f + rg||} \in S$ or $\dfrac{f - rg}{||f - rg||} \in S$ for all positive r depending on the sign of $\mathrm{Re}\langle (A-\delta)f, g\rangle$.

Example 4.12 Another example of such an S is

$$S = \left\{ f \in H : ||f|| = 1, ||Af|| \geq \delta \right\}$$

where A is any operator.

This is obvious from example 4.11 by noting that $||Af|| \geq \delta$ is equivalent to $\langle A^*Af, f\rangle \geq \delta^2$.

Theorem 4.13 *Let S be a set with properties (i) and (ii) mentioned above. Then $W_S(T)$ is a convex set in the complex plane.*

Proof Let $||f|| = ||g|| = 1$, $f, g \in S$. For any complex scalar $z = x + iy$ and $0 < t < 1$, consider the equation

$$\frac{\langle T(f+zg), f + zg\rangle}{||f + zg||^2} = t\langle Tf, f\rangle + (1-t)\langle Tg, g\rangle . \qquad \ldots (4.1)$$

Equation (4.1) on simplification yields an expression of the form

$$|z|^2 + Cz + D\bar{z} - \frac{1 - t}{t} = 0$$

where C, D are complex numbers, in general dependent on t.

Separating real and imaginary parts we get

$$x^2 + y^2 + 2ax + 2by - \frac{1 - t}{t} = 0 \qquad \ldots(4.2)$$

and

$$cx + dy = 0 \qquad \ldots(4.3)$$

where a, b, c, d are some real numbers independent of x and y.

Since $\frac{1 - t}{t} > 0$, (4.2) gives an equation of a circle containing the origin and (4.3) gives a straight line through the origin. Hence there will be two values of z of the form $r_1 e^{i\theta}$ and $r_2 e^{i\theta}$ satisfying equations (4.2) and (4.3). But by our assumption either

$$\frac{f + r_1 e^{i\theta} g}{\|f + r_1 e^{i\theta} g\|} \in S \quad \text{or} \quad \frac{f - r_2 e^{i\theta} g}{\|f + r_2 e^{i\theta} g\|} \in S .$$

Thus there exists an element $h \in S$, $\|h\| = 1$ such that

$$\langle Th,h\rangle = t\langle Tx,x\rangle + (1-t)\langle Ty,y\rangle$$

and the proof is complete.

□

By taking $S = H$ we have

Corollary 4.14 *$W(T)$ is convex.*

Indeed a similar technique was used in theorem 1.2 to prove the convexity of $W(T)$.

Since the sets in examples 4.11 and 4.12 have the properties required in theorem 4.13, we have

Corollary 4.15 *The set*

$$\left\{ \langle Tf,f\rangle \;:\; \|f\| = 1, \quad \langle Af,f\rangle \geq \delta, \quad A = A^* \right\}$$

is convex for any operator T.

Proof Take $S = \left\{ f \in H \;:\; \|f\| = 1, \ \langle Af,f\rangle \geq \delta, \ A = A^* \right\}$ and apply theorem 4.13.

□

Corollary 4.16 (Kyle) *The set*

$$\left\{ \langle Tf,f\rangle \;:\; \|f\| = 1 \ \text{ and } \ \|Af\| \geq \delta \right\}$$

is convex for any two operators A and T.

Proof Take $S = \left\{ f \in H \;:\; \|f\| = 1, \quad \|Af\| \geq \delta \right\}$ and apply theorem 4.13.

□

Corollary 4.17 *$W_\delta(T)$ is convex.*

Proof Obvious from corollary 4.16 by taking $A = T$.

□

It is worth noting that using similar proofs it can be shown that $W_S(T)$ is convex if S is equal to any of the sets given below.

$$S_1 = \left\{ f \in H : \|f\| = 1, \quad \|Tf\| \leqslant \delta \right\} ,$$

$$S_2 = \left\{ f \in H : \|f\| = 1, \quad \|Tf\| \gneqq \delta \right\} ,$$

$$S_3 = \left\{ f \in H : \|f\| = 1, \quad \|Tf\| \lneqq \delta \right\} .$$

In this chapter we saw that any extreme point of $W(T)^-$ which is approached by $<Tx_n, x_n>$ with the unit vectors x_n weakly converging to x, must be attained if the weak limit is not zero. For other points on the boundary this result need not in general hold. For such points we obtained an upper bound for the weak limit.

To prove this result we used an inequality in terms of limit supremum as given in theorem 2.14. Since the limits in question do in fact exist, we could have used an inequality (given in lemma 2.9) in terms of f, where f has the properties detailed in lemma 2.7. This provides yet another example of the usefulness of Berberian's technique as described in Chapter 2.

We also saw how an argument based on path-connectedness proved the convexity of $W_\delta(T)$. Then we defined a restricted numerical range $W_S(T)$ and imposed certain conditions on the set S to make $W_S(T)$ convex. This provided another method of obtaining the convexity of $W_\delta(T)$.

CONCLUSION

We have considered the numerical range $W(T)$ of an operator T on a Hilbert space H as a convex set in the complex plane and associated subsets $M_z(T)$ of vectors from H with every point z of the numerical range. We saw that $M_z(T)$ is a subspace if and only if z is an extreme point of $W(T)$. When this is not the case, we obtained results in terms of the linear span of $M_z(T)$. This led to the characterization of $W(T)$ in terms of the subsets $M_z(T)$ as given by Embry. Since this characterization excludes the case of unattained boundary points of $W(T)$, we generalized these results to achieve a characterization of the closure of the numerical range in terms of subsets $N_z(T)$ consisting of bounded sequences of vectors. We saw that the sets $M_z(T)$ and $N_z(T)$ have similar properties and that the two characterizations are also similar, though not exactly alike.

Two Cauchy-Schwartz type inequalities of Embry associated with the union of $M_z(T)$ over all points z on a line of support of $W(T)$ were given and orthogonality of vectors from these subsets was observed. These results were again generalized in terms of sequences of vectors.

We proved these results, sometimes by direct methods, but often using a modification of a technique given by Berberian which involved a change of Hilbert space and operator via a construction based on normalized positive linear functionals.

In Chapter 3 we gave various known results concerning attained boundary points of the numerical range of seminormal and convexoid operators. These results were then extended to unattained boundary points. Again Berberian's technique proved useful in the proofs.

Embry showed that the subsets $M_z(T)$ behave in a particular fashion for several special types of operator. We observed that the generalized subsets $N_z(T)$ retain these characteristics.

Finally, we considered convexity of different numerical ranges. We defined a restricted numerical range $W_S(T)$ and attached certain properties to the set S so that $W_S(T)$ is convex. As a corollary we obtained the convexity of Stampfli's numerical range $W_\delta(T)$, a result proved differently by Kyle.

In Chapter 2 several areas for further investigation suggested themselves. For example, in section 2.2 we proved that corresponding to a line of support L of $W(T)$, the sets $N_L(T)$ and $N(T)$ are closed. Moreover, $N_L(T)$ is a subspace. Is the same true for $N(T)$? Homogeneity being obvious only linearity has to be verified. If we could prove the linearity of $N(T)$, in theorem 2.19 (ii) we would have been able to show $\gamma N_z(T) = N(T)$ where $z \in L$ is a nonextreme boundary point of $W(T)^-$. This is similar to the corresponding result for $M_z(T)$ given in theorem 1.11 (ii).

In lemma 2.18, we showed $N'_a(T) \subset \gamma N_z(T)$ where z lies in the interior of a line segment with end points $a, b \in W(T)^-$. We had to assume $\|x_n\|$ is bounded away from zero as a requirement for the proof. Is it possible to omit this condition and get a result of the form $N_a(T) \subset \gamma N_z(T)$? If this were true this lemma could be used in the proofs of both parts (ii) and (iii) of theorem 2.19.

Embry (1970) gave a theorem in terms of the intersection of maximal subspaces of $M_z(T)$. It is worth investigating whether similar results hold for the intersection of maximal subspaces of $N_z(T)$. If this were possible we would have the following result as a corollary:

If T is hyponormal and z is a boundary point of $W(T)^-$, we have

$$\cap \left\{ \text{maximal subspaces of } N_z(T) \right\}$$
$$= \left\{ (x_n) \in \ell_\infty(H) : Tx_n - zx_n \to 0 \text{ and } T^*x_n - \bar{z}x_n \to 0 \right\} .$$

This, in turn, would lead to an alternative proof of the known result:

For hyponormal T, if z is an extreme point of $W(T)^-$, then z is an approximate eigenvalue of T.

Another question of interest is whether or not the separating functional in lemma 2.7, in addition, can be assumed to satisfy (vi). *f is multiplicative on ℓ_∞ with respect to* *the pointwise algebraic product.* Were this the case, this could lead to new results as well as provide shorter proofs for several results given in the thesis.

These and similar questions seem worthy of further investigation, an investigation which I hope to undertake in the near future.

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