UNATTAINED BOUNDARY POINTS OF THE NUMERICAL RANGE OF HILBERT SPACE OPERATORS

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I certify that the substance of this thesis has not already been submitted for any degree and is not being currently submitted for any other degree.

I certify that any help received in preparing this thesis, and all sources used, have been acknowledged in this thesis.

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(Signature)

ERRATA

"Schwarz" has been consistently miss-spelt as Schwartz.
- Apologies to L. Schwartz for the unnecessary accolade and to H. Schwarz for this echoing of an error in Embry.

CONTENTS

ABSTRACT

The numerical range of an operator on a complex Hilbert space is considered and we introduce various known results associated with different points of the numerical range. By the Toeplitz-Hausdorff theorem, the numerical range is a convex set in the complex plane, though not necessarily closed. Thus these results, though interesting, are inapplicable to the unattained boundary points of the numerical range. Therefore, generalizations of these results which may hold for all points of the closure of the numerical range seem to be called for.

We follow the definitions of M.R. Embry for certain subsets consisting of vectors from the Hilbert space and then extend these definitions to subsets consisting of bounded sequences of vectors. As observed in this thesis, these two types of subsets are very similar in properties. Embry has characterized the numerical range by means of subsets of vectors and we attempt to characterize the closure of the numerical range by means of subsets consisting of sequences. These characterizations, though similar, are not exactly alike. For example, the linearity of the subset still holds when the point associated with it is an extreme point of the closure of the numerical range, but for nonextreme boundary points, the expression for the linear span of the associated subset is somewhat different from that given by Embry.

To prove some of these results we use a modification of a technique developed by Berberian, which involves a change of Hilbert space and operator via a construction based on normalized positive linear functionals. This technique has proved very useful. Throughout our dissertation we will make use of it to obtain generalizations of results by interpreting the results themselves in the new space and operator.

We extend two Cauchy-Schwartz type inequalities of Embry to the unattained boundary points of the numerical range and obtain a result by K.C. Das and B.D. Craven as a corollary. The orthogonal tendency of sequences from subsets associated with the boundary is also noted.

Embry has shown that the subsets associated with points of the numerical range behave in a particular fashion if the operator has special characteristics and vice versa. We achieve some easy generalizations of these results for subsets consisting of sequences. Several results of C.S. Lin, J.G. Stampfli and G. de Barra concerning seminormal and convexoid operators are then extended to the unattained boundary points of the numerical range.

K.C. Das and G. Garske gave a theorem concerning weak convergence to zero at the unattained extreme points of the closure of the numerical range. Das and Craven also gave a bound for the norm of the weak limit of sequences corresponding to points on a line segment on the boundary of the numerical range. We achieve all these results as a simple corollary of a generalized Cauchy-Schwartz inequality.

In the concluding part of our thesis we investigate whether convexity holds for other numerical ranges as well. A restricted numerical range is defined and certain conditions are imposed so that this newly defined numerical range is convex. As a corollary to this result, we deduce the convexity of Stampfli's numerical range, a result proved differently by J. Kyle.

INTRODUCTION

The purpose of this thesis is to first survey different known results for points of the numerical range of Hilbert space operators and then extend these results to include the case of boundary points which fail to lie in the numerical range itself.

Let $W(T)$ denote the numerical range of an operator (that is, a bounded linear transformation) T on a complex Hilbert space H. By the Toeplitz-Hausdorff theorem $W(T)$ is a convex set in the complex plane, but it is not necessarily closed. Thus results which hold for all points of $W(T)$ need not hold for all points of $W(T)^-$, the closure of $W(T)$ and it is worth investigating whether generalizations of these results applicable to $W(T)^-$ are possible.

M.R. Embry (1970) associated certain subsets $M_z(T)$ with each point z of $W(T)$ and showed that if $M_z(T)$ is linear, then z is an extreme point of $W(T)$. J.G. Stampfli (1966) proved the converse of this result. For other points of $W(T)$, Embry (1970) gave results in terms of the linear span of $M_z(T)$ and thus obtained a characterization of the points of $W(T)$ in terms of $M_z(T)$ and its linear span.

K.C. Das and B.D. Craven extended the definitions of these subsets to define similar subsets $N_z(T)$ of bounded sequences of vectors and showed that z is an extreme point of $W(T)^-$ if and only if $N_z(T)$ is a subspace. This suggested that the results of Embry for other points of $W(T)$ might be similarly generalized. We obtain results of this type which are similar but not exactly like those given by Embry.

The proof given by Das and Craven for linearity of $N_z(T)$ with z an extreme point of $W(T)^-$ and for its converse is very much computational in nature. A conceptual proof would have been welcome and we obtain such a proof using a modified technique first developed by Berberian (1962). This technique involves a change of Hilbert space and operator via a construction based on normalized positive linear functionals. These functionals have all the properties of Banach-Mazur generalized limits as used by Berberian except translation invariance. It is important for our proofs that there be a sufficient supply of such functionals to allow us to separate any given bounded strictly positive sequence from the subspace of null sequences.

Since the numerical ranges of T and the new operator are related, in fact the numerical range of the newly constructed operator is equal to $W(T)^-$ [Berberian and Orland (1967)], this technique allows us to apply all the known results on numerical ranges to the new operator. Calculation then enables us to come back to the original space and operator and thus obtain results for $W(T)^-$. This technique has proved very useful and can be applied to many cases where extensions of results to unattained boundary points are required.

Since the properties of $M_z(T)$ and $N_z(T)$ are very similar, it is natural to expect that the Cauchy-Schwartz type inequalities given by Embry (1975) for vectors from $M_z(T)$ and related subspaces with z an attained boundary point of $W(T)$ may be extended to unattained boundary points. We obtain such generalizations by use of Berberian's technique. These results together with an alternative proof of the result of Das and Craven for an extreme point of $W(T)^-$ are to appear in a joint paper by B. Sims and the author.

The first chapter of the thesis is basically a collection of results for points of $W(T)$ as given by Embry. In some cases we provide modified or alternative proofs. These results are generalized in Chapter 2 where we obtain a characterization of $W(T)^-$ in terms of the subsets $N_z(T)$. The lemma used in the proof of this characterization yields a simple demonstration of the convexity of $W(T)$.

The generalized Cauchy-Schwartz inequalities are also included in Chapter 2. The use of limit supremum and limit infinmum in the arguments of the proof strengthens these inequalities and we thus obtain two sharper inequalities. The result of Das and Craven is seen to follow as a corollary to one of these inequalities. Another corollary shows the orthogonal tendency of a pair of sequences which corresponds to the result given by Embry (1975).

Embry (1971) has shown that the subsets $M_z(T)$ behave in a particular fashion if the operator T has special properties and vice versa. She considered, for example, isometries, unitary operators and normal operators and gave conditions in terms of the subsets $M_z(T)$ which characterize these operators. These results appear in Chapter 3. We then generalize them in terms of the subsets $N_z(T)$.

Several results of J.G. Stampfli (1966), G. de Barra (1981) and C.S. Lin (1975) concerning seminormal and convexoid operators are also given in Chapter 3 and then extended to the unattained boundary points of $W(T)$. In proving some of these results we again resort to Berberian's technique. The same technique provides an alternative proof of the known result that a seminormal operator is convexoid.

K.C. Das (1977) proved a theorem concerning the weak limit of sequence of vectors associated with the unattained bare points of $W(T)^-$. G. Garske (1979) extended this result to include all the extreme points of $W(T)^-$. He also gave an example to show that the result is not true for arbitrary boundary points. Das and Craven gave a bound for the norm of the weak limit of sequences associated with such nonextreme boundary points. We explain these results in Chapter 4 and then show how they follow as a simple corollary to one of the generalized Cauchy-Schwartz inequalities.

Stampfli (1970) defined a modified numerical range $W_\delta(T)$ and asked if it was convex. J. Kyle (1977) proved the convexity of $W_\delta(T)$ using ideas which are improvements on basic ideas of N.P. Dekker (1969). Chapter 4 gives his proof as well as an alternative proof. We define a restricted numerical range $W_S(T)$ and impose conditions on the set S to ensure convexity, the convexity of both $W(T)$ and $W_\delta(T)$ follows from that of $W_S(T)$.

In this thesis, especially in the second chapter, several areas for further investigation have suggested themselves. We have mentioned such possibilities in the conclusion. Some of the material presented in the thesis will appear in joint papers by K.C. Das, B. Sims and the author (1, 2, 3).

NOTATION

We use the following notations, the less familiar of which are followed by the page number where the definition first occurs.

$B(H)$	Set of operators on H
$B(T)$	Set of bare points of $W(T)^-$
c	Set of real convergent sequences
c_0	Set of real null sequences
co A	Convex hull of a set A
$d(z_0,W(T))$	Distance of a point z_0 from $W(T)$
$E(T)$	Set of extreme points of $W(T)^-$
γA	Linear span of a set A
H	A complex Hilbert space
ℓ_∞	Set of real bounded sequences
ℓ_∞^*	Dual of ℓ_∞
ℓ_∞^+	Set of bounded nonnegative sequences
ℓ_∞^-	Set of bounded nonpositive sequences
$\ell_\infty(H)$	Set of bounded sequences of vectors from H
$M(T)$	p. 7
$M_z(T)$	p. 7
$N(T)$	p. 29
$N'(T)$	p. 82
$N_L(T)$	p. 29
$N_z(T)$	p. 28

$N'_z(T)$	p. 40
R	Set of real numbers
R^+	Set of nonnegative real numbers
R^p	p-dimensional cartesian space
r(T)	Spectral radius of T
$\sigma(T)$	Spectrum of T
$\sigma_{ap}(T)$	Approximate point spectrum of T
$\sigma_p(T)$	Point spectrum of T
T	An operator (that is, a bounded linear transformation) on H
T*	Hilbert space adjoint of T
W(T)	Numerical range of T
$W(T)^-$	Closure of W(T)
$\partial W(T)$	Boundary of W(T)
$W_\delta(T)$	Stampfli's numerical range, p. 120
$W_S(T)$	Restricted numerical range
w(T)	Numerical radius of T
$<,>$	Inner product
$\| \ \|$	Norm