Chapter 1

Preliminaries

1.1 Basic theory of differential equations

The main mathematical tools used in this thesis are from the theory of differential equations. The whole theory of differential equation is so vast that we can only choose to present here those material that is most relevant to our study here.

1.1.1 The existence and uniqueness theorem of ODE

We will frequently use the existence and uniqueness theorem of ordinary differential equations (ODEs for short) and parabolic initial-boundary value problems.

Consider the initial value problem of an ODE system

$$\begin{cases} \dfrac{dx}{dt} = f(t,x),\ (t,x) \in D, \\ x(t_0) = x_0, \end{cases} \tag{1.1.1}$$

where $D = \{(t,x) \in \mathbb{R}^{N+1} | |t - t_0| < a,\ |x - x_0| < b\}$, a, b are positive constants,

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}, \qquad x_0 = \begin{pmatrix} x_1^0 \\ \vdots \\ x_N^0 \end{pmatrix}, \qquad f = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}, \qquad \frac{dx}{dt} = \begin{pmatrix} \dfrac{dx_1}{dt} \\ \vdots \\ \dfrac{dx_N}{dt} \end{pmatrix}.$$

Theorem 1.1.1 *Suppose $f(t, x)$ is a continuous function on D and there is a constant $L > 0$ such that*

$$|f(x) - f(y)| \leq L|x - y|.$$

Then there exists a positive constant $\delta = \delta(L)$ such that (1.1.1) *has a unique solution $x = x(t)$ defined in $|t - t_0| \leq \delta$.*

For a parabolic initial-boundary value problem, often we can use the contraction mapping theorem together with the L^p estimates and Sobolev embedding to obtain the local existence and uniqueness. Detailed discussion will be presented in the main text where it is necessary.

1.1.2 Function spaces

We collect here some function spaces that most commonly used in the theory of partial differential equations.

Let Ω be a bounded domain in $\mathbb{R}^N$. The Banach space $C(\overline{\Omega})$ is defined as the set of uniformly continuous functions on $\overline{\Omega}$ with the norm

$$\|u\|_\infty = \|u\|_{C(\overline{\Omega})} = \sup_{x \in \Omega} |u(x)|.$$

The Banach space $C^m(\overline{\Omega})$ is the set of functions that are m times continuously differentiable over $\overline{\Omega}$ with the norm

$$\|u\|_{C^m(\overline{\Omega})} = \sum_{\beta \leq m} \|D^\beta u\|_{C(\overline{\Omega})},$$

where $\beta = (\beta_1, \beta_2, \cdots, \beta_N)$ is the multi-index, $|\beta| = \sum_{j=1}^N \beta_j$, and

$$D^\beta u = \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \cdots \partial x_N^{\beta_N}}.$$

A function u is Hölder continuous with exponent $\alpha \in (0, 1]$, if for all $x, y \in \Omega$,

$$|u(x) - u(y)| \leq C|x - y|^\alpha$$

for some constant C.

The Hölder space $C^{m,\alpha}(\overline{\Omega})$ is a Banach space consisting of functions that are m-times continuously differentiable and whose m-th partial derivatives are α-Hölder continuous, and endowed with the norm

$$\|u\|_{C^{m,\alpha}(\overline{\Omega})} = \|u\|_{C^m(\overline{\Omega})} + \sum_{|\beta|=m} [D^\beta u]_\alpha,$$

where

$$[D^\beta u]_\alpha = \sup_{x,y\in\Omega, x\neq y} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}.$$

Denote by $C_0^\infty(\Omega)$ the set of infinitely differentiable functions with compact support in Ω. Let u be a locally integrable function on Ω. A locally integrable function v is the β-th weak partial derivative of u if

$$\int_\Omega \phi v = (-1)^{|\beta|} \int_\Omega u D^\beta \phi$$

for all $\phi \in C_0^\infty(\Omega)$.

For $p \in [1,\infty]$ and m a nonnegative integer, the Sobolev space $W^{m,p}(\Omega)$ is defined as the set of functions whose first m orders weak partial derivatives are L^p-integrable, endowed with the norm

$$\|u\|_{m,p} = \begin{cases} \left(\sum_{|\beta|\leq m} \int_\Omega |D^\beta u|^p \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \sum_{|\beta|\leq m} \text{ess sup}_{x\in\Omega} |D^\beta u| & \text{if } p = \infty. \end{cases}$$

We say that a domain Ω is smooth if $\partial\Omega$ can be locally expressed as the graph of a smooth function.

A Banach space X_1 is said to be continuously embedded into a Banach space X_2, written $X_1 \hookrightarrow X_2$, if $X_1 \subset X_2$ and the injection mapping from X_1 to X_2 is continuous. The embedding is said to be compact, written $X_1 \hookrightarrow\hookrightarrow X_2$, if the injection mapping is also compact.

There are vast literatures on the embedding and compact embedding between different function spaces, we refer readers to e.g. [1], [27], [45] and the references therein for details. We only quote here a result that we will use in the main text.

Theorem 1.1.2 (Theorem 7.26 of [27]) *Suppose Ω is a smooth domain. Then*

$$W^{m,p}(\Omega) \hookrightarrow\hookrightarrow C^{k,\alpha}(\overline{\Omega}) \;\; \text{if} \;\; 0 \leq k < m - N/p < k+1, \;\; \alpha < m - N/p - k.$$

From the above theorem, if in particular, Ω is an open interval, $m = 2$ and $p > 1$, we have

$$W^{2,p}(\Omega) \hookrightarrow\hookrightarrow C^1(\overline{\Omega}).$$

For parabolic problems we have the Sobolev spaces involving spatial and temporal variables. Let $Q = Q_T = \Omega \times (0,T)$ where Ω is smooth bounded domain in $\mathbb{R}^N$ and $0 < T < \infty$. $W^{2,1;p}(Q)$ is the space of functions $u \in L^p(Q)$ satisfying $u_t, D_x u, D_x^2 u \in L^p(Q)$, endowed with the norm

$$\|u\|_{2,1;p} = \|u\|_{2,1;p;Q} := \|u\|_{p;Q} + \|u_t\|_{p;Q} + \|D_x u\|_{p;Q} + \|D_x^2 u\|_{p;Q}.$$

Given $\alpha \in (0,1]$, we set

$$[u]_{\alpha;Q_T} = \sup\left\{\frac{|u(x,t)-u(y,s)|}{|x-y|^\alpha + |t-s|^{\alpha/2}} : x, y \in \Omega, t, s \in (0,T), (x,t) \neq (y,s)\right\}.$$

Let k be a nonnegative integer, $\alpha \in (0,1)$ and $a = k + \alpha$. Then if we put

$$\|u\|_{a,a/2;Q_T} := \sum_{|\beta|+2j \le k} \max_{\overline{Q_T}} |D_x^\beta D_t^j u| + \sum_{|\beta|+2j=k} [D_x^\beta D_t^j u]_{\alpha;Q_T}$$

and

$$C^{a,a/2}(\overline{Q_T}) := \{u : \|u\|_{a,a/2;Q_T} < \infty\},$$

then $C^{a,a/2}(\overline{Q_T})$ is a Banach space with the norm $\|\cdot\|_{a,a/2;Q_T}$. We have the well-known imbedding theorem:

Theorem 1.1.3 (Lemmas II.3.3, II.3.4 of [45]) *If $p > N+2, a < 2 - (N+2)/p$ and Ω is a smooth domain in $\mathbb{R}^N$, then*

$$W^{2,1;p}(Q_T) \hookrightarrow C^{a,a/2}(\overline{Q_T}).$$

1.1.3 Basic theory of second order linear elliptic and parabolic equations

Suppose Ω is a smooth bounded domain in $\mathbb{R}^N (N \ge 1)$. A second order linear (uniform) elliptic partial differential equation is of the form

$$Lu = \sum_{i,j=1}^N a_{ij}(x)\partial_i\partial_j u + \sum_{i=1}^N b_i(x)\partial_i u + c(x)u = f(x), \tag{1.1.2}$$

where a_{ij}, b_i,c and f are measurable functions satisfying the uniform elliptic condition

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \text{ for some } \lambda > 0 \text{ and all } x \in \Omega,\ \xi \in \mathbb{R}^N.$$

It is usually also assumed there exists a constant $\Lambda > 0$ such that

$$\sum_{i,j=1}^{N} |a_{ij}| + \sum_{i=1}^{N} |b_i| + |c| \leq \Lambda.$$

A second order linear (uniform) parabolic equation is of the form

$$\partial_t u - Au = f(x,t),$$

where

$$Au = \sum_{i,j=1}^{N} a_{ij}(x,t)\partial_i\partial_j u + \sum_{i=1}^{N} b_i(x,t)\partial_i u + c(x,t)u, \tag{1.1.3}$$

and there exist constant $\lambda > 0$, $\Lambda > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \geq \lambda|\xi|^2 \text{ for all } (x,t) \in Q_T,\ \xi \in \mathbb{R}^N,$$

$$\sum_{i,j=1}^{N} |a_{ij}(x,t)| + \sum_{i=1}^{N} |b_i(x,t)| + |c(x,t)| \leq \Lambda \text{ for all } (x,t) \in Q_T.$$

Theorem 1.1.4 (Theorem 15.2 of [2]) *Let Ω be a bounded domain with C^2 boundary $\partial\Omega$. Suppose $a_{ij} \in C(\overline{\Omega}), b_i, c \in L^\infty(\Omega)$ and $f \in L^p(\Omega)$. Then for any $u \in W^{2,p}(\Omega)$ satisfying*

$$Lu = f \text{ in } \Omega,\ \partial_\nu u = 0 \text{ on } \partial\Omega, \tag{1.1.4}$$

we have

$$\|u\|_{W^{2,p}(\Omega)} \leq C\left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}\right),$$

where C depends on L and Ω only.

From this theorem, if in particular, $u \in W^{2,p}(\Omega)$ for some $p > 1$ and $f \in L^\infty(\Omega)$, we will have $u \in W^{2,p}(\Omega)$ for any $p > 1$. Henceforth, if in addition, Ω is an open interval, we have from Theorem 1.1.2, $u \in C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in (0,1)$.

Theorem 1.1.5 (Theorem 7.35 of [48]) *Let Ω be a bounded domain with C^2 boundary $\partial\Omega$. Suppose $a_{ij} \in C(\overline{Q_T})$ and b_i, c are bounded functions, $f \in L^p(Q_T)$. Then for any $u \in W^{2,1;p}(Q_T)$ satisfying*

$$u_t - Au = f \ \text{ in } \ Q_T, \ \ \partial_\nu u = 0 \ \text{ on } \ \partial\Omega \times (0,T), \ \ u(x,0) = 0, \tag{1.1.5}$$

we have

$$\|u\|_{2,1;p;Q_T} \leq C\left(\|u\|_{p;Q_T} + \|f\|_{p;Q_T}\right), \tag{1.1.6}$$

where C depends on A and Q_T only.

With Theorem 1.1.5, one can use the same method as in the proof of Theorem 7.32 of [48](for the existence and uniqueness of $W^{2,1;p}$ solution of initial Dirichlet value problem) the following

Theorem 1.1.6 *Under the assumption of Theorem 1.1.5, the problem*

$$u_t - Au = f \ \text{ in } \ Q_T, \ \ \partial_\nu u = 0 \ \text{ on } \ \partial\Omega \times (0,T), \ \ u(x,0) = 0,$$

has a unique solution $u \in W^{2,1;p}(Q_T)$ and moreover

$$\|u\|_{2,1;p;Q_T} \leq C\|f\|_{p;Q_T},$$

with C depends on A and Q_T only.

1.2 Principal eigenvalues and maximum principles

Concerning the eigenvalues of elliptic operators, we have the following two well-known theorems:

Theorem 1.2.1 (Theorem A3.1 of [17]) *Let L be uniformly elliptic with $C^\alpha(\overline{\Omega})$ coefficients. Suppose that Ω is $C^{2,\alpha}$ and $\sigma \in C^{1,\alpha}(\partial\Omega)$ when B is of Robin type. Then the eigenvalue problem*

$$Lu = \lambda u \ \text{ in } \ \Omega, \quad Bu = 0 \ \text{ on } \ \partial\Omega, \tag{1.2.1}$$

has a nonzero $(C^{2,\alpha}(\overline{\Omega}))$ solution if and only if $\lambda \in \Sigma$, where Σ is a sequence of complex numbers $\{\lambda_n\}$, called eigenvalues, with λ_1 real, and satisfying $\lambda_1 < \inf_{k\geq 2} Re(\lambda_k)$. Moreover, complex *eigenvalues come in pairs of the form* $\eta+\xi i$*, and by a nonzero solution of* (1.2.1) *with* $\lambda = \eta+\xi i$*, we mean a function of the form* $u = v + wi$*, where* v *and* w *are real valued functions. Clearly* $v - wi$ *is a nonzero solution with* $\lambda = \eta - \xi i$

Theorem 1.2.2 (Theorem 1.3 of [17]) *Under the conditions of Theorem 1.2.1. The first eigenvalue* λ_1 *of* (1.2.1)*, also called the principal eigenvalue of* (1.2.1)*, is simple and corresponds to a positive eigenfunction; none of the other eigenvalues corresponds to a positive eigenfunction.*

Theorem 1.2.3 (The weak maximum principle, Corollary 3.2 of [27]) *Let* L *be elliptic in the bounded domain* Ω*. Suppose that in* Ω*,* $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *satisfies*

$$Lu \geq 0 (\leq 0), \quad c \leq 0.$$

Then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega}\{u, 0\} (\inf_{\Omega} u \geq \inf_{\partial\Omega}\{u, 0\}).$$

Theorem 1.2.4 (The Hopf boundary lemma, Lemma 3.4 of [27]) *Suppose that* L *is uniformly elliptic,* $c = 0$ *and* $Lu \geq 0$ *in* Ω*, where* $u \in C^2(\Omega)$*. Let* $x_0 \in \partial\Omega$ *be such that*

(i) u *is continuous at* x_0*;*

(ii) $u(x_0) > u(x)$ *for all* $x \in \Omega$*;*

(iii) $\partial\Omega$ *satisfies an interior sphere condition at* x_0*.*

Then

$$D_\nu u(x_0) > 0$$

whenever the directional derivative exists, where ν *is a unit vector pointing outward of* Ω *at* x_0*. If* $c \leq 0$ *and* c/λ *is bounded, the same conclusion holds provided* $u(x_0) \geq 0$*, and if* $u(x_0) = 0$ *the same conclusion holds irrespective of the sign of* c*.*

Theorem 1.2.5 (The strong maximum principle, Theorem 3.5 of [27]) *Let* L *be uniformly elliptic,* $c = 0$ *and* $Lu \geq 0 (\leq 0)$ *in a domain* Ω *(not necessarily bounded). Then if* u *achieves its maximum (minimum) in the interior of* Ω*,* u *is a constant. If* $c \leq 0$ *and* c/λ *is bounded, then* u

cannot achieve a non-negative maximum (non-positive minimum) in the interior of Ω *unless it is a constant.*

The classic maximum principles have natural extensions to operators in divergence form:

$$\mathcal{L}u = D_i(a^{ij}(x)D_j u + b^i(x)u) + c^i(x)D_i u + d(x)u$$

with the assumptions

$$\begin{cases} a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \text{ for some } \lambda > 0 \text{ and all } \xi \in \mathbb{R}^N, \\ a^{ij}, b^i, c^i, d \in L^\infty(\Omega), \end{cases}$$

and

$$\int_\Omega (dv - b^i D_i v)dx \leq 0 \text{ for all non-negative } v \in C_0^\infty(\Omega). \tag{1.2.2}$$

We call a function $u \in W^{1,2}(\Omega)$ satisfies $\mathcal{L}u = 0(\geq 0, \leq 0)$ respectively in Ω, if

$$\int_\Omega \{a_{ij}D_j u + b_i u)D_i v - (c_i D_i u + du)v\}\, dx = 0(\leq 0, \geq 0).$$

for any $v \geq 0, v \in C_0^\infty(\Omega)$.

Theorem 1.2.6 (Theorem 8.1 of [27]) *Let* $u \in W^{1,2}(\Omega)$ *satisfy* $\mathcal{L}u \geq 0(\leq 0)$ *in* Ω*. Then*

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ (\inf_\Omega u \geq \inf_{\partial\Omega} u^-),$$

where $\sup_{\partial\Omega} u := \inf\{k \in \mathbb{R}^1 : (u-k)^+ \in W_0^{1,2}(\Omega)\}$*;* $\inf_{\partial\Omega} u = -\sup_{\partial\Omega}(-u)$*.*

Theorem 1.2.7 (Theorem 8.19 of [27]) *Let* $u \in W^{1,2}(\Omega)$ *satisfy* $\mathcal{L}u \geq 0$ *in* Ω*. Then, if for some ball* $B \subset\subset \Omega$*, we have* $\sup_B u = \sup_\Omega u \geq 0$*, the function* u *must be constant in* Ω *and equality holds in* (1.2.2) *when* $u \not\equiv 0$*.*

For the parabolic operator $\partial_t u - Au$ we have the following maximum principle:

Theorem 1.2.8 (Theorem 2 of [54]) *Assume* $c(x,t) \geq 0$ *in* Q_T*. Let* $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ *and satisfies*

$$\partial_t u - Au \leq 0(\geq 0) \quad \textit{in} \quad Q_T.$$

If $u \le M(u \ge m)$ *on* $\overline{Q_T}$ *and there is* $(x_1, t_1) \in Q_T$ *such that* $u(x_1, t_1) = M(u(x_1, t_1) = m)$ *and* $M \ge 0(m \le 0)$ *if* $c(x,t) \not\equiv 0$*. Then*

$$u(x,t) \equiv M \;\; (u(x,t) \equiv m).$$

We also have the parabolic Hopf's lemma.

Theorem 1.2.9 (Theorem 3 of [54]) *Assume* $c(x,t) \ge 0$ *in* Q_T*. Let* $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ *and satisfy*

$$\partial_t u - Au \le 0 (\ge 0) \quad in \;\; Q_T.$$

If $u \le M(u \ge m)$ *on* $\overline{Q_T}$ *and there is* $(x_2, t_2) \in \partial\Omega \times (0, T]$ *such that* $u(x_2, t_2) = M(u(x_1, t_1) = m)$ *and* $M \ge 0(m \le 0)$ *if* $c(x,t) \not\equiv 0$*. If* $\partial_\nu u(x_2, t_2)$ *exists and* $\partial\Omega$ *satisfies the interior ball condition at* x_2*, then*

$$\partial_\nu u(x_2, t_2) > 0 \; (\partial_\nu u(x_2, t_2) < 0).$$

As a consequence of the parabolic maximum principle and the Hopf's lemma, we have the following comparison principle

Theorem 1.2.10 (Theorem 8 of [54]) *Let* $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ *and satisfy*

$$\partial_t u - Au \ge 0 \;\; in \; Q_T, \quad a\partial_\nu u + b(x,t)u \ge 0 \;\; on \; (\overline{Q_T} \setminus (\Omega \times \{T\}) \;\; and \;\; u(x,0) \ge 0 \;\; in \;\; \Omega,$$

where $a = 0, b = 1$*, or* $a = 1$ *and* $b(x,t) \ge 0$*. Then*

$$u(x,t) \ge 0 \quad in \;\; Q_T.$$

If further $u(x,0) \not\equiv 0$ *in* Ω*, then*

$$u(x,t) > 0 \quad in \;\; Q_T.$$

1.3 Topological degree and bifurcation theorems

Let X be a real Banach space and $\Omega \subset X$ be open bounded. Denote by $K(\overline{\Omega})$ the set of completely continuous operators mapping $\overline{\Omega}$ into X. Let

$$M = \{(I - F, \Omega, y) : F \in K(\overline{\Omega}) \text{ and } y \notin (I - F)(\partial\Omega)\}.$$

Then there exists exactly one function $deg : M \to \mathbf{Z}$ satisfying

(d1) $deg(I, \Omega, y) = 1$ for $y \in \Omega$;

(d2) $deg(I - F, \Omega, y) = deg(I - F, \Omega_1, y) + deg(I - F, \Omega_1, y)$ where Ω_1 and Ω_2 are disjoint open subsets of Ω such that $y \notin \overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)$;

(d3) $deg(I - H(t, \cdot), \Omega, y(t))$ is independent of $t \in [0, 1]$ whenever $H : [0, 1] \times \overline{\Omega} \to X$ is completely continuous, $y : [0, 1] \to X$ is continuous and $y(t) \notin (I - H(t, \cdot))(\partial\Omega)$ on $[0, 1]$.

If we replace the Banach space X with a positive cone P in X and the completely continuous mapping is from P to P, we can define topological degree on cones. If in the definition of topological degree, the open bounded domain is a neighborhood of an isolated fixed point (or a compact set of fixed points), then we call the topological degree the fixed point index of the isolated fixed point (the compact set of fixed points).

We will use in the thesis the following global bifurcation theorem, which is a variation of that appeared in [55].

Theorem 1.3.1 (Theorem 29.2 of [15]) *Let X be a real Banach space, $K \subset X$ a cone, $T \in L(X)$ positive and $T|_K$ compact, $G : \mathbb{R}^+ \times K \to X$ completely continuous and $G(\lambda, x) = o(|x|)$ as $x \to 0$ uniformly in λ from compact subsets of $\mathbb{R}^+$. Suppose also that*

(a) $x = 0$ is the only fixed point of $G(0, \cdot)$ and $F_0(\lambda, x) = \lambda Tx + G(\lambda, x) \in K$ on $\mathbb{R}^+ \times K$;

(b) $T|_K$ has characteristic values $\lambda_1, \cdots, \lambda_m$, for some $m \geq 1$.

Then at least one $(\lambda_i; 0)$ is a positive bifurcation point for $x - F_0(\lambda, x) = 0$ and the component of $\bar{M}$ containing this point is unbounded, where

$$M = \{(\lambda, x) \in \mathbb{R}^+ \times K : x = F_0(\lambda, x), \lambda > 0 \text{ and } x \in K \setminus \{0\}\}.$$

In this thesis, usually $X = C([0, h])$ and $K = \{u \in C([0, h]) : u(x) \geq 0 \text{ for all } x \in [0, h]\}$ and T_K has only one characteristic value λ_1. By this theorem, we have an unbounded continuum bifurcating from $(\lambda_1, 0)$ and remains in K.

The following fixed point index calculation theorem plays an important role in the thesis.

Theorem 1.3.2 (Lemma 2.1 of [14]) *Let $E_1,\ E_2$ be ordered Banach spaces with positive cones $C_1,\ C_2$, respectively, $E = E_1 \times E_2$ and $C = C_1 \times C_2$. Let D be an open set in C containing 0 and $S_i : \overline{D} \to C_i$ be completely continuous operators, $i = 1, 2$. Denote by (u, v) a general element in C with $u \in C_1,\ v \in C_2$ and $S(u, v) = (S_1(u, v), S_2(u, v)),\ C_2(\varepsilon) = \{v \in C_2 : \|v\|_{E_2} < \varepsilon\}$. Suppose $U \subset C_1 \cap D$ is relatively open and bounded, and*

$$S_1(u, 0) \neq u \ \ \textit{for} \ \ u \in \partial U, \ \ S_2(u, 0) \equiv 0 \ \ \textit{for} \ \ u \in \overline{U}.$$

Suppose $S_2 : \ D \to C_2$ extends to a continuously differentiable mapping of a neighborhood of D into E_2, $C_2 - C_2$ is dense in E_2 and $\Sigma = \{u \in U : \ u = S_1(u, 0)\}$. Then the following conclusions are true:

(i) if for any $u \in \Sigma$, the spectral radius $r(S_2'(u, 0)|_{C_2}) > 1$ and 1 is not an eigenvalue of $S_2'(u, 0)|_{C_2}$ corresponding to a positive eigenvector, then for $\varepsilon > 0$ small, $\deg_C(I - S, U \times C_2(\varepsilon), 0)$ is defined and equals 0,

(ii) if for any $u \in \Sigma$, $r(S_2'(u, 0)|_{C_2}) < 1$, then $\deg_C(I - S, U \times C_2(\varepsilon), 0)$ and $\deg_{C_1}(I - S_1|_{C_1}, U, 0)$ are both defined for $\varepsilon > 0$ small and they are equal.

Chapter 2

Background and outline of research

Phytoplankton are photosynthesizing microscopic organisms that inhabit the upper sunlit layer of almost all oceans and bodies of fresh water. Through photosynthesis they create organic compounds from carbon dioxide dissolved in the water, a process that sustains the aquatic food web. Phytoplankton are also responsible for much of the oxygen present in the Earth's atmosphere.

Phytoplankton species typically compete for nutrients and light in aquatic environments ([20, 21, 37, 42, 63]. But in oligotrophic ecosystems with ample supply of light, they compete only for nutrients [43, 49], and in eutrophic environments with ample nutrients supply, they compete only for light [24, 35]. In a water column, a phytoplankton population diffuses due to water turbulence, but they also sink or buoy. Most phytoplankton are heavier than water and have a tendency to sink. On the other hand, some important phytoplankton, like some cyanobacteria, green algae, have a lower density than water and will float [24].

2.1 The models and existing mathematical works

The simplest situation in phytoplankton dynamics is when the nutrients supply is ample and the phytoplankton compete only for lights. In 1949, the classic work of Riley et al. [56] investigated phytoplankton concentration in a vertical water column using a simple linear partial differential equation. In 1995, Huisman and Weissing developed a theory of interspecific competition for light that assumes complete mixing of phytoplankton species. This theory is based on a system of ordinary differential equations. In 1999 and 2006, Huisman et al. [35, 37] introduced a reaction-diffusion model which is a very promising basic model for studying phytoplankton dynamics: consider a water column with a cross section of one unit area and with n phytoplankton species. Let x denote the depth within the water column where x runs from 0 (top) to h (bottom). And let $u_i(x,t)$ denote the population density of a phytoplankton species i at depth x and time t. The rate of changes in phytoplankton densities is described by the following system of reaction-diffusion equations:

$$(u_i)_t = (D_i(x)\,(u_i)_x - \alpha_i(x)u_i)_x + (g_i(I(x,t)) - d_i)\,u_i,\ i = 1, \cdots, n, \tag{2.1.1}$$

where $g_i(I(x,t))$ is the specific growth rate of phytoplankton species i as a function of light intensity $I(x,t)$, the continuous functions $D_i(x) > 0$ is the diffusion coefficients, $\alpha_i(x)$ is the sinking ($\alpha_i(x) > 0$) or buoyant ($\alpha_i(x) < 0$) speed, and d_i is the loss rate of the phytoplankton species i. The water column is assumed to be closed, and thus the zero-flux boundary conditions are imposed:

$$D_i(x)(u_i)_x(x,t) - \alpha_i(x)u_i(x,t) = 0,\ x = 0, h,\ t \geq 0,\ \ i = 1, \cdots, n. \tag{2.1.2}$$

The initial conditions are

$$u_i(x,0) = u_i^0(x) \gneqq 0,\ 0 \leq x \leq h,\ \ i = 1, \cdots, n. \tag{2.1.3}$$

The specific growth rate $g_i(I)$ satisfies

$$g_i(0) = 0,\ \ g_i'(I) > 0 \text{ for } I \geq 0,$$

and there are positive constants c_i, γ_i such that

$$g_i(I) \leq c_i I^{\gamma_i} \ \text{ for any } \ I \geq 0.$$

The light intensity takes the form

$$I(x,t) = I_0 e^{-k_0 x} \exp\left(- \int_0^x \left[k_1 u_1(s,t) + \cdots + k_n u_n(s,t)\right] ds \right),$$

where I_0 is the incident light intensity, k_0 is the background turbidity and k_i is the specific light attenuation coefficient of the phytoplankton species i.

Typical growth functions $g_i(I)$ take the form

$$g_i(I) = \frac{m_i I}{\gamma_i + I},$$

or

$$g_i(I) = \frac{m_i}{\gamma_i}(1 - e^{-\gamma_i I}),$$

or

$$g_i(I) = I^{\gamma_i},$$

where m_i, γ_i are positive constants for all $i = 1, \cdots, n$.

The first part of the thesis is a continuation of the recent works [22], [33], and the much earlier research [40]. These works are mainly concerned with the one species phytoplankton case. In this case (2.1.1)-(2.1.3) becomes

$$\begin{cases} u_t = J_x(x,t) + \left[g\left(e^{-k_0 x - k\int_0^x u(s,t)ds}\right) - d\right]u, & 0 < x < h, \quad t > 0, \\ J(x,t) = D(x)u_x(x,t) - \alpha(x)u(x,t) = 0, & x = 0 \text{ or } h, \quad t > 0, \\ u(x,0) = u_0(x) \gneqq 0, & 0 \leq x \leq h. \end{cases} \tag{2.1.4}$$

In [22], the special case $D(x) \equiv D$ (a positive constant) and $\alpha(x) \equiv 0$ was considered. It was shown that for this special case, there exists a critical death rate $d^* > 0$ such that for $0 < d < d^*$, (2.1.4) has a unique positive steady state $u^*(x)$, and it has no positive steady state when $d \geq d^*$; moreover, if $u(x,t)$ is the unique solution of (2.1.4), then regardless of what initial function, $u_0(x) \gneqq 0$, is taken, one always has

$$\lim_{t\to\infty} u(x,t) = \begin{cases} 0 & \text{if } d \geq d^*, \\ u^*(x) & \text{if } 0 < d < d^*. \end{cases}$$

However, no analysis was given in [22] on the change of behavior of the model as the other parameters vary.

In [33], the case $D(x) \equiv D > 0$ and $\alpha(x) \equiv \alpha \in (-\infty, \infty)$ was investigated. In this case, again it was shown that there exists a critical death rate d^* so that for $0 < d < d^*$, (2.1.4) has a unique positive steady state, and it has no positive steady state when $d \geq d^*$. The dynamical behavior of (2.1.4) was not considered in [33] but Hsu and Lou carefully examined how the critical death rate d^* changes as the other parameters D, α and h vary, and certain interesting critical values of D, α and h were obtained through the analysis of d^*. The asymptotic profile of the positive steady state was studied for the case $\alpha \to +\infty$ and the case $\alpha \to -\infty$. Several open questions arise from [33]. For example, with D and α suitably fixed and regarding d^* as a function of h: $d^* = d^*(h)$, Hsu and Lou proved that $d^*(h)$ is a strictly decreasing function of h with $d^*_\infty := \lim_{h\to\infty} d^*(h) \in [0, \infty)$. They showed that $d^*_\infty > 0$ when $\alpha < \alpha_0$ for some $\alpha_0 > 0$ but the question whether $d^*_\infty > 0$ for all $\alpha \in (-\infty, \infty)$ was left open. We will give a complete answer to this question: There exists a unique $\alpha^* > 0$ such that $d^*_\infty > 0$ if and only if $\alpha < \alpha^*$.

In [40], the case $D(x) \equiv D > 0$, $\alpha(x) \equiv \alpha > 0$, and $h = \infty$, was studied, where the boundary condition at $x = h$ should be replaced by $u(x, t) \to 0$ as $x \to \infty$. For such a case, sharp necessary and sufficient conditions were given for the existence and uniqueness of positive steady state, and it was shown that the unique positive steady state is the global attractor of (2.1.4) (with $h = \infty$) whenever it exists. The case $D(x) \equiv D > 0, \alpha(x) \equiv \alpha > 0$ and $h < \infty$ was considered in [41] with a sketch of the proof given, using a very different approach from the one used in this thesis.

[22] also considered the two species $(n = 2)$ case. In this case, [22] studied the necessary condition for the existence of positive steady state and identified a set of (d_1, d_2) for which (2.1.1)-(2.1.3) has at least one positive steady state. [22] also studied the conditions on which the positive solution of (2.1.1)-(2.1.3) will persist or extinct. In their paper, Du and Hsu assumed $D(x) = D$, a positive constant, and $\alpha(x) \equiv 0$.

While much mathematical theory has been established for one or two species phytoplankton systems, little is achieved in this direction for three or more species models. In multi-species micro-organism competitions, it was predicted in Huisman and Weissing [38, 39, 62] that inter-specific competition in well-mixed environment leads to competitive exclusion, similar to that in [31, 32]. However it is widely observed in multi-species phytoplankton communities that these communities often appear to violate the competitive exclusion principle [4]. This phenomenon is the so called paradox of plankton, see [30] and the reference therein. Coexistence and persistence of two competing phytoplankton species have been established mathematically in [22]. We will extend the results of [22] on two species case to more than two species case.

2.2 Outline of research

In the main part of this thesis, Chapter 3-Chapter 5, we study model (2.1.1)-(2.1.3) for one species ($n = 1$) case, as well as multi-species ($n \geq 2$) case.

In Chapter 3, we consider the one species case, namely, (2.1.4). Our first result is the existence and uniqueness of positive steady state solution.

Theorem 2.2.1 (Theorem 3.2.1) *Problem* (2.1.4) *has no positive steady state solution for* $d \notin (0, d^*)$ *and it has a unique positive steady state solution for* $d \in (0, d^*)$*, where* $d^* = -\lambda_1(-g(e^{-k_0 x}))$ *and* $\lambda_1(\Psi)$ *is the principal eigenvalue of the eigenvalue problem*

$$\begin{cases} (-D(x)\phi' + \alpha(x)\phi)' + \Psi(x)\phi = \lambda\phi, & 0 < x < h, \\ D(x)\phi'(x) - \alpha(x)\phi(x) = 0, & x = 0, h. \end{cases}$$

The second result is concerned with the properties of the unique steady state solution.

Theorem 2.2.2 (Theorem 3.2.2) *If we denote the uniqueness positive steady state solution of* (2.1.4) *by* u_d*, then*

(i) $d \to u_d$ *is continuous from* $(0, d^*)$ *to* $C^2([0, h])$,

(ii) $0 < d_1 < d_2 < d^*$ *implies* $u_{d_1}(0) > u_{d_2}(0)$,

(iii) $0 < d_1 < d_2 < d^*$ *implies* $\int_0^x u_{d_1}(s)ds > \int_0^x u_{d_2}(s)ds$ *for all* $x \in (0, h]$.

(iv) $u_d \to 0$ *uniformly in* $[0, h]$ *as* $d \to d^*$, $du_d \to \frac{e^{R(x)}}{\int_0^h e^{R(s)}ds} \int_0^\infty g(e^{-ks})ds$ *uniformly in* $[0, h]$ *as* $d \to 0$*, where* $R(x) = \int_0^x \frac{\alpha(s)}{D(s)}\,ds$.

The third result of Chapter 3 is about the dynamic behavior of (2.1.4).

Theorem 2.2.3 (Theorem 3.3.1) *If $d \geq d^*$, then the solution $u(x,t)$ of* (2.1.4) *converges to 0 uniformly for $x \in [0,h]$ as $t \to \infty$.*

If $0 < d < d^*$, then $u(x,t)$ converges to the unique steady state $u_d(x)$ uniformly for $x \in [0,h]$ as $t \to \infty$.

The above three theorems extend the results of [22] from the special case that $D(x) \equiv D > 0$ and $\alpha(x) \equiv 0$ to the general case that $D(x)$ and $\alpha(x)$ are arbitrary positive functions with continuous derivatives. We note that vertical diffusion in a phytoplankton water column caused by the wind and wave actions is in general inhomogeneous. Remarkably, in a recent paper [57], Ryabov, Rudolf and Blasius made use of a variable but continuous diffusion rate $D(x)$ in their model to demonstrate through numerical simulation that for some parameter ranges, the phytoplankton dynamics exhibits a bistable behavior: depending on the initial state, the phytoplankton population may stabilize at a steady state with maximum in an upper mixed layer (UML), or at a steady state with maximum below the UML, the latter representing a deep chlorophyll maxima (DCM). This is in agreement with earlier numerical simulation results obtained by Yoshiyama and Nakajima [64] based on a more simplified model where $D(x)$ was assumed to be ∞ near the water surface (representing a complete mixing layer above a seasonal thermocline) and it is assumed to be finite and positive below the seasonal thermocline. In contrast, our results here show that the phytoplankton always stabilizes at a unique steady state, regardless of its initial state. However, this is not in contradiction to [57] and [64], since in our model ample nutrients supply is assumed while in both [57] and [64] the phytoplankton is nutrient limited. Thus our theoretical results here imply that limitation of nutrients is a necessary condition for the bistable behavior of phytoplankton dynamics demonstrated in [57] and [64]; inhomogeneous diffusion and sinking alone cannot cause such bistable behavior.

Though the ideas in the proof of our general results on the dynamical behavior follow [22], significant changes are needed in the detailed arguments.

In Section 3.3, we study the asymptotic profiles of the steady state solution in several important limiting cases; namely, small diffusion, large diffusion, and deep water column. In the discussion of these results, we limit ourselves to $D(x) = D$ and $\alpha(x) = \alpha$, two positive constants, for simplicity of presentation. For the small diffusion case, we show that the phytoplankton population concentrates at the bottom of the water column:

Theorem 2.2.4 (Theorem 3.4.8) *Let $d \in (0, g(e^{-k_0 h}))$. Then for all small $D > 0$, the unique positive solution $u_D(x)$ of* (2.1.4) *is strictly increasing in* $[0, h]$. *Moreover, as $D \to 0$,*

$$\max_{x\in\left[0,\, h-\frac{2D}{\alpha}|\ln D|\right]} \left| u_D(x) - \tau^* D^{-1} e^{\alpha(x-h)/D} \right| \to 0, \tag{2.2.1}$$

and

$$\int_0^h u_D(x)dx \to \tau_*/\alpha, \tag{2.2.2}$$

where τ_* is uniquely determined by the equation

$$d = \int_0^1 g\left(e^{-k_0 h - k\tau_* x/\alpha}\right) dx.$$

While for large diffusion, we prove that the population tends to distribute evenly in the water column:

Theorem 2.2.5 (Theorem 3.4.9) *As $D \to \infty$,*

$$u_D(x) \to c^* \quad \textit{uniformly on} \quad [0, h], \tag{2.2.3}$$

where c^* is uniquely determined by the equation

$$d = \frac{1}{h}\int_0^h g\left(e^{-k_0 x - kc^* x}\right) dx.$$

The concentration result in the small diffusion case appears to correspond to the widely observed deep chlorophyll maxima (DCM). Our result that with large diffusion the phytoplankton distribution tends to be homogeneous in the water column is naturally expected, and it lends further support to the practice used in the modeling of phytoplankton in completely mixed water columns (e.g. [28]).

Our theoretical results also reveal that, when all the other factors are the same, in a water column with positive background turbidity, the total biomass is bigger in the large diffusion case than in the small diffusion case, and in a water column with zero (or negligible) background turbidity, the total biomass tends to the same limit in both cases. We do not know whether this phenomenon has been observed before.

When the water column depth goes to infinity, as expected, we prove that the population distribution approaches that obtained in [40] with infinite water depth, and the population density reaches a maximum at a certain finite depth.

Theorem 2.2.6 (Theorem 3.4.18) *Suppose* $d^*_\infty = \lim_{h\to\infty} d^*_h > 0$*, and* $0 < d < d^*_\infty$.

(a) Suppose either $k_0 > 0$*, or* $k_0 = 0$ *and* $0 < d < g(1) - \alpha^2/4D$*. Let* $u_h(x)$ *be the unique positive solution to* (2.1.4)*. Then, as* $h \to \infty$*,*

$$u_h(x) \to u_\infty(x) \text{ in } C^1_{loc}([0,\infty)), \tag{2.2.4}$$

where $u_\infty(x)$ *is the unique positive solution to*

$$\begin{cases} -Du''_\infty + \alpha u'_\infty = \left[g(e^{-k_0x - k\int_0^x u_\infty(s)ds}) - d\right] u_\infty, \quad x \in (0,\infty), \\ u'(0) = \dfrac{\alpha}{D} u(0). \end{cases} \tag{2.2.5}$$

Moreover $u_\infty(x)$ *has the following properties:*

(i) There exists $x_\infty \in (0,\infty)$ *such that*

$$u'_\infty(x_\infty) = 0,\ u'_\infty(x) > 0 \text{ for } x \in [0, x_\infty),\ u'_\infty(x) < 0 \text{ for } x \in (x_\infty, \infty);$$

(ii) There exists positive constants C*,* L*, and* α_0 *such that*

$$u_\infty(x) \le Ce^{-\alpha_0 x} \text{ for any } x \in [L, \infty). \tag{2.2.6}$$

(b) Suppose $k_0 = 0$ *and* $d \ge g(1) - \alpha^2/4D$*. Then* $u_h \to 0$ *in* $C^1_{loc}([0,\infty)$.

Here the limit $\lim_{h\to\infty} d^*_h$ exists, because according to a result in [33], d^*_h is decreasing with respect to h.

As a by product of the last result, we are able to completely answer a problem left open in [33] as mentioned in the last section:

Theorem 2.2.7 (Theorem 3.4.19) *Let $k_0 > 0$. Then there exists a positive constant α^* such that $d_\infty^* > 0$ if and only if $\alpha < \alpha^*$.*

For the multi-species case, we first extend the results in [22] for the two species case to the case where $D(x) > 0$ and $\alpha(x)$ are smooth general functions. The proofs are very similar to those of [22]. But by the same reason as mentioned above for the one species case, and since it makes the thesis more complete, we decide to give a detailed presentation here. This is the main task of Chapter 4. Actually we prove in Chapter 4 the following result:

Theorem 2.2.8 (Theorem 4.1.1) *Let $u_{d_i}^*, d_i \in (0, d_i^*), i = 1, 2,$ be the unique positive solution of the problem*

$$\begin{cases} (-D_i(x)u' + \alpha_i(x)u)' = [g_i(e^{-k_0 x - k_i \int_0^x u(s)\,ds}) - d_i]u, \;\; 0 < x < h, \\ D_i(0)u'(0) - \alpha_i(0)u(0) = D_i(h)u'(h) - \alpha_i(h)u(h) = 0. \end{cases}$$

If

$$\begin{cases} 0 < d_1 < -\lambda_1^{(1)}[-g_1(e^{-k_0 x - k_2 \int_0^x u_{d_2}^*(s)\,ds})] \\ 0 < d_2 < -\lambda_1^{(2)}[-g_2(e^{-k_0 x - k_1 \int_0^x u_{d_1}^*(s)\,ds})], \end{cases} \tag{2.2.7}$$

where $\lambda_1^{(i)}(\Psi)$ is the principal eigenvalue of the eigenvalue problem

$$(-D_i(x)\phi' + \alpha_i(x)\phi)' + \Psi(x)\phi = \lambda\phi, \quad D_i(0)\phi'(0) - \alpha_i(0)\phi(0) = D_i(h)\phi'(h) - \alpha_i(h)\phi(h) = 0,$$

then (2.1.1)-(2.1.3) *has at least one positive steady state solution.*

The sufficient condition for the existence of positive steady state solution (2.2.7) is rather implicit. We have a discussion at the end of Chapter 4 that it is not a necessary condition. Actually, it is also shown in Chapter 4 that a necessary condition for (2.1.4) to have a positive steady state solution is both

$$0 < d_1 < -\lambda_1^{(1)}[-g_1(e^{-k_0 x})] \quad \text{and} \quad 0 < d_2 < -\lambda_1^{(2)}[-g_2(e^{-k_0 x})]$$

hold. However we have the following

Proposition 2.2.9 (Proposition 4.1.5) *For fixed $d_1 \in (0, d_1^*)$, if $\delta > 0$ is small enough, then* (2.1.1)-(2.1.3) *has no positive steady state solution if $d_2 \notin (\delta, d_2^* - \delta)$.*

To explain (2.2.7) more clearly, we need more preparations. We leave this task to Chapter 5.

In Chapter 5, we extend the results for the two species case to the more than two species cases to reveal a general phenomenon. For the more than two species case, for simplicity we assume $D_i(x) = D_i$, and $\alpha_i(x) = \alpha_i$, where $D_i > 0$ and $\alpha_i \in (-\infty, \infty)$ are constants. For three species, we show that when the turbulence diffusion rates $D_i, i = 1, 2, 3$, are very large, generically there are no possibility of coexistence of multiple phytoplankton species in the same water column:

Theorem 2.2.10 (Theorem 5.2.1) *(i) If $d_i > \int_0^1 g_i(e^{-k_0 x})dx$ for some $i \in \{1, 2, 3\}$, then there exists a constant $D > 0$, such that if $\min\{D_1, D_2, D_3\} \geq D$ then* (2.1.1)-(2.1.3) *has only the trivial steady state solution.*

(ii) If $d_i \in (0, \int_0^1 g_i(e^{-k_0 x})dx]$ for all $i = 1, 2, 3$, then there exists a positive constant D such that if $\min\{D_1, D_2, D_3\} \geq D$, (2.1.1)-(2.1.3) *has no positive steady state solution except possibly when the following exceptional situation occurs: there exists a constant $c \geq 0$ such that*

$$c_1 = c_2 = c_3 = c,$$

where c_i is uniquely determined by

$$d_i = \int_0^1 g_i(e^{-(k_0 + c_i)x})dx. \tag{2.2.8}$$

However when the turbulence diffusion is not large, multiple phytoplankton species can coexist in the same water column, in certain parameter ranges:

Theorem 2.2.11 (Theorem 5.2.2) *Suppose that*

$$\begin{aligned}
0 &< d_1 < -\lambda_1^{(1)}\left[-g_1(\sigma_{d_2 d_3}(x))\right], \\
0 &< d_2 < -\lambda_1^{(2)}\left[-g_2(\sigma_{d_1 d_3}(x))\right], \\
0 &< d_3 < -\lambda_1^{(3)}\left[-g_3(\sigma_{d_1 d_2}(x))\right],
\end{aligned} \tag{2.2.9}$$

where

$$\sigma_{d_2 d_3}(x) = e^{-k_0 x - \int_0^x [u_{d_2}(y) + u_{d_3}(y)]dy}, \quad x \in [0, 1],$$

$$\sigma_{d_1 d_3}(x) = e^{-k_0 x - \int_0^x [u_{d_1}(y) + u_{d_3}(y)]dy}, \quad x \in [0, 1],$$

$$\sigma_{d_1d_2}(x) = e^{-k_0x - \int_0^x [u_{d_1}(y) + u_{d_2}(y)]dy}, \quad x \in [0,1],$$

where $u_{d_i}(x)$ *is the unique positive solution of*

$$-D_iu'' + \alpha_iu' = [g_i(e^{-k_0x - \int_0^x u(s)ds}) - d_i]u, \quad D_iu'(0) - \alpha_iu(0) = D_iu'(1) - \alpha_iu(1) = 0.$$

Then (2.1.1)-(2.1.3) *has at least one positive steady state solution.*

Theorem 2.2.12 (Theorem 5.2.3) *Suppose* $\alpha_1, \alpha_2, \alpha_3 \in R^1$ *and at least two of the three* $\alpha_i (i = 1, 2, 3)$ *are nonpositive. Then there exist suitable* (D_1, D_2, D_3) *and* (d_1, d_2, d_3) *such that* (2.2.9) *holds, and hence there is at least one positive steady state solution to* (2.1.1)-(2.1.3).

Theorem 2.2.13 (Theorem 5.2.4) *Suppose* D_1, D_2, D_3 *are fixed. We can choose suitable* $\alpha_1, \alpha_2, \alpha_3$ *and* d_1, d_2, d_3 *such that* (2.2.9) *holds, and hence there exists at least one positive steady state solution to* (2.1.1)-(2.1.3).

The proof of Theorem 2.2.11 is rather long. We use a fixed point index calculation technique developed in Dancer and Du [14], where they treated the classical Lotka-Volterra type multi-species reaction-diffusion systems. See also results in Du [18] concerning the existence of positive periodic solutions for a competitor-competitor-mutualist model with diffusion. Other results concerning three species systems can be found in [29, 46, 47] and the references therein. For our system, the mathematical treatment is much more difficult. The difficulties are twofold. One is the nonlocal terms involved, which, among other difficulties, makes the usual comparison principle difficult to use. The other is the fact there are fewer parameters in the system than in the classical Lotka-Volterra type competition models, and hence sufficient conditions for coexistence are more difficult to formulate. Because of these difficulties, we need much more involved analysis on the solutions of the system in order to make the abstract tools applicable.

The proofs of Theorems 2.2.12 and 2.2.13 rely on a development of an idea in [33] concerning the fine properties of certain eigenvalue problems and a detailed a priori estimates on the positive steady state solutions of (2.1.1)-(2.1.3). The exact ranges of parameters for the existence of positive steady states can be found in the proofs of these two theorems in Chapter 5.

In the last section of Chapter 5, we extend the results for the two and three species to the general $n(\geq 2)$ species case.

Theorem 2.2.14 (Theorem 5.4.2) *(i) If* $d_i > \int_0^1 g_i(e^{-k_0x})dx$ *for some* $i \in \{1, \cdots, n\}$*, then there exists a constant* $D > 0$*, such that if* $\min\{D_1, D_2, \cdots, D_n\} \geq D$ *then* (2.1.1)-(2.1.3) *has only the trivial steady state solution.*

(ii) If $d_i \in (0, \int_0^1 g_i(e^{-k_0x})dx]$ *for all* $i = 1, 2, \cdots, n$*, then there exists a positive constant* D *such that if* $\min\{D_1, D_2, \cdots, D_n\} \geq D$*,* (2.1.1)-(2.1.3) *has no positive steady state solution except possibly when the following exceptional situation occurs:*
there exists a constant $c \geq 0$ *such that*

$$c_1 = c_2 = \cdots = c_n = c,$$

where c_i *is uniquely determined by*

$$d_i = \int_0^1 g_i(e^{-(k_0+c_i)x})dx, \ i = 1, 2, \cdots, n. \tag{2.2.10}$$

Theorem 2.2.15 (Theorem 5.4.3) *Suppose that*

$$0 < d_i < -\lambda_1^{(i)}[-g_i(\kappa_i(x))], \ i = 1, 2, \cdots, n, \tag{2.2.11}$$

where

$$\kappa_i(x) = e^{-k_0x} \exp\left(\sum_{j \neq i} \int_0^x u_{d_j}(y)dy\right), \ i = 1, 2, \cdots, n,$$

and u_{d_j} *is the unique positive solution of the equation*

$$-D_j u'' + \alpha_j u' = \left[g_j\left(e^{-k_0x - \int_0^x u(y)dy}\right) - d_j\right] v, \ D_j u'(0) - \alpha_j u(0) = D_j u'(1) - \alpha_j u(1) = 0.$$

Then (2.1.1)-(2.1.3) *has at least one positive steady state solution.*

Theorem 2.2.16 (Theorem 3.4.27) *Suppose that* $\alpha_1, \alpha_2, \cdots, \alpha_n \in R^1$ *and at least* $n-1$ *of the* $\alpha_i (i = 1, 2, \cdots, n)$ *are nonpositive. Then we can choose suitable* $(D_1, D_2, \cdots, D_n)$ *and* $(d_1, d_2, \cdots, d_n)$ *such that* (2.2.11) *holds, and hence there is at least one positive steady state solution to* (2.1.1)-(2.1.3).

Theorem 2.2.17 (Theorem 5.4.5) *Suppose* $D_1, D_2, \cdots, D_n$ *are fixed. We can choose suitable* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *and* $d_1, d_2, \cdots, d_n$ *such that* (2.2.11) *holds, and hence there exists at least one positive steady state solution to* (2.1.1)-(2.1.3).

We end the introduction by mentioning some other closely related mathematical research. In [20, 21] a reaction-diffusion-advection model proposed by Klausmeier and Litchman [42] was studied, where both nutrient and light limitations were present, and the focus was on examining the biomass concentration under the assumption that, apart from passive diffusion caused by currents movement, the species actively moves towards an optimal spatial position that maximizes its use of both light and nutrient. In [65], a phytoplankton-nutrient model proposed in [37] was studied. This is also a reaction-diffusion-advection model, but the reaction term is different from the model of [42] considered in [20, 21], and no active movement of the phytoplankton is assumed. In [44], like in [58], the case $k_0 = 0$ was considered, where $D(x)$ and $\sigma(x)$ are both positive constants, and h can be either finite or infinite. Under suitable conditions, the authors proved the existence and uniqueness of a positive steady state, and its local stability; the asymptotic profile of the positive steady state was also considered for some limiting situations. In [50], the phytoplankton-nutrient model of [42] was considered. Instead of considering the case of large advection as in [20, 21], [50] studied the small diffusion case.

Chapter 3

The one species case

3.1 Introduction

In this chapter we consider model (2.1.1)-(2.1.3) in the single species case. That is (2.1.4):

$$\begin{cases} u_t = J_x(x,t) + \left[g\left(e^{-k_0 x - k\int_0^x u(s,t)ds}\right) - d\right]u, & 0 < x < h, \quad t > 0, \\ J(x,t) = D(x)u_x(x,t) - \alpha(x)u(x,t) = 0, & x = 0 \text{ or } h, \quad t > 0, \\ u(x,0) = u_0(x) \gneqq 0, & 0 \le x \le h, \end{cases} \tag{3.1.1}$$

where $u(x,t)$ is the phytoplankton density at depth x and time t, $D(x)$ is the diffusion caused by water turbulence, $\alpha(x)$ is the sinking ($\alpha(x) > 0$) or buoyant ($\alpha(x) < 0$) velocity, $d > 0$ is the death rate of the phytoplankton species,

$$I(x,t) = e^{-k_0 x - k\int_0^x u(s,t)ds}$$

is the light density function in the water column with $k_0 > 0$ the background turbidity coefficient and $k > 0$ the light absorption coefficient of the phytoplankton species, the growth function $g(I)$ is a smooth function satisfying

$$g(0) = 0, \quad g'(I) > 0 \quad \text{for all} \quad I > 0.$$

We also suppose there are constants $c, \gamma > 0$ such that

$$g(I) \le cI^\gamma$$

so that

$$\int_0^\infty g(e^{-\xi})\,d\xi < \infty.$$

As mentioned in the Introduction, this one species model has been studied by many authors, notably [22, 33] and [40]. We first generalize the results in [22] and [33] (concerning the constant coefficient case) to the varying coefficient case. Then we consider how the phytoplankton behavior changes when the parameters of the system change. More precisely, we consider the cases of small diffusion, large diffusion, and deep water depth and find out how the phytoplankton distributions have in these cases. As a byproduct, we solve a problem left open in [33].

3.2 Existence and uniqueness of positive steady state solution

This subsection is concerned with the positive steady state of (3.1.1). Let

$$v(x,t) = u(x,t)e^{-R(x)}$$

with

$$R(x) = \int_0^x \frac{\alpha(s)}{D(s)}\,ds.$$

Then $v(x,t)$ satisfies

$$\begin{cases} v_t = e^{-R(x)}\left[D(x)e^{R(x)}v_x\right]_x + \left[g\left(e^{-k_0x-k\int_0^x e^{R(s)}v(s,t)ds}\right) - d\right]v, & 0<x<h, t>0,\\ v_x(0,t) = v_x(h,t) = 0, & t>0,\\ v(x,0) = u_0(x)e^{-R(x)} =: v_0(x) \gneqq 0, & 0\le x\le h. \end{cases} \tag{3.2.1}$$

The corresponding steady state equation is

$$\begin{cases} -e^{-R(x)}\left[D(x)e^{R(x)}v'\right]' = \left[g\left(e^{-k_0x-k\int_0^x e^{R(s)}v(s)ds}\right) - d\right]v, & 0<x<h,\\ v'(0) = v'(h) = 0. \end{cases} \tag{3.2.2}$$

For a function $\Psi \in C([0,h])$, let $\lambda_1(\Psi)$ denote the smallest eigenvalue of

$$-e^{-R(x)}\left[D(x)e^{R(x)}\varphi'\right]' + \Psi\varphi = \lambda\varphi \ \text{ in } \ (0,h), \quad \varphi'(0) = \varphi'(h) = 0. \tag{3.2.3}$$

It is well known (see e.g. [6]) that $\lambda_1(\Psi)$ is a continuous function of Ψ in $C([0,h])$ and $\lambda_1(\Psi_1) > \lambda_1(\Psi_2)$ for $\Psi_1 \geq \Psi_2$ and $\Psi_1 \not\equiv \Psi_2$; $\lambda_1(\Psi)$ is the only eigenvalue of (3.2.3) that corresponds to an eigenfunction which does not change sign; moreover, $\lambda_1(\Psi)$ has a variational formulation:

$$\lambda_1(\Psi) = \inf_{\varphi \in H^1((0,h)),\, \varphi \neq 0} \frac{\int_0^h e^{R(x)} \left[D(x)|\varphi'(x)|^2 + \Psi(x)\varphi^2(x)\right] dx}{\int_0^h e^{R(x)} \varphi^2(x) dx}.$$

Clearly $\lambda_1(0) = 0$ with a corresponding eigenfunction $\varphi_1(x) \equiv 1$.

Define

$$\Phi_0(x) = -g(e^{-k_0 x}), \quad d^* := -\lambda_1(\Phi_0).$$

Then $0 < d^* < \infty$.

With d^* thus defined, we have the following existence and uniqueness result

Theorem 3.2.1 *Problem* (3.2.2) *has no positive solution for* $d \notin (0, d^*)$ *and it has a unique positive solution for* $d \in (0, d^*)$.

Proof. The first equation in (3.2.2) can be rewritten as

$$-e^{-R(x)} \left[D(x) e^{R(x)} v'\right]' - g\left(e^{-k_0 x - k\int_0^x e^{R(s)} v(s) ds}\right) v = -dv.$$

Hence, if v is a positive solution, then

$$d = -\lambda_1 \left[-g\left(e^{-k_0 x - k\int_0^x e^{R(s)} v(s) ds}\right)\right].$$

By the monotonicity property of the principal eigenvalue, we obtain

$$\lambda_1\left[-g\left(e^{-k_0 x}\right)\right] < \lambda_1 \left[-g\left(e^{-k_0 x - k\int_0^x e^{R(s)} v(s) ds}\right)\right] < \lambda_1(0) = 0.$$

That is

$$d^* > d > 0.$$

Hence (3.2.2) has no positive solution for $d \notin (0, d^*)$.

We next prove there is at least one positive solution for each $d \in (0, d^*)$. Let K be the set of positive functions in $C([0,h])$. For $v \in K$ define

$$G(v) = g\left(e^{-k_0 x - k\int_0^x e^{R(s)} v(s) ds}\right) v + (d^* - d + 1)v.$$

For $f \in C([0,1])$, let L be the solution operator of

$$-e^{-R(x)}\left[D(x)e^{R(x)}v'\right]' + (d^*+1)v = f(x), \quad v'(0) = v'(h).$$

It is clear that $L(v)$ is a linear completely continuous operator from K to K. Since $G(d,v)$ is clearly a continuous operator from $\mathbb{R}^+ \times K$ to K, we have $F := L \circ G$ is a completely continuous operator from $\mathbb{R}^+ \times K$ to K; moreover $F(d,\cdot)$ maps K° into K°, where K° denotes the interior of K. Thus to find a positive solution of (3.2.2) is equivalent to find a point v in K° such that

$$v = F(d,v).$$

We can easily verify that F satisfies the conditions of Theorem 1.3.1. Thus we know (3.2.2) has an unbounded connected positive solution branch, $\Gamma = \{(d,v)\}$, that bifurcates from the trivial solution branch $\{(d,0)\}$ at $(d^*,0)$. We prove that Γ can only become unbounded when $d \to 0$. In such a case, due to the connectedness of Γ, (3.2.2) has at least one positive solution for any $d \in (0,d^*)$. Assume there exist a sequence d_n and corresponding positive solution sequence v_n such that as $n \to \infty$,

$$d_n \to d_0, \qquad \|v_n\|_\infty \to \infty.$$

Set $\hat{v}_n = v_n/\|v_n\|_\infty$. Then we have

$$\begin{cases} -\left[D(x)e^{R(x)}\hat{v}_n'\right]' = \left[g\left(e^{-k_0x-k\int_0^x v_n(s)e^{R(s)}\,ds}\right) - d_n\right]e^{R(x)}\hat{v}_n \ \text{ in } \ (0,h), \\ \hat{v}_n'(0) = \hat{v}_n'(h) = 0. \end{cases}$$

Since $f_n(x) := g(e^{-k_0x}e^{-k\int_0^x v_n(s)e^{R(s)}\,ds})$ is a bounded sequence in $L^\infty([0,h])$, we have that $\{\hat{v}_n\}$ and $\{\hat{v}_n''\}$ are both bounded sequences in $L^\infty([0,h])$. By the Sobolev compact embedding theorem, we may assume, by passing to a subsequence, that $\hat{v}_n \to \hat{v}$ in $C^1([0,h])$. Moreover, $f_n \to f$ weakly in $L^2([0,h])$, and $0 \le f \le g(1)$ since $0 \le f_n \le g(1)$ for each n. It follows that $\hat{v}$ is a weak solution of

$$-\left[D(x)e^{R(x)}\hat{v}'\right]' = (f-d_0)e^{R(x)}\hat{v} \ \text{ in } \ (0,h), \ \ \hat{v}'(0) = \hat{v}'(h) = 0.$$

We also have $\hat{v} \ge 0$, $\|\hat{v}\|_\infty = 1$. Since $(f-d_0) \in L^\infty([0,h])$, by the strong maximum principle we have $\hat{v} > 0$ on $[0,h]$ and $-d_0 = \lambda_1(-f)$.

On the other hand, from $\hat{v}_n \to \hat{v} > 0$ uniformly in $[0, h]$ and $\|v_n\|_\infty \to \infty$, we have that $v_n \to \infty$ uniformly on $[0, h]$. It follows that

$$e^{-k\int_0^x v_n(s)e^{R(s)}\,ds} \to 0$$

uniformly on any compact subset of $(0, h]$. This means that $f \equiv 0$ and hence

$$-d_0 = \lambda_1(-f) = 0.$$

Therefore Γ can only become unbounded through the existence of a sequence $(d_n, v_n) \in \Gamma$ such that $d_n \to 0$ and $\|v_n\|_\infty \to \infty$. Moreover, the above proof shows that in such a case, $v_n \to \infty$ uniformly on $[0, h]$. As a consequence of the connectedness of Γ, we conclude that (3.2.2) has at least one positive solution for each $d \in (0, d^*)$.

We next prove the uniqueness conclusion. Suppose by way of contradiction that for some $d \in (0, d^*)$, (3.2.2) has two positive solutions v_1 and v_2. We first observe that $v_1 - v_2$ must change sign in $(0, h)$. Otherwise we may assume that $v_1 \le v_2$ and $v_1 \not\equiv v_2$. From this and the equations for v_1 and v_2 we obtain

$$-d = \lambda_1\left[-g\left(e^{-k_0x}e^{-k\int_0^x v_1(s)e^{R(s)}\,ds}\right)\right] < \lambda_1\left[-g\left(e^{-k_0x}e^{-k\int_0^x v_2(s)e^{R(s)}\,ds}\right)\right] = -d,$$

a contradiction. Therefore $v_1 - v_2$ changes sign in $(0, h)$.

We claim that $v_1(0) \ne v_2(0)$. Otherwise, we denote $\xi_i(x) = \int_0^x v_i(s)e^{R(s)}\,ds$, $\eta_i(x) = e^{R(x)}v_i'(x)$, for $i = 1, 2$, and find that (v_i, ξ_i, η_i) are solutions of the initial value system

$$\begin{cases} (v', \xi', \eta') = \left(e^{-R(x)}\eta, e^{R(x)}v, -[g(e^{-k_0x}e^{-k\xi}) - d]e^{R(x)}v\right), \\ (v(0), \xi(0), \eta(0)) = (v_1(0), 0, 0). \end{cases}$$

By the well-known existence and uniqueness theorem of ODE, we find that $(v_1, \xi_1, \eta_1) = (v_2, \xi_2, \eta_2)$ in a small neighborhood $[0, \delta)$. We may then repeat this argument to conclude that $v_1 \equiv v_2$ as long as they are defined, which is a contradiction to our assumption that they are different solutions of (3.2.2). Therefore $v_1(0) \ne v_2(0)$.

For definiteness we assume that $v_1(0) < v_2(0)$. Since $v_1 - v_2$ changes sign in $(0, h)$, there exists $x_0 \in (0, h)$ such that $v_2(x) > v_1(x)$ in $[0, x_0)$ and $v_1(x_0) = v_2(x_0)$. Thus we have

$$\int_0^{x_0} [-(D(x)e^{R(x)}v_1')'v_2]\,dx = \int_0^{x_0} [g(e^{-k_0x}e^{-k\int_0^x v_1(s)e^{R(s)}\,ds}) - d]e^{R(x)}v_1v_2\,dx.$$

By integration by parts, we deduce

$$-D(x)e^{R(x)}v_1'v_2\Big|_0^{x_0} + \int_0^{x_0} D(x)e^{R(x)}v_1'v_2'\,dx$$
$$= \int_0^{x_0} g(e^{-k_0x}e^{-k\int_0^x v_1(s)e^{R(s)}\,ds})e^{R(x)}v_1v_2\,dx - d\int_0^{x_0} e^{R(x)}v_1v_2\,dx.$$

Similarly, we have

$$\int_0^{x_0} [-D(x)(e^{R(x)}v_2')'v_1]\,dx = \int_0^{x_0} [g(e^{-k_0x}e^{-k\int_0^x v_2(s)e^{R(s)}\,ds}) - d]e^{R(x)}v_1v_2\,dx,$$

and

$$-D(x)e^{R(x)}v_2'v_1\Big|_0^{x_0} + \int_0^{x_0} D(x)e^{R(x)}v_1'v_2'\,dx$$
$$= \int_0^{x_0} g(e^{-k_0x}e^{-k\int_0^x v_2(s)e^{R(s)}\,ds})e^{R(x)}v_1v_2\,dx - d\int_0^{x_0} e^{R(x)}v_1v_2\,dx.$$

Therefore

$$D(x)e^{R(x)}[v_1v_2' - v_2v_1']\Big|_0^{x_0}$$
$$= \int_0^{x_0} [g(e^{-k_0x}e^{-k\int_0^x v_1(s)e^{R(s)}\,ds}) - g(e^{-k_0x}e^{-k\int_0^x v_2(s)e^{R(s)}\,ds})]e^{R(x)}v_1v_2\,dx. \tag{3.2.4}$$

Since $v_1'(0) = v_2'(0) = 0$ by the boundary condition, and $v_1(x_0) = v_2(x_0) > 0$, $v_1'(x_0) \geq v_2'(x_0)$, we have

$$D(x)e^{R(x)}[v_1v_2' - v_2v_1']\Big|_0^{x_0} = D(x_0)e^{R(x_0)}v_1(x_0)[v_2'(x_0) - v_1'(x_0)] \leq 0.$$

Therefore (3.2.4) implies that

$$\int_0^{x_0} [g(e^{-k_0x}e^{-k\int_0^x v_1(s)e^{R(s)}\,ds}) - g(e^{-k_0x}e^{-k\int_0^x v_2(s)e^{R(s)}\,ds})]e^{R(x)}v_1v_2\,dx \leq 0.$$

But on the other hand, from $v_1(x) < v_2(x)$ in $(0, x_0)$ we have

$$\int_0^{x_0} [g(e^{-k_0x}e^{-k\int_0^x v_1(s)e^{R(s)}\,ds}) - g(e^{-k_0x}e^{-k\int_0^x v_2(s)e^{R(s)}\,ds})]e^{R(x)}v_1v_2\,dx > 0.$$

This contradiction proves our uniqueness conclusion. □

Theorem 3.2.2 *If we denote the unique positive solution of* (3.2.2) *by* v_d*, then*

(i) $d \to v_d$ *is continuous from* $(0, d^*)$ *to* $C^2([0, h])$*,*

(ii) $0 < d_1 < d_2 < d^*$ *implies* $v_{d_1}(0) > v_{d_2}(0)$*,*

(iii) $0 < d_1 < d_2 < d^*$ *implies* $\int_0^x e^{R(s)} v_{d_1}(s)ds > \int_0^x e^{R(s)} v_{d_2}(s)ds$ *for all* $x \in (0, h]$*.*

(iv) $v_d \to 0$ *uniformly in* $[0, h]$ *as* $d \to d^*$*,* $dv_d \to \int_0^\infty g(e^{-ks})ds \Big/ \int_0^h e^{R(s)}ds$ *uniformly in* $[0, h]$ *as* $d \to 0$ *.*

Proof. To prove that $d \to v_d$ is continuous from $(0, d^*)$ to $C^2([0, h])$, we use the standard compactness and uniqueness argument. If $d_n \to d_0 \in (0, d^*)$, then a subsequence of v_{d_n} converges in $C^1([0, h])$ to a positive solution of (3.2.2) with $d = d_0$. The uniqueness assumption means this positive solution must be v_{d_0}. Therefore the entire sequence converges to v_{d_0}. From the equation of v_{d_0} we see that $v_{d_n} \to v_{d_0}$ in $C^1([0, h])$ implies that the convergence also holds in $C^2([0, h])$. This prove conclusion (i).

We now turn to conclusion (ii). We will write $v_1 = v_{d_1}, v_2 = v_{d_2}$ for simplicity.

Argue indirectly. Assume that for some $0 < d_1 < d_2 < d^*$, $v_1(0) \leq v_2(0)$ holds. Consider firstly the case $v_1(0) < v_2(0)$. Then we can show $v_1 - v_2$ changes sign and define $[0, x_0]$ as in the uniqueness proof of Theorem 3.2.1. We similarly have

$$e^{R(x)}[v_1 v_2' - v_2 v_1']\Big|_0^{x_0} \leq 0.$$

On the other hand,

$$\begin{aligned}
e^{R(x)}[v_1 v_2' - v_2 v_1']\Big|_0^{x_0} &= \int_0^{x_0} [-v_2(e^{R(x)}v_1')' + v_1(e^{R(x)}v_2')']\, dx \\
&= \int_0^{x_0} [g(e^{-k_0 x} e^{-k\int_0^x e^{R(s)} v_1(s)\, ds}) - g(e^{-k_0 x} e^{-k\int_0^x e^{R(s)} v_2(s)\, ds})] e^{R(x)} v_1 v_2\, dx \\
&\quad + (d_2 - d_1)\int_0^{x_0} e^{R(x)} v_1 v_2\, dx \\
&> 0,
\end{aligned}$$

a contradiction.

Consider now the case $v_1(0) = v_2(0)$. From the equation and $d_2 > d_1$ and $v_1'(0) = v_2'(0)$ we find that $v_2''(0) > v_1''(0)$. It follows that $v_2(x) > v_1(x)$ for $x > 0$ small. Thus we can still find an interval $(0, x_0)$ as above and derive a contradiction. Therefore $v_1(0) > v_2(0)$. Conclusion (ii) is now proved.

To prove conclusion (iii), we first observe that if v is a positive solution of (3.2.2) then $z(x) := \int_0^x e^{R(s)}v(s)\,ds$ satisfies

$$
\begin{aligned}
&-D(x)z'' + \alpha(x)z' = -dz + \int_0^x g(e^{-k_0 s - kz(s)})e^{R(s)}v(s)\,ds \\
&= -dz + k^{-1}\int_0^x g(e^{-k_0 s-kz(s)})d(k_0 s + kz(s)) - k_0 k^{-1}\int_0^x g(e^{-k_0 s-kz(s)})\,ds \\
&= -dz + k^{-1}\int_0^{k_0 x + kz(x)} g(e^{-\xi})\,d\xi - k_0 k^{-1}\int_0^x g(e^{-k_0 s - kz(s)}) \\
&= -dz + G(k_0 x + kz(x)) - k_0 k^{-1}\int_0^x g(e^{-k_0 s - kz(s)})\,ds,
\end{aligned}
$$

where

$$G(\eta) := k^{-1}\int_0^\eta g(e^{-\xi})\,d\xi.$$

For $d \in (0, d^*)$, we set $z_d(x) := \int_0^x e^{R(s)}v_d(s)\,ds$. Fix any $d_2 \in (0, d^*)$, we will show that $z_{d_1}(x) > z_{d_2}(x)$ for all $x \in (0, h]$ if $0 < d_1 < d_2$. Write $v_i = v_{d_i}$ and $z_i = z_{d_i}$. We know from the proof of Theorem 3.2.1, that $v_d(x)$ can only become unbounded through $d \to 0$ and $v_d(x) \to \infty$ uniformly as $d \to 0$. Thus for d_1 small, we have $v_1 > v_2$ on $[0, h]$ and hence $z_1 > z_2$ for $x \in (0, h]$. If the desired conclusion does not hold, then we can find a maximal $d_1 < d_2$ such that $z_d(x) > z_{d_2}(x)$ in $(0, h]$ for $d \in (0, d_1)$. Then we have $z_{d_1} \geq z_{d_2}$. We claim that $z_{d_1}(x) = z_{d_2}(x)$ holds for some $x \in (0, h]$. Otherwise $z_1(x) > z_2(x)$ for all $x \in (0, h]$. Fix $d_0 \in (d_1, d_2)$. By (ii), for any $d \in [d_1, d_0]$, $v_d(0) \geq v_{d_0}(0) > v_{d_2}(0)$. By (i), there exists $C > 0$ such that $\|v_d\|_{C^2([0,h])} < C$ for all $d \in [d_1, d_0]$. Therefore we can find $\delta > 0$ small enough such that $z_d(x) > z_{d_2}(x)$ for $d \in [d_1, d_0]$ and $x \in (0, \delta]$. Since $z_1(x) > z_2(x)$ in $[\delta, h]$, by (i) we can find $\tilde{d}_1 \in (d_1, d_0]$ such that $z_d(x) > z_{d_2}(x)$ for $d \in [d_1, \tilde{d}_1]$ and $x \in [\delta, h]$. Thus $z_d(x) > z_{d_2}(x)$ for $d \in (0, \tilde{d}_1]$ and $x \in (0, h]$, contradicting the maximality of d_1. This proves our claim that $z_{d_1}(x) = z_{d_2}(x)$ holds for some $x \in (0, h]$. We show this leads to a contradiction.

Consider firstly the possibility that $x = h$, i.e., $z_1(h) = z_2(h)$. Since, for $i = 1, 2$,

$$D(h)z_i''(h) - \alpha(h)z_i'(h) = D(h)e^{R(h)}v_i'(h) = 0,$$

we deduce from the above equation for z that,

$$d_i z_i(h) = G(k_0 h + kz_i(h)) - k_0 k^{-1}\int_0^h g(e^{-k_0 s - kz_i(s)})\,ds.$$

Using the assumption $z_1(h) = z_2(h)$, we obtain from the above identity

$$(d_2 - d_1)z_1(h) = k_0k^{-1} \int_0^h [g(e^{-k_0s-kvz_1(s)}) - g(e^{-k_0s-kz_2(s)})]\,ds.$$

It follows the right side of the above identity is less than or equal to 0, but the left side is positive. We arrive at a contradiction. Hence we have $z_1(h) > z_2(h)$.

Consider next the remaining possibility that $x \in (0, h)$. Denoting $w := z_1 - z_2$ we obtain

$$\begin{aligned}
-D(x)w'' + &\alpha(x)w' \\
&= d_2z_2 - d_1z_1 + C(x)w - k_0k^{-1} \int_0^x [g(e^{-k_0s-kz_1(s)}) - g(e^{-k_0s-kz_2(s)})]\,ds \\
&\geq [C(x) - d_1]w, \\
w(0) &= 0, \quad w(h) > 0,
\end{aligned}$$

where $C(x) = G'(k_0x + k\theta(x))$ for some $\theta(x) \in [z_2(x), z_1(x)]$. By the maximum principle we have $w(x) > 0$ in $(0, h]$, again reaching a contradiction.

We now prove (iv). The conclusion $v_d(x) \to 0$ uniformly as $d \to d^*$ is a direct result from the standard bifurcation analysis. To prove $dv_d(x) \to \int_0^\infty g(e^{-k_0x})dx \left(\int_0^h e^{R(s)}ds\right)^{-1}$ in $C([0, h])$ as $d \to 0$, we denote $z_d(x) = \int_0^x v_d(s)e^{R(s)}ds$. Multiply (3.2.2) by $e^{R(x)}$ and integrate it over $[0, h]$. By taking into account the boundary conditions, we obtain

$$\begin{aligned}
dz_d(h) &= k^{-1} \int_0^h g(e^{-k_0s-kz_d(s)})d(k_0s + kz_d(s)) - k_0k^{-1} \int_0^h g(e^{-k_0s-kz_d(s)})ds \\
&= k^{-1} \int_0^{k_0h+kz_d(h)} g(e^{-s})ds - k_0k^{-1} \int_0^h g(e^{-k_0s-kz_d(s)})ds
\end{aligned} \tag{3.2.5}$$

From the proof of Theorem 3.2.1, we know that $\|v_d\|_\infty \to \infty$ and $\hat{v}_d(x) := v_d(x)/\|v_d\|_\infty \to v_0(x) > 0$ in $C^1[0, h]$ as $d \to 0$. Note that $0 < v_0(x) \leq 1$. We will prove that $v_0(x) \equiv 1$ on $[0, h]$. We note that $v_d(x) \to \infty$ uniformly in $[0, h]$ as $d \to 0$. As a result $z_d(x) \to \infty$ in any compact subset of $(0, h]$ as $d \to 0$, and hence by the dominated convergence theorem, we obtain

$$\int_0^h g(e^{-k_0s-kz_d(s)})ds \to 0,$$

and,

$$dz_d(h) \to k^{-1} \int_0^\infty g(e^{-s})ds \quad \text{as} \quad d \to 0.$$

We also have as $d \to 0$

$$f_d(x) = g\left(e^{-k_0 x - k\int_0^x v_d(s)e^{R(s)}ds}\right) \to 0 \text{ uniformly in } [\delta, h],$$

for any $0 < \delta < h$.

Consider the equation for $\hat{v}_d$:

$$\begin{cases} -\left[D(x)e^{R(x)}\hat{v}_d'\right]' = \left[g\left(e^{-k_0 x - k\int_0^x v_d(s)e^{R(s)}\,ds}\right) - d\right] e^{R(x)}\hat{v}_d \text{ in } (0, h), \\ \hat{v}_d'(0) = \hat{v}_d'(h) = 0. \end{cases}$$

Integrate the above equation over $[0, h']$, we obtain

$$-D(h')e^{R(h')}v_d'(h') = \int_0^{h'} \left[g\left(e^{-k_0 x - k\int_0^x v_d(s)e^{R(s)}\,ds}\right) - d\right] e^{R(x)}\hat{v}_d(x)dx$$

Letting $d \to 0$,

$$D(h')e^{R(h')}\hat{v}_0'(h') = 0.$$

That is

$$v_0(h') = c_0, \ h' \in (0, h],$$

for some constant $0 < c_0 \leq 1$. By the continuity of v_0,

$$v_0(x) = c_0, \ x \in [0, h].$$

By the definition of v_0, clearly $c_0 = 1$. That means

$$\hat{v}_d(x) \to 1 \text{ in } C^1([0, h]) \text{ as } d \to 0.$$

Consequently from

$$dv_d \int_0^h \hat{v}_d(s)e^{R(s)}ds = \frac{dv_d}{\|v_d\|_\infty}\int_0^h v_d(s)e^{R(s)}ds \to k^{-1}\int_0^\infty g(e^{-s})ds \quad \text{as } d \to \infty$$

we obtain that

$$dv_d \to k^{-1}\int_0^\infty g(e^{-s})ds \Big/ \int_0^h e^{R(s)}ds.$$

This proves conclusion (iv).

The proof of Theorem (3.2.2) is now complete. □

3.3 The global stability of the corresponding parabolic equation

In this subsection, we consider the global dynamical behavior of equation (3.1.1), or equivalently, equation (3.2.1). Assume $v_0 \in C^2([0,h])$. We use the contraction mapping theorem, the standard L^p estimate and Sobolev embeddings to show that there exist a unique solution $v(x,t)$ of (3.2.1) for small $t > 0$.

Define $\Delta_T = [0,h] \times [0,T]$ and $D_T = \{v \in C(\Delta_T) : v(x,0) = v_0(x), \|v - v_0\|_{C(\Delta_T)} \le 1\}$. Then D_T is a complete metric space with the usual metric $d(u,v) = \|u - v\|_{C(\Delta_T)}$.

By Theorem 1.1.6 we find that for any $v \in D_T$ the following initial boundary value problem:

$$\begin{cases} \overline{v}_t = e^{-R(x)}\left[D(x)e^{R(x)}\overline{v}_x\right]_x + \left[g\left(e^{-k_0x-k\int_0^x e^{R(s)}v(s,t)ds}\right) - d\right]v, & 0 < x < h, t > 0, \\ \overline{v}_x(0,t) = \overline{v}_x(h,t) = 0, & t > 0, \\ \overline{v}(x,0) = v_0(x), & 0 \le x \le h, \end{cases} \tag{3.3.1}$$

admits a unique solution $v \in W^{2,1;p}(\Delta_T)$ for any $p > 1$; moreover,

$$\|\overline{v} - v_0\|_{2,1;p;\Delta_T} \le C\|G(x)\|_{p;\Delta_T},$$

where $G(x) = \left[g\left(e^{-k_0x-k\int_0^x e^{R(s)}v(s,t)ds}\right) - d\right]v(x)$.

Hence there is a positive constant $\bar{C}$ such that $\|\overline{v} - v_0\|_{2,1;p;\Delta_T} \le \bar{C}$ for any $v \in D_T$. By Theorem 1.1.3, we have $\overline{v} - v_0 \in C^{1+\alpha,(1+\alpha)/2}(\Delta_T)$ and for any $v \in D_T$,

$$\|\overline{v} - v_0\|_{C^{1+\alpha,(1+\alpha)/2}(\Delta_T)} \le C_1, \tag{3.3.2}$$

where C_1 is a constant dependent on h, α, and $\|v_0\|_{C^2([0,h])}$.

Now define $\mathcal{F} : D_T \to C(\Delta_T)$ by

$$\mathcal{F}(v) = \overline{v}.$$

Then $v \in D_T$ is a fixed point of $\mathcal{F}$ if and only if it solves (3.2.1) for $0 \le t \le T$.

By (3.3.2) we have

$$\|\overline{v} - v_0\|_{C(\Delta_T)} \le \|\overline{v} - v_0\|_{C^{0,(1+\alpha)/2}(\Delta_T)} T^{(1+\alpha)/2} \le C_1 T^{(1+\alpha)/2}.$$

Therefore if we take $T \leq C_1^{-2/(1+\alpha)}$, then $\mathcal{F}$ maps D_T into itself.

Next we show $\mathcal{F}$ is a contraction mapping on D_T for T sufficiently small. Indeed, let $v_i \in D_T(i = 1, 2)$ and $\overline{v}_i = \mathcal{F}(v_i)$. Set $V = \overline{v}_1 - \overline{v_2}$. Then V satisfies

$$\begin{aligned} V_t - e^{-R(x)}\left[D(x)e^{R(x)}V_x\right]_x = &\left[g\left(e^{-k_0 x - k\int_0^x e^{R(s)}v_1(s,t)ds}\right) - d\right]v_1 \\ &- \left[g\left(e^{-k_0 x - k\int_0^x e^{R(s)}v_2(s,t)ds}\right) - d\right]v_2, \end{aligned}$$

$$V(x,0) = 0, \quad 0 < x < h, \;\; V_x(0,t) = V_x(h,0) = 0, t > 0.$$

We note that

$$\begin{aligned} &\left[g\left(e^{-k_0 x - k\int_0^x e^{R(s)}v_1(s,t)ds}\right) - d\right]v_1 - \left[g\left(e^{-k_0 x - k\int_0^x e^{R(s)}v_2(s,t)ds}\right) - d\right]v_2 \\ &\leq d|v_2 - v_1| + \sup_{0\leq s\leq 1} g(s)|v_1 - v_2| + k\sup_{0\leq s\leq 1}|g'(s)|\|v_2\|_{C(\Delta_T)}\left|\int_0^x (v_1 - v_2)dx\right| \qquad (3.3.3) \\ &\leq C_2\|v_1 - v_2\|_{C(\Delta_T)}. \end{aligned}$$

and using the L^p estimates for parabolic equations and Sobolev embedding theorems we obtain

$$\|\overline{v}_1 - \overline{v_2}\|_{C^{1+\alpha,(1+\alpha)/2}(\Delta_T)} \leq C_3\|v_1 - v_2\|_{C(\Delta_T)} \qquad (3.3.4)$$

Fix a T_0 and $C_2 = C_2(T_0), C_3 = C_3(T_0)$ such that (3.3.3) and (3.3.4) hold for $T = T_0$. We note that (3.3.3) and (3.3.4) hold for any $0 < T \leq T_0$ for $C_2 \equiv C_2(T_0)$ and $C_3 \equiv C_3(T_0)$. Therefore

$$\|\overline{v}_1 - \overline{v}_2\|_{C(\Delta_T)} \leq \|\overline{v}_1 - \overline{v}_2\|_{C^{0,(1+\alpha)/2}(\Delta_T)}T^{(1+\alpha)/2} \leq C_3 T^{(1+\alpha)/2}\|v_1 - v_2\|_{C(\Delta_T)}.$$

Hence for $T \leq \min\{C_1^{-2/(1+\alpha)}, (2C_3)^{-2/(1+\alpha)}\}$, $\mathcal{F}$ maps D_T into itself and is a contraction mapping. By the contraction mapping theorem $\mathcal{F}$ has a unique fixed point in D_T. We thus proved the local existence and uniqueness for (3.2.1).

By the comparison principle $u \leq \phi$, where ϕ is the unique positive solution of the problem

$$\begin{cases} \phi_t = e^{-R(x)}[D(x)\phi_x(x,t)]_x + [g(1) - d]\phi, & 0 < x < h, \;\; t > 0, \\ \phi_x(x,t) = 0, & x = 0 \text{ or } h, \;\; t > 0, \\ \phi(x,0) = \|v_0(x)\|_\infty, & 0 \leq x \leq h. \end{cases}$$

It is clear that ϕ is defined for all $t > 0$. Hence $v(x,t)$ has local a priori bounds. By using the contraction mapping theorem repeatedly one shows that $v(x,t)$ can be uniquely extended to define on all $t > 0$. By the maximum principle, $v(x,t) > 0$ for all $t > 0$ and $x \in [0,h]$.

The main result of this subsection is the following theorem.

Theorem 3.3.1 *If $d \geq d^*$, then the solution $v(x,t)$ of* (3.2.1) *converges to 0 uniformly for $x \in [0,h]$ as $t \to \infty$.*

If $0 < d < d^*$, then $v(x,t)$ converges to the unique positive steady state $v_d(x)$ uniformly for $x \in [0,h]$ as $t \to \infty$.

Proof. We first consider the case $d > d^* = -\lambda_1(-g(e^{-k_0 x}))$. In such a case, we have

$$\begin{aligned} v_t =& e^{-R(x)}\left[D(x)e^{R(x)}v_x\right]_x + \left[g\left(e^{-k_0x-k\int_0^x e^{R(s)}v(s,t)ds}\right) - d\right]v \\ \leq& e^{-R(x)}\left[D(x)e^{R(x)}v_x\right]_x + \left[g\left(e^{-k_0x}\right) - d\right]v. \end{aligned}$$

Let $\phi_1(x)$ be a principal eigenfunction corresponding to d^* satisfying $v_0(x) \leq \phi_1(x)$ in $[0,h]$. Then we obtain, by the comparison principle,

$$v(x,t) \leq e^{-(d-d^*)t}\phi_1(x) \to 0 \text{ as } t \to 0.$$

Thus

$$\lim_{t\to\infty} v(x,t) = 0 \text{ uniformly for } x \in [0,h] \text{ if } d > d^*.$$

We now consider the case $0 < d < d^*$. By Theorem 3.2.1, (3.2.1) has a unique positive steady state solution $v_d(x)$. We want to show that $v(x,t) \to v_d(x)$ uniformly for $x \in [0,h]$ as $t \to \infty$. For this purpose, we need two key lemmas.

Let

$$z(x,t) = \int_0^x e^{R(s)}v(s,t)ds.$$

Then $z(0,t) = 0$ and

$$\begin{aligned} z_t =& D(x)z_{xx} - \alpha(x)z_x - dz + \int_0^x g\left(e^{-k_0s-kz(s,t)}\right)e^{R(s)}v(s,t)ds \\ =& D(x)z_{xx} - \alpha(x)z_x - dz + G(k_0x + kz(x,t)) - k_0k^{-1}\int_0^x g\left(e^{-k_0s-kz(s,t)}\right)ds, \end{aligned} \tag{3.3.5}$$

where $G(\eta) = k^{-1}\int_0^\eta g(e^{-\xi})\,d\xi$.

Lemma 3.3.2 *Suppose $d \in (-\infty,\infty)$ and $v, \tilde{v} \in C^{2,1}([0,h]\times(0,\infty))$ satisfy*

$$\begin{cases} v_t \leq e^{-R(x)}\left[D(x)e^{R(x)}v_x\right]_x + \left[g\left(e^{-k_0x-k\int_0^x e^{R(s)}v(s,t)ds}\right) - d\right]v, 0 < x < h, t > 0, \\ v_x(0,t) = v_x(h,t) = 0,\ t > 0, \end{cases}$$

and

$$\begin{cases} \tilde{v}_t \geq e^{-R(x)} \left[D(x)e^{R(x)}\tilde{v}_x\right]_x + \left[g\left(e^{-k_0x-k\int_0^x e^{R(s)}\tilde{v}(s,t)ds}\right) - d\right]\tilde{v}, 0 < x < h, t > 0, \\ \tilde{v}_x(0,t) = \tilde{v}_x(h,t) = 0,\ t > 0. \end{cases}$$

If $v(x,t) < \tilde{v}(x,t)$ *for* $x \in [0,h]$ *and all small* $t \geq 0$*, then* $z(x,t) < \tilde{z}(x,t)$ *for all* $t > 0$ *and* $x \in (0,h]$*, where*

$$z(x,t) = \int_0^x e^{R(s)}v(s,t)ds, \quad \tilde{z}(x,t) = \int_0^x e^{R(s)}\tilde{v}(s,t)ds.$$

Proof. From $v(x,t) < \tilde{v}(x,t)$ for $x \in [0,h]$ and all small $t \geq 0$ we have

$$z(x,t) < \tilde{z}(x,t) \text{ for } x \in (0,h] \text{ and } t \geq 0 \text{ small.} \tag{3.3.6}$$

Suppose the conclusion of Lemma 3.3.2 is not true. Then there exists a finite maximal time denoted by t^* such that (3.3.6) holds for every $t \in [0,t^*)$. Clearly $z(x,t^*) \leq \tilde{z}(x,t^*)$ for all $x \in [0,h]$. We claim that

$$z(x,t^*) = \tilde{z}(x,t^*) \text{ for some } x \in (0,h]. \tag{3.3.7}$$

Otherwise we have $z(x,t^*) < \tilde{z}(x,t^*)$ for all $x \in (0,h]$. Let

$$w(x,t) = \tilde{z}(x,t) - z(x,t).$$

Then $w(x,t) \geq 0$ for all $0 \leq t \leq t^*$ and $0 \leq x \leq h$, and

$$\begin{aligned} w_t \geq & D(x)w_{xx} - \alpha(x)w_x - dw + C(x,t)w \\ & + k_0k^{-1}\int_0^x \left[g\left(e^{-k_0s-kz(s,t)}\right) - g\left(e^{-k_0s-k\tilde{z}(s,t)}\right)\right]ds \\ \geq & D(x)w_{xx} - \alpha(x)w_x + [C(x,t)-d]w \ \text{ for } 0 \leq x \leq h,\ t \in (0,t^*], \\ & w(0,t) = 0,\ w(h,t) > 0 \ \text{ for } t \in (0,t^*], \\ & w(x,0) > 0 \ \text{ for } 0 < x \leq h, \end{aligned} \tag{3.3.8}$$

where

$$C(x,t) = kG'(k_0x + k\theta(x,t)),\ \theta(x,t) \in [z(x,t), \tilde{z}(x,t)].$$

We use the strong maximum principle and Hopf boundary lemma to conclude that $w(x,t) > 0$ for $t \in (0, t^*]$ and $x \in (0, h]$, and $w_x(0, t^*) > 0$. Then by the smoothness of $w(x,t)$, we have $w_x(x,t) > 0$ for all t close to t^* and x close to 0. Thus from $w(0,t) \equiv 0$ we conclude $w(x,t) > 0$ for $0 < x \le \delta$, $t^* \le t \le t^* + \delta$ for some small $\delta > 0$. From $w(x, t^*) > 0$ for $x \in [\delta, h]$, we can find $\delta_0 \in (0, \delta)$ such that $w(x,t) > 0$ for $x \in [\delta, h]$ and $t \in [t^*, t^* + \delta_0]$. Thus $w(x,t) > 0$ for $x \in (0, h]$ and $t \in (0, t^* + \delta_0]$, contradicting the maximality of t^*. This proves (3.3.7).

Thus there exists $x_0 \in (0, h]$ such that $w(x_0, t^*) = 0$. If $x_0 = h$, then $w_t(h, t^*) \le 0$. By the boundary conditions, we have

$$D(h)z_{xx}(h,t^*) - \alpha(h)z_x(h,t^*) = D(h)u_x(h,t^*) - \alpha(h)u(h,t^*) = 0,$$
$$D(h)\tilde{z}_{xx}(h,t^*) - \alpha(h)\tilde{z}_x(h,t^*) = D(h)\tilde{u}_x(h,t^*) - \alpha(h)\tilde{u}(h,t^*) = 0.$$

Thus we have $D(h)w_{xx}(h,t^*) - \alpha(h)w_x(h,t^*) = 0$, and from (3.3.8) we obtain

$$0 \ge w_t(h,t^*) \ge k_0 k^{-1} \int_0^h \left[g\left(e^{-k_0 s - kz(s,t)}\right) - g\left(e^{-k_0 s - k\tilde{z}(s,t)}\right) \right] ds.$$

Since $z(x,t^*) \le \tilde{z}(x,t^*)$ in $[0,h]$, the above inequality holds only if $z(x,t^*) \equiv \tilde{z}(x,t^*)$.

From (3.3.8), $w(x,t)$ is an upper solution of the problem

$$\begin{cases} \tilde{w}_t = D(x)\tilde{w}_{xx} - \alpha(x)\tilde{w}_x - d\tilde{w}, & 0 < x < h, \ 0 < t \le t^*, \\ \tilde{w}(0,t) = \tilde{w}(h,t) = 0, & 0 < t \le t^*, \\ \tilde{w}(x,0) = w(x,0) > 0, & 0 < x < h. \end{cases}$$

By the strong maximum principle, $\tilde{w}(x,t) > 0$ for $x \in (0,h)$ and $0 < t \le t^*$. On the other hand, by the comparison principle, we have $w(x,t) \ge \tilde{w}(x,t)$ for $x \in (0,h)$ and $0 < t \le t^*$. Hence $w(x,t^*) > 0$ for $x \in (0,h)$. This contradicts our earlier conclusion that $w(x,t^*) \equiv 0$. Therefore we must have $w(h,t^*) > 0$. We may now apply the strong maximum principle to (3.3.8) to conclude that $w(x,t^*) > 0$ for $x \in (0,h]$, which contradicts (3.3.7). The proof is now complete. □

Lemma 3.3.3 *Suppose $d > 0$, and let $v(x,t)$ be the unique solution of* (3.2.1). *Then there exists $C > 0$ such that*

$$v(x,t) \le C \ \text{ for all } x \in [0,h], \ t > 0. \tag{3.3.9}$$

Proof. Our assumption on the function g implies that

$$g(I) \leq \beta I^{\gamma} \text{ for some } \beta > 0, \gamma > 0 \text{ and all } I \in [0,1].$$

Therefore with

$$I(x,t) = \exp\left(-k_0 x - k\int_0^x e^{R(s)} v(s,t) ds\right)$$

we have

$$g(I(x,t)) \leq \beta I^{\gamma}(x,t) \leq \beta \exp\left(-\gamma k \int_0^x e^{R(s)} v(s,t) ds\right).$$

Then from the equation for v we obtain

$$e^{R(x)} v_t \leq \left[D(x) e^{R(x)} v_x\right]_x + \left[\beta \exp\left(-\gamma k \int_0^x e^{R(x)} v(s,t) ds\right) - d\right] e^{R(x)} v.$$

Integrating this inequality for x over $[0,h]$ we obtain

$$\frac{d}{dt}\left[\int_0^h e^{R(x)} v dx\right] \leq \beta \int_0^h \exp\left(-\gamma k \int_0^x e^{R(s)} v(s,t) ds\right) e^{R(x)} v dx - d\int_0^h e^{R(x)} v dx.$$

Denote

$$w(t) = \int_0^h e^{R(x)} v(x,t) dx, \quad z(x,t) = \int_0^x e^{R(s)} v(s,t) ds.$$

We obtain

$$\int_0^h \exp\left(-\gamma k \int_0^x e^{R(s)} v(s,t) ds\right) e^{R(x)} v dx = \int_0^h e^{-\gamma k z} z_x dx$$
$$= (\gamma k)^{-1}\left[e^{-\gamma k z(0,t)} - e^{-\gamma k z(h,t)}\right] = (\gamma k)^{-1}\left[1 - e^{-\gamma k w(t)}\right].$$

Consequently

$$w_t \leq \beta(\gamma k)^{-1}\left[1 - e^{-\gamma k w(t)}\right] - dw,$$

and

$$w_t + dw \leq C_0 := \beta(\gamma k)^{-1}.$$

Thus we obtain

$$\left(e^{dt} w\right)_t \leq C_0 e^{dt},$$

and

$$w(t) \leq w(0) e^{-dt} + C_0 e^{-dt}\int_0^t e^{ds} ds \leq C := w(0) + C_0/d. \tag{3.3.10}$$

To prove the boundedness of $v(x,t)$ we set

$$W(t) := \max_{x\in[0,h],s\in[0,t]} v(x,s).$$

Clearly $W(t)$ is non-decreasing. Suppose for contradiction that $W(t)\to\infty$ as $t\to\infty$. We are going to derive a contradiction.

Since $W(t)\to\infty$, we can find $t_n\to\infty$ such that $W(t_n)=\max_{x\in[0,h]} v(x,t_n)\to\infty$. Define

$$z_n(x,t)=\frac{v(x,t+t_n-1)}{W(t_n)}.$$

Clearly z_n satisfies

$$\begin{cases} (z_n)_t = e^{-R(x)}\left[D(x)e^{R(x)}(z_n)_x\right]_x + c_n z_n, \\ (z_n)_x(0,t)=(z_n)_x(h,t)=0,\ t>0, \\ (z_n)(x,0)\in[0,1], \end{cases}$$

where $c_n(x,t)=g(I(x,t+t_n-1))-d$. We also have $|c_n|\le M_0:=\max_{I\in[0,1]}|g(I)-d|$. A simple comparison consideration gives

$$0\le z_n(x,t)\le e^{M_0 t}\ \text{ for } x\in[0,h] \text{ and } t\ge 0.$$

By standard parabolic regularity we know that z_n is bounded in $C^{1+\alpha,\alpha}([0,h]\times[1/2,2])$, for any $\alpha\in(0,1)$. Therefore by passing to a subsequence if necessary we have $z_n\to z^*$ in $C^{1,0}([0,h]\times[1/2,2])$. Since $|c_n|\le M_0$, by passing to a further subsequence, we may assume that $c_n\to c$ weakly in $L^2([0,h]\times[1/2,2])$. Clearly we have $|c|\le M_0$. It follows that z^* is a weak solution to

$$\begin{cases} z^*_t = e^{-R(x)}\left[D(x)e^{R(x)}z^*_x\right]_x + cz^*, & x\in[0,h],\ t\in[1/2,2], \\ z^*_x(0,t)=z^*_x(h,t)=0, & x=0 \text{ or } h,\ t\in[1/2,2], \\ z^*(x,t)\in[0,e^{2M_0}], & x\in[0,h],\ t\in[1/2,2]. \end{cases}$$

Since $\max_{x\in[0,h]} z_n(x,1)=1$, we have $\max_{x\in[0,h]} z^*(x,1)=1$ and hence z^* is not identically zero. By the strong maximum principle we have $z^*(x,1)\ge\delta_0>0$ in $[0,h]$. It follows that $z_n(x,1)\ge\delta_0/2$ for all large n and $x\in[0,h]$. Consequently

$$v(x,t_n)\ge(\delta_0/2)W(t_n)\ \text{ for all large } n \text{ and } x\in[0,h].$$

This means

$$v(x,t_n)\to\infty \ \text{ as } n\to\infty,$$

since $W(t_n)\to\infty$ as $n\to\infty$. We thus reach a contradiction to (3.3.10). Therefore there exists a constant C such that

$$v(x,t)\le C \ \text{ for all } x\in[0,h] \text{ and } t>0.$$

This completes the proof. □

We are now ready to prove the conclusion of Theorem 3.3.1 for the case $0<d<d^*$. We want to show that for such d, the solution $v(x,t)$ of (3.2.1) converges to the unique positive steady state $v_d(x)$ uniformly for $x\in[0,h]$ as $t\to\infty$.

We may assume the initial data $v_0>0$ in $[0,h]$. Otherwise we can replace $v_0(x)$ by $v(x,1)$ and $v(x,t)$ by $v(x,t+1)$.

Since $d<d^*=-\lambda_1(\Phi_0)$ and

$$\Phi_\delta(x):=-g(e^{-(k_0+k\delta)x})\to\Phi_0(x)$$

uniformly in $[0,h]$ as $\delta\to 0$, we can find $\delta>0$ sufficiently small such that $d<-\lambda_1(\Phi_\delta)$. Fix such a δ and let ϕ be a positive eigenfunction corresponding to $\lambda_1(\Phi_\delta)$. Then we choose $\epsilon>0$ small so that $\epsilon\phi(x)<v_0(x)$ and $\epsilon\phi(x)e^{R(x)}<\delta$ in $[0,h]$. Let $\underline{v}(x,t)$ be the unique solution of (3.2.1) with initial data $\underline{v}(x,0)=\epsilon\phi(x)$. Then we can find $\tau>0$ small such that

$$0<e^{R(x)}\underline{v}(x,t)<\delta \ \text{ for } t\in(0,\tau] \text{ and } x\in[0,h].$$

Hence for $t\in(0,\tau]$,

$$\begin{aligned}\underline{v}_t =&e^{-R(x)}\left[D(x)e^{R(x)}\underline{v}_x\right]_x+\left[g\left(e^{-k_0x-k\int_0^x e^{R(s)}\underline{v}(s,t)ds}\right)-d\right]\underline{v}\\ \ge&e^{-R(x)}\left[D(x)e^{R(x)}\underline{v}_x\right]_x+\left[-\Phi_\delta(x)-d\right]\underline{v}\\ >&e^{-R(x)}\left[D(x)e^{R(x)}\underline{v}_x\right]_x+\left[-\Phi_\delta(x)+\lambda_1(\Phi_\delta)\right]\underline{v}\end{aligned}$$

It follows that

$$
\begin{cases}
(\underline{v}-\epsilon\phi)_t > e^{-R(x)}\left[D(x)e^{R(x)}(\underline{v}-\epsilon\phi)_x\right]_x \\
\qquad\qquad + \left[-\Phi_\delta(x)+\lambda_1(\Phi_\delta)\right](\underline{v}-\epsilon\phi), & x\in[0,h],\ t\in(0,\tau], \\
(\underline{v}-\epsilon\phi)_x = 0, & x=0 \text{ or } h,\ t\in(0,\tau], \\
\quad \underline{v}-\epsilon\phi = 0, & x\in[0,h],\ t=0.
\end{cases}
$$

By the strong maximum principle we deduce $\underline{v}-\epsilon\phi>0$ for $x\in[0,h]$ and $t\in(0,\tau]$. Hence for any fixed $s\in(0,\tau]$ we have

$$\underline{v}(x,s)>\underline{v}(x,0) \text{ in } [0,h].$$

By continuity,

$$\underline{v}(x,s+t)>\underline{v}(x,t) \text{ in } [0,h] \text{ for all small } t\ge 0.$$

By Lemma 3.3.2 we have $\underline{z}(x,s+t)>\underline{z}(x,t)$ for $x\in(0,h]$ and all $t>0$, where $\underline{z}(x,t)=\int_0^x e^{R(s)}\underline{v}(s,t)ds$. This means $\underline{z}(x,t)$ is monotone increasing in t.

By Lemma 3.3.3, $\underline{z}(x,t)\le C$ for all $x\in[0,h]$ and $t>0$ for some $C>0$. Hence $\lim_{t\to\infty}\underline{z}(x,t)=z_*$ exists. On the other hand, by Lemma 3.3.3 $\|\underline{v}(\cdot,t)\|_\infty$ is also bounded. Thus we can apply the standard parabolic regularity to (3.2.1) to conclude that for any sequence $t_n\to\infty$, $\{\underline{v}(\cdot,t_n)\}$ has a subsequence that converges in $C^1([0,h])$: $\underline{v}(\cdot,t_{n_k})\to v_*$. Since $\underline{z}(\cdot,t_n)\to z_*$, we must have $z_*(x)=\int_0^x e^{R(s)}v_*(s)ds$. Hence $v_*=e^{-R(x)}z_*'$. This implies that $\lim_{t\to\infty}\underline{v}(x,t)$ exists and equals $e^{-R(x)}z_*'(x)$. Hence $e^{-R(x)}z'$ must be a nonnegative steady state of (3.2.1). Since $z_*(0)=0$ and z_* is the limit of an increasing sequence, we have $z_*(x)>0$ for $x\in(0,h]$ and $z_*'\not\equiv 0$. Therefore $e^{-R(x)}z_*'$ is a nontrivial nonnegative steady state of (3.2.1). By the strong maximum principle it is positive and hence we can use Theorem 3.2.1 to conclude that $e^{-R(x)}z_*'\equiv v_d$.

Next we consider $d_K=-\lambda_1(\Phi_K)$ with $K>0$ large. Recall that $\Phi_K(x)=-g(e^{-(k_0+kK)x})$. Let $\phi_K(x)$ be the positive eigenfunction corresponding to $\lambda_1(\Phi_K)$ with $\|\phi_K\|_\infty=1$. It is easy to see by a regularity and compactness argument that as $K\to\infty$, $\lambda_1(\Phi_K)\to 0$ and $\phi_K\to 1$ in $C^1([0,h])$. Therefore we can find $K_0>0$ large so that

$$d>-\lambda_1(\Phi_K),\ \ \frac{1}{2}<\phi_K(x) \ \text{ for } K\ge K_0.$$

We now fix $K > K_0$ such that $K > v_0(x)e^{R(x)}$ in $[0, h]$. Then

$$v_0(x) < 2K\phi_K(x), \quad K < 2K\phi_K(x) \text{ for } x \in [0, h].$$

Let $\bar{v}(x,t)$ be the solution of (3.2.1) with initial data $\bar{v}(x,0) = 2K\phi_K(x)$. Then we can find $\delta_0 > 0$ small so that $v_0(x) < \bar{v}(x,t), K < \bar{v}(x,t)e^{R(x)}$ for $t \in (0,\delta_0]$ and $x \in [0,h]$. Hence for $t \in (0,\delta_0]$, we have

$$\begin{aligned}
\bar{v}_t =& e^{-R(x)}\left[D(x)e^{R(x)}\bar{v}_x\right]_x + \left[g\left(e^{-k_0x-k\int_0^x e^{R(s)}\bar{v}(s,t)ds}\right) - d\right]\bar{v} \\
\le& e^{-R(x)}\left[D(x)e^{R(x)}\bar{v}_x\right]_x + \left[-\Phi_K(x) - d\right]\bar{v} \\
<& e^{-R(x)}\left[D(x)e^{R(x)}\bar{v}_x\right]_x + \left[-\Phi_K(x) + \lambda_1(\Phi_K)\right]\bar{v}
\end{aligned}$$

Thus for $w(x,t) = \bar{v}(x,t) - 2K\phi_K(x)$, we have

$$\begin{cases}
w_t < e^{-R(x)}\left[D(x)e^{R(x)}w_x\right]_x + \left[-\Phi_K(x) + \lambda_1(\Phi_K)\right]w, & x \in [0,h],\ t \in (0,\delta_0], \\
w_x = 0, & x = 0 \text{ or } h,\ t \in (0,\delta_0], \\
w = 0, & x \in [0,h],\ t = 0.
\end{cases}$$

By the strong maximum principle we deduce $w = \bar{v} - 2K\phi_K(x) < 0$ for $t \in (0,\delta_0]$ and $x \in [0,h]$. It follows that $\bar{v}(x,s) < \bar{v}(x,0)$ for $0 < s \le \delta_0$. Using the same argument as before, we conclude that

$$\bar{z}(x,t) := \int_0^x e^{R(s)}\bar{v}(s,t)ds$$

is monotone decreasing in t. Moreover, from Lemma 3.3.2 it follows that $\bar{z}(x,t) > z(x,t) := \int_0^x e^{R(s)}v(s,t)ds > \underline{z}(x,t)$ for all $t > 0$ and $x \in (0,h]$. Hence $\lim_{t\to\infty}\bar{z}(x,t) = z^*(x) \ge \int_0^x e^{R(s)}v_d(s)ds$. We may then use parabolic regularity much as before to deduce that $\bar{v}(x,t) \to e^{-R(x)}(z^*)'(x)$ in $C^1[0,h]$, and $e^{-R(x)}(z^*)'(x)$ is a positive steady state of (3.2.1). Thus we must have $e^{-R(x)}(z^*)'(x) \equiv v_d(x)$.

Since $\underline{z} \le z \le \bar{z}$, and $\lim_{t\to\infty}\underline{z}(x,t) = \lim_{t\to\infty}\bar{z}(x,t) = \int_0^x e^{R(s)}v_d(s)ds$, we necessarily have

$$\lim_{t\to\infty} z(x,t) = \int_0^x e^{R(s)}v_d(s)ds.$$

Thus we can repeat the above argument to conclude that $v(x,t) \to v_d(x)$ uniformly for $x \in [0,h]$ as $t \to \infty$. This proves what we wanted.

Finally we consider the case $d = d^*$. We want to show that in this case the solution $v(x,t)$ to (3.2.1) converges to 0 uniformly for $x \in [0,h]$ as $t \to \infty$.

This follows from a simple modification of the proof for the case $0 < d < d^*$ given above. Indeed, let $\bar{v}(x,t)$ be defined exactly as above. Then we know that $\bar{z}(x,t) := \int_0^x e^{R(s)}\bar{v}(s,t)ds > 0$ is strictly decreasing in t. Hence $\lim_{t\to\infty} \bar{z}(x,t) = z^*(x) \geq 0$ exists. By the same consideration as in the proof for the case $0 < d < d^*$ we can show that $\bar{v}(x,t) \to e^{-R(x)}(z^*)'(x)$ in $[0,h]$ as $t \to \infty$, and hence $e^{-R(x)}(z^*)'(x)$ is a nonnegative steady state of (3.2.1). However, since $d = d^*$, by Theorem 3.2.1, the only nonnegative steady state of (3.2.1) is the trivial solution 0. Hence $\bar{v}(x,t) \to 0$ uniformly for $x \in [0,h]$ as $t \to \infty$, and $\bar{z}(x,t) \to 0$ as $t \to \infty$.

Using Lemma 3.3.2, we have $0 < z(x,t) < \bar{z}(x,t)$, which implies that $z(x,t) \to 0$ as $t \to \infty$. Using this fact and the parabolic regularity as before, we deduce $\lim_{t\to\infty} v(\cdot,t)$ exists in $C^1([0,h])$, and the limit is a nonnegative steady state of (3.2.1). Since $d = d^*$, this limit must be 0. This finishes the proof.

□

Remark 3.3.4 *If $d \leq 0$, it is easy to show that the unique solution $v(x,t)$ of* (3.2.1) *satisfies $\lim_{t\to\infty} v(x,t) = \infty$ uniformly for $x \in [0,h]$. This case is not of biological interest though.*

3.4 The limiting profile of the positive steady state solution

In this section, we study the asymptotic profile of the phytoplankton distribution in three limiting cases: (a) small diffusion rate, (b) large diffusion rate, and (c) deep water column. For simplicity we assume that $D(x) \equiv D > 0$, $\alpha(x) \equiv \alpha > 0$ throughout this section.

3.4.1 The small diffusion case

With $D(x) \equiv D$ and $\alpha(x) \equiv \alpha$, the original equation becomes

$$\begin{cases} u_t = J_x(x,t) + \left[g\left(e^{-k_0 x - k\int_0^x u(s,t)ds}\right) - d\right] u, & 0 < x < h, \quad t > 0, \\ J(x,t) = Du_x(x,t) - \alpha u(x,t) = 0, & x = 0 \text{ or } h, \quad t > 0, \\ u(x,0) = u_0(x) \gneqq 0, & 0 \le x \le h. \end{cases} \tag{3.4.1}$$

The corresponding steady state equation is

$$\begin{cases} -Du'' + \alpha u' = \left[g(e^{-k_0 x - k\int_0^x u(s)ds}) - d\right] u, \quad x \in (0,h), \\ Du'(0) = \alpha u(0), \quad Du'(h) = \alpha u(h). \end{cases} \tag{3.4.2}$$

We will study the asymptotic profile of the positive solution of (3.4.2) as the diffusion rate D approaches 0.

Denote by $\lambda_1^D(\Psi)$ the first eigenvalue of the eigenvalue problem

$$-D\phi'' + \alpha\phi' + \Psi\phi = \lambda\phi, \quad D\phi'(0) = \alpha\phi(0), \quad D\phi'(h) = \alpha\phi(h).$$

Let

$$d_D^* = -\lambda_1^D\left(-g\left(e^{-k_0 x}\right)\right). \tag{3.4.3}$$

By Theorem 3.2.1, for any $0 < d < d_D^*$, there exists a unique positive solution u_D to (3.4.2). By Theorems 3 and 6 in [33], we have $\lim_{D\to 0} d_D^* = g\left(e^{-k_0 h}\right)$, and moreover, for fixed $0 < d < g\left(e^{-k_0 h}\right)$, (3.4.2) has a unique positive solution for any small $D > 0$. Therefore, in the following we always assume

$$0 < d < g\left(e^{-k_0 h}\right).$$

For convenience, we rewrite our equation. Choose $x_D \in [0,h]$ such that

$$u(x_D) = \max_{x\in[0,h]} u(x) = \|u\|_\infty.$$

Set

$$v(x) := u(x)\exp\left[-\frac{\alpha(x - x_D)}{2D}\right].$$

Then we obtain an equivalent problem to (3.4.2):

$$\begin{cases} -Dv'' + \dfrac{\alpha^2}{4D}v = \left[g\left(e^{-k_0x-k\int_0^x v(s)\exp\left[\frac{\alpha(s-x_D)}{2D}\right]ds}\right) - d\right]v, \quad x \in (0,h), \\ v'(0) = \dfrac{\alpha}{2D}v(0), \quad v'(h) = \dfrac{\alpha}{2D}v(h). \end{cases} \tag{3.4.4}$$

Let

$$x = Dy + x_D \qquad \text{and} \qquad z(y) = v(x_D + Dy).$$

We then have

$$-z'' + \frac{\alpha^2}{4}z = D\left[g\left(e^{-k_0(x_D+Dy)-kD\int_{-x_D/D}^{y} z(s)e^{\alpha s/2}ds}\right) - d\right]z \tag{3.4.5}$$

in the interval $(-x_D/D, (h-x_D)/D)$, and z satisfies the boundary condition

$$z' = \frac{\alpha}{2}z \quad \text{for } \; y = -x_D/D \text{ and } (h-x_D)/D. \tag{3.4.6}$$

Concerning the location of x_D, we have the following result.

Lemma 3.4.1 *There exists $D_0 > 0$ such that for any $0 < D \leq D_0$, the unique solution $u(x)$ of (3.4.2) is strictly increasing in $[0,h]$, and hence for such D we have $x_D = h$.*

Proof. Since the function $g\left(e^{-k_0(x_D+Dy)-kD\int_{-x_D/D}^{y} z(s)e^{\alpha s/2}ds}\right) - d$ is bounded, we may choose D_0 so small that $D_0\left|\left[g\left(e^{-k_0(x_D+Dy)-kD\int_{-x_D/D}^{y} z(s)e^{\alpha s/2}ds}\right) - d\right]\right| \leq \frac{\alpha^2}{4}$. It follows from (3.4.5) that $z''(y) \geq 0$ for any $y \in (-x_D/D, (h-x_D)/D)$ with $0 < D \leq D_0$. On the other hand, by (3.4.6), we have $z'(-x_D/D) = \frac{\alpha}{2}z(-x_D/D) > 0$. Hence we have $z'(y) > 0$ on $[-x_D/D, (h-x_D)/D]$ and $z(y)$ is strictly increasing on $[-x_D/D, (h-x_D)/D]$. This means $v(x)$ is strictly increasing on $[0,h]$. It follows $u(x) = v(x)e^{\alpha(x-x_D)/D}$ is strictly increasing on $[0,h]$. This completes the proof of the lemma. □

In the following, we always assume $0 < D \leq D_0$ and $x_D = h$.

Denote by $v_D(x)$ the unique positive solution to (3.4.4). Set $\tilde{z}_D(y) = z_D(y)/\|z_D\|_\infty$. By the definition of $z_D(y)$, we have $\|z_D\|_\infty = z_D(0) = v_D(h) = u_D(h)$. Furthermore, the following holds.

Lemma 3.4.2

$$As\ D \to 0,\ \tilde{z}_D(y) \to e^{\frac{\alpha}{2}y} \quad in\ \ C^1(\Lambda)\ \ for\ any\ finite\ interval\ \ \Lambda \subset (-\infty, 0].$$

Proof. Let $\{D_n\}$ be an arbitrary sequence of positive numbers converging to 0 as $n \to \infty$. We know from Lemma 3.4.1 that $x_n := x_{D_n} = h$ for sufficiently large n. Thus we have $\frac{h-x_n}{D_n} = 0$ and $-\frac{x_n}{D_n} \to -\infty$ as $n \to \infty$. Moreover, using (3.4.5) and the standard elliptic regularity, we find that by passing to a subsequence if necessary, $\tilde{z}_n := \tilde{z}_{D_n} \to z_0$ in $C^1(\Lambda)$ for any finite interval $\Lambda \subset (-\infty, 0]$ and some function $z_0(y)$ defined on $(-\infty, 0]$, and z_0 satisfies (in the weak sense and hence classical sense)

$$\begin{cases} -z_0'' + \dfrac{\alpha^2}{4} z_0 = 0, \quad y \in (-\infty, 0), \\ z_0(0) = 1, \quad z_0'(0) = \dfrac{\alpha}{2}. \end{cases}$$

It is readily checked that the unique solution of this initial value problem is given by $z_0(y) = e^{\frac{\alpha}{2}y}$. This implies that $\tilde{z}_D(y) \to e^{\frac{\alpha}{2}y}$ in $C^1(\Lambda)$ for any finite interval $\Lambda \subset (-\infty, 0]$ as $D \to 0$. The lemma is proved. □

Set $\tilde{v}_D(x) = v_D(x)/\|v_D\|_\infty$. We have

Lemma 3.4.3 $\|\tilde{v}_D(\cdot) - \exp\left[\frac{\alpha}{2D}(\cdot - h)\right]\|_{L^\infty([0,h])} \to 0$ *as* $D \to 0$.

Proof. Let D be sufficiently small. By Lemma 3.4.2, $|\tilde{v}_D(x) - \exp[\frac{\alpha}{2D}(x-h)]|$ remains bounded for those $x \in [0, h]$ such that $(x-h)/D$ remains bounded. Using the fact that $\lim_{y \to -\infty} e^{\alpha y/2} = 0$ and the monotonicity of $\tilde{z}_D(y)$ for small D, we readily deduce that the function $|\tilde{v}_D(x) - \exp[\frac{\alpha}{2D}(x-h)]|$ at the remaining $x \in [0, h]$ is also small. This proves the lemma. □

We now consider the function

$$\hat{u}_D(x) = D^{-1}\tilde{v}_D(x) \exp\left[\frac{\alpha(x-h)}{2D}\right] = \frac{u_D(x)}{D\|v_D\|_\infty} = \frac{u_D(x)}{D\|u_D\|_\infty}.$$

We will show that, for small D, $\hat{u}_D$ behaves like a δ-function concentrating at $x = h$. Indeed we have the following result.

Lemma 3.4.4 *For any given small $\delta > 0$, $x \in [0, h-\delta]$ implies*

$$0 < \hat{u}_D(x) \le D^{-1} \exp\left[-\frac{\alpha\delta}{2D}\right] \to 0 \text{ as } D \to 0. \tag{3.4.7}$$

Moreover,

$$\lim_{D\to 0} \int_0^h \hat{u}_D(x)dx = \frac{1}{\alpha}. \tag{3.4.8}$$

Proof. The inequality (3.4.7) is clear if we notice that $0 < \tilde{v}_D(x) \le 1$ for $x \in [0, h]$. To prove (3.4.8), we let $\eta > 0$ be any fixed large number. Let $D \in (0, D_0]$ be sufficiently small such that $h/D > \eta$. By Lemma 3.2, $\tilde{z}_D \to e^{\frac{\alpha y}{2}}$ in $C^1([-\eta, 0])$. We have

$$\begin{aligned}\int_0^h \hat{u}_D(x)dx &= \int_0^h D^{-1}\tilde{v}_D(x)e^{\alpha(x-h)/(2D)}dx \\ &= \int_{-h/D}^0 \tilde{z}_D(y)e^{\frac{\alpha}{2}y}dy \\ &= \int_{-\eta}^0 \tilde{z}_D(y)e^{\frac{\alpha}{2}y}dy + \int_{-h/D}^{-\eta} \tilde{z}_D(y)e^{\frac{\alpha}{2}y}dy.\end{aligned}$$

Now

$$0 < \int_{-h/D}^{-\eta} \tilde{z}_D(y)e^{\frac{\alpha}{2}y}dy \le \int_{-\infty}^{-\eta} e^{\frac{\alpha}{2}y}dy = \frac{2}{\alpha}e^{-\alpha\eta/2}.$$

And as $D \to 0$,

$$\int_{-\eta}^0 \tilde{z}_D(y)e^{\frac{\alpha}{2}y}dy \to \int_{-\eta}^0 e^{\alpha y}dy = \frac{1-e^{-\alpha\eta}}{\alpha}.$$

It follows that

$$\lim_{D\to 0}\int_0^h \hat{u}_D(x)dx = 1/\alpha + o_\eta(1),$$

with $o_\eta(1) \to 0$ as $\eta \to \infty$. Letting $\eta \to \infty$, we readily have (3.4.8). This completes the proof of the lemma. □

Denote $\tau_D := D\|v_D\|_\infty = D\|u_D\|_\infty$. Then we have

$$u_D(x) = \tau_D \hat{u}_D(x).$$

Moreover, $\hat{u}_D(x)$ satisfies

$$\begin{cases} -D\hat{u}_D'' + \alpha\hat{u}_D' = \left[g\left(e^{-k_0x-k\tau_D\int_0^x \hat{u}_D ds}\right) - d\right]\hat{u}_D, & x \in (0, h), \\ D\hat{u}_D' = \alpha\hat{u}_D, & x = 0, h. \end{cases} \tag{3.4.9}$$

Lemma 3.4.5 *τ_D is bounded from above.*

Proof. Suppose there exists a subsequence $D_n \to 0$ such that $\tau_n := \tau_{D_n} \to \infty$. If we denote $\hat{u}_n := \hat{u}_{D_n}$ and $g_n(x) = g\left(e^{-k_0 x - k\tau_n \int_0^x \hat{u}_n ds}\right)$, and integrate equation (3.4.9) over $[0, h]$, we will have

$$\int_0^h [g_n(x) - d]\hat{u}_n(x)dx = 0,$$

and hence

$$\begin{aligned} d\int_0^h \hat{u}_n(x)dx &= \int_0^h g_n(x)\hat{u}_n(x)dx \\ &\leq \int_0^h g(e^{-k\tau_n \int_0^x \hat{u}_n(s)ds})\hat{u}_n(x)dx \\ &= \frac{1}{\tau_n}\int_0^{\tau_n \int_0^h \hat{u}_n(x)dx} g(e^{-k\eta})d\eta. \end{aligned}$$

Since $\int_0^h \hat{u}_n(x)dx \to \frac{1}{\alpha}$ (Lemma3.4.4), it follows that

$$\limsup_{n\to\infty} \tau_n \leq \frac{\alpha}{d}\int_0^\infty g\left(e^{-k\eta}\right)d\eta < \infty,$$

which contradicts the initial assumption that $\tau_n \to \infty$. This finishes the proof. □

Lemma 3.4.6 *τ_D is bounded away from* 0.

Proof. Assume there exist a sequence $D_n \to 0$ so that $\tau_n \to 0$. Much as before, we integrate (3.4.9) over $[0, h]$ to get

$$\int_0^h [g_n(x) - d]\hat{u}_n(x)dx = 0. \tag{3.4.10}$$

By the mean value theorem we have

$$\int_0^h g_n(x)\hat{u}_n(x)dx = g_n(x_n^0)\int_0^h \hat{u}_n(x)dx$$

for some $x_n^0 \in [0, h]$.

We may assume, subject to a subsequence, that $x_n^0 \to x^0 \in [0, h]$. It follows that

$$g_n(x_n^0) = g\left(e^{-k_0 x_n^0 - k\tau_n \int_0^{x_n^0} \hat{u}_n(x)dx}\right) \to g(e^{-k_0 x^0}) \text{ as } n \to \infty.$$

Thus letting $n \to \infty$ in (3.4.10) and using Lemma 3.4.4, we obtain

$$d = g(e^{-k_0 x_0}),$$

which contradicts our assumption that $0 < d < g(e^{-k_0 h})$. This completes the proof of the lemma. $\square$

Lemma 3.4.7

$$\lim_{D \to 0} \tau_D \to \tau_*,$$

where $\tau_* > 0$ is uniquely determined by the equation

$$d = \int_0^1 g\left(e^{-k_0 h - k\tau_* x/\alpha}\right) dx. \tag{3.4.11}$$

Proof. Since τ_D is bounded away from 0 and ∞, for any sequence $D_n \to 0$, subject to a subsequence, we may assume that $\tau_n := \tau_{D_n} \to \tau_* \in (0, \infty)$. We prove that τ_* is uniquely determined by (3.4.11).

To this end we fix a $\delta \in (0, h)$. Integrating the equation for $\hat{u}_n := \hat{u}_{D_n}$, namely, (3.4.9), over $[0, h]$ we obtain

$$\int_0^h [g_n(x) - d]\hat{u}_n(x)dx = 0.$$

By Lemma 3.4.4, as $n \to \infty$,

$$\int_0^h \hat{u}_n(x)dx \to \frac{1}{\alpha}, \qquad \int_0^\delta \hat{u}_n(x)dx \to 0,$$

and

$$\int_0^\delta g_n(x)\hat{u}_n(x)dx \le g(1) \int_0^\delta \hat{u}_n(x)dx \to 0.$$

Therefore

$$\begin{aligned} \frac{d}{\alpha} &= \int_\delta^h g_n(x)\hat{u}_n(x)dx + o(1) \\ &= \int_\delta^h g\left[e^{-k_0 x - k\tau_n \int_0^x \hat{u}_n(s)ds}\right] \hat{u}_n(x)dx + o(1) \\ &= \frac{1}{k\tau_n} \int_{k_0\delta + k\tau_n \int_0^\delta \hat{u}_n dx}^{k_0 h + k\tau_n \int_0^h \hat{u}_n dx} g(e^{-s})ds - \frac{k_0}{k\tau_n} \int_\delta^h g_n(x)dx + o(1) \end{aligned} \tag{3.4.12}$$

Since $0 < g_n(x) \le g(1)$ for all n and $x \in [0, h]$, we may assume, by passing to a subsequence, $g_n(x) \to g_*(x)$ weakly in $L^2([0, h])$ with $\|g_*\|_\infty \le g(1)$. Hence letting $n \to \infty$ in (3.4.12), we obtain

$$\frac{d}{\alpha} = \frac{1}{k\tau_*} \int_{k_0\delta}^{k_0 h + k\tau_*/\alpha} g\left(e^{-x}\right) dx - \frac{k_0}{k\tau_*} \int_\delta^h g_*(x)dx.$$

Letting $\delta \to h$, we obtain

$$\frac{dk\tau_*}{\alpha} = \int_{k_0 h}^{k_0 h + k\tau_*/\alpha} g\left(e^{-x}\right) dx.$$

That is,

$$d = \int_0^1 g\left(e^{-k_0 h - k\tau_* x/\alpha}\right) dx.$$

Since τ_* is uniquely determined this way, we have

$$\tau_D \to \tau_* \text{ as } D \to 0.$$

This finishes the proof. □

Summing up the above discussion, we have the following result.

Theorem 3.4.8 *Let $d \in (0, g(e^{-k_0 h}))$. Then for all small $D > 0$, the unique positive solution $u_D(x)$ of* (3.4.2) *is strictly increasing in* $[0, h]$. *Moreover, as $D \to 0$,*

$$\max_{x \in \left[0,\, h - \frac{2D}{\alpha}|\ln D|\right]} \left|u_D(x) - \tau^* D^{-1} e^{\alpha(x-h)/D}\right| \to 0, \tag{3.4.13}$$

and

$$\int_0^h u_D(x) dx \to \tau_*/\alpha, \tag{3.4.14}$$

where τ_* is uniquely determined by (3.4.11).

Proof. These conclusions follow directly from Lemmas 3.4.1, 3.4.3, 3.4.4 and 3.4.7. We explain how (5.3.12) is obtained; the other conclusions are obvious.

From the definitions we obtain

$$u_D(x) = e^{\frac{\alpha(x-h)}{2D}} \|v_D\|_\infty \tilde{v}_D(x) = \tau_D D^{-1} e^{\frac{\alpha(x-h)}{2D}} \tilde{v}_D(x).$$

For $x \in [0, h - \frac{2D}{\alpha}|\ln D|]$, we have

$$D^{-1} e^{\alpha(x-h)/(2D)} \le 1,$$

and hence (5.3.12) follows readily from Lemmas 3.4.3 and 3.4.7. □

Let us observe that (5.3.12) and (3.4.14) imply that for small D, $u_D(x)$ behaves like a δ-function concentrating at $x = h$.

3.4.2 The large diffusion case

By Theorem 6 in [33], $d_D^* \to \frac{1}{h}\int_0^h g(e^{-k_0x})dx$ as $D \to \infty$. By Theorem 3.2.1, we know that for any fixed $d \in (0, \frac{1}{h}\int_0^h g(e^{-k_0x})dx)$, (3.4.2) has a unique positive solution $u_D(x)$ for every large D. We now show that the asymptotic profile of $u_D(x)$ is given by the following result.

Theorem 3.4.9 *As $D \to \infty$,*

$$u_D(x) \to c^* \quad \textit{uniformly on} \;\; [0,h], \tag{3.4.15}$$

where c^* is uniquely determined by the equation

$$d = \frac{1}{h}\int_0^h g\left(e^{-k_0x-kc^*x}\right)dx.$$

Proof. Setting $v_D(x) := u_D(x)\exp\left(-\frac{\alpha x}{2D}\right)$, we have

$$\begin{cases} -v_D'' + \dfrac{\alpha^2}{4D^2}v_D = \dfrac{1}{D}\left[g\left(e^{-k_0x-k\int_0^x v_D(s)\exp\left(\frac{\alpha s}{2D}\right)ds}\right) - d\right]v_D, \quad x \in (0,h), \\ v_D'(0) = \dfrac{\alpha}{2D}v_D(0), \;\; v'(h) = \dfrac{\alpha}{2D}v_D(h). \end{cases} \tag{3.4.16}$$

Denote $\tilde{v}_D = v_D/\|v_D\|_\infty$. We then have

$$\begin{cases} -\tilde{v}_D'' + \dfrac{\alpha^2}{4D^2}\tilde{v}_D = \dfrac{1}{D}\left[g\left(e^{-k_0x-k\|v_D\|_\infty\int_0^x \tilde{v}_D(s)\exp\left(\frac{\alpha s}{2D}\right)ds}\right) - d\right]\tilde{v}_D, \\ \tilde{v}_D'(0) = \dfrac{\alpha}{2D}\tilde{v}_D(0), \;\; \tilde{v}_D'(h) = \dfrac{\alpha}{2D}\tilde{v}_D(h). \end{cases} \tag{3.4.17}$$

The right side of the first equation of (3.4.17) is clearly uniformly bounded on $[0,h]$ for all large D. Hence $\tilde{v}_D$ and $\tilde{v}_D''$ are both uniformly bounded on $[0,h]$ for large D. Thus along any sequence of D going to ∞, we can choose a subsequence, say D_n, such that $D_n \to \infty$, and $\tilde{v}_n := \tilde{v}_{D_n}$ converges in $C^1([0,h])$ to a function v_0. Clearly v_0 satisfies (in the weak sense and hence classical sense)

$$v_0'' = 0 \;\text{ in } (0,h), \;\; v_0'(0) = v_0'(h) = 0, \;\; \|v_0\|_\infty = 1, \tag{3.4.18}$$

which implies $v_0 \equiv 1$. It follows that $\tilde{v}_D \to 1$ in $C^1([0,h])$ as $D \to \infty$.

On the other hand, since $\tilde{v}_D$ satisfies (3.4.17), we can multiply the first equation of (3.4.17) by $\exp\left(\frac{\alpha x}{2D}\right)$ and integrate it over $[0,h]$ to obtain

$$\int_0^h \left[g\left(e^{-k_0x-k\|v_D\|_\infty\int_0^x \tilde{v}_D(s)\exp\left(\frac{\alpha s}{2D}\right)ds}\right) - d\right]\tilde{v}_D(x)\exp\left(\frac{\alpha x}{2D}\right)dx = 0.$$

Using our assumption on d and the fact that $\tilde{v}_D \to 1$ in $C^1([0,h])$, we easily see from the above identity that $\|v_n\|_\infty$ is bounded away from ∞ and 0. Thus by passing to a subsequence, we may assume that $\|v_n\|_\infty \to c^*$ for some $c^* \in (0,\infty)$. Letting $n \to \infty$ in the above equation (with $D = D_n$), we obtain

$$\int_0^h \left[g\left(e^{-k_0 x - kc^* x}\right) - d \right] dx = 0. \tag{3.4.19}$$

It follows that

$$d = \frac{1}{h}\int_0^h g\left(e^{-k_0 x - kc^* x}\right) dx.$$

As this identity uniquely determines the value of c^*, we must have

$$u_n(x) := v_n(x)\exp(x/2D_n) \to c^* \text{ uniformly on } [0,h].$$

Hence $u_D(x) \to c^*$ uniformly on $[0,h]$ as $D \to \infty$. The proof is now complete. □

Remark 3.4.10 From Theorem 3.4.9 we clearly have

$$\int_0^h u_D(x)dx \to c^* h \text{ as } D \to \infty.$$

Comparing

$$d = \frac{1}{h}\int_0^h g\left(e^{-k_0 x - kc^* x}\right) dx = \int_0^1 g(e^{-k_0 hx - kc^* hx})dx,$$

with (3.4.11), namely

$$d = \int_0^1 g(e^{-k_0 h - k\tau_* x/\alpha})dx,$$

we easily deduce that

$$c^* h > \tau_*/\alpha \text{ if } k_0 > 0; \;\; c^* h = \tau_*/\alpha \text{ if } k_0 = 0.$$

That is, when all the other parameters are the same and the water column has positive background turbidity ($k_0 > 0$), the total biomass of the phytoplankton in the large diffusion case is bigger than that in the small diffusion case. In view of the profiles of the phytoplankton distribution given in Theorems 3.4.8 and 3.4.9, the above conclusion appears biologically reasonable, as in the small diffusion case the population concentrates near the bottom of the water column, and hence intuitively its overall use of light would be less than in the large diffusion case, where the population distribution is rather even over the water column. The fact that the total biomass tends to the same limit in both cases when $k_0 = 0$ appears less intuitive.

3.4.3 The deep water column case

From the definition,

$$d^* = -\lambda_1(\Phi_0) \text{ with } \Phi_0(x) = -g(e^{-k_0 x}).$$

Hence

$$d^* = g(1) \text{ if } k_0 = 0;\ d^* < g(1) \text{ if } k_0 > 0.$$

This indicates that $d^* > 0$ is independent of h when $k_0 = 0$, and d^* depends on h when $k_0 > 0$. We will write $d^* = d^*_h$ to stress its dependence on h when $k_0 > 0$. In fact, by Lemma 6.1 of [33], when $k_0 > 0$, $d^* = d^*_h > 0$ is strictly decreasing in h. Therefore

$$d^*_\infty := \lim_{h\to\infty} d^*_h$$

always exists and $d^*_\infty \in [0, g(1))$. We assume for the moment that

$$d^*_\infty > 0, \tag{3.4.20}$$

and will find out the conditions on the other parameters D and α guaranteeing this condition later.

From now on, we assume that (3.4.20) holds and fix $d \in (0, d^*_\infty)$. It follows that $0 < d < d^*_h$ for every $h > 0$. By Theorem 3.2.1, (3.4.2) has a unique positive solution $u(x)$ for every $h > 0$. To stress its dependence on h, we denote $u(x) = u_h(x)$ and will examine the asymptotic profile of $u_h(x)$ as $h \to \infty$. It turns out that once this profile is known, then one can use the results in [40] to find out the exact conditions on D and α such that (3.4.20) holds. This will answer a question left open in [33].

Let

$$u_h(x) = w_h(x) e^{\alpha x/D}.$$

Then $w_h(x)$ satisfies the equation

$$\begin{cases} -\left[De^{\alpha x/D} w'\right]' = \left[g\left(e^{-k_0 x - k\int_0^x w(s)e^{\alpha x/D} ds}\right) - d\right] w e^{\alpha x/D}, \ \ 0 < x < h, \\ w'(0) = w'(h) = 0. \end{cases} \tag{3.4.21}$$

Lemma 3.4.11 $w_h'(x) < 0$ *for* $x \in (0, h)$.

Proof. Integrating the first equation in (3.4.21) over $[0, h]$, we obtain

$$\int_0^h \left[g\left(e^{-k_0 x - k \int_0^x e^{\alpha s/D} w_h(s) ds} \right) - d \right] w_h(x) e^{\alpha x/D} dx = -De^{\alpha x/D} w_h'(x) \Big|_0^h = 0. \qquad (3.4.22)$$

Since the function $f(x) := g\left(e^{-k_0 x - k \int_0^x e^{\alpha s/D} w_h(s) ds}\right) - d$ is strictly decreasing, the above identity implies $f(0) > 0$, $f(h) < 0$, and $f(x)$ has a unique zero $x_h \in (0, h)$. Therefore, by (3.4.21), $e^{\alpha x/D} w_h'(x)$ is strictly decreasing in $[0, x_h]$, and strictly increasing in $[x_h, h]$. Since $w_h'(0) = w_h'(h) = 0$, it follows that

$$w_h'(x) < 0 \quad \text{for all } \ x \in (0, h).$$

□

By the above lemma, we have $\|w_h\|_\infty = w_h(0)$. Denote

$$\tilde{w}_h(x) := w_h(x)/\|w_h\|_\infty = w_h(x)/w_h(0) \text{ and } z_h(x) := \frac{\tilde{w}_h'(x)}{\tilde{w}_h(x)}.$$

Then clearly

$$z_h(0) = z_h(h) = 0 \text{ and } z_h(x) < 0 \text{ for } x \in (0, h). \qquad (3.4.23)$$

Moreover, the following conclusions hold.

Lemma 3.4.12

$$\liminf_{h \to \infty} \int_0^h \tilde{w}_h(x) e^{\alpha x/D} dx > 0, \qquad (3.4.24)$$

and

$$0 \geq z_h(x) \geq -\frac{\alpha + \sqrt{\alpha^2 + 8dD}}{2D} \textit{ for all } h > 0 \textit{ and } x \in [0, h]. \qquad (3.4.25)$$

Proof. Suppose there is some $\bar{h} > 0$ and $\bar{x} \in [0, \bar{h}]$ such that

$$z_{\bar{h}}(\bar{x}) < -\frac{\alpha + \sqrt{\alpha^2 + 8dD}}{2D}.$$

Clearly $\bar{x} \neq 0, \bar{x} \neq \bar{h}$ since $z_{\bar{h}}(0) = z_{\bar{h}}(\bar{h}) = 0$.

Since $z_{\bar{h}}(\bar{h}) = 0$, we can find a point $\hat{x} \in (\bar{x}, \bar{h})$ such that

$$z_{\bar{h}}(\hat{x}) = -\frac{\alpha + \sqrt{\alpha^2 + 8dD}}{2D}$$

and for all $x \in (\bar{x}, \hat{x})$,

$$z_{\bar{h}}(x) < -\frac{\alpha + \sqrt{\alpha^2 + 8dD}}{2D}.$$

It follows that

$$z'_{\bar{h}}(\hat{x}) \geq 0.$$

On the other hand, by (3.4.21) we have

$$\begin{aligned} Dz'_{\bar{h}}(\hat{x}) + d &= -g\left(e^{-k_0\hat{x} - k\int_0^{\hat{x}} e^{\alpha s/D} w_{\bar{h}}(s)ds}\right) - Dz^2_{\bar{h}}(\hat{x}) - \alpha z_{\bar{h}}(\hat{x}) + 2d \\ &\leq -Dz^2_{\bar{h}}(\hat{x}) - \alpha z_{\bar{h}}(\hat{x}) + 2d = 0. \end{aligned}$$

Thus

$$z'_{\bar{h}}(\hat{x}) \leq -d/D < 0.$$

We reach a contradiction. This proves that (3.4.25) holds. It follows that

$$\tilde{w}_h(x) = \exp\left(\int_0^x z_h(s)ds\right) \geq \exp\left(-\frac{\alpha + \sqrt{\alpha^2 + 8dD}}{2D}x\right).$$

Therefore for $h > 0$,

$$\int_0^h \tilde{w}_h(x)e^{\alpha x/D}dx \geq \int_0^h \exp\left(\frac{\alpha - \sqrt{\alpha^2 + 8dD}}{2D}x\right)dx.$$

(3.4.24) follows readily from this inequality. □

Lemma 3.4.13 *$w_h(0)$ is bounded away from ∞.*

Proof. Integrating the first equation in (3.4.21) over $[0, h]$ and dividing the result by $w_h(0)$, we obtain

$$\int_0^h \left[g\left(e^{-k_0 x - kw_h(0)\int_0^x \tilde{w}_h(s)e^{\alpha s/D}ds}\right) - d\right]\tilde{w}_h(x)e^{\alpha x/D}dx = 0.$$

Therefore

$$\begin{aligned} d\int_0^h \tilde{w}_h(x)e^{\alpha x/D}dx \leq& \int_0^h g\left(e^{-kw_h(0)\int_0^x \tilde{w}_h(s)e^{\alpha s/D}ds}\right)\tilde{w}_h(x)e^{\alpha x/D}dx \\ =& \frac{1}{w_h(0)}\int_0^{w_h(0)\int_0^h \tilde{w}_h(x)e^{\alpha x/D}dx} g\left(e^{-k\eta}\right)d\eta \\ \leq& \frac{1}{w_h(0)}\int_0^\infty g\left(e^{-k\eta}\right)d\eta \end{aligned}$$

Lemma 3.4.13 then follows from Lemma 3.4.12 and the fact that $\int_0^\infty g\left(e^{-k\eta}\right)d\eta < \infty$. □

Lemma 3.4.14 *$w_h(0)$ is bounded away from* 0 *if* $0 < d < d^*_\infty$ *and* $k_0 > 0$.

Proof. We argue indirectly. Assume that $0 < d < d^*_\infty$ and $k_0 > 0$, but there exists $h_n \to \infty$ such that $w_n(0) := w_{h_n}(0) \to 0$. Let $\phi_n(x) > 0$ be the (L^∞-normalized) first eigenfunction corresponding to $d^*_n := d^*_{h_n}$. Multiplying the first equation in (3.4.21) by $\phi_n(x)/\|w_n\|_\infty$ and integrating it over $[0, h_n]$, we obtain

$$-D\int_0^{h_n} e^{\alpha x/D}\tilde{w}_n'\phi_n' dx = \int_0^{h_n}\left[g\left(e^{-k_0x-kw_n(0)\int_0^x \tilde{w}_n(s)e^{\alpha s/D}ds}\right) - d\right]\tilde{w}_n\phi_n e^{\alpha x/D}dx. \quad (3.4.26)$$

Multiplying the equation for ϕ_n by $\tilde{w}_n e^{\alpha x/D}$ and integrating it over $[0, h_n]$ we obtain

$$-D\int_0^{h_n} e^{\alpha x/D}\tilde{w}_n'\phi_n' = \int_0^{h_n}\left[g\left(e^{-k_0x}\right) - d^*_n\right]\tilde{w}_n\phi_n e^{\alpha x/D}dx. \quad (3.4.27)$$

From (5.4.4) and (3.4.27) we deduce

$$\begin{aligned}&(d^*_n - d)\int_0^{h_n}\tilde{w}_n\phi_n e^{\alpha x/D}dx\\ =&\int_0^{h_n}\left[g\left(e^{-k_0x}\right) - g\left(e^{-k_0x-kw_n(0)\int_0^x \tilde{w}_n(s)e^{\alpha s/D}ds}\right)\right]\tilde{w}_n\phi_n e^{\alpha x/D}dx.\end{aligned} \quad (3.4.28)$$

Therefore

$$\int_0^{h_n}\Delta_n(x)\tilde{w}_n\phi_n e^{\alpha x/D}dx = 0, \quad (3.4.29)$$

where

$$\Delta_n(x) := d^*_n - d - \left[g\left(e^{-k_0x}\right) - g\left(e^{-k_0x-kw_n(0)\int_0^x \tilde{w}_n(s)e^{\alpha s/D}ds}\right)\right].$$

Clearly

$$d^*_n - d \geq d^*_\infty - d > 0$$

and

$$0 < g\left(e^{-k_0x-kw_n(0)\int_0^x \tilde{w}_n(s)e^{\alpha s/D}ds}\right) < g\left(e^{-k_0x}\right) \to 0 \text{ as } x \to \infty.$$

Thus we can find a sufficiently large T such that

$$\Delta_n(x) > 0 \text{ for all } x \geq T \text{ and } n \geq 1. \quad (3.4.30)$$

For $x \in [0, T]$, since $w_n(0) \to 0$, we have

$$g\left(e^{-k_0x}\right) - g\left(e^{-k_0x-kw_n(0)\int_0^x \tilde{w}_n(s)e^{\alpha s/D}ds}\right) \to 0$$

uniformly in $[0,T]$. It follows that, as $n\to\infty$,

$$\Delta_n(x)\to d^*_\infty - d > 0 \text{ uniformly for } x\in[0,T].$$

Thus for all large n, $\Delta_n(x)>0$ in $[0,T]$. In view of (3.4.30), we find that $\Delta_n(x)>0$ in $[0,h_n]$ for all large n. But this clearly contradicts (3.4.29). This proves the lemma. $\square$

Lemma 3.4.15 *For* $k_0=0$, $w_h(0)$ *is bounded away from* 0 *if* $0<d<g(1)-\frac{\alpha^2}{4D}$, *but* $\lim_{h\to\infty} w_h(0)=0$ *when* $d\geq g(1)-\frac{\alpha^2}{4D}$.

Proof. Let $k_0=0$. First we let $0<d<g(1)-\frac{\alpha^2}{4D}$ and assume there is a subsequence of h, say h_n, such that $h_n\to\infty$ and $w_n(0):=w_{h_n}(0)\to 0$. We prove this is impossible. Using elliptic regularity, much as before, we deduce by standard argument that, subject to a subsequence, $\tilde{w}_n(x):=w_n(x)/w_n(0)\to w_0(x)$ in $C^1(\Lambda)$ for any finite interval $\Lambda\subset[0,\infty)$, and w_0 is positive and satisfies

$$\begin{cases} -Dw_0''-\alpha w_0'=[g(1)-d]w_0, & 0<x<\infty,\\ w_0(0)=1,\ \ w_0'(0)=0. \end{cases}$$

A direct calculation yields

$$w_0(x)=e^{-\frac{\alpha}{2D}x}\Big[\cos(\theta x)+\frac{\alpha}{2D\theta}\sin(\theta x)\Big],$$

with $\theta=(2D)^{-1}\sqrt{4D[g(1)-d]-\alpha^2}$. But this is absurd since the above expression changes sign in $(0,\infty)$.

We next prove that $w_h(0)\to 0$ when $d\geq g(1)-\alpha^2/(4D)$. Assume on the contrary that $d\geq g(1)-\alpha^2/4D$ but there is a subsequence $h_n\to\infty$ such that $w_n(0):=w_{h_n}(0)\to\tau\in(0,\infty)$. As before we may assume that $\tilde{v}_n(x):=w_{h_n}(x)e^{\alpha x/2D}\to v_0(x)$ in $C^1_{\text{loc}}([0,\infty))$ and $v_0(x)$ is a weak hence classic positive solution to

$$\begin{cases} -Dv_0''+\dfrac{\alpha^2}{4D}\alpha v_0=\left[g\left(e^{-k\int_0^x v_0(s)e^{\alpha s/2D}ds}\right)-d\right]v_0, & 0<x<\infty,\\ v_0(0)=\tau,\ \ v_0'(0)=\alpha\tau/2D. \end{cases}$$

From $d\geq g(1)-\alpha^2/(4D)$ we deduce $v_0''(x)\geq 0$ for $x>0$. Hence $v_0'(x)\geq\alpha\tau/2D$ and $v_0(x)\geq\tau$ for any $x\geq 0$. This implies

$$\int_0^\infty v_0(x)e^{\alpha x/2D}dx=\infty.$$

On the other hand, from (3.4.22)(with $k_0 = 0$), we have

$$d\int_0^h w_h(x)e^{\alpha x/D}dx = \int_0^h g\left(e^{-k\int_0^x w_h(s)e^{\alpha s/D}ds}\right) w_h(x)e^{\alpha x/D}dx$$
$$\leq \int_0^\infty g(e^{-k\eta})d\eta.$$

It follows that, for any given $M > 0$,

$$d\int_0^M v_0(x)e^{\alpha x/(2D)}dx = \lim_{n\to\infty} d\int_0^M \tilde{v}_n(x)e^{\alpha x/(2D)}dx$$
$$\leq \lim_{n\to\infty} d\int_0^{h_n} w_{h_n}(x)e^{\alpha x/D}dx$$
$$\leq \int_0^\infty g(e^{-k\eta})d\eta.$$

Hence

$$\int_0^\infty v_0(x)e^{\alpha x/(2D)}dx \leq d^{-1}\int_0^\infty g(e^{-k\eta})d\eta < \infty.$$

This contradiction proves the lemma.

□

Let

$$v_h(x) = w_h(x)e^{\alpha x/2D} = u_h(x)e^{-\alpha x/2D}.$$

Then $v_h(x)$ is the unique positive solution of the boundary value problem

$$\begin{cases} -Dv'' + \dfrac{\alpha^2}{4D}v = \left[g\left(e^{-k_0 x - k\int_0^x v(s)\exp\left[\frac{\alpha s}{2D}\right]ds}\right) - d\right]v, \quad x \in (0,h), \\ v'(0) = \dfrac{\alpha}{2D}v(0), \quad v'(h) = \dfrac{\alpha}{2D}v(h). \end{cases} \tag{3.4.31}$$

By Lemma 3.4.11, for any fixed $T > 0$ and $x \in [0,T]$,

$$0 < v_h(x) = w_h(x)e^{\alpha x/2D} \leq w_h(0)e^{\alpha x/2D} \leq w_h(0)e^{\alpha T/2D} < \infty.$$

Since $v_h(x)$ satisfies (3.4.31), we find that $v_h''(x)$ is uniformly bounded on $[0,T]$. Thus by the Sobolev compact embedding theorems and a diagonal argument, from any given sequence of h going to infinity, we can choose a subsequence $\{h_n\}$, such that $v_n(x) := v_{h_n}(x)$ satisfies

$$v_n(x) \to v_\infty(x) \text{ in } C^1(\Lambda) \text{ for any finite interval } \Lambda \in [0,\infty).$$

By Lemmas 3.4.13 and 3.4.14 , if $k_0 > 0$, or if $k_0 = 0$ and $0 < d < g(1) - \alpha^2/4D$, we have

$$\tau_\infty := v_\infty(0) = \lim_{n\to\infty} w_{h_n}(0) \in (0, \infty).$$

On the other hand, if $k_0 = 0$ and $d \geq g(1) - \alpha^2/4D$, then

$$\tau_\infty := v_\infty(0) = \lim_{h\to\infty} w_h(0) = 0.$$

Moreover, v_∞ satisfies (in the weak sense and hence classical sense)

$$\begin{cases} -Dv_\infty'' + \dfrac{\alpha^2}{4D} v_\infty = \left[g\left(e^{-k_0 x - k\int_0^x v_\infty(s)\exp\left(\frac{\alpha s}{2D}\right) ds} \right) - d \right] v_\infty, \quad x \in (0, \infty), \\ v_\infty(0) = \tau_\infty, \;\; v_\infty'(0) = \dfrac{\alpha \tau_\infty}{2D}. \end{cases} \tag{3.4.32}$$

Thus for $k_0 = 0$ and $d \geq g(1) - \alpha^2/4D$, we have $\tau_\infty = 0$ and hence $v_\infty \equiv 0$ by the uniqueness results of ODE. Hence in this case we have

$$\lim_{h\to\infty} v_h(x) = 0 \quad \text{in} \;\; C^1_{loc}([0, \infty)),$$

which implies

$$\lim_{h\to\infty} u_h(x) = 0 \quad \text{in} \;\; C^1_{loc}([0, \infty)).$$

In the following, we assume $k_0 > 0$, or $k_0 = 0$ and $0 < d < g(1) - \alpha^2/4D$. In this case $\tau_\infty \in (0, \infty)$. By the strong maximum principle we have $v_\infty(x) > 0$ for any $x \in [0, \infty)$. We will prove that such v_∞ is unique, which implies that $v_h \to v_\infty$ in $C^1_{loc}([0, \infty))$ as $h \to \infty$. We need the following lemma.

Lemma 3.4.16 *There exist positive constants C, L and $\alpha' > \alpha$ such that*

$$v_\infty(x) \leq C e^{-\frac{\alpha'}{2D} x} \;\; \textit{for } x \in [L, \infty). \tag{3.4.33}$$

Proof. First we consider the case $k_0 > 0$.

Fix $x_d \in (0, \infty)$ such that

$$g\left(e^{-k_0 x}\right) \leq \frac{d}{2} \;\; \text{for } x \in [x_d, \infty).$$

Then we have, for $x \in [x_d, \infty)$,

$$v''_\infty(x) \geq \left(\frac{\alpha^2}{4D^2} + \frac{d}{2D}\right) v_\infty.$$

Denote

$$\xi = \sqrt{\frac{\alpha^2}{4D^2} + \frac{d}{2D}},$$

and for any small $\epsilon > 0$ define

$$z_\epsilon(x) = v_\infty(x_d)\big[(1-\epsilon)e^{-\xi(x-x_d)} + \epsilon e^{\xi(x-x_d)}\big].$$

One readily checks that z_ϵ satisfies

$$z'' = \xi^2 z,\ z(x_d) = v_\infty(x_d),\ \lim_{x\to\infty} \frac{z(x)}{\exp\left(\frac{\alpha}{2D}x\right)} = +\infty.$$

Since $v_\infty(x) \leq v_\infty(0)e^{\frac{\alpha}{2D}x}$ for $x > 0$ (as each v_h has this property due to the monotonicity of $w_h(x)$), we may now apply the comparison principle to conclude that $v_\infty \leq z_\epsilon$ in $[x_d, \infty)$. Letting $\epsilon \to 0$, we deduce

$$v_\infty(x) \leq v_\infty(x_d)e^{-\xi(x-x_d)} \text{ for } x \geq x_d.$$

Taking $C = v_\infty(x_d)e^{x_d}$, $L = x_d$ and $\alpha' = 2D\xi$, we readily have (3.4.33). This finishes the proof for the case $k_0 > 0$.

Next we consider the case $k_0 = 0$. If

$$G(x) := g\left(e^{-k\int_0^x v_\infty(s)e^{\frac{\alpha s}{2D}}\,ds}\right) \geq d$$

for all $x \geq 0$, then from (3.4.32) we deduce

$$v''_\infty(x) \leq \frac{\alpha^2}{4D^2} v_\infty,$$

and we can use the comparison theorem to deduce

$$v_\infty(x) \geq \tau_\infty e^{-\frac{\alpha}{2D}x} - \epsilon e^{\frac{\alpha}{2D}x}$$

for all $x > 0$ and $\epsilon > 0$. Letting $\epsilon \to 0$ we deduce

$$v_\infty(x) \geq \tau_\infty e^{-\frac{\alpha}{2D}x}$$

for all $x \geq 0$. But then we deduce, as $x \to \infty$,

$$\int_0^x v_\infty(s) e^{\frac{\alpha}{2D} s} ds \to \infty$$

and

$$g\left(e^{-k \int_0^x v_\infty(s) e^{\frac{\alpha s}{2D}} ds}\right) \to 0 < d,$$

a contradiction. Therefore there exists $x_d > 0$ such that

$$G(x) \leq G(x_d) < d$$

for $x \geq x_d$. We may now repeat the argument used for the case $k_0 > 0$ to deduce that

$$v_\infty(x) \leq v_\infty(x_d) e^{-\xi'(x - x_d)} \text{ for } x \geq x_d,$$

with $\xi' = \sqrt{\frac{\alpha^2}{4D^2} + \frac{d - G(x_d)}{D}}$.

The proof of the lemma is now complete. □

Remark 3.4.17 *Clearly Lemma 3.4.16 implies $u_\infty(x) \to 0$ as $x \to \infty$. Moreover, from* (3.4.25), *we obtain*

$$0 \geq \frac{w'_\infty(x)}{w_\infty(x)} \geq -\frac{\alpha + \sqrt{\alpha^2 + 8dD}}{2D} \text{ for all } x \geq 0. \tag{3.4.34}$$

This fact will be useful later.

Now we are in a position to state and prove the main result of this subsection.

Theorem 3.4.18 *Let* (3.4.20) *hold, and $0 < d < d^*_\infty$.*

(a) Suppose either $k_0 > 0$, or $k_0 = 0$ and $0 < d < g(1) - \alpha^2/4D$. Let $u_h(x)$ be the unique positive solution to (3.4.2)*. Then, as $h \to \infty$,*

$$u_h(x) \to u_\infty(x) \text{ in } C^1_{loc}([0, \infty)), \tag{3.4.35}$$

where $u_\infty(x)$ is the unique positive solution to

$$\begin{cases} -Du''_\infty + \alpha u'_\infty = \left[g(e^{-k_0 x - k\int_0^x u_\infty(s)ds}) - d\right] u_\infty, & x \in (0, \infty), \\ u'(0) = \dfrac{\alpha}{D} u(0). \end{cases} \tag{3.4.36}$$

Moreover $u_\infty(x)$ has the following properties:

(i) There exists $x_\infty \in (0,\infty)$ such that

$$u'_\infty(x_\infty) = 0,\ u'_\infty(x) > 0 \textit{ for } x \in [0, x_\infty),\ u'_\infty(x) < 0 \textit{ for } x \in (x_\infty, \infty);$$

(ii) There exists positive constants C, L, and α_0 such that

$$u_\infty(x) \le Ce^{-\alpha_0 x} \textit{ for any } x \in [L, \infty). \tag{3.4.37}$$

(b) Suppose $k_0 = 0$ and $d \ge g(1) - \alpha^2/4D$. Then $u_h \to 0$ in $C^1_{loc}([0,\infty))$.

Proof. Part (b) follows from the discussion right before Lemma 3.4.16. For part (a), let v_h and v_∞ be defined as above. We prove that $v_h(x) \to v_\infty(x)$ in $C^1_{loc}([0,\infty))$. If this is proved, we set

$$u_\infty(x) = v_\infty(x)e^{\alpha x/2D}.$$

Then (3.4.35) and (3.4.36) follow readily. (3.4.37) is also obvious from Lemma 3.4.16 by taking $\alpha_0 = \frac{\alpha' - \alpha}{2D}$.

We already know that for any sequence of h converging to ∞, there is a subsequence $\{h_n\}$ such that $v_{h_n} \to v_\infty$ in $C^1_{loc}([0,\infty))$, and v_∞ is a positive solution to (3.4.32). To prove that $\lim_{h\to\infty} v_h = v_\infty$, it suffices to show that the limit v_∞ is unique. We argue indirectly. Assume there is another sequence of h other that h_n, say $\bar{h}_n$, such that

$$\bar{v}_n := v_{\bar{h}_n} \to \bar{v}_\infty \not\equiv v_\infty \text{ in } C^1_{\text{loc}}([0,\infty)).$$

Clearly $\bar{v}_\infty$ satisfies

$$\begin{cases} -D\bar{v}''_\infty + \dfrac{\alpha^2}{4D}\bar{v}_\infty = \left[g\left(e^{-k_0 x - k\int_0^x \bar{v}_\infty(s)\exp\left(\frac{\alpha s}{2D}\right)ds}\right) - d\right]\bar{v}_\infty, \quad x \in (0,\infty), \\ \bar{v}_\infty(0) = \bar{\tau}_\infty,\ \ \bar{v}'_\infty(0) = \dfrac{\alpha\bar{\tau}_\infty}{2D}. \end{cases} \tag{3.4.38}$$

We prove that $v_\infty(x) = \bar{v}_\infty(x)$ for all $x \in [0,\infty)$. This contradiction would imply the uniqueness of v_∞.

If $\tau_\infty = \bar{\tau}_\infty$, by the uniqueness theorem of the initial value problem of ODEs, we readily have $v_\infty(x) = \bar{v}_\infty(x)$ for all $x \in [0,\infty)$.

Suppose $\tau_\infty \ne \bar{\tau}_\infty$. For definiteness, we assume $\tau_\infty < \bar{\tau}_\infty$. We now have two cases:

1. *there exists* $c \in (0, \infty)$ *such that* $v_\infty(x) < \bar{v}_\infty(x)$ *for* $x \in [0, c)$ *and* $v_\infty(c) = \bar{v}_\infty(c)$*;*

2. $v_\infty(x) < \bar{v}_\infty(x)$ *for all* $x \in [0, \infty)$.

If case (1) happens, we multiply (3.4.32) by $\bar{v}_\infty$ and integrate it over $[0, c]$ to obtain

$$-D[v'_\infty \bar{v}_\infty]\Big|_0^c + D\int_0^c v'_\infty \bar{v}'_\infty + \left(\frac{\alpha^2}{4D} + d\right)\int_0^c v_\infty \bar{v}_\infty = \int_0^c g_\infty v_\infty \bar{v}_\infty,$$

where $g_\infty = g\left(e^{-k_0 x - k\int_0^x v_\infty(s)\exp\left(\frac{\alpha s}{2D}\right)ds}\right)$.

Similarly, we multiply (3.4.38) by v_∞ and integrate it over $[0, c]$ to obtain

$$-D[v_\infty \bar{v}'_\infty]\Big|_0^c + D\int_0^c v'_\infty \bar{v}'_\infty + \left(\frac{\alpha^2}{4D} + d\right)\int_0^c v_\infty \bar{v}_\infty = \int_0^c \bar{g}_\infty v_\infty \bar{v}_\infty,$$

with $\bar{g}_\infty = g\left(e^{-k_0 x - k\int_0^x \bar{v}_\infty(s)\exp\left(\frac{\alpha s}{2D}\right)ds}\right)$.

From these two identities, we deduce

$$D[v_\infty \bar{v}'_\infty - v'_\infty \bar{v}_\infty]\Big|_0^c = \int_0^c [g_\infty - \bar{g}_\infty] v_\infty \bar{v}_\infty.$$

As

$$D[v_\infty \bar{v}'_\infty - v'_\infty \bar{v}_\infty]\Big|_0^c = D[\bar{v}'_\infty(c) - v'_\infty(c)]v_\infty(c) \le 0,$$

and

$$\int_0^c [g_\infty - \bar{g}_\infty] v_\infty \bar{v}_\infty > 0 \ \text{ since } g_\infty > \bar{g}_\infty \text{ on } (0, c),$$

we reach a contradiction. Hence case (1) can not happen.

We prove case (2) can not happen either. Suppose on the contrary case (2) happens. Then for any $b \in (0, \infty)$, we have similar to case (1) that

$$D[v_\infty(b)\bar{v}'_\infty(b) - v'_\infty(b)\bar{v}_\infty(b)] = \int_0^b [g_\infty - \bar{g}_\infty] v_\infty \bar{v}_\infty.$$

As $b \to \infty$, the right side of this identity is bounded from below by a positive constant (it is positive and increasing in b), while by Lemma 3.16 and Remark 3.17,

$$D[v_\infty(b)\bar{v}'_\infty(b) - v'_\infty(b)\bar{v}_\infty(b)] \le D\left(\left|\frac{v'_\infty(b)}{v_\infty(b)}\right| + \left|\frac{\bar{v}'_\infty(b)}{\bar{v}_\infty(b)}\right|\right) v_\infty(b)\bar{v}_\infty(b) \to 0.$$

We again reach a contradiction. This means that case (2) can not happen either.

So in all possible cases we arrive at a contradiction. This proves the uniqueness of v_∞ and hence (3.4.35).

It remains to prove property (i), which will follow if we can show that

$$z(x) := u'_\infty(x)/u_\infty(x)$$

has a unique zero in $[0, \infty)$.

It is readily checked that z satisfies

$$Dz' = -Dz^2 + \alpha z - G(x) + d, \ 0 < x < \infty.$$

where $G(x) = g\left(e^{-k_0 x - k\int_0^x u_\infty(s)ds}\right)$ is a strictly decreasing function of x.

Since $u_\infty(0) = \tau_\infty > 0$, $u'_\infty(0) = \alpha\tau_\infty/D > 0$ and $\lim_{x\to\infty} u_\infty(x) = 0$, we can find $x_\infty \in (0, \infty)$ such that $u'(x_\infty) = 0$. Obviously x_∞ is a zero for z in $(0, \infty)$. We prove that there is no other zeros of z.

We may assume that $z(x)$ has no zero in $[0, x_\infty)$. It follows that $z'(x_\infty) \leq 0$ and $G(x_\infty) - d \geq 0$. Assume $x_* > x_\infty$ is also a zero of z. Without loss of generality we may assume that there is no other zeros in (x_∞, x_*). (Note that $z'(x_\infty) = 0$ implies $Dz''(x_\infty) = -G'(x_\infty) > 0$.) Since $G(x)$ is strictly decreasing, we have

$$z'(x_*) - z'(x_\infty) = D^{-1}(G(x_\infty) - G(x_*)) > 0.$$

Hence we have either $z'(x_\infty) < 0$ or $z'(x_\infty) = 0$ and $z'(x_*) > 0$. This implies that $z(x)$ is negative in (x_∞, x_*). By the mean value theorem we can find $\bar{x} \in (x_\infty, x_*)$ such that $z'(\bar{x}) = 0$ and $z(\bar{x}) < 0$. Consequently,

$$G(\bar{x}) - d = -Dz'(\bar{x}) - Dz^2(\bar{x}) + \alpha z(\bar{x}) < 0.$$

Since $G(x)$ is strictly decreasing, we have

$$G(x) - d < 0 \text{ for all } x \in [\bar{x}, \infty).$$

By (3.4.34), $w'_\infty(x) \leq 0$ for $x \geq 0$. That is

$$\left[u_\infty(x)e^{-\alpha x/D}\right]' \leq 0 \ \forall x \geq 0.$$

It follows that

$$z(x) \le \alpha/D \; \forall x \in [0, \infty).$$

Now we have

$$z(x_*) = 0, \;\; z'(x) = -z^2(x) + \frac{\alpha}{D} z(x) + \frac{1}{D}(d - G(x)) > z(x)\,(\alpha/D - z(x)) \;\; \forall x \in [x_*, \infty).$$

This implies that $z(x) > 0$ for all $x \in (x_*, \infty)$. Hence

$$u'_\infty(x) > 0 \;\; \text{for all } x \in (x_*, \infty).$$

But this is in contradiction to (3.4.37).

The proof of the theorem is complete. □

Finally we use Theorem 3.4.18 and some results in [40] to find out exactly when (3.4.20) holds. By Theorem 3.4.18 we find that if $k_0 > 0$, if $d^*_\infty > 0$ and $0 < d < d^*_\infty$, then (3.4.36) has a positive solution that converges to 0 as $x \to \infty$. By Theorem 3.1 of [40], this implies that condition (B) there holds, which is equivalent to, by Theorem 6.2 in [40],

$$\frac{\alpha^2}{4D} - g(1) + d < 0 \text{ and } k_0 < K_c(d, \alpha), \tag{3.4.39}$$

where $K_c : B_0 \to (0, \infty)$ is a continuous function, with

$$B_0 := \left\{ (d, \alpha) \in R^2 : d > 0, \alpha > 0, \; \frac{\alpha^2}{4D} - g(1) + d < 0 \right\},$$

that has the following properties (Proposition 6.1 of [40]):

(a) $K_c(d, \alpha)$ is strictly decreasing in d and in α;

(b) $K_c(d, \alpha) \to 0$ as (d, α) approaches a point on the curve $\frac{\alpha^2}{4D} - g(1) + d = 0$;

(c) $K_c(d, 0^+) < \infty$ for $d \in (0, g(1))$, $K_c(0^+, \alpha) < \infty$ for $\alpha \in (0, 2\sqrt{Dg(1)}\,)$;

(d) $K_c(0^+, 0^+) = +\infty$.

Therefore $d^*_\infty > 0$ and $0 < d < d^*_\infty$ imply $(d, \alpha) \in B_0$ and $k_0 < K_c(d, \alpha)$. In particular, this implies that

$$\alpha < 2\sqrt{Dg(1)}.$$

Thus $d^*_\infty = 0$ whenever $\alpha \geq 2\sqrt{Dg(1)}$.

For fixed $k_0 > 0$ and small $\epsilon > 0$, we consider the function $K_c(\epsilon, \alpha)$, which is strictly decreasing and takes values between 0 (when $\alpha = 2\sqrt{D(g(1) - \epsilon)}$) and $K_c(\epsilon, 0^+)$. For $\epsilon > 0$ small enough, by property (d) we find that $K_c(\epsilon, 0^+) > k_0$, and hence we can find a unique $\alpha_\epsilon \in (0, 2\sqrt{D(g(1) - \epsilon)})$ such that $k_0 = K_c(\epsilon, \alpha_\epsilon)$. By the monotonicity of K_c we find that α_ϵ is decreasing in ϵ and

$$\alpha^* := \lim_{\epsilon \to 0} \alpha_\epsilon \in (0, 2\sqrt{Dg(1)}\,]. \tag{3.4.40}$$

We show that $d^*_\infty = 0$ whenever $\alpha \geq \alpha^*$. Arguing indirectly we assume that $d^*_\infty > 0$ for some $\alpha \geq \alpha^*$. Then for $d \in (0, d^*_\infty)$, by Theorem 3.4.18, (3.4.36) has a positive solution that converges to 0 as $x \to \infty$. Thus we can use [40] to conclude that (3.4.39) holds. In particular, $k_0 < K_c(d, \alpha)$. On the other hand, our earlier analysis shows that for sufficiently small $\epsilon \in (0, d)$, $\alpha \geq \alpha^* > \alpha_\epsilon$ and

$$k_0 = K_c(\epsilon, \alpha_\epsilon) > K_c(d, \alpha).$$

This contradiction proves that we must have $d^*_\infty = 0$ when $\alpha \geq \alpha^*$.

Next we prove that $d^*_\infty > 0$ if $\alpha < \alpha^*$. Note that by [33], we already know that $d^*_\infty > 0$ if $\alpha \leq 0$. Fix a $\alpha \in (0, \alpha^*)$. By the definition of α^*, $\alpha < \alpha_\epsilon$ for all small $\epsilon > 0$. We fix such an $\epsilon > 0$. Then

$$k_0 = K_c(\epsilon, \alpha_\epsilon) < K_c(\epsilon, \alpha).$$

By [40], for $d = \epsilon$, there exists a unique (smooth) positive function u_d defined on $[0, \infty)$ such that $w(x) := u_d(x)e^{-\alpha x/D}$ satisfies

$$\begin{cases} -Dw'' - \alpha w' = \left[g\left(e^{-k_0 x - k\int_0^x w(s)e^{\alpha s/D}ds}\right) - d\right] w, \ 0 < x < \infty, \\ w'(0) = 0. \end{cases} \tag{3.4.41}$$

Moreover, $w(x)e^{\alpha x/D} \to 0$ exponentially as $x \to \infty$.

For h large, consider the variational characterization of the eigenvalue $-d_h^*$:

$$-d_h^* = \inf_{\phi \in H^1(0,h)} \frac{\int_0^h e^{(\alpha/D)x}\left[D\phi_x^2 - g\left(e^{-k_0 x}\right)\phi^2\right]dx}{\int_0^h e^{(\alpha/D)x}\phi^2 dx}.$$

Using $w(x) := u_d(x)e^{-\alpha x/D}$ (restricted on $[0,h]$) as a test function we have

$$-d_h^* \leq \frac{\int_0^h e^{(\alpha/D)x}\left[Dw_x^2 - g\left(e^{-k_0 x}\right)w^2\right]dx}{\int_0^h e^{(\alpha/D)x}w^2 dx}.$$

Multiplying the first equation in (3.4.41) by $we^{(\alpha/D)x}$ and integrating it over $[0,h]$ we obtain

$$-Dw_x we^{(\alpha/D)x}\Big|_0^h + D\int_0^h w_x^2 e^{(\alpha/D)x}dx = \int_0^h \left[g\left(e^{-k_0 x - k\int_0^x w(s)e^{\alpha s/D}ds}\right) - d\right]w^2 e^{\alpha x/D}dx.$$

Therefore

$$\begin{aligned}
-d_h^* &\leq \frac{\int_0^h e^{(\alpha/D)x}\left[Dw_x^2 - g\left(e^{-k_0 x}\right)w^2\right]dx}{\int_0^h e^{(\alpha/D)x}w^2 dx} \\
&\leq \frac{\int_0^h e^{(\alpha/D)x}\left[g\left(e^{-k_0 x - k\int_0^x w(s)e^{\alpha s/D}ds}\right) - g\left(e^{-k_0 x}\right) - d\right]w^2 dx + Dw_x we^{\alpha x/D}\Big|_0^h}{\int_0^h e^{(\alpha/D)x}w^2 dx} \\
&\leq -d + \frac{Dw_x we^{\alpha x/D}\Big|_0^h}{\int_0^h e^{(\alpha/D)x}w^2 dx}.
\end{aligned}$$

By the exponential decay property of $w(x)e^{\alpha x/D}$, we have, as $h \to \infty$,

$$\left|Dw_x we^{\alpha x/D}\Big|_0^h\right| = \left|Dw_x(h)w(h)e^{\alpha h/D}\right| \leq D\sup_{s\in[0,\infty)}|w_x(s)|w(h)e^{\alpha h/D} \to 0,$$

since by Remark 3.4.17, we have $\sup_{s\in[0,\infty)}|w_x(s)| < \infty$. Hence

$$-d_\infty^* = \lim_{h\to\infty} -d_h^* \leq -d.$$

That is

$$d_\infty^* \geq d = \epsilon > 0.$$

We have thus proved the following result.

Theorem 3.4.19 *Let* $k_0 > 0$ *and suppose that* α^* *is defined by* (3.4.40). *Then* $d_\infty^* > 0$ *if and only if* $\alpha < \alpha^*$.

Chapter 4

The two species case

4.1 Steady-states of the two species model

In this chapter we study the steady states of the two species model, namely, for $i = 1, 2$,

$$\begin{cases} (u_i)_t = (D_i(x)(u_i)_x - \alpha_i(x)u_i)_x + (g_i(I(x,t)) - d_i)u_i, \ \ 0 < x < h, \ t > 0 \\ D(0)(u_i)_x(0,t) - \alpha_i(0)u_i(0,t) = D_i(0)(u_i)_x(h,t) - \alpha_i(h)u_i(h,t) = 0, \ \ t \geq 0, \\ u_i(x,0) = u_i^0(x) \gneqq 0, \ \ 0 \leq x \leq h, \end{cases} \tag{4.1.1}$$

where $g_i \in C^1([0,\infty))$ satisfies

$$g_i(0) = 0 \text{ and } g_i \text{ is strictly increasing,} \tag{4.1.2}$$

$$I(x,t) = e^{-k_0 x} \exp\left(-\int_0^x [k_1 u_1(s,t) + k_2 u_2(s,t)]\, ds \right), \tag{4.1.3}$$

$D_i(x), \alpha_i(x) \in C^1([0,h])$ are positive functions. k_0, k_1, k_2 are positive constants and $d_1, d_2 \in (0,\infty)$ are parameters.

For $i = 1, 2$, let

$$u_i(x,t) = e^{R_i(x)} v_i(x,t)$$

with

$$R_i(x) = \int_0^x \frac{\alpha_i(s)}{D_i(s)}\, ds.$$

Then (4.1.1) becomes

$$\begin{cases} (v_i)_t = e^{-R_i(x)}(D_i(x)e^{R_i(x)}(v_i)_x)_x + (g_i(I(x,t)) - d_i)v_i, & 0 < x < h,\ t > 0, \\ (v_i)_x(0,t) = (v_i)_x(h,t) = 0, & t \geq 0, \\ v_i(x,0) = e^{-R_i(x)}u_i^0(x) =: v_i^0(x) \gneq 0, & 0 \leq x \leq h, \end{cases} \tag{4.1.4}$$

where

$$I(x,t) = e^{-k_0 x} \exp\left(-\int_0^x [k_1 e^{R_1(s)} v_1(s,t) + k_2 e^{R_2(s)} v_2(s,t)]\, ds \right).$$

The corresponding steady state system is

$$\begin{cases} -e^{-R_1(x)}(D_1(x)e^{R_1(x)}v_1')' = (g_1(I(x)) - d_1)v_1, & 0 < x < h, \\ -e^{-R_2(x)}(D_2(x)e^{R_2(x)}v_2')' = (g_2(I(x)) - d_2)v_2, & 0 < x < h, \\ v_i'(0) = v_i'(h) = 0, & i = 1, 2, \end{cases} \tag{4.1.5}$$

where

$$I(x) = e^{-k_0 x} \exp\left(-\int_0^x [k_1 e^{R_1(s)} v_1(s) + k_2 e^{R_2(s)} v_2(s)]\, ds \right).$$

We want to find sufficient conditions for (4.1.5) to have at least one positive solution.

For a function $\Psi \in C([0,h])$, Let $\lambda_1^{(i)}(\Psi), i = 1, 2$, be the first eigenvalue of the following eigenvalue problem

$$-e^{-R_i(x)}(D_i(x)e^{R_i(x)}\varphi')' + \Psi(x)\varphi = \lambda\varphi, \quad 0 < x < h, \quad \varphi'(0) = \varphi'(h) = 0. \tag{4.1.6}$$

Clearly $\lambda_1^{(i)}(0) = 0, i = 1, 2$.

Define

$$d_i^* = -\lambda_1^{(i)}(-g_i(e^{-k_0 x})).$$

Nonnegative solutions of (4.1.5) can be classified into three classes: The unique trivial solution $(v_1, v_2) = (0, 0)$, which exists for all $d_1, d_2 \in R$. Two semitrivial solutions $(v_1, v_2) = (0, v_{d_2}^*)$ and $(v_1, v_2) = (v_{d_1}^*, 0)$, the former exists for $d_2 \in (0, d_2^*)$ and the latter exists for $d_1 \in (0, d_1^*)$, where $v_{d_1}^*, v_{d_2}^*$ denote the unique positive steady state for the v_1 and v_2 equations respectively, guaranteed by Theorem 3.2.1. The third class are positive solutions (v_1, v_2) with $v_1 > 0$ and $v_2 > 0$ in $[0, 1]$, which are the main interest here.

A necessary condition for the existence of a positive solution to (4.1.5) can be easily observed. Suppose that (v_1, v_2) is a positive solution of (4.1.5). Then from the equation for v_1 we obtain

$$-d_1 = \lambda_1^{(1)}(-g_1(e^{-k_0x-\int_0^x[k_1e^{R_1(s)}v_1(s)+k_2e^{R_2(s)}v_2(s)]ds})) \in (-d_1^*, 0).$$

That is $d_1 \in (0, d_1^*)$. Similarly from the equation for v_2 we deduce $d_2 \in (0, d_2^*)$. Thus for (4.1.5) to have a positive solution we necessarily have

$$0 < d_1 < d_1^*, \quad 0 < d_2 < d_2^*. \tag{4.1.7}$$

On the other hand, we have the following

Theorem 4.1.1 *Let $v_{d_i}^*, d_i \in (0, d_i^*), i = 1, 2$ be the unique positive solution of the problem*

$$\begin{cases} -e^{-R_i(x)}(D_i(x)e^{R_i(x)}v')' = [g_i(e^{-k_0x-k_i\int_0^x e^{R_i(s)}v(s)\,ds}) - d_i]v, \ \ 0 < x < h, \\ v'(0) = v'(h) = 0. \end{cases}$$

If

$$\begin{cases} 0 < d_1 < -\lambda_1^{(1)}[-g_1(e^{-k_0x-k_2\int_0^x e^{R_2(s)}v_{d_2}^*(s)\,ds})] =: \tilde{d}_1, \\ 0 < d_2 < -\lambda_1^{(2)}[-g_2(e^{-k_0x-k_1\int_0^x e^{R_1(s)}v_{d_1}^*(s)\,ds})] =: \tilde{d}_2, \end{cases} \tag{4.1.8}$$

then (4.1.5) *has at least one positive solution.*

To prove Theorem 4.1.1, let $E = C([0, h])$ and let P be the usual positive cone in E: $P = \{v \in E : v(x) \geq 0 \text{ in } [0, h]\}$. We define

$$A(v_1, v_2) = (A_1(v_1, v_2), A_2(v_1, v_2)),$$

where

$$A_1(v_1, v_2) = L_1 \circ G_1(d_1, v_1, v_2), \ \ A_2(v_1, v_2) = L_2 \circ G_2(d_2, v_1, v_2),$$

$$G_1(d_1, v_1, v_2)(x) = [d_1^* - d_1 + g_1(e^{-k_0x-\int_0^x(k_1e^{R_1}v_1+k_2e^{R_2}v_2)ds})]e^{R_1(x)}v_1(x),$$

$$G_2(d_2, v_1, v_2)(x) = [d_2^* - d_2 + g_2(e^{-k_0x-\int_0^x(k_1e^{R_1}v_1+k_2e^{R_2}v_2)ds})]e^{R_2(x)}v_2(x),$$

and for $i = 1, 2$, L_i is the solution operator for the problem

$$-(D_i(x)e^{R_i(x)}v')' + d_i^*e^{R_i(x)}v = f_i(x), \ \ v_i'(0) = v_i'(h) = 0,$$

namely $v = L_i(f_i)$. It is easily seen that (v_1, v_2) solves (4.1.5) if and only if $(v_1, v_2) = A(v_1, v_2)$.

By standard elliptic regularity theory we know that $A : E \times E \to E \times E$ is completely continuous. Moreover, by the strong maximum principle and the fact that

$$d_i^* - d_i + g_i(e^{-k_0 x - \int_0^x (k_1 e^{R_1} v_1 + k_2 e^{R_2} v_2) ds}) > 0 \text{ in } [0, h],$$

we find that $v_i \in \dot{P} := P \setminus \{0\}$ implies $A_i(v_1, v_2) \in P^\circ := \{v \in P : v(x) > 0 \text{ in } [0, 1]\}$. Thus we have

$$A(P \times P) \subset P \times P, \quad A(\dot{P} \times \dot{P}) \subset P^\circ \times P^\circ,$$
$$A(\dot{P} \times P) \subset P^\circ \times P, \quad A(P \times \dot{P}) \subset P \times P^\circ.$$

To use the topological degree (fixed point index) argument, we need some preparation.

Lemma 4.1.2 *Let $\bar{d}_1 \in (0, d_1^*), \bar{d}_2 \in (0, d_2^*)$ be two constants. Then there exist positive constants $C_1 = C_1(\bar{d}_1, \bar{d}_2), C_2 = C_2(\bar{d}_1, \bar{d}_2)$such that for any nonnegative solution (v_1, v_2) of* (4.1.5) *corresponding to (d_1, d_2) with $d_1 \geq \bar{d}_1, d_2 \geq \bar{d}_2$, one has the estimate*

$$\|v_i\|_\infty \leq C_i, \quad i = 1, 2. \tag{4.1.9}$$

Proof. Argue indirectly. Suppose there is a sequence of (d_1, d_2), say (d_{1n}, d_{2n}) and the corresponding nonnegative solutions v_{1n}, v_{2n} of (4.1.5) such that $\|v_{1n}\|_\infty + \|v_{2n}\|_\infty \to \infty$. Without loss of generality, we assume $\|v_{1n}\|_\infty \to \infty$. Set $\tilde{v}_{1n} = v_{1n}/\|v_{1n}\|_\infty$. Then $\tilde{v}_{1n}$ satisfies

$$-e^{R_1(x)}(D_1(x) e^{R_1(x)} \tilde{v}_{1n}')' = [g_1(I_n(x)) - d_{1n})] \tilde{v}_{1n}, \quad \tilde{v}_{1n}'(0) = \tilde{v}_{1n}'(h) = 0, \tag{4.1.10}$$

where

$$I_n(x) = e^{-k_0 x} \exp\left(-k_1 \|v_{1n}\|_\infty \int_0^x e^{R_1(s)} \tilde{v}_{1n}(s)\, ds - k_2 \int_0^x e^{R_2(s)} v_{2n}(s)\, ds\right).$$

The right hand side of (4.1.10) is clearly uniformly bounded. By the standard elliptic regularity, we may assume, by passing to a subsequence, $\tilde{v}_{1n} \to v_0$ in $C^1([0, h])$. We may also assume $g_1(I_n) \to g_0$ weakly in $L^2((0, h))$, $d_{1n} \to d_0 \geq \bar{d}_1$. Moreover v_0 satisfies (in the weak sense)

$$-e^{-R_1(x)}(D_1(x) e^{R_1(x)} v_0')' = [g_0(x) - d_0] v_0, \quad v_0'(0) = v_0'(h) = 0, \quad \|v_0\|_\infty = 1.$$

By the maximum principle $\min_{x\in[0,h]} v_0(x) > c_0 > 0$ for some constant c_0. Therefore $\lim_{n\to\infty} v_{1n}(x) = \infty$ uniformly, and $g_0(x) = \lim_{n\to\infty} g_1(I_n(x)) = 0$. It follows $-d_0 = -\lambda_1(0) = 0$, contradicting $d_0 \geq \bar{d}_1$. The contradiction proves the lemma. $\square$

We will use Theorem 1.3.2 to get sufficient conditions for the existence of positive steady states. The Frechet derivative of $A(v_1, v_2)$ with respect ot (v_1, v_2) at $(v^*_{d_1}, 0)$ and at $(0, v^*_{d_2})$, and the associated eigenvalue problems play a crucial role. We will denote these derivatives by $A'_{(v_1,v_2)}(v^*_{d_1}, 0)$ and $A'_{(v_1,v_2)}(0, v^*_{d_2})$, respectively, and the associated eigenvalue problems are

$$A'_{(v_1,v_2)}(v^*_{d_1}, 0)(m_1, m_2) = \xi(m_1, m_2), \tag{4.1.11}$$

and

$$A'_{(v_1,v_2)}(0, v^*_{d_2})(m_1, m_2) = \eta(m_1, m_2). \tag{4.1.12}$$

A direct calculation show that $\eta = 1$ is an eigenvalue of (4.1.12) if and only if the following problem has a solution $(m_1, m_2) \neq (0, 0)$:

$$\begin{cases} -e^{-R_1(x)}(D_1(x)e^{R_1(x)}m_1')' = [g_1(\sigma_2(x)) - d_1]m_1, & x \in (0, h), \\ -e^{-R_2(x)}(D_2(x)e^{R_2(x)}m_2')' = [g_2(\sigma_2(x)) - d_2]m_2 - \delta_2(x), & x \in (0, h), \\ m_1' = m_2' = 0, & x = 0, h, \end{cases} \tag{4.1.13}$$

where

$$\delta_2(x) = g_2'(\sigma_2(x))\sigma_2(x)v^*_{d_2}(x)\int_0^x [k_1 e^{R_1(s)}m_1(s) + k_2 e^{R_2(s)}m_2(s)]ds,$$
$$\sigma_2(x) = e^{-k_0 x - \int_0^x k_2 e^{R_2(s)} v^*_{d_2}(s)ds}.$$

Similarly, if we define

$$\sigma_1(x) = e^{-k_0 x - \int_0^x k_1 e^{R_1(s)} v^*_{d_1}(s)ds},$$

then $\xi = 1$ is an eigenvalue of (4.1.12) if and only if the following problem has a solution $(m_1, m_2) \neq (0, 0)$:

$$\begin{cases} -e^{-R_1(x)}(D_1(x)e^{R_1(x)}m_1')' = [g_1(\sigma_1(x)) - d_1]m_1 - \delta_1(x), & x \in (0, h), \\ -e^{-R_2(x)}(D_2(x)e^{R_2(x)}m_2')' = [g_2(\sigma_1(x)) - d_2]m_2, & x \in (0, h), \\ m_1' = m_2' = 0, & x = 0, h, \end{cases} \tag{4.1.14}$$

where

$$\delta_1(x) = g_1'(\sigma_1(x))\sigma_1(x)v_{d_1}^*(x)\int_0^x [k_1 e^{R_1(s)} m_1(s) + k_2 e^{R_2(s)} m_2(s)]ds$$

The following lemma holds the key for solving (4.1.13) and (4.1.14).

Lemma 4.1.3 *Let* $i \in \{1, 2\}$*. If* $\psi \in C^2([0, h])$ *satisfies*

$$\begin{cases} -e^{-R_i(x)}(D_i(x)e^{R_i(x)}\psi')' = [g_i(\sigma_i(x)) - d_i]\psi \\ \qquad\qquad - g_i'(\sigma_i(x))\sigma_i(x)v_{d_i}^*(x)\int_0^x k_i e^{R_i(s)}\psi(s)ds, \quad x \in (0, h), \\ \psi'(0) = \psi'(h) = 0, \end{cases} \tag{4.1.15}$$

then $\psi \equiv 0$.

Proof. We argue indirectly. Suppose $\psi \not\equiv 0$ solves (4.1.15). We first claim that $\psi(0) \neq 0$. Otherwise, define

$$\xi(x) = \int_0^x e^{R_i(s)}\psi(s)ds, \quad \eta(x) = D_i(x)e^{R_i(x)}\psi'(x)$$

Then $(\xi(x), \psi(x), \eta(x))$ is a solution of the ODE system

$$\begin{cases} \xi' = e^{R_i(x)}\psi, \\ \psi' = D_i^{-1}e^{-R_i(x)}\eta, \\ \eta' = -(x)[g_i(\sigma_i(x)) - d_i]e^{R_i(x)}\psi + g_i'(\sigma_i(x))\sigma_i(x)v_{d_i}^*(x)e^{R_i(x)}k_i\xi, \end{cases} \tag{4.1.16}$$

with the initial condition $(\xi(0), \psi(0), \eta(0)) = (0, 0, 0)$. Clearly $(\xi, \psi, \eta) \equiv (0, 0, 0)$ is the unique solution of this initial value ODE problem. Hence $\psi \equiv 0$, contradicting our assumption that $\psi \not\equiv 0$.

Without loss of generality we may assume that $\psi(0) > 0$. Next we claim that $\psi(x)$ changes sign in $(0, h)$. Otherwise $\psi(x) \geq, \not\equiv 0$ in $[0, h]$. Multiplying the first equation in (4.1.15) by $e^{R_i(x)}v_{d_i}^*$ and integrating it over $[0, h]$, we deduce

$$\begin{aligned} \int_0^h e^{R_i(x)}\psi'(x)(v_{d_i}^*)'(x)dx = & \int_0^h [g_i(\sigma_i(x)) - d_i]e^{R_i(x)}\psi(x)v_{d_i}^*(x)dx \\ & - \int_0^1 g_i'(\sigma_i(x))\sigma_i(x)e^{R_i(x)}[v_{d_i}^*(x)]^2\int_0^x k_i e^{R_i(s)}\psi(s)dsdx. \end{aligned} \tag{4.1.17}$$

$v_{d_i}^*$ satisfies

$$\begin{cases} -e^{-R_i(x)}(D_i(x)e^{R_i(x)}(v_{d_i}^*)')' = (g_i(\sigma_i(x)) - d_i)v_{d_i}^*, \quad 0 < x < h, \\ (v_{d_i}^*)'(0) = (v_{d_i}^*)'(h) = 0. \end{cases} \tag{4.1.18}$$

Multiplying the first equation of (4.1.18) by $e^{R_i(x)}\psi$ and integrating it over $[0, h]$, we deduce

$$\int_0^h e^{R_i(x)}\psi'(x)(v_{d_i}^*)'(x)dx = \int_0^h [g_i(\sigma_i(x)) - d_i]e^{R_i(x)}\psi(x)v_{d_i}^*(x)dx. \tag{4.1.19}$$

From (4.1.17) and (4.1.19) we readily have

$$\int_0^h g_i'(\sigma_i(x))\sigma_i(x)e^{R_i(x)}[v_{d_i}^*(x)]^2 \int_0^x k_i e^{R_i(s)}\psi(s)dsdx = 0.$$

But the integrand function in the last identity is clearly nonnegative and not identically zero in $[0, h]$. Hence the integral should be positive. This contradiction shows that $\psi(x)$ changes sign in $(0, h)$

Let $x_0 \in (0, h)$ be the first zero of $\psi(x)$, namely $\psi(x) > 0$ in $[0, x_0)$ and $\psi(x_0) = 0$. We now consider the eigenvalue problem

$$-e^{-R_i(x)}(D_i(x)e^{R_i(x)}\phi')' = [g_i(\sigma_i(x)) - d_i]\phi + \lambda\phi, \text{ in } (0, x_0), \ \ \phi'(0) = \phi(x_0) = 0. \tag{4.1.20}$$

We claim that the first eigenvalue λ_1 of this problem is positive. Indeed, let ϕ_1 be a positive eigenfunction corresponding to λ_1. Multiplying the first equation in (4.1.20) (with $\lambda = \lambda_1$, $\phi = \phi_1$)by $e^{R_i(x)}v_{d_i}^*$ and integrating it over $[0, x_0]$ we obtain

$$\begin{aligned} & -e^{R_i(x_0)}\phi_1'(x_0)v_{d_i}^*(x_0) + \int_0^{x_0} e^{R_i(x)}\phi_1'(x)(v_{d_i}^*)'(x)dx \\ = & \int_0^{x_0} [g_i(\sigma_i(x)) - d_i]e^{R_i(x)}\phi_1(x)v_{d_i}^*(x)dx + \lambda_1 \int_0^{x_0} e^{R_i(x)}\phi_1(x)v_{d_i}^*(x)dx. \end{aligned}$$

On the other hand, multiplying (4.1.18) by $e^{R_i(x)}\phi_1$ and integrating it over $[0, x_0]$, we obtain

$$\int_0^{x_0} e^{R_i(x)}\phi_1'(x)(v_{d_i}^*)'(x)dx = \int_0^{x_0} [g_i(\sigma_i(x)) - d_i]e^{R_i(x)}\phi_1(x)v_{d_i}^*(x)dx.$$

From the last two identities, we arrive at

$$\lambda_1 \int_0^{x_0} e^{R_i(x)}\phi_1(x)v_{d_i}^*(x)dx = -e^{R_i(x_0)}\phi_1'(x_0)v_{d_i}^*(x_0)$$

Hence $\lambda_1 > 0$ is clear, since $v^*_{d_i}(x_0) > 0$, $\int_0^{x_0} e^{R_i(x)}\phi_1(x)v^*_{d_i}(x)dx > 0$ and $\phi_1'(x_0) < 0$ (by the Hopf lemma).

To obtain the desired contradiction, we now multiply the first equation in (4.1.20) (with $\lambda = \lambda_1$, $\phi = \phi_1$) by $e^{R_i(x)}\psi$ and then integrate it over $[0, x_0]$. Consequently we deduce

$$\int_0^{x_0} e^{R_i(x)}\phi_1'(x)\psi'(x)dx = \int_0^{x_0} [g_i(\sigma_i(x)) - d_i]e^{R_i(x)}\phi_1(x)\psi(x)dx + \lambda_1 \int_0^{x_0} e^{R_i(x)}\phi_1(x)\psi(x)dx.$$

On the other hand, multiplying the first equation in (4.1.15) by $e^{R_i(x)}\phi_1$ and integrating it over $[0, x_0]$, we deduce

$$\begin{aligned}\int_0^{x_0} e^{R_i(x)}\phi_1'(x)\psi'(x)dx &= \int_0^{x_0} [g_i(\sigma_i(x)) - d_i]e^{R_i(x)}\phi_1(x)\psi(x)dx \\ &\quad - \int_0^{x_0} g_i'(\sigma_i(x))\sigma_i(x)v^*_{d_i}(x)e^{R_i(x)}\phi_1(x)\int_0^x k_i e^{R_i(s)}\psi(s)dsdx.\end{aligned}$$

From the last two equations we arrive at

$$\lambda_1 \int_0^{x_0} e^{R_i(x)}\phi_1(x)\psi(x)dx = -\int_0^{x_0} g_i'(\sigma_i(x))\sigma_i(x)v^*_{d_i}(x)e^{R_i(x)}\phi_1(x)\int_0^x k_i e^{R_i(s)}\psi(s)dsdx. \tag{4.1.21}$$

Since $\lambda_1 > 0$ and $\psi(x) > 0$ in $[0, x_0)$, the left side of the above identity is positive. However, the integrand function in the right side of (4.1.21) is nonnegative and hence the right side of (4.1.21) is not positive. This contradiction completes the proof. □

Lemma 4.1.4 *Problem* (4.1.13) *has a solution* $(m_1, m_2) \neq (0, 0)$ *if and only if* $m_1 \neq 0$ *and*

$$-e^{-R_1(x)}(D_1(x)e^{R_1(x)}m_1') = [g_1(\sigma_2(x)) - d_1]m_1, \quad m_1'(0) = m_1'(h) = 0.$$

Moreover, with m_1 *given,* m_2 *can be uniquely solved from the second equation in* (4.1.13) *togethor with the Neumann boundary conditions.*

Similarly, (4.1.14) *has a solution* $(m_1, m_2) \neq (0, 0)$ *if and only if* $m_1 \neq 0$ *and*

$$-e^{-R_2(x)}(D_2(x)e^{R_2(x)}m_2')' = [g_2(\sigma_1(x)) - d_2]m_2, \quad m_2'(0) = m_2'(h) = 0.$$

Moreover, with m_2 *given,* m_1 *can be uniquely solved from the first equation in* (4.1.14) *togethor with the Neumann boundary conditions.*

Proof. We only consider the statement for (4.1.13); the proof of that for (4.1.14) is analogous. Let (m_1, m_2) solves (4.1.13). If $m_1 = 0$, then by Lemma 4.1.3 we deduce $m_2 = 0$. Suppose now $m_1 \neq 0$. Then we can apply the Fredholm alternative for compact operators and Lemma 4.1.31 to conclude that the second equation in (4.1.13) together with the Neumann boundary conditions in uniquely solvable for any given m_1. □

We are now ready to prove our main theorem in this Chapter, namely Theorem 4.1.1.

Proof of Theorem 4.1.1. We use the fixed point calculation technique of Theorem 1.3.2. Define $\Omega = \Lambda \times U \times V$ with

$$U = \{v_1 \in P : \|v_1\|_\infty < C\}, \quad V = \{v_2 \in P : \|v_2\|_\infty < C\},$$

where $C > 0$ is large enough such that (4.1.9) holds and $\|v^*_{d_1}\|_\infty < C$.

Let B_1 be a small ball in E containing $v^*_{d_1}$. Since $v^*_{d_1} \in P^\circ$, we may assume that $B_1 \subset P^\circ$. Then by Theorem 1.3.2, we have

$$\text{index}_{P\times P}(A(d_2, \cdot), (v^*_{d_1}, 0)) = \begin{cases} 0 & \text{if } r(L) > 1, \\ \deg_P(I - A_1(\cdot, 0), B_1) & \text{if } r(L) < 1, \end{cases}$$

where $L = (A_2)'_{v_2}(v^*_{d_1}, 0)$ and $r(L)$ denotes the spectral radius of the linear operator L.

It is easily checked that $r(L) > 1$ if $d_2 < -\lambda_1^{(1)}(-g_2(\sigma_1(x))) = \tilde{d}_2$, and $r(L) < 1$ if $d_2 > -\lambda_1^{(1)}(-g_2(\sigma_1(x))) = \tilde{d}_2$. Thus

$$\text{index}_{P\times P}(A(d_2, \cdot), (v^*_{d_1}, 0)) = \begin{cases} 0 & \text{if } d_2 < \tilde{d}_2, \\ \deg_P(I - A_1(\cdot, 0), B_1) & \text{if } d_2 > \tilde{d}_2. \end{cases}$$

We show next that

$$\deg_P(I - A_1(\cdot, 0), B_1) = 1.$$

Since $(v^*_{d_1}, 0)$ is the only fixed point of $A_1(\cdot, 0)$ in $B_1 \cap P^\circ$, we clearly have

$$\deg_P(I - A_1(\cdot, 0), B_1) = \text{index}_P(A_1(\cdot, 0), v^*_{d_1}).$$

We will use a homotopy argument to $A_1(\lambda, v_1, 0) = L_1 \circ G_1(\lambda, v_1, 0)$ with $\lambda \in [d_1^*, d_1^* + 1]$. By Theorem 3.2.1 we know that for $\lambda \in [d_1, d_1^*)$ the equation $A_1(\lambda, v, 0) = v$ has exactly two solutions in P: The trivial solution $v = 0$ and the unique positive solution $v = v_\lambda > 0$. For $\lambda \in [d_1^*, d_1^* + 1]$, there is one solution in P: $v = 0$. Moreover, one easily sees that 0 is a linearized stable fixed point of $A_1(\lambda, \cdot, 0)$ when $\lambda > d_1^*$, and it is a linearized unstable fixed point when $\lambda < d_1^*$. It follows that

$$\text{index}_P(A_1(\lambda, \cdot, 0), 0) = \begin{cases} 0 & \text{for } \lambda < d_1^*, \\ 1 & \text{for } \lambda > d_1^*. \end{cases}$$

Choose $C_0 > 0$ large enough such that $\|v_\lambda\|_\infty < C_0$ for $\lambda \in [d_1, d_1^*)$, and denote $P_{C_0} := \{v \in P : \|v\|_\infty < C_0\}$. Then by the homotopy invariance property of the topological degree, we find that $\deg_P(I - A_1(\lambda, \cdot, 0), P_{C_0})$ is well defined and its value does not depend on λ for $\lambda \in [d_1, d_1^* + 1]$ By the additivity of the topological degree we have

$$\begin{aligned} \deg_P(I - A_1(\lambda, \cdot, 0), P_{C_0}) &= \text{index}_P(A_1(\lambda, \cdot, 0), 0) + \text{index}_P(A_1(\lambda, \cdot, 0), v_\lambda) \\ &= \text{index}_P(A_1(\lambda, \cdot, 0), v_\lambda) \end{aligned}$$

for $\lambda \in [d_1, d_1^*)$, and

$$\deg_P(I - A_1(\lambda, \cdot, 0), P_{C_0}) = \text{index}_P(A_1(\lambda, \cdot, 0), 0) = 1$$

for $\lambda \in (d_1^*, d_1^* + 1]$. It follows that

$$\text{index}_P(A_1(\lambda, \cdot, 0), v_\lambda) = 1$$

for $\lambda \in [d_1, d_1^*)$. Taking $\lambda = d_1$ we obtain

$$\deg_P(I - A_1(\cdot, 0), B_1) = \text{index}_P(A_1(\lambda, \cdot, 0), v_{d_1}^*) = 1.$$

The proof of Theorem 4.1.1 is complete. □

Before ending this chapter, we give some discussion on condition (4.1.8). The condition is rather implicit. We have showed that $d_1 \in (0, d_1^*)$ and $d_2 \in (0, d_2^*)$ is necessary for (4.1.5) to have a positive solution. Condition (4.1.8) is more restrictive than this necessary condition. Indeed we have

Proposition 4.1.5 *For fixed* $d_1 \in (0, d_1^*)$*, if* $\delta > 0$ *is small enough, then* (4.1.5) *has no positive solution if* $d_2 \notin (\delta, d_2^* - \delta)$*.*

Proof. Otherwise, we can find $d_2^n \downarrow 0$ or $d_2^n \uparrow d_2^*$ and a positive solution (v_1^n, v_2^n) with $d_2 = d_2^n$. In the first case we define $\hat{v}_2^n = v_2^n/\|v_2^n\|_\infty$ and

$$f_n = \exp\left(-k_0 x - k_1 \int_0^x e^{R_1(s)} v_1^n(s) ds - k_2 \int_0^x e^{R_2(s)} v_2^n(s) ds\right),$$

and as before find that by passing to a subsequence $\hat{u}_i^n \to \hat{v}_i$ in $C^1([0,1])$ for $i = 1, 2$, $f_n \to f$ and $g_2(f_n) \to g_2(f)$ weakly in $L^2([0,1])$, and $\hat{v}_2$ is a positive solution to

$$-e^{-R_2(x)}(D_2(x)e^{R_2(x)}\hat{v}_2')' = g_2(f)\hat{v}_2, \quad \hat{v}'(0) = \hat{v}'(h) = 0. \tag{4.1.22}$$

Multiplying the first equation in (4.1.22) by $e^{R_2(x)}$ and integrating the resultant equation over $[0, h]$ we get

$$\int_0^h g_2(f)(x) e^{R_2(x)} \hat{v}_2(x)\, dx = 0.$$

Since $\hat{u} > 0$ in $[0, h]$ and $g(f) \geq 0$ in $[0, h]$, the above identity implies $g_2(f) = 0$ a.e. in $[0, h]$. It follows that $f(x) = 0$ a.e. in $[0, h]$.

Now define $\hat{v}_1^n = v_1^n/\|v_1^n\|_\infty$ and we obtain from the equation for v_1^n that

$$-e^{-R_1(x)}(D_1(x)e^{R_1(x)}(\hat{v}_1^n)')' = [g_1(f_n) - d_1]\hat{v}_1^n, \quad (\hat{v}_1^n)'(0) = (\hat{v}_1^n)'(h) = 0.$$

As before by elliptic regularity and passing to a subsequence, $\hat{v}_1^n \to \hat{v}_1$ in $C^1([0, h])$ and $g_1(f_n) \to g_1(f) = 0$ weakly in $L^2([0, h])$, and $\hat{v}_1$ is a positive solution to

$$-e^{-R_1(x)}(D_1(x)e^{R_1(x)}\hat{v}_1')' = -d_1\hat{v}_1, \quad \hat{v}_1'(0) = \hat{v}_1'(h) = 0.$$

This implies $d_1 = 0$, a contradiction to our assumption $d_1 \in (0, d_1^*)$.

Next we consider the case $d_2^n \to d_2^*$. We define $\hat{v}_1^n$, $\hat{v}_1^n$ and f_n as above. By the same argument we know that by passing to a subsequence, $\hat{v}_1^n \to \hat{v}_1$ and $\hat{v}_2^n \to \hat{v}_2$ in $C^1([0, h])$, $f_n \to f$ and $g_i(f_n) \to g_i(f)$ in $L^2([0, h])$, and $\hat{v}_2$, $\hat{v}_1$ are positive solutions to

$$-e^{-R_2(x)}(D_2(x)e^{R_2(x)}\hat{v}_2') = [g_2(f) - d_2^*]\hat{v}_2, \quad \hat{v}_2'(0) = \hat{v}_2'(h) = 0, \tag{4.1.23}$$

and

$$-e^{-R_1(x)}(D_1(x)e^{R_1(x)}\hat{v}_1')' = [g_1(f) - d_1]\hat{v}_1, \quad \hat{v}_1'(0) = \hat{v}_1'(h) = 0, \tag{4.1.24}$$

respectively.

Let us now look at the sequence $\{\|v_1^n\|_\infty\}$. If this sequence is not bounded, then by passing to a subsequence we have $\{\|v_1^n\|_\infty\} \to \infty$ and hence $v_1^n = \|v_1^n\|_\infty \hat{v}_1^n \to \infty$ uniformly in $[0,h]$. This implies that $f \equiv 0$ and (4.1.24) becomes

$$-e^{-R_1(x)}(D_1(x)e^{R_1(x)}\hat{v}_1')' = -d_1\hat{v}_1, \quad \hat{v}_1'(0) = \hat{v}_1'(h) = 0,$$

which implies $d_1 = 0$, a contradiction. Thus $\{\|v_1^n\|_\infty\}$ is bounded. For the same reason, $\{\|v_2^n\|_\infty\}$ is bounded. So we may assume that

$$\|v_1^n\|_\infty \to \sigma_1 \geq 0, \quad \|v_2^n\|_\infty \to \sigma_2 \geq 0.$$

It then follows that

$$f_n(x) \to e^{-k_0x - \int_0^x (k_1\sigma_1 e^{R_1}\hat{v}_1 + k_2\sigma_2 e^{R_2}\hat{v}_2)ds} \text{ uniformly in } [0,h].$$

Thus

$$f(x) = e^{-k_0x - \int_0^x (k_1\sigma_1 e^{R_1}\hat{v}_1 + k_2\sigma_2 e^{R_2}\hat{v}_2)ds} \leq e^{-k_0x}$$

with equality holding if and only if $\sigma_1 = \sigma_2 = 0$. It follows that $g_2(f(x)) \leq g_2(e^{-k_0x})$ with equality holding for all $x \in [0,h]$ if and only if $\sigma_1 = \sigma_2 = 0$. Form this and (4.1.23) we deduce

$$d_2^* = -\lambda_1^{(2)}(g_2(f)) \leq -\lambda_1^{(2)}(-g_2(e^{-k_0x})),$$

with equality holding if and only if $\sigma_1 = \sigma_2 = 0$. Thus in view of the definition of d_2^*, we necessarily have $\sigma_1 = \sigma_2 = 0$ and thus $f(x) = e^{-k_0x}$. We now use (4.1.24) and find

$$d_1 = -\lambda_1^{(1)}(-g_1(f)) = -\lambda_1^{(1)}(-g_1(e^{-k_0x})).$$

That is, $d_1 = d_1^*$, a contradiction to our assumption on d_1. $\square$

On the other hand, we can find (d_1, d_2) such that (4.1.8) is satisfied, and hence (4.1.5) has a positive solution. We will leave the detailed discussion to the next chapter, where we treat the general multiple species model.

Chapter 5

The more than two species case

5.1 The multi-species case

In this Chapter, we study an n-species reaction-diffusion-advection system proposed by Huisman et al. in [37] modeling the growth of competitive phytoplankton species in a vertical water column

$$(u_i)_t = D_i(u_i)_{xx} - \alpha_i(u_i)_x + [g_i(I(x,t,u)) - d_i]\, u_i, \ \ 0 < x < h, \ \ t > 0, \ \ i = 1, 2, \cdots, n, \tag{5.1.1}$$

with no-flux boundary conditions

$$D_i(u_i)_x(0,t) - \alpha_i u_i(0,t) = D_i(u_i)_x(L,t) - \alpha_i u_i(h,t) = 0, \ \ t \geq 0, \ i = 1, 2, \cdots, n, \tag{5.1.2}$$

and initial conditions

$$u_i(x,0) = u_i^0(x) \geq\not\equiv 0, \ \ 0 \leq x \leq h, \ i = 1, 2, \cdots, n. \tag{5.1.3}$$

Here u_i is the population density of the phytoplankton species i, $D_i > 0$ is the diffusion coefficient caused by the water turbulence, $\alpha_i \in R^1$ is the sinking ($\alpha_i > 0$) or buoyant ($\alpha_i < 0$) velocity, $d_i > 0$ is the loss rate, $L > 0$ is the depth of the water column. The light distribution function $I(x,t,u)$ takes the form

$$I(x,t,u) = I_0 e^{-k_0 x} \exp\left(- \sum_{j=1}^{n} k_j \int_0^x u_j(s,t)ds \right),$$

where $I_0 > 0$ is the incident light intensity, $k_0 > 0$ is the background turbidity, k_i is the absorption coefficient of the phytoplankton species i. The term $g_i(I)$ represents the specific growth rate of the phytoplankton species i, which is a function of the light intensity $I(x,t,u)$. In this model ample nutrient supply is assumed so that the phytoplankton growth is only limited by the light intensity. $g_i (i = 1, 2, \cdots, n)$ are smooth functions satisfying

$$\begin{cases} g_i(0) = 0, \quad g_i'(I) > 0 \ \text{ for } I > 0, \\ \text{there exist constants } \sigma_i > 0 \text{ such that } g_i(I) \leq \sigma_i I \text{ for small } I > 0. \end{cases} \tag{5.1.4}$$

Under assumption (5.1.4), we have

$$\int_0^\infty g_i\left(e^{-\sigma x}\right) dx < \infty \ \text{ for any constant } \sigma > 0.$$

Before continuing our discussion, we do some rescaling to simplify our system (5.1.1)-(5.1.3). Replacing x with hx, $u_i(\cdot)$ with $k_i^{-1}u_i(L\cdot)$, D_i with L^2D_i, α_i with $L\alpha_i$, k_i with k_iL and $g_i(I_0\cdot)$ with $g_i(\cdot)$, we may assume that u_i satisfies the modified system

$$(u_i)_t = D_i(u_i)_{xx} - \alpha_i(u_i)_x + [g_i(I(x,t,u)) - d_i]\, u_i, \ \ 0 < x < 1, \ \ t > 0, \ \ i = 1, 2, \cdots, n, \tag{5.1.5}$$

with no-flux boundary conditions

$$D_i(u_i)_x(0,t) - \alpha_i u_i(0,t) = D_i(u_i)_x(1,t) - \alpha_i u_i(1,t) = 0, \ \ t \geq 0, \ i = 1, 2, \cdots, n, \tag{5.1.6}$$

and initial conditions

$$u_i(x,0) = u_i^0(x) \geq\not\equiv 0, \ \ 0 \leq x \leq 1, \ i = 1, 2, \cdots, n, \tag{5.1.7}$$

with

$$I(x,t) = I(x,t,u) = e^{-k_0 x} \exp\left(- \int_0^x [\sum_{j=1}^n u_j(s,t)] ds \right).$$

Through the change of variables

$$u_i(x,t) = v_i(x,t) e^{(\alpha_i/D_i)x},$$

we arrive at an equivalent system

$$(v_i)_t = D_i(v_i)_{xx} + \alpha_i(v_i)_x + [g_i(I(x,t)) - d_i]\, v_i, \ \ 0 < x < 1, \ \ t > 0, \ \ i = 1, 2, \cdots, n, \tag{5.1.8}$$

with homogeneous Neumann boundary conditions

$$(v_i)_x(0,t) = (v_i)_x(1,t) = 0, \ \ t \geq 0, \ i = 1, 2, \cdots, n, \tag{5.1.9}$$

and initial conditions

$$v_i(x,0) = v_i^0(x) \geq\not\equiv 0, \ \ 0 \leq x \leq 1, \ i = 1, 2, \cdots, n, \tag{5.1.10}$$

where

$$I(x,t) = e^{-k_0 x} \exp\left(-\int_0^x [\sum_{j=1}^n e^{(\alpha_j/D_j)s} v_j(s,t)] ds \right).$$

In the rest of this chapter, as mentioned in Chapter 2, among other things we use the fixed index calculation technique developed by Dancer and Du in [14] to obtain a group of sufficient conditions for the existence of positive steady states for (5.1.1)-(5.1.3), or equivalently (5.1.5)-(5.1.7), then we use the fine properties of certain eigenvalues to find more explicit sufficient condition for the existence of positive steady state solutions. For the purpose of clarity, we will only treat the $n = 3$ case in detail. The extension to $n \geq 3$ is given in Section 5.4.

The main structure of the chapter is as following. Section 5.2 is devoted to the study of the existence and nonexistence of positive steady state of our system. In Section 5.3 we use fixed point index calculation to prove a group of explicit conditions for the existence of positive solutions, on which Section 5.2 is based. In the final section, Section 5.4, we extend the results for the three species case to the general $n(\geq 3)$ species case.

5.2 Existence and non-existence of positive steady state solutions

In this section, we consider the positive steady-state solutions of (5.1.8)-(5.1.10) ($n = 3$). That is we study the positive solutions of the system

$$\begin{cases} -D_1 v_1'' - \alpha_1 v_1' = [g_1(I(x)) - d_1]v_1, & 0 < x < 1, \\ -D_2 v_2'' - \alpha_2 v_2' = [g_2(I(x)) - d_2]v_2, & 0 < x < 1, \\ -D_3 v_3'' - \alpha_3 v_3' = [g_3(I(x)) - d_3]v_3, & 0 < x < 1, \\ \ v_i'(0) = v_i'(1) = 0, \ \ i = 1, 2, 3, \end{cases} \tag{5.2.1}$$

where

$$I(x) = e^{-k_0 x} \exp\left(-\int_0^x \left[v_1(y)e^{(\alpha_1/D_1)y} + v_2(y)e^{(\alpha_2/D_2)y} + v_3(y)e^{(\alpha_3/D_3)y}\right] dy\right). \tag{5.2.2}$$

We first find a necessary condition for (5.2.1) to have a positive solution. For that, we denote by $\lambda_1^{(i)}(\Psi), i = 1, 2, 3$ the principal eigenvalue of the eigenvalue problem

$$\begin{cases} -D_i\phi'' - \alpha_i\phi' + \Psi(x)\phi = \lambda\phi, \ \ 0 < x < 1, \\ \ \phi'(0) = \phi'(1) = 0. \end{cases}$$

It is well known (see [6]) that $\lambda_1^{(i)}(\Psi)$ is a continuous function of Ψ in $C([0, 1])$ and $\lambda_1^{(i)}(\Psi_1) \geq \lambda_1^{(i)}(\Psi_2)$ for $\Psi_1 \geq \Psi_2$ and equality holds only if $\Psi_1 \equiv \Psi_2$.

Define

$$d_i^* = -\lambda_1^{(i)}\left(-g_i\left(e^{-k_0 x}\right)\right), \ \ i = 1, 2, 3. \tag{5.2.3}$$

Then if (5.2.1) has a positive solution (v_1, v_2, v_3), we have

$$-d_i = -\lambda_1^{(i)}\left(g_i(I(x))\right) \in (-d_i^*, 0).$$

We thus obtain a necessary condition for (5.2.1) to have a positive solution:

$$d_i \in (0, d_i^*), \ \ i = 1, 2, 3. \tag{5.2.4}$$

Moreover, for a generic triple $d_i \in (0, d_i^*), i = 1, 2, 3$, (5.2.1) generally does not have a positive solution if the diffusion coefficients $D_i, i = 1, 2, 3$ are large, as can be seen from the following theorem.

Theorem 5.2.1 *(i) If there is an $i \in \{1, 2, 3\}$ such that $d_i > \int_0^1 g_i(e^{-k_0 x})dx$, then there exists a constant $D > 0$, such that if $\min\{D_1, D_2, D_3\} \geq D$ then* (5.2.1) *has only the trivial solution.*

(ii) If $d_i \in (0, \int_0^1 g_i(e^{-k_0 x})dx]$ for all $i = 1, 2, 3$, then there exists a positive constant D such that if $\min\{D_1, D_2, D_3\} \geq D$, (5.2.1) *has no positive solution except possibly when the following exceptional situation occurs:*
there exists a constant $c \geq 0$ such that

$$c_1 = c_2 = c_3 = c,$$

where c_i is uniquely determined by

$$d_i = \int_0^1 g_i(e^{-(k_0+c_i)x})dx. \tag{5.2.5}$$

Proof. Denote by $(v_{1D}, v_{2D}, v_{3D}) \in C([0,1]) \times C([0,1]) \times C([0,1])$ a positive solution of (5.2.1) with $D = (D_1, D_2, D_3)$. Suppose there is a sequence of $D = (D_1, D_2, D_3)$, say $D_n = (D_{1n}, D_{2n}, D_{3n})$, such that $\min\{D_{1n}, D_{2n}, D_{3n}\} \to \infty$ and that (5.2.1) has a positive solution with $D = D_n$. Set $v_{in} = v_{iD_n}$ and $\tilde{v}_{in} = v_{in}/\|v_{in}\|_\infty, i = 1, 2, 3$. Then we have

$$\begin{cases} -\tilde{v}_{1n}'' - \dfrac{\alpha_1}{D_{1n}}\tilde{v}_{1n}' = \dfrac{1}{D_{1n}}\left[g_1\left(I_n(x)\right) - d_1\right]\tilde{v}_{1n}, \\ -\tilde{v}_{2n}'' - \dfrac{\alpha_2}{D_{2n}}\tilde{v}_{2n}' = \dfrac{1}{D_{2n}}\left[g_2\left(I_n(x)\right) - d_2\right]\tilde{v}_{2n}, \\ -\tilde{v}_{3n}'' - \dfrac{\alpha_3}{D_{3n}}\tilde{v}_{3n}' = \dfrac{1}{D_{3n}}\left[g_3\left(I_n(x)\right) - d_3\right]\tilde{v}_{3n}, \\ (\tilde{v}_{1n}', \tilde{v}_{2n}', \tilde{v}_{3n}')(0) = (\tilde{v}_{1n}', \tilde{v}_{2n}', \tilde{v}_{3n}')(1) = 0, \end{cases} \tag{5.2.6}$$

where

$$I_n(x) = e^{-k_0 x - \int_0^x [\|v_{1n}\|_\infty \tilde{v}_{1n}(s)e^{(\alpha_1/D_{1n})s} + \|v_{2n}\|_\infty \tilde{v}_{2n}(s)e^{(\alpha_2/D_{2n})s} + \|v_{3n}\|_\infty \tilde{v}_{3n}(s)e^{(\alpha_3/D_{3n})s}]ds}. \tag{5.2.7}$$

The right hand side of the equations in (5.2.6) is clearly bounded. By the standard elliptic regularity and Sobolev embedding, subject to a subsequence, $\tilde{v}_{in} \to v_{i0}$ in $C^1([0,1])$ and

$$v_{i0}'(x) = \lim_{n\to\infty} v_{in}'(x) = \lim_{n\to\infty}\left(-\frac{1}{D_{in}}\int_0^x \left[g_i\left(I_n(s)\right) - d_i\right]\tilde{v}_{in}(s)ds - \frac{\alpha_i}{D_{in}}\tilde{v}_{in}(x)\right) = 0,\ i = 1,2,3.$$

Hence $\tilde{v}_{in} \to 1$ in $C^1([0,1]), i = 1,2,3$.

Now multiplying the equation for $\tilde{v}_{in}$ by $e^{(\alpha_i/D_{in})x}$ and integrating over $[0,1]$ we deduce

$$\begin{aligned}&\int_0^1 g_i\left(e^{-k_0x-\int_0^x[v_{1n}(s)e^{(\alpha_1/D_{1n})s}+v_{2n}(s)e^{(\alpha_2/D_{2n})s}+v_{3n}(s)e^{(\alpha_3/D_{3n})s}]ds}\right)\tilde{v}_{in}(x)e^{(\alpha_i/D_{in})x}dx\\ &\qquad = d_i\int_0^1 \tilde{v}_{in}(x)e^{(\alpha_i/D_{in})x}dx.\end{aligned} \tag{5.2.8}$$

From (5.2.8) we have

$$d_i\int_0^1 \tilde{v}_{in}(x)e^{(\alpha_i/D_{in})x}dx \le \int_0^1 g_i\left(e^{-\|v_{in}\|_\infty \tilde{v}_{in}(s)e^{(\alpha_i/D_{in})s}]ds}\right)\tilde{v}_{in}(x)e^{(\alpha_i/D_{in})x}dx. \tag{5.2.9}$$

We claim $\|v_{in}\|_\infty$ is bounded away from ∞ for each $i = 1,2,3$. Otherwise, subject to a subsequence, one can let $n \to \infty$ in (5.2.9) and deduce $d_i \le 0$, which is impossible. Hence we can assume, subject to a subsequence, $\|v_{in}\|_\infty \to \tau_i \in [0,\infty)$ as $n \to \infty$.

Now letting $n \to \infty$ in (5.2.8) we deduce

$$d_i = \int_0^1 g_i\left(e^{-(k_0+c_i)x}\right)dx,\ i = 1,2,3, \tag{5.2.10}$$

where $v_{jn} \to \tau_j \in [0,\infty), j = 1,2,3$, and $c_i = \tau_1 + \tau_2 + \tau_3, i = 1,2,3$.

From the above discussion, conclusion (ii) of the theorem follows readily. From (5.2.10), we have $d_i \le \int_0^1 g_i\left(e^{-k_0x}\right)dx$ and hence conclusion (i) of the theorem also follows. □

From Theorem 5.2.1 we find that when the diffusion coefficients are very large, the phytoplankton species can not coexist generically. To see this point more clearly, we look at the widely used nonlinearity

$$g_i(I) = \frac{m_iI}{\delta_i + I}, i = 1,2,3,$$

where m_i, δ_i are positive constants. In this case (5.2.10) becomes

$$\frac{m_i}{d_i}\ln\left(\frac{\delta_i + 1}{\delta_i + e^{-(k_0+c)}}\right) = k_0 + c,\ i = 1,2,3. \tag{5.2.11}$$

Clearly (5.2.11) implies rather severe restrictions on the parameters. For example even under the restriction $\delta_1 = \delta_2 = \delta_3 = \delta$, we still need

$$\frac{m_1}{d_1} = \frac{m_2}{d_2} = \frac{m_3}{d_3}$$

to guarantee (5.2.11).

From the above discussion, we notice that even in the two species case, when the diffusion coefficients are large, phytoplankton species can hardly coexist.

In sharp contrast, when the diffusion coefficients of the system are small, we will prove coexistence of the three phytoplankton is possible.

For that, we consider the equation, for each $i \in \{1, 2, 3\}$,

$$\begin{cases} -D_i v'' - \alpha_i v' = \left[g_i\left(e^{-k_0 x - \int_0^x v(s) e^{(\alpha_i/D_i)s} ds}\right) - d_i\right] v, \; 0 < x < 1, \\ v'(0) = v'(1) = 0. \end{cases} \tag{5.2.12}$$

From [23] (see [22] for the case $\alpha_i = 0$), (5.2.12) has a positive solution if and only if $d_i \in (0, d_i^*)$, where d_i^* is defined by (5.2.3); moreover, the positive solution is unique. We denote by v_{d_i} the positive solution corresponding to $d_i (\in (0, d_i^*))$.

By using topological degree argument, we can prove the following

Theorem 5.2.2 *Suppose that*

$$\begin{aligned} 0 < d_1 < -\lambda_1^{(1)}\left[-g_1(\sigma_{d_2 d_3}(x))\right], \\ 0 < d_2 < -\lambda_1^{(2)}\left[-g_2(\sigma_{d_1 d_3}(x))\right], \\ 0 < d_3 < -\lambda_1^{(3)}\left[-g_3(\sigma_{d_1 d_2}(x))\right], \end{aligned} \tag{5.2.13}$$

where

$$\sigma_{d_2 d_3}(x) = e^{-k_0 x - \int_0^x [e^{(\alpha_2/D_2)y} v_{d_2}(y) + e^{(\alpha_3/D_3)y} v_{d_3}(y)] dy}, \quad x \in [0, 1],$$

$$\sigma_{d_1 d_3}(x) = e^{-k_0 x - \int_0^x [e^{(\alpha_1/D_1)y} v_{d_1}(y) + e^{(\alpha_3/D_3)y} v_{d_3}(y)] dy}, \quad x \in [0, 1],$$

$$\sigma_{d_1 d_2}(x) = e^{-k_0 x - \int_0^x [e^{(\alpha_1/D_1)y} v_{d_1}(y) + e^{(\alpha_2/D_2)y} v_{d_2}(y)] dy}, \quad x \in [0, 1].$$

Then (5.2.1) *has at least one positive solution.*

The proof of this theorem will be given in section 3 of this chapter. We notice that condition (5.2.13) is rather implicit. Our next theorem gives a specific range of parameters for which (5.2.13) is satisfied.

Theorem 5.2.3 *Suppose $\alpha_1, \alpha_2, \alpha_3 \in R^1$ and at least two of the three $\alpha_i (i = 1, 2, 3)$ are nonpositive. Then there exist suitable (D_1, D_2, D_3) and (d_1, d_2, d_3) such that* (5.2.13) *holds, and hence there is at least one positive solution to* (5.2.1).

The range of (D_1, D_2, D_3) and (d_1, d_2, d_3) where (5.2.13) holds will become clear in the proof.

Proof. Let α be a nonpositive constant, D be a positive parameter, $\pi(x)$ be a continuous, strictly increasing function on $[0, 1]$. Denote by $\lambda_D(\pi(x))$ the principal eigenvalue of the eigenvalue problem

$$\begin{cases} -D\varphi'' - \alpha\varphi' + \pi(x)\varphi = \lambda\varphi, \quad 0 < x < 1, \\ \varphi'(0) = \varphi'(1) = 0. \end{cases}$$

By the same technique as used in the proof of Theorem 3.6 of [33], we can prove $\lambda_D(\pi(x))$ is strictly increasing as a function of $D \in (0, \infty)$, moreover,

$$\lim_{D\to 0} \lambda_D(\pi(x)) = \pi(0), \qquad \lim_{D\to\infty} \lambda_D(\pi(x)) = \int_0^1 \pi(x)\,dx. \tag{5.2.14}$$

Consider the equation

$$\begin{cases} -Dv'' - \alpha v' = \left[g\left(e^{-k_0 x - \int_0^x v(s)e^{(\alpha/D)s}\,ds}\right) - d\right] v, \quad 0 < x < 1, \\ v'(0) = v'(1) = 0, \end{cases} \tag{5.2.15}$$

where g satisfies the conditions for $g_i, i = 1, 2, 3$ in (5.1.4).

By Theorem 3.1 of [33], $d \in \left(0, -\lambda_D\left(-g\left(e^{-k_0 x}\right)\right)\right)$ is a necessary and sufficient condition for (5.2.15) to have a positive solution $v_{D,d}$, moreover the positive solution is unique. Clearly if $0 < d < g\left(e^{-k_0}\right)$, then (5.2.15) has a unique positive solution for any $D \in (0, \infty)$ and $\alpha \in R^1$.

Multiplying (5.2.15) by $e^{(\alpha/D)x}$ and integrating the resultant equation over $[0,1]$, we have

$$\begin{aligned} d\int_0^1 e^{(\alpha/D)x}v(x)dx &= \int_0^1 g\left(e^{-k_0x-\int_0^x e^{(\alpha/D)s}v(s)ds}\right)e^{(\alpha/D)x}v(x)dx \\ &\leq \int_0^1 g\left(e^{-\int_0^x e^{(\alpha/D)s}v(s)ds}\right)e^{(\alpha/D)x}v(x)dx \\ &\leq \int_0^{\int_0^1 e^{(\alpha/D)x}v(x)dx} g\left(e^{-s}\right)ds \leq C := \int_0^\infty g(e^{-s})ds. \end{aligned} \tag{5.2.16}$$

Define

$$c_d = \sup_{\alpha\in R^1, D>0}\int_0^1 e^{(\alpha/D)x}v_{D,\alpha}(x)dx. \tag{5.2.17}$$

We show that

$$c_d \to 0 \ \text{ as } d \to g(1). \tag{5.2.18}$$

Assume (5.2.18) does not hold. Then there exists $D_n > 0, \alpha_n \in R^1$ and $d_n \to g(1)$ such that

$$\int_0^1 v_n(x)e^{(\alpha_n/D_n)x}dx \to I_* > 0.$$

It follows from (5.2.16) that

$$g(1)I_* \leq \int_0^{I_*} g\left(e^{-s}\right)ds = g\left(e^{-s_*}\right)I_* \ \text{ for some } s_* \in (0, I_*),$$

which is impossible.

We are now ready to complete the proof of the theorem. Without loss of generality, we may assume $\alpha_2 \leq 0, \alpha_3 \leq 0$. We first fix $D_1 > 0$ and choose d_2 such that

$$0 < d_2 < g_2\left(1\right).$$

Let c_{d_2} be a constant given by (5.2.17), but with (D, α, d, g) in (5.2.15) replaced by $(D_2, \alpha_2, d_2, g_2)$. Choose

$$d_1 \in \left(0, g_1\left(e^{-k_0-c_{d_2}}\right)\right).$$

By (5.2.17) we have

$$0 < d_1 < -\lambda_{D_1}\left[-g_1\left(e^{-k_0x-\int_0^x v_{D_2,d_2}(s)e^{(\alpha_2/D_2)s}ds}\right)\right].$$

Now by (5.2.14), we have

$$\lim_{D_2\to 0} \lambda_{D_2} \left[-g_2\left(e^{-k_0 x - \int_0^x v_{D_1,d_1}(s)e^{(\alpha_1/D_1)s}ds}\right)\right] = -g_2(1).$$

We can choose D_2 sufficiently small such that

$$0 < d_2 < -\lambda_{D_2} \left[-g_2\left(e^{-k_0 x - \int_0^x v_{D_1,d_1}(s)e^{(\alpha_1/D_1)s}ds}\right)\right].$$

Choose $\epsilon > 0$ such that

$$0 < d_1 < -\lambda_{D_1} \left[-g_1\left(e^{-k_0 x - \int_0^x v_{D_2,d_2}(s)e^{(\alpha_2/D_2)s}ds - \epsilon}\right)\right],$$

$$0 < d_2 < -\lambda_{D_2} \left[-g_2\left(e^{-k_0 x - \int_0^x v_{D_1,d_1}(s)e^{(\alpha_1/D_1)s}ds - \epsilon}\right)\right].$$

By (5.2.17) and (5.2.18), we can choose $d_3 > 0$ such that $g_3(1) - d_3 > 0$ is small enough such that

$$\int_0^1 v_{D_3,d_3}(x)e^{(\alpha_3/D_3)x}dx \le \epsilon. \qquad (5.2.19)$$

Hence

$$0 < d_1 < -\lambda_{D_1} \left[-g_1\left(e^{-k_0 x - \int_0^x v_{D_2,d_2}(s)e^{(\alpha_2/D_2)s}ds - \int_0^x v_{D_3,d_3}(s)e^{(\alpha_3/D_3)s}ds}\right)\right], \qquad (5.2.20)$$

$$0 < d_2 < -\lambda_{D_2} \left[-g_2\left(e^{-k_0 x - \int_0^x v_{D_1,d_1}(s)e^{(\alpha_1/D_1)s}ds - \int_0^x v_{D_3,d_3}(s)e^{(\alpha_3/D_3)s}ds}\right)\right]. \qquad (5.2.21)$$

Finally by (5.2.14) again, we can choose D_3 small enough such that

$$0 < d_3 < -\lambda_{D_3} \left[-g_3\left(e^{-k_0 x - \int_0^x v_{D_1,d_1}(s)e^{(\alpha_1/D_1)s}ds - \int_0^x v_{D_2,d_2}(s)e^{(\alpha_2/D_2)s}ds}\right)\right].$$

Note that (5.2.17), (5.2.19) and hence (5.2.20) and (5.2.21) are not affected by the choice of D_3. The proof is complete.

□

Theorem 5.2.3 tells us that if at least two of the three species are buoyant, when the death rates and the turbulence diffusions of the species are sufficiently small, the three species can coexist in the same water column.

Theorem 5.2.4 *Suppose D_1, D_2, D_3 are fixed. We can choose suitable $\alpha_1, \alpha_2, \alpha_3$ and d_1, d_2, d_3 such that* (5.2.13) *holds, and hence there exists at least one positive solution to* (5.2.1).

Proof. Let $\pi(x)$ be any continuous strictly increasing function on $[0,1]$. Let λ_α be the principal eigenvalue of the eigenvalue problem

$$\begin{cases} -D\varphi'' - \alpha\varphi' + \pi(x)\varphi = \lambda\varphi, \quad 0 < x < 1, \\ \varphi'(0) = \varphi'(1) = 0. \end{cases}$$

By the same technique as in the proof of Theorem 3.2 of [33], $\lambda_\alpha(\pi(x))$ is strictly increasing as a function of $\alpha \in (-\infty, \infty)$, moreover,

$$\lim_{\alpha\to-\infty} \lambda_\alpha(\pi(x)) = \pi(0), \qquad \lim_{\alpha\to\infty} \lambda_\alpha(\pi(x)) = \pi(1). \tag{5.2.22}$$

We also have the same c_d as in (5.2.17) with c_d independent of $\alpha \in R^1, D > 0$:

$$c_d = \sup_{\alpha\in R^1, D>0} \int_0^1 e^{(\alpha/D)x} v(x)dx$$

and

$$c_d \to 0, \quad \text{as } d \to g(1).$$

We can now begin to choose suitable $\alpha_1, \alpha_2, \alpha_3$ and d_1, d_2, d_3 such that (5.2.13) holds. Choose α_2 and d_1 such that $0 < d_1 < g_1\left(e^{-k_0}\right)$. Let c_{d_1} be a constant chosen according to (5.2.17). Choose $d_2 \in \left(0, g_2\left(e^{-k_0 - c_{d_1}}\right)\right)$. Now we can choose α_1 sufficiently negative to ensure

$$\begin{aligned} 0 < d_1 < -\lambda_{\alpha_1}\left[-g_1\left(e^{-k_0 x - \int_0^x v_{\alpha_2,d_2}(s)e^{(\alpha_2/D_2)s}ds}\right)\right], \\ 0 < d_2 < -\lambda_{\alpha_2}\left[-g_2\left(e^{-k_0 x - \int_0^x v_{\alpha_1,d_1}(s)e^{(\alpha_1/D_1)s}ds}\right)\right]. \end{aligned}$$

Choose $\epsilon > 0$ such that

$$\begin{aligned} 0 < d_1 < -\lambda_{\alpha_1}\left[-g_1\left(e^{-k_0 x - \int_0^x v_{\alpha_2,d_2}(s)e^{(\alpha_2/D_2)s}ds - \epsilon}\right)\right], \\ 0 < d_2 < -\lambda_{\alpha_2}\left[-g_2\left(e^{-k_0 x - \int_0^x v_{\alpha_1,d_1}(s)e^{(\alpha_1/D_1)s}ds - \epsilon}\right)\right]. \end{aligned} \tag{5.2.23}$$

Choose $d_3 > 0$ such that $g_3(1) - d_3 > 0$ is small enough such that

$$\int_0^1 v_{\alpha_3,d_3}(s)e^{(\alpha_3/D_3)s}ds \le \epsilon. \tag{5.2.24}$$

By (5.2.23) and (5.2.24) we have

$$\begin{aligned}
0 < d_1 &< -\lambda_{\alpha_1}\left[-g_1\left(e^{-k_0x-\int_0^x v_{\alpha_2,d_2}(s)e^{(\alpha_2/D_2)s}ds-\int_0^x v_{\alpha_3,d_3}(s)e^{(\alpha_3/D_3)s}ds}\right)\right],\\
0 < d_2 &< -\lambda_{\alpha_2}\left[-g\left(e^{-k_0x-\int_0^x v_{\alpha_1,d_1}(s)e^{(\alpha_1/D_1)s}ds-\int_0^x v_{\alpha_3,d_3}(s)e^{(\alpha_3/D_3)s}ds}\right)\right].
\end{aligned} \tag{5.2.25}$$

Finally choose α_3 negative enough such that

$$0 < d_3 < -\lambda_{\alpha_3}\left[-g\left(e^{-k_0x-\int_0^x v_{\alpha_1,d_1}(s)e^{(\alpha_1/D_1)s}ds-\int_0^x v_{\alpha_2,d_2}(s)e^{(\alpha_2/D_2)s}ds}\right)\right]. \tag{5.2.26}$$

Note that (5.2.25) still holds. The proof is complete. □

Remark 5.2.5 *In Theorem 5.2.3, we assume that at least two of the three α_i, $i = 1, 2, 3$, are nonpositive. We guess it is purely technical. It would be interesting to find conditions that do not require the signs of these α_is.*

5.3 The proof of Theorem 5.2.2

In this section, we use topological degree (fixed point index) theory to prove Theorem 5.2.2. Similar techniques have been used in [14] and [16] in treating classic competition or predator-prey systems. Here we need some a prior estimates specific to our nonlocal problem.

We begin by proving the following boundedness lemma.

Lemma 5.3.1 *Let $t \in [0, 1]$. Fix $d = (d_1, d_2, d_3)$, $d_i \in (0, d_i^*)$ $(i = 1, 2, 3)$. Suppose that (v_1, v_2, v_3) is a nonnegative solution of*

$$\begin{cases}
-D_1v_1'' - \alpha_1v_1' + d_1^*v_1 = t[g_1(I(x)) + d_1^* - d_1]v_1, & 0 < x < 1,\\
-D_2v_2'' - \alpha_2v_2' + d_2^*v_2 = t[g_2(I(x)) + d_2^* - d_2]v_2, & 0 < x < 1,\\
-D_3v_3'' - \alpha_3v_3' + d_3^*v_3 = t[g_3(I(x)) + d_3^* - d_3]v_3, & 0 < x < 1,\\
\quad (v_1', v_2', v_3')(0) = (v_1', v_2', v_3')(1) = (0, 0, 0).
\end{cases} \tag{5.3.1}$$

Then there exists a constant $B = B_d$ dependent on d, but independent of t such that

$$\|v_1\|_\infty + \|v_2\|_\infty + \|v_3\|_\infty \le B_d.$$

Proof. Argue indirectly. Assume (v_{1n}, v_{2n}, v_{3n}) is a sequence of nonnegative solutions to (5.3.1) with $t = t_n$ satisfying

$$\|v_{1n}\|_\infty + \|v_{2n}\|_\infty + \|v_{3n}\|_\infty \to \infty \ \text{ as } \ n \to \infty.$$

Without loss of generality, we assume $\|v_{1n}\|_\infty \to \infty$ as $n \to \infty$. Setting $\tilde{v}_{1n} := v_{1n}/\|v_{1n}\|_\infty$, we have

$$\begin{cases} -D_1\tilde{v}_{1n}'' - \alpha_1\tilde{v}_{1n}' + d_1^*\tilde{v}_{1n} = t_n[g_1(I_n(x)) + d_1^* - d_1]\tilde{v}_{1n}, \\ \tilde{v}_{1n}'(0) = \tilde{v}_{1n}'(1) = 0, \end{cases}$$

where

$$I_n(x) = \exp\left(-k_0x - \int_0^x [e^{(\alpha_1/D_1)y}v_{1n}(y) + e^{(\alpha_2/D_2)y}v_{2n}(y) + e^{(\alpha_3/D_3)y}v_{3n}(y)]dy\right).$$

Note that $[g_1(I_n(x)) + d_1^* - d_1]\tilde{v}_{1n}$ is bounded in $C([0,1])$ with respect to n. By the standard elliptic regularity and Sobolev embedding theorems, subject to a subsequence, $\tilde{v}_{1n} \to v_0$ in $C^1([0,1])$. We may also assume $g_1(I_n) \to g_0$ weakly in $L^2([0,1])$ and $t_n \to t_0 \in [0,1]$. Then v_0 satisfies

$$\begin{cases} -D_1v_0'' - \alpha_1v_0' + d_1^*v_0 = t_0[g_0(x) + d_1^* - d_1]v_0, \\ v_0'(0) = v_0'(1) = 0, \ \ v_0(x) \geq 0, \ \ \|v_0\|_\infty = 1. \end{cases}$$

By the strong maximum principle we have

$$v_0(x) > 0 \ \text{ in } \ [0,1],$$

and hence

$$v_{1n}(x) = \|v_{1n}\|_\infty\tilde{v}_{1n}(x) \to \infty \ \text{ uniformly in } \ [0,1].$$

Thus

$$g_0(x) = \lim_{n\to\infty} g_1\left(e^{-k_0x - \int_0^x [e^{(\alpha_1/D_1)y}v_{1n}(y) + e^{(\alpha_2/D_2)y}v_{2n}(y) + e^{(\alpha_3/D_3)y}v_{3n}(y)]dy}\right) = 0.$$

Therefore

$$\begin{cases} -D_1v_0'' - \alpha_1v_0' + d_1^*v_0 = t_0[d_1^* - d_1]v_0, \\ v_0'(0) = v_0'(1) = 0, \ \ v_0(x) > 0, \ \ \|v_0\|_\infty = 1. \end{cases}$$

This implies

$$t_0[d_1^* - d_1] = \lambda_1^{(1)}(d_1^*) = d_1^*.$$

From $d_1 \in (0, d_1^*)$ we then have

$$t_0 = \frac{d_1^*}{d_1^* - d_1} > 1,$$

a contradiction. The contradiction finishes the proof. □

Let

$$K = \{v \in C([0,1]) : v(x) \geq 0 \text{ in } [0,1]\},$$

$$E = C([0,1]) \times C([0,1]) \times C([0,1]) \quad \text{and} \quad C = K \times K \times K,$$

Clearly K is a cone in $C([0,1])$.

Let L_i $(i = 1,2,3)$ be the solution operator of

$$-D_i v'' - \alpha_i v' + d_i^* v = f_i(x) \quad (f_i \in C([0,1])), \;\; v'(0) = v'(1) = 0;$$

$G_i(d_i, v_1, v_2, v_3)$ $(i = 1,2,3)$ be the operator defined by

$$G_i(d_i, v_1, v_2, v_3) = [g_i(I) + d_i^* - d_i]\, v_i.$$

Let

$$\Omega = \{(v_1, v_2, v_3) \in C : \|v_1\| + \|v_2\| + \|v_3\| \leq B_d + 1\}.$$

Define the operator $A: \;\Omega \to E$ by

$$A(v_1, v_2, v_3) = (A_1(v_1, v_2, v_3), A_2(v_1, v_2, v_3), A_3(v_1, v_2, v_3)) \;\text{ for any } (v_1, v_2, v_3) \in C,$$

where

$$A_i(v_1, v_2, v_3) = L_i \circ G_i(d_i, v_1, v_2, v_3), \;\; i = 1,2,3.$$

Then it is easy to see that (v_1, v_2, v_3) solves (5.2.1) if and only if it is a fixed point of A. Clearly $A : \Omega \to C$ is a completely continuous operator. Its derivative operator at $v = (v_1, v_2, v_3)$ is

$$A_i'(v)h = L_i \circ G_i'(v)h \; (i = 1,2,3), \;\; h = (h_1, h_2, h_3)^T,$$

where $(h_1, h_2, h_3)^T$ denotes the transpose of the row matrix (h_1, h_2, h_3) and

$$\begin{cases} G'_{iv_j}(v)h = \delta_{ij}\left(g_i(I(x)) + d_i^* - d_i\right)h_j - g_i'(I(x))I(x)v_i \displaystyle\int_0^x \sum_{j=1}^{3} e^{(\alpha_j/D_j)y} h_j(y)dy, \\ h_j'(0) = h_j'(1) = 0, \quad i, j = 1, 2, 3, \end{cases}$$

where $\delta_{ij} = 1$, for $i = j$; $\delta_{ij} = 0$, for $i \neq j$.

For $d_i \in (0, d_i^*)$ $(i = 1, 2, 3)$, we have the following result.

Lemma 5.3.2

$$\deg_C(I - A, \Omega, 0) = 1. \tag{5.3.2}$$

Proof. In fact, for $t \in [0, 1]$, $(v_1, v_2, v_3) = tA(v_1, v_2, v_3)$ $((v_1, v_2, v_3) \in \overline{\Omega})$ is equivalent to

$$\begin{cases} -D_1 v_1'' - \alpha_1 v_1' + d_1^* v_1 = t[g_1(I(x)) + d_1^* - d_1]v_1, & 0 < x < 1, \\ -D_2 v_2'' - \alpha_2 v_2' + d_2^* v_2 = t[g_2(I(x)) + d_2^* - d_2]v_2, & 0 < x < 1, \\ -D_3 v_3'' - \alpha_3 v_3' + d_3^* v_3 = t[g_3(I(x)) + d_3^* - d_3]v_3, & 0 < x < 1, \\ \quad (v_1', v_2', v_3')(0) = (v_1', v_2', v_3')(1) = (0, 0, 0). \end{cases} \tag{5.3.3}$$

By Lemma 5.3.1, there is no nonnegative solution of (5.3.3) satisfying

$$\|v_1\| + \|v_2\| + \|v_3\| = B_d + 1. \tag{5.3.4}$$

This implies for any (v_1, v_2, v_3) satisfying (5.3.4), $t \in [0, 1]$, we have

$$v \neq tAv.$$

By the homotopy invariant property of the topological degree, we have

$$\deg_C(I - A, \Omega, 0) = \deg_C(I, \Omega, 0) = 1.$$

□

By the strong maximum principle, nonnegative solutions of (5.2.1) can be classified into three classes:

(I) The unique trivial solution $(v_1, v_2, v_3) = (0, 0, 0)$, which exists for all d_1, d_2 and d_3.

(II) Three semitrivial solutions $(v_1, v_2, v_3) = (v_{d_1}, 0, 0)$ for $d_1 \in (0, d_1^*)$, $(v_1, v_2, v_3) = (0, v_{d_2}, 0)$ for $d_2 \in (0, d_2^*)$ and $(v_1, v_2, v_3) = (0, 0, v_{d_3})$ for $d_3 \in (0, d_3^*)$.

(III) Semitrivial solutions of (5.2.1) which have exactly one component identically zero.

Let $(v_1, 0, 0)$ be the nonnegative solution of (5.2.1) and ξ be a number such that there are positive $h_i \in k, i = 1, 2, 3$ such that

$$A'_{(v_1,0,0)}(h_1, h_2, h_3)^T = \xi(h_1, h_2, h_3)^T. \tag{5.3.5}$$

Clearly if $d_i \in (0, d_i^*)$, we have $\xi > 0$ if it exists. That is we want to find $(h_1, h_2, h_3) \neq (0, 0, 0)$ such that

$$\begin{cases} -D_1 h_1'' - \alpha_1 h_1' + (1 - \xi^{-1}) d_1^* h_1 = \xi^{-1} \left[g_1(\sigma_1(x)) - d_1 \right] h_1 \\ \qquad - \xi^{-1} g_1'(\sigma_1(x)) \sigma_1(x) v_1(x) \displaystyle\int_0^x \sum_{j=1}^3 e^{(\alpha_j/D_j)y} h_j(y) dy, & x \in (0,1), \\ -D_2 h_2'' - \alpha_2 h_2' + (1 - \xi^{-1}) d_2^* h_2 = \xi^{-1} \left[g_2(\sigma_1(x)) - d_2 \right] h_2, & x \in (0,1), \\ -D_3 h_3'' - \alpha_3 h_3' + (1 - \xi^{-1}) d_3^* h_3 = \xi^{-1} \left[g_3(\sigma_1(x)) - d_3 \right] h_3, & x \in (0,1), \\ h_1' = h_2' = h_3' = 0, & x = 0, 1, \end{cases} \tag{5.3.6}$$

where

$$\sigma_i(x) = e^{-k_0 x - \int_0^x e^{(\alpha_i/D_i)y} v_i(y) dy}, \quad i = 1, 2, 3.$$

For later use, we need to calculate ξ. First we find the condition needed for $\xi = 1$. That is we need to solve

$$\begin{cases} -D_1 h_1'' - \alpha_1 h_1' = \left[g_1(\sigma_1(x)) - d_1 \right] h_1 \\ \qquad - g_1'(\sigma_1(x)) \sigma_1(x) v_1(x) \displaystyle\int_0^x \sum_{j=1}^3 e^{(\alpha_j/D_j)y} h_j(y) dy, & x \in (0,1), \\ -D_2 h_2'' - \alpha_2 h_2' = \left[g_2(\sigma_1(x)) - d_2 \right] h_2, & x \in (0,1), \\ -D_3 h_3'' - \alpha_3 h_3' = \left[g_3(\sigma_1(x)) - d_3 \right] h_3, & x \in (0,1), \\ h_1' = h_2' = h_3' = 0, & x = 0, 1. \end{cases} \tag{5.3.7}$$

Similar as in Chapter 3 we can prove the following

Lemma 5.3.3 *Let $i \in \{1,2,3\}$, $h \in C^2([0,1])$ satisfy*

$$\begin{cases} -D_i h'' - \alpha_i h' = [g_i(\sigma_i(x)) - d_i]h - g_i'(\sigma_i(x))\sigma_i(x)v_i(x)\displaystyle\int_0^x e^{(\alpha_i/D_i)y}h(y)dy, \\ h'(0) = h'(1) = 0. \end{cases} \tag{5.3.8}$$

Then $h \equiv 0$.

Denote by v_{d_i} $(0 < d_i < d_i^*,\ i = 1,2,3)$ the unique positive solution of the problem

$$\begin{cases} -D_i v'' - \alpha_i v' = \left[g_i\left(e^{-k_0 x - \int_0^x e^{(\alpha_i/D_i)y}v(y)dy}\right) - d_i\right]v, \quad 0 < x < 1, \\ v'(0) = v'(1) = 0. \end{cases} \tag{5.3.9}$$

Let

$$\sigma_{d_i}(x) = e^{-k_0 x - \int_0^x e^{(\alpha_i/D_i)y}v_{d_i}(y)dy}, \quad i = 1,2,3.$$

Then it is clear that

$$\begin{cases} 0 < -\lambda_1^{(i)}\left[-g_i\left(\sigma_{d_1}(x)\right)\right] < -\lambda_1^{(i)}\left[-g_i\left(e^{-k_0 x}\right)\right] = d_i^*\ (i = 2,3), \\ 0 < -\lambda_1^{(j)}\left[-g_j\left(\sigma_{d_2}(x)\right)\right] < -\lambda_1^{(j)}\left[-g_j\left(e^{-k_0 x}\right)\right] = d_j^*\ (j = 1,3), \\ 0 < -\lambda_1^{(k)}\left[-g_k\left(\sigma_{d_3}(x)\right)\right] < -\lambda_1^{(k)}\left[-g_k\left(e^{-k_0 x}\right)\right] = d_k^*\ (k = 1,2). \end{cases}$$

Lemma 5.3.4 *Suppose $d_1 \in (0, d_1^*), d_2 \in (0, d_2^*), d_3 \in (0, d_3^*)$,*

$$\begin{aligned} d_i &\neq -\lambda_1^{(i)}\left[-g_i\left(\sigma_{d_1}(x)\right)\right]\ (i = 2,3), \\ d_j &\neq -\lambda_1^{(j)}\left[-g_j\left(\sigma_{d_2}(x)\right)\right]\ (j = 1,3), \\ d_k &\neq -\lambda_1^{(k)}\left[-g_k\left(\sigma_{d_3}(x)\right)\right]\ (k = 1,2). \end{aligned}$$

Then $(v_{d_1}, 0, 0), (0, v_{d_2}, 0), (0, 0, v_{d_3})$ are all isolated solutions of (5.2.1) and

$$\begin{aligned} &\mathrm{index}_C\left(A, (v_{d_1}, 0, 0)\right) \\ &= \begin{cases} 0, & \text{if } d_2 \in (0, -\lambda_1^{(2)}\left[-g_2\left(\sigma_{d_1}(x)\right)\right]) \text{ or } d_3 \in (0, -\lambda_1^{(3)}\left[-g_3\left(\sigma_{d_1}(x)\right)\right]), \\ 1, & \text{if } d_2 \in (-\lambda_1^{(2)}\left[-g_2\left(\sigma_{d_1}(x)\right)\right], d_2^*) \text{ and } d_3 \in (-\lambda_1^{(3)}\left[-g_3\left(\sigma_{d_1}(x)\right)\right], d_3^*); \end{cases} \end{aligned}$$

$$\begin{aligned} &\mathrm{index}_C\left(A, (0, v_{d_2}, 0)\right) \\ &= \begin{cases} 0, & \text{if } d_1 \in (0, -\lambda_1^{(1)}\left[-g_1\left(\sigma_{d_2}(x)\right)\right]) \text{ or } d_3 \in (0, -\lambda_1^{(3)}\left[-g_3\left(\sigma_{d_2}(x)\right)\right]), \\ 1, & \text{if } d_1 \in (-\lambda_1^{(1)}\left[-g_1\left(\sigma_{d_2}(x)\right)\right], d_1^*) \text{ and } d_3 \in (-\lambda_1^{(3)}\left[-g_3\left(\sigma_{d_2}(x)\right)\right], d_3^*); \end{cases} \end{aligned}$$

$\text{index}_C\left(A,(0,0,v_{d_3})\right)$

$$=\begin{cases} 0, & \text{if } d_1 \in (0,-\lambda_1^{(1)}\left[-g_1\left(\sigma_{d_3}(x)\right)\right]) \text{ or } d_2 \in (0,-\lambda_1^{(2)}\left[-g_2\left(\sigma_{d_3}(x)\right)\right]), \\ 1, & \text{if } d_1 \in (-\lambda_1^{(1)}\left[-g_1\left(\sigma_{d_3}(x)\right)\right], d_1^*) \text{ and } d_2 \in (-\lambda_1^{(2)}\left[-g_2\left(\sigma_{d_3}(x)\right)\right], d_2^*). \end{cases}$$

Proof. We only prove the conclusion holds for $(v_{d_1},0,0)$. The proofs for $(0,v_{d_2},0)$ and $(0,0,v_{d_3})$ are similar. Suppose that $(v_{d_1},0,0)$ is not an isolated solution, then there exists a sequence $\{(v_{1n},v_{2n},v_{3n})\}$ of nonnegative solutions for (5.2.1), such that $(v_{1n},v_{2n},v_{3n}) \to (v_{d_1},0,0)$ as $n\to\infty$. As $(v_{d_1},0,0)$ is the unique type II solution of (5.2.1) with the second and the third components identically zero, we may assume that $\|v_{2n}\|_\infty > 0$ (or $\|v_{3n}\|_\infty > 0$).

Set $\tilde{v}_{2n} = \frac{v_{2n}}{\|v_{2n}\|_\infty}$. We then have

$$\begin{cases} -D_2\tilde{v}_{2n}'' - \alpha_2\tilde{v}_{2n}' = [g_2(I_n(x)) - d_2]\tilde{v}_{2n}, & 0<x<1, \\ \tilde{v}_{2n}'(0) = \tilde{v}_{2n}'(1) = 0, \ \ 0<\tilde{v}_{2n}\le 1, \ \ \|\tilde{v}_{2n}\|_\infty = 1, \end{cases} \tag{5.3.10}$$

where

$$I_n(x) = \exp\left(-k_0x - \int_0^x [e^{(\alpha_1/D_1)y}v_{1n}(y) + e^{(\alpha_2/D_2 y}v_{2n}(y) + e^{(\alpha_3/D_3)y}v_{3n}(y)]dy\right).$$

The right hand side of (5.3.10) is bounded. By the standard L^p theory of elliptic regularity and Sobolev embedding theorems, we may assume, by passing to a subsequence, $\tilde{v}_{2n} \to v_0$ in $C^1([0,1])$. Moreover, v_0 satisfies

$$\begin{cases} -D_2v_0'' - \alpha_2v_0' = [g_2(e^{-k_0x-\int_0^x e^{(\alpha_1/D_1)y}v_{d_1}(y)dy}) - d_2]v_0, & 0<x<1, \\ v_0'(0) = v_0'(1) = 0, \ \ 0\le v_0\le 1, \ \ \|v_0\|_\infty = 1. \end{cases}$$

By the strong maximum principle $v_0>0$ in $[0,1]$. Hence $d_2 = -\lambda_1^{(2)}\left[-g_2\left(\sigma_{d_1}(x)\right)\right]$, a contradiction. Thus $(v_{d_1},0,0)$ is an isolated solution.

By Lemma 5.3.3, it is easy to check that

$$\begin{cases} -D_2h_2'' - \alpha_2h_2' + (1-\xi^{-1})d_2^*h_2 = \xi^{-1}\left[g_2(\sigma_1(x)) - d_2\right]h_2, & x\in(0,1), \\ -D_3h_3'' - \alpha_3h_3' + (1-\xi^{-1})d_3^*h_3 = \xi^{-1}\left[g_3(\sigma_1(x)) - d_3\right]h_3, & x\in(0,1), \\ h_2' = h_3' = 0, & x=0,1 \end{cases} \tag{5.3.11}$$

has an eigenvalue $\xi > 1$ if and only if

$$d_2 \in (0, -\lambda_1^{(2)}\left[-g_2\left(\sigma_{d_1}(x)\right)\right]) \text{ or } d_3 \in (0, -\lambda_1^{(3)}\left[-g_3\left(\sigma_{d_1}(x)\right)\right]);$$

and (5.3.11) has the eigenvalue $\xi < 1$ if and only if

$$d_2 \in (-\lambda_1^{(2)}\left[-g_2\left(\sigma_{d_1}(x)\right)\right], d_2^*) \text{ and } d_3 \in (-\lambda_1^{(3)}\left[-g_3\left(\sigma_{d_1}(x)\right)\right], d_3^*).$$

Now a use of Theorem 1.3.2 completes the proof of this lemma. □

Next we consider semitrivial solutions of (5.2.1) which have exactly one component identically zero. For $i = 1, 2, 3$, we denote by T_i the set of semitrivial solutions which have the ith component zero and the other two components positive.

For simplicity, we also write $w_0 = (0, 0, 0)$, $w_1 = (v_{d_1}, 0, 0)$, $w_2 = (0, v_{d_2}, 0)$, $w_3 = (0, 0, v_{d_3})$. Then evidently the set

$$M = \left(\bigcup_{i=1}^{3} T_i\right) \bigcup \left(\bigcup_{i=0}^{3} \{w_i\}\right)$$

contains all the nonnegative solutions of (5.2.1) which are not positive.

Lemma 5.3.5 *T_1 is a compact set in C if w_2 and w_3 are both isolated in Ω; T_2 is a compact set in C if w_3 and w_1 are both isolated in Ω; T_3 is a compact set in C if w_1 and w_2 are both isolated in Ω.*

Proof. We only give the proof for T_3, the cases for T_1 and T_2 are similar.

We first prove that $w_0 = (0, 0, 0)$ is isolated. Otherwise, let $\{(v_{1n}, v_{2n}, v_{3n})\}$ be a sequence of nonnegative solutions of (5.2.1) that converges to $(0, 0, 0)$. Then, by choosing a subsequence, we may assume that $v_{1n} > 0$ for all $n = 1, 2, \cdots$. Set $\tilde{v}_{1n} = v_{1n}/\|v_{1n}\|_\infty$. Then $\tilde{v}_{1n}$ satisfies

$$\begin{cases} -D_1\tilde{v}_{1n}'' - \alpha_1\tilde{v}_{1n}' = \left[g_1(I_n(x)) - d_1\right]\tilde{v}_{1n}, \\ \tilde{v}_{1n}'(0) = \tilde{v}_{1n}'(1) = 0, \end{cases} \tag{5.3.12}$$

where

$$I_n(x) = e^{-k_0 x}\exp\left(-\int_0^x [e^{(\alpha_1/D_1)y}v_{1n}(y) + e^{(\alpha_2/D_2)y}v_{2n}(y) + e^{(\alpha_3/D_3)y}v_{3n}(y)]dy\right).$$

Clearly $g_1(I_n(x)) \to g_1(e^{-k_0 x})$ in $L^2([0,1])$. The right hand side of (5.3.12) is bounded. Thus by elliptic regularity we may assume, by passing to a subsequence, that $\tilde{v}_{1n} \to v_0$ in $C^1([0,1])$. Moreover v_0 satisfies

$$\begin{cases} -D_1 v_0'' - \alpha_1 v_0' = [g_1(e^{-k_0 x}) - d_1] v_0, \\ v_0'(0) = v_0'(1) = 0, \quad 0 \le v_0 \le 1, \quad \|v_0\|_\infty = 1. \end{cases}$$

By the strong maximum principle, we have $v_0(x) > 0$ for $x \in [0,1]$. This implies

$$d_1 = -\lambda_1^{(1)}[-g_1(e^{-k_0 x})] = d_1^*,$$

contradicting to $d_1 \in (0, d_1^*)$. Hence $w_0 = (0,0,0)$ is isolated.

Now suppose w_1 and w_2 are also isolated. Let $(v_{1n}, v_{2n}, 0) \in T_3$. By Lemma 5.3.1, $\{(v_{1n}, v_{2n}, 0) \in T_3\}$ is precompact. Subject to a subsequence, we may assume $(v_{1n}, v_{2n}, 0) \to (v_1, v_2, 0)$. Now that w_0, w_1, w_2 are all isolated, hence $v_1 \not\equiv 0, v_2 \not\equiv 0$. By the strong maximum principle $v_1 > 0, v_2 > 0$. Hence $(v_1, v_2, 0) \in T_3$. Thus T_3 is compact. □

Let $E_1 = C([0,1]) \times C([0,1])$, $E_2 = C([0,1])$ and $E = E_1 \times E_2$, $C_1 = K \times K$, $C_2 = K$ and $C = C_1 \times C_2$, where K is the set of positive functions in $C([0,1])$. Then E is an ordered Banach space with positive cone C.

Define $S : \Omega \to C$ by

$$S((v_1, v_2), v_3) = (S_1((v_1, v_2), v_3), S_2((v_1, v_2), v_3)) = ((A_1(v_1, v_2, v_3), A_2(v_1, v_2, v_3)), A_3(v_1, v_2, v_3)).$$

In order to use Theorem 1.3.2, we choose a neighbourhood $U \subset C_1 \cap \Omega$ of $T_3 \cap C_1$ such that $(v_{d_1}, 0), (0, v_{d_2}) \notin \bar{U}$. Now $S_1(v_1, v_2, 0) = (v_1, v_2)$ with $(v_1, v_2) \in \bar{U}$ if and only if $(v_1, v_2, 0) \in T_3$.

Fix $d_1 \in (0, d_1^*), d_1 \neq -\lambda_1^{(1)} [-g_1(\sigma_{d_i}(x))]$ $(i = 2, 3)$; $d_2 \in (0, d_2^*), d_2 \neq -\lambda_1^{(2)} \left[-g_2\left(\sigma_{d_j}(x)\right)\right]$ $(j = 1, 3)$ and $d_3 \in (0, d_3^*), d_3 \neq -\lambda_1^{(3)} [-g_3(\sigma_{d_k}(x))]$ $(k = 1, 2)$, then $\deg_{C_1}(I - S_1|_{C_1}, U, 0)$ is defined and moreover

Lemma 5.3.6

$$\deg_{C_1}\left(I - S_1|_{C_1}, U, 0\right)$$
$$= \begin{cases} 1, & \text{if } d_1 \in (0, -\lambda_1^{(1)}\left[-g_1\left(\sigma_{d_2}(x)\right)\right]) \text{ and } d_2 \in (0, -\lambda_1^{(2)}\left[-g_2\left(\sigma_{d_1}(x)\right)\right]), \\ -1, & \text{if } d_1 \in (-\lambda_1^{(1)}\left[-g_1\left(\sigma_{d_2}(x)\right)\right], d_1^*) \text{ and } d_2 \in (-\lambda_1^{(2)}\left[-g_2\left(\sigma_{d_1}(x)\right)\right], d_2^*), \\ 0, & \text{if } [d_1 + \lambda_1^{(1)}\left[-g_1\left(\sigma_{d_2}(x)\right)\right]] \cdot [d_2 + \lambda_1^{(2)}\left[-g_2\left(\sigma_{d_1}(x)\right)\right]] < 0. \end{cases}$$

Proof. By Theorem 2.1 in [23], $(0,0)$, $(v_{d_1}, 0)$ and $(0, v_{d_2})$ are the only semitrivial solutions to the equation in C_1:

$$\begin{cases} -D_1 v_1'' - \alpha_1 v_1' = [g_1(\sigma_{12}(x)) - d_1]v_1, & 0 < x < 1, \\ -D_2 v_2'' - \alpha_2 v_2' = [g_2(\sigma_{12}(x)) - d_2]v_2, & 0 < x < 1, \\ \quad v_i'(0) = v_i'(1) = 0, \;\; i = 1, 2, \end{cases} \tag{5.3.13}$$

where $\sigma_{12}(x) = \exp\left(-k_0 x - \int_0^x [e^{(\alpha_1/D_1)y} v_1(y) + e^{(\alpha_2/D_2)y} v_2(y)]dy\right)$. Now choose neighborhoods U_0, U_1, U_2 of $(0,0), (v_{d_1}, 0), (0, v_{d_2})$ in C_1 respectively, such that $\bar{U}_0, \bar{U}_1, \bar{U}_2$ and $\bar{U}$ are disjoint. It is clear that (5.3.13) does not have any nonnegative solutions in $\bar{\Omega} \cap C_1$. Therefore $\deg_{C_1}(I - S_1|_{C_1}, \Omega \cap C_1, 0)$, $\deg_{C_1}(I - S_1|_{C_1}, U, 0)$ and $\sum_{i=0}^2 \deg_{C_1}(I - S_1|_{C_1}, U_i, 0)$ are all defined and

$$\deg_{C_1}(I - S_1|_{C_1}, \Omega \cap C_1, 0) = \deg_{C_1}(I - S_1|_{C_1}, U, 0) + \sum_{i=0}^{2} \deg_{C_1}(I - S_1|_{C_1}, U_i, 0).$$

It is clear that $\deg_{C_1}(I - S_1|_{C_1}, U_0, 0) = 0$. By Lemma 5.3.2, $\deg_{C_1}(I - S_1|_{C_1}, \Omega \cap C_1, 0) = 1$. We thus have

$$\deg_{C_1}(I - S_1|_{C_1}, U, 0) = 1 - \sum_{i=1}^{2} \deg_{C_1}(I - S_1|_{C_1}, U_i, 0).$$

A use of Lemma 5.3.4 then finishes the proof. □

We call this degree the face index of T_3 and denote it by $\text{index}_f(A, T_3)$. This is well defined because this degree does not depend on the particular choice of the neighbourhood U. Thus we have

$$
\begin{aligned}
&\operatorname{index}_f(A, T_3) \\
&= \begin{cases} 1, & \text{if } d_1 \in (0, -\lambda_1^{(1)}\left[-g_1\left(\sigma_{d_2}(x)\right)\right]) \text{ and } d_2 \in (0, -\lambda_1^{(2)}\left[-g_2\left(\sigma_{d_1}(x)\right)\right]), \\ -1, & \text{if } d_1 \in (-\lambda_1^{(1)}\left[-g_1\left(\sigma_{d_2}(x)\right)\right], d_1^*) \text{ and } d_2 \in (-\lambda_1^{(2)}\left[-g_2\left(\sigma_{d_1}(x)\right)\right], d_2^*), \\ 0, & \text{if } [d_1 + \lambda_1^{(1)}\left[-g_1\left(\sigma_{d_2}(x)\right)\right]] \cdot [d_2 + \lambda_1^{(2)}\left[-g_2\left(\sigma_{d_1}(x)\right)\right]] < 0. \end{cases}
\end{aligned} \tag{5.3.14}
$$

Similarly, we have the results for T_1 and T_2:

$$
\begin{aligned}
&\operatorname{index}_f(A, T_1) \\
&= \begin{cases} 1, & \text{if } d_2 \in (0, -\lambda_1^{(2)}\left[-g_2\left(\sigma_{d_3}(x)\right)\right]) \text{ and } d_3 \in (0, -\lambda_1^{(3)}\left[-g_3\left(\sigma_{d_2}(x)\right)\right]), \\ -1, & \text{if } d_2 \in (-\lambda_1^{(2)}\left[-g_2\left(\sigma_{d_3}(x)\right)\right], d_2^*) \text{ and } d_3 \in (-\lambda_1^{(3)}\left[-g_3\left(\sigma_{d_2}(x)\right)\right], d_3^*), \\ 0, & \text{if } [d_2 + \lambda_1^{(2)}\left[-g_2\left(\sigma_{d_3}(x)\right)\right]] \cdot [d_3 + \lambda_1^{(3)}\left[-g_3\left(\sigma_{d_2}(x)\right)\right]] < 0. \end{cases}
\end{aligned} \tag{5.3.15}
$$

$$
\begin{aligned}
&\operatorname{index}_f(A, T_2) \\
&= \begin{cases} 1, & \text{if } d_1 \in (0, -\lambda_1^{(1)}\left[-g_1\left(\sigma_{d_3}(x)\right)\right]) \text{ and } d_3 \in (0, -\lambda_1^{(3)}\left[-g_3\left(\sigma_{d_1}(x)\right)\right]), \\ -1, & \text{if } d_1 \in (-\lambda_1^{(1)}\left[-g_1\left(\sigma_{d_3}(x)\right)\right], d_1^*) \text{ and } d_3 \in (-\lambda_1^{(3)}\left[-g_3\left(\sigma_{d_1}(x)\right)\right], d_3^*), \\ 0, & \text{if } [d_1 + \lambda_1^{(1)}\left[-g_1\left(\sigma_{d_3}(x)\right)\right]] \cdot [d_3 + \lambda_1^{(3)}\left[-g_3\left(\sigma_{d_1}(x)\right)\right]] < 0. \end{cases}
\end{aligned} \tag{5.3.16}
$$

For any $(\bar{v}_1, \bar{v}_2, 0) \in T_3$, we can easily show that $r(S_2'(\bar{v}_1, \bar{v}_2, 0)|_{C_2}) > 1$ if and only if $d_3 < -\lambda_1^{(3)}\left[-g_3(\bar{\sigma}(x))\right]$, where

$$
\bar{\sigma}(x) = e^{-k_0 x - \int_0^x [e^{(\alpha_1/D_1)y}\bar{v}_1(y) + e^{(\alpha_2/D_2)y}\bar{v}_2(y)]dy}, \quad x \in [0, 1],
$$

and $r(A_2'(\bar{v}_1, \bar{v}_2, 0)|_{C_2}) < 1$ if and only if $d_3 > -\lambda_1^{(3)}\left[-g_3(\bar{\sigma}(x))\right]$. Therefore, by Theorem 1.3.2, we have

$$
\begin{aligned}
&\deg_C\left(I - A, U \times C_2(\varepsilon), 0)\right) \\
&= \begin{cases} 0, & \text{if } d_3 < -\lambda_1^{(3)}\left[-g_3(\bar{\sigma}(x)\right], \text{ for any } (\bar{v}_1, \bar{v}_2, 0) \in T_3, \\ \operatorname{index}_f(A, T_3), & \text{if } d_3 > -\lambda_1^{(3)}\left[-g_3(\bar{\sigma}(x)\right], \text{ for any } (\bar{v}_1, \bar{v}_2, 0) \in T_3. \end{cases}
\end{aligned}
$$

Since the above degree does not depend on the particular choices of U and ε, we call this degree the index of T_3 and denote it by $\text{index}_C(A, T_3)$. We can define $\text{index}_C(A, T_1)$ and $\text{index}_C(A, T_2)$ similarly. So we have

$$\begin{aligned}&\text{index}_C(A, T_3)\\ &= \begin{cases} 0, & \text{if } d_3 < -\lambda_1^{(3)}\left[-g_3(\bar{\sigma}(x)\right], \text{ for any } (\bar{v}_1, \bar{v}_2, 0) \in T_3, \\ \text{index}_f(A, T_3), & \text{if } d_3 > -\lambda_1^{(3)}\left[-g_3(\bar{\sigma}(x)\right], \text{ for any } (\bar{v}_1, \bar{v}_2, 0) \in T_3; \end{cases}\end{aligned} \tag{5.3.17}$$

$$\begin{aligned}&\text{index}_C(A, T_2)\\ &= \begin{cases} 0, & \text{if } d_2 < -\lambda_1^{(2)}\left[-g_2(\hat{\sigma}(x))\right], \text{ for any } (\hat{v}_1, 0, \hat{v}_3) \in T_2, \\ \text{index}_f(A, T_2), & \text{if } d_2 > -\lambda_1^{(2)}\left[-g_2(\hat{\sigma}(x))\right], \text{ for any } (\hat{v}_1, 0, \hat{v}_3) \in T_2; \end{cases}\end{aligned} \tag{5.3.18}$$

$$\begin{aligned}&\text{index}_C(A, T_1)\\ &= \begin{cases} 0, & \text{if } d_1 < -\lambda_1^{(1)}\left[-g_1(\tilde{\sigma}(x)\right], \text{ for any } (0, \tilde{v}_2, \tilde{v}_3) \in T_1, \\ \text{index}_f(A, T_1), & \text{if } d_1 > -\lambda_1^{(1)}\left[-g_1(\tilde{\sigma}(x)\right], \text{ for any } (0, \tilde{v}_2, \tilde{v}_3) \in T_1, \end{cases}\end{aligned} \tag{5.3.19}$$

where

$$\begin{aligned}\hat{\sigma}(x) &= e^{-k_0 x - \int_0^x [e^{(\alpha_1/D_1)y}\hat{v}_1(y) + e^{(\alpha_3/D_3)y}\hat{v}_3(y)]dy}, \\ \tilde{\sigma}(x) &= e^{-k_0 x - \int_0^x [e^{(\alpha_2/D_2)y}\tilde{v}_2(y) + e^{(\alpha_3/D_3)y}\tilde{v}_3(y)]dy}.\end{aligned}$$

Lemma 5.3.7 *Suppose* $\text{index}_C(A, w_i)$ *and* $\text{index}_C(A, T_i)$ $(i = 1, 2, 3)$ *are well defined, and*

$$\sum_{i=1}^{3} \text{index}_C(A, w_i) + \sum_{i=1}^{3} \text{index}_C(A, T_i) \neq 1.$$

Then (5.2.1) *has at least one positive solution.*

Proof. Suppose that (5.2.1) has no positive solution and $\text{index}_C(A, w_i)$ $(i = 0, 1, 2, 3)$ and $\text{index}_C(A, T_j)$ $(j = 1, 2, 3)$ are well defined. Then by the additivity of the degree

$$\deg_C(I - A, \Omega, 0) = \sum_{i=0}^{3} \text{index}_C(A, w_i) + \sum_{j=1}^{3} \text{index}_C(A, T_j).$$

It follows from Lemma 5.3.2 and $\text{index}_C\,(A,(0,0,0)) = 0$ that

$$1 = \sum_{i=1}^{3} \text{index}_C\,(A, w_i) + \sum_{i=1}^{3} \text{index}_C\,(A, T_i)\,.$$

This leads to a contradiction. □

Before going further, we prove the following comparison lemma.

Lemma 5.3.8 *Suppose that* v_{d_i} $(0 < d_i < d_i^*,\ i = 1,2,3)$ *is the unique positive solution of* (5.3.9). *Then for any* $(v_1, v_2, v_3) \in T_1 \cup T_2 \cup T_3$*, we have*

$$\int_0^x e^{(\alpha_i/D_i)y} v_i(y)dy \le \int_0^x e^{(\alpha_i/D_i)y} v_{d_i}(y)dy \ \ \textit{for any}\ \ x \in [0,1],\ \ i = 1,2,3. \tag{5.3.20}$$

Proof. We prove the case for $i = 1$. The proofs for $i = 2$ and $i = 3$ are similar. Note that v_{d_1} satisfies

$$\begin{cases} -D_i v_{d_1}'' - \alpha_1 v_{d_1}' = \left[g_1\left(\exp\left[-k_0 x - \int_0^x e^{(\alpha_1/D_1)y} v_{d_1}(y)dy\right]\right) - d_1\right] v_{d_1},\ 0 < x < 1, \\ v_{d_1}'(0) = v_{d_1}'(1) = 0. \end{cases}$$

For any $(v_1, v_2, v_3) \in T_1 \cup T_2 \cup T_3$, v_1 satisfies

$$\begin{cases} -D_1 v_1'' - \alpha_1 v_1' \le \left[g_1\left(\exp\left[-k_0 x - \int_0^x e^{(\alpha_1/D_1)y} v_1(y)dy\right]\right) - d_1\right] v_1,\ \ 0 < x < 1, \\ v_1'(0) = v_1'(1) = 0. \end{cases}$$

Choose $\theta(x) > v_1(x)$ for all $x \in [0,1]$. It is not difficulty to prove that the parabolic problem

$$\begin{cases} v_t = D_1 v'' + \alpha_1 v' + \left[g_1\left(\exp\left[-k_0 x - \int_0^x e^{(\alpha_1/D_1)y} v(y)dy\right]\right) - d_1\right] v, \\ v_x(0,t) = v_x(1,t) = 0,\ \ v(x,0) = \theta(x) \end{cases} \tag{5.3.21}$$

has a unique positive solution $v(x,t)$ for all $t \ge 0$. We may now find a small $\delta > 0$ such that $v_1(x) < v(x,t)$ for all $x \in [0,1]$ and $t \in [0,\delta)$. Note that $v_1(x,t) \equiv v_1(x)$ satisfies

$$\begin{cases} (v_1)_t \le D_1 (v_1)_{xx} + \alpha_1 (v_1)_x = \left[g_1\left(\exp\left[-k_0 x - \int_0^x e^{(\alpha_1/D_1)y} v_1(y)dy\right]\right) - d_1\right] v_1, \\ (v_1)_x(0,t) = (v_1)_x(1,t) = 0,\ \ v_1(x,0) = v_1(x). \end{cases} \tag{5.3.22}$$

By Lemma 4.1 of [23] we have

$$\int_0^x e^{(\alpha_1/D_1)y} v_1(y)dy < \int_0^x e^{(\alpha_1/D_1)y} v(y,t)dy \quad \text{for all } \ t \geq 0 \ \text{ and } \ x \in (0,1]. \tag{5.3.23}$$

By Theorem 2.2 of [23] we have

$$\lim_{t\to\infty} v(x,t) = v_{d_1}(x) \ \text{ uniformly for } \ x \in [0,1].$$

Then (5.3.20) follows by letting $t \to \infty$ in (5.3.23). □

We are now ready to prove Theorem 5.2.2.

Proof. [Proof of Theorem 2.2] From (5.2.13), we have

$$0 < d_i < -\lambda_1^{(i)}[-g_i(\sigma_{d_1}(x))], \ i = 2,3,$$
$$0 < d_j < -\lambda_1^{(j)}[-g_j(\sigma_{d_2}(x))], \ j = 3,1,$$
$$0 < d_k < -\lambda_1^{(k)}[-g_k(\sigma_{d_3}(x))], \ k = 1,2.$$

By Lemma 5.3.4 we have $w_i, i = 1,2,3$ are all isolated. Therefore $\text{index}_C\,(A, w_i)\,, i = 1,2,3$ are all well-defined and

$$\text{index}_C\,(A, w_i) = 0, \ \ i = 1,2,3.$$

Moreover, by Lemma 5.3.5 T_1, T_2 and T_3 are all compact sets hence $\text{index}_C\,(A, T_i)\,, i = 1,2,3$ are all well-defined.

For any $(v_1, v_2, v_3) \in T_1 \cup T_2 \cup T_3$, we have by Lemma 5.3.8

$$0 \leq \int_0^x e^{(\alpha_i/D_i)y} v_i(y) \leq \int_0^x e^{(\alpha_i/D_i)y} v_{d_i}(y), \ \ x \in [0,1], \ \ i = 1,2,3. \tag{5.3.24}$$

Hence

$$d_3 < -\lambda_1^{(3)}\left[-g_3(\sigma_{d_1d_2}(x))\right] \leq -\lambda_1^{(3)}\left[-g_3(\bar{\sigma}(x))\right], \ \text{ for any } \ (\bar{v}_1, \bar{v}_2, 0) \in T_3.$$

Thus, (5.3.17) implies

$$\text{index}_C\,(A, T_3) = 0.$$

In the same way, we have

$$\text{index}_C\,(A, T_1) = \text{index}_C\,(A, T_2) = 0.$$

Therefore,

$$\sum_{i=1}^{3} \text{index}_C\,(A, w_i) + \sum_{i=1}^{3} \text{index}_C\,(A, T_i) = 0 \neq 1.$$

It follows from Lemma 5.3.7 that (5.2.1) has at least one positive solution.

The proof of Theorem 5.2.2 is now complete. □

5.4 The n-species case

Our methods for the three-species system can be generalized to the general n-species $(n \geq 3)$ case. We only list the important results and give necessary explanations. The proofs of these results are simple extensions of those of the three species case.

Theorem 5.4.1 *If* (5.1.8) *has a positive steady state solution, then*

$$d_i \in (0, d_i^*) \;\; \textit{where} \;\; d_i^* = -\lambda_1^{(i)}[-g_i(e^{-k_0 x})], \; i = 1, 2, \cdots, n.$$

Theorem 5.4.2 *(i) If there is an $i \in \{1, 2, \cdots, n\}$ such that $d_i > \int_0^1 g_i(e^{-k_0 x})dx$, then there exists a constant $D > 0$, such that if $\min\{D_1, D_2, \cdots, D_n\} \geq D$ then* (5.1.8) *has only the trivial steady state solution.*

(ii) If $d_i \in (0, \int_0^1 g_i(e^{-k_0 x})dx]$ for all $i = 1, 2, \cdots, n$, then there exists a positive constant D such that if $\min\{D_1, D_2, \cdots, D_n\} \geq D$, (5.1.8) *has no positive steady state solution except possibly when the following exceptional situation occurs:*
there exists a constant $c \geq 0$ such that

$$c_1 = c_2 = \cdots = c_n = c,$$

where c_i is uniquely determined by

$$d_i = \int_0^1 g_i(e^{-(k_0+c_i)x})dx, \; i = 1, 2, \cdots, n. \tag{5.4.1}$$

Theorem 5.4.3 *Suppose that*

$$0 < d_i < -\lambda_1^{(i)}[-g_i(\kappa_i(x))],\ i = 1, 2, \cdots, n, \tag{5.4.2}$$

where

$$\kappa_i(x) = e^{-k_0 x} \exp\left(\sum_{j \neq i} \int_0^x e^{(\alpha_j/D_j)y} v_{d_j}(y)dy\right),\ i = 1, 2, \cdots, n,$$

and v_{d_j} is the unique positive solution of the equation

$$-D_j v'' - \alpha_j v' = \left[g_j\left(e^{-k_0 x - \int_0^x e^{(\alpha_j/D_j)y} v(y)dy}\right) - d_j\right] v,\ v'(0) = v'(1) = 0,\ j = 1, 2, \cdots, n.$$

Then (5.1.8) *has at least one positive steady state solution.*

Theorem 5.4.4 *Suppose that $\alpha_1, \alpha_2, \cdots, \alpha_n \in R^1$ and at least $n-1$ of the $\alpha_i (i = 1, 2, \cdots, n)$ are nonpositive. Then we can choose suitable $(D_1, D_2, \cdots, D_n)$ and $(d_1, d_2, \cdots, d_n)$ such that* (5.4.2) *holds, and hence there is at least one positive steady state solution to* (5.1.8).

Proof. We use induction to prove the theorem. Without loss of generality, we may assume $\alpha_j \leq 0, j = 2, 3, \cdots, n$.

By Theorem 5.2.3, Theorem 5.4.4 is valid for $n = 3$. Suppose Theorem 5.4.4 is valid for $m (\geq 3)$, we show it is also valid for $m + 1$. Theorem 5.4.4 is valid for m means there exist suitable $(D_1, \cdots, D_m)$ and $(d_1, \cdots, d_m)$ such that

$$0 < d_i < -\lambda_{D_i}\left[-g_i\left(\exp\left[-k_0 x - \sum_{j=1,\cdots,m, j\neq i} \int_0^x v_{D_j,d_j}(s) e^{(\alpha_j/D_j)s} ds\right]\right)\right],\ i = 1, \cdots, m. \tag{5.4.3}$$

Choose $\epsilon > 0$ such that

$$0 < d_i < -\lambda_{D_i}\left[-g_i\left(\exp\left[-k_0 x - \sum_{j=1,\cdots,m, j\neq i} \int_0^x v_{D_j,d_j}(s) e^{(\alpha_j/D_j)s} ds - \epsilon\right]\right)\right],\ i = 1, \cdots, m.$$

By (5.2.17) and (5.2.18) we can choose $d_{m+1} > 0$ such that $g_{m+1}(1) - d_{m+1} > 0$ is small enough such that

$$\int_0^1 v_{D_{m+1},d_{m+1}}(x) e^{(\alpha_{m+1}/D_{m+1})x} dx \leq \epsilon \text{ for any } D_{m+1} > 0.$$

Thus for any $D_{m+1} > 0$ we have for all $i = 1, \cdots, m$,

$$0 < d_i < -\lambda_{D_i}\left[-g_i\left(\exp\left[-k_0 x - \sum_{j=1,\cdots,m+1, j\neq i}\int_0^x v_{D_j,d_j}(s)e^{(\alpha_j/D_j)s}ds\right]\right)\right]. \qquad (5.4.4)$$

By (5.2.14), with $D_1, \cdots, D_m, d_1, \cdots, d_m, d_{m+1}$ fixed, we have

$$\lim_{D_{m+1}\to 0} -\lambda_{D_{m+1}}\left[-g_{m+1}\left(\exp\left[-k_0 x - \sum_{j=1,\cdots,m}\int_0^x v_{D_j,d_j}(s)e^{(\alpha_j/D_j)s}ds\right]\right)\right] = g_{m+1}(1).$$

Note that $d_{m+1} \in (0, g_{m+1}(1))$. We can choose D_{m+1} sufficiently small such that

$$0 < d_{m+1} < -\lambda_{D_{m+1}}\left[-g_{m+1}\left(\exp\left[-k_0 x - \sum_{j=1,\cdots,m}\int_0^x v_{D_j,d_j}(s)e^{(\alpha_j/D_j)s}ds\right]\right)\right].$$

Since (5.4.4) is not affected by the choice of D_{m+1}, we have for all $i = 1, \cdots, m+1$,

$$0 < d_i < -\lambda_{D_i}\left[-g_i\left(\exp\left[-k_0 x - \sum_{j=1,\cdots,m+1, j\neq i}\int_0^x v_{D_j,d_j}(s)e^{(\alpha_j/D_j)s}ds\right]\right)\right]. \qquad (5.4.5)$$

The proof is complete. □

Similarly we can extend Theorem 5.2.4 to obtain

Theorem 5.4.5 *Suppose $D_1, D_2, \cdots, D_n$ are fixed. We can choose suitable $\alpha_1, \alpha_2, \cdots, \alpha_n$ and $d_1, d_2, \cdots, d_n$ such that* (5.4.2) *holds, and hence there exists at least one positive steady state solution to* (5.1.8).

Bibliography

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] S. Agmon, A. Douglis, L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary condtions I*, Comm. Pure Appl. Math. 12 (1959), 623-727.

[3] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Review, 18 (1976), 620-709.

[4] R. A. Armstrong, R. McGree, *Competitive exclusion*, Amer. Naturalist, 115 (1980), 151-170.

[5] J. Blat, K. J. Brown, *Global bifurcation of positive solutions in some systems of elliptic equations*, SIAM J. Math. Anal., 17 (1986), 1339-1353.

[6] F. Belgacem, Elliptic Boundary Value Problems with Indefinite Weights: Variational Formulations of the Principal Eigenvalue and Applications, Pitman Res, Notes Math. Ser. 368, Longman Sci., 1997.

[7] R. S. Cantrell, C. Cosner, Spatial Ecology via Reaction-Diffusion Equations, Series in Mathematical and Computational Biology, John Wiley and Sons, Chichester, UK, 2003.

[8] X. Chen, Y. Lou, *Principal eigenvalue and eigenfunctions of an elliptic operator with large advection and its application to a competition model*, Indiana Univ. Math. J., 57 (2008), 627-658.

[9] E. A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill Book Company, Inc., New York. 1955.

[10] M. G. Crandall, P. H. Rabinowitz, *Bifurcation from simple eigenvalues*, J. Funct. Anal., 8 (1971), 321-340.

[11] E. N. Dancer, *On the indices of fixed points of mappings in cones and applications*, J. Math. Anal. Appl., 91 (1983), 131-151.

[12] E. N. Dancer, *On positive solutions of some pairs of differential equatios*, Trans. Amer. math. Soc., 284 (1984), 729-743.

[13] E. N. Dancer, *On positive solutions of some pairs of differential equatios, II*, J. Differential Equations, 60 (1985), 236-258.

[14] E. N. Dancer, Y. Du, *Positive solutions for a three-species competition system with diffusion – I. General existence results*, Nonlinear analysis, 24 (1995), 337-357.

[15] K. Deimling, *Nonlinear Functional Analysis*, Springer-verlag, Berlin, 1985.

[16] Y. Du, *Positive periodic solutions of a competitor-competitor-mutualist model*, Differential and Integral Equations, 9 (1996), 1043-1066.

[17] Y. Du, *Order Structure and Topological Methods in Nonlinear Partial Differential Equations, Vol 1: Maximum Principles and Applications*, World Scientic Publishing Co. Pte. Ltd., 2006.

[18] Y. Du, *Bifurcation from semitrivial solution bundles and applications to certain equation systems*, Nonlinear Anal., 27 (1996), 1407-1435.

[19] Y. Du, *A degree theoretic approach to N-species periodic competition systems on the whole R^N*, Proc. Roy. Soc. Edinburgh Sect. A, 129 (1999), 295-318.

[20] Y. Du, S.-B. Hsu, *Concentration phenomena in a nonlocal quasi-linear problem modeling phytoplankton I: Existence*, SIAM J. Math. Anal., 40 (2008), 1419-1440.

[21] Y. Du, S.-B. Hsu, *Concentration phenomena in a nonlocal quasi-linear problem modeling phytoplankton II: Limiting profile*, SIAM J. Math. Anal., 40 (2008), 1441-1470.

[22] Y. Du, S.-B. Hsu, *On a nonlocal reaction-diffusion problem arising from the modeling of phytoplankton growth*, SIAM J. Math. Anal., 42 (2010), 1305-1333.

[23] Y. Du, L. Mei, *On a nonlocal reaction-diffusion-advection equation modelling phytoplankton dynamics*, Nonlinearity, 24 (2011), 319-349.

[24] U. Ebert, M. Arrayas, N. Temme, B. Sommeijer, J. Huisman, *Critical condition for phytoplankton blooms*, Bull. Math. Biol., 63 (2001), 1095-1124.

[25] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice Hall, Eaglewood Cliffs, NJ, 1964.

[26] D. J. Gerla, W.M. Wolf, J. Huisman, *Photoinhibition and the assembly of light-limited phytoplankton communities*, Oikos (in press).

[27] D. Gilberg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 1998.

[28] J. K. Hale, P. Waltman, *persistence in infinite-dimensional systems*, SIAM J. Math. Anal., 20 (1989).

[29] L. Hsiao, P. De Mottoni, *Persistence in reaction-diffusion systems: interaction of two predators and one prey*, Nonlinear Anal., 11 (1987), 877-891.

[30] G. E. Hustchison, *The paradox of the plankton*, Amer. Naturalist, 95 (1961), 137-145.

[31] S.-B. Hsu, *Limiting behavior for competing species*, SIAM J. Appl. Math., 34 (1978), 760-763.

[32] S.-B. Hsu, S. Hubbell, P. Waltman, A mathematical theory for single-nutrient competition in continuous cultures of micro-organisms, SIAM J. Appl. Math., 32 (1977), 366-383.

[33] S.-B. Hsu, Y. Lou, *Single phytoplankton species growth with light and advection in a water column*, SIAM J. Appl. Math., 70 (2010), 2942-2974.

[34] J. Huisman, M. Arrayas, U. Ebert, B. Sommeijer, *How do sinking phytoplankton species manage to persist*? American Naturalist, 159 (2002), 245-254.

[35] J. Huisman, P. van Oostveen, F.J. Weissing, *Species dynamics in phytoplankton blooms: incomplete mixing and competition for light*, American Naturalist, 154 (1999), 46-67.

[36] J. Huisman, P. van Oostveen, F.J. Weissing, *Critical depth and critical turbulence: Two different mechanisms for the development of phytoplankton blooms*, Limnol. Oceanogr., 44 (1999), 1781-1787.

[37] J. Huisman, N. N. Pham Thi, D.M. Karl, B. Sommeijer, *Reduced mixing generates oscillations and chaos in oceanic deep chlorophyll maxima*, Nature, 439 (2006), 322-325.

[38] J. Huisman, F. J. Weissing, *Light-limited growth and cometition for light in well-mixed aquatic environments: An elementary model*, Ecology, 75 (1994), 507-520.

[39] J. Huisman, F. J. Weissing, *Competition for nutrients and light in a mixed water column: a theoretical analysis*, American Naturalist, 146 (1995), 536-564.

[40] H. Ishii, I. Takagi, *Global stability of stationary solutions to a nonlinear diffusion equation in phytoplankton dynamics*, J. Math. Biology, 16 (1982), 1-24.

[41] H. Ishii, I. Takagi, *A nonlinear diffusion equation in phytoplankton dynamics with self-shading effect*, Mathematics in Biology and Medicine (Bari, 1983), Lecture Notes in Biomath., 57, Springer, Berlin, 1985, pp66-71.

[42] C. A. Klausmeier, E. Litchman, *Algal games: The vertical distribution of phytoplankton in poorly mixed water columns*, Limnol. Oceanogr., 46 (2001), 1998-2007.

[43] C. A. Klausmeier, E. Litchman, S. A. Levin, *Phytoplankton growth and stoichiometry under multiple nutrient limitation*, Limnol. Oceanogr., 49 (2004), 1463-1470.

[44] T. Kolokolnikov, C. H. Ou, Y. Yuan, *Phytoplankton depth profiles and their transitions near the critical sinking velocity*, J. Math. Biol., 59 (2009), 105-122.

[45] O. A. Ladyzenskaja, V. A. Solonnikov, N. N. Uralceva, *Linear and Quasilinear Equations of Parabolic Type*, AMS, Rhode Island: Providence, 1968.

[46] N. Lakos, *Existence of steady-state solutions for a one-predator two-prey system*, SIAM J. Math. Anal., 21 (1990), 647-659.

[47] A. Leung, *A study of 3-species prey-predator reaction-diffusions by monotone schemes*, J. Math. Anal. Appl., 100 (1984), 583-604.

[48] G. M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific, Singapore, 2005.

[49] E. Litchman, C. A. Klausmeier, J.R. Miller, O.M. Schofield, P.G. Falkowski, *Multinutrient, multi-group model of present and future oceanic phytoplankton communities*, Biogeosciences, 3 (2006), 585-606.

[50] L. Mei, X. Zhang, *On a nonlocal reaction-diffusion-advection system modeling phytoplankton growth with light and nutrients*, Discrete and continous dynamical systems-B, 17 (2012), 221 - 243

[51] J. P. Mellard, K. Yoshiyama, E. Litchman, C.A. Klausmeier, *The vertical distribution of phytoplankton in stratified water columns*, J. Theor. Biology, 269 (2011), 16-30.

[52] J. D. Murray, *Mathematical Biology*, 2nd Edition, Springer-Verlag, 1993.

[53] C. V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York, 1992.

[54] M. H. Protter, H. F. Weinberger, *Maximum Principles in Differential Equations*, 2nd Edition, Springer-Verlag, Berlin, 1984.

[55] P. H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal., 7 (1971), 487-513.

[56] G. A. Riley, H. Stommel, D. F. Bumpus, *Quantitative ecology of the plankton of the western North Atlantic*, Bull. Bingham oceanogr. Collect. 12 (1949), 1-169.

[57] A. B. Ryabov, L. Rudolf and B. Blasius, *Vertical distribution and composition of phytoplankton under the influence of an upper mixed layer*, J. Theor. Biol., 263 (2010), 120-133.

[58] N. Shigesada and A. Okubo, *Analysis of the self-shading effect on algal vertical distribution in natural waters*, J. Math. Biol., 12(1981), 311-326.

[59] H. Smith, *Monotone Dynamic Systems: an introduction to the theory of competitive and cooperative systems*, Mathematical Surveys and Monographs, Vol 41, American Mathematical Society, RI, 1995.

[60] D. Tilman, *Resource Competition and Community Structure*, Princeton University Press, Princeton, NJ, 1982.

[61] S. Torato, *Mutual shading effect on algal distribution: A nonlinear problem*, Nonlinear Anal., 13 (1989), 969-986.

[62] F. J. Weissing, J. Huisman, Growth and competition in a lighted gradient, J. Theoret. Biol., 168 (1994), 323-336.

[63] K. Yoshiyama, J. P. Mellard, E. Litchman and C. A. Klausmeier, *Phytoplankton competition for nutrients and light in a stratified water column*, American Naturalist, 174 (2)(2009), 190-203.

[64] K. Yoshiyama and H. Nakajima, *Catastrophic transition in vertical distributions of phytoplankton: alternative equilibria in a water column*, J. Theor. Biol., 216 (2002), 397-408.

[65] A. Zagaris, A. Doelman, N. N. Pham Thi and B.P. Sommeijer, *Blooming in a nonlocal, coupled phytoplankton-nutrient model*, SIAM J. Appl. Math., 69(2009), 1174-1204.

[66] X. Q. Zhao, *Dynamical Systems in Population Biology*, Springer-Verlag, New York, 2003.