REACTION DIFFUSION EQUATIONS ARISING FROM PHYTOPLANKTON DYNAMICS

By

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DECLARATION

I certify that the substance of this thesis has not already been submitted for any degree and is not currently being submitted for any other degree.

I certify that to the best of my knowledge, any help received in preparing this thesis, and all sources used, have been acknowledged in this thesis.

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Publications

Journal papers published

1. Wei Dong, Linfeng Mei, *Multiple solutions for an indefinite superlinear elliptic problem on* $\mathbb{R}^N$, Nonlinear Analysis: Theory, Methods & Applications, 73 (2010), 2056-2070.

2. Yihong Du, Linfeng Mei, *On a nonlocal reaction-diffusion-advection equation modelling phytoplankton dynamics*, Nonlinearity, 24 (2011), 319-349.

3. Linfeng Mei, Xiaoyan Zhang, *On a nonlocal reaction-diffusion-advection system modeling phytoplankton growth with light and nutrients*, Discrete and Continuous Dynamical Systems - Series B, 17 (2012), 221-243.

4. Linfeng Mei, Xiaoyan Zhang, *Existence of positive stationary solutions for a diffusive variable-territory predator-prey model*, IMA Journal of Applied Mathematics, published online December 2011, doi:10.1093/imamat/hxr059.

Papers submitted/in preparation

1. Linfeng Mei, Xiaoyan Zhang, *Coexistence and competitive exclusion in multi-species phytoplankton dynamics*, submitted

Academic activities

(1) The Inaugural Pacific Rim Mathematical Association (PRIMA) Congress, held in University of New South Wales, Sydney, Australia, July 6-10, 2009.

(2) 20 minutes contributed talk in Workshop On PDE Models Of Biological Process, held in National Hsing-Hua University, Taiwan, December 13-17, 2010.

Talk title: Limiting profiles of a phytoplankton model.

(3) The 8th East China Partial Differential Equations Conference & The 2th International Workshop On Reaction-Diffusion Models And Mathematical Biology, held in Xi'an, China, July 11-14.

(4) 30 minutes contributed talk in the 55th Annual Meeting of the Australian Mathematical Society, University of Wollongong, Wollongong, Australia, September 26-29, 2011.

Talk Title: On a nonlocal equation modeling phytoplankton growth.

Abstract

Partial differential equation theory, especially the theory of reaction-diffusion equations, has long been found extremely useful in the qualitative study of theoretical ecology. The interplay between partial differential equations and mathematical ecology not only benefits ecology greatly, it is also one of the important sources from which many beautiful and challenging mathematical models have been derived and henceforth helps to develop the theory of partial differential equations.

In this thesis, we are concerned with the mathematical study of a basic model proposed by ecologists describing the ecological behaviors of phytoplankton in eutrophic environments. The model is viewed by some ecologists as a very realistic model to describe the behavior of phytoplankton. On the other hand, the model involves a nonlocal term and is mathematically challenging, and not much mathematical research has been carried out on it. This thesis is an attempt to further the understanding of this model. It consists of five chapters.

In Chapter 1, we collect some basic mathematical facts that the thesis will be based on. These facts include the existence and uniqueness theorem of ordinary differential equations, the definitions of Hölder spaces and Sobolev spaces and the embedding theorems, the L^p estimates of partial differential equations, the principal eigenvalues and the maximum principles, as well as topological degree theory and bifurcation theorems.

In Chapter 2, we first give a brief introduction on the biological background of our model and its mathematical formulation. Then we recall the existing mathematical works on the model and briefly outline the work we have done.

In Chapter 3. We study the one species model:

$$\begin{cases} u_t = J_x(x,t) + \left[g\left(e^{-k_0 x - k\int_0^x u(s,t)ds}\right) - d\right]u, & 0 < x < h, \ \ t > 0, \\ J(x,t) = D(x)u_x(x,t) - \alpha(x)u(x,t) = 0, & x = 0 \text{ or } h, \ \ t > 0, \\ u(x,0) = u_0(x) \gneqq 0, & 0 \le x \le h, \end{cases}$$

where $g(I)$ is a smooth function satisfying

$$g(0) = 0, \qquad g'(I) > 0 \ \text{ for } \ I > 0,$$

and for any $\gamma > 0$

$$\int_0^\infty g(e^{-\gamma x})dx < \infty.$$

We prove there is a critical d^* such that when $0 < d < d^*$, the above equation has a unique positive steady state u_d^*, and when $d \ge d^*$, it has no positive steady state. Moreover, when $0 < d < d^*$, u_d^* is the global attractor, when $d \ge d^*$, 0 is the global attractor. We also prove in this chapter that when the diffusion coefficient approaches zero the phytoplankton species will behave like a δ-function, concentrating at the bottom of the water column; when the diffusion coefficient is very large, the phytoplankton species will distribute evenly in the water column. We aslo study in this chapter the behavior of the phytoplankton species when the depth of the water column goes to infinite, and find that the distribution of the phytoplankton species converges to that studied by Ishii and Takagi [40]. As a byproduct, we solved a problem left open in Hsu and Lou [33].

In Chapter 4 and Chapter 5, we study the behavior of multiple species phytoplankton that live in the same vertical water column:

$$(u_i)_t = (D_i(x)\,(u_i)_x - \alpha_i(x)u_i)_x + (g_i(I(x,t)) - d_i)\,u_i, \ i = 1, \cdots, n, \tag{0.0.1}$$

with zero-flux boundary conditions

$$D_i(x)(u_i)_x(x,t) - \alpha_i(x)u_i(x,t) = 0, \ x = 0, h, \ t \ge 0, \ \ i = 1, \cdots, n,$$

and initial conditions

$$u_i(x,0) = u_i^0(x) \gneqq 0, \ 0 \le x \le h, \ \ i = 1, \cdots, n,$$

$g_i(I)$ satisfies

$$g_i(0) = 0, \;\; g_i'(I) > 0 \text{ for } I \geq 0,$$

and there are positive constants c_i, γ_i such that

$$g_i(I) \leq c_i I^{\gamma_i} \;\text{ for any }\; I \geq 0,$$

$$I(x,t) = I_0 e^{-k_0 x} \exp\left(- \int_0^x [k_1 u_1(s,t) + \cdots + k_n u_n(s,t)]\, ds \right).$$

In Chapter 4, we study the existence of positive steady states for the two species model. Among other things, we use the topological degree argument to find a group of sufficient conditions under which the model has at least one positive steady state. We also study the necessary conditions for the existence of positive steady states and show that the sufficient conditions are not necessary.

In Chapter 5, we study the existence of positive steady state of the more than two species model. As in Chapter 4, we find a group of sufficient conditions for the existence of the positive steady state for the multiple species mode in terms of principal eigenvalues:

$$0 < d_i < -\lambda_1^{(i)}[-g_i(\kappa_i(x))], \; i = 1, 2, \cdots, n, \tag{0.0.2}$$

where

$$\kappa_i(x) = e^{-k_0 x} \exp\left(\sum_{j \neq i} \int_0^x u_{d_j}(y) dy \right), \; i = 1, 2, \cdots, n,$$

$\lambda_1^{(i)}(\Psi)$ is the principal eigenvalue of the eigenvalue problem

$$-D_j \phi'' + \alpha_i \phi' + \Psi \phi = \lambda \phi, \;\; D_j \phi'(0) - \alpha_j \phi'(0) = D_j \phi'(1) - \alpha_j u \phi'(1) = 0,$$

and u_{d_j} is the unique positive solution of the equation

$$-D_j u'' + \alpha_j u' = \left[g_j \left(e^{-k_0 x - \int_0^x u(y) dy} \right) - d_j \right] u, \; D_j u'(0) - \alpha_j u'(0) = D_j u'(1) - \alpha_j u'(1) = 0.$$

Here we assume for simplicity, $D_j(x) \equiv D_j, \alpha_j(x) \equiv \alpha_j$ for $j = 1, \cdots, n$. The derivation of this sufficient condition is more complicated, and cannot use the topological degree argument in Chapter 4 directly. It is derived from a fixed point index calculation technique developed by Dancer and Du in [14].

A considerable part of Chapter 5 is devoted to the explanation of the rather implicit sufficient condition (0.0.2). By developing an idea from Hsu and Lou [33], we use the fine properties of the eigenvalues to find two groups of explicit conditions under which (0.0.2) is satisfied. Interestingly, one group of conditions require the diffusions of the system to be sufficiently small, as opposed to the situations that when the diffusions of the system is sufficiently large, the system cannot have any positive steady states. This explains from one angle the famous paradox of plankton.

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