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A uniformity criterion for vector bundles on complex projective spaces

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1. Introduction

Let $E \to \mathbf{CP}_n, n \geq 2$, be a holomorphic vector bundle of rank r. For each projective line $l \subset \mathbf{CP}_n$, the splitting theorem of Grothendieck indicates that

$$E\mid_l \cong \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbf{P}_1}(a_i),$$

where $a_E(l) = (a_1, ..., a_r) \in \mathbf{Z}^r$, is uniquely determined such that $a_1 \geq a_2 \geq ... \geq a_r$. Define

$$a_E = \inf_{l \subset \mathbf{CP}_n} a_E(l)$$

under the lexicographic ordering of r-tuples $a_E(l)$. If $\mathbf{Gr}(2, n+1)$ denotes the Grassmann manifold of complex two-planes, let $\mathbf{F}$ be the associated incidence manifold of the " standard construction" (cf. eg., [8]):

$$\begin{array}{ccc} & \mathbf{F} & \\ {\scriptstyle \mu} \swarrow & & \searrow {\scriptstyle \nu} \\ \mathbf{CP}_n & & \mathbf{Gr}(2, n+1) \end{array}$$

where, for each $x \in \mathbf{Gr}(2, n+1)$, $\mu \circ \nu^{-1}(x) = l \subset \mathbf{CP}_n$. Moreover (cf. [8], Lemma 3.2.2),

$$A = \{x \in \mathbf{Gr}(2, n+1) \mid a_E(\mu \circ \nu^{-1}(x)) > a_E\}$$

is a closed analytic subset, called the "jumping locus" of E. In particular, E is said to be *uniform* if $A = \emptyset$. A key idea in the following section will be to consider the complex manifold $\mathcal{X} = \mathbf{P}(\mu^* E)$, and let

$$\mathcal{X} \xrightarrow{f} \mathbf{Gr}(2, n+1)$$

be the holomorphic family induced by ν, such that for all $x \in \mathbf{Gr}(2, n+1)$,

$$f^{-1}(x) = \mathbf{P}(\mu^* E \mid_{\nu^{-1}(x)}).$$

The main result of this note is the following, which addresses the question of uniformity for rank-two bundles E in terms of the complex codimension c of A:

Theorem: *If $E \to \mathbf{CP}_n$ is a holomorphic vector bundle of rank two, then the codimension of the jumping locus $A \subset \mathbf{Gr}(2, n+1)$ is at most two.*

In fact, when E is a semistable vector bundle of rank two over $\mathbf{CP}_n$, considerably more information about the jumping locus is already known. For example, if the first chern class, $c_1(E) = 0$, then a theorem of W. Barth (cf. [8], theorem 2.2.3) indicates that A supports a determinantal divisor, of which the degree is given by $c_2(E)$. When E is stable, the codimension of A is one for $c_1(E)$ even, and at most two for $c_1(E)$ odd, unless in either case $A = \emptyset$ (cf. L. Gruson and C. Peskine [5]). Examples of stable rank-two vector bundles with $c_1(E) = -1$ for which A has codimension two were first given by Hulek [6]. Related results on uniformity for bundles of higher rank over $\mathbf{CP}_n$ may be found in Ballico [1], Brieskorn [2] and Leiterer [7]. Without the assumption of semi-stability, the uniformity of a holomorphic bundle E is seen to be a corollary of the following:

Theorem: *Let $\mathbf{F} \xrightarrow{\pi} X$ be a holomorphic $\mathbf{CP}_1$-bundle over a complex manifold X, and $A \subset X$ a closed analytic subset. If $E \to \mathbf{F}$ is a holomorphic vector bundle of rank two such that for all $x \in X \setminus A$, $E \mid_{\pi^{-1}(x)}$ has uniform splitting type, (a, b) with $a > b$ then $c = codim_X(A) \geq 3 \Rightarrow E$ splits uniformly on $\pi^{-1}(x)$ for all $x \in X$.*

This result in turn is a straightforward consequence of the local triviality of the projectivised family $\mathcal{X} = \mathbf{P}(E) \xrightarrow{f} X$ when restricted to $X \setminus A$, a theorem of Scheja [9] on extension of cohomology classes, and Hartogs' removable singularities theorem applied to holomorphic vector fields.

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2. Uniformity criterion

We will begin with the following

Lemma 1. *Let X be a complex manifold, and $\mathbf{F} \xrightarrow{\pi} X$ a holomorphic $\mathbf{CP}_1$-bundle with relative ample sheaf $\mathcal{O}_{\mathbf{F}}(1)$ such that*

$$\mathcal{O}_{\mathbf{F}}(1) \mid_{\pi^{-1}(x)} \cong \mathcal{O}_{\mathbf{P}_1}(1).$$

Let $A \subset X$ be a closed analytic subset of complex codimension $c \geq 3$, and $\mathcal{V} \to \pi^{-1}(X \setminus A)$ a locally free sheaf such that $\mathcal{V} \mid_{\pi^{-1}(x)}$ has uniform splitting type $(a_1, ..., a_r)$, where $a_i \geq 0 \Rightarrow a_i > a_{i+1}$, on each fibre. Then for each smooth point $x \in A$ there is a Stein open neighbourhood U_x such that the direct image $R^0\pi_*(\mathcal{V}) \mid_{U_x \setminus A} \cong \mathcal{O}^m$, for some non-negative integer m.

Proof. Induction on the rank s of $\mathcal{V}$; the case $s = 0$ is trivial. Suppose the lemma is true for any locally free sheaf $\mathcal{V}$ of rank $s - 1$ on $\pi^{-1}(X \setminus A)$. Now consider

$$\mathcal{V} \mid_{\pi^{-1}(x)} \cong \bigoplus_{1 \leq i \leq s} \mathcal{O}_{\mathbf{P}_1}(a_i)$$

for all $x \in X \setminus A$. If $a_1 < 0$ there is nothing to prove, so assume $a_1 \geq 0$ and consider

$$\mathcal{H}om(\mathcal{O}_{\mathbf{F}}(a_1), \mathcal{V}) \cong \mathcal{V} \otimes \mathcal{O}_{\mathbf{F}}(-a_1).$$

Note that $a_1 > a_2 \Rightarrow$

$$\mathcal{V} \otimes \mathcal{O}_{\mathbf{F}}(-a_1) \mid_{\pi^{-1}(x)} \cong \mathcal{O} \oplus \bigoplus_{2 \leq i \leq s} \mathcal{O}_{\mathbf{P}_1}(a_i - a_1)$$

for all $x \in X \setminus A$, and hence

$$\mathcal{F} = R^0\pi_*(\mathcal{V} \otimes \mathcal{O}_{\mathbf{F}}(-a_1))$$

is locally free of rank one. For any smooth $x \in A$, let U_x be a Stein open neighbourhood. From a theorem of Scheja [9], and a simple application of the Künneth formula, it follows that for U_x sufficiently small, both

$$H^1(U_x \setminus A, \mathcal{O}) = 0 \quad \text{and} \quad H^2(U_x \setminus A, \mathbf{Z}) = 0,$$

hence the solvability of the second "Cousin problem" implies that $\mathcal{F}\mid_{U_x\setminus A}\cong \mathcal{O}$ (cf. eg., [4]). Now let $\sigma : U_x \setminus A \to \mathcal{F}$ be a non-vanishing section, which induces a non-vanishing homomorphism

$$\varphi \in \Gamma(\pi^{-1}(U_x \setminus A), \mathcal{V} \otimes \mathcal{O}_{\mathbf{F}}(-a_1)).$$

Hence there exists an exact sequence

$$0 \to \mathcal{O}_{\mathbf{F}}(a_1) \xrightarrow{\varphi} \mathcal{V} \to \mathcal{V}_1 \to 0$$

over $\pi^{-1}(U_x \setminus A)$. Note that $\mathcal{V}_1 = coker(\varphi)$ is locally free of rank $s-1$. Moreover, $x' \in U_x \setminus A \Rightarrow \varphi\mid_{\pi^{-1}(x')}$ must be a non-zero constant section, so that

$$\mathcal{V}_1\mid_{\pi^{-1}(x')}\cong \bigoplus_{2\le i\le s} \mathcal{O}_{\mathbf{P}_1}(a_i) \qquad \text{for all} \qquad x' \in U_x \setminus A.$$

Now from the induction hypothesis applied to $\mathcal{V}_1 \to \pi^{-1}(U_x \setminus A)$, it follows that

$$R^0\pi_*(\mathcal{V}_1)\mid_{U'_x\setminus A}\cong \mathcal{O}^m$$

for some sufficiently small Stein open neighbourhood $U'_x \subseteq U_x$. Moreover

$$a_1 \ge 0 \Rightarrow R^1\pi_*(\mathcal{O}_{\mathbf{F}}(a_1)) = 0,$$

and hence there is a short exact sequence of direct images

$$0 \to R^0\pi_*(\mathcal{O}_{\mathbf{F}}(a_1)) \to R^0\pi_*(\mathcal{V}) \to \mathcal{O}^m \to 0 \qquad (*)$$

over $U'_x \setminus A$. Note that $R^0\pi_*(\mathcal{O}_{\mathbf{F}}(a_1))\mid_{U'_x}\cong \mathcal{O}^{a_1+1}$ assuming U'_x is a sufficiently small Stein neighbourhood of x, hence the obstruction to splitting of (*) lies in

$$H^1(U'_x \setminus A, \mathcal{H}om(\mathcal{O}^m, \mathcal{O}^{a_1+1})) \cong H^1(U'_x \setminus A, \mathcal{O})^{m(a_1+1)}.$$

But now, according to the theorem of Scheja cited above, $c \ge 3 \Rightarrow$

$$H^1(U'_x \setminus A, \mathcal{O}) = 0,$$

and so $R^0\pi_*(\mathcal{V})\mid_{U'_x\setminus A}\cong \mathcal{O}^{m'}$, where $m' = m + a_1 + 1$.

Let $E \to \mathbf{F}$ be a holomorphic vector bundle of rank two, and denote by $\mathcal{X}$ the compact complex manifold corresponding to the projectivization $\mathbf{P}(E)$, then $\mathcal{X} \xrightarrow{\varpi} \mathbf{F}$ is a $\mathbf{P}_1$-bundle over the fibre bundle $\mathbf{F}$. Moreover, the proper, holomorphic submersion $f = \pi \circ \varpi$ determines a holomorphic family $\mathcal{X} \xrightarrow{f} X$, in which the fibre $f^{-1}(x) \cong \mathbf{P}(E\mid_{\pi^{-1}(x)})$ for each $x \in X$. Let

Θ denote the subsheaf of $\mathcal{TX}$ (the holomorphic tangent sheaf) corresponding to the kernel of $f_* : \mathcal{TX} \to \mathcal{T}X$, and consider the long exact sequence of direct images

$$... \to R^0 f_*(\mathcal{TX}) \to \mathcal{T}X \xrightarrow{\rho} R^1 f_*(\Theta) \to ...$$

where ρ corresponds to the Kodaira-Spencer homomorphism (cf. [3]). If $E\mid_{\pi^{-1}(x)}$ splits uniformly for all $x \in X \setminus A$ then $f\mid_{X\setminus A}$ is *locally trivial*, ie., there exists an open cover $\{U_i\}_{i\in I}$ of $X \setminus A$, such that $f^{-1}(U_i) \cong U_i \times \mathbf{P}(E\mid_{\pi^{-1}(x)})$, for each $x \in U_i$ and all $i \in I$, hence $\rho\mid_{X\setminus A} \equiv 0$ and there is an exact sequence

$$0 \to R^0 f_*(\Theta)\mid_{X\setminus A} \to R^0 f_*(\mathcal{TX})\mid_{X\setminus A} \to \mathcal{T}X\mid_{X\setminus A} \to 0.$$

For any smooth $x \in A$, let U_x be a sufficiently small Stein open neighbourhood, and consider the corresponding long exact cohomology sequence

$$\begin{aligned} ... &\to H^0(U_x \setminus A, R^0 f_*(\mathcal{TX})) \to H^0(U_x \setminus A, \mathcal{T}X) \\ &\to H^1(U_x \setminus A, R^0 f_*(\Theta)) \to ... \ . \end{aligned}$$

Note that $c \geq 2$ implies any global holomorphic vector field on $f^{-1}(U_x \setminus A)$ admits a unique extension to $f^{-1}(U_x)$ by Hartogs' theorem, hence

$$\begin{aligned} H^0(U_x \setminus A, R^0 f_*(\mathcal{TX})) &\cong H^0(f^{-1}(U_x \setminus A), \mathcal{TX}) \\ &\cong H^0(f^{-1}(U_x), \mathcal{TX}) \ . \end{aligned}$$

In particular, $H^1(U_x \setminus A, R^0 f_*(\Theta)) = 0$ implies any nonvanishing vector field ξ on U_x may be lifted to a global vector field $f^*(\xi)$ on $f^{-1}(U_x)$, such that integration of $f^*(\xi)$ induces a global trivialization

$$\Phi : f^{-1}(U_x) \cong U_x \times \mathbf{P}(E\mid_{\pi^{-1}(x)}) \ .$$

If this is the case for all smooth $x \in A$, then the family determined by $f\mid_{X\setminus A^s}$ is in fact locally trivial, where A^s denotes the singular locus of A. By descending induction on the dimension of smooth components in the singular stratification of A, $\mathcal{X}$ itself is then seen to be a locally trivial family.

Now let $\Sigma \subseteq \Theta$ be a locally free sheaf corresponding to the kernel of $\varpi_* : \mathcal{TX} \to \mathcal{T}\mathbf{F}$, hence

$$0 \to \Sigma \to \Theta \to \Theta/\Sigma \to 0 \ .$$

Consider the corresponding long exact sequence of direct image sheaves with respect to ϖ, noting that $R^i\varpi_*(\Sigma) \equiv 0, i \geq 1$, and $R^0\varpi_*(\Theta/\Sigma)$ is isomorphic to that invertible subsheaf $\mathcal{S} \subset \mathcal{T}\mathbf{F}$, whose sections lie tangent to the fibres $\pi^{-1}(x) \cong \mathbf{P}_1$ for all $x \in X$. Hence

$$0 \to R^0\varpi_*(\Sigma) \to R^0\varpi_*(\Theta) \to \mathcal{S} \to 0. \qquad (1)$$

Note that for all $x \in X$, $H^0(\pi^{-1}(x), R^0\varpi_*(\Theta)) \cong H^0(f^{-1}(x), \Theta)$ corresponds to the Lie algebra of holomorphic vector fields on the rational ruled surface $f^{-1}(x)$, for which it may be shown by direct computation that

$$H^0(f^{-1}(x), \Theta) \cong H^0(f^{-1}(x), \Sigma) \oplus H^0(\pi^{-1}(x), \mathcal{S}).$$

From the Nakayama lemma it then follows that the long exact sequence of direct images of (1) with respect to π may be replaced by a short exact sequence

$$0 \to R^0 f_*(\Sigma) \to R^0 f_*(\Theta) \to R^0\pi_*(\mathcal{S}) \to 0 \tag{2}$$

over $X \setminus A$. Now construct the twisted Euler sequence,

$$0 \to \mathcal{O}_{\mathcal{X}} \to \varpi^*\mathcal{E}^* \otimes \mathcal{O}_{\mathcal{X}}(1) \to \Sigma \to 0 \ ,$$

where $\mathcal{E}^*$ denotes the dual of the locally free sheaf $\mathcal{E}$ associated with E , and $\mathcal{O}_{\mathcal{X}}(1)$ restricts to the hyperplane bundle on each fibre of $\mathcal{X}$. Note that

$$R^0\varpi_*(\mathcal{O}_{\mathcal{X}}) \cong \mathcal{O}_{\mathbf{F}}, \qquad R^0\varpi_*(\mathcal{O}_{\mathcal{X}}(1)) \cong \mathcal{E},$$

and $R^i\varpi_*(\mathcal{O}_{\mathcal{X}}) = 0, i \geq 1$, yields a short exact sequence of direct images

$$0 \to \mathcal{O}_{\mathbf{F}} \to \mathcal{E}^* \otimes \mathcal{E} \to R^0\varpi_*(\Sigma) \to 0\,. \tag{3}$$

For any $p \in \mathbf{F}$, let $\varphi \in (\mathcal{E}^* \otimes \mathcal{E})_p = \mathcal{E}nd(E)_p$, and write $\varphi = \psi I_E + \omega$, with

$$\psi = \frac{1}{2}trace(\varphi) \in \mathcal{O}_{\mathbf{F},p}, \quad \text{and} \quad \omega = \varphi - \frac{1}{2}trace(\varphi) I_E \in \mathcal{E}nd_0(E)_p,$$

where $\mathcal{E}nd_0(E)$ denotes the subsheaf of traceless endomorphisms of E. From (3) it follows that

$$R^0\varpi_*(\Sigma) \cong \mathcal{E}nd(E)/\mathcal{O}_{\mathbf{F}} \cong \mathcal{E}nd_0(E).$$

We are now ready to conclude

Theorem 1. *If $\mathbf{F} \xrightarrow{\pi} X$ is a holomorphic $\mathbf{CP}_1$-bundle, and $E \to \mathbf{F}$ a holomorphic vector bundle of rank two such that for all $x \in X \setminus A$, $E\mid_{\pi^{-1}(x)}$ has uniform splitting type (a, b) with $a > b$, then $c \geq 3 \Rightarrow E$ splits uniformly on $\pi^{-1}(x)$ for all $x \in X$.*

Proof. From the preceding discussion, the sequence (2) above may be rewritten

$$0 \to R^0\pi_*(\mathcal{E}nd_0(E)) \to R^0 f_*(\Theta) \to R^0\pi_*(\mathcal{S}) \to 0,$$

noting that $\mathcal{E}\mid_{\pi^{-1}(x)} \cong \mathcal{O}(a) \oplus \mathcal{O}(b)$ for all $x \in X \setminus A \Rightarrow$

$$\mathcal{E}nd_0(E)\mid_{\pi^{-1}(x)} \cong \mathcal{O}(\alpha) \oplus \mathcal{O} \oplus \mathcal{O}(-\alpha)$$

where $\alpha = a - b > 0$. But then $\mathcal{V} = \mathcal{E}nd_0(E)$ satisfies the hypothesis of Lemma 1, ie.,

$$R^0\pi_*(\mathcal{E}nd_0(E)) \mid_{U_x \setminus A} \cong \mathcal{O}_{U_x}^{\alpha+2}$$

for any smooth $x \in A$ and sufficiently small Stein neighbourhood U_x. Moreover, $R^0\pi_*(\mathcal{S})$ admits a locally free extension from $U_x \setminus A$ to U_x, hence $R^0\pi_*(\mathcal{S}) \mid_{U_x} \cong \mathcal{O}^3$. From the long exact sequence

$$\begin{aligned} H^1(U_x \setminus A, R^0\pi_*(\mathcal{E}nd_0(E))) &\to H^1(U_x \setminus A, R^0 f_*(\Theta)) \\ &\to H^1(U_x \setminus A, \mathcal{O}^3) \end{aligned}$$

it now follows (once again via the cohomology extension theorem of Scheja) that $H^1(U_x \setminus A, R^0 f_*(\Theta)) = 0$ when $c \geq 3$. Consequently $f^{-1}(U_x)$ corresponds to a trivial holomorphic family, such that

$$f^{-1}(U_x) = \mathbf{P}(E \mid_{\pi^{-1}(U_x)}) \Rightarrow E \mid_{\pi^{-1}(x')} \cong \mathcal{O}_{\mathbf{P}_1}(a) \oplus \mathcal{O}_{\mathbf{P}_1}(b)$$

for all $x' \in U_x \setminus A$, and

$$E \mid_{\pi^{-1}(x')} \cong (\mathcal{O}_{\mathbf{P}_1}(a) \oplus \mathcal{O}_{\mathbf{P}_1}(b)) \otimes \mathcal{O}_{\mathbf{P}_1}(\lambda)$$

for all $x' \in A \cap U_x$. But the chern class $c_1(E \mid_{\pi^{-1}(x)})$ is constant for all $x \in X$, therefore $\lambda = 0$. Hence $A \cap U_x = \emptyset \Rightarrow A = \emptyset$.

Now consider $E \to \mathbf{CP}_n$, a holomorphic vector bundle of rank two. For all $x \in \mathbf{Gr}(2, n+1) = X$, recall that $\mu \circ \nu^{-1}(x) \cong \mathbf{CP}_1$ and $a_E(\mu \circ \nu^{-1}(x)) \in \mathbf{Z}^2$, under the lexicographic ordering. Moreover, if

$$a_E = \inf_{x \in X} a_E(\mu \circ \nu^{-1}(x)), \qquad \text{then}$$

$$A = \{x \in X \mid a_E(\mu \circ \nu^{-1}(x)) > a_E\}$$

is a closed analytic subset of X (cf. [8], Lemma 3.2.2). Note that the case $a_E = (b, b)$ reduces, modulo the tensor product $E \otimes \mathcal{O}_{\mathbf{CP}_n}(-b)$, to the case $a_{E(-b)} = (0, 0)$, which implies that $E(-b)$ is semistable. The theorem of Barth mentioned above is then sufficient to conclude that A is a hypersurface, unless it is empty. Hence as an immediate consequence of the preceding theorem, simply replacing π with ν, and taking the vector bundle over $\mathbf{F}$ to correspond to $\mu^* E$, we have

Corollary 1. *If $E \to \mathbf{CP}_n$ is a rank-two vector bundle of uniform splitting type on $\mu \circ \nu^{-1}(X \setminus A)$, then $c \geq 3 \Rightarrow E$ is uniform on $\mathbf{CP}_n$.*

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