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A removable singularities theorem for families of ruled surfaces

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1 Introduction

Given a holomorphic family, $\mathcal{M} \xrightarrow{f} D$, with D a contractible Stein domain in $\mathbf{C}^m$, and $Z \subset D$ an analytic subset, such that the restricted family $f^{-1}(D \setminus Z)$ is locally trivial, under what circumstances does f define a trivial deformation over D ? An example of this type of problem is a conjecture of Kodaira and Spencer [9], recently proved by Siu [11]: if $D \subset \mathbf{C}$, and $f^{-1}(t) \cong \mathbf{P}_n$, (the complex projective space), for all $t \in D \setminus \{0\}$, then $f^{-1}(0) \cong \mathbf{P}_n$. In [5] a first step was taken in answering this question, when $\mathcal{M}$ is a family of compact complex surfaces, $M_t = f^{-1}(t)$ isomorphic, for all $t \in D \setminus Z$, to a holomorphic fibre bundle

$$\begin{array}{ccc} F & \hookrightarrow & M \\ & & \downarrow \\ & & \mathbf{P}_1 \end{array}$$

such that F is a Riemann surface of genus g. Moreover M_t may be considered to be any compact surface which "blows down" to a holomorphic fibre bundle of the above type, under a finite sequence of bimeromorphic transformations. In particular, when $F \cong \mathbf{P}_1$ (ie., M is a rational ruled surface), then M_t may represent any rational surface. Under these conditions, it was shown that $\mathcal{M}$ extends as a trivial family over D when the complex codimension of Z in D is at least four. Conversely, from the work of Kodaira and Spencer [9], on deformation of complex structures, numerous examples have arisen of "structure-jumping" across subsets, Z, of codimension at most two (cf. [7], [9]). The purpose of the present note is to extend the result of [5] to families $\mathcal{M}$, in which the generic manifold $M_t, t \in D \setminus Z$, is a ruled surface of positive genus, ie., the role of fibre and base space have been reversed, so that

$$\begin{array}{ccc} \mathbf{P}_1 & \hookrightarrow & M \\ & & \downarrow \\ & & X \end{array}$$

determines a ruling in which the Riemann surface X has genus greater than or equal to one. All ruled surfaces are of the form

$$M = \mathbf{P}(E) = E^* \setminus \{0\}/\mathbf{C}^*,$$

where $E \xrightarrow{\rho} X$ is a holomorphic vector bundle of rank two, such that $\mathbf{P}(E) \cong \mathbf{P}(E')$ if and only if there exists a line bundle $L \longrightarrow X$ with $E' = E \otimes L$. In addition, M admits a section, $s : X \hookrightarrow M$, corresponding to a rank-one subbundle, $\Lambda \subset E^*$ (or, by duality, a quotient line bundle, $E \longrightarrow \Lambda^* \longrightarrow 0$). Let $X_0 = s(X) \subset M$, then the self-intersection is given by the adjunction formula

$$K_X^2 = X_0 \cdot (X_0 + K_M),$$

where K_X and K_M are the canonical divisors of X and M respectively. In particular, there exists a section $\Sigma \subset M$, such that the corresponding line bundle

$$\mathscr{L}(\Sigma) \cong \mathscr{O}_M(1)$$

(cf [6], proposition V.2.8), and $K_M \in Pic(M)$ is linearly equivalent to

$$-2\Sigma + \rho^*(K_X + \Delta),$$

where $\Delta \in Pic(X)$ is the divisor corresponding to the determinant $\bigwedge^2(E)$ (cf. [6], lemma V.2.10). Now

$$\begin{aligned} X_0^2 &= K_X^2 - X_0 \cdot K_M \\ &= (2g_X - 2) - X_0 \cdot (-2\Sigma + \rho^*(K_X) + \rho^*(\Delta)) \\ &= (2g_X - 2) + 2deg(\Lambda^*) - deg(K_X) - deg(\Delta) \\ &= 2deg(\Lambda^*) - deg(E). \end{aligned}$$

Hence, if $X_0^2 < 0$, it follows that E is not semistable, since

$$deg(\Lambda^*) < \frac{1}{2} deg(E).$$

Given that the conclusion is the same if E is replaced by $E \otimes L$, for any line bundle L, it will be convenient to refer to $M = \mathbf{P}(E)$ as an *unstable* ruled surface when $X_0^2 < 0$.

The statement of the main result may be summarised as follows.

Theorem. *If $\mathscr{M} \xrightarrow{f} D$ is a holomorphic family of compact, complex surfaces, locally trivial over $D \setminus Z$, such that $f^{-1}(t)$ is isomorphic, for all $t \in D \setminus Z$, to*

(a) an unstable ruled surface M, of genus $g_X \geq 1$,
(b) any compact surface having, as minimal model under blowing down, a surface of the above type, then $codim_D(Z) \geq 4 \Rightarrow \mathcal{M}$ is a trivial family.

Local triviality implies that over sufficiently small neighbourhoods, $U \subseteq D \setminus Z$, holomorphic vector fields in D may be lifted (non-uniquely) to the total space $\mathcal{M}$, inducing a local equivalence of structures along the fibres. If the complex codimension of Z in the parameter space D is at least two, then 'patching together' these local liftings gives rise to obstructions in $H^1(D \setminus Z, R^0f_*(\Theta))$, where $R^0f_*(\Theta)$ denotes the direct image of the sheaf $\Theta \subset \mathcal{T}\mathcal{M}$ of germs of holomorphic vector fields lying parallel to the fibres of $\mathcal{M}$. When the obstruction vanishes, the lifting extends globally from $D \setminus Z$ to $\mathcal{M} \setminus f^{-1}(Z)$. But the holomorphic vector field so defined on the total space must now extend across codimension at least two, by Hartogs' Theorem, hence $\mathcal{M} \xrightarrow{f} D$ is trivial. The goal of sections two and three will be to show that for a family, $\mathcal{M} \xrightarrow{f} D$, of compact surfaces, the generic fibres being ruled, of genus $g_X \geq 2$, the corresponding locally free sheaf, $R^0f_*(\Theta)\,|_{D\setminus Z}$, will split into a direct sum of invertible sheaves, when $codim_D(Z) \geq 4$. A modified version of the procedure developed in [5] will be used to show that elements of the Lie algebra, $H^0(M_t, \Theta_t)$, are determined by their 'values' at a finite number of uniformly marked points. From this the splitting of $R^0f_*(\Theta)\,|_{D\setminus Z}$ will follow. Vanishing of obstructions will then be a consequence of Scheja's work on extension of cohomology groups [10], and hence the triviality of the extension of $\mathcal{M}$. Some additional argument is required for the case of $g_X = 1$, which is supplied in section four, while in section five, an argument already developed in [5] is applied to include families of compact surfaces M_t, which have as minimal model under a finite sequence of bimeromorphic transformations, any ruled surface of the above type. Although the case of $X_0^2 = 0$ has been treated partially in [5], under the heading of "Cartesian Products", a general argument for the case of semistable surfaces (ie., $X_0^2 \geq 0$) has not yet been found. However, when E admits only $\mathbf{C}^*$-automorphisms, of the form λI (as occurs, for example, when E is stable), it is seen in section six that structure-jumping in $\mathcal{M}$ is negated for $codim_D(Z) \geq 2$. Moreover, a classification is given for those remaining semistable surfaces which lie outside the scope of the present treatment.

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2 Classification of vector fields

Let M be a compact, complex surface, ruled with respect to a curve $X \subset M$, corresponding to a Riemann surface of genus $g \geq 1$. M is consequently of the form $\mathbf{P}(E)$, where E is a holomorphic vector bundle of rank two over X. If $\mathscr{T}M$ denotes the holomorphic tangent sheaf, let $\theta \subseteq \mathscr{T}M$ correspond to the sheaf of germs of sections lying tangent to the fibres of M, hence

$$0 \longrightarrow \theta \longrightarrow \mathscr{T}M \longrightarrow \mathscr{T}M/\theta \longrightarrow 0.$$

Compactness of the fibres of $M \xrightarrow{\nu} X$ implies

$$R^0\nu_*(\mathscr{T}M/\theta) \cong \mathscr{T}X,$$

hence there is a long exact sequence

$$0 \longrightarrow R^0\nu_*(\theta) \longrightarrow R^0\nu_*(\mathscr{T}M) \xrightarrow{\nu_*} \mathscr{T}X \longrightarrow R^1\nu_*(\theta) \longrightarrow ...,$$

where, for all $p \in X$,

$$H^1(\nu^{-1}(p), \theta\,|_{\nu^{-1}(p)}) \cong H^1(\mathbf{P}_1, \mathscr{T}\mathbf{P}_1) = 0 \Rightarrow R^1\nu_*(\theta) \equiv 0.$$

Consequently, there exists a long exact cohomology sequence

$$0 \longrightarrow H^0(X, R^0\nu_*(\theta)) \longrightarrow H^0(X, R^0\nu_*(\mathscr{T}M)) \longrightarrow H^0(X, \mathscr{T}X) \longrightarrow ..., \qquad (*)$$

in which the last term is isomorphic to $\mathbf{C}$ when $g = 1$, and is zero for $g \geq 2$. This latter case implies that there is in fact an isomorphism

$$H^0(M, \theta) \cong H^0(M, \mathscr{T}M),$$

of complex Lie Algebras, which for convenience will henceforth be denoted $\mathscr{G}_\theta$ and $\mathscr{G}_M$ respectively. Note that the ruling of M is unique, hence the definition of $\mathscr{G}_\theta$ in terms of holomorphic vector fields lying tangent to the fibres is canonical (cf. [2], Proposition V.4.3). Let F_p denote the unique fibre passing through a given $p \in M$, and suppose that the complex rank of $\mathscr{G}_\theta$ is r.

Proposition 1 *For any Stein open subset $U \subseteq M$, corresponding to a coordinate neighbourhood, there exist points $p_i \in U$, $1 \leq i \leq r$, such that the map*

$$\psi : \mathscr{G}_\theta \longrightarrow \mathbf{C}^r \qquad \sigma \mapsto (\sigma(p_1), ..., \sigma(p_r))$$

is an isomorphism.

Proof. Let $V_0 \subseteq M$ be the vanishing locus of $\mathscr{G}_\theta$, noting that $\mathscr{G}_\theta \neq 0 \Rightarrow V_0 \neq M$. Choose $p_1 \in U \setminus V_0$, and define

$$\psi_1 : \mathscr{G}_\theta \longrightarrow T_{p_1}F_{p_1}$$

such that for all $\sigma \in \mathscr{G}_\theta, \psi_1(\sigma) = \sigma(p_1)$. Note that ψ_1 is well defined, since U is a coordinate neighbourhood. Let $\mathscr{G}_1 = ker(\psi_1)$, then

$$p_1 \notin V_0, \qquad dim_{\mathbf{C}}T_{p_1}F_{p_1} = 1 \Rightarrow rank_{\mathbf{C}}\mathscr{G}_1 = r - 1.$$

Now let V_1 (such that $V_0 \subseteq V_1 \subseteq M$) be the vanishing locus of $\mathscr{G}_1$. Choose $p_2 \in U \setminus V_1$, and define

$$\psi_2 : \mathscr{G}_1 \longrightarrow T_{p_2}F_{p_2}, \qquad \sigma \mapsto \sigma(p_2).$$

Let $\mathscr{G}_2 = ker(\psi_2)$, then

$$p_2 \notin V_1, \qquad dim_{\mathbf{C}} T_{p_2}F_{p_2} = 1 \Rightarrow rank_{\mathbf{C}}(\mathscr{G}_2) = r - 2.$$

.....

Let V_{r-1} ($V_0 \subseteq V_1 \subseteq ... \subseteq V_{r-1} \subset M$) be the vanishing locus of $\mathscr{G}_{r-1}$. Choose $p_r \in U \setminus V_{r-1}$, and define

$$\psi_r : \mathscr{G}_{r-1} \longrightarrow T_{p_r}F_{p_r}, \qquad \sigma \mapsto \sigma(p_r).$$

Let $\mathscr{G}_r = ker(\psi_r)$, then

$$p_r \notin V_{r-1}, \qquad dim_{\mathbf{C}} T_{p_r}F_{p_r} = 1 \Rightarrow rank_{\mathbf{C}}(\mathscr{G}_r) = 0.$$

Hence there exists a linear injective map

$$\psi : \mathscr{G}_\theta \longrightarrow \mathbf{C}^r, \qquad \sigma \mapsto (\sigma(p_1), ..., \sigma(p_r)),$$

which is also surjective by comparison of dimension. □

The following lemma will provide information about the vanishing loci, $V_i \subset M, 0 \leq i \leq r-1$, for use in the next section.

Lemma 1 *Each of the vanishing loci, $V_i \subset M, 0 \leq i \leq r-1$, is contained in a finite union of smooth curves in M.*

Proof. Consider $0 \neq \sigma \in \mathscr{G}_\theta$, and let $V(\sigma) \subset M$ denote the vanishing locus of σ. Note that $V(\sigma)$ corresponds to an effective divisor of the form

$$\mathscr{D}_\sigma = aX_0 + \Sigma_{j=1}^{\mu} b_j F_j + D_\sigma, \qquad a, b_j \in \mathbf{Z}_+,$$

where the F_j are fibres, $1 \leq j \leq \mu$, and D_σ corresponds to the residue of algebraically irreducible components which are neither X_0 nor fibres of M. If it is assumed that the first Chern class of the normal bundle, $c_1(\mathscr{N}_{X_0}) = X_0^2 < 0$, then

$$H^0(X_0, \mathscr{N}_{X_0}) = 0.$$

Hence

$$\sigma\,|_{X_0} \in H^0(X_0, \mathscr{N}_{X_0}) \Rightarrow \sigma\,|_{X_0} \equiv 0, \qquad \text{ie.,} \qquad a \geq 1.$$

Conversely, for all $p \in M$,

$$\sigma\,|_{F_p} \in H^0(\mathbf{P}_1, \mathscr{T}\mathbf{P}_1) \Rightarrow a \leq 2.$$

Now suppose there exists an irreducible $C \subseteq D_\sigma$ which admits a singularity at p. Let $U \subseteq M$ be an open neighbourhood of p such that the locus of $C \cap U$ is given locally by $g = 0$, for some $g \in \Gamma(U, \mathscr{O}_M)$. Moreover, let $g = g_1 ... . g_\gamma$ denote the factorisation of g into analytically irreducible elements of $\mathscr{O}_{M,p}$ (unique up to multiplication by an invertible element in this ring). Now choose local coordinates

(z, ζ) on U such that the locus of $F_p \cap U$ is determined by $z = 0$, and let d_k denote the Weierstrass degree of $g_k, 1 \leq k \leq \gamma$. Hence the intersection multiplicity

$$F_p \cdot C \geq m_p(C) = \Sigma_{k=1}^{\gamma} d_k \geq 2.$$

But $F_p \cdot X = 1, m_p(C) \geq 2 \Rightarrow \sigma \mid_{F_p} \equiv 0$, and for all $q \in U$ sufficiently small,

$$F_q \cdot C \geq m_p(C), F_q \cdot X = F_p \cdot X \Rightarrow \sigma \mid_U \equiv 0,$$

which contradicts the assumption that $\sigma \neq 0$. Hence C must be smooth, and D_σ corresponds to a union of smooth curves, such that

$$V(\sigma) = X \cup_{1 \leq j \leq \mu} F_j \cup_{1 \leq l \leq \lambda} C_l.$$

Finally, consider $\mathscr{G}_i \subseteq \mathscr{G}_0, 1 \leq i \leq r-1$, corresponding to the subalgebras of proposition one, with $V_i \subset M$ the associated vanishing loci. Suppose that $\sigma_1, \ldots, \sigma_{r-i}$ generate $\mathscr{G}_i$, then

$$V_i = \cap_{1 \leq k \leq r-i} V(\sigma_k).$$

If V_i contains isolated points (ie., $dim_{\mathbf{C},p}(V_i) = 0$), then each of these is contained in a fibre of M, otherwise $dim_{\mathbf{C},p}(V_i) > 0 \Rightarrow p$ belongs to a smooth curve in $V(\sigma_k)$ for some k. Now let χ_i denote the appropriate union of smooth curves such that $V_i \subseteq \chi_i \subset M, 1 \leq i \leq r-1$. □

3 The removable singularities theorem for hyperelliptic X

Consider a holomorphic family, $\mathscr{M} \xrightarrow{f} D$, where $D \subseteq \mathbf{C}^m$ is a contractible Stein domain, and $Z \subset D$ is an analytic subset. Moreover, suppose that for all $t \in D \setminus Z$, the fibres $M_t = f^{-1}(t)$ are isomorphic to a compact surface M, which is ruled with respect to a smooth curve X, of genus greater than or equal to two. Hence the subfamily, $\tilde{\mathscr{M}} = f^{-1}(D \setminus Z)$ is locally trivial, ie., there exists an open cover, $\mathscr{U} = \{U_i\}_{i \in I}$ of $D \setminus Z$, such that for each $i \in I$ there is a biholomorphism

$$\Phi_i : U_i \times M \rightarrow f^{-1}(U_i),$$

where $\Phi_i(t, M) = M_t$, for all $t \in U_i$. Let $\vec{v} \in \Gamma(D, \mathscr{T}D)$ be a non-zero holomorphic vector field, then the vanishing of the Kodaira-Spencer map,

$$\rho : \mathscr{T}D \rightarrow R^1 f_*(\Theta) \mid_{D \setminus Z}$$

(where Θ denotes the subsheaf of $\mathscr{T}\mathscr{M}$ corresponding to germs of holomorphic vector fields which lie tangent to the fibres M_t, for all $t \in D$), implies the existence of a local lifting,

$$f^*(\vec{v} \mid_{U_i}) \in \Gamma(f^{-1}(U_i), \mathscr{T}\mathscr{M}),$$

for each $i \in I$, which induces Φ_i. Now for all pairs, $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$, the expressions

$$f^*(\vec{v}\,|_{U_i}) - f^*(\vec{v}\,|_{U_j}) \in \Gamma(U_i \cap U_j, R^0 f_*(\Theta))$$

determine, in the direct limit, a cohomology class,

$$\xi \in H^1(D \setminus Z, R^0 f_*(\Theta)),$$

which measures the obstruction to global lifting of the vector field $\vec{v}\,|_{D\setminus Z}$. Let $\underline{\mathscr{G}}_{\tilde{\mathcal{M}}}$ denote the holomorphic vector bundle having $R^0 f_*(\Theta)\,|_{D\setminus Z}$ as the corresponding locally free sheaf of sections. Hence, for all $t \in D \setminus Z$, the fibres

$$\underline{\mathscr{G}}_{\tilde{\mathcal{M}},t} \cong R^0 f_*(\Theta)_t / \mathbf{m}_t \cdot R^0 f_*(\Theta)_t \cong H^0(M_t, \mathscr{T} M_t),$$

where $\mathbf{m}_t$ denotes the maximal ideal in $\mathscr{O}_D$ of t, and

$$H^0(M_t, \mathscr{T} M_t) \cong \mathscr{G}_M.$$

Moreover, recall from the previous section that $g_X \geq 2 \Rightarrow \mathscr{G}_M \cong \mathscr{G}_\theta$. Before applying the argument of proposition one uniformly to the fibres of $\underline{\mathscr{G}}_{\tilde{\mathcal{M}}}$, however, the following result is needed. (From now on, $f^{-1}(Z)$ will be denoted by $\mathscr{Z} \subset \mathcal{M}$.)

Lemma 2 *Let $\mathscr{H} \subset \tilde{\mathcal{M}}$ be an analytic hypersurface, such that for all $t \in D \setminus Z, H_t = \mathscr{H} \cap M_t$ is a smooth curve. If $d = codim_D(Z) \geq 4$, then the closure, $\bar{\mathscr{H}}$, of $\mathscr{H}$ in $\mathcal{M}$ is an analytic hypersurface such that $\bar{\mathscr{H}} \cap M_t$ is smooth for all $t \in D$. Moreover $\bar{\mathscr{H}} \cap \mathscr{Z}$ is a proper subvariety of $\mathscr{Z}$.*

Proof. See [5], Lemma 1. □

Our aim now is to define a "marking" procedure, by which points corresponding to the set of $p_i \in M, 1 \leq i \leq r$, in proposition one may be chosen uniformly on each submanifold $M_t \subset \tilde{\mathcal{M}}$. Assume henceforth that $d \geq 4$.

Step 1: For each $t \in D \setminus Z$, let $V_{0,t} \subset M_t$ correspond to the vanishing locus of the Lie Algebra $\underline{\mathscr{G}}_{\tilde{\mathcal{M}},t}$, and define

$$\mathscr{V}_0^{\cdot} = \cup_{t \in D\setminus Z} V_{0,t}.$$

Hence $\mathscr{V}_0^{\cdot} \subset \tilde{\mathcal{M}}$ determines a subvariety such that $\mathscr{V}_0^{\cdot} \cap M_t \cong V_0 \subset M$ (cf. proposition one) for all $t \in D \setminus Z$. Recall from lemma two that there exists a finite union $\chi_0 \subset M$ of smooth curves such that $V_0 \subseteq \chi_0$. In particular, for each isolated $p \in V_0$, there is a unique fibre $F_p \subseteq \chi_0$. Similarly, since $\tilde{\mathcal{M}}$ is locally trivial, and fibres are preserved under biholomorphic transformations of M, due to the uniqueness of the ruling, it follows that each smooth component $H_{0,j} \subseteq \chi_0$ determines a subvariety $\mathscr{H}_{0,j} \subset \tilde{\mathcal{M}}$. For each $i \in I$, and for each j, the family of curves corresponding to $\mathscr{H}_{0,j} \cap f^{-1}(U_i)$ defines a topological space, which is homeomorphic to U_i, hence the collection of $\mathscr{H}_{0,j}$ determines a covering space of $D \setminus Z$. But $d \geq 2 \Rightarrow D \setminus Z$ is simply connected, hence the covering space is trivial, and there is a one to one correspondence between $H_{0,j}$ and $\mathscr{H}_{0,j}$, $1 \leq j \leq \rho$. Let

$$\mathscr{X}_0 = \cup_{1\leq j\leq \rho}\mathscr{H}_{0,j} \subset \tilde{\mathscr{M}}.$$

Now apply lemma three to each $\mathscr{H}_{0,j} \subset \tilde{\mathscr{M}}, 1 \leq j \leq \rho$, for which the closure, $\bar{\mathscr{H}}_{0,j} \subset \mathscr{M}$ intersects $\mathscr{Z}$ as a proper subvariety. Hence the extended hypersurface

$$\bar{\mathscr{X}_0} = \cup_j \bar{\mathscr{H}}_{0,j} \subset \mathscr{M}$$

intersects $\mathscr{Z}$ as a proper subvariety. Choose a smooth point, $p \in \mathscr{Z}\setminus \bar{\mathscr{X}_0}$, and a Stein open neighbourhood, $\mathscr{U}_p \subset \mathscr{M}\setminus \bar{\mathscr{X}_0}$, such that $\mathscr{U}_p \cap M_t$ is a coordinate neighbourhood for all $t \in U_p = f(\mathscr{U}_p) \subseteq D$. Now define a holomorphic map

$$\varphi_1 : U_p \longrightarrow \mathscr{U}_p \quad \text{such that} \quad f \circ \varphi_1 = id_{U_p}.$$

Step 2: Let $\mathbf{F}_1$ denote the holomorphic line bundle over $U_p \setminus Z$, such that for all $t \in U_p \setminus Z$,

$$\mathbf{F}_{1,t} = T_{\varphi_1(t)}F_{\varphi_1(t)}.$$

Define a bundle morphism,

$$\Psi_1 : \underline{\mathscr{G}}_{\tilde{\mathscr{M}}}|_{U_p\setminus Z} \longrightarrow \mathbf{F}_1,$$

such that for all $t \in U_p \setminus Z, \sigma \in \underline{\mathscr{G}}_{\tilde{\mathscr{M}},t}, \Psi_{1,t}(\sigma) = \sigma(\varphi_1(t))$, and let $\underline{\mathscr{G}}_1 \subseteq \underline{\mathscr{G}}_{\tilde{\mathscr{M}}}|_{U_p\setminus Z}$ be the sub-bundle of complex rank $r-1$ corresponding to $ker(\Psi_1)$. For each $t \in U_p \setminus Z$, let $V_{1,t} \subset M_t$ correspond to the vanishing locus of the Lie Algebra $\underline{\mathscr{G}}_{1,t}$, and define

$$\mathscr{V}_1 = \cup_{t\in U_p\setminus Z} V_{1,t}.$$

Hence $\mathscr{V}_1 \subset \mathscr{U}_p \setminus \mathscr{Z}$ determines a subvariety such that $\mathscr{V}_1 \cap M_t \cong V_1 \subset M$ (cf. proposition one) for all $t \in U_p \setminus Z$. Once again, if $V_1 \subseteq \chi_1$ is the union of smooth curves specified in lemma one, the argument of step 1 above may be applied verbatim to obtain a union

$$\mathscr{X}_1 = \cup_j \mathscr{H}_{1,j} \subset \mathscr{U}_p \setminus \mathscr{Z},$$

such that $\mathscr{V}_1 \subseteq \mathscr{X}_1$, and the extended hypersurface, $\bar{\mathscr{X}_1} \subset \mathscr{U}_p$ intersects $\mathscr{Z}$ as a proper subvariety. Now define a holomorphic map

$$\varphi_2 : U_p \longrightarrow \mathscr{U}_p \setminus \bar{\mathscr{X}_1}, \quad \text{such that} \quad f \circ \varphi_2 = id_{U_p}.$$

.....

Step r: Let $\mathbf{F}_{r-1}$ denote the holomorphic line bundle over $U_p \setminus Z$, such that for all $t \in U_p \setminus Z$,

$$\mathbf{F}_{r-1,t} = T_{\varphi_{r-1}(t)}F_{\varphi_{r-1}(t)}.$$

Define a bundle morphism,

$$\Psi_{r-1} : \underline{\mathscr{G}}_{r-2} \longrightarrow \mathbf{F}_{r-1},$$

such that for all $t \in U_p \setminus Z, \sigma \in \underline{\mathscr{G}}_{r-2,t}, \Psi_{r-1,t}(\sigma) = \sigma(\varphi_{r-1}(t))$, and let $\underline{\mathscr{G}}_{r-1} \subseteq \underline{\mathscr{G}}_{r-2}(\subseteq \ldots \subseteq \underline{\mathscr{G}}_{\tilde{\mathscr{M}}}|_{U_p\setminus Z})$ be the sub-bundle of complex rank 1 corresponding to $ker(\Psi_{r-1})$. Let $\mathscr{V}_{r-1}^{\cdot} \subset \mathscr{U}_p \setminus \mathscr{Z}$ be the family of vanishing loci, and $\mathscr{X}_{r-1}^{\cdot}$ the associated union of smooth hypersurfaces, which extends appropriately as $\bar{\mathscr{X}_{r-1}^{\cdot}} \subset \mathscr{U}_p$. Note that

$$V_0 \subseteq V_1 \subseteq ... \subseteq V_{r-1} \Rightarrow \bar{\mathscr{X}_0^{\cdot}} \subseteq \bar{\mathscr{X}_1^{\cdot}} \subseteq ... \subseteq \bar{\mathscr{X}_{r-1}^{\cdot}},$$

hence define

$$\varphi_r : U_p \longrightarrow \mathscr{U}_p \setminus \bar{\mathscr{X}_{r-1}^{\cdot}},$$

which completes the marking of each fibre in $f^{-1}(U_p \setminus Z)$. Now in a manner similar to that of proposition one, there exists a bundle morphism,

$$\Psi : \underline{\mathscr{G}}_{\tilde{\mathscr{U}}} |_{U_p \setminus Z} \longrightarrow \bigoplus_{1 \leq i \leq r} \mathbf{F}_i,$$

where, for all $t \in U_p \setminus Z, \sigma \in \underline{\mathscr{G}}_{\tilde{\mathscr{U}},t}$,

$$\Psi_t(\sigma) = (\sigma(\varphi_1(t)), ..., \sigma(\varphi_r(t))),$$

which is an isomorphism. Descending to the associated locally free sheaves, $R^0 f_*(\Theta)$ (corresponding to $\underline{\mathscr{G}}_{\tilde{\mathscr{U}}}$), and $\mathscr{F}_i$ (corresponding to $\mathbf{F}_i, 1 \leq i \leq r$) it follows that

$$R^0 f_*(\Theta) |_{U_p \setminus Z} \cong \bigoplus_{1 \leq i \leq r} \mathscr{F}_i.$$

Proposition 2 $R^0 f_*(\Theta) |_{U_p \setminus Z}$ *extends freely across* Z *in a neighbourhood of* $f(p)$.

Proof. Recall that $f(p)$ is smooth, ie., $U_p \cap Z$ is a manifold. Choose $U_p' \subset\subset U_p$ such that $U_p' \cong \Delta \times \Delta'$, with Δ a polydisc in $\mathbf{C}^d$ ($d = codim_D(Z) \geq 4$), and $Z \cap U_p' \cong \Delta'$, a polydisc in $\mathbf{C}^{m-d}$, both centred on $f(p)$. Hence

$$U_p' \setminus Z \cong \Delta' \times (\Delta \setminus \{f(p)\}).$$

Now, since $d \geq 3$, it follows from a result of Frenkel (cf. Scheja [9], and Siu and Trautmann [12]) that

$$H^1(U_p' \setminus Z, \mathscr{O}) = 0.$$

The remainder of the argument is concerned with the exact sequence

$$H^1(U_p' \setminus Z, \mathscr{O}) \rightarrow H^1(U_p' \setminus Z, \mathscr{O}^*) \rightarrow H^2(U_p' \setminus Z, \mathbf{Z}),$$

by means of which the relevant "Cousin Problem" may be solved. From the Künneth Formula, it follows that

$$H^2(\Delta' \times (\Delta \setminus \{f(p)\}), \mathbf{Z}) \cong \bigoplus_{i+j=2} [H^i(\Delta', \mathbf{Z}) \otimes H^j(\Delta \setminus \{f(p)\}, \mathbf{Z})].$$

where $i = 1, 2 \Rightarrow H^i(\Delta', \mathbf{Z}) = 0$, since Δ' is contractible, and

$$i = 0 \Rightarrow H^2(\Delta \setminus \{f(p)\}, \mathbf{Z}) = 0,$$

since $\Delta \setminus \{f(p)\}$ has the homotopy type of a sphere, $\mathbf{S}^{2d-1}$, hence

$$H^2(\Delta' \times (\Delta \setminus \{f(p)\}), \mathbf{Z}) = 0.$$

Thus $H^1(U_p' \setminus Z, \mathcal{O}^*) = 0$, ie., all line bundles over $U_p' \setminus Z$ are trivial, and extend freely, via Hartogs' Theorem, to U_p', hence so does $R^0 f_*(\Theta)|_{U_p \setminus Z}$. □

An immediate corollary of proposition 2, and the results of Scheja cited above, is that $H^1(U_p \setminus Z, R^0 f_*(\Theta)|_{U_p \setminus Z}) = 0$. It now remains simply to state the main result of this note.

Theorem 1 *If $\mathcal{M} \xrightarrow{f} D$ is a holomorphic family of compact surfaces, locally trivial over $D \setminus Z$, such that $f^{-1}(t) \cong M$, an unstable surface, ruled with respect to a curve $X \subset M$, of genus $g \geq 2$, for all $t \in D \setminus Z$, then $d = codim_D(Z) \geq 4 \Rightarrow \mathcal{M}$ is a trivial family over D.*

Proof. Suppose $p \in \mathcal{Z} \setminus \bar{\mathcal{X}_0}$ is smooth, then there exist neighbourhoods $\mathcal{U}_p \subseteq \mathcal{M}, U_p' \subset\subset U_p$, such that $d \geq 4 \Rightarrow H^1(U_p' \setminus Z, R^0 f_*(\Theta)|_{U_p' \setminus Z}) = 0$, hence the obstruction to a global lifting, $f^*(\vec{v})$, of $\vec{v} \in \Gamma(U_p', \mathcal{T} D)$ to $f^{-1}(U_p') \setminus \mathcal{Z}$ vanishes. From Hartogs' theorem it follows that the holomorphic vector field $f^*(\vec{v})$ must extend uniquely across $f^{-1}(U_p') \cap \mathcal{Z}$, thereby negating structure jumping in the neighbourhood U_p' of t. It follows that $\mathcal{M} \xrightarrow{f} D$ must be locally trivial on the complement in D of an analytic locus, $Z' \subset Z$, which is a subset of codimension at least five. Now continue the argument inductively on the dimension of each smooth component of the stratification of Z'. Hence $\mathcal{M}$ must be locally trivial on D, and since $D \subseteq \mathbf{C}^m$ is Stein, it follows that $\mathcal{M}$ is in fact trivial. □

4 The removable singularities theorem for elliptic X

Recall the exact sequence (*), derived in section two, for the case $g_X = 1$:

$$0 \longrightarrow \mathcal{G}_\theta \xrightarrow{\eta_*} \mathcal{G}_M \xrightarrow{\nu_*} \mathbf{C} \longrightarrow \ldots$$

Note that if ν_* is the zero map, then $\mathcal{G}_\theta \cong \mathcal{G}_M$, and the entire argument of section three extends to include this case in the preceding theorem. A similar inclusion occurs if $\eta_* = 0$, since $\mathcal{G}_M \cong \mathbf{C} \Rightarrow R^0 f_*(\Theta)|_{D \setminus Z}$ is locally free of rank one, and extends freely to D when $d \geq 3$. Hence it will be assumed that

$$0 \longrightarrow \mathcal{G}_\theta \xrightarrow{\eta_*} \mathcal{G}_M \xrightarrow{\nu_*} \mathbf{C} \longrightarrow 0$$

is an exact sequence, with $\mathcal{G}_\theta \neq 0$. By analogy with the previous section, $V_0 \subset M$ will denote the vanishing locus of $\mathcal{G}_\theta$.

Lemma 3 *Given $p \in M \setminus V_0$, and $U \subseteq M$ a Stein neighbourhood of p, then for all $\delta \in T_p M$ there exists $\sigma \in \mathcal{G}_M$ such that $\sigma|_U(p) = \delta$.*

Proof. Suppose there exists $\delta \in T_p M$ such that for all $\sigma \in \mathcal{G}_M, \sigma(p) \neq \delta$, and consider the linear map,

$$\psi : \mathcal{G}_M \longrightarrow T_p M, \qquad \sigma \mapsto \sigma(p).$$

Now $rank_{\mathbf{C}}(T_pM) = 2 \Rightarrow rank_{\mathbf{C}}(im(\psi)) \leq 1$. If F_p once more denotes the fibre of M through p, then

$$im(\psi) \neq T_pF_p \Rightarrow \psi \circ \eta_*(\mathscr{G}_\theta) = 0. \qquad (**)$$

But $p \in M \setminus V_0$, therefore (**) implies that $\mathscr{G}_\theta = 0$, which contradicts one of the initial assumptions. If instead it is assumed that $im(\psi) = T_pF_p$, then $\nu_*(\sigma(p)) = 0$, ie., $\nu_*(\sigma) = 0$, for all $\sigma \in \mathscr{G}_M$, hence $\nu_* = 0$, which contradicts the other initial assumption, hence there can be no such δ. □

Keeping in mind that $\mathscr{G}_\theta$ is a canonical subalgebra of $\mathscr{G}_M$, let $\mathbf{F}_\theta \subseteq \underline{\mathscr{G}}_{\tilde{\mathscr{M}}}$ denote the corresponding sub-bundle of rank r, and $\tilde{\mathscr{V}_0} \subset \tilde{\mathscr{M}}$ the vanishing subvariety associated with $\mathbf{F}_\theta$. Moreover, $\tilde{\mathscr{X}_0} \subset \tilde{\mathscr{M}}$ will denote the union of smooth hypersurfaces containing $\tilde{\mathscr{V}_0}$, which extends to $\bar{\mathscr{X}_0} \subset \mathscr{M}$. Recall that for each $i \in I$, the biholomorphism

$$\Phi_i : U_i \times M \longrightarrow f^{-1}(U_i)$$

is induced by liftings $f^*(\vec{v}\,|_{U_i})$ for all $\vec{v} \in \Gamma(D, \mathscr{T}D)$. Now choose $p \in \mathscr{Z} \setminus \bar{\mathscr{X}_0}$, and a Stein open neighbourhood $\mathscr{U}_p \subseteq \mathscr{M} \setminus \bar{\mathscr{X}_0}$, with $U_p = f(\mathscr{U}_p)$. Let

$$\varphi : U_p \longrightarrow \mathscr{U}_p$$

be a holomorphic map such that $f \circ \varphi = id_{U_p}$, and for all $t \in \tilde{U}_i = U_i \cap U_p$, consider

$$\delta(t) = (\varphi)_*(\vec{v}(t)) - f^*(\vec{v}\,|_{\tilde{U}_i})_{\varphi(t)} \in T_{\varphi(t)}M_t.$$

Let σ_i denote a holomorphic t-parameter family of vector fields in $\underline{\mathscr{G}}_{\tilde{\mathscr{M}}}\,|_{\tilde{U}_i}$ (ie., a section belonging to $\Gamma(\tilde{U}_i, R^0 f_*(\Theta))$), such that

$$\sigma_i(\varphi(t)) = \delta(t).$$

Now $\hat{f}^*(\vec{v}\,|_{\tilde{U}_i}) = f^*(\vec{v}\,|_{\tilde{U}_i}) + \sigma_i$ has the property that

$$(\varphi)_*(\vec{v}(t)) - \hat{f}^*(\vec{v}\,|_{\tilde{U}_i})_{\varphi(t)} = 0.$$

Similarly, let $\hat{f}^*(\vec{v}\,|_{\tilde{U}_j}) = f^*(\vec{v}\,|_{\tilde{U}_j}) + \sigma_j$ for $j \in I$ such that $\tilde{U}_j \cap \tilde{U}_i \neq \emptyset$, then for all $t \in \tilde{U}_i \cap \tilde{U}_j$,

$$(\hat{f}^*(\vec{v}\,|_{\tilde{U}_i}) - \hat{f}^*(\vec{v}\,|_{\tilde{U}_j}))_{\varphi(t)} =$$

$$[\hat{f}^*(\vec{v}\,|_{\tilde{U}_i})_{\varphi(t)} - (\varphi)_*(\vec{v}(t))] - [\hat{f}^*(\vec{v}\,|_{\tilde{U}_j})_{\varphi(t)} - (\varphi)_*(\vec{v}(t))] = 0.$$

Let $\mathscr{S} \subseteq R^0 f_*(\Theta)\,|_{U_p \setminus Z}$ denote the locally free sheaf of rank $r-1$ corresponding to the kernel of the bundle morphism

$$\Psi : \mathbf{F}_\theta \longrightarrow \mathbf{F}_1, \qquad \sigma \mapsto \sigma(\varphi(t)),$$

for all $\sigma \in \mathbf{F}_{\theta,t} \cong \mathscr{G}_\theta, t \in U_p \setminus Z$. Hence

$$\hat{f}^*(\vec{v}\,|_{\tilde{U}_i}) - \hat{f}^*(\vec{v}\,|_{\tilde{U}_j}) \in \Gamma(\tilde{U}_i \cap \tilde{U}_j, \mathscr{S}), \qquad i, j \in I,$$

determines a cohomology class $\xi \in H^1(U_p \backslash Z, \mathscr{S})$. Now, continuing the marking argument of the preceding section, we have

$$\mathscr{S} \cong \bigoplus_{2 \leq i \leq r} \mathscr{T}_i,$$

so that the argument of theorem one can now be applied verbatim to conclude

Theorem 2 *If $\mathscr{M} \xrightarrow{f} D$ is a holomorphic family of compact surfaces, locally trivial over $D \setminus Z$, such that $f^{-1}(t) \cong M$, an unstable surface, ruled with respect to an elliptic curve $X \subset M$, for all $t \in D \setminus Z$, then $d = codim_D(Z) \geq 4 \Rightarrow \mathscr{M}$ is a trivial family over D.*

5 Bimeromorphic transforms

Suppose now that N is any compact complex surface, having as a minimal model any ruled surface M of the above type. Let

$$\pi : N \longrightarrow M$$

be a bimeromorphic map consisting of a finite sequence of blow-ups, ie.,

$$\pi = \pi_1 \circ \ldots \circ \pi_\mu,$$

such that for some finite set, $S \subset M$, the divisor, $A = \pi^{-1}(S)$, is a union of exceptional divisors in N. If R denotes the union of all smooth rational curves of negative self-intersection in N, then $A \subseteq R$. Moreover, R has countably many irreducible components, since each negatively self-intersecting smooth rational curve lies in a distinct homology class of $H_2(N, \mathbf{Z})$, and it is one of the basic properties of blowing-up that

$$H^2(\pi_i^{-1}(N_{i-1}), \mathbf{Z}) \cong \pi_i^*(H^2(N_{i-1}, \mathbf{Z})) \oplus \mathbf{Z}[e], \qquad 1 \leq i \leq \mu$$

(cf., eg. [2]), where $N_{i-1} = \pi_{i-1}^{-1} \circ \ldots \circ \pi_1^{-1}(M)$, and e is dual to the exceptional set of π_i. Hence R is a nowhere dense subset of N.

If $\hat{\mathscr{G}} = H^0(N, \mathscr{T}N)$, then the biholomorphism $N \setminus A \cong M \setminus S$ induces an injective linear map,

$$\pi_* : \hat{\mathscr{G}} \hookrightarrow \mathscr{G}.$$

Consider a holomorphic family of compact, complex surfaces, $\mathscr{N} \xrightarrow{g} D \subseteq \mathbf{C}^m$, and an analytic subset $Z \subset D$, such that the fibres $N_t = g^{-1}(t) \cong N$, where N is as defined above, for all $t \in D \setminus Z$. Let $\hat{\mathscr{Z}} = g^{-1}(Z)$, and assume that $g|_{\mathscr{N} \setminus \hat{\mathscr{Z}}}$ is locally trivial, ie., there exists a cover $\{U_i\}_{i \in I}$ of $D \setminus Z$, such that for each $i \in I$, there is a biholomorphism

$$\Phi_i : U_i \times N \longrightarrow g^{-1}(U_i),$$

with $\Phi_i(t, N) = N_t$ for all $t \in U_i$. If $R_t = \Phi_i(t, R), t \in U_i, i \in I$, then

$$\mathscr{R} = \cup_{t \in D \setminus Z} R_t \subset \mathscr{N} \setminus \hat{\mathscr{Z}}$$

will restrict to a countable union of smooth hypersurfaces in N_t, for all $t \in D \setminus Z$.

Our aim will be to find, for some smooth $t_0 \in Z$, a neighbourhood $U_{t_0} \subseteq D$ such that there exists a holomorphic map

$$\varpi : g^{-1}(U_{t_0}) \longrightarrow \mathscr{M}_{t_0},$$

with the following properties:

(i) $\mathscr{M}_{t_0} \xrightarrow{f} U_{t_0}$ is a trivial family of minimal compact surfaces, of one of the types previously discussed,

(ii) The diagram

$$\begin{array}{ccc} g^{-1}(U_{t_0}) & \xrightarrow{\quad \varpi \quad} & \mathscr{M}_{t_0} \\ & {\scriptstyle g} \searrow \quad \swarrow {\scriptstyle f} & \\ & U_{t_0} & \end{array}$$

commutes. More specifically, ϖ 'simultaneously blows-down' the fibres of $g^{-1}(U_{t_0})$, ie., for all $t \in U_{t_0}$, the diagram

$$\begin{array}{ccc} N_t & \cong & N \\ {\scriptstyle \varpi_t} \downarrow & & \downarrow {\scriptstyle \pi} \\ M_t & \cong & M \end{array}$$

commutes. Although for a given surface N, the non-uniqueness of minimal model implies non-uniqueness of ϖ, the construction does not depend on the specific choice of minimal M. Details of the construction of ϖ, along with a proof of the following theorem, may be found in [5] (cf. theorem 4, section 5).

Theorem 3 *If $\mathscr{N} \xrightarrow{g} D$ is a holomorphic family of compact complex surfaces, locally trivial over $D \setminus Z$, such that $N_t = g^{-1}(t) \cong N$ a compact surface having as minimal model a ruled surface of one of the types previously discussed, then $d \geq 4 \Rightarrow \mathscr{N}$ is trivial over D.*

6 Some remarks on holomorphic vector bundles

A holomorphic vector bundle E, of rank two on X, is said to be "simple" if the only endomorphisms $E \xrightarrow{\alpha} E$ are complex-scalar, ie., for all $p \in X, \alpha_p = \lambda I \in End(E_p)$, for some $\lambda \in \mathbf{C}$ (note that λ is independent of p, since X is compact). More concisely, the criterion for simplicity may be expressed as

$$dim_{\mathbf{C}} H^0(X, E \otimes E^*) = 1.$$

Conversely, given $dim_{\mathbf{C}} H^0(X, E \otimes E^*) > 1$, it is always possible to find an endomorphism such that the image has rank one. For suppose α is a non-scalar automorphism of E, and choose $p \in X$. Let ε be an eigenvalue of α_p, ie.,

$$\delta(p) = det(\alpha_p - \varepsilon I) = 0.$$

Now the holomorphic function δ must be constant, once more due to compactness of X, hence $\delta \equiv 0$. Hence there is an endomorphism

$$\varphi = \alpha - \varepsilon I$$

such that $L = im(\varphi)$ has rank one, for otherwise $\alpha = \varepsilon I \Rightarrow \alpha$ is scalar (cf. [1], proposition 16). Suppose that $\varphi^2 = 0$, then $im(\varphi) \subseteq ker(\varphi)$. But $\varphi \neq 0 \Rightarrow K = ker(\varphi)$ has rank less than or equal to one, therefore $L = K$, yielding a short exact sequence

$$0 \longrightarrow L \longrightarrow E \xrightarrow{\varphi} L \longrightarrow 0. \qquad (1)$$

Consequently, each such vector bundle E corresponds to an extension class

$$\xi \in H^1(X, \mathcal{O}_X).$$

If, on the other hand, $\varphi^2 \neq 0$, then $L \cong \varphi(L) \subset E \Rightarrow 0 \longrightarrow K \longrightarrow E \xrightarrow{\varphi} L \longrightarrow 0$ is a split exact sequence, ie.,

$$E \cong K \oplus L, \qquad (2)$$

(cf. [3], lemma 4.4). In particular, $dim_{\mathbf{C}} H^0(X, E \otimes E^*) > 1 \Rightarrow E$ cannot be stable, otherwise

$$L \subset E \Rightarrow deg(L) < \frac{1}{2} deg(E),$$

and

$$E \longrightarrow L \longrightarrow 0 \Rightarrow deg(L) > \frac{1}{2} deg(E),$$

a contradiction. Moreover, E is semistable implies $deg(L) = deg(K) = \frac{1}{2} deg(E)$ by the same argument, applied to (1) or (2). Now, if $M \cong \mathbf{P}(E)$ is a hyperelliptic ruled surface, we may draw the following

Corollary 1 *If $\mathcal{M} \xrightarrow{f} D$ is a holomorphic family of compact complex surfaces, locally trivial over $D \setminus Z$, such that $f^{-1}(t)$ is isomorphic, for all $t \in D \setminus Z$, to $M = \mathbf{P}(E)$, where E is stable, then $codim_D(Z) \geq 2 \Rightarrow \mathcal{M}$ is a trivial family.*

Proof. Suppose $\sigma \in \mathcal{G}_\theta$ is a holomorphic vector field on $M = \mathbf{P}(E)$. Integration of σ gives rise to a complex τ-parameter family of fibre-preserving automorphisms of M, which lifts to a family of automorphisms α_τ of E, uniquely determined up to multiplication by a constant scalar, $\lambda_\tau \in \mathbf{C}^*$. But by the above argument, E stable implies $dim_{\mathbf{C}} H^0(X, E \otimes E^*) = 1$, hence $\alpha_\tau = \lambda_\tau I$, for all $\tau \in \mathbf{C} \Rightarrow \sigma \equiv 0$. Consequently

$$\mathcal{G}_\theta = 0 \Rightarrow R^0 f_*(\Theta) \mid_{D \setminus Z} = 0 \qquad (g_X \geq 2),$$

hence each vector field $\vec{v} \in \Gamma(D, \mathcal{T}D)$ admits a global lifting $f^*(\vec{v} \mid_{D \setminus Z})$, which extends to $\mathcal{M}$ by Hartogs' theorem, when $codim_D(Z) \geq 2$. □

Remark. For the case $g_X = 1$, note that E stable implies $R^0 f_*(\Theta) \mid_{D \setminus Z}$ is a holomorphic line bundle, hence triviality of $\mathcal{M}$ requires $codim_D(Z) \geq 3$.

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