Masters in Education by Research Thesis

There is More Than One Way to Skin a Cat: An Exploration of Flexible Mental Multiplication Strategies with Pre-Service Teachers

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Abstract

This research project investigates possible strategies that satisfy Australian and NSW curriculum outcomes for the Year 5 topic of mental multiplication and teaching strategies that use a flexible approach to mental multiplication that have the potential to enhance student number sense. Two cohorts of pre-service teachers (n=36) at the University of New England participated in the three phases of the project. The initial phase involved timed testing of their mental multiplication skills and untimed reflection on strategy use for the test questions. A questionnaire explored their attitudes towards, and knowledge of, mental computation in this context. Participants were introduced to a range of possible mental multiplication strategies and a teaching approach emphasising flexibility, adaptivity and multiplicative thinking. The final phase of the project sought to measure changes in the performance and attitudes of the participants who had taken part in the intervention. This was contrasted with the results and opinions of those who had only been involved in the testing and questionnaire components and not the intervention. The project uses a case study design with a mixed methods approach.

The participants demonstrated a poor knowledge of, and performance in, mental multiplication at the level required of Year 5 students. Considering that the majority had experienced schooling in the last 20 years when curriculum planners had emphasised the importance of mental computation skills, there was a distinct lack of knowledge of appropriate strategies for this topic. The use of Hill, Ball and Schilling's (2008) Mathematical Knowledge for Teaching (MKT) framework identified deficits in the participants' knowledge of the topic and their preparedness to teach it. Whilst the intervention was well received, its short duration meant that significant gains could not be made in the participants' own SCK. The data did show an increase in the range and flexibility of the strategies used to solve mental multiplication problems and an improvement in timed testing results. There was an attitudinal shift of the participants away from a traditional algorithmic approach to the topic towards more flexible, number sensible approaches.

The research is significant as it adds to the literature concerning mental multiplication strategies and possible teaching programs in the later primary years. Whilst there is considerable research concerning mental computation, it has focussed largely on the early

years of primary and, in particular, addition and subtraction. Studies of mental multiplication with pre-service and practising teachers are less common as are specific recommendations for appropriate strategies to include in teaching programs (Hartnett, 2007). Both the Australian and NESA outcomes and support documents are vague in this area with few examples of actual strategy use for mental multiplication. It is hoped that this study promotes further discussion of this topic leading to improved curriculum guidelines and flexible teaching approaches.

Style

The style and format adopted in this thesis is that employed by the American Psychological Association as recommended in their Publication Manual (American Psychological Association, 2010). To assist with readability the font size of a number of the larger tables and figures has been reduced. Table Captions are listed on one line to allow a list of tables to be compiled.

Certification of Dissertation

I certify that the ideas, experimental work, results, analyses, software and conclusions reported in this dissertation are entirely my own effort, except where otherwise acknowledged. I also certify that the work is original and has not been previously submitted for any other award, except where otherwise acknowledged.

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19/2/2019

Signature of Candidate — Date

ENDORSEMENT

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Glossary of Terms

General

ACARA	Australian Curriculum, Assessment and Reporting Authority
ACMNA	Australian Curriculum: Mathematics Number and Algebra`
AEC	Australian Education Council
DfES	Department for Education and Skills, UK
NDIA	National Disability Insurance Agency
NESA	NSW Education Standards Authority
PST	Pre-service Teacher

Mental Multiplication Strategies

Code	Descriptor	Examples
BDF	Basic or Derived Fact	5 x 8 = 40, 50 x 8 = 400, 50 x 80 = 4000
COU	Counting – skip counting, doubling +/-	3 x 24 = 24+24+24 = 48+24 =72 3x24=2x24+24 = 48+24 = 72
BFS	Basic Fact Shortcuts – shortcuts for multipliers from 2 to 12, 15 and 20	x5 = ½ x 10 x8 = double, double, double x9 = x10 – x1 x11 = x10 + x1 x12 = x10 + double x15 = x10 + x5
LRS	Left to Right Separated – using the Distributive Law	7x23: 7x20=140, 7x3=21, 140+21=161
RLS	Right to Left Separated – visualising the formal algorithm	7x23: 7x3=21write 1 and carry 2, 7x2=14, 14+2=16. Answer=161
FAC	Factors	6x32: 6=2x3, 32x3=96, 96x2=192
PAR	Partitioning (Non-standard)	8 x 21 = 8x10+8x11 = 80 + 88 = 168 19x34 = 20x34 – 1x34 = 680 – 34 = 646
CMP	Compensation	17x25 = 17x100 ÷ 4 = 425 15x24 = 30x12 = 360
CMB	A Combination of the above strategies	21 x 23 = 11 x 23 + 10 x 23 (PAR) = 253 (BFS) + 230 = 483

Strategies used in Whitacre's (2007) Study

Code	Descriptor	Examples
AD	Additive Distributive	8 x 21 = 8x10+8x11 = 80 + 88 = 168
AP	Aliquot Parts	17x25 = 17x100 ÷ 4 = 425
DER	Derived	21x23 = 21x22 + 1x21 = 21x2x11 + 21 = 462 + 21 = 483
FAC	Factors	6x32: 6=2x3, 32x3=96, 96x2=192
MASA	Mental Analogue of a Standard Algorithm	7x23: 7x3=21write 1 and carry 2, 7x2=14, 14+2=16. Answer=161
PP	Partial Products	21x23 = 20x20 + 20x3 + 1 x 20 + 1x3
SD	Subtrcative Distributive	19x34 = 20x34 – 1x34 = 680 – 34 = 646

Addition and Subtraction Strategies

Code	Descriptor	Examples
1010	Splitting both numbers into tens and ones and dealing with the parts separately	19 + 23 = 10 + 20 + 9 + 3
N10	Splitting one number into tens and ones and adding to the other number	19 + 23 = 19 + 20 + 3
A10	One number is split to bridge to a multiple of ten and then the remainder is added	23 + 12 = 23 + 7 + 5 = 30 + 5
N10C	One number is rounded up or down to a multiple of ten then added. An adjustment for the rounding is then made	19 + 23 = 20 + 23 - 1

Acronyms for Hill, Ball and Schilling's (2008) Framework

MKT Mathematical Knowledge for Teaching

SMK Subject Matter Knowledge, which is subdivided into:

CCK Common Content Knowledge – knowledge in common with other professions

SCK Specialised Content Knowledge – knowledge particular to mathematics, solutions, rules and procedures

KMH Knowledge at the Mathematical Horizon – how that topic fits with other topics, strands and the program of learning

PCK Pedagogical Content Knowledge, which is subdivided into:

KCS Knowledge of Content and Students – what motivates students, conceptions and misconceptions, where their ideas fit in a topic, what is difficult or easy

KCT Knowledge of Content and Teaching – the sequencing, examples and explanations used to teach a topic

KCC Knowledge of Content and Curriculum – the strands, outcomes and recommendations of the curriculum.

Definitions of Terms

Adaptivity the selection of the most appropriate strategy for a particular problem

Algorithm a process or set of rules to be followed to complete a calculation. Algorithms may be formal – a common recognition of a procedure for completing a calculation; or informal – one of many procedures that complete a calculation.

Aliquot parts a quantity which can be divided into another a whole number of times e.g. 25 is an aliquot part of 100.

Approximation a value or quantity than is nearly but not exactly correct

Automaticity an ability to recall facts accurately without occupying working memory with the required calculation

Distributive Law the law relating the operations of multiplication and addition, stated symbolically, $a(b + c) = ab + ac$; that is, the monomial factor a is distributed, or separately applied, to each term of the binomial factor $b + c$, resulting in the product $ab + ac$.

Estimation an approximate calculation of the value of a numerical operation

Fluency the ability to choose appropriate procedures; carry out procedures flexibly, accurately, efficiently and appropriately; and recall factual knowledge and concepts readily.

Flexibility the use of a variety of strategies

Mental arithmetic the recall of basic number facts learnt through drill and practice

Mental computation the ability to calculate exact numerical answers without the aid of external devices

Multiplicative thinking: an awareness of the relationship between multiplying and dividing, factors and multiples, times bigger and smaller, the use of materials and arrays for modeling products and the effect of multiplying and dividing by powers of ten

Number sense general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgments and to develop useful and efficient strategies for managing numerical situations

Chapter 1 – Introduction to Research

1.1 Introduction

Mark Twain used the phrase: 'there's more than one way to skin a cat' in his 1889 novel, 'A Connecticut Yankee in King Arthur's Court'. The phrase was in common use, at that time, in the southern states of the US and suggested that there was more than one way to solve a particular problem. The 'cat' referred to the local catfish that had to be skinned prior to cooking and not some bizarre southern ritual with felines. The premise of looking for alternate methods of solution is at the very heart of this research project. All too often, educators see Mathematics as a lock-step discipline with only one correct method leading to one correct answer. Students are done a great disservice if they are not given the necessary skills to explore alternate pathways to solutions and evaluate the effectiveness of these alternatives. The twin concepts of flexibility and adaptivity in strategy choice (Verschaffel, Luwel, Torbeyns & Van Doren, 2009) are at the very heart of this project.

This research project takes the concept of mental multiplication by one- and two-digit numbers and explores it in depth to show that there may be alternative strategies that warrant inclusion in current primary school mathematics curricula. Initial research demonstrated that the methods recommended for mental multiplication in the Australian and complementary NSW Education Standards Authority (NESA, 2015) curriculum documents may not always be the best pathway to a solution and there may be other strategies and algorithms that warrant inclusion in curriculum planning.

1.2 Reasons for Research

An interest in this topic stems from the candidate's extensive experience with teaching Mathematics to primary and secondary students in NSW schools and a particular interest in mental multiplication. Experience with the delivery of professional development activities for primary school and pre-service teachers has also led to concerns relating to the specific content knowledge of both groups in relation to mental multiplication and their attitudes towards its importance in the curriculum.

These reflections, coupled with anecdotal evidence from the classroom, resulted in the candidate researching alternate ways of teaching mental multiplication with a problem solving emphasis that builds on student-developed strategies and enhances number sense. There also appears to be a significant deficit in the range of mental multiplication strategies that are mentioned in either the Australian (2012) or NSW (NESA, 2015) curriculum documents for this Year 5 outcome The mental multiplication strategies mentioned are limited to left-to-right multiplication (highest powers of ten first), use of the Distributive Law, partitioning and factors. Although the formal algorithm is mentioned, it would not be regarded by researchers as a strategy appropriate for mental multiplication (Clark, 2005). Examples of how these strategies may be applied are also limited as is any guidance in the order of teaching the varied written, mental and digital competencies required by the outcome.

These initial observations have resulted in the formulation of the research problem.

1.3 The Research Problem

There is a need for teachers and pre-service teachers to understand the relevance of mental computation to students' number sense as well as strategies for teaching mental multiplication as a problem solving exercise. As Hartnett (2007) points out, the present generation of primary teachers, usually experienced traditional algorithmic procedures in their own education and have little knowledge of mental computation strategies. Although many can see the benefits of including mental computation strategies in their programs they lack the confidence, ideas and knowledge to proceed.

Hence the decision to conduct pilot research in the specific area of Year 5 (NESA, 2015) mental multiplication with two cohorts of pre-service teachers at the University of New England. This involved an initial assessment of their prior knowledge of the topic using Hill, Ball and Schilling's (2008) framework of Mathematical Knowledge for Teaching (MKT). This was followed by intervention and analysis of changes in participants' knowledge through testing and questionnaire.

As the research literature suggests, the project includes opportunities to enhance participants' flexible and adaptive solutions to mental multiplication problems. The extensive use of the Distributive Law in non-standard partitioning also focuses on developing participants' multiplicative thinking. A taxonomy of mental multiplication strategies was developed by the researcher from strategies suggested by the research literature. This was trialled with the pre-service teachers resulting in their recommendations for a teaching program for the Year 5 topic of mental multiplication. The final outcome of the project is a set of recommendations for future curriculum planning in the area of mental multiplication.

1.4 The Research Questions

The following research questions were formulated for investigation within a pre-service teaching program:

1. What was the prior knowledge of mental multiplication displayed by pre-service teachers?
2. How did this change following exposure to strategies suggested in the research literature and presented through the intervention?
3. What strategies do pre-service teachers propose for a teaching program that addresses the mental multiplication component of the Year 5 Australian Curriculum outcome ACMNA100 and its NSW counterpart?
4. What implications for practice are suggested by the study for possible inclusion in curriculum documentation?

1.5 The Significance and Innovation of the Research

The project will be significant as it will allow for the publication of this thesis and subsequent journal articles addressing gaps in the academic literature relating to:

- the content knowledge of mental multiplication that a particular cohort of pre-service teachers exhibit prior to intervention;
- changes in pedagogical content knowledge post intervention; and

- what pre-service teachers consequently value as mental multiplication strategies in their own teaching program for this topic.

In addition, it will provide a valuable resource for primary school teachers by presenting a teaching approach to mental multiplication that focuses on flexibility and adaptivity in a way that changes simple calculations into problem solving activities (Trafton & Thiessen, 2004). The extensive use of the Distributive Law through partitioning also enhances multiplicative thinking in students and encourages number sensible solutions to problems. The project will identify mental multiplication strategies not currently included in curriculum documents and make recommendations concerning the accompanying notes and examples in curriculum documentation.

1.6 The Aims and Objectives of the Research

The primary aim of the research is to investigate strategies for mental multiplication, particularly by one- and two-digit numbers and trial these with a group of pre-service teachers. As mentioned, the approach chosen should also have the capacity to enhance students' flexible and adaptive use of number sensible strategies to solve mental multiplication problems.

The objectives of the research are to provide an insight into the prior knowledge of pre-service teachers in relation to mental multiplication and determine their acceptance of a suggested teaching strategy for this topic. The resultant thesis should also inspire further journal articles and conference presentations. The project may also result in recommendations to curriculum planners in the form of specific strategies to include, examples to clarify existing strategies and notes on the order of teaching this Year 5 topic.

1.7 The Thesis Structure

This research is reported over six chapters. This chapter has outlined the context of the study and described the research problem and related research questions that will be addressed by the study. Its significance to educational practice in the area of mental computation has been described, as have the overall aims and objectives of the project. Chapter 2 will provide a summary of the research literature concerning mental computation and number sense and some of the “big ideas” (Hurst, 2015) that relate to these topics. It begins with an outline of the importance of mental computation in curriculum planning documents with particular reference to the current Australian curriculum (ACARA, 2012) and its NSW counterpart (NESA, 2015). The relationships between mental computation, mental arithmetic and number sense are examined and definitions provided in the research literature will be described. The development of teaching approaches to the topic in the last three decades is considered as are the current trends in research that include multiplicative thinking, flexibility and adaptivity and number sensible strategies for mental computation. The specific knowledge required to teach the topic is related to established frameworks for the mathematical knowledge required for teaching. Particular reference is made to studies involving pre-service teachers and their knowledge of number sense, mental computation techniques and their beliefs and attitude towards mathematics. The chapter also includes a summary of mental computation strategies proposed by various researchers and a proposed taxonomy for mental multiplication strategies.

Chapter 3 outlines the research paradigm, theoretical framework, research design and methodology. This study is an explanatory case study (Yin, 2015) using a mixed methods approach involving predominately quantitative data supported by qualitative, free response questions. Data collection instruments and data analysis are also outlined in this chapter. Finally, ethical considerations are mentioned.

Chapter 4 involves analysis of the data collected during Phase 1 of the project. The data collection instruments used include a timed test of 15 mental multiplication questions, an untimed test coding sheet where the participants suggest possible strategies for the test questions and a questionnaire. One section of the questionnaire allows them to expand on their knowledge of mental multiplication strategies through four questions repeated from the initial test. The questionnaire is designed to provide information concerning the participants’ knowledge of mental multiplication using five of the six strands of the MKT framework. Test results are analysed and compared by cohort and correlated with items from the questionnaire when significant.

Chapter 5 continues the data analysis, looking at the Test 2 results in relation to Test 1 results and the two groupings resulting from attendance at the intervention. Strategy use is compared between tests and between the same two groups. This is also related to Whitacre’s

(2007) study of mental multiplication strategy use with PSTs. The questionnaire is used to identify any changes to the knowledge and attitudes of participants who were involved in all three phases of the project.

Chapter 6 relates the data analysis of the two previous chapters to the research problem and questions. Specific conclusions and implications for practice are made in relation to the research questions. The chapter also considers the limitations of the project and makes recommendations for possible changes for future researchers in this area. Opportunities for future research are also considered.

1.8 Chapter Conclusion

This project investigates possible strategies that satisfy Australian and NSW curriculum documents for Year 5 mental multiplication and teaching strategies that use mental multiplication as a problem solving exercise that has the potential to enhance students' number sense, multiplicative thinking and knowledge of flexible and adaptive approaches to mental multiplication. The research problem and associated questions outline a focus for the research that is pursued through the remaining five chapters of this study.

It is expected that recommendations will be made for the possible inclusion of some of these mental multiplication strategies in the present Australian curriculum based upon the trial with two cohorts of pre-service teachers at the University of New England. The results of this trial will be analysed and reported through conference papers and peer referenced journal articles and suggestions for future research will be outlined.

Chapter 2 - Literature Review

2.1 Introduction

This chapter presents a review of literature concerning mental computation, its strategies and algorithms and studies concerning the teaching of mental calculation skills and number sense. Studies of various mental computation strategies and algorithms lead naturally towards a taxonomy of these, followed by research involving the best ways to introduce these strategies to students and the benefits derived from them developing skills in mental calculation. There is also discussion of teacher and pre-service teacher conceptions and knowledge of mental computation and number sense. Recent research foci of flexible and adaptive use of number sensible strategies that enhance multiplicative thinking in students are also reported.

Mental computation is viewed as an important life skill for adults. Northcote and McIntosh (1999) found that adults used mental computation for over three quarters of calculations involved in their everyday lives. Written calculations and calculator use was responsible for less than 15% of these calculations. This was reinforced in a more recent Australian study by Northcote and Marshall (2016) who found that the adults in their survey predominantly used mental calculations in 86% of daily calculations reported. Written methods only featured in 10% of calculations and calculator and phone use were rarely reported.

One consequence of this research is the realisation that mental calculation involves much more than the basic recall of number facts. Adults regularly complete accurate mental calculations and estimations that involve two and three-digit numbers, fractions, decimals and percentages. As a result, a number of local and overseas curriculum documents have shown increasing support for mental computation because of its perceived value as a future life skill for adults and for its contribution to number sense.

2.2 National and international support for mental computation

Those national and international documents that have shown increased support for mental computation to have a higher profile in their mathematics curricula, include the Netherlands

(Treffers & DeMoor, 1990), the United States *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000), the United Kingdom *Primary Framework for Literacy and Mathematics* (DfES, 2007), and the *National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991). The AEC suggested that students

> should regard mental arithmetic as a first resort in many situations where a calculation is needed. Strategies associated with mental computation should be developed explicitly throughout the schooling years, and should not be restricted to the recall of basic facts (p. 109).

This was supported strongly in the current Australian Curriculum (ACARA, 2012) with many references to mental computation. For example, the outcome ACMNA100 (v8.3 F-10 Mathematics) is a Year 5 Mathematics outcome that asks students to:

> Solve problems involving multiplication of large numbers by one- or two-digit numbers using efficient mental, written strategies and appropriate digital technologies.

The wording of this outcome and its accompanying notes, like all elements of the Australian Curriculum, gives State bodies and teachers the freedom to choose the combination of strategies that they feel best achieve the outcome. Of those mentioned, partitioning and the use of the Distributive Law are the only ones that relate to mental multiplication (see Appendix 7).

The NESA K-10 Mathematics Curriculum (2015) interprets this outcome through its omnibus outcome MA 3-6 NA. It suggests that to achieve the outcome the student:

> Selects and applies appropriate strategies for multiplication and division, and applies the order of operation to calculations involving more than one operation.

This is further delineated in the Content section of Multiplication and Division 1 (see Appendix 7), which suggests the use of standard partitioning (working left to right), the formal algorithm for single digit multipliers (working right to left) and factorization for mental multiplications.

Background Information to this outcome (see Appendix 7) adds one new strategy of basic fact shortcuts to the mental multiplication repertoire, although the suggestion to “memorise multiples of 11, 12, 15, 20 and 25” is contrary to research literature (Menon, 2003, Whitacre, 2007) which suggests techniques for calculating these multiples based the Distributive Law and Aliquot parts.

The National Numeracy Learning Progression (NESA, 2018) through its sub-element, Multiplicative Strategies (see Appendix 7) suggests using the strategies of factorization, standard partitioning (left to right) and basic fact shortcuts (x8 is double x4) for mental multiplication. A later section of this chapter relates these suggestions to others suggested in the research literature.

Analysis of both the ACARA and NESA documents mentioned above suggests that teachers in NSW schools would be expected to place a greater emphasis on the formal algorithm than originally implied in the ACARA outcomes and supporting information. There is also little reference in the NESA syllabus concerning the use of partitioning and the Distributive Law other than standard partitioning from left to right or right to left through the short form of the formal algorithm. Both ACARA and NESA documents provide little guidance relating to the order and sequence of teaching this outcome and do not explicitly discriminate between mental and written strategies.

The Australian Curriculum: Mathematics (ACARA, 2012) also suggests, through its Proficiency Strands (see Appendix 7), that there should be a focus on developing increasingly sophisticated and refined skills of mathematical understanding, fluency, reasoning and problem solving. Fluency is not defined solely as the recall of basic facts and procedures but extends to the choice of appropriate strategies that are then applied flexibly, accurately and efficiently to arrive at robust solutions to problems and computations. This fits well with the concepts of flexibility and adaptivity mentioned later in this chapter.

The NESA Mathematics syllabus also has a central theme of Working Mathematically (see Appendix 7) that translates into specific outcomes for each Stage. Like the ACARA proficiencies, these outcomes relate to the progressive development of mathematical understanding and fluency through inquiry, exploring and connecting mathematical concepts, choosing and applying problem-solving skills and mathematical techniques, communication and reasoning. The definition of fluency uses identical language to that used in the ACARA proficiencies.

2.3 Mental arithmetic, mental calculation and number sense

The historical development of concepts in this area initially defined mental computation as the ability to calculate exact numerical answers without the aid of external devices (Reys, Reys & Hope, 1993; Sowder, 1988). Exact answers refer to the difference between mental calculation and estimation and approximation, both of which are facilitated by strong mental calculation skills (Reys, 1984). Trafton (1978) further suggests that mental calculation may involve the use of non-traditional algorithms and strategies, thus implying that there are specific strategies for mental computation that are different from traditional pen and paper algorithms.

Threlfall (2002) states that these strategies occur "where students can be correct by constructing a sequence of transformations of a number problem to arrive at a solution as opposed to just knowing, simply counting or making a mental representation of a 'paper and pencil' method" (p. 30). The concept of ' a sequence of transformations' used to arrive at a solution suggests that mental computation is much more than familiarity with basic number facts. It implies a higher order thinking skill (Reys & Nohda, 1994) involving choices for the student between viable pathways, all of which lead to an exact solution of the problem being investigated. Similarly, Reys, Reys and Barger (1994) describe mental computation as a "vehicle for promoting thinking, conjecturing, and generalizing based on conceptual understanding rather than as a set of skills that serve as an end of instruction" (p. 31). This mirrors the concept of fluency as defined in the Australian Curriculum Proficiency Strands and outlined in the previous section.

This is far removed from the behaviourist conception of mental arithmetic as the drill and practice of basic number facts until recall is automatic (Shoenfeld, 2011). This concept of 'errorless learning' (Resnick, 1983, p. 7) still persists today in the form of programmed instruction and is best evidenced through Kumon Maths (Russell, 1996) and similar commercial and electronic tutoring packages. McIntosh (1995) reported a study by Biggs (1967) involving 5000 UK students who experienced regular practice of basic facts. Biggs found that their proficiency in mental arithmetic bore no relationship to the extent of regular practice. In fact, practising basic facts under time constraints tended to increase number anxiety in students.

The recall of basic number facts does, however, have an important part to play in mental calculation. Automaticity with basic number facts, particularly using varied formats of presentation, is still a valued goal for students. Cowan, Donlan, Shepherd, Cole-Fletcher, Saxton, and Hurry (2011) comment that research shows that competence with basic number facts covaries with achievement in mathematics. They explain that having basic fact solutions stored in long term memory frees working memory for more sophisticated calculations, particularly involving larger numbers. Wong and Evans (2007) also make the point that the absence of automaticity may cause learning to stall while the student searches for the required fact. Rote recall of basic facts, however, should not be confused with mental computation.

Heirdsfield and Cooper (2002) showed that efficient mental computation involved much more than the recall of basic number facts. Students proficient in mental computation had integrated understanding of number facts that allowed them to select between appropriate strategies depending on the number combinations in problems. "These proficient students also exhibited some metacognitive strategies and beliefs, and affects (e.g., beliefs about self and teaching) that supported their mental computation." (p. 45) Similarly, Barody (2006) argued that proficiency in mental calculation involved the use of efficient strategies to accurately solve problems rather than simple recall of basic facts. He found that this was dependent on students' number sense involving an understanding of number operations, patterns and principles. Heirdsfield, Cooper, Mulligan and Irons (1999) recommended that the teaching of mental computation should focus on number understanding, principles of numbers and operations, and effects of operations on numbers; and encourage variety in the use of number sensible strategies (p. 92)

Number sense is described by McIntosh, Reys, Reys, Bana, and Farrell (1997, p. 324) as a student's "general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgments and to develop useful and efficient strategies for managing numerical situations". McIntosh, Reys and Reys (1992) developed a number sense framework based on six strands, one of which was computing and counting strategies. Hence, mental computation is an important subset of number sense. Research suggests that it not only relies on number sense, but, that it can also make an important contribution to the development of number sense, As Hartnett (2007) points out, "needing number sense for efficient use of computation strategies, and the development of number sense by using such strategies, are very closely interrelated" (p. 346). This is also supported by the National Numeracy Learning Progression (NESA, 2018) that sees Multiplicative Strategies as a sub-element of Number Sense and Algebra.

To achieve this contribution to number sense, research suggests that it is necessary to develop proficiency in mental computation through the acquisition of self-developed or spontaneous strategies rather than memorisation of number facts and procedures associated with mental arithmetic (Kamii, Lewis, & Livingston, 1993; McIntosh, De Nardi & Swan, 1994; Reys & Barger, 1994). This implies a reduction of emphasis on written algorithms, a growth in instruction time spent on arithmetical properties and alternative computational strategies, and a change to more child-centred and flexible approaches to teaching operations. Hence, much of the research in the last 30 years has focused on these twin concepts of mental computation and number sense with a strong recommendation for constructivist approaches in the classroom that utilize more effective and creative approaches (McIntosh, 1995).

2.4 The teaching of mental computation

Early research into the teaching of mental computation strongly supported the transition to student-developed strategies and student-centred teaching. The initial problem, however, was a lack of research data on general levels of competency in mental computation, the types of problems that students would attempt mentally and the nature of strategies used. Reys, Reys and Hope (1993) conducted a longitudinal study of three school grades across two school districts (Texas, USA and Saskatchewan, Canada) testing twice in the year with timed items delivered with a mix of verbal and visual presentation. The results indicated a 'dismal' level of performance and prompted the comment that progress in mental computation performance was dependent both on teachers recognising its importance and allocating time for students to develop their own computational alternatives based on their own mathematical knowledge. (p.314)

Their recommendations for flexible thinking that results from learner-centred techniques led to follow-up studies by Bell, Balfanz, Carroll, Hartfield, McBride and Saecker (1999) and Carroll (1996) that compared the performance of students experiencing the US Reform curriculum, ‘Everyday Maths’, with Reys’ sample. Students doing the Reform Curriculum had been encouraged to invent and discuss their own solutions to mental problems in a constructivist classroom atmosphere. They also experienced traditional algorithms as per curriculum requirements. Both studies showed a marked improvement in performance of mental computation without any drop in competency for pen and paper algorithms.

Given evidence of poor performance in mental computation by cohorts experiencing traditional teaching methods and formal algorithms, the historical research took two directions. The first involved a focus on student-developed, intuitive strategies involving the use of concrete materials and concepts and teacher facilitation (Carroll, 1997; Heirdsfield, 2002, 2003), while the second considered teacher-led introduction of efficient strategies that supported and streamlined student efforts and still involved classroom discussion (Belshuizen, 2001; Threlfall, 2002).

Carroll (1997) supported the first approach and reported on the types and the range of student-developed strategies in the younger years, particularly in addition and subtraction. He commented that “whilst students did not always use the most efficient strategy the strength of their number sense seems to have been enhanced by allowing them to use their own strategies” (p. 96). Belshuizen (2001), however, was concerned that an incomplete set of strategies may lead to efficient strategies not being available for use because they had not been taught. Anghileri (1989) and Mulligan and Mitchelmore (1997) produced evidence that, even though multiplication and division were harder concepts, students were still able to produce their own strategies for one- and two-digit multiplication and division problems. Clarke (2001) commented that “occasionally, teachers claim that, ‘only the brighter children can create their own algorithms’. Those involved in projects that encourage children to create their own algorithms dispute this, but even if it were true, the encouragement for children to do so will likely yield a range of algorithms. These can be shared publicly and discussed, and children who are unable to create a written method of their own will at least have a range of options from which to choose for their own use” (p. 96).

Heirdsfield, Cooper, Mulligan and Irons (1999) summarised the early research in this area, identifying the following points:

- Children should be encouraged to invent their own computational procedures, as they develop better understanding of the effects of operations on number, and place value.
- If computational strategies are introduced by teachers, they should build on student-developed strategies.
- Children can better take responsibility for their own learning through being allowed to investigate their own solution strategies to problems.
- Number sense is enhanced by students developing their own computation strategies.
- Whatever the teaching approach, time should be spent on students describing and discussing various solution strategies for problems.
- Student-developed strategies should be valued equally for the understanding displayed as well as the misconceptions. (p. 96)

Another common theme in research documents has been the conviction that formal pen and paper algorithms should not be taught until students had thoroughly experienced developing their own solution strategies. Narode, Board and Davenport (1993) summarised their findings:

> We believe that by encouraging students to use only one method (algorithmic) to solve problems, they lose some of their capacity for flexible and creative thought. They become less willing to attempt problems in alternative ways, and they become afraid to take risks. Furthermore, there is a high probability that the students will lose conceptual knowledge in the process of gaining procedural knowledge (p. 260).

Similarly, Carpenter and Moser (1982) found that once children learned formal arithmetic procedures they stopped analysing the addition and subtraction problems they had previously been able to solve. Clark (2001) believes "that there is no place for formally introducing conventional algorithms to children in the first five years of school" (p. 96). The 2002 NSW Board of Studies Curriculum Notes seemed to support this opinion when it stated that "formal written algorithms are introduced after students have gained a firm understanding of basic concepts including place value, and have developed mental strategies for computing with two-digit and three- digit numbers". (BOS, 2002, p. 9)

Despite this, and similar warnings from curriculum planners, Verschaffel, Greer and De Corte ((2007) found that while most researchers advocated for students in later years of primary to be exposed to both mental and written strategies, an inordinate amount of time was still spent practising traditional algorithms. Gravemeijer, van Galen, Boswinkel and van den Heuvel-Panhuizen (2011) went further by suggesting that formal algorithms could be replaced by student-developed semi-informal routines "provided they are grounded in well-developed number sense" (p. 126). They argued that if appropriate problem models are used that suit informal routines then students can develop flexible pathways to the solution. These informal routines can later be developed into conventional algorithms if needed. This would seem to be supported in the current Curriculum Notes from the NESA (2015) K – 10 syllabus which states that "students may find recording (writing out) informal mental strategies to be more efficient than using formal written algorithms, particularly in the case of multiplication" (NESAm 2015 Background Information to Mathematics and Division 1, para 2)

An example of an informal routine, provided by Gravemeijer et al. (2011, p133), is a ratio table that can then be linked to the formal algorithm.

Example: Consider 152 x 242. This results in the ratio table below.

1	2	50	100	152
242	484	12100	24200	36784

This can then be used to model the formal algorithm as shown below.

```
  242 x
  152
  ---
  484
12100
24200
-----
36784
```

Trafton and Thiessen (2004) argue that rather than seeing computation as a set of rules and procedures to be learnt and remembered, it needs to be approached as a problem solving activity. If students are encouraged to make decisions about a problem and choose from a diverse range of alternate strategies, then simple calculations can become problems.

Threlfall (2002) advocated the need for teaching that fostered flexibility and choice. He maintained that, for flexibility in mental calculation, a deep understanding of number and operation relationships and knowledge of basic facts and fact families was required. This concept is central to this research project as are the associated concepts of flexibility and adaptivity – a focus of recent research in this field.

Verschaffel, Luwel, Torbeyns and Van Doren (2009) distinguish between these terms, defining flexibility as the use of a variety of strategies and adaptivity as the selection of the most appropriate strategy for a particular problem. Flexibility is encouraged by the establishment of inquiry-based classroom cultures wherein students discuss whether a strategy is reasonable, identify its weaknesses and then further strengthen their arguments by considering input from others. Boaler (2015) reinforces this view by stating, "the core of mathematics is reasoning - thinking through why methods make sense and talking about reasons for the use of different methods" (p. 4).

In the particular area of multiplication and division, recent research has focused on the concept of multiplicative thinking. Hurst and Hurrell (2015, 2016) see the component parts of multiplicative thinking as an awareness of the relationship between multiplying and dividing, factors and multiples, times bigger and smaller, the use of materials for modeling problems and the effect of multiplying and dividing by powers of ten. Siemon, Bleckley and Neal (2012) maintain that multiplicative thinking underpins important concepts of ratio and proportion, fractions and algebraic thinking and is one of the 'big ideas' in mathematics education. This has bearing on this project as multiplicative thinking deals with working flexibly with a wide range of numbers and solving multiplication (and/or division) problems then reporting the solutions in different ways. The intervention in this project sought ways that students could be flexible in their solution strategies and find alternate ways to represent their solutions. Proponents of multiplicative thinking have also suggested the importance of students' understanding and use of the Distributive Law and this was central to the strategies used in this project. Hurst and Hurrell (2017) described the distributive property as "a critically important understanding that numbers can be partitioned to make operating with them easier" (p. 28).

Whilst the focus on flexibility, adaptivity and multiplicative thinking has been applauded, progress has been dependent on teachers' beliefs and their specific content and pedagogical knowledge of computation. The next section deals with the knowledge required for teaching with particular reference to primary mathematics.

2.5 The Knowledge Required to Teach Mental Computation

As Hartnett (2007) points out, the present generation of primary teachers usually experienced traditional algorithmic procedures in their own education and have little knowledge of mental computation strategies. Although many can see the benefits of including mental computation strategies in their programs they lack the confidence, ideas and knowledge to proceed. This is particularly true when faced with harder computations involving multi-digit numbers (Verschaffel, Greer & De Corte, 2007).

Menon (2009) speaks of the three phases of mathematical subject knowledge: traditional, pedagogical and reflective. Teachers in the traditional phase rely on subject knowledge encountered when they were students and pre-service teachers. This is modified by experience with learners into the pedagogical phase where teaching approaches are modified by individual learner needs. The final phase of reflective subject knowledge sees teachers reflect on the actual subject matter through experiences of different teaching approaches. These three phases fit well with Hartnett's (2007) work with practicing teachers. She found that when working with teachers who acknowledged the need for a change of emphasis to mental computation there was a range of questions that needed answering. Questions such as the sequence of teaching, best teaching approaches, a taxonomy of strategies with clear examples, and appropriate resources were vital before progress in the classroom could occur and teachers could move from a traditional to a pedagogical phase.

Hill, Rowan and Ball (2005) demonstrated through parallel testing of teacher knowledge and student achievement in standardized tests that "teachers' performance on (our) knowledge for teaching questions—including both common and specialized content knowledge—significantly predicted the size of student gain scores [in their testing]" (p. 44). The authors argued that teachers needed a greater depth of understanding that went beyond the pedagogical questions of Hartnett's sample. They illustrated this through the formal algorithm for multiplication showing that mere knowledge of the strategy was not enough for teaching. Teachers needed to be able to explain it in a number of ways (partial products and arrays) suited to individual learner needs and questions and then needed to understand students' incorrect attempts at using the algorithm. These conclusions reinforced similar statements from the US Conference Board of the Mathematical Sciences (2001):

> Although almost all teachers remember traditional computation algorithms, their mathematical knowledge in this domain generally does not extend much further. ... In fact, in order to interpret and assess the reasoning of children learning to perform arithmetic operations, teachers must be able to call upon a richly integrated understanding of operations, place value, and computation in the domains of whole numbers, integers, and rationals. (p. 58)

As Sullivan (2008) has pointed out, it is not just a question of knowledge of mathematics, but also one of knowledge for teaching mathematics and knowledge of pedagogy. This is not a new area of research. In his seminal work of 1986, Shulman proposed that there was a "special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding" (Shulman, 1986, p. 8). He saw this as an intersection of Content Knowledge, Pedagogical Knowledge and Contextual Knowledge combining knowledge of subject matter, ways of teaching that topic and knowledge of the conceptions, preconceptions and misconceptions that learners of different ages bring to the classroom (Shulman, 1986, p. 9). Figure 2.1 models these three domains of teacher knowledge.

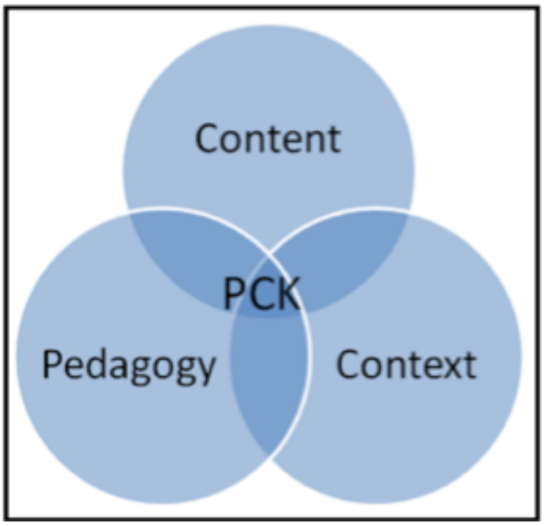


Figure 2.1 Shulman's (1986) Domains of Pedagogical Content Knowledge

Shulman (1987, p. 8) expanded his initial model of teacher knowledge to include seven categories:

- Content knowledge
- General pedagogical knowledge – strategies of classroom management and organization
- Curriculum knowledge – materials and programs that serve as tools of the trade
- Pedagogical content knowledge
- Knowledge of learners and their characteristics
- Knowledge of educational contexts from the classroom, to the governance of school districts, of communities and cultures
- Knowledge of educational ends, purposes and values and their philosophical and historical grounds

Shulman's model was modified by Hill, Ball and Schilling (2008) into a framework for Mathematical Knowledge for Teaching (MKT) shown below in Figure 2.2.

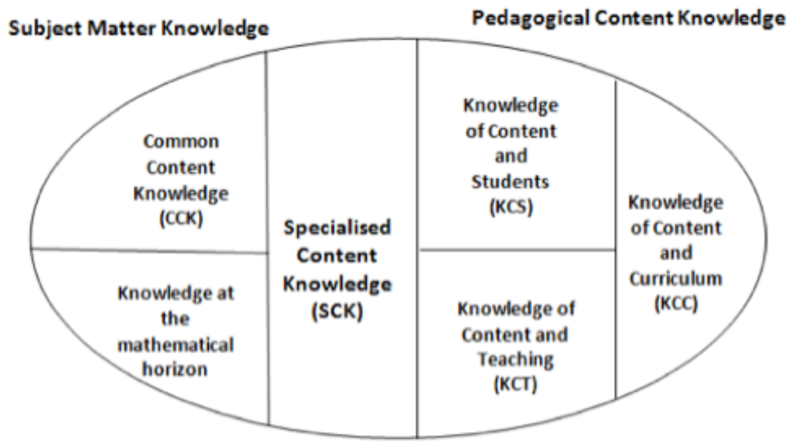


Figure 2.2 Mathematical Knowledge for Teaching (Hill et al., 2008 p. 377)

Divided into Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK), the authors identified six areas of knowledge. Subject Matter Knowledge is subdivided into:

- Common Content Knowledge (CCK) – knowledge in common with other professions
- Specialised Content Knowledge (SCK) – knowledge particular to mathematics, solutions, rules and procedures
- Knowledge at the Mathematical Horizon (KMH) – how that topic fits with other topics, strands and the program of learning

PCK keeps faith with Shulman's concept and is divided into:

- Knowledge of Content and Students (KCS) – what motivates students, conceptions and misconceptions, where their ideas fit in a topic, what is difficult or easy
- Knowledge of Content and Teaching (KCT) – the sequencing, examples and explanations used to teach a topic
- Knowledge of Content and Curriculum (KCC) – the strands, outcomes and recommendations of the curriculum. (pp. 377-8)

Shulman's framework is particularly suited to Mathematics and to teacher assessment and professional development and pre-service teacher training. It has been used by researchers in the field of number sense (Heirdsfield & Lamb, 2005), PCK of teachers (An, Kulm & Wu 2004; Baker & Chick 2006; Huckstep, Rowland & Thwaites 2003; Yang, Reys & Reys 2009) and pre-service teacher training (Cochran, DeRuiter & King 1993); Turnuklu & Yesildere 2007; Whitacre 2007). In this research, the Mathematical Knowledge for Teaching (MKT) framework (Hill et al., 2008) will guide the types of data collected, data collection methods and the analysis of that data as outlined in Chapter 3.

Askew (2008) adds another dimension to the debate on teacher knowledge through his concept of mathematical sensibilities. Like Davis and Summit (2006), he suggests that "the question is not one of whether or not classrooms are learner- centred or teacher-centred but whether or not they are mathematics centred" (p. 29). To foster in-depth discussions of mathematical sensibilities, he suggests that schools need to develop learning communities within their staff, thus avoiding a reliance on textbooks as "surrogate classroom partners" (p. 31). The sensibilities that he proposes underpin the science of mathematics and include concepts such as precision in working through a problem, being able to move from the specific to general, a curiosity about how things work and a care for the discipline that recognizes the "beauty and romance" (p. 25) that others see in it. Hurst (2014, p. 288) comments that "teachers who have such 'sensibility' are likely to be better able to make mathematical connections explicit for their students".

The concept of a set of mathematical sensibilities unique to the discipline is not unlike Ma's (1999) idea of profound mathematical knowledge. In her study comparing the knowledge of Chinese and US teachers of Mathematics, she related the profound knowledge of fundamental mathematics (PUFM) exhibited by Chinese teachers with their ability to see connections within the discipline and understand the conceptual underpinnings of the subject. Ma concluded that no amount of general pedagogical knowledge could make up for ignorance of particular mathematical concepts. One of Ma's examples of PUFM was being able to explain the formal algorithm for long multiplication through the distributive property.

Similarly, Hurst (2014) suggests that a focus on the "big ideas" (p. 287) of mathematics is needed to promote a more connected view of mathematics. He advocates that mathematical knowledge be re-conceptualised "in terms of the myriad connections and links that exist within and between mathematical ideas" (p.288). A focus on these 'big ideas" will them allow teachers to see the links between ideas and plan instruction to highlight these important connections. This project seeks to incorporate the big ideas of multiplicative thinking, flexibility and adaptivity and number sensible use of strategies into its design as outlined in Chapter 3.

The discussion of knowledge now returns to the specifics of Harnett's (2007) study relating to the need for a taxonomy of strategies with clear examples to aid teachers to engage fully with the topic of mental computation.

2.6 Mental Computation Strategies

The research into student-developed mental strategies has largely focussed on the younger years and, in particular, addition and subtraction (Caney, 2008). Beishuizen and others have been prominent in categorising four distinct strategies for addition and subtraction of numbers to 100 (Beishuizen 1985; Beishuizen, van Putten & van Mulken 1997; Klein, Beishuizen & Treffors 1998):

- Splitting both numbers into tens and ones and dealing with the parts separately (1010).
- Splitting one number into tens and ones and adding to the other number (N10).
- One number is split to bridge to a multiple of ten and then the remainder is added (A10).
- One number is rounded up or down to a multiple of ten then added. An adjustment for the rounding is then made (N10C).

In Australian research, Cooper, Heirdsfield and Irons (1996) also developed a five-point categorisation of addition and subtraction strategies:

- Counting.
- Separation (1010) but split into left to right and right to left.
- Aggregation (N10) also split into left to right and right to left.
- Wholistic which included N10C and similar variations where both numbers were adjusted and then the answer readjusted.
- Visualisation of the formal algorithm.

Student–developed multiplication and division strategies have been reported by Hierdsfield, Cooper, Mulligan and Irons (1999):

Multiplication

- Counting (CO) including repeated addition, skip counting, doubling (x2).
- Basic fact (BF) and derived fact (BDF) such as 5x9 or 50x9.
- Right to left separated into place values (RLS) 45x6 = 5x6 + 40x6.
- Left to right separated into place values (LRS) 45x6 = 40x6 + 5x6.
- Wholistic (WH). Numbers are treated as wholes 5x19 = 5x20 -5.

Division

- Counting (CO) including repeated subtraction, skip counting, halving (÷2).
- Basic fact and derived fact (BF) such as 45÷9 or 450÷9.
- R/L separated into place values (RLS) 100÷5: 0÷5 = 0, 10÷5 =2, hence 20.
- L/R separated into place values (LRS) 100÷5: 10÷5 = 2, 0÷5 =0, hence 20.
- Wholistic (WH). Numbers are treated as wholes 100÷5: 100÷10=10. 10x2=20

Additional multiplication strategies mentioned in research include:

- Aliquot parts (multiplication by factors) 17x25 = 17÷4x100 = 425 (Whitacre, 2007).
- Partitioning (non-standard separation) 21x23 = 11x23 + 10x23 (Cooper et al., 1996).
- Subtractive Distributive 19 x 23 = 20 x 23 – 23 (Hierdsfield et al.,1999; Whitacre, 2007).
- Repeated doubling (x4 and x8) (Wigley, 1996).

- Near doubles (x2 – 1, x2 +1) (McIntosh & Dole, 2005).
- Commutative Law (3x4 = 4x3) (Anghilieri, 1999).
- Compensation (Askew, 2008).

In discussing mental computation strategies, Cooper et al. (1996) comment that

> There appears to be a need, in multiplication and division mental computation as well as estimation, for assistance to be given to children to use strategies different from those associated with traditional computation (e.g., trial and error and wholistic, and, maybe, some forms of non-standard separation). (p. 96)

Menon (2003) contributed work on students' use of basic fact shortcuts in mental multiplication. He stated that some might argue that these shortcuts encourage rote learning and therefore detract from conceptual understanding, however, in his experience he found that the shortcuts motivated students and that these shortcuts could be taught using number relationships in such a way as to enhance number sense and conceptual understanding (p. 476). Both concepts of non-standard separation and basic fact shortcuts are important facets of the teaching strategy proposed in this research.

Askew (2008) suggests the use of arrays to illustrate how the strategies work and to introduce them to students thus building on existing knowledge. For example, a compensation strategy transforms the calculation 45 x 24 into a more manageable 90 x 12. Using arrays, as shown in Figure 2.3, it is easy to see how the strategy works.

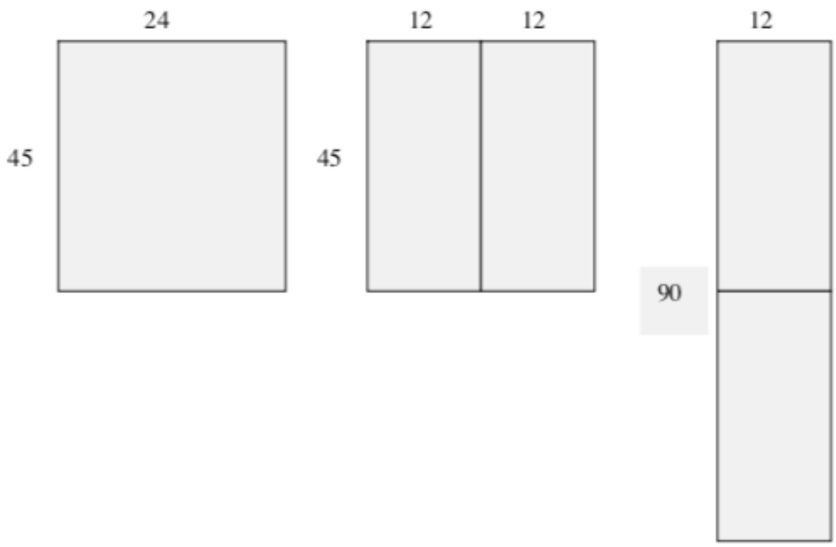


Figure 2.3 The Use of Arrays to Illustrate a Compensation Strategy for 45 x 24

Arrays are also recommended by Hurst and Hurrell (2017) for explanations of the Distributive Law, leading to strategies such as left right separated and partitioning. In a separate paper (Hurst & Hurrell, 2016), they also argue for a more appropriate explanation of multiplying by powers and multiples of ten (BDF) that references a shift in place value and the use of zero as a place holder when needed. They strongly recommend not using phrases such as “add a zero” when multiplying by powers of ten or doing basic derived fact questions. The latter suggestions have been incorporated in the notes of the intervention phase of this research.

As can be seen above, mental computation is largely strategy based as opposed to the use of defined algorithmic approaches to a particular operation. Whilst the formal algorithm for short multiplication is included, it is viewed rather as a strategy using left to right multiplication. Menon (2003) suggests that there may be place for informal algorithmic approaches in basic fact shortcuts to improve calculation efficiency, however, it is still possible to allow students to invent and choose between alternate strategies for all basic fact shortcuts. The important point is that mental computation involves a unique set of strategies as opposed to a single algorithmic process. Similarly, it cannot be viewed simply as a mental analogue of a pen and paper algorithm (Markovits and Sowder (1994).

Table 2.1 suggests a composite taxonomy of alternate strategies appropriate to the curriculum outcomes for mental multiplication at a Year 5 level that seeks to summarise the above research with references to sources.

Table 2.1 *Taxonomy of Mental Multiplication Strategies for Year 5*

Strategy	**Examples**	
Basic or Derived Fact	5 x 8 = 40, 50 x 8 = 400, 50 x 80 = 4000	Heirdsfield, Cooper, Mulligan and Irons (1999).
Counting – skip counting, doubling +/-	3 x 24 = 24+24+24 = 48+24 =72 3x24=2x24+24 = 48+24 = 72	Heirdsfield et al. (1999); McIntosh & Dole (2005); Wigley (1996).
Basic Fact Shortcuts	x3 = x2 + 1 x4 = double, double x5 = ½ x 10 x6 = x2 then x3 x8 = double, double, double x9 = x10 – x1 x11 = x10 + x1 Trachtenberg shortcuts	Belscuizen, van Putten & van Mulken (1997); Cutler and McShane (1962); McIntosh and Dole (2005); Menon (2003); Wigley (1996).
Right Left Separated	7x23: 7x3=21write 1 and carry 2, 7x2=14, 14+2=16. Answer=161	Heirdsfield et al.(1999) McIntosh & Dole (2005).
Left Right Separated	7x23: 7x20=140, 7x3=21, 140+21=161	Heirdsfield et al. (1999). Belscuizen et al. (1997) McIntosh & Dole (2005).
Factors	6x32: 6=2x3, 32x3=96, 96x2=192	Belscuizen et al. (1997).
Partitioning (non standard)	8 x 21 = 8x10+8x11 = 80 + 88 = 168 19x34 = 20x34 – 1x34 = 680 – 34 = 646	Belscuizen et al. (1997); Heirdsfield et al. (1999); McIntosh & Dole (2005); Verschaffel, Greer, and De Corte (2007).
Compensation	17x25 = 17x100 ÷ 4 = 425 15x24 = 30x12 = 360	Askew (2008); Belscuizen et al. (1997); Heirdsfield et al. (1999); Verschaffel et al. (2007).

Markovits and Sowder (1994) propose an alternate approach to a taxonomy of strategies for mental computation that uses broad headings and can be applied to each of the basic operations.

Their taxonomy has only four categories:

1. *Standard:* The student used a mental analogue of a standard paper-and-pencil algorithm (MASA).
2. *Transition:* The student continued to be somewhat bound to the standard algorithm. However, more attention was given to the numbers being computed and less to algorithmic procedures.
3. *Nonstandard with no reformulation:* A left-to-right process was used.
4. N*onstandard with reformulation:* The numbers were reformulated to make the computation easier. (p. 14)

The relationship between the two taxonomies is shown in Table 2.2.

Table 2.2 *Markovits and Sowder's Taxonomy Compared to Table 2.1*

Markovits and Sowder	MASA	Transition	Nonstandard	Nonstandard with reformation
Table 2.1	Right to Left	Partial Products using LRS or RLS	Left to Right, Basic Fact Shortcuts	Non-standard Partitioning, Factors, Compensation and combinations of strategies

The advantage of Markovits and Sowder's (1994) taxonomy is that it shows a progression from less flexible to more flexible strategies or as Whitacre (2007) describes it, a progression in number sensible strategies. This taxonomy is mentioned further in Whitacre's study with PSTs (Section 2.7) and comparisons made between that study and this research project in Chapter 5.

2.7 Pre-service Teachers' Knowledge of Mental Computation and Number Sense

Research on number sense exhibited by pre-service primary teachers in a number of countries (Alajmi & Reys, 2007; Ghazali, Othman, Alias & Saleh, 2010; Tsao, 2004; Veloo, 2010; Yang, Reys & Reys, 2009), has shown that their knowledge on this topic is generally poor and close to a secondary level student. The use of standard algorithms is a predominant feature of their calculations and mental computation is generally weaker than written. Handal (2003) comments that "pre-service teachers bring into their education program mental structures overvaluing the role of memorization of rules and procedures in the learning and teaching of school mathematics" (p. 49). Menon (2009) speaks of pre-service teachers being in the “traditional” phase where their knowledge is a product of their past experience as students. He cites a particular example of the calculation 456 x 78 where 95% of participants in his study could answer the question correctly but only 25% could present a valid word problem to illustrate it. Further, only 18% could explain how they would introduce the algorithm used to students through manipulatives or other means. He concluded that “they could not transform the knowledge they had as students …. to teachers, who had to have a much deeper knowledge, to teach math” (p. 3).

Yang, Reys and Reys (2009), in a study of 280 Taiwanese PSTs found that only one-fifth applied number sense based strategies to computational problems as opposed to some algorithmic procedure. A comparison study by Almeida, Bruno and Perdomo-Díaz (2016) of 67 Spanish pre-service secondary students (studying a Mathematics degree) showed a better use of number sense strategies but still a large dependence on algorithmic procedures. Both studies argued for increased emphasis on number sense in pre-service teacher training.

A study by Ineson (2008) of 170 trainee teachers in a post-graduate certificate program in the UK similarly looked at the incidence of informal procedures as opposed to algorithmic strategies and whether students could justify results. Again the results showed little evidence of ‘connectedness’ (Ma, 1999) with the trainee teachers largely viewing computational mathematics as a set of disparate rules and processes. This echoes Benbow’s (1993) findings that PSTs thought of mathematics as a discipline based on the memorisation of rules and procedures that are “applied in exactly one correct way to arrive at the only correct answer to a problem” (p 9).

A South African study (Courtney-Clarke & Wessels, 2014) of 47 PSTs showed a lack of number sense-based strategies coupled with a lack of conceptual understanding and basic facts fluency both of which are essential for mental computation. There was a strong correlation between the results of the number sense test and mental computation test. Testing of the four operations found that the PSTs were more competent in addition and subtraction compared to multiplication and division. Performance of this sample in written multiplication was poor with a mean of 37.6% while the mean for mental multiplication questions was only 15.2%. The study also reinforced earlier contentions that without appropriate intervention, PSTs would replicate their experiences as students in their teaching. However, Jordan (2007) remarks that, if provided with appropriate intervention at tertiary level that includes number sense activities with a focus on mental computation and estimation, PSTs are capable of a marked improvement in computational performance.

An example of an effective intervention was reported by Whitacre (2007) after working with 50 PSTs in the US over a semester in number sense and mental computation involving all four basic operations. The participants showed a marked decrease in reliance on traditional algorithms and a consequent increase in alternative strategies. The students also exhibited greater flexibility in the range of strategies used and those used were more number sensible. Using Markovits and Sowder's (1994) taxonomy of mental calculation strategies, he reported significant gains in the flexible use of multiplication strategies as shown in Figure 2.4.

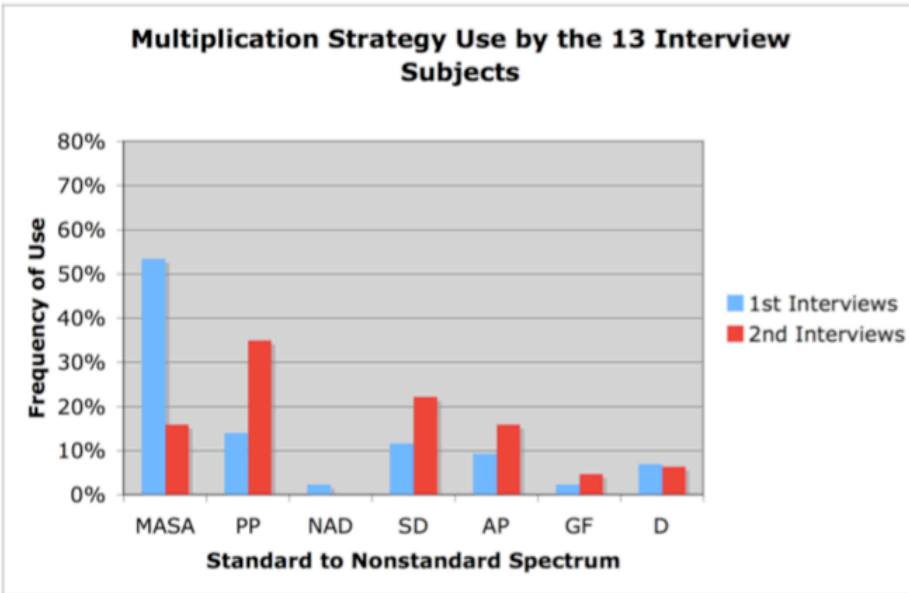


Figure 2.4 Multiplication Strategy Use in Whitacre's Study (2007, p. 14)

Note: MASA stands for a mental analogue of a standard algorithm, PP – Partial Products, NAD – nonstandard additive distribution, SD – subtractive distribution, AP – Aliquot parts, GF = general factors, D – derived.

There was a marked decline in the use of the standard algorithm and a consequent increase in the use of more flexible strategies of subtractive distribution, Aliquot parts, factors and combinations of strategies (derived). In follow-up studies with a second cohort of 37 PSTs, Whitacre (2016) extended the program to include rational numbers and again found that "the participants made more correct comparisons, reasoned more flexibly, and came to favor less conventional and more sophisticated strategies after the semester long course" (p. 57).

In recent years, there has been a growing emphasis on using Mathematical Knowledge for Teaching (MKT) as a framework for teaching both procedural understanding and mathematical fluency to pre-service teachers (Delaney, Ball, Hill, Schilling & Zopf, 2008; Hine, 2015). In relation to number sense, this means showing PSTs how to adapt mental computation into meaningful problem solving activities in a classroom atmosphere that supports reflection and communication as a shared activity (Trafton & Thiessen, 2004). Wigley (1996) suggests that the correct approach to teaching mental computation recognises that there are a few big ideas that should be taught to everyone. The aim of this research project is to identify those big ideas with specific regard to mental multiplication and provide pre-service teachers with the opportunity to evaluate them for use in their own future teaching programs. It hopes to move one cohort of pre-service teachers from Menon's (2009) traditional phase to, at least, awareness of the pedagogical phase.

2.8 Chapter Conclusions

National and international curriculum papers have supported the increase in emphasis of mental computation because of its benefits to students' number sense, as a higher order thinking skill, and for its importance in facilitating everyday life skills such as estimation and approximation. Research shows that, to achieve these benefits, students must be allowed to develop and discuss their own solutions to computation problems well in advance of the introduction of any formal algorithms.

There is consensus that a constructivist approach to teaching mental computation in the early years of schooling will allow students to gain confidence with number operations and eventually lead to a more meaningful introduction of formal algorithms in the latter years of primary education. Taxonomies of student-developed solution strategies for all four operations with number have been developed but, individually, they fall short in addressing the full scope of curriculum outcomes, particularly for mental multiplication.

Research has also shown a serious lack of subject knowledge and pedagogical knowledge of teachers and pre-service teachers in the areas of number sense and mental computation. Thus, research is needed to identify appropriate strategies and specific mental computation algorithms that support and build upon student-developed strategies thus allowing students to address the full range of problems outlined in the syllabus documents. To that end, this research project looks at a possible program of instruction for mental multiplication in Year 5, trialling this with a cohort of pre-service teachers. The methodology of this project is outlined in detail in the following chapter.

Chapter 3 - The Research Project Methodology

3.1 Introduction

This project investigated pre-service teachers' knowledge of mental multiplication. A range of strategies displaying varying degrees of flexibility were introduced to two cohorts of pre-service teachers at the University of New England resulting in suggestions from the participants for teaching programs that address this Year 5 outcome. Following the intervention, tests and questionnaires elicited information concerning any changes in participants' knowledge of mental multiplication. The research paradigm, theoretical framework and the research design and methodology to support this research process are described below. Data collection instruments and data analysis are also outlined in this chapter. Finally, ethical considerations are mentioned.

3.2 Theoretical Framework

The theoretical framework for this project is Hill, Ball and Schilling's (2008, p. 377) framework for Mathematical Knowledge for Teaching (MKT) as shown in Figure 3.1. This is a modified version of Shulman's (1987) heuristic for Pedagogical Content Knowledge (PCK) as outlined in Chapter 2.

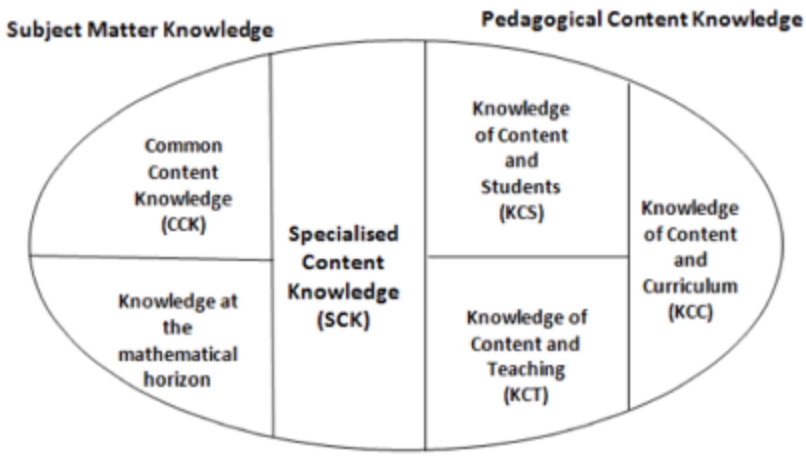


Figure 3.1. Mathematical knowledge for teaching

Divided into Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK), the Hill, Ball and Schilling model identified six areas of knowledge. Each category has been annotated with examples particular to this research project.

Subject Matter Knowledge was subdivided into:

- Common Content Knowledge (CCK) – knowledge in common with other professions e.g. the use of mental computation in estimation and approximation, particularly by adults in work situations.
- Specialised Content Knowledge (SCK) – knowledge particular to mathematics, solutions, rules and procedures e.g. strategies and algorithms specific to mental computation.
- Knowledge at the mathematical horizon (KMH) – how that topic fits with other topics, strands and the program of learning e.g. how mental multiplication fits with other areas of mental and written computation and how it relates to number sense and multiplicative thinking.

The definition of PCK kept faith with Shulman's (1987) concept and was divided into:

- Knowledge of Content and Students (KCS) – what motivates students, conceptions and misconceptions, where their ideas fit in a topic, what is difficult or easy e.g. how students develop mental computation strategies, their nature, taxonomy and common errors. What elements of multiplicative thinking are important in this topic? Which multipliers and products cause difficulties for students?
- Knowledge of Content and Teaching (KCT) – the sequencing, examples and explanations used to teach a topic e.g. how to best facilitate student-developed mental computation strategies and how to build on those strategies.
- Knowledge of Content and Curriculum (KCC) – the strands, outcomes and recommendations of the curriculum e.g. the importance of mental computation in curriculum documents, outcomes mentioning it and its place in the K-10 continuum.

The MKT framework is particularly suited to teacher assessment and professional development and pre-service teacher training. It guided the types of data collected, data collection methods and the analysis of that data.

3.3 Research Paradigm

This research is firmly located in the constructivist paradigm (Creswell, 2014) because of the consistent references in the research literature to constructivist approaches in the classroom and the role of the participants in constructing their own understanding of the topic. This is reinforced by the subjective stance of the researcher in relation to mental computation and the researcher's involvement in the study (Cohen et al., 2007). The use of qualitative data methods (in a concurrent mixed methods approach) that attempted to elicit participants' individual and collective conceptions concerning mental computation strategies and programs of instruction (Benzies & Allen, 2001) and the participants' conceptions of mental multiplication strategies based on age, culture and experiences as learners and then later, as adults (Mack, 2010) support the use of this paradigm.

In this study, the researcher has adopted symbolic interactionism as the philosophical perspective (Mack, 2010). Students learn by constructing knowledge based on their social interactions with the teacher and their peers. The value they give to what they learn, including the strategies of mental multiplication that are appropriate in a given situation, are taken from that social interaction. That is, if the teacher says this is the way to do it, or validates other students' methods, then that is what they accept. The teacher occupies an important symbolic role and the students' interaction with that important symbolic person is what influences them to learn as they do.

Epistemology is concerned with "the nature and forms of knowledge" (Cohen et al., 2007, p. 7). Symbolic interactionism sees meaning or knowledge from three related perspectives (Blumer, 1969):

- Individuals attach meaning to things and act on the basis of that meaning.
- Meaning is arrived at through a process of symbolic interaction or the use of language to interact with others and derive meaning.
- That meaning is fluid and changes through interaction with the environment and others.

From an ontological perspective, the world exists apart from the individual's perception of it but it is this subjective perception that guides the actions, beliefs and behavior of the individual (Blumer, 1969). Symbolic interactionism links the individual to the context of environment (Benzies & Allen, 2001). Hence, there will be “multiple and different perspectives of the participants in the research” (Mack, 2010, p. 8).

3.4 Research Design

The research design is a case study – the case involved two cohorts of pre-service teachers encountering mental computation through face-to-face lectures at UNE. Creswell et al. (2007) define a case study as a methodology in which “the investigator explores a bounded system (a case) or multiple bounded systems (cases) over time through detailed, in-depth data collection involving multiple sources of information and reports a case description and case-based themes” (p. 253).

Firstly, this study qualifies as a bounded study in that the participants all come from two cohorts of pre-service teachers studying by attendance, rather than distance education mode. The study occurred at a time in their course when this Year 5 Mathematics topic was of relevance but had not been previously encountered (other than as students themselves or as written or graphical algorithms for multiplication). It was also bounded by time, and content – mental multiplication strategies and algorithms – which are a subset of the unit on mental computation strategies.

The particular case under study considered how these two cohorts of pre-service teachers interacted, both individually and collectively, with the six types of knowledge in the MKT theoretical framework to develop a teaching program for mental multiplication at a Year 5 level. It is an in-depth study as it uses a mix of quantitative and qualitative research methods to collect rich data relating to the six types of knowledge that the MKT framework suggests are necessary to successfully support learning and understanding of this topic. It is a concurrent design using multiple units of analysis with a mix of quantitative and qualitative methods (Yin, 2015).

Yin (2015) would further describe it as an explanatory case study (as opposed to exploratory or descriptive) in that it sought to explain the causal links between the decisions made by the cohort of pre-service teachers prior to and following intervention. Stake (1995) would see it as an instrumental case study (as opposed to intrinsic or collective) as it provides insight into the issue (strategies used for mental multiplication) rather than the particular case being studied. Although the case plays a supportive role and is still looked at in depth, it will not be seen as representative of other case studies of strategy choice for mental multiplication – for example, teachers and students.

3.5 Research Methods

A concurrent mixed methods approach was used where quantitative data was enriched by qualitative methods. Harrison et al. (2017, para 30) state that:

> the use of multiple methods to collect and analyze data are encouraged and found to be mutually informative in case study research where together they provide a more synergistic and comprehensive view of the issue being studied.

The research literature shows that this is a 'signature' design for research in mathematics education, and particularly teacher and pre-service teacher assessment of MKT, with numerous researchers using the approach of testing, questionnaires, intervention, retesting and further questionnaires (Callingham, 2005; Caney, 2008; McIntosh & Dole, 2005; Whitacre, 2007; Yang, Reys & Reys, 2009). As such, it may be described as an iterative and cyclic approach (Teddlie & Tashakkori, 2012).

Using Cresswell's (2014) categorization of mixed methods designs, this matches the criteria for a Concurrent Triangulation design in that data collection is concurrent and the various forms of data collection support and cross-validate the findings of the study. It also has the advantage of using qualitative data to overcome a possible weakness in the quantitative data. An example of this may be the open-ended questions in the questionnaire supporting data obtained from the timed testing.

Examples of quantitative data methods used in this study are tests and closed questions such as sex, age, level of mathematics education. Qualitative data is supplied through participant observation and open-ended questions. These are discussed in Section 3.7

3.6 Data Collection

3.6.1 Participants

The study participants came from two cohorts (n=56) of pre-service teachers studying on-campus at UNE. The participants were enrolled in the units EDME145 and EDME 369. The study occurred at a time in their course when this Year 5 mathematics topic was of relevance but had not been previously encountered, although the students had studied written algorithms and graphical solutions to multiplication problems.

Initially, 36 students agreed to participate in the project and completed consent forms, however, the participation rate declined through the three phases of the project as the students experienced competing demands on their time from assignments and exams. Each participant was asked to devise his or her own identifier made up of digits and numbers to be used in all three phases of the study. It was suggested that they use the initials of their mother's maiden name and the two-digit year of their mother's birth.

3.6.2 Three Phase Model

The three phases of the research design were completed in consecutive weeks in the allocated timeslots for lectures for each group. However, the intervention occurred in the second week when lectures did not normally occur as the two courses had a scheduled one-week break from lectures for this week. All three phase were facilitated by the researcher who was not the normal course presenter. Strudents not participating in the research moved to a secondary location with their normal lecturer and began work on that week's course content.

Phase 1 (September 17/18)

In the week prior to the first phase, participants received a Participant Information Sheet, via their class Moodle sites, outlining the nature and structure of the research project and ethical considerations (Appendix 1). The project was outlined to each group of pre-service teachers by the researcher and those who agreed to participate then completed a voluntary consent form administered by the Unit Co-ordinators, Dr Schmude and Dr Wodolko. Those students not wishing to be involved left with Dr Schmude or Dr Wodolko to complete class activities at another venue. A total of 36 students agreed to participate, and completed the necessary Consent Form (Appendix 1) in the presence of the two Unit Co-ordinators. Sixteen of these came from EDME 145 and a further 20 from EDME 369.

A pre-test (Appendix 2) was then given to participants, in each group timeslot, via a timed PowerPoint presentation totalling 15 questions. As is common with this type of test, the questions were delivered both visually and verbally with a fifteen second delay between questions (Caney, 2008; McIntosh, 2006). After completing the test, participants were then asked to summarise the strategy they could have used for each question. This was completed on a separate sheet with questions listed (Appendix 3). This activity was untimed.

An online questionnaire (Appendix 4) was then completed to gauge each participant's background knowledge and pre-conceptions of mental computation. Items were multiple choice or multi-part Likert-type items with open-ended questions providing participants with the opportunity to clarify or elaborate on their responses. There was also an untimed set of four mental multiplication questions (from the original test) where alternate strategies were canvassed.

Phase 1 was completed in 30 minutes and participants then rejoined the class with time to complete the scheduled coursework for that lecture.

Phase 2 (September 25/26)

The tests in Phase 1 were marked and coded and a program of instruction developed that elaborated on strategies recorded by the participants and used examples from these to highlight new concepts and directions. The coding taxonomy (with examples) used to discriminate between strategies is shown in Appendix 3 and was developed in Chapter 2. In particular, basic number fact shortcuts (BFS), standard partitioning (LRS), non-standard partitioning (PAR) and compensation (CMP) methods were the focus of the intervention. There was also discussion of combinations of methods, listed as CMB in the test coding. Lecture notes (Appendix 5) and the PowerPoint presentation were both posted on the two course Moodle sites after the lectures had been completed (see Chapter 5). The intervention was delivered by the researcher as an interactive workshop and varied for each group depending on their responses to exercises in the presentation slides and their questions and comments. Each workshop was completed within the allocated two-hour timeslot.

Phase 3 (October 2/3)

Those not participating from each cohort left with Dr Schmude. Participants then completed a retest of similar structure and question type to the first test (Appendix 2). They were again asked to record a strategy they would have used to answer each question if time had permitted; this was untimed (Appendix 3). Finally, a second online questionnaire completed the cycle of data collection (Appendix 4). This session was completed in 30 minutes and participants were able to rejoin the class and complete the course work assigned.

Because of the placement of the data collection and intervention (towards the end of the Trimester), attendance varied greatly from session to session. Also the optional nature of the lectures in the intervention week resulted in a substantial drop in attendance. The final week of data collection, just prior to exams and final assignment deadlines also saw a reduced attendance of all three groups. While this posed some problems for data analysis, the number of those participating in Phase 1 provided a rich data source.

The test-retest design also allowed for a measure of practice effect through unsystematic variation of one group compared to systematic variation of a second group that experienced the intervention. This design increased the reliability of the results (Field, 2009).

3.6.3 Data Collection Instruments – Questionnaire 1 and 2

Both questionnaires used multiple choice items, multi-part Likert-type items or general response questions. Likert-type items were mostly based on a 5-point scale, with responses varying from 1. Strongly disagree to 5. Strongly agree. This style is common in questionnaire design when data regarding the attitudes of participants are sought (Cohen, Manion & Morrison, 2007).

Questionnaire 1 began with nine background questions related to participant demographics and experience with mental multiplication. They were asked their age (in age bands), sex and highest level of mathematics studied at school. They were also asked for their major teaching area and a second specialisation (if one existed). Two Likert-type questions then asked about their enjoyment of mathematics at school and their perceived ability in the subject. Two questions asked about their experience with mental computation at school and as an adult (SCK). Items 10 to 21 were common to both questionnaires and examined five of the six framework subheadings as shown in Table 3.1.

Table 3.1 *Classification of Questions 8 to 21 based on the MKT Framework.*

Question	Prompt	Relation to Framework
8	I learned a number of mental computation strategies at school	SCK
9	As an adult I use mental computation regularly	SCK
10	I am aware of a range of strategies for mental computation	SCK
11	I am confident in using a range of strategies for mental computation	SCK
12	I am aware of the place of mental multiplication in the curriculum	KCC
13	I feel confident about teaching mental computation to Year 5	KCT
14	I think mental computation is an essential skill for school students	KCC
15	I think mental computation is an essential skill for adults	CCK
16	Mental computation is best learned in primary school	KCC
17	Mental computation should be taught before formal methods of written computation	KCC
18	All students have the potential to be good at mental computation	KCT
19	It is important for students to have as wide a range of mental multiplication strategies as possible to allow them to solve different problems efficiently	KCT
20	Mental multiplication promotes critical thinking and problem solving	KMH
21	Mental multiplication promotes number sense	KMH

Items 8 to 21 were chosen specifically for their relationship to sub-elements of the MKT framework but were also guided by previous studies with practising teachers (Caney, 2008; Reys, Reys, Nohda and Emori, 1995). The survey concluded with five open-ended questions that asked participants to list as many alternate strategies for each multiplication problem posed as possible. Four of those questions were the same as the test questions. This was untimed and designed to provide further evidence relating to the participants' SCK. Finally, two free response questions allowed participants to comment on what they hoped would be achieved through their involvement in the study and any observations that they cared to make.

Both questionnaires were administered electronically and participants could access them through a website address. Laptops were provided if necessary.

The final questionnaire started with items 10 to 21 from the first questionnaire and a similar set of five free response questions asking for alternate solutions to problems, four of which were from the second test. In Section 3, there were four questions relating to their attendance at the intervention and their evaluation of it and whether they had read the notes and a similar evaluation on a five-point Likert scale. The final section dealt with evaluations of the strategies presented and what they would include in a teaching program and its duration. The questionnaire finished with a free response question inviting further comments about the project.

Whilst both questionnaires were untimed, there was some pressure from the course co-ordinators to complete it within 30 minutes so they could resume their normal program. As a consequence, three participants in Phase 1 started the questionnaire but did not complete it before the non-participants had rejoined the group and the lecture had begun. This similarly occurred in the Phase 3 questionnaire with five participants not completing the questionnaire. The website used did not record partial data for all those participants.

3.6.4 Data Collection Instruments – Tests 1 and 2

The selection of questions was based on the syllabus requirements of the NESA (2015) curriculum for K-10 Mathematics expressed as outcomes for this Year 5 topic. Sources from the literature review, particularly McIntosh (1995, 2006), provided examples of questions used in testing with Australian student cohorts from Years 3 to 10. Whitacre's (2007) assessments with pre-service teachers were also consulted. Table 3.2 details the questions and related NESA outcomes.

Table 3.2 *Distribution of Test Questions by Category*

Test			Retest	
No.	Question	NESA Outcome	No.	Question
1	40 x 90	2 x 2 digit	1	50 x 90
2	5 x 8000	1 x 4 digit	2	6 x 7000
3	18 x 6	1 x 2 digit	3	22 x 6
4	5 x 19	1 x 2 digit	4	5 x 23
5	126 x 4	1 x 3 digit	5	217 x 4
6	9 x 45	1 x 2 digit	6	9 x 37
7	3 x 195	1 x 3 digit	7	3 x 295
8	143 x 7	1 x 3 digit	8	321 x 8
9	19 x 25	2 x 2 digit	9	23 x 25
10	23 x 12	2 x 2 digit	10	32 x 12
11	15 x 18	2 x 2 digit	11	14 x 17
12	21 x 23	2 x 2 digit	12	32 x 31
13	123 x 11	3 x 2 digit	13	432 x 11
14	45 x 24	2 x 2 digit	14	24 x 35
15	34 x 35	2 x 2 digit	15	46 x 45

Whilst the questions tested similar outcomes, it could be argued that Test 2 was slightly more difficult than Test 1. Ten of the questions had the same multipliers but the digits of the multiplicand were generally greater in Test 2. Experience would suggest that the higher the number the more difficult the multiplication. For example, consider Question 12, Test 1

21 x 23 = 20 x 23 + 1 x 23 = 460 + 23 = 483

This uses LRS. The same strategy applied to Question 12, Test 2 produces:

32 x 31 = 30 x 32 + 1 x 32 = 960 + 32 = 992

Tripling would normally be seen as more difficult than doubling so the second question would be considered slightly more difficult.

The two tests were administered in the same time slot using PowerPoint slides with a 15 second delay between slides. The questions were bold, Calibri 96 point in a horizontal format. Each question was read as it appeared on the screen. Test answers were recorded on a blank answer sheet with limited space for each question that discouraged any working. The delay between questions matched that used by McIntosh (2006) for ‘long’ questions and was slightly longer than that used by Caney (2006) with Middle School students.

3.6.5 Data Collection Instruments – Coding Sheets 1 and 2

Following each test, the participants were asked to record the strategy that they used, or would have used if time permitted, for each question. They were asked just to record their working and that the answer was not required. This section was untimed and the participants individually moved on to the questionnaire when they were finished. All test participants completed their coding sheets for both tests. The original plan called for the participants to form small focus groups and code each other’s sheet using the taxonomy shown in Table 3.3.

Table 3.3 *Coding Taxonomy with Examples*

Code	Descriptor	Examples
BDF	Basic or Derived Fact	5 x 8 = 40, 50 x 8 = 400, 50 x 80 = 4000
COU	Counting – skip counting, doubling +/-	3 x 24 = 24+24+24 = 48+24 =72 3x24=2x24+24 = 48+24 = 72
BFS	Basic Fact Shortcuts – shortcuts for multipliers from 2 to 12, 15 and 20	x5 = ½ x 10 x8 = double, double, double x9 = x10 – x1 x11 = x10 + x1 x12 = x10 + double x15 = x10 + x5 Trachtenberg and Hall shortcuts (see Appendix 5)
LRS	Left to Right Separated – using the Distributive Law	7x23: 7x20=140, 7x3=21, 140+21=161
RLS	Right to Left Separated – visualising the formal algorithm	7x23: 7x3=21write 1 and carry 2, 7x2=14, 14+2=16. Answer=161
FAC	Factors	6x32: 6=2x3, 32x3=96, 96x2=192
PAR	Partitioning (Non-standard)	8 x 21 = 8x10+8x11 = 80 + 88 = 168 19x34 = 20x34 – 1x34 = 680 – 34 = 646
CMP	Compensation	17x25 = 17x100 ÷ 4 = 425 15x24 = 30x12 = 360
CMB	A Combination of the above strategies	21 x 23 = 11 x 23 + 10 x 23 (PAR) = 253 (BFS) + 230 = 483

Unfortunately, time restraints on the data collection phases meant that the focus groups were not able to be used and the researcher coded all the test responses. All test responses were able to be coded using the taxonomy.

3.6.6 The Intervention

A two-hour time slot was allocated for the intervention workshop with each of the cohorts. This occurred in the week following Phase 1 when there were no scheduled lectures for either EDME145 or EDME369. That meant that attendance was voluntary and resulted in reduced numbers for this phase, as outlined in Chapter 5.

In designing the intervention, there were a number of specific objectives that dictated content. Firstly, a broad range of strategies needed to be included that were flexible enough to solve the full spectrum of problems outlined in the syllabus. These had to be conceptual in nature rather than procedural and be based around a strong understanding of the Distribution Law (multiplicative thinking). Rather than limit partitioning to LRS, non standard partitioning, including additive and subtractive distributions, was seen as the vehicle for increased flexibility along with Basic Fact Shortcuts to speed calculation.

These objectives resulted in the inclusion of the following strategies: LRS, BFS, PAR, FAC, CMP and CMB. It was assumed that BDF and COU strategies would be taught in Year 4. The focus on flexibility and adaptivity, required that participants be shown multiple pathways to each problem solution. A review of Test 1 questions was used to highlight adaptivity with optimal pathways discussed. Multiplicative thinking was mentioned in relation to the PAR strategy which uses the Distributive Law to create multiple solution pathways and also in relation to multiplying by powers and multiples of ten (see Chapter 2). The Commutative law was related to the use of factors in examples. To better understand this focus, the notes from the intervention have been reproduced in Appendix 5 with the addition of multiple solutions to the questions from Test 1 that were discussed at the end of the tutorial.

3.7 Data Analysis

The variables in this study were the knowledge strands associated with the MKT model and, in particular, the multiplication strategies identified in the SCK strand. All questions were related to these knowledge strands and coded strategies with the exception of some biographical data that added to the narrative of the case study. The wide range of data collection instruments allowed for triangulation to occur between data sets.

3.7.1 Questionnaire 1 Analysis

Questions were multiple choice, multi-part Likert-type items or general response questions. Likert-type questions produced data that was ordinal in nature resulting in analysis that precluded most statistical measures of centre and dispersion. As Cohen et al (2007) state, the distance between categories of ‘strongly disagree’ and ‘disagree’ may be completely different from any two other adjacent points on the scale. The questionnaire was analysed through frequency distribution and items compared with non-parametric correlation.

Multiple choice items used in personal profile questions are forms of nominal data. As such, data analysis of frequency distribution was used. With age related questions, a form of ordinal data, frequency distribution was used along with median and modal class. Similarly, a frequency distribution compared levels of mathematics studied at school as a group and by cohort. Enjoyment of mathematics at school and perceived ability were compared by cohort and correlated. Mann-Whitney U tests gauged significance.

Items 8 to 21 were grouped by framework categories and compared by frequency distribution with Spearman’s correlations. A cross-category correlation between items 11 and 13 was explored as both questions related to confidence with mental multiplication strategies, one as a teacher, the other as an adult.

Open-ended questions included in the questionnaire were analysed using a clustering procedure (Miles & Huberman, 1994). This helped to identify emergent themes related to the areas of knowledge in the MKT framework.

3.7.2 Mental Multiplication Assessment 1

Participant scores on both tests involved ratio (or numerical) data and hence had an absolute zero, and meaning attached to differences and the ratio of scores. A full range of statistical tools was available including measures of central tendency (e.g. mean, mode, median) and measures of dispersion (e.g. range, standard deviation). Number of correct responses, mean and standard deviation were calculated for the first test as an entire group and also for each of the cohorts (EDME145 and EDME369). The difference in test means for the two cohorts was explored in relation to the level of mathematics studied at school, enjoyment and perceived ability in the subject, use and confidence with mental computation strategies as an adult. These measures were correlated with the test results to determine significance.

Questions 9 to 15 of the test were analysed separately as they involved two-digit x two digit multiplications. These had proved to be the most difficult section of the test. The analysis was by frequency distribution and strategy use.

3.7.3 Coding Sheet 1 and Free Response Items (Questionnaire 1)

Coding of the untimed section where working was shown for the test questions is nominal data and was recorded by frequency distribution and column graph. There were added variables of the number of incorrect uses of each strategy and blank entries to consider in the display of data. The use of strategies and their distribution were analysed for the whole group and by cohort. Additionally, comparisons were made with strategy use for the two-digit x two-digit questions. The alternate strategies canvassed in the free response section of the questionnaire were also coded and compared by cohort.

Strategy use was then categorised according to its potential to answer a range of test questions i.e. its flexibility. This was loosely based on Markovits and Sowder's (1994) taxonomy as reported in Whitacre's (2007) study of mental computation with pre-service teachers. Whitacre's taxonomy is compared with the taxonomy used in this study in Table 3.4.

Table 3.4 *Comparison of Whitacre's Taxonomy for Multiplication Strategies and the Taxonomy Used in This Study*

Categorisation	**Standard**	**Transition**	**Non Standard**	**Non Standard with Reduction**
Specific coding (Whitacre)	MASA	Partial Products	Additive Distributive	Non-standard Partitioning (SD and AD), Aliquot Parts, Derived, Factors
Specific Coding (Hall)	RLS	Partial Products	LRS, BFS	PAR, FAC, CMP, CMB

MASA stands for Mental Analogue of a Standard Algorithm that is classified as RLS in this study. Partial Products is consistent with LRS for single-digit multipliers. Subtractive Distributive has been grouped with Additive Distributive under a single classification of Non-Standard Partitioning (PAR). Factors are the same, CMP includes Aliquot Parts and Derived and is similar to CMB. The classifications in this study were chosen to be consistent with Australian syllabus terminology. The strategies of BDF and COU were not used in Whtiacre's study.

As Whitacre (2007) points out, the taxonomy

> suggests a progression from one end of a spectrum to the other. Students who rely heavily on Standard methods evidence poor number sense. Their understanding of an operation seems to be tied to symbol manipulation, so that they lack flexibility. At the other end of the spectrum, students who readily employ Non-standard methods, especially Non-standard with reformulation, exhibit good number sense. Their understanding of the operations is independent from any particular algorithm, so that they have good flexibility. Those primarily using Transition strategies can, indeed, be seen as in transition from Standard to Nonstandard, from tied to the algorithms to independent, from inflexible to flexible. (p. 7)

This results in two categorisations of strategies based on flexibility:

- inflexible and appropriate for a limited range of questions (BDF, COU and RLS for one-digit multipliers)
- flexible and appropriate for a wider range of questions (LRS, PAR, FAC, CMP, CMB)

Strategy use was compared using these two categories with a separate category for blank and incorrect entries that was not relevant in Whitacre's study but is in this project. Strategy use was also compared with the findings from Whitacre's (2007) study. It was also compared by the range of strategies used by each participant although the degree of flexibility of the strategies was not taken into account.

3.7.4 Questionnaire 2 Analysis

Initially, items relating to the efficacy of the intervention and its notes were analysed by frequency distribution using a five-point Likert scale. Strategies were assessed by each participant for inclusion in a teaching program and these results were also displayed by frequency distribution. Recommendations for the length of a teaching program were similarly displayed.

Analysis then moved on to items 10 to 21 of the questionnaire and these were again grouped under the same framework headings and displayed by frequency distribution. Comparisons between the participants' responses to the first questionnaire and these responses were made for the group that attended the intervention. Again, correlations were used as appropriate and when significant. Paired *t*-test and Wilcoxon test were used due to the small sample size (de Winter, 2013). Free response questions were grouped by theme.

3.7.5 Mental Multiplication Assessment 2

Test 2 results were reported by frequency distribution, mean and standard deviation. They were compared and correlated with Test 1 results. The participants were then split into two groups, those who had attended the intervention and those who had not. These results were again compared by mean and standard deviation and correlated with each group's Test 1 results and tested for significance. The difference between the results of the two groups was also tested for significance.

Improvement in results of the group not attending the intervention were analysed in terms of the change in candidature, possible variation in the difficulty of the second test and practice effect.

3.7.6 Coding Sheet 2 and Free Response Items (Questionnaire 2)

The use of strategies and their distribution were analysed for the whole group and by attendance at the intervention. Additionally, comparisons were made with strategy use for the two-digit x two-digit questions. The alternate strategies canvassed in the free response section of the questionnaire were also coded and compared by group. All of these distributions were compared by groups.

Strategy flexibility and the range of strategies used were compared by group based on attendance at the intervention. The free response questions were coded and compared with Phase 1 results and by group. Comparisons were made with the findings from Whitacre's (2007) study.

3.8 Ethical Considerations

Walliman (2010, p. 43) cites two aspects of ethical issues in research:

1. The individual values of the researcher relating to honesty, frankness and personal integrity. In relation to this:

    - The candidate has completed the Academic Integrity Module as required.
    - This document will be subject to the Turnitin self check and final check before submission.
    - All quotes and sources of information have been properly cited in this document using APA guidelines (Sites, A. W. APA STYLE GUIDE 2009; VandenBos, G. R. (Ed), 2010).
    - Ethics application has been approved with no conditions. Approval number HE18-135 valid from June 26, 2018 to June 26, 2019 (See Appendix 6).

2. The researcher's treatment of other people involved in the research, relating to informed consent, confidentiality, anonymity and courtesy.

To this extent, the researcher produced a document for potential participants (Appendix 1), outlining:

- the purposes, contents and procedures of the research.
- any foreseeable risks and negative outcomes, discomfort or consequences and how they will be handled; benefits that might derive from the research and incentives to participate.
- the right of voluntary non-participation, withdrawal and rejoining the project, rights and obligations to confidentiality and non-disclosure of the participants' identities and research outcomes.

- informed consent form for participation.
- the benefits of participating in the research.

  (Cohen et. al. 2013, p. 55)

Participant identities were not known to the researcher. Each participant was asked to select an identifier, unique to each individual, that was used for all tie data collection instruments allowing responses to be analysed across phases. Their participation and performance had no bearing on assessment of the units in which they were enrolled.

Chapter 4 - Data Collection and Analysis – Phase One

4.1 Introduction

This chapter provides an outline of the first phase of data collection with particular reference to the participants and the background information supplied by items in the first Questionnaire (Appendix 4). Initial test data are presented to provide an overview of the participants' ability in timed mental multiplication, their school experience of mental computation and their mathematical background at HSC level. Analysis of items eight to twenty-one of the first questionnaire are related to five of the six elements of Hill, Ball and Schilling's Mathematical Knowledge for Teaching framework (2008), outlined in Chapter 3.

This is followed by extensive analysis of the participants' untimed responses to the test questions and coding of the strategies they listed for these questions. Coding of strategies used the coding taxonomy developed in the literature review and outlined in the research design. Each participant's coded strategies are further supplemented by four free response items in the questionnaire that canvassed alternate strategies to a selection of test problems. The combination of the two, leads to conclusions about the participants' knowledge of mental multiplication (SCK) and their preparedness to teach the topic to a Year 5 cohort (KCT) using the guidelines set out in the Australian and NSW curriculum documents (KCC).

4.2 Data Collection

Three groups were involved in data collection: one cohort of EDME 369 attending lectures on Tuesdays between 2pm and 5pm and two classes of EDME 145 attending lectures on Tuesdays between 11am and 1pm and Wednesdays between 9am and 11am. The data collection occurred in the last three weeks of Trimester 2 from September 18 to October 3, 2018 inclusive.

The number of participants in each data collection phase is detailed below in Table 4.1.

Table 4.1 *Participation in the Various Phases of Data Collection*

Phase 1		Phase 2		Phase 3	
Task	Number of participants	Task	Number of participants	Task	Number of participants
Test 1	36	Attended Tutorial	11	Test 2	23
Coding Sheet	36	Accessed Tutorial Notes	8	Coding Sheet	23
Questionnaire 1	33	Attended and Accessed Notes	7	Questionnaire 2	18

Of the 36 possible participants, 23 attempted both tests, while 13 attempted Test 1 only. Only eight participants were involved in all three Phases of data collection and one of those accessed the tutorial notes but did not attend the tutorial.

4.3 Participant Demographics

Of those participating in Phase 1 ($n = 36$), 16 were studying EDME145 and 20 were studying EDME369. All students were progressing towards a BEd majoring in K-6 Primary. Although the students in EDME369 had experienced a greater number of courses in primary mathematics methods, neither group had experienced specific tuition in mental computation strategies. Both groups had received instruction in written and graphical methods of multiplication by one- and two-digit numbers.

Of those completing Questionnaire 1 ($n = 33$), 7 were male (21%) and 26 were female (79%). The percentage of males participating was slightly higher than the national average for male primary school teachers of 16.5% (ABS, 2017)

The median and modal ages for participants were contained in the 18 to 25 years class with only three students outside that class; two in the 26 – 35 year range and one in the 36 – 45 year range. This is consistent with previous on-campus candidatures for this course.

The highest level of mathematics attained at High School is detailed in Table 4.2 with percentages shown in brackets. Given the age of participants, it is unlikely that they have encountered any further mathematics education other than their present course work.

Table 4.2 *Highest Level of Mathematics Studied at High School by Participants (n=33)*

Course	Year 10 (%)	Year 11 (%)	Maths General (%)	Mathematics 2U (%)	Ext 1 or better (%)
Number	2 (6%)	1 (3%)	19 (58%)	6 (18%)	5 (15%)

In NSW, HSC mathematics courses offered include Mathematics General, Mathematics (2U), Extension 1 and Extension 2. Those studying Mathematics (2U) or higher encounter a stronger emphasis on algebra, calculus, deductive geometry and trigonometry. They have also been involved in a selection process that began in Year 8. Students in the top sets of Year 8 are chosen to study the 5.3 course in Years 9 and 10. From there, it is possible to choose any mathematics course for the HSC but those choosing Mathematics (2U) or higher are expected to have completed the Stage 5.3 course successfully. This creates a distinct division in terms of mathematical exposure between those studying Mathematics 2U or higher and those opting for Mathematics General in Years 11 and 12.

Of those nominating a specialty area(s) within the Primary Methods Bachelor of Education program, the following distribution (Table 4.3) was observed.

Table 4.3 *Specialty Areas of Pre-Service TeachersEenrolled in EDME 145 and EDME 369 (n=36)*

Specialty	English (%)	Maths (%)	HSIE (%)	STEM (%)	PDHPE (%)	Creative Arts (%)
Number	13 (36%)	6 (17%)	6 (17%)	5 (14%)	1 (3%)	5 (14%)

It was interesting to note that only one of the 11 studying HSC mathematics at Mathematics (2U) or higher, nominated mathematics as a specialty area. Four more nominated Science or STEM as their area of specialisation. The other five nominating mathematics as a specialty area all studied Mathematics General which would have been considered as appropriate for primary teaching. The predominance of English/HSIE majors matches that of teachers currently in Primary Schools where, in 2014, 46.6% nominated English as their specialty area and a further 16.8% saw themselves as HSIE specialists (National Teaching Workforce Dataset, 2014).

The two items in the first questionnaire related to perceived aptitude in mathematics and enjoyment of mathematics at school were:

Q6 At school I was good at maths

Q7 I enjoyed doing maths at school

These elicited the full range of responses as shown in Figure 4.1.

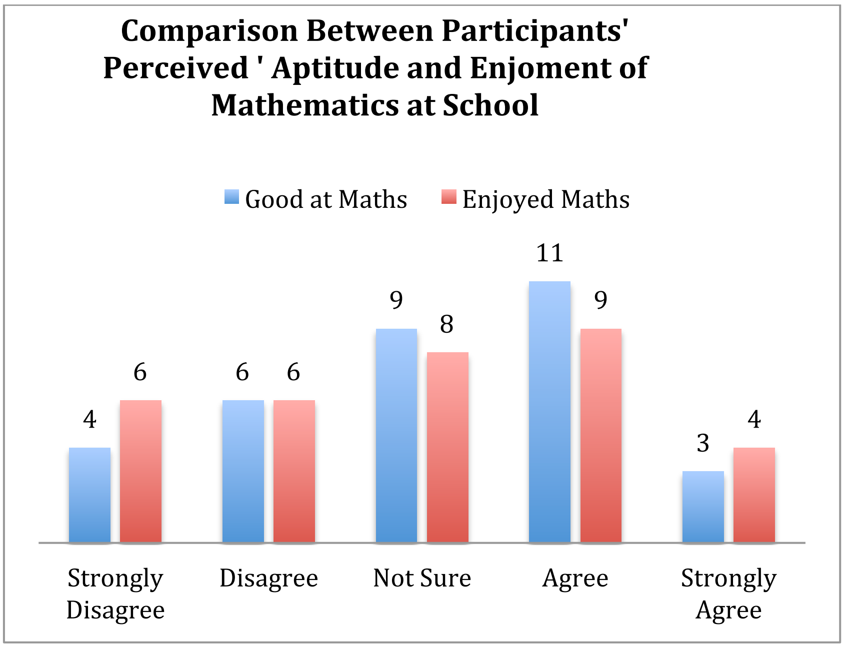


Figure 4.1 Participants' Perceived Aptitude and Enjoyment of Mathematics at School *(n = 33)*

The data indicates that 12 of the participants (36%) did not enjoy mathematics at school and 10 participants (30%) felt that they did not do well in the subject. Surprisingly, one of the six students nominating mathematics as a specialty area disagreed with both statements and a second was unsure in both cases. The Spearman's correlation between these items ($\rho = 0.889, p < 0.01$) demonstrates a strong association between their enjoyment of mathematics at school and their perception of aptitude. Also, as the literature points out (Hill, Rowan & Ball, 2005; Hill & Rowe, 1998) there is a significant correlation between teachers' ability in mathematics and their students' performance in the subject. To have such a large percentage suggesting that they did not do well in mathematics at school and that they did not enjoy the subject is of concern and raises questions about the capacity of a substantial number of the participants in creating a classroom environment that would result in high quality achievement in mathematics.

This data suggests that a substantial number of the participants would possess limited mental computation skills, which would be associated with low levels of KCT and SCK (Hill et al., 2008) in this area. It also suggests that many may have a negative or ambivalent attitude towards instruction in this area (Askew, 2008; Handel, 2003; Sullivan, 2008).

Table 4.4 shows responses separated by course for Items 6 and 7 of the Questionnaire and includes the results of a Mann-Whitney *U* test to identify an significant differences based on course enrolment.

Table 4.4 *Responses to Items 6 and 7 by course (n = 33)*

Item	Course	Strongly disagree	Disagree	Not Sure	Agree	Strongly agree	*Mean rank*	*U*	*p*	n
6	EDME145	2	4	6	2	2	15.44	129.5	0.382	16
	EDME369	2	2	3	9	1	18.47			17
7	EDME145	3	3	5	3	2	15.00	104.0	0.238	16
	EDME369	3	3	3	6	2	18.88			17

A comparison of the mean ranks indicates that, for these participants, those in EDME369 responded with a higher level of agreement on both items, but this difference was not statistically significant.

Table 4.5 illustrates the highest level of mathematics studied by participants at High School (Item 3), separated by cohort (EDME145 and EDME369)

Table 4.5 *HSC Courses Studied at High School split by cohort (EDME145/EDME369)*

Item	Prompt	Course	Year 10	Year 11	Maths General	Mathematics 2U	Ext 1 or higher	n
3	What level of mathematics did you complete in High School?	EDME145	1	0	12	1	2	16
		EDME369	1	1	7	5	3	17

A greater number of participants from EDME369 studied more difficult mathematics courses at High School than did those from EDME145. Forty seven percent of EDME369 participants studied 2U or higher compared to 19% of EDME145 participants. While comparison of the two data sets does not show a statistically significant difference ($U = 129.5, p = 0.794$) it is worth considering the performance of those studying higher levels of mathematics in relation to their mental multiplication test scores.

4.4 Initial Test Results

As mentioned in Chapter 3, the first test involved 15 mental multiplication questions satisfying the requirements of the Year 5 curriculum outcome ACMNA 100 (ACARA, 2012). The questions were delivered both visually and verbally with a 15-second delay between questions. The participants were asked simply to record their answer and the answer sheet left little room for working.

The results for Test 1 are summarised in Figure 4.2.

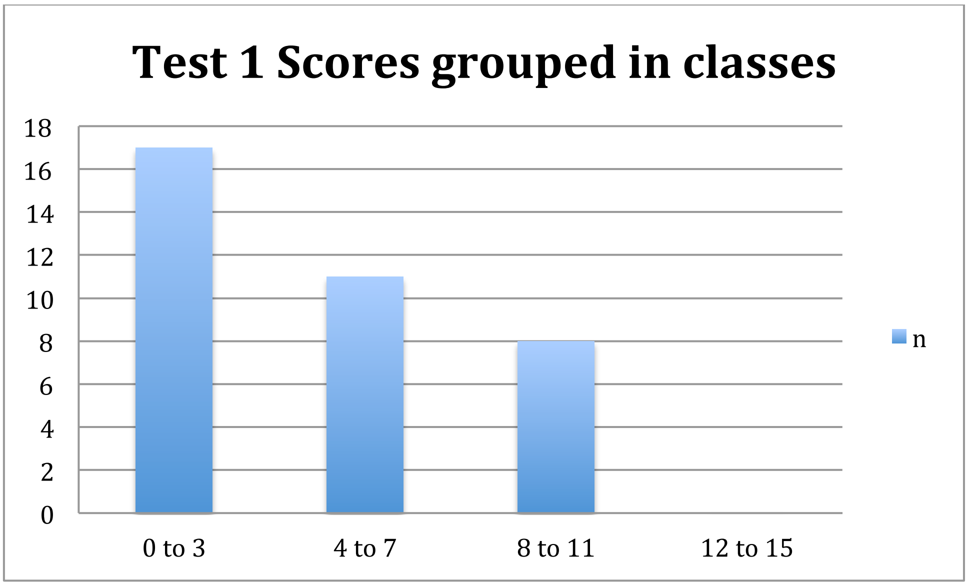


Figure 4.2 Distribution of Scores (out of 15) in Test 1 (n = 36).

Considering that this was designed as an exit test for a Year 5 cohort after experiencing tuition in mental multiplication techniques, it was expected that the performance of the pre-service teachers would be substantially better than a mere pass. The test scores ranged from 0 to 11 with only eight students (22%) scoring 8/15 or better. The participant with the highest score of 11/15 was the oldest of the group, had studied General Mathematics at the HSC and felt she had done well at school though not sure if she enjoyed the subject. There was agreement with the statements suggesting that she used mental multiplication as an adult and was aware and confident with a range of strategies.

The mean and standard deviation of the test (all participants) were, respectively $\mu = 4.3$, $\sigma = 2.8$ and showed a distinct lack of SCK in a timed situation. If split into the two courses, EDME145 and EDME369, the frequency distribution, test means and standard deviations were as follows in Figure 4.3 and Table 4.6

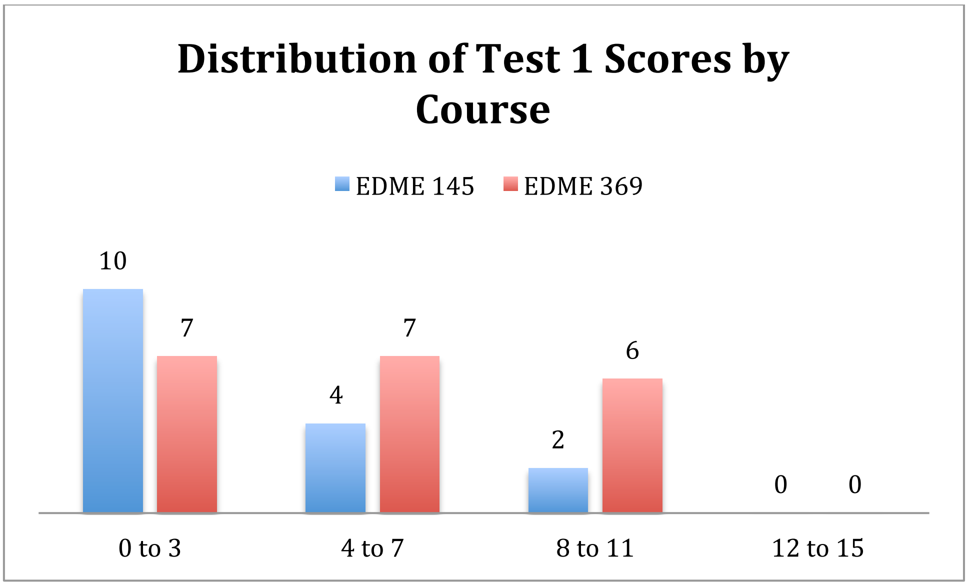


Figure 4.3 Distribution of scores (out of 15) in Test 1 by Course

Table 4.6 *Means and Standard Deviations of Test 1 by cohort.*

Statistic	All Participants $n = 36$	EDME 145 $n = 16$	EDME 369 $n = 20$
Mean	4.3	3.6	4.9
Std Dev	2.8	2.4	3.0

The comparison between the two cohorts shows a better performance by the third-year group, however the difference was not statistically significant ($t = -1.44, df = 34, p = 0.160$). While it could be argued that the improved results on the test for participants in EDME369 were better due to increased exposure to Primary Mathematics Methods courses, caution would need to be exercised in reaching this conclusion without further research, perhaps with a larger dataset. It should also be noted that the third year cohort reported a higher, although non-statistical significant, perception of their ability in, and enjoyment of, mathematics at high school.

In relation to the level of mathematics studied at High School, those opting for the higher levels of mathematics seemed to perform better in the test. If split into groups of Mathematics (2U) and above, General Mathematics and no HSC mathematics, the means of each group increased with the difficulty of the course studied as shown in Table 4.7

Table 4.7 *Test 1 Means Based on Course Studied at High School*

Item	Prompt	Year 10			Year 11			Maths General			Mathematics 2U			Ext 1 or higher		
		$\bar{X}$	σ	n	$\bar{X}$	σ	n	$\bar{X}$	σ	n	$\bar{X}$	σ	n	$\bar{X}$	σ	n
3	Level of Mathematics in High School	1	0	2	4	na	1	4.32	2.75	19	5.33	3.08	6	5.4	3.58	5

Having said that, the choice of course at the HSC was not strongly associated with performance in the mental multiplication test. The correlation between the two factors was relatively low, but close to statistical significance ($\rho = 0.314, p = 0.075$). It was interesting to note that students in the 2U and above group scored both the second highest score of 9/15 and the lowest score of 0/15. General Mathematics, although devoid of calculus and higher-level algebra, trigonometry and abstract geometry, would still be viewed as highly appropriate for the needs of Primary pre-service teachers, particularly in relation to basic computations. This is supported by previous research by Askew (2006) that suggests that the procedural knowledge needed to pass higher level mathematics courses has little relationship to the conceptual knowledge needed to teach and understand number sense based activities such as mental computation.

Similarly, subject specialisation was not a strong predictor of success in timed testing, Those choosing mathematics or STEM as a subject specialisation (*n*=11) had a test mean and standard deviation of $\mu = 4.9,\ \sigma = 3.2$. The larger standard deviation resulted from a bi-modal distribution of scores with six of this group scoring between 0 and 4 and the remaining five scoring between 7 and 9.

In terms of performance in the mental multiplication timed test, results did, to differing extents, model the level of mathematics studied at school, perceived ability in mathematics and enjoyment of the subject, with the best predictor being the candidates' enjoyment of the subject at school although none of these comparisons were statistically significant. It would seem that these three elements were certainly factors in the higher level of performance of the third year group in the timed test, although their results would still be considered as poor in terms of their overall SCK.

One question not asked of participants that could have proved useful was a self-assessment of their confidence with times table facts from 2 to 10. Responses in the coding sheets and open-ended comments showed a weakness for some in this area. For example, two responses to Question 16 (optional comments) were:

> "I'm very horrible at anything above 5 time tables other then 10"

> "These mathematical questions are extremely difficult to do in someone's head, after a long period of time not doing time tables (especially larger digits)"

If this research were to continue into primary school classrooms, initial testing of times table facts would be an element of the research design to consider.

4.5 Participant Confidence in the Use of Mental Multiplication Strategies

Items 8 to 11 of the first questionnaire dealt with the participants' personal knowledge of mental computation strategies and their confidence in using these strategies (SCK). Table 4.8 details their responses to the three questions.

Table 4.8 *Responses to Questions 8 to 11 ($n = 33$).*

Item No	Text	Strongly Disagree	Disagree	Not Sure	Agree	Strongly Agree
8	I learned a number of mental computation strategies at school	5	7	8	10	3
9	As an adult I use mental computation regularly	4	6	7	11	5
10	I am aware of a range of strategies for mental computation	4	3	11	13	2
11	I am confident in using a range of strategies for mental computation	4	12	9	6	2

These responses are presented in an alternate tabulation such as Table 4.9, where responses indicating agreement and disagreement have been amalgamated.

Table 4.9 *Grouped Percentages Showing Degree of Agreement with Questions 8 to 11 Prompts.*

Question	Prompt	Disagree %	Not Sure %	Agree %
8	I learned a number of mental computation strategies at school	36%	24%	39%
9	As an adult I use mental computation regularly	30%	21%	48%
10	I am aware of a range of strategies for mental computation	21%	33%	45%
11	I am confident in using a range of strategies for mental computation	48%	27%	24%

Responses to Question 8 suggest that the minority (39%) recall specific teaching of mental computation at school with 36% disagreeing with the prompt to some extent. Considering the focus placed on mental computation since the early 1990's in curriculum planning documents (Australian Education Council, 1991) and in current curriculum documents (ACARA, 2012), this suggests that there is still considerable work to be done in schools in this important area of numeracy and number sense. This sample of pre-service teachers had a median age in the 18 to 25 years bracket and should all have encountered mental computation as part of their schooling.

Nearly half of the respondents (48%) agreed that they used mental computation as adults regularly and a similar percentage were aware of a range of strategies for mental computation (45%). However, a low 24% felt confident about using these with almost half (48%) lacking confidence in this area.

Correlations between Test 1 results and these four questions are shown in Table 4.10.

Table 4.10 *Correlation Matrix for Test 1 and Responses to Items 8 to 11 of the Questionnaire (n=33 T1, n=34 Q's 8, 9, 10, 11)*

Spearman's Correlations	Test 1	Q8	Q9	Q10	Q11
Test 1	1.000	.333	.632**	.639**	.634**
Sig (2-tailed)		.058	.000	.000	.000
Q8		1.000	.326	.437**	.592**
Sig (2-tailed)			.060	.010	.000
Q9			1.000	.776**	.646**
Sig (2-tailed)				.000	.000
Q10				1.000	.659**
Sig (2-tailed)					.000
Q11					1.000
Sig (2-tailed)					

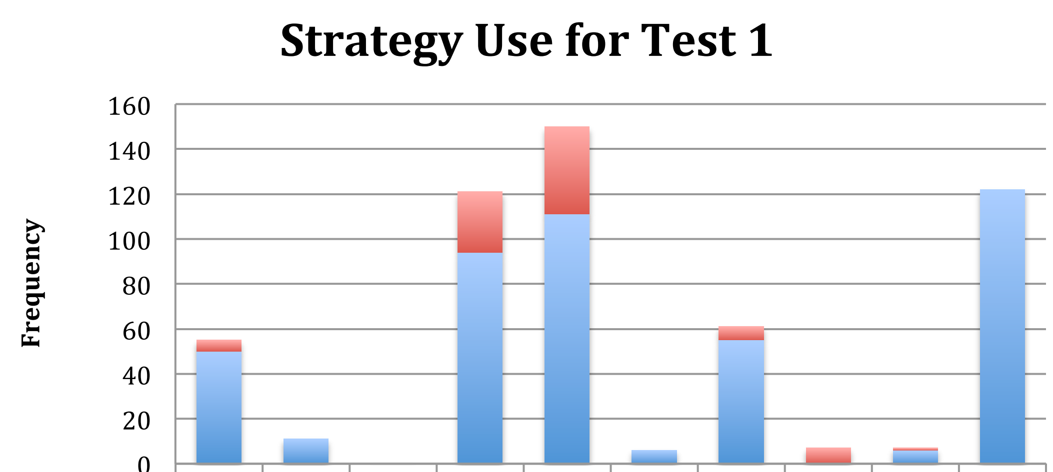
** Correlation is significant at the 0.01 level (2-tailed)

This table shows that the prompt concerning strategies learned at school (Q8) correlates poorly with the test results and has no statistical significance. The other three prompts (Q9 to 11) all correlate in the moderate to strong range with the test results. This may suggest that participant's awareness and perceived confidence as adults with mental multiplication is a better predictor of test results than those previously mentioned i.e. participant's experience with mental computation strategies at school, the level of mathematics studied at school and their subject specialisations and their enjoyment and perceived ability in mathematics at school. One interpretation of these findings is that schooling has had limited effect on pre-service teachers' capacities with mental computation. Experience as adults, seems to be the better predictor of results in this area although the low test scores would suggest that this experience has been superficial to date.

Whilst test results provided some initial insight into the question of participants' knowledge of mental computation strategies, their untimed record of possible strategies that they would use for each test question provided additional data for this question. The next section considers that data.

4.6 Participant Competence in the Use of Mental Multiplication Strategies

Analysis now turns to untimed coding from the first test and the resultant use of strategies is shown in Figure 4.4.



	BDF	COU	BFS	RLS	LRS	FAC	PAR	CMP	CMB	Blank
Incorrect	5	0	0	27	39	0	6	7	1	
Correct	50	11	0	94	111	6	55	0	6	122

Figure 4.4 Strategies Suggested for Test 1 Questions

Figure 4.4 illustrates a data set of 540 entries resulting from 36 participants coding 15 questions each. A number of participants could suggest strategies for the single-digit multipliers but were unable to suggest any for double-digit multipliers. This resulted in 122 (22.6%) blank entries. The remaining 418 strategies suggested were coded using the taxonomy that was outlined in Chapters 2 and 3 (Appendix 4). All strategies could be coded using the taxonomy.

If the strategy suggested could have led to a correct solution of the problem, it was coded as correct regardless of any concerns about the complexity of the calculation. For example, a possible Right Left Separated strategy for 45 x 24 is simply to list it as a vertical multiplication using the formal algorithm. The participants were not asked to show that they could have correctly done this multiplication.

Eighty-five entries (15.7%) suggested strategies that could be coded, but would have led to an incorrect answer. A common example of this was to suggest a Left Right Separated strategy for 21 x 23 as 20 x 20 + 1 x 3. This can be coded as LRS because it suggests working from the highest power of ten to the units digits; however it would not lead to a correct answer as two of the partial products have been left out. An incorrect partitioning strategy for 40 x 90 was 40 x 100 – 90 and an incorrect Basic Derived Fact strategy for the same question was '4 x 9 = 45 add 2 zeroes'. One student offered seven Compensation strategies for double-digit multipliers. All would have led to approximations but not the correct answer. For example '21 x 23 = 20 x 24' is a Compensation strategy that produces a reasonable approximation but not the exact answer to the multiplication.

Generally errors made in calculations were conceptual rather than procedural. McIntosh (2006) distinguishes between these types of errors in his testing of students from grades 3 to 10 in Tasmanian and ACT schools between 2001 and 2003. Procedural errors occur when the student can demonstrate a strategic understanding of the concept involved but makes careless numerical errors in applying the correct strategy (e.g. 40x90: 4x9=45 add two zertoes). Conceptual errors involve a more fundamental lack of knowledge concerning the concept needed to complete the calculation, for example, the Distributive Law. As the students in this study were not asked to complete the calculations provided in their untimed strategy analysis section, less procedural errors were evident.

For double-digit multiplication questions (Q 9-15), 29 entries (5.4%) suggested using the formal algorithm (RLS). This could not have been achieved in the time or without using pen and paper for working - it is not an appropriate mental multiplication strategy for two-digit multipliers (Clark, 2005).

This meant that 43.7% of possible entries were blank, obviously incorrect or not viable for that problem. That does not imply that the remaining 56.3% of suggested strategies would have definitely led to a correct answer (as mentioned above); they simply had the potential to do so. The predominant strategy suggestion in this category was Left Right Separated (LRS) with 111 instances or 20.6% of all strategy suggestions. A further 39 instances suggested LRS but applied it incorrectly.

The strategies that were used appropriately (56.3%), could be split into two categories:

- inflexible and appropriate to a limited range of problems (BDF, COU and RLS for one-digit multipliers)

- flexible and appropriate for a wider range of questions (LRS, PAR, FAC, CMP, CMB)

The former totalled 126 entries or 23.3%. The latter totalled 178 entries or 33.0%. This means that only one third of the strategies proposed by the participants had the potential to answer the majority of questions involving one- and two-digit multipliers as required by the Australian curriculum.

The next useful comparison is the range of strategies used, remembering that the Question 10 prompt was 'I am aware of a range of strategies for mental computation'. Table 4.11 summarises the range of strategies used by each individual participant for the 15 test questions.

Table 4.11 *Variety of Strategies Used by Each Participant in the 15 Test Questions*

Number of Strategies Used	1	2	3	4	5
Number of Participants	6	12	7	8	3
% of Participants	17%	33%	19%	22%	8%

The number of strategies use by participants correlated significantly with the result on the first test ($\rho = 0.402, p = 0.020$), but the correlation was only moderate, perhaps because some participants using three or four strategies may have chosen from the simplest strategies of Counting (COU), Basic Derived Fact (BDF) and Right Left Separated (RLS) and still not scored well because of the limited application of these strategies.

However, those students who scored 8 or better in the test, all used three or four strategies correctly and these were generally the more flexible of the options. This included Partitioning (PAR), but mostly subtractive partitioning such as 19 x 25 = 20 x 25 – 25. Also evident were six examples of combination strategies such as $15 \times 18 = 15^2 + 3 \times 15$ (BFS + PAR), 19 x 25 = 16 x 25 + 3 x 25 = 400 +75 (PAR + BFS) and 15 x 18 = 15 x 10 + 15 x 8 = 15 x 10 + 30 x 4 (CMP + LRS). There was limited use of Basic Fact Shortcuts, Factors, Compensation or Non-Standard Additive Partitioning. One example of Non-Standard Additive Partitioning was 126 x 4 = 125 x 4 + 4.

Half of the participants were only aware of one or two strategies and these were often the simpler ones. This supports the design of the intervention that introduces a broad range of strategies to the participants that include those mentioned in both the Australian (ACARA, 2012) and NSW (NESA, 2015) curriculum documents as well as others suggested in the literature (Caney, 2008; Heirdsfield, Cooper, Mulligan & Irons, 1999; Markovits & Sowder, 1994; Menon, 2003; Whitacre, 2007).

When Questions 9 to 15 of the timed test (two-digit x two-digit) were analysed separately, the disparity between knowledge of appropriate mental multiplication strategies and their use in a timed test became greater. Figure 4.5 shows the frequency of correct responses to these seven questions.

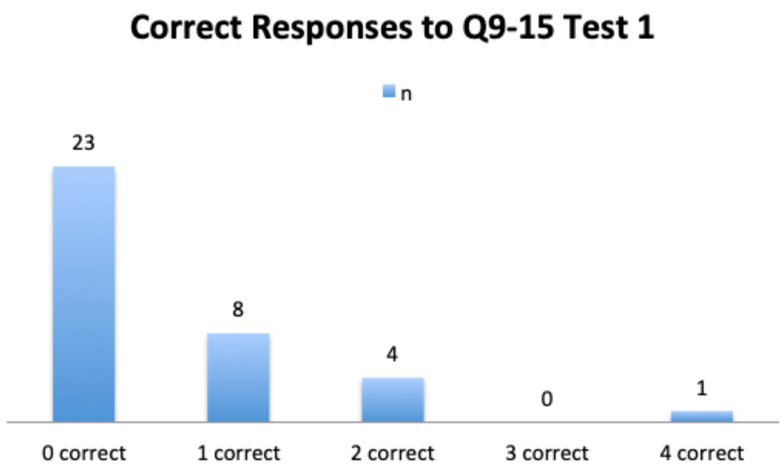


Figure 4.5 Correct Test Responses to Questions Involving Two-digit x Two-digit Problems from Timed Test 1 (Q 9 – 15), (n=36)

Of those attempting questions with two-digit multipliers, the highest score was 4/7 by one student with no other student scoring more than 2/7. Sixty-four percent of participants were unable to answer any of these questions correctly. In terms of appropriate strategies suggested in the coding sheets, there were only 73 suggested of a possible 252 (29%). The remainder (71%) were left blank (38%), incorrect (21%) or inappropriate for two-digit multipliers (12%). Of the 73 appropriate strategies, the majority were LRS (48 occurrences), non-standard partitioning (18 occurrences), FAC (1 occurrence) and CMB (6 occurrences). This suggests a lack of appropriate strategies that specifically handle two-digit multipliers and a strong reliance on the LRS strategy for these problems.

Four items of the questionnaire sought to elicit further information about the participants' knowledge of a wide range of strategies. They reiterated four of the two-digit x two-digit questions and asked for as many alternate ways of doing the problem as possible.

Consider, for example, 45 x 24. Possible strategies that could be suggested to solve it are as follows:

LRS 45 x 20 + 45 x 4 = 900 + 180 = 1080
LRS 40 x 24 + 5 x 24 = 960 + 120 = 1080
CMP 90 x 12 = 1080 (double the first and half the second)
CMP 120 x 9 = 1080 (5 x 24 and one fifth of 45)
PAR 20 x 24 + 25 x 24 = 480 + 600 = 1080
PAR 50 x 24 – 5 x 24 = 1200 – 120 = 1080
FAC 24 x 5 x 9 = 120 x 9 = 1080
FAC 45 x 4 x 3 x 2 = 180 x 3 x 2 = 540 x 2 = 1080
CMB 45 x 12 x 2 = (45 x 10 + 90) x 2 = 540 x 2 = 1080 (LRS + FAC)
CMB 90 x 3 x 4 = 270 then double double (CMP + FAC + BFS)

There are many more ways to solve the problem; and the four items in the questionnaire sort similar responses from the participants. This section was untimed and there was ample room to include a range of strategies. Participants were asked to outline each strategy without having to complete the solution. Again, there was a considerable difference between the responses of the third year group (EDME369) and the first year group (EDME145). Of the 16 participants from EDME145, only five could suggest alternative strategies to any of the four problems and only seven strategies were suggested. Of these, three were inappropriate to the question, suggesting the formal algorithm be used. Of the other four, all were LRS.

The EDME369 cohort had 15 of the 17 participants suggesting a total of 50 alternative strategies. Again, the bulk were RLS (21 responses) or LRS (7 responses). There were also 18 non-standard partitioning (PAR) strategies suggested with the majority being an expanded LRS strategy with four partial products starting with the highest power of ten. There were also three strategies using factors (FAC). Six participants who had not been able to list a possible strategy for the problem on their initial coding sheets now managed to find one. However, 16 of these alternatives were the formal algorithm (RLS).

Some of the alternate suggestions from both groups, and their codings, were as follows:

19 x 25

= 16 x 25 + 3 x 25 (PAR)

= 20 x 25 – 25 (PAR)

= 10 x 25 + 9 x 25 (LRS)

= 10 x 20 +10 x 5 + 9 x 20 + 9 x 5 (PAR)

23 x 12

= 25 x 12 – 2 x 12 (PAR)

= 20 x 12 + 3 x 12 (LRS)

= 10 x 23 + 2 x 23 (LRS)

= 20 x 10 + 20 x 2 + 3 x 10 + 3x 2 (PAR)

21 x 23

= 20 x 23 + 1 x 23 (LRS)

= 20 x 20 + 20 x 3 + 1 x 23 (PAR)

= 20 x 20 + 20 x 3 + 1 x 20 + 1 x 3 (PAR)

= 10 x 23 x 2 + 1 x 23 (FAC+LRS)

45 x 24

= 45 x 20 + 45 x 4 (LRS)

= 45 x 25 - 45 (PAR)

= 40 x 20 + 40 x 4 + 5 x 20 + 5 x 4 (PAR)

= 45 x 4 x 6 (FAC)

= 45 x 10 x 2 + 4 x 40 + 5 x 4 (FAC + PAR)

The greater range of strategies suggested by the third year group suggests a better SCK for this group and may have been a factor in their better test performance. Still, the range of suggested strategies from both groups were fairly limited with a majority of blank, incorrect or inappropriate strategies suggested. The group's responses as a whole fall short of the level suggested by the Australian curriculum for Year 5 students and hence the goal for our pre-service teachers. It is obvious from the above results that this group of pre-service teachers are not at that level as adults even though their responses to Items 9 and 10 of the questionnaire may have suggested otherwise.

As a final comment on the participants SCK, Table 4.12 repeats the correlation coefficient matrix for Items 9 to 11 from the Questionnaire.

Table 4.12 *Correlation Coefficient Matrix for Items 9 to 11 of Questionnaire*

Spearman's Correlations	Q9	Q10	Q11
Q9	1.000	.776**	.646**
Sig (2-tailed)		.000	.000
Q10		1.000	.659**
Sig (2-tailed)			.000
Q11			1.000
Sig (2-tailed)			

** Correlation is significant at the 0.01 level (2-tailed)

There was a strong correlation between participants' responses to these three related questions regarding their knowledge and use of a range of mental computation strategies. However, when it came to the harder problems involving two-digit x two-digit multiplications, the successful use of appropriate strategies markedly declined. This suggests that while half the participants had a knowledge of mental multiplication strategies, most of these allowed them to only attempt the simpler multiplications by one-digit numbers. Few had the ability to suggest appropriate strategies for harder problems (33%) and their success in applying them was poor.

Future programs of instruction in mental computation need to focus strongly on Basic Fact Shortcuts to reduce computation time. The shortcuts suggested in the intervention appear to be unknown to the participants as none were used in the coding sheets or questionnaire. The more flexible strategies of Left Right Separation and Partitioning were used for 31% of responses but there was little use of Factors, Compensation and Combinations of strategies. These would all be seen as essential components of a teacher's SCK for this topic.

4.7 Knowledge of the Place of Mental Computation in the Curriculum

Having considered content knowledge in relation to appropriate mental multiplication strategies (SCK) in the previous section, analysis moves on to the participants' knowledge of the place of mental multiplication in the curriculum. Items 12, 14, 16 and 17 of the questionnaire addressed that area, with participant responses summarised in Figure 4.6.

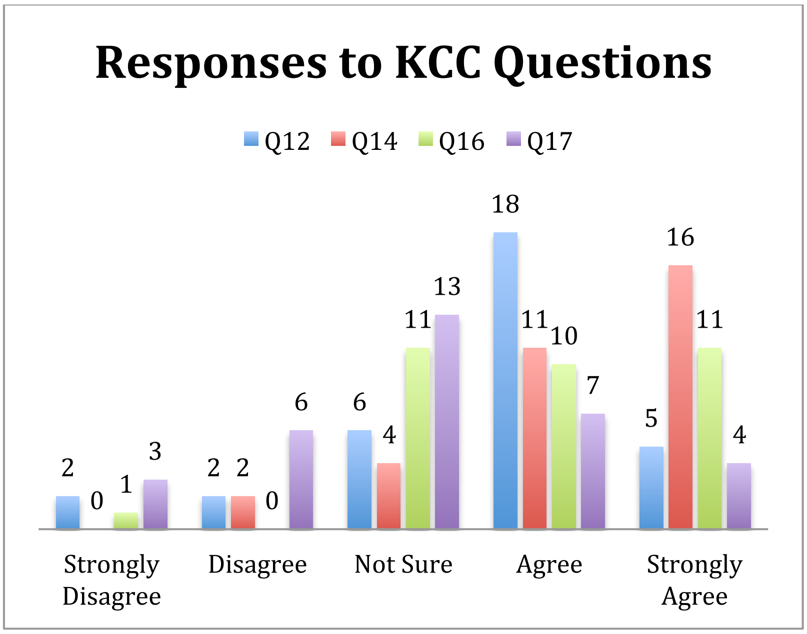


Figure 4.6 Responses to KCC Items (n=33)

Again, tabulated as percentages in Table 4.13:

Table 4.13 *Responses to KCC Items by Percentage*

Item	Prompt	Strongly Disagree	Disagree	Not Sure	Agree	Strongly Agree
12	I am aware of the place of mental multiplication in the curriculum.	6%	6%	18%	55%	16%
14	I think mental computation is an essential skill for school students	0%	6%	12%	33%	48%
16	Mental computation is best learned in primary school	3%	0%	33%	30%	33%
17	Mental computation should be taught before formal methods of written computation	9%	18%	39%	21%	12%

Items 14 and 16 indicate quite a strong belief in mental computation being taught to school students but slightly less acceptance that its best place is in the primary curriculum. One third of participants were unsure that this topic should be placed in primary. This contradicts the 71% of participants who felt that they knew its place in the curriculum. So in terms of KCC, this suggests some deficits as the Australian curriculum clearly places it in Year 5.

The other comparison of interest is between Items 12 and 17. Having suggested that a large majority know the place of mental multiplication in the curriculum (71%), this contrasts with participants' views about teaching mental computation before formal algorithms. As demonstrated in the previous section, there was a considerable reliance on the formal algorithm as a mental strategy. Fifteen of the 36 participants (42%) used it as one of their main strategies for the coding sheet. Also, there were 121 entries for the RLS strategy in coding sheets (22%). It is not surprising that 27% would disagree that mental computation should be taught before formal algorithms if they believe RLS to be a viable mental multiplication strategy. This may also reflect the fact that whilst they have not experienced mental multiplication in their current course, they have encountered the formal algorithm in its written and graphical forms.

4.8 Knowledge of Content and Teaching

Questions relating to KCT were Items 13, 18 and 19 in the questionnaire with participant responses summarised in Figure 4.7.

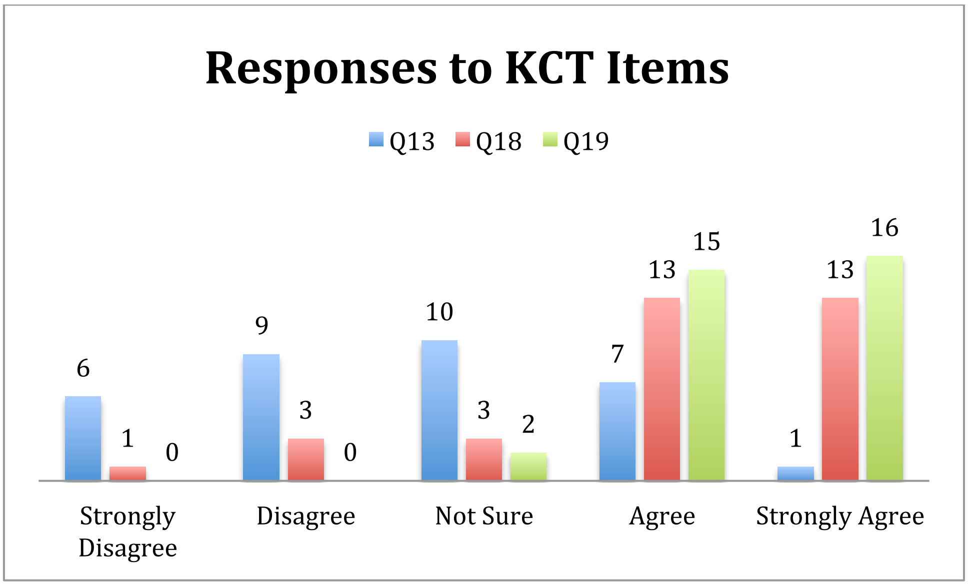


Figure 4.7 Responses to KCT Items from Questionnaire (n = 33)

Table 4.14 summarises the data in terms of the level of agreement with each prompt.

Table 4.14 *Responses to KCT Grouped by Percentage (n=33)*

Question	Prompt	Strongly Disagree	Disagree	Not Sure	Agree	Strongly Agree
13	I feel confident about teaching mental computation to Year 5	18%	27%	30%	21%	3%
18	All students have the potential to be good at mental computation	3%	9%	9%	39%	39%
19	It is important for students to have as wide a range of mental multiplication strategies as possible to allow them to solve different problems efficiently.	0%	0%	6%	46%	49%

Beginning with item 13, there was a strong lack of confidence in teaching this topic – 45% disagreed with the prompt and a further 30% were unsure. Only 24% seemed to have some degree of confidence in their ability to teach this topic. Following some ordinary timed test results and coding sheets where only 56.3% of strategy suggestions may have led to a correct answer, the participants' lack of confidence in teaching the topic seems to be justified. An interesting comparison is how responses to this item relate to responses for item 11 (I am confident using a range of strategies for mental computation). Table 4.15 compares percentage responses for items 11 and 13.

Table 4.15 *Comparison Between KCT and SCK Responses to Q11 and Q13*

Question	Prompt	Strongly Disagree %	Disagree %	Not Sure %	Agree %	Strongly Agree %
13	I feel confident about teaching mental computation to Year 5	18%	27%	30%	21%	3%
11	I am confident in using a range of strategies for mental computation	12%	36%	27%	18%	6%

Whilst the data might seem to be similar, there was little correlation between the two items ($\rho = 0.158, p = 0.381$) as further demonstrated by the scatter plot shown in Figure 4.8.

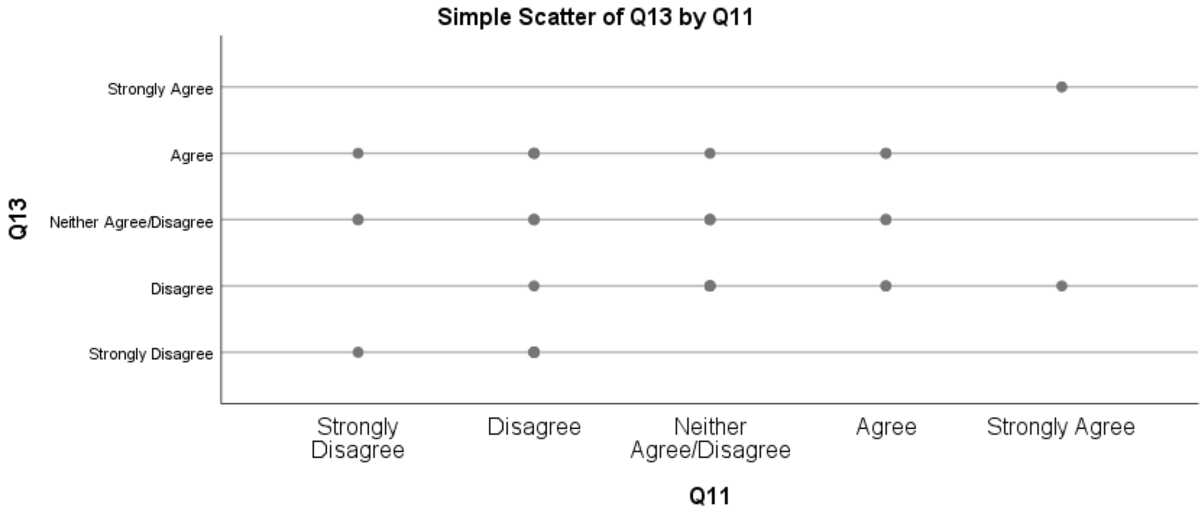


Figure 4.8 Scatter Plot of Responses to Items 11 and 13 (n=33).

Only three participants felt both confident with their knowledge and ability to teach the subject. In one case, this confidence would seem to be misplaced as the participant scored 3 in the test and could only suggest LRS and BDF for strategies and offered only one alternate strategy for the four problems in the questionnaire. The other two participants both scored 8/15 in the test and had a far greater range of strategies at their disposal and offered alternatives for the four questionnaire problems.

A second group ($n = 5$) seemed to have realistic expectations of themselves and their ability to teach this topic. They all felt confident; strongly agree ($n = 1$) or agree ($n = 4$); in their knowledge of a range of strategies and this was supported by their test scores of between 7 and 11 ($\mu = 8.4$). Yet, they were unsure ($n = 2$) or disagreed ($n = 3$) with the KCT prompt. With appropriate assistance, they should be capable of teaching this topic.

The next group of interest ($n = 5$) were not confident with their ability to use mental computation strategies (strongly disagree=1, disagree =3, unsure=1) but were confident in their ability to teach the topic (agree=5). This group is of concern as their test average ($\mu = 3.2$) indicates a substantial weakness in SCK but they still felt confident in teaching the topic. This concern was compounded when none of the group took up the opportunity of attending the intervention or reading the notes.

The final group of 20 participants is consistent in their assessment of their ability and confidence in teaching the topic. They were either unsure or lacked confidence in their ability in mental computation and expressed similar concerns about teaching the topic. This is supported by this group's test average (($\mu = 3.4$) and coding sheets. The fact that they are realistic about their abilities suggests a willingness to seek help to overcome these weaknesses and, consequently, seven members of this group attended the intervention.

The response to item 18 demonstrated strong acceptance (79% strongly agree or agree) of the statement that all students had the potential to learn mental computation strategies thereby implying that it was not just an extension topic for the more able student. This certainly reflects the spirit of the current syllabus (NESA, 2015) that asks teachers to practise mental computation skills on a regular basis with all students and sees it as important outcome for all students.

Item 19 strongly suggests that the participants are receptive to learning and teaching a wide range of strategies rather than looking for a few strategies that will answer most questions. This is consistent with research literature (see Section 2.6) that does suggest the flexible use of a wide range of strategies. The NSW syllabus objectives (NESA, 2015) however, seem to suggest a more limited range of strategies, listing only RLS, LRS and FAC in their notes and examples. Based on coding results, the participants are relying on a limited range of strategies that include RLS, LRS and PAR with the latter mainly involving Subtractive Distributive suggestions. It will be of interest to see if their recommendations for their own teaching programs suggest a wider range of strategies after they have had experience of other strategies through the intervention.

4.9 Knowledge at the Mathematical Horizon

Items 20 and 21 deal with knowledge that is a component of other topics and strands in the syllabus. Participant responses for these questions are shown in Figure 4.9.

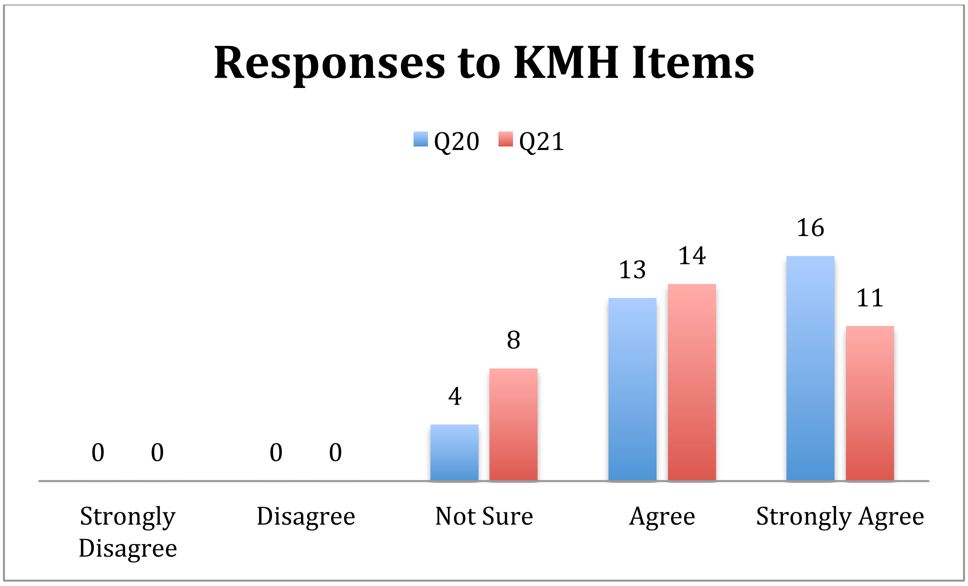


Figure 4.9 Responses to Items 20 and 21 of the Questionnaire (n=33)

Table 4.16 summarises the data in terms of the level of agreement with each prompt.

Table 4.16 *Responses to KMH Items Expressed as Percentages (n=33)*

Question	Prompt	Strongly Disagree	Disagree	Not Sure	Agree	Strongly Agree
20	Mental multiplication promotes critical thinking and problem solving.	0%	0%	12.1%	39.4%	48.5%
21	Mental multiplication promotes number sense.	0%	0%	24.2%	42.4%	33.3%

Items 20 and 21 show a strong belief in the use of mental computation as a problem solving activity (87.9% agreement), and to enhance students' number sense, (75.9% agreement). These two questions have a moderately strong correlation ($\rho = 0.645, p < 0.01$). This was a surprisingly positive response to what is a central message of the intervention. The use of non-standard partitioning (PAR) features in the tutorial notes as a way to enhance flexibility in mental computation by suggesting multiple solution pathways. Using this technique in combination with other strategies, such as BFS, assists students to solve harder problems and find alternative solutions. It is this approach that turns mental computation into a problem solving activity where multiple solution pathways can be explored and assessed. Examples of this approach were provided in Section 4.5. Research mentioned in the literature review also strongly supports the role of mental computation in enhancing number sense, whilst also being seen as part of number sense (Barody, 2006; Hartnett, 2007).

4.10 Common Content Knowledge

The last item in the questionnaire to require analysis was Item 15 and it was the only question in the category of Common Content Knowledge (CCK) that deals with knowledge common to other professions. Participant responses are shown in Figure 4.10.

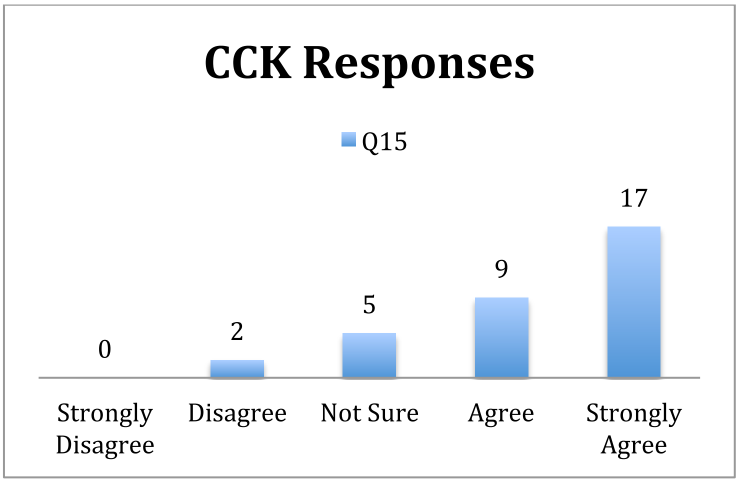


Figure 4.10 Common Content Knowledge Responses

This data is summarised in Table 4.17.

Table 4.17 *Percentage Comparison of Item 15 Responses*

Question	Prompt	Strongly Disagree	Disagree	Not Sure	Agree	Strongly Agree
15	I think mental computation is an essential skill for adults	0%	6.1%	15.2%	27.3%	51.5%

There was strong support (78.8% agree) for the premise that mental computation is an essential skill for all adults. This supports the literature that suggests that mental computation is an essential precursor to estimation and approximation and used by adults in preference to written algorithms (Northcote & Marshall, 2016).

4.11 Chapter Conclusions

The participation rates from the two cohorts were initially encouraging with 36 of the total 56 candidates agreeing take part in the research. The timing of the research, however, meant that numbers declined in Phases 2 and 3 as the students faced competing demands on their time. Phase 1, which involved the initial timed test, test coding and the first questionnaire, did provide a rich source of data that has been analysed in this chapter.

Despite a willingness to participate and some knowledge of mental computation skills, initial testing showed deficiencies across both cohorts when it came to implementing strategies in a timed situation. The third year cohort showed a slightly better aptitude (but not statistically significant), as did those who had expressed greater confidence in their mathematical ability during school and greater enjoyment of the subject. Results were not dependent on the level of mathematics studied at High School or on the participants' school experience with mental computation strategies. Rather their experience as adults, their perceived awareness and confidence with mental computation proved to be a strong indicator of test performance.

The performance in the first test demonstrated a need for further investigation to ascertain just what strategies the pre-service teachers knew and could use in their own computations as adults and in teaching the topic to their prospective students. To this end, Section 4.6 investigated the pre-service teachers' knowledge and use of mental multiplication strategies (SCK) in untimed situations. The strategies used in both untimed sections (test coding sheet and questionnaire) fell into three categories:

- Blank, incorrect or inappropriate to the problem (43.7%)
- Inflexible and appropriate to a limited range of problems (23.3%)
- Flexible and appropriate to a wider range of problems (33.0%)

This is somewhat consistent with Whitacre's (2007) findings of number sensible mental strategies that were discussed in the literature review in Chapter 2. Whitacre found that prior to intervention there was a heavy reliance on the use of the formal algorithm (MASA) as a mental multiplication strategy (53.5%) with Partial Products being used in 14% of strategy suggestions. This study found a lower, but still substantial, reliance on the formal algorithm

(23% usage) with 28% suggesting LRS (Partial Products). This study also had the added dimension of blank entries where no strategy could be suggested (23%) and strategies applied incorrectly (16%).

The use of the more flexible strategies of FAC, BFS, PAR, CMP and CMB is similarly limited in both studies. Whitacre (2007) found that NAD was used once, SD accounted for 11.6% of strategy use, AP 9.3%, FAC was used once and Derived was used three times. This study saw PAR (NAD and SD) used 10% of the time with Subtractive Distributive (SD) being the dominant component. FAC was used on six occasions (1%). CMP (contains AP) was not used and CMB (Derived) was used six times.

The analysis of the seven questions with two-digit multipliers gave an added dimension to this study as it highlighted a significant weakness in this area. Only one participant scored above 2/7 for these questions and there were only 73 appropriate strategies suggested (29%). The remainder were blank, incorrect or RLS. The section on alternate strategies using four of the seven two-digit x two-digit problems from the test, yielded 57 alternate strategies with the dominant suggestions being RLS (24 or 42%) and PAR (18 or 32%). The only other strategies suggested as alternatives were LRS (11 or 19%) and FAC (3 or 5%).

Having highlighted significant deficiencies in the participants' SCK, it was then necessary to assess their knowledge of the place of mental computation in the curriculum. Although 71% of participants felt that they knew the topic's place in the curriculum, 36% were unsure or disagreeing with the prompt that it is best learned in primary school, even though the curriculum firmly places it in Year 5. A further 66% were unsure or disagreeing with the prompt that it should be taught prior to the formal algorithm despite strong advice to the contrary in curriculum documents and research literature.

Clark (2005) summarises research that points to the potential dangers of teaching formal algorithms to primary school students prior to them experiencing their own informal strategies that include mental computation. He also cites an issue of relevance as the formal algorithm does not correspond to the way adults think about numbers. He suggests it is less efficient as a computation strategy and encourages students to give up their own thinking leading to a blind acceptance of results and over zealous application of the strategy. The extensive use of RLS as a mental strategy by the participants would suggest that one recommendation of this research would be to suggest that the following clause that appeared in the 2002 BOS syllabus be added to the present curriculum notes to avoid confusion in this area:

> formal written algorithms are introduced after students have gained a firm understanding of basic concepts including place value, and have developed mental strategies for computing with two-digit and three-digit numbers (BOS, 2002, p. 9).

The next area of interest and analysis was the participants' knowledge of content and teaching (KCT). In summary, most had a realistic concept of their own abilities in relation to mental computation and this translated into an appropriate assessment of their readiness to teach the topic. Twenty participants (61%) expressed concern, or were unsure regarding their mastery of a range of strategies and ability to teach the topic. There were five participants, all with low test scores, who expressed concern about their ability but were still confident in teaching the topic and did not attend the intervention or access the notes. As the literature points out (Hill, Rowan & Ball, 2005; Hill & Rowe, 1998) there is a significant correlation between teachers' ability in mathematics and their students' performance in the subject. As a microcosm, this would suggest that this group may have significant difficulty in imparting the important concepts in this topic to their students.

The section dealing with KMH showed strong support for contention that mental computation skills enhanced number sense and problem solving. The research also shows an increase in flexibility and adaptability when students are encouraged to seek alternate pathways to mental multiplication problems, a feature of the intervention.

Carroll (1996) found that a focus on students discussing alternate pathways to solutions using informal and invented strategies led to an increase in number sense and a greater capacity for mental calculations.

> Teaching students strategies that facilitate mental computation, estimation and number sense and allowing students to investigate, explore and develop alternatives for numerical computations are essentially the suggested avenues to improve student mental computation skills. (p. 142)

Rathgeb-Schnierer and Green (2013) speak of flexibility in mental computation as a highly desirable goal in elementary (primary) education. They define flexibility as "the knowledge of multiple solutions as well as the ability and tendency to selectively choose the most appropriate ones for a given problem and a particular problem-solving goal" (p. 354).

Mental computation has the ability to enhance number sense and problem solving ability in students and the approach suggested in the next chapter may facilitate that goal. The strong support of the participants for these two propositions is encouraging although early evidence presented in this chapter suggests they are unsure of their own ability in this regard and the teaching strategies required to promote both objectives.

The ability to use mental computation in conjunction with estimation and approximation is an important skill for adults and common knowledge in a wide range of trades and professions. The strong support for this contention was not reflected in their own experience with mental computation as adults. Only 48% agreed that they used the skill regularly although four of the five participants who strongly agreed with the prompt scored 8/15 or better in the first test. The correlation between test results and participants use of mental computation as an adult was moderately strong at $\rho = 0.645$ ($p < 0.01$).

The next chapter examines strategies presented to the participants and any resultant changes in their test and coding performance. It then proceeds to look at subsequent changes to the participants' perceived knowledge for teaching the topic again using the same set of questions that focussed on five of the six strands of MKT in the first questionnaire.

Chapter 5 Phase 2 and 3: The Intervention and its Effect

5.1 Introduction

This chapter outlines the intervention or Phase 2 of the research project and analyses any resultant effect in terms of test scores, strategy use and attitude towards mental multiplication and its place in the curriculum as evidenced in Phase 3. Participants were given the option of attending a two-hour tutorial, presented by the researcher, on the strategies suggested for mental multiplication. The tutorial took the form of an interactive workshop and was offered in each cohort's usual lecture slot. Due to the voluntary nature of the tutorial and its placement just prior to assignment deadlines and exams, attendance was reduced with 11 attending from the original 36 participants – six from EDME145 and five from EDME369.

The tutorial was presented via PowerPoint slides and each slide had review questions for the participants to attempt during the session. These slides were posted on each cohort's Moodle site along with a comprehensive set of notes and further practice questions. These notes and the final session of the tutorial appear in Appendix 5. One participant from EDME145 accessed the notes but did not attend the tutorial and five more accessed the notes following attendance at the tutorial. Four participants who attended Phases 1 and 2 did not attend Phase 3 of the project and one completed the test and coding sheet but not the questionnaire. Three participants completed the questionnaire and answered that they had attended the lectures and read the notes. As they were not on the tutorial roll and their coding sheets showed no evidence of the use of new strategies encountered in the notes and tutorial, their input was not included with the above group.

The notes provide examples of the strategies suggested for this topic and the breadth of Basic Fact Shortcuts suggested. Due to the limited time available for the intervention, many of the basic fact shortcuts appear as procedures or, at best, alternatives. Given more time, a more constructivist approach would have been used allowing students to suggest their own strategies and build on these suggestions through class discussion. Some procedures are necessary to shorten calculation time such as Trachenberg and Hall shortcuts (see Apprndix

7, pp. 182 – 186). The notes demonstrate a teaching approach for the topic that is designed to facilitate flexibility and adaptivity in students (through the use of non-standard partitioning) and present the topic with twin foci of number sensible strategies in a problem solving setting.

Twenty-five participants completed the test and coding sheet but two of those had not participated in the initial phases of the project and had not signed consent forms. Their data is not included in this study. Of the 23 participants who completed the test, 18 then completed the questionnaire. This included seven participants who were involved in all three phases of the project. The bulk of analysis in this chapter considers the latter group's responses and test and coding sheet results.

Comparisons were made between Test 1 and Test 2 for all participants and then split into the three groupings dependent on participation in some or all of the Phases. A similar analysis of coding sheet results looked at the two groups who completed both tests with the difference being participation in Phase 2. Comparisons of the questionnaire items were again split into the relevant components of the MKT framework outlined in Chapter 3. This section uses data exclusively from the seven participants who completed all three Phases. As this is a small sample, it was difficult to apply traditional measures of significance to comparisons between the data sets obtained in Phases 1 and 3. De Winter (2013) argues that the *t*-test is acceptable provided effect sizes are large. He recommends use of the paired samples *t*-test and Wilcoxon test for small populations and these appear in the analysis.

The lack of any significant difference between the two cohorts of EDME145 and EDME369 demonstrated in Chapter 4, resulted in comparisons between groupings based on Phase attendance rather than by cohort in this chapter.

5.2 Participant Evaluation and Recommendations from Phase 2

Eleven participants attended the intervention and six of those also completed all of Phase 3. One other participant attended the lecture and completed the test and coding sheet but not the final questionnaire. Another participant accessed the notes without attending the lecture and completed Phase 3. Table 5.1 shows the responses of the six participants who were involved in both Phase 2 and 3.

Table 5.1 *Combined Ratings of Tutorial and Notes (n = 6)*

Item	Prompt	Of No Value	Of Little Value	Of Moderate Value	Of Considerable Value	Of Great Value
28	If you attended the tutorial, please rate it	0	0	2	0	4
30	If you studied the notes, please rate them	0	0	0	3	3

Both the tutorial and the notes rated highly amongst the participants who accessed them, with the notes receiving slightly higher ratings than the two-hour tutorial. There were no participants who found them of little or no value. The participants were then asked which of the strategies they encountered in the tutorial or notes should be taught to a Year 5 class. Six participants completed this question and their choices are shown in Table 5.2. It was assumed that BDF and COU strategies would have been taught in Year 4 and, hence, they were not included in this list.

Table 5.2 *Choice of Strategies for a Year 5 Class (n=6)*

Strategy	BFS	LRS	RLS	FAC	PAR	CMP
Frequency	6	6	0	1	6	6

Two important points come from analysis of this item. Firstly, the complete lack of support for the formal algorithm as a mental multiplication strategy (RLS). Secondly, the unanimous support for Basic Fact Shortcuts (BFS), Left Right Separated (LRS), non-standard Partitioning (PAR) and Compensation (CMP) as essential components of a teaching program for this topic. Compared with CMP, Factors (FAC) received little support, even though they are complementary strategies. Since the two strategies are closely related, it may be possible to combine them in future research into one heading containing factors and compensation. The preference for CMP over FAC in the participants' teaching program is also of interest as FAC and LRS are the only strategies mentioned in the current NESA (2015) syllabus documents that relate to mental multiplication at this level.

Participants were then asked to rate each strategy separately with the results shown in Table 5.3

Table 5.3 *Participant Ratings of Individual Strategies from the Intervention (n=6)*

Code	Strategy	Of No Value	Of Little Value	Of Moderate Value	Of Considerable Value	Of Great Value
BFS	Basic Fact Shortcuts – shortcuts for multipliers from 2 to 12, 15 and 20.	0	0	1	1	4
LRS	Left to right separated – standard partitioning	0	0	1	1	4
RLS	Right to left separated – visualising the formal algorithm	0	2	3	1	0
FAC	Factors	0	0	3	3	0
PAR	Partitioning (Non-standard)	0	0	0	2	4
CMP	Compensation	0	1	0	1	4

From these responses, the relative value of the strategies to the participants can be expressed in a hierarchical order from most valued to least valued as shown in Table 5.4. To achieve this, rating options were allocated a value from 0 for Of No Value to 4 for Of Great Value. A total summated score was calculated for each strategy, with the strategies ordered according to the total score from highest to lowest.

Table 5.4 *Participant Ratings of Strategies in Order*

Rating	Most Valued				Least Valued
Strategy	PAR	BFS/LRS	CMP	FAC	RLS

The participant who saw CMP being of little value would still include it in their teaching program. The participant who mentioned RLS as being of considerable value would not have had it in their teaching program but would have used it as an extension strategy. Those strategies chosen as extension strategies are shown below in Table 5.5.

Table 5.5 *Strategies Participants Would Use as Extensions to Their Program (n=7)*

Strategy	BFS	LRS	RLS	FAC	PAR	CMP
Frequency	3	3	5	4	5	6

This question may have been confusing to participants. Two suggested that they would use all six strategies for an extension group having left out some in their mainstream teaching program. One suggested the two strategies as extensions that were left out of her teaching program. Those choosing RLS as an extension did not have it as part of their mainstream program. This, perhaps, indicates that they view it as more appropriate for able students after encountering an initial program of mental multiplication. All those choosing BFS, LRS, PAR and CMP as extensions also had them as part of their mainstream teaching program.

A final question asked participants how they would pace their instruction for this topic (with a Year 5 class) with the results shown in Table 5.6.

Table 5.6 *Suggested Duration of Instruction for Year 5 (n=7)*

Duration of Instruction	Two to Three Weeks	Over One Term	Over Two Terms	Over Three or Four Terms
Frequency	1	2	2	2

The mixed response to this question suggests there may be some uncertainty amongst participants about the time needed to teach this topic to Year 5. However, the majority would see it as taking at least one term to complete and possibly more. This is in keeping with the current NESA (2012) syllabus recommendation MA3-6NA that states:

> In Stage 3, mental strategies need to be continually reinforced
> (NESA, 2015: Multiplication and Division 1, Backgound Information, para 2)

Four participants also chose to comment on the intervention in the free response section of the questionnaire.

Comment 1:

> I found that the lecture on the mental computation to be interesting since I was unaware of the range of mental strategies that could be used and how to teach them.

The comment above indicates a growth in this participant's SCK and KCT as a result of the intervention and a lack of knowledge of mental multiplication strategies from previous school and tertiary experiences.

Comment 2:

> The content learnt is very valuable to young children as the different strategies help visualise the process of multiplication and overall improve a child's number sense and ability to modify numerals and equations to suit the task at hand.

This comment demonstrates a similar growth in this participant's KCT as well as an understanding of the topic's contribution to number sense and flexibility in problem solving.

Comment 3:

> Extremely useful tool for teachers as well when generating equations for children. I think that coming from not being able to do any mental multiplication larger than single digits, this has given me a much broader range of skills. Given more time to learn and consolidate the strategies I would be confident that I would be able to answer the test questions efficiently.

The above comment indicates a growth in this participant's SCK. It also highlights the short duration of the intervention and the need for more time to fully digest the information.

Comment 4:

> I am not a strong mathematician and have struggled with it since High School. Personally, the very first thing my brain goes to is always the formal algorithm. After participating in this trial I am definitely aware that my difficulties in maths stem from being taught mental skills at school, then being taught the formal algorithm. After I learnt the formal algorithm, I decided that this was obviously the correct and 'adult' way to do multiplication so this is what I used. And so I forgot all the important mental skills. I also have trouble visualising and manipulating

> algorithms to make them easier to complete. Although intellectually I always know it is an option, to me it feels wrong to be able to simply change the question to an easier one. As a result I now don't really know where to start when doing these manipulations.

The last comment provides insight into the effect of the formal algorithm that has concerned many researchers, particularly Clark (2005), who contends that the formal algorithm "encourages children to give up their own thinking, leading to a loss of 'ownership of ideas' and tends to lead to blind acceptance of results and over-zealous applications" (p. 97). Handal (2003) comments that "pre-service teachers bring into their education program mental structures overvaluing the role of memorization of rules and procedures in the learning and teaching of school mathematics" (p. 49). Whilst this participant has certainly identified that as a problem, the short intervention was not sufficient to allow him to feel confident with alternate pathways to solutions. He did, however, improve his score in the second test and used a greater range of strategies in more questions in his second coding sheet.

5.3 Comparisons of Participant Test Performance

There was a two-week gap between Test 1 and Test 2 and the number of participants attempting Test 2 (n = 23) declined from Test 1 (n = 36). Hence the comparisons in Figure 5.1 of distributions by class in both tests are expressed in terms of percentages.

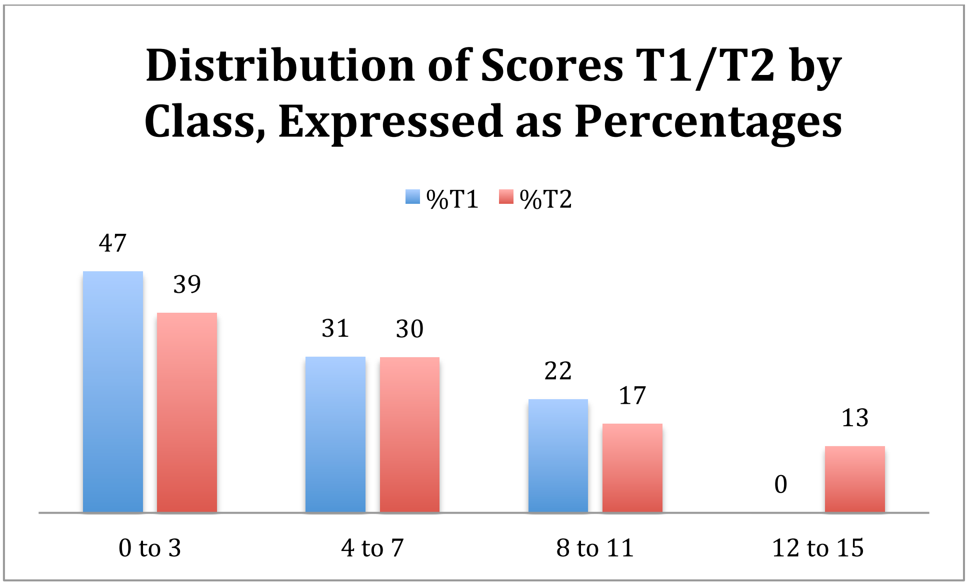


Figure 5.1 Distribution of Test Scores by Class Expressed as %

Scores ranged from 0 to 11 in Test 1 compared with a range between 1 and 13 in Test 2. While the pass rate increased from 22% in Test 1 to 30% in Test 2, as mentioned previously, a mere pass could not be considered as mastery of mental multiplication at a Year 5 exit level. The 13% who scored 12 or more could be reasonably satisfied with their performance in relation to their own SCK. A dependent samples *t*-test was conducted for the 23 participants who had completed both Test 1 and Test 2, with a statistically significant improvement in performance identified: ($\bar{\bar{X}}_{Test\ 1} = 4.43, \sigma_{Test\ 1} = 3.31, \bar{\bar{X}}_{Test\ 2} = 5.81, \sigma_{Test\ 1} = 3.70, df = 20, t = -3.185, p = 0.005$).

The improvement in test means could have resulted from the intervention but other factors may also have had a bearing on this. For example, the second test may have been easier, there could have been a practice effect, and/or the quality of the testing cohort may have changed. The high correlation of Test 1 and 2 scores, $\rho = 0.82$, may indicate that there was consistency in the test design, administration and environment and, indeed, that was true. The tests were administered in the same location, at the same time of day and using the same presentation mode (PowerPoint slides of 15 second duration). The test questions were also similar as shown in Table 5.7

Table 5.7 *Distribution of Test Questions by Category*

Test			Retest	
No.	Question	NSW Outcome	No.	Question
1	40 x 90	2 x 2 digit	1	50 x 90
2	5 x 8000	1 x 4 digit	2	6 x 7000
3	18 x 6	1 x 2 digit	3	22 x 6
4	5 x 19	1 x 2 digit	4	5 x 23
5	126 x 4	1 x 3 digit	5	217 x 4
6	9 x 45	1 x 2 digit	6	9 x 37
7	3 x 195	1 x 3 digit	7	3 x 295
8	143 x 7	1 x 3 digit	8	321 x 8
9	19 x 25	2 x 2 digit	9	23 x 25
10	23 x 12	2 x 2 digit	10	32 x 12
11	15 x 18	2 x 2 digit	11	14 x 17
12	21 x 23	2 x 2 digit	12	32 x 31
13	123 x 11	3 x 2 digit	13	432 x 11
14	45 x 24	2 x 2 digit	14	24 x 35
15	34 x 35	2 x 2 digit	15	46 x 45

The table demonstrates consistency in the categories of each question asked in the two tests. There were nine questions with the same multiplier, while others were of a similar difficulty. The products for the two sets of questions were larger in Test 2 with only two exceptions. This suggests, that the second test may have been slightly more difficult that the first. For example, consider Question 12, Test 1

21 x 23 = 20 x 23 + 1 x 23 = 460 + 23 = 483

This uses LRS. The same strategy applied to Question 12, Test 2 produces:

32 x 31 = 30 x 32 + 1 x 32 = 960 + 32 = 992

Tripling would normally be seen as more difficult than doubling so the second question would be considered slightly more difficult. Given that, the second test does not appear easier than the first.

Practice effect is defined by the APA Dictionary of Psychology (2007) as "any change or improvement that results from practice or repetition of task items or activities". The Dictionary then expands further:

The practice effect is of particular concern in experimentation involving within-subjects designs as participants' performance on the variable of interest may improve simply from repeating the activity rather than from any study manipulation imposed by the researcher (VandenBos, 2007).

Twenty-one students completed both the pre- and post-test, with seven of these completing the intervention. A paired samples *t*-test completed for those who did not complete the intervention indicated an improvement on the second test that was close to being statistically significant: ($\bar{X}_{Pre-test} = 4.71, \sigma_{Pre-test} = 3.54, \bar{X}_{Post-test} = 5.79, \sigma_{Post-test} = 3.53, t = -1.883, p = 0.082, df = 10$, suggesting that there may have been some training effect. Future research designs would benefit from a larger time gap between testing, thus allowing for a longer intervention as mentioned below.

The seven participants who completed the intervention and both tests demonstrated a statistically significant improvement: $\bar{X}_{Pre-test} = 3.86, \sigma_{Pre-test} = 2.99, \bar{X}_{Post-test} = 5.86, \sigma_{Post-test} = 4.30, t = -3.240, p = 0.018, df = 6$. A comparison of the improvement for the two groups was done by subtracting the pre-test result from the post-test result for the 23 students and then performing an independent samples *t*-test based upon whether the intervention had been completed. While those who completed the intervention demonstrated more improvement, the difference was not statistically significant: $\bar{X}_{No\ Intervention} = 1.07, \sigma_{No\ Intervention} = 2.13, \bar{X}_{Intevention} = 2.00, \sigma_{Intevention} = 1.63, t = -1.010, p = 0.325, df = 19$.

This analysis indicates that, while those students who completed the intervention have obtained improved results, a longer intervention would be required to achieve statistically significant improvements in performance. The intervention in this study was limited in length, largely due to time constraints arising from unit requirements.

To consider this further, we need to split the test results into groupings shown in Table 5.8.

Table 5.8 *Test Statistics Based on Phase Groupings*

Test	All groups			Phase 1 only Group 1			Phases 1 and 3 Group 2			All Phases Group 3		
	$\bar{X}$	σ	n	$\bar{X}$	σ	n	$\bar{X}$	σ	n	$\bar{X}$	σ	n
1	4.3	2.8	36	4.2	2.4	13	4.6	1.3	16	3.9	3.0	7
2	5.7	3.7	23				5.6	3.5	16	5.9	4.3	7

The 16 participants completing both tests but not attending the intervention had a 21.6% improvement in their mean result. The seven participants involved in all three Phases had a 51.9% increase in their mean result. This suggests that there was a substantial net effect of the intervention in improving the test scores of the seven participants involved in all three Phases.

It is hypothesised that the improvement in scores resulted from a combination of practice effect and the intervention with the latter having a substantial effect on the scores of the seven participants who completed it. It is also interesting to note that this group of seven started with the lowest test mean of the three groupings and ended with the highest test mean. This is directly related to a low SCK in five of the seven participants and a lack of confidence in their mental computation abilities as shown in Section 4.5

The seven two-digit x two-digit problems (Q9 to Q15) continued to be difficult for both groups who attempted the second test. The group experiencing the intervention (n=7) were able to lift their total score in this area from four correct to 10, but one participant accounted for six of those 10. The group not experiencing the intervention (n=15) lifted their score from 10 to 15 with one participant scoring five of those 15. A greater improvement by the intervention group but certainly not significant to the study as the level of competency of both groups remained low (apart from two participants).

Whilst the improvement in scores was encouraging, the means of all groupings including those who had experienced the intervention were still well below the level that would indicate some mastery of the topic knowledge (SCK). An analysis of strategy use by the two groups who did both tests may provide some further insight into the efficacy of the intervention program

5.4 Comparison of Strategy Use by Participants Completing Both Tests

A group of 15 participants completed Test 1 and Test 2 but not the intervention. Their strategy suggestions for the 15 questions in both tests are shown in Table 5.9.

Table 5.9 *Strategy Suggestions of Group 2 (n=15)*

Strategies	BDF	COU	RLS	LRS	PAR	FAC	BFS	CMP	CMB	Blank	Total
T1 correct	24	9	28	25	28	0	0	0	5	51	170
T1 incorrect	1	0	17	28	2	0	0	7	0		55
Subtotals	25	9	45	53	30	0	0	7	5	51	225
T2 correct	24	1	38	46	20	0	1	0	0	39	169
T2 incorrect	1	0	10	37	2	0	0	6	0		56
Subtotals	25	1	48	83	22	0	1	6	0	39	225

Strategy usage in the second test showed a slightly improved awareness of mental multiplication that could be attributed to practice effect i.e. the opportunity to reflect and interact with other participants after Test 1 may have been responsible for the few major changes in the choice of strategies. Certainly, the number of blank entries decreased from 22.7% to 17.3%. There was an increase in LRS, one of the preferred mental strategies in syllabus documents, however, the large number of incorrect applications of that strategy indicates that the participants are not confident with its use particularly with two-digit multipliers. This was also evident in Test 1. The use of RLS persisted showing a misplaced reliance on the formal algorithm for mental multiplication, particularly with two-digit multipliers (15 strategy suggestions). There was, again, little use of sophisticated strategies such as BFS, CMP and CMB. The same participant again applied an incorrect compensation strategy to six of the two-digit x two-digit problems.

Using the same breakdown as in the previous chapter, the strategies used fell into three categories as shown in Table 5.10.

Table 5.10 *Strategy Suggestions of Group 2 by Category (n=15)*

Strategy Category	Test 1	Test 2
Blank, incorrect or inappropriate to the problem	49.3%	48.9%
Inflexible and appropriate to a limited range of problems	24.9%	21.3%
Flexible and appropriate for a wider range of problems	25.8%	29.8%

This was a slight improvement on strategy suggestions by Group 2 (no intervention) but still poor in terms of the participants' SCK and, as a consequence, their ability to teach the topic effectively. Almost 50 % of the strategy suggestions from this group in both tests were either blank, incorrect or inappropriate to the problem.

The strategy use of the eight participants that completed the intervention and both tests is shown in Table 5.11.

Table 5.11 *Strategy Suggestions Group 3 (n=8)*

Strategies	BDF	COU	RLS	LRS	PAR	FAC	BFS	CMP	CMB	Blank	Subtotals
T1 correct	10	0	36	34	13	6	0	0	1	14	114
T1 incorrect	1	0	1	3	1	0	0	0	0		6
Subtotals	11	0	37	37	14	6	0	0	1	14	120
T2 correct	14	0	4	31	10	3	18	1	2	31	114
T2 incorrect	0	0	0	4	2	0	0	0	0		6
Subtotals	14	0	4	35	12	3	18	1	2	31	120

The second set of strategy choices of these eight participants did show a marked change from Test 1 choices. The almost complete abandonment of RLS as a choice was encouraging and the four instances of it were all with single digit multipliers. There was a considerable increase in the use of BFS, all applied appropriately and correctly. Other strategy choices remained largely consistent although there was an increase in blank entries from 14 (11.7%) to 31 (25.8%). Without interview, the latter would be difficult to explain.

However, three of the four participants who left blanks on their coding sheet were later able to suggest alternate strategies to four of those questions repeated in the questionnaire.

Using the same criteria as above, the strategies used fell into three categories as shown in Table 5.12.

Table 5.12 *Strategy Suggestions of Group 3 by Category (n=8)*

Strategy Category	Test 1	Test 2
Blank, incorrect or inappropriate to the problem	30.0%	30.8%
Inflexible and appropriate to a limited range of problems	25.0%	15.0%
Flexible and appropriate for a wider range of problems	45.0%	54.2%

This again, demonstrates an improvement in this group's strategy use above that of the comparison group not experiencing the intervention or notes. Whilst the timed test scores for this group remains low, their knowledge of, and use of, more flexible strategies suggest that their performance could improve with practice. Over 50% of strategies suggested by this group were of the more flexible category and appropriate for a wider range of problems.

The next useful comparison is the range of strategies used, remembering that the Question 10 prompt was 'I am aware of a range of strategies for mental computation'. Table 5.13 summarises the range of strategies used by each individual participant for the 15 test questions.

Table 5.13 *Variety of Strategies Used by Each Participant in the 15 Test Questions*

Group 2	Number of Strategies Used	1	2	3	4	5	6	Total
Phases 1 and 3 only	Number of Participants	3	4	6	2			15
	% of Participants	20%	27%	40%	13%			100%
Group 3	Number of Strategies Used	1	2	3	4	5	6	
All 3 Phases	Number of Participants		1	3	2	1	1	8
	% of Participants		12.5%	37.5%	25%	12.5%	12.5%	100%

The range of strategies suggested, even amongst the weaker participants, increased markedly for the group encountering all three phases of the study. The top three scores in the test all used four or more strategies with the highest score using six strategies. By contrast, the range of strategies used by the group who did not experience the intervention stayed the same and top scores used between two and four strategies.

The questionnaire repeated four questions from the test and asked for alternate solutions to the problems. The questions were of similar types to those used in the first questionnaire and the responses from both questionnaires for Groups 1 and 2 are shown in Table 5.14.

Table 5.14 *Comparison of Alternate Strategy Suggestions Between Tests and Groups*

Test 1	RLS	LRS	PAR	FAC	BFS	CMP	CMB	Total
Group 2 (n=10)	6	7	3	0	0	0	1	17
Group 3 (n=7)	12	6	7	0	0	0	0	25
Test 2								
Group 2 (n=10)	0	12	6	0	0	1	0	19
Group 3 (n=7)	0	6	9	4	3	1	0	23

In both groups, the use of RLS disappeared entirely in Test 2 alternate suggestions. Group 3 containing participants who experienced the intervention were able to suggest more alternates in Test 1 but they had a heavy reliance on RLS (12 of 25 suggestions) with only two other strategy types offered. In the second test they also offered more alternatives that Group 1 but now had a much wider strategy repertoire of five different strategies compared to three from Group 1.

To illustrate the approach when coding solutions, some of the responses, from both groups are shown. Trachtenberg's and Hall's methods are detailed in the tutorial notes (Appendix 5, pp. 182 – 186) and are shortcuts that apply to mulitpliers of 11, 12 and 9 respectively.

23 x 25
= 23 ÷ 4 x 100 (CMP)
= 25 x 25 – 2 x 25 (PAR)
= 20 x 25 + 3 x 25 (LRS)

= 20 x 20 +20 x 5 + 3 x 20 + 3 x 5 (PAR)

= 23 x 5 x 5 = 115 x 5 (FAC)

32 x 12

= 32 x 11 + 32 (PAR)

= Trachtenberg's method (BFS)

= 30 x 12 + 2 x 12 (LRS)

= 10 x 32 + 2 x 32 (LRS)

= 30 x 10 + 30 x 2 + 2 x 10 + 2 x 2 (PAR)

14 x 17

= 14 x 10 + 14 x 7 (LRS)

= 17 x 10 + 4 x 17 (LRS)

= 20 x 14 – 3 x 14 (PAR)

= 15 x 17 – 17 (PAR)

= 14 x 7 x 2= 98 x 2 (FAC)

= 12 x 17 + 28 (BFS + PAR)

9 x 37

= 9 x 30 + 9 x 7 (LRS)

= 10 x 37 - 37 (PAR)

= Hall's method (BFS)

= 37 x 3 x 3 (FAC)

= 30 x 3 x 3 + 9 x 7 (FAC + PAR)

= 9 x 40 – 3 x 9 (PAR)

Note: For explanations of Trachenberg's methods and Hall's method see workshop notes in Appendix 5, pp. 182 -186.

5.5 Changes in Participant Knowledge of Mental Computation

5.5.1 Introduction

This section deals with any changes in the knowledge of those participants who experienced the intervention and answered both questionnaires (n=7). The changes are related to Hill, Ball and Schilling's (2008) framework for MKT.

5.5.2 Changes in Participants' Perceived SCK

Items 10 (I am aware of a range of strategies for mental computation) and 11 (I am confident in using a range of strategies for mental computation) of both questionnaires dealt with the participants' personal knowledge of mental computation strategies and their confidence in using these strategies (SCK). Table 5.15 summarises their responses to these questions. The result of a Wilcoxon Signed Ranks test is also shown.

Table 5.15 *Responses to Items 10 and 11 for Both Questionnaires* ($n = 7$)

Item No	Strongly Disagree	Disagree	Not Sure	Agree	Strongly Agree	*Z*	*p*
10 Q1	1	2	0	3	1	-1.508	0.132
10 Q2	0	0	1	3	3		
11 Q1	1	3	0	2	1	-1.469	0.142
11 Q2	1	1	0	3	2		

Note: Q1 stands for Questionnaire 1 and Q2 for Questionnaire 2

While the differences are not statistically significant, largely due to the small sample size, this shows a substantial improvement in awareness of strategies and confidence in using them.

5.5.3 Changes in Participants' Perceived KCC

Having considered content knowledge in relation to appropriate mental multiplication strategies (SCK) in the previous section, analysis moves on to the participants' knowledge of the place of mental multiplication in the curriculum (KCC). Items 12, 14, 16 and 17 of both questionnaires addressed this area, with participant responses summarised in Table 5.16. The result of a Wilcoxon Signed Ranks test is also shown.

Table 5.16 *Responses to KCC Items in Both Questionnaires (n=7)*

Item	Prompt	Str'ly Disagree	Dis-agree	Not Sure	Agree	Str'ly Agree	*Z*	*p*
12 Q1	I am aware of the place of mental multiplication in the curriculum.			1	4	2	-1.732	0.083
12 Q2					3	4		
14 Q1	I think mental computation is an essential skill for school students			1	2	4	-1.732	0.083
14 Q2					1	6		
16 Q1	Mental computation is best learned in primary school			1	2	4	-0.577	0.564
16 Q2				1	1	5		
17 Q1	Mental computation should be taught before formal methods of written computation			5	1	1	-2.271	0.023
17 Q2					3	4		

Note: Q1 stands for Questionnaire 1 and Q2 for Questionnaire 2

The participants all agree that mental computation is an essential skill for school students. There is an increased awareness of the place of mental computation in the curriculum and all see it being taught prior to the formal algorithm. The latter statement is encouraging as many were previously uncertain about this. The placement of the topic in the primary curriculum was supported by six of the seven participants.

5.5.4 Changes in Participants' Perceived KCT

Questions relating to KCT were Items 13, 18 and 19 in both questionnaires with participant responses summarised in Table 5.17. The result of a Wilcoxon Signed Ranks test is also shown.

Table 5.17 *Frequency of Responses to KCT Items in Both Questionnaires (n=7)*

Item	Prompt	Strongly Disagree	Disagree	Not Sure	Agree	Strongly Agree	Z	p
13 Q1	I feel confident about teaching mental computation to Year 5	1	2	3		1	-1.841	0.066
13 Q2				3	3	1		
18 Q1	All students have the potential to be good at mental computation				3	4	-1.000	0.317
18 Q2					2	5		
19 Q1	It is important for students to have as wide a range of mental multiplication strategies as possible to allow them to solve different problems efficiently.				3	4	-.0577	0.564
19 Q2					2	5		

Note: Q1 stands for Questionnaire 1 and Q2 for Questionnaire 2

The participants all agreed that a wide range of strategies should be taught which was earlier supported by their selection of LRS, PAR, BFS and CMP for teaching programs. They also believed that all students have the potential to access this topic which is the intention of the syllabus documents.

Their confidence in teaching the topic would seem to have improved after the intervention. Table 5.18 compares this with previous items 10 and 11 also dealing with awareness and confidence in using strategies (as opposed to teaching them).

Table 5.18 *Comparison Between KCT and SCK Responses to Items 10, 11 and 13 from Questionnaire 2 (n=7)*

Question	Prompt	Strongly Disagree	Disagree	Not Sure	Agree	Strongly Agree
13	I feel confident about teaching mental computation to Year 5			3	3	1
10	I am aware of a range of strategies for mental computation			1	3	3
11	I am confident in using a range of strategies for mental computation			3	3	1

Whilst awareness of strategies improved markedly, both confidence with those strategies and the ability to teach them had only improved moderately. This is possibly a reflection of the lack of time to practise the strategies, continued weak test scores for the lower achievers in the group and continued difficulty with two-digit multipliers.

5.5.5 Changes in Participants' Perceived KMH

Questions relating to KMH were Items 20 and 21 in both questionnaires with participant responses summarised in Table 5.19.

Table 5.19 *Responses to KMH Items in Both Questionnaires (n=7)*

Question	Prompt	Strongly Disagree	Disagree	Not Sure	Agree	Strongly
20 Q1	Mental multiplication promotes critical thinking and problem solving.			1	2	4
20 Q2					2	5
21 Q1	Mental multiplication promotes number sense.			1	2	4
21 Q2					2	5

Note: Q1 stands for Questionnaire 1 and Q2 for Questionnaire 2

Items 20 and 21 showed a continued strong belief in the use of mental computation as a problem solving activity and to enhance students' number sense. All participants agreed, to some extent, with both prompts. Having experienced the intervention, this could be regarded as an affirmation of the approach used.

5.5.6 Changes in Participants' Perceived CCK

The last item in the questionnaire to require analysis was Item 15 and it was the only question in the category of Common Content Knowledge (CCK). Participant responses for both questionnaires are shown in Table 5.20.

Table 5.20 *Comparison of Item 15 Responses Between Questionnaires (n=7)*

Question	Prompt	Strongly Disagree	Disagree	Not Sure	Agree	Strongly Agree
15 Q1	I think mental computation is			1	2	4
15 Q2	an essential skill for adults			1		6

Note: Q1 stands for Questionnaire 1 and Q2 for Questionnaire 2

There was strong support for the premise that mental computation is an essential skill for all adults, with the exception of one participant who remained unsure. This supports the literature that suggests that mental computation is an essential precursor to estimation and approximation and used by adults in preference to written algorithms (Clark, 2005).

5.6 Chapter Conclusions

Despite the short duration of the intervention and reduced numbers, there was considerable agreement about the efficacy of the program and the strategies recommended to the participants. The tutorial and, more particularly, the notes received strong support from the participants. Similarly, strategies of LRS, PAR, BFS and CMP were strongly recommended for inclusion in a teaching program for Year 5. There was little support for the use of RLS as a mental strategy although some saw it as an extension to the topic. FAC also received little support for inclusion in teaching programs although this may be a result of it closely resembling CMP which had the added strategy of Aliquot Parts i.e. it may have been seen as a subset of CMP.

Recommendations from the most recent Australian (2012) and NESA (2015) curriculum documents for this topic include LRS, PAR and FAC, although partitioning is only mentioned in the Australian curriculum and the only example of non-standard partitioning in either document involves subtractive partitioning. The participants also recommended that BFS should be considered in teaching programs. The NESA (2015) curriculum does make mention of BFS in its Background Information to the outcome MA3-6NA, which states:

> Students could extend their recall of number facts beyond the multiplication facts to 10 × 10 by memorising multiples of numbers such as 11, 12, 15, 20 and 25.
> (NESA, 2015: Mathematics K to 10, Stage 3, Multiplication and Division 1, Background Information, para 1)

However, the techniques used in the intervention for this range of multipliers did not rely on memory but rather on strategy use. For example, x 12 could be seen as x 10 + double (LRS) or as x 11 + x 1 (PAR). The use of Trachtenberg and Hall shortcuts would involve memorising the procedure.

The majority of participants favoured an extended treatment for this topic rather than condensing it into a short block of two to three weeks. Again, this was supported by statements in the NESA (2015) curriculum, mentioned above.

The strong acceptance of the material presented in the intervention also reflects on the approach taken to the topic – that of teaching a broad range of strategies that can then be combined to allow multiple pathways to the solution. This provides a practical example of how flexibility and adaptivity can be incorporated to enhance students' number sense and give the topic a problem solving focus. Although flexibility/adaptivity are often treated as synonymous terms in the research literature (see Chapter 2), this study uses Verschaffel, Luwel, Torbeyns and Van Doren's (2009) distinction between 'flexibility' as the use of a variety of strategies and 'adaptivity' as the selection of the most appropriate strategy for a particular problem. Whilst the notes (Appendix 5) focus on the flexible use of strategies in mental multiplication, the tutorial had the additional focus of the most appropriate pathway to a solution. For example, each of the sets of reflection exercises in the basic fact section had one example that was best done using subtractive partitioning and this was emphasized in the tutorial. The review of Test 1 problems at the end of the tutorial also looked at the relative value of solution approaches to achieve a result in the given timeframe. This section has been added to the original notes in Appendix 5.

The study does not, however, take into consideration the socio/cultural aspect of the intervention – its voluntary nature, its placement against competing demands on participants' time and the lack of opportunities for participants to interact significantly. These considerations would require a greater timeframe than was possible, more in keeping with the recommendations of the participants for teaching this topic to Year 5.

The participants who completed all three phases of the project began with test scores that identified them as the weakest of three groups – those completing Phase 1 only, those completing Phases 1 and 3 and those completing all three phases. Table 5.21 is repeated below to facilitate further discussion.

Table 5.21 *Test Statistics Based on Phase Groupings*

Test	All groups			Phase 1 only Group 1			Phase 1 and 3 Group 2			All Phases Group 3		
	$\bar{X}$	σ	n	$\bar{X}$	σ	n	$\bar{X}$	σ	n	$\bar{X}$	σ	n
1	4.3	2.8	36	4.2	2.4	13	4.6	1.3	16	3.9	3.0	7
2	5.7	3.7	23				5.6	3.5	16	5.9	4.3	7

Whilst the average achievement of the two groups completing both tests (Groups 2 and 3) remained low, the group participating in all three phases (Group 3) ended as the group with the highest average and included the best-performed participant. Although the difference in test means was not statistically significant, the increase for Group 3 was. With such a short intervention, test results are not the best measure of the participants' changes in SCK for this topic. A better measure would appear to be their use of strategies in the untimed section on suggested test strategies and open-ended questions that canvassed alternate strategy suggestions.

Group 3's range and knowledge of strategies improved considerably from six strategies in Test 1 to eight in Test 2. They also reduced their reliance on RLS from 37 instances (30.8%) to four instances (3.3%) and in the latter, RLS was appropriately applied to single digit multipliers. LRS became its most popular strategy suggestion for Group 3 with 35 instances (29.2%) and was reasonably consistent with its use in Test 1 (37 instances or 30.8%). Their strategy suggestions that could have produced a correct answer to test problems improved from 60.0% to 69.2% (see Table 5.11) and they had only 5% incorrect suggestions. The use of more flexible strategies increased from 45% to 54.2% and the largest increase was for BFS (0 to 18 correct uses). The seven participants offered 23 viable alternate strategies for the four test questions in the questionnaire using a range of five strategies and did not include RLS in their suggestions.

In contrast, Group 2 did not make similar gains in flexible strategy use. Six strategies were used by this group in both tests. They increased their reliance on RLS from 20.0% (45 instances) to 21.3% (48 instances). Their use of their modal strategy, LRS, improved from 23.6% (53 instances) to 36.9% (83 instances), however there were 16.4% (37 instances) incorrect applications of this strategy compared to 12.4% (28 instances) in Test 1. They knew it was a more appropriate strategy but could not apply it effectively in 44.6% of problems where it was suggested. Their strategy suggestions that could have produced a correct answer to test problems improved marginally from 50.7% to 51.1% (see Table 5.10) and the more flexible use of strategies increased from 25.8% to 29.8%. The ten participants offered 19 viable alternate strategies for the four test questions in the questionnaire using a range of three strategies and also did not include RLS in their suggestions.

Recall Table 3.4 that compared Whitacre's (2007) taxonomy with that of this study. It is now repeated as Table 5.22.

Table 5.22 *Comparison of Whitacre's Taxonomy for Multiplication Strategies and the Taxonomy Used in This Study*

Categorisation	**Standard**	**Transition**	**Non Standard**	**Non Standard with Reduction**
Specific coding (Whitacre)	MASA	Partial Products	Additive Distributive	Non-standard Partitioning (SD and AD), Aliquot Parts, Derived, Factors
Specific Coding (Hall)	RLS	Partial Products	LRS, BFS	PAR, FAC, CMP, CMB

Figures 5.2 and 5.3 compare the strategy use of participants in this study and Whitacre's (2007) study before and after their intervention.

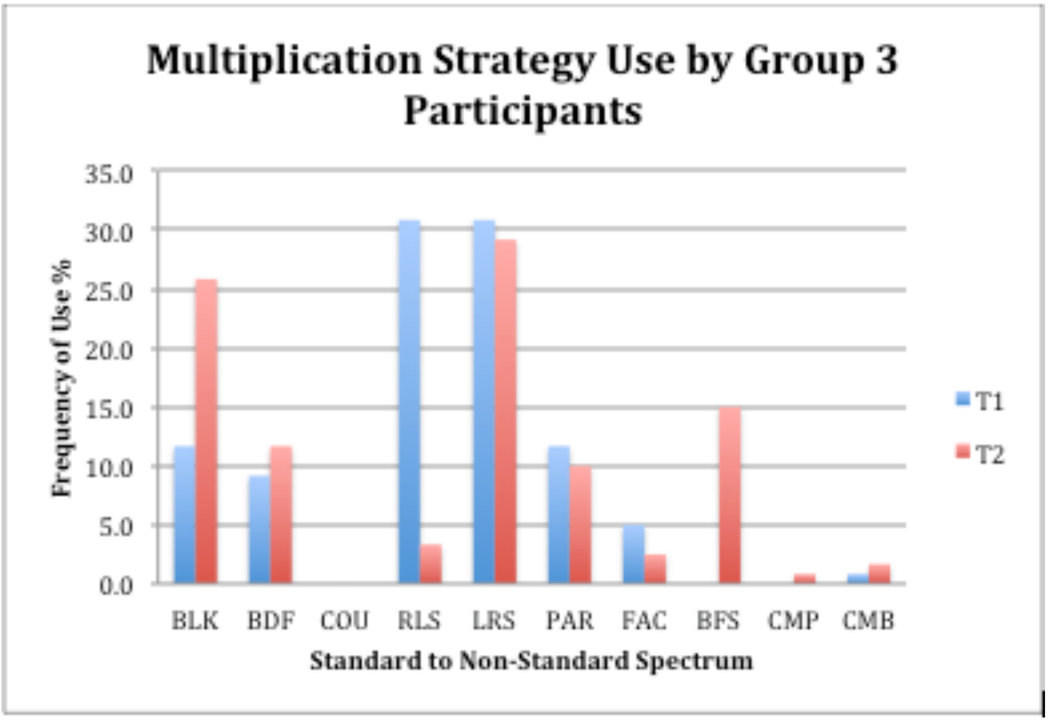


Figure 5.2 Strategy Use of Group 3 Participants in this Study (n=7)

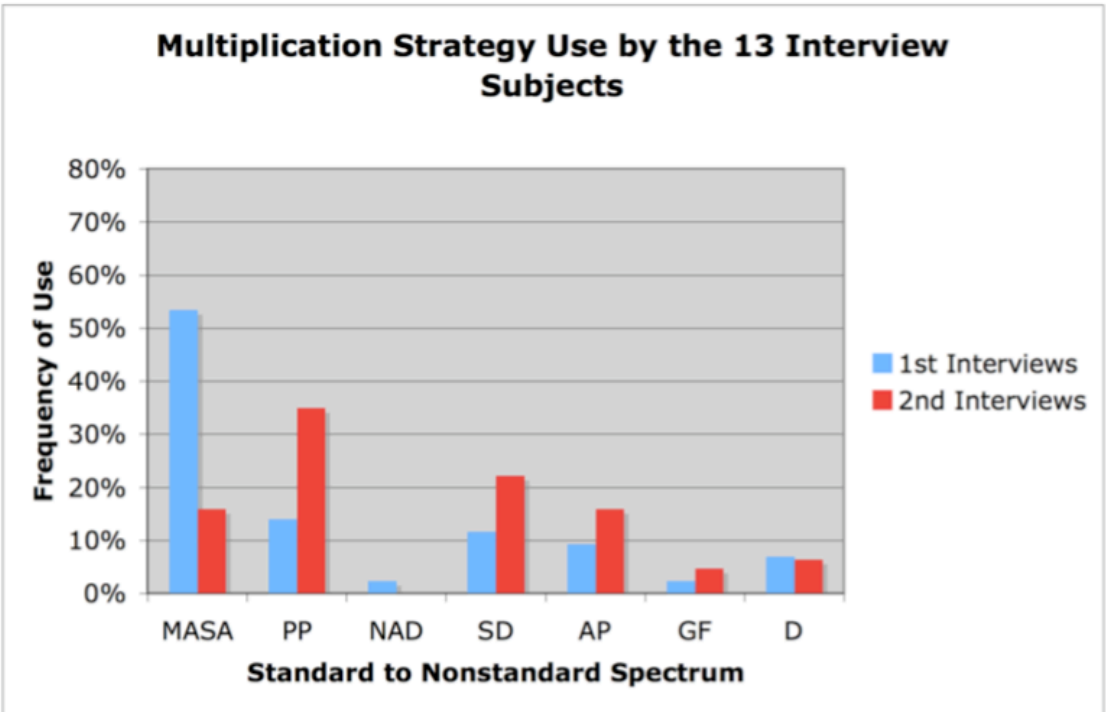


Figure 5.3 Strategy Use of Participants in Whitacre;s (2007) Study (n=13)

In relation to Whitacre's (2007) study, he found that, following intervention,
the frequency of use of the MASA (RLS) dropped to less than one-third of that seen in first interviews (15.9%), In this study the frequency use of RLS dropped from 30.8% to 3.3%. In Whitacre's study, Partial Products (LRS) became the most common strategy for multiplication, with students using it 34.9% of the time. In this study, LRS also became the most popular strategy with 29.2% usage. The frequency of use of six of the seven alternative strategies increased in Whitacre's study and increased in five of six alternate strategies in this study. In Whiacre's study, SD accounted for 22.2% of strategy uses, AP for 15.9%, FAC 4.8%, and Derived 6.4%. In this study, BFS (including AP) accounted for 15% of strategy uses, PAR (including SD) for 10%, FAC and CMP (similar to FAC) for 3.3% and CMB (Derived) for 1.7%.

A similar comparison of Group 2's strategy use in Test 1 and 2 is shown in Figure 5.4

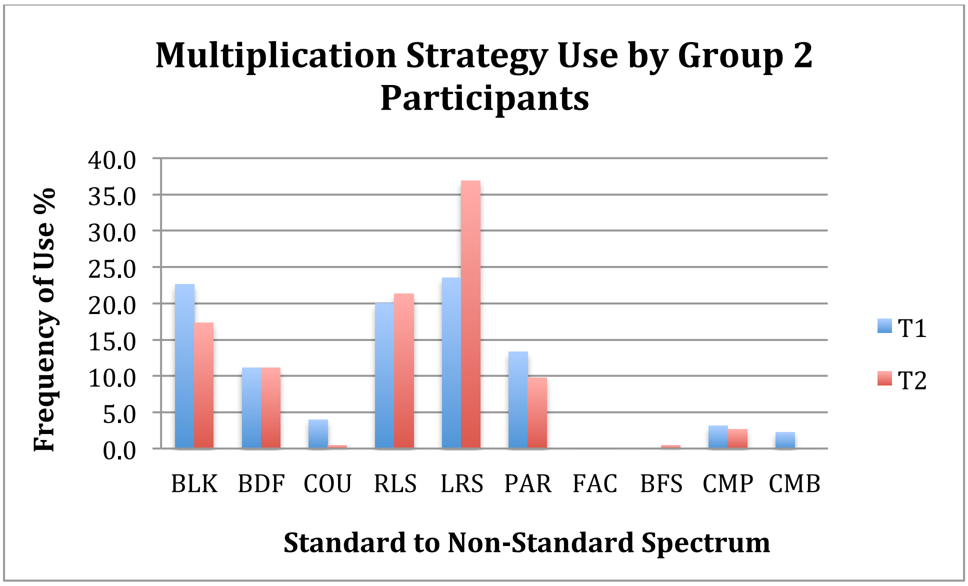


Figure 5.4 Strategy Use of Group 2 Participants in this Study (n=15)

The movement by participants from inflexible to more flexible strategy use in Whitacre's study and Group 3 in this study was not reflected in the results of Group 2 – the group that did Phases 1 and 3 but not the intervention. Whilst their use of LRS increased, so too did their use of RLS. The more flexible number sensible strategies of BFS, FAC, CMP and CMB were barely used and PAR declined in its usage. This suggests that the intervention, despite its short timeframe, had a positive effect on Group 3's choice of more number-sensible, flexible strategies.

Items 10 to 21 of the two questionnaires sought responses from participants involving all six categories of the MKT framework. For Group 3, there was a strong improvement in awareness of strategies (although not statistically significant) and a moderate improvement in confidence in using them (SCK). Group 3 participants all agreed that mental computation is an essential skill for school students and most saw it being taught in primary with only one participant unsure about that statement. There was an increased awareness of the place of mental computation in the curriculum and all saw it being taught prior to the formal algorithm (KCC).

Whilst knowledge of strategies had improved markedly, confidence in teaching them had only improved moderately. This is possibly a reflection of the lack of time to practise the strategies, continued weak test scores for the lower achievers in the group and continued difficulty with two-digit multipliers. There was a strong belief in the use of mental computation to enhance students' critical thinking and number sense (KMH). The participants all agreed that a wide range of strategies should be taught which was earlier supported by their selection of LRS, PAR, BFS and CMP for teaching programs. They also believed that all students have the potential to access this topic (KCT). There was strong support for the premise that mental computation is an essential skill for all adults, with the exception of one participant who remained unsure (CCK).

The final chapter relates the data analysis presented in this and the previous chapter to the research questions outlined in Chapter 1.

Chapter 6 Thesis Summary, Conclusions and Implications for Practice

6.1 Introduction

As the title of the thesis indicates, this project explored the flexible use of a range of mental multiplication strategies to solve problems that satisfy the outcomes of the NESA and Australian curriculum documents for the Year 5 outcome of mental multiplication by one- and two-digit numbers. The participants in this case study were drawn from two cohorts of pre-service teachers at the University of New England (n=36) who had not previously encountered the topic but had knowledge of graphical and written methods of multiplication.

The project aimed to establish the participants' knowledge of mental multiplication and whether any changes to this knowledge resulted from exposure to a proposed teaching strategy that involved a flexible approach to mental multiplication using a range of number sensible teaching strategies. Data relating to changes in the participants' knowledge of mental multiplication were collected and analysed using a concurrent mixed methods approach that involved two rounds of timed testing, reflection and a questionnaire separated by an intervention in the form of a two-hour tutorial. The research design that dictated data collection and analysis was based on Hill, Ball and Schilling's (2008) framework of Mathematical Knowledge for Teaching (MKT) as outlined in Chapter 3. Chapters 4 and 5 reported the data analysis for Phases 1, 2 and 3 respectively. This chapter summarises those statistics and relates them to the research questions outlined in Chapter 1.

The project aimed to address four related research questions:

1. What was the prior knowledge of mental multiplication displayed by pre-service teachers?

2. How did this change following exposure to strategies suggested in the research literature and presented through the intervention?

3. What strategies do pre-service teachers propose for a teaching program that addresses the mental multiplication component of the Year 5 Australian Curriculum outcome ACMNA100 and its NSW counterpart?

4. What implications for practice are suggested by the study for possible inclusion in curriculum documentation?

6.2 Conclusions

This section deals with conclusions that answer the first three Research Questions. Each conclusion is followed by argument supporting the conclusion with evidence developed in Chapters 4 and 5. Conclusions 6.2.1 to 6.2.3 relate to the first research question; Conclusions 6.2.4 and 6.2.5 relate to the second research question and Conclusions 6.2.6 and 6.2.7 relate to the third research question.

6.2.1 Conclusion 1

It was concluded that the mental multiplication capacities of the participants were not sufficient for most participants to teach this area of the syllabus at a Year 5 level. Both the timed test results and untimed strategy suggestions of participants demonstrated a poor knowledge of mental multiplication strategies (specific content knowledge - SCK) that could be described as traditional (Menon, 2009) and lacking in flexible and number sensible strategies (Whitacre, 2007). The participants were particularly weak in two-digit x two-digit calculations and associated strategy suggestions, with many suggesting the use of right-left separated (mental image of the formal algorithm). The majority of participants had a limited repertoire of strategy choices.

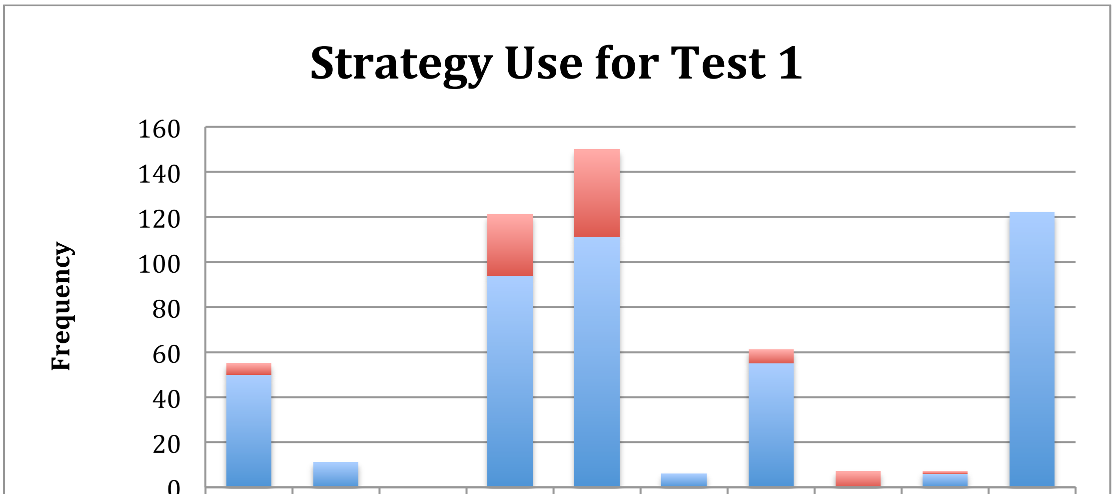
The test results for the entire group (n=36) and the two cohorts are repeated in Table 6.1.

Table 6.1 *Means and Standard Deviations of Test 1 by cohort*

Statistic	All Participants $n = 36$	EDME 145 $n = 16$	EDME 369 $n = 20$
Mean	4.3	3.6	4.9
Std Dev	2.8	2.4	3.0

Considering that the 15-question test was designed as an exit test for Year 5 after experiencing tuition in mental multiplication techniques, it was expected that the performance of the pre-service teachers would be substantially better than the results shown above. The test scores ranged from 0 to 11 with only eight students (22%) scoring 8/15 or better with one student recording a score that could be considered as competent (11/15). The comparison between the two cohorts shows a slightly better performance by the third-year group, however the difference was not statistically significant ($t = -1.44, df = 34, p = 0.160$). The entire group was particularly weak in two-digit x two digit problems with only one participant scoring above 2/7.

Whilst timed testing provided some useful insights in answering the first research question, the untimed suggestions of strategy use by participants enhanced the data from Test 1. Figure 6.1 repeats the coding summary of those strategies reported in Chapter 4.



	BDF	COU	BFS	RLS	LRS	FAC	PAR	CMP	CMB	Blank
Incorrect	5	0	0	27	39	0	6	7	1	
Correct	50	11	0	94	111	6	55	0	6	122

Figure 6.1 Strategies Suggested for Test 1 Questions

As outlined in Chapter 4, these can be split into three categories based on the flexibility of the strategies (Whitacre, 2007) as shown in Table 6.2.

Table 6.2 *Strategy Suggestions for Test 1 Questions by Category (n=36)*

Strategy Category	Strategies	Frequency	%
Blank, incorrect or inappropriate to the problem	RLS was considered inappropriate for use with two-digit multipliers	236	43.7%
Inflexible and appropriate to a limited range of problems	BDF, COU, RLS for one-digit multipliers	126	23.3%
Flexible and appropriate for a wider range of problems	LRS, PAR, FAC, CMP, BFS, CMB	178	33.0%

Table 6.2 shows that only 56.3% of strategy suggestions had the potential to solve the target problem, and a low 33% could be described as flexible and capable of solving the full range of problems encountered in the test.

The modal strategy suggestion was left-right separated (150 instances or 27.8%) although 39 of these suggestions would have led to incorrect solutions. The latter were mainly associated with two-digit x two-digit problems. Right-left separated (RLS), a mental version of the formal algorithm, was suggested 121 times (22.4%) although 27 would have led to incorrect solutions and a further 29 were inappropriate suggestions for two-digit x two-digit problems. That left 65 suggestions of RLS for one-digit multipliers that are appropriate but considered slightly more difficult than LRS (Hierdsfield, Cooper, Mulligan & Irons, 1999).

Of the more flexible strategies, only non-standard partitioning (PAR) received much support with 49 correct suggestions (9.1%). Most of these were subtractive distributive with few instances of non-standard additive partitioning. There was only one instance of a combination of strategies and no use of any basic fact shortcuts. The seven suggestions of compensation (CMP) by one participant would all have led to incorrect answers.

The large number of blank entries (122 or 22.6%) prompted a closer examination of the two-digit x two-digit problems (Q 9 to 15). These seven questions had only 73 strategy suggestions that might have produced a correct answer. This was only 29% of possible suggestions; the remainder being left blank or leading to an incorrect answer. Twenty-three of the 33 participants (69.7%) scored 0/7 for this section and only one participant scored above 2/7. This demonstrates a particular problem with this type of mental multiplication.

When asked for alternate solutions to four of the seven two-digit x two-digit problems, there were only a total of 32 appropriate suggestions, most of those using LRS or an expanded version of the formal algorithm (4 partial products).

In terms of the range of strategies suggested by participants, 50% could only suggest one or two strategies, while the best-performed students used either three or four strategies. There were only four of 33 participants (12.1%) who could suggest appropriate strategies for all 15 test problems.

6.2.2 Conclusion 2

The experience of participants with Mathematics at school seemed to have little or no bearing on their mental computation skills. This included both specific tuition in mental computation and the level of Maths studied for the HSC. Similarly, any subject specialisation in their current course was not a significant predictor of performance in this topic. The better predictors of performance were the participants' own assessment of their knowledge and use of mental computation as adults although the results from the test and strategy suggestions demonstrated that this experience was superficial with few exceptions.

All but three participants had studied Mathematics for their HSC and the dominant course of study was General Maths (58%). A further 18% had studied Mathematics 2U and 14% had studied Ext 1 or higher. As a predictor of test results, the level of Maths attempted at the HSC showed no significant correlation with the test scores ($\rho = 0.314, p = 0.075$).

In relation to their experience with mental computation, only 39% could recall specific teaching of mental computation strategies at school with 36% disagreeing with the prompt to some extent. This prompt had a similarly weak correlation with test results that was not statistically significant ($\rho = 0.333, p = 0.058$).

Nearly half of the respondents (48%) agreed that they used mental computation as adults regularly and a similar percentage expressed awareness of a range of strategies for mental computation (45%). However, a low 24% felt confident about using mental computation strategies with almost half (48%) lacking confidence in this area. The last three prompts – knowledge of mental computation as an adult, their regular use and participants' confidence in using them all showed a significant correlation with test results in the moderate to strong range (Chapter 4, Table 4.10).

6.2.3 Conclusion 3

The participants expressed a general lack of confidence in their ability to teach this topic and some deficits were evidenced in curriculum and content knowledge associated with the placement of this topic in the primary curriculum and its relationship to the teaching of the formal algorithm. It is strongly recommended that this topic be viewed as a separate topic in any pre-service mathematics course and taught in conjunction with graphical methods, such as arrays, using a constructivist approach. The topic should have its own unique set of strategies and be placed firmly in the Stage 3 curriculum (NESA, 2015). In addition, it should be taught in advance, and separate from, the formal algorithm for long multiplication.

After examining participants' SCK, the questionnaire focussed on their knowledge and attitude towards four other areas of the MKT framework. In relation to the knowledge of curriculum and content (KCC), items 14 and 16 indicated quite a strong belief in mental computation being taught to school students (81% agreement) but slightly less acceptance that its best place is in the primary curriculum (66% agreement). One third of participants were unsure that this topic should be placed in primary. This contradicts the 71% of participants who felt that they knew its place in the curriculum (item 12). So in terms of KCC, this suggests some deficits as the Australian curriculum clearly places mental multiplication by one- and two-digit numbers in Year 5.

The other comparison of interest was between items 12 and 17. Having suggested that a large majority knew the place of mental multiplication in the curriculum (71%), this contrasted with participants' views about teaching mental computation before formal algorithms. As demonstrated in the argument following Conclusion 6.2.1, there was a considerable reliance on the formal algorithm as a mental strategy. Fifteen of the 36 participants (42%) used it as one of their main strategies for the coding sheet and there were 121 suggestions of the RLS strategy (22.4% of all strategy suggestions). It is not surprising that 27% would disagree that mental computation should be taught before formal algorithms if a considerable number of participants believe RLS to be a viable mental multiplication strategy. This raises concerns about their own tertiary mathematics methods courses that have already treated graphical and formal methods of multiplying prior to any mention of mental strategies.

Items 13, 18 and 19 were related to participants' knowledge of content and teaching (KCT). Beginning with item 13, there was a strong lack of confidence in teaching this topic – 45% disagreed (to some extent) with the prompt and a further 30% were unsure. Only eight participants (24%) expressed any confidence in their preparedness to teach this topic. Whilst this item did not correlate significantly with item 11 (I am confident in using a range of strategies for mental computation), analysis of the two sets of responses did provoke some interesting comparisons. Only three participants expressed confidence in both their use of mental computation strategies and their ability to teach the topic. In one case, this confidence was misplaced as he scored 3/15 in the test and could suggest only two strategies (BDF and LRS) for the 15 test questions. The other two participants scored 8/15 and had a broader range of strategy suggestions. Another group of five participants expressed a lack of confidence in their own ability in mental computation but professed confidence in teaching the topic. Again this confidence seemed to be misplaced as the test mean of this group was only 3.2. In summary, only two participants expressed a confidence in teaching the topic that could be supported by test results and strategy suggestions.

The response to item 18 demonstrated strong acceptance (79% strongly agree or agree) of the statement that all students had the potential to learn mental computation strategies thereby implying that it was not just an extension topic for the more able student. Responses to item 19 strongly suggested (94% strongly agree or agree) that the participants are receptive to teaching a wide range of strategies rather than looking for a few strategies that will answer most questions.

There was strong acceptance of the relationship between mental computation and number sense (76% strongly agree or agree) and 88% thought it had the potential to promote problem solving and critical thinking (KMH). A large majority saw it as an essential skill for adults (78.8%) and school students (81%) alike (CCK and KCC).

6.2.4 Conclusion 4

Both Test 2 results and the associated strategy suggestions demonstrated a strong improvement in the SCK of participants completing the intervention (Group 3) that was not replicated, to the same extent, in the results and strategy suggestions of Group 2 (Phases 1 and 3 only). There was a substantial decrease in Group 3's reliance on RLS while Group 2 increased their use of the mental image of the algorithm.

While 36 participants completed the first test and strategy suggestions, the number completing the second test and associated strategy suggestions reduced to 23 participants due to competing demands on their time. Seven of those also completed the intervention and that resulted in three distinct groups involved in the analysis of results. Table 6.3 (repeated) shows the difference in test means and standard deviations for the three groups.

Table 6.3 *Test Statistics Based on Phase Groupings*

Test	All groups			Phase 1 only Group 1			Phase 1 and 2 Group 2			All Phases Group 3		
	$\bar{X}$	σ	n	$\bar{X}$	σ	n	$\bar{X}$	σ	n	$\bar{X}$	σ	n
1	4.3	2.8	36	4.2	2.4	13	4.6	1.3	16	3.9	3.0	7
2	5.7	3.7	23				5.6	3.5	16	5.9	4.3	7

There was a 32.6% improvement between Test 1 and Test 2 for all those completing both tests. Various factors may have influenced this improvement: the improvement of Group 3 scores possibly resulting from exposure to the intervention, a practice effect resulting from the proximity of the two tests and an improvement in the candidature's strength resulting from some of the weaker students not participating beyond Phase 1.

The 16 participants (Group 2) completing both tests but not attending the intervention had a 21.6% improvement in their mean result. A paired samples *t*-test completed for this group indicated that this improvement was close to being statistically significant: It is hypothesised that this improvement was due mainly to practice effect combined with a small improvement due to changes in candidature (2.3%).

The seven participants involved in all three Phases had a 51.9% increase in their mean result, demonstrating a statistically significant improvement: A comparison of the improvement for the two groups was done by subtracting the pre-test result from the post-test result for the 23 students and then performing an independent samples *t*-test based upon whether the intervention had been completed. While those who completed the intervention demonstrated more improvement, the difference was not statistically significant:

The strategy suggestions, however, did demonstrate a strong improvement in the SCK of participants completing the intervention that was not replicated in the strategy suggestions of Group 2 (Phases 1 and 3 only). Figure 6.1 shows the change in strategy suggestions for Group 2 (no intervention)

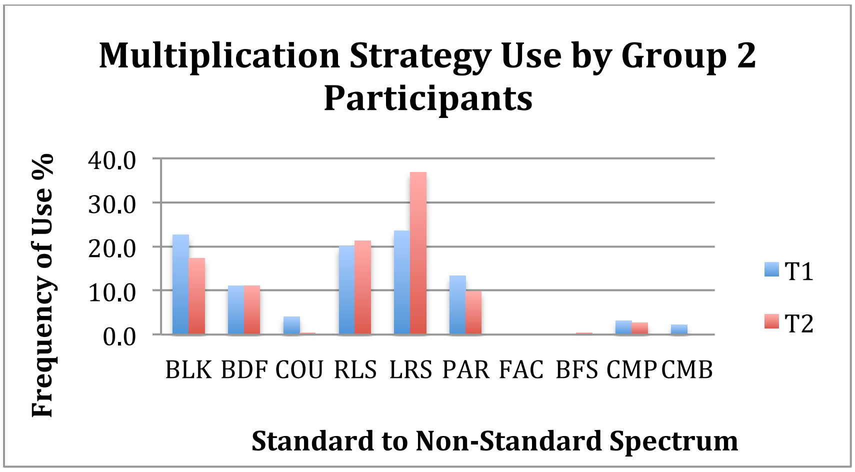


Figure 6.2 Strategy Use of Group 2 Participants in this Study by Percentage (n=15)

Group 2 used a total of six strategies in both tests. They increased their reliance on RLS from 20.0% (45 instances) to 21.3% (48 instances). Their use of their modal strategy, LRS, improved from 23.6% (53 instances) to 36.9% (83 instances), however there were 16.4% (37 instances) of incorrect applications of this strategy compared to 12.4% (28 instances) in Test 1. They knew it was an appropriate strategy but could not apply it effectively in 44.6% of problems where it was suggested. As in Test 1, there was little use of the more flexible strategies of BFS, FAC and CMB. The seven suggestions of CMP would all have led to incorrect answers.

Their strategy suggestions that could have produced a correct answer to test problems improved marginally from 50.7% to 51.1% (see Table 5.10) and the more flexible use of strategies increased from 25.8% to 29.8%. The ten participants offered 19 viable alternate strategies for the four test questions in the questionnaire using a range of three strategies and did not include RLS in their suggestions.

Figure 6.3 shows the change in strategy suggestions for Group 3 (all three phases).

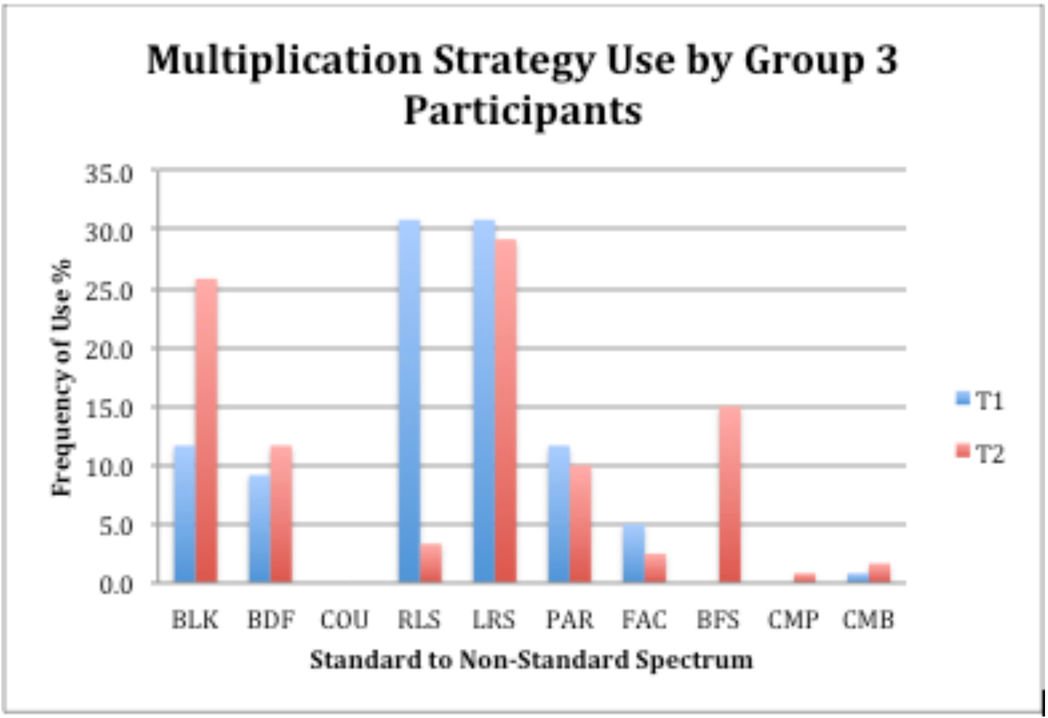


Figure 6.3 Strategy Use of Group 3 Participants in this Study by Percentage (n=7)

Group 3's range and knowledge of strategies improved considerably from six strategies in Test 1 to eight in Test 2. They also reduced their reliance on RLS from 37 instances (30.8%) to four instances (3.3%) and these four instances were appropriately applied to single digit multipliers. LRS became the modal strategy suggestion for Group 3 with 35 instances (29.2%) and this was reasonably consistent with its use in Test 1 (37 instances or 30.8%).

Their strategy suggestions that could have produced a correct answer to test problems improved from 60.0% to 69.2% (see Table 5.11) and they had only 5% incorrect suggestions. The use of more flexible strategies increased from 45% to 54.2% and the largest increase was for BFS (0 to 18 correct uses) The seven participants offered 23 viable alternate strategies for the four test questions in the questionnaire using a range of five strategies and did not include RLS in their suggestions.

6.2.5 Conclusion 5

There was a strong improvement in awareness of strategies (although not statistically significant) and a moderate improvement in confidence in using them (SCK) for those experiencing all three phases of the project (Group 3). There was also an increase in confidence about teaching this topic (KCT) and an increased awareness of the place of mental computation in the curriculum and all now agreed that it should be taught prior to the formal algorithm (KCC).

Group 3 participants all agreed that mental computation is an essential skill for school students and most saw it being taught in primary with only one participant unsure about that statement. There was an increased awareness of the place of mental computation in the curriculum and all saw it being taught prior to the formal algorithm (KCC). There was also an increase in confidence about teaching this topic and all agreed that it was appropriate for all students and that a wide range of strategies should be taught (KCT). There was unanimous agreement that mental computation had the potential to improve students' number sense and problem solving ability (KMH). All but one participant agreed strongly that it was an essential skill for adults (CCK).

6.2.6 Conclusion 6

Following the intervention, there was unanimous support from Group 3 for BFS, LRS, PAR and CMP as essential components of a teaching program for this topic. In particular, non-standard partitioning (PAR) was seen as the most valuable strategy. There was no support for the formal algorithm (RLS) as a mental multiplication strategy and little support for FAC.

The strong acceptance of the intervention and accompanying notes by the participants was encouraging. This led to the following choice of strategies for an ungraded Year 5 class is shown in Table 6.4.

Table 6.4 *Choice of Strategies for an Ungraded Year 5 (n=6)*

Strategy	BFS	LRS	RLS	FAC	PAR	CMP
Frequency	6	6	0	1	6	6

Participant rankings of those strategies is shown in Table 6.5

Table 6.5 *Participant Ratings of Strategies in Order*

Rating	Most Valued				Least Valued
Strategy	PAR	BFS LRS	CMP	FAC	RLS

The preference for CMP over FAC in the participants' teaching program is also of interest as FAC and LRS are the only strategies mentioned in the current NESA syllabus documents (2015) that relate to mental multiplication at this level. FAC could reasonably be seen as a subset of CMP and hence the latter may have been more valued in a teaching program.

6.2.7 Conclusion 7

The participants strongly suggested that this Year 5 topic be spread over an extended period of at least one term and not treated as a two-week topic. As the syllabus background information recommends, "mental strategies need to be continually reinforced". (NESA, 2015: Mathematics K – 10, Stage 3, Multplication and Divisoion 1, Background Information, para 2) This suggests a program over at least one term where strategies are introduced incrementally and practised before progressing to the next strategy. Following terms would regularly revise mental multiplication, along with the other three operations.

The final question in the second questionnaire asked participants how they would pace their instruction for this topic with the results shown in Table 5.6.

Table 6.6 *Suggested Duration of Instruction (n=7)*

Duration of Instruction	Two to Three Weeks	Over One Term	Over Two Terms	Over Three or Four Terms
Frequency	1	2	2	2

The mixed response to this question suggests there may be some uncertainty amongst participants about the time needed to teach this topic. However, the majority would see it as taking at least one term to complete and possibly more.

6.3 Implications for Practice

The first implication for practice relates to the final research question, while the second relates to pre-service teacher courses in Mathematics.

6.3.1 Implication 1

The first implication of the research is that the NSW (NESA, 2015) syllabus outcomes, notes and background information for this topic should be expanded and clarified as follows:

- Outcomes should differentiate between mental, written and graphical strategies for this topic and include specific examples of non-standard subtractive partitioning and non-standard additive partitioning. These examples should include two-digit multipliers. Compensation should also be included as an appropriate strategy for mental multiplication with an example such as 18 x 35 = 9 x 70 = 630.
  Factors would still be mentioned for simple one-digit multipliers such as 6.
- It is also suggested that the curriculum notes for this outcome should include the previous statement contained in the BOS (2002) syllabus notes:

  > formal written algorithms are introduced after students have gained a firm understanding of basic concepts including place value, and have developed mental strategies for computing with two-digit and three- digit numbers (BOS 2002 p 9).

- Background Information for the NESA K-10 Mathematics Curriculum (2015) outcome MA 3-6 NA states, in part, that "students could extend their recall of number facts beyond the multiplication facts to 10 × 10 by memorising multiples of numbers such as 11, 12, 15, 20 and 25. It is suggested that the phrase “by memorising multiples of numbers such as 11, 12, 15, 20 and 25” be replaced with “ by calculating multiples of numbers such as 11, 12, 15, 20 and 25 using mental multiplication strategies of partitioning and compensation”.

6.3.2 Implication 2

A second implication of the research involves the following suggestions for the programming of pre-service teacher courses in Mathematics.

- Mental computation strategies, as recommended in the syllabus and research literature (Clark, 2005), need to be introduced prior to formal algorithms for each operation. Graphical methods, particularly arrays, should be used to illustrate the mental multiplication strategies.
- The strategies used need to build upon student-developed strategies and address the number sensible elements of flexibility, adaptivity and multiplicative thinking (Heirdsfield & Cooper, 2004). The research supports the inclusion of LRS, PAR, FAC, CMP and CMB as appropriate number sensible strategies for introducing mental multiplication to pre-service teachers.

6.4 Limitations of the Research Project

Schoenfeld (2002 p 497) argues that there are three fundamental dimensions of research findings: generality, trustworthiness and importance. An exploration of these three dimensions reveals any limitations of the research project:

• *Generality, or Scope.* The small sample size, even for the first survey completion, limits the capacity to generalise the results (Field, 2009). It was difficult to apply traditional measures of significance to comparisons between the data sets obtained in Phases 1 and 3. De Winter (2013) argues that the t-test is acceptable provided effect sizes are large. He recommends use of the paired samples t-test or the Wilcoxon test for small populations and both of these were used in Chapter 5. Correlations between item responses have not been used as a consequence of the small sample size. However, there are still useful comparisons to be made between the data sets, particularly with strategy selection and these comparisons primarily use frequency distribution for analysis.

The demographics of this cohort compared with a distance cohort of students may be different as would a larger cohort in city universities. The epistemology of symbolic interactionalism also suggests that each student will bring his or her own constructed views of mental computation to the project based on age, prior experience and culture.

At best, the study is one case of a cohort of pre-service teachers experiencing mental multiplication for the first time with planned intervention aimed at broadening their conceptions of mental multiplication strategies. Given a different researcher, the strategies suggested in the intervention may be completely different (Whitacre, 2007).

• *Trustworthiness.* Given the limitations of the case in question, any claims concerning preferred strategies or a hierarchy of strategies are but one example of opinion from one cohort of pre-service teachers. The study will add to the body of knowledge and perhaps suggest strategies worthy of further consideration in research with teacher and student samples.

• *Importance:* The call for research into pre-service teacher MKT (Hine 2015) specifically related to number sense (Whitacre, 2016) means that this research offers concrete suggestions for the selection of possible strategies for mental multiplication based on the experiences of one cohort of pre-service teachers. The elements of flexibility, adaptivity and multiplicative thinking that are incorporated into the design of the project add to the body of research in these areas with particular reference to number sensible mental multiplication strategies.

An additional limitation is one of structure. Given that the research took place within a program of instruction dictated by the relevant department and course co-ordinators, the research needed to fit in with their timeframes and the demands of other units of work and assessments. This led to a compacted timeframe and competing demands for the participants. The voluntary nature of their participation meant that some may not have placed great emphasis on the material or the discussions and may have resulted in decreased participation in the intervention phase. The intervention also occurred in a week when normal lectures were suspended just prior to final exams, hence affecting attendance. Given only a two-week gap between testing, practice effect had to be considered when analyzing test data.

6.5 Recommendations for Future Studies

In relation to future studies of mental multiplication with pre-service teachers, the following recommendations are suggested:

- that the topic be encountered over an extended period of time, and prior to the introduction of formal algorithmic procedures
- that the participants be taught coding of strategies and provided with opportunities to discuss each other's strategies and code them
- that some measure of participants' knowledge of basic multiplication facts up to 10 x 10 be made initially
- that FAC and CMP be combined into one category
- that subtractive and additive non-standard partitioning be separated to enable a better measure of additive partitioning
- that BFS include multiples from 2 to 12, 15, 20 and 25, moving Aliquot parts from CMP

In relation to future research into the topic of mental multiplication, research with existing teachers and their students would be the logical next step. This might involve:

- Initial testing of basic multiplication facts to 10 with a Year 5 cohort
- Post-testing of mental multiplication with a Year 6 cohort (having experienced it in Year 5).
- Coding of strategies used by Year 6 with teachers.
- Development of a learning community within each school to design teaching programs for the topic, and remedial programs for basic number facts as needed.
- Implementation of the program over a term.
- Records of teachers in the form of diaries.
- Post-testing of Year 5, coding and comparison with Year 6 initial testing.

- Regular colleagial discussions of progress amongst teachers and with the researcher.

It might also be useful to work with at least three schools, possibly involving different sectors – Government, Catholic and Private.

6.6 Chapter Conclusion

Despite the limited time frame for the study, Phase 1 provided some rich data for the first of four research questions. The 36 participants in this phase had a severely limited knowledge of mental multiplication that relied on procedural methods at the expense of any flexible, number-sensible strategies. Their poor performance in both timed testing and strategy suggestions showed a lack of SCK required to teach the topic. Most of their limited knowledge seemed to come from experiences with mental multiplication as an adult rather than their school or university experience of the topic. There was confusion regarding the formal algorithm – whether it was a useful strategy for mental multiplication and its placement in regard to this topic. As a group, they generally lacked confidence in their own mental computation ability and their ability to teach the topic. Conclusions 1 to 3 address that first research question.

The second research question was addressed by Conclusions 4 and 5. A reduced group of 23 participants were involved in Phase 3 and seven of those experienced the intervention. The small group size made comparisons using significance and correlation difficult, but some useful data, particularly in relation to strategy choice was obtained. The group experiencing all three phases of the project had a significant improvement in test scores and a measurable change in strategy use. Their use of inflexible strategies almost disappeared, except in cases where those strategies were appropriate. They adopted more flexible strategies and were able to offer more alternative solutions to problems using these. The group not experiencing the intervention did not make the same gains in strategy use and their small improvement in test scores was not significant.

In response to the third research question, the seven participants strongly suggested a teaching program that involved PAR, BFS, CMP and LRS strategies. They saw this being taught over an extended period and could see value in the program in terms of students' number sense and problem solving. In general, they embraced the concept of flexible number sensible strategies with multiple pathways to solutions.

The first implication for practice relates to the NSW syllabus outcomes, notes and background information for multiplication by one- and two-digit numbers. Unless schools have strong learning communities, it is felt that the syllabus needs to be more prescriptive in this area and provide a wider range of mental multiplication strategies with appropriate examples. If number sense and mental computation are to be seen as important, they need a stronger focus in curriculum documents that clearly details the extent and treatment of the topic.

The second implication for practice suggests that this topic receive at least equal prominence with graphical and algorithmic approaches to mental multiplication in pre-service teacher mathematical methods courses. It should be encountered prior to the formal algorithm, using graphical methods to support its introduction. Possible strategies for inclusion in pre-service teacher courses would include PAR, LRS, BFS, CMP and CMB. The concepts of flexibility and adaptivity would be addressed through the use of multiple pathways to the solution of mental multiplication problems that used a combination of number sensible strategies as listed above.

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Appendix 1 – Participant Information Sheet and Consent Form

Information Sheet for Participants

School Of Education
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Australia
Phone: (02) 6773 5981
Email: phall2@myune.edu.au
www.une.edu.au/education

Information Sheet For Participants

My name is Peter Hall and I am conducting research into effective mental multiplication strategies for primary school students as part of a Research Masters in Education through the School of Education, UNE. My supervisors are Dr Robert Whannell and A/Prof Pep Serow. This paper outlines the reasons for the research, its format and anticipated outcomes and the benefits to participants. Ethical considerations are mentioned and a formal written consent form is attached should you agree to participate. Please return the consent form to the researcher at the first tutorial before research commences. Your participation in this research is voluntary.

Research Project	Investigating strategies for one- and two-digit mental multiplication used by pre-service teachers.
Aim of the Research	The research initially aims to explore pre-service teachers' conceptions of mental multiplication. These conceptions are tested against their initial knowledge before the students are introduced to a range of possible classroom strategies. Following this intervention, the conceptions of the cohort are again surveyed and compared with a second test. The main aim of the research is to discern appropriate classroom strategies for mental multiplication that are supported by this cohort of pre-service teachers, studying EDME145 and EDME 369 in second trimester, 2018.
Questionnaires and tests	Participation will be done in scheduled on-campus tutorial sessions allocated to EDME145 and EDME 369 in trimester 2, 2018 but in a week where lectures do not take place. Participants will be required to complete the following during three on-campus tutorials, requiring a total of three hours

	of time commitment: • Initial Questionnaire: 28 questions aimed at eliciting students' pre-conceptions of mental computation. • Test 1 (pre-test): Timed assessment using 15 mental multiplication questions involving one- and two-digit multipliers. • Test 1 coding: students annotate coding sheet with their working for the test questions which are then coded by the research team using a coding taxonomy developed by the researcher. • Intervention: the researcher introduces a number of strategies for consideration. • Test 2 (post-test): a similar 15 question test. • Test 2 coding: students annotate coding sheet with their working for the test questions which are then coded by the research team using a coding taxonomy developed by the researcher. • Questionnaire 2: 3 Sections relating to strategy use, choice of strategies for a possible teaching program and their relative value and sequencing. Note that the research study participants will complete all of these steps. Those students not participating in the research will not complete the initial and final tests and questionnaires, but may participate in the intervention. Students who are not participating in the research will be provided with alternate activities by the course co-ordinator, Dr Schmude.
Confidentiality	Any personal details gathered in the course of the study will remain confidential. No individual will be identified by name in any publication of the results. All names will be replaced by pseudonyms; this will ensure your anonymity. If you agree I would like to quote some of your responses from the questionnaires. This will also be done in a way that ensure that you are not identifiable.
Reciprocity	Participants will have access to detailed lecture notes outlining the Mathematical Knowledge for Teaching (MKT) framework as it relates to mental multiplication, all suggested strategies with examples, the strategy taxonomy and test questions (after the second test).
Participation is Voluntary	Please understand that your involvement in this study is voluntary and I respect your right to withdraw from the study at any time without consequence and without needing to provide an explanation.
Other Ethical Considerations	The research project will be conducted as part of EDME145 and EDME 369 with the approval of the Unit Coordinator, Dr Martin Schmude. Participation in the research project is voluntary but the project is part of the

	EDME145 and EDME 369 course curricula. Therefore, the intervention phase of the research project will be available to the entire cohort. Students who decide not to participate will not be required to complete the two tests and questionnaires. It is unlikely that there will be any additional risks associated with the research as it will be conducted in normal lecture venues using the existing course timetable. No data from the research will be used in determining results or grades for either course and data will only be available to the researchers mentioned above.
Use of Information	I will use information from the various data collection instruments as part of my Masters thesis, which I expect to complete in early 2019. Information from the various data collection instruments may also be used in academic journal articles and conference presentations after this date. At all times, I will safeguard your identity by presenting the information in a way that will not allow you to be identified.
Upsetting Issues	It is unlikely that this research will raise any personal or upsetting issues but if it does you may wish to contact your local Community Health on 6776 9600 on weekdays between the hours of 8.30 am and 5.00 pm. You may also contact Student Support at UNE, located in the West Wing of the TC Lamble Building, or by phone (02 6773 2897) or email (studentsupport@une.edu.au).
Storage of Information	Data from the research will be stored securely, initially in the une.cloud.une account of Mr Hall, and then later in Dr Whannell's account. Hard copy data will be stored in locked filing cabinets, separate to any identifiers, in the School of Education.
Disposal of Information	All the data collected in this research will be kept for a minimum of five years after successful submission of my thesis, after which it will be disposed of by deleting relevant computer files, and destroying or shredding hardcopy materials.
Approval	This project has been approved by the Human Research Ethics Committee of the University of New England (Approval No. HE18-135 Valid to 26/6/2019).
Researchers Contact Details	Feel free to contact me with any questions about this research by email at phall2@myune.edu.au. You may also contact my supervisors. My Principal supervisor's name is Dr Robert Whannell and he can be contacted by email at rwhannel@une.edu.au or by phone on 02 6773 5981 and my Co-supervisor's name is A/Prof Pep Serow and her email address is

	pserow2@une.edu.au and phone number is 02 6773 2378
Complaints	Should you have any complaints concerning the manner in which this research is conducted, please contact: Mrs Jo-Ann Sozou Research Ethics Officer Research Services University of New England Armidale, NSW 2351 Tel: (02) 6773 3449 Email: ethics@une.edu.au

Thank you for considering this request and I look forward to further contact with you.

Regards,

Peter Hall

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CONSENT FORM for PARTICIPANTS

Research Project: Investigating strategies for one- and two-digit mental multiplication used by pre-service teachers.

I, ………………………………………………………………………., have read the information contained in the Information Sheet for Participants and any questions I have asked have been answered to my satisfaction. Yes/No

I agree to participate in this activity, realising that I may withdraw at any time. Yes/No

I agree that research data gathered for the study may be quoted and published using a pseudonym. Yes/No

I am older than18 years of age.

Yes/No

……………………………….. ……………………………

Participant Date

……………………………….. ……………………………

Researcher Date

Appendix 2 - Test and Retest

Table A.1

Questions Used for Tests 1 and 2

Test			**Retest**	
No.	Question	NSW Outcome	No.	Question
1	40 x 90	2 x 2 digit	1	50 x 90
2	5 x 8000	1 x 4 digit	2	6 x 7000
3	18 x 6	1 x 2 digit	3	22 x 6
4	5 x 19	1 x 2 digit	4	5 x 23
5	126 x 4	1 x 3 digit	5	217 x 4
6	9 x 45	1 x 2 digit	6	9 x 37
7	3 x 195	1 x 3 digit	7	3 x 295
8	143 x 7	1 x 3 digit	8	321 x 8
9	19 x 25	2 x 2 digit	9	23 x 25
10	23 x 12	2 x 2 digit	10	32 x 12
11	15 x 18	2 x 2 digit	11	14 x 17
12	21 x 23	2 x 2 digit	12	32 x 31
13	123 x 11	3 x 2 digit	13	432 x 11
14	45 x 24	2 x 2 digit	14	24 x 35
15	34 x 35	2 x 2 digit	15	46 x 45

Test Answer Sheet

Write answers only.

These are mental questions with no written working allowed.

There is a 15 second delay between questions.

Student Identifier:

Test # No.	Answer
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	

Appendix 3 - Taxonomy of Mental Multiplication Strategies and Coding Sheets

Table A.2

Coding Taxonomy with Examples

Code	Descriptor	Examples
BDF	Basic or Derived Fact	5 x 8 = 40, 50 x 8 = 400, 50 x 80 = 4000
COU	Counting – skip counting, doubling +/-	3 x 24 = 24+24+24 = 48+24 =72 3x24=2x24+24 = 48+24 = 72
BFS	Basic Fact Shortcuts – shortcuts for multipliers from 2 to 12, 15 and 20.	x5 = ½ x 10 x8 = double, double, double x9 = x10 – x1 x11 = x10 + x1 x12 = x10 + double x15 = x10 + x5 Trachtenberg, teen multipliers, squares ending in 5
LRS	Left to right separated – using the Distributive Law	7x23: 7x20=140, 7x3=21, 140+21=161
RLS	Right to left separated – visualising the formal algorithm	7x23: 7x3=21write 1 and carry 2, 7x2=14, 14+2=16. Answer=161
FAC	Factors	6x32: 6=2x3, 32x3=96, 96x2=192
PAR	Partitioning (Non-standard)	8 x 21 = 8x10+8x11 = 80 + 88 = 168 19x34 = 20x34 – 1x34 = 680 – 34 = 646
CMP	Compensation	17x25 = 17x100 ÷ 4 = 425 15x24 = 30x12 = 360
CMB	A combination of the above strategies	21 x 23 = 11 x 23 + 10 x 23 (PAR) = 253 (BFS) + 230 = 483

Test 1

Question No.	Question	Working	Code	Research Comment
Example	18 x 7	10 x7 + 8x7	LRS	
1	40 x 90			
2	5 x 8000			
3	18 x 6			
4	5 x 19			
5	126 x 4			
6	9 x 45			
7	3 x 195			
8	143 x 7			
9	19 x 25			
10	23 x 12			
11	15 x 18			
12	21 x 23			
13	123 x 11			
14	45 x 24			
15	34 x 35			

Test 2

Question No.	Question	Working	Code	Research Comment
Example	18 x 7	10 x7 + 8x7	LRS	
1	50 x 90			
2	6 x 7000			
3	22 x 6			
4	5 x 23			
5	217 x 4			
6	9 x 37			
7	3 x 295			
8	321 x 8			
9	23 x 25			
10	32 x 12			
11	14 x 17			
12	32 x 31			
13	432 x 11			
14	35 x 24			
15	46 x 45			

Appendix 4 - Questionaire 1 and 2

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Questionnaire 1

Research Project: Investigating strategies for one- and two-digit mental multiplication used by pre-service teachers.

Please return to the Researcher, Mr Peter Hall at the first tutorial session.

For any questions regarding this research project, please contact:

Dr Rob Whannell	Principal Supervisor	02 6773 5981 email: rwhannel@une.edu.au
Mr Peter Hall	Research Masters Candidate	02 6775 1161 or 0427 255 685 email: phall2@myune.edu.au

Information:

This questionnaire has 28 questions and should take approximately 20 minutes to complete. If you have agreed to participate in the research, I would appreciate it if you could complete a hard copy of the questionnaire and return it to me at the first tutorial session. Alternately, you could email an electronic copy to me prior to the first tutorial.

The questionnaire has some demographic questions followed by responses regarding your present conceptions of mental arithmetic. Questions 6 to 21 are based on a common 5-point Likert scale ranging from Strongly Disagree to Strongly Agree. You only need to tick one box that best reflects your opinion. Questions 22 to 26 provide an opportunity to demonstrate your knowledge of mental multiplication strategies without time constraints. Question 28 provides an option of further comment to clarify any of your answers.

The student identifier aims to protect your identity. It is a two letter/two-digit identifier of your choice that you will need to use on all data collection instruments during the research. An example could be PH27 – the choice is entirely yours but make it something easy to remember.

Participant Identifier: ____________

Q1. What is your gender? Male ☐ Female ☐ Other ☐

Q2. What is your age group? 18 – 25 ☐ 26 – 35 ☐ 36 – 45 ☐ > 45 ☐

Q3. What level of mathematics did you complete at High School?

Year 10 ☐ Year 11 ☐ Maths General ☐ MA 2 ☐ Ext 1 or higher ☐

Q4. What is your major teaching area? ______________________________

Q5. What is your specialisation teaching area? ______________________________

For questions 6 to 21 nominate your level of agreement with the prompt by ticking the relevant box

No	Prompt	Strongly Disagree	Disagree	Not Sure	Agree	Strongly Agree
Q6	At school I was good at Maths	☐	☐	☐	☐	☐
Q7	I enjoyed doing Maths at school	☐	☐	☐	☐	☐
Q8	I learned a number of mental computation strategies at school	☐	☐	☐	☐	☐
Q9	As an adult I use mental computation regularly	☐	☐	☐	☐	☐
Q10	I am aware of a range of strategies for mental computation	☐	☐	☐	☐	☐
Q11	I am confident in using a range of strategies for mental computation	☐	☐	☐	☐	☐
Q12	I am aware of the place of mental multiplication in the curriculum.	☐	☐	☐	☐	☐
Q13	I feel confident about teaching mental computation to Year 5	☐	☐	☐	☐	☐
Q14	I think mental computation is an essential skill for school students	☐	☐	☐	☐	☐
Q15	I think mental computation is an essential skill for adults	☐	☐	☐	☐	☐
Q16	Mental computation is best learned in primary school	☐	☐	☐	☐	☐
Q17	Mental computation should be taught before formal methods of written computation	☐	☐	☐	☐	☐
Q18	All students have the potential to be good at mental computation	☐	☐	☐	☐	☐

Q19	It is important for students to have as wide a range of mental multiplication strategies as possible to allow them to solve different problems efficiently.	☐	☐	☐	☐	☐
Q20	Mental multiplication promotes critical thinking and problem solving.	☐	☐	☐	☐	☐
Q21	Mental multiplication promotes number sense.	☐	☐	☐	☐	☐

For Questions 22 to 26, write down as many different mental strategies as you can to answer each question.

No.	Question	Strategies	Coding (leave blank)
Q 22	19 x 25		
Q 23	23 x 12		
Q 24	15 x 18		
Q 25	21 x 23		
Q 26	45 x 24		

Q27 By participating in this research, what do you hope to gain or achieve? (free response)

Q28 Do you have any optional comments to make regarding any of the questions above? (free response)

Thank you for completing this questionnaire.

Peter Hall

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Final Questionnaire

Research Project: Investigating strategies for one- and two-digit mental multiplication used by pre-service teachers.

Please return to the Researcher, Mr Peter Hall at the last tutorial session.

For any questions regarding this research project, please contact:

Dr Rob Whannell	Principal Supervisor	02 6773 5981 email: rwhannel@une.edu.au
Mr Peter Hall	Research Masters Candidate	02 6775 1161 or 0427 255 685 email: phall2@myune.edu.au

Information:

The final questionnaire has 3 sections and should take approximately 20 minutes to complete. If you have agreed to participate in the research, I would appreciate it if you could complete a hard copy of the questionnaire and return it to me at the last tutorial session. Alternately, you could email an electronic copy to me after the project has finished.

In Section 1, Questions 1 to 9 have not been included as they were completed in the first survey. Questions 10 to 21 are similar to those in the first questionnaire and are also based on a 5-point Likert scale ranging from Strongly Disagree to Strongly Agree. You only need to tick one box that best reflects your opinion.

Section 2 provides the opportunity to demonstrate your knowledge of alternate strategies for mental multiplication.

Section 3 seeks your opinion on the range of strategies for mental multiplication that you would ideally include in a Year 5 teaching program and the timing of that program. There is a final opportunity to comment on the research and expand upon any answers to previous questions.

Again, please use your student identifier to protect your identity. It is a two letter/two-digit identifier of your choice that you will need to use on all data collection instruments during the research. An example could be PH27 – the choice is entirely yours but it must be the same as previously used.

Section 1 Student Identifier: ___________________ (Same as questionnaire 1 to allow data matching)

No	Prompt	Strongly Disagree	Disagree	Not Sure	Agree	Strongly Agree
Q10	I am aware of a range of strategies for mental computation	☐	☐	☐	☐	☐
Q11	I am confident in using a range of strategies for mental computation	☐	☐	☐	☐	☐
Q12	I am aware of the place of mental multiplication in the curriculum.	☐	☐	☐	☐	☐
Q13	I feel confident about teaching mental computation to Year 5	☐	☐	☐	☐	☐
Q14	I think mental computation is an essential skill for school students	☐	☐	☐	☐	☐
Q15	I think mental computation is an essential skill for adults	☐	☐	☐	☐	☐
Q16	Mental computation is best learned in primary school	☐	☐	☐	☐	☐
Q17	Mental computation should be taught before formal methods of written computation	☐	☐	☐	☐	☐
Q18	All students have the potential to be good at mental computation	☐	☐	☐	☐	☐
Q19	It is important for students to have as wide a range of mental multiplication strategies as possible to allow them to solve different problems efficiently.	☐	☐	☐	☐	☐
Q20	Mental multiplication promotes critical thinking and problem solving.	☐	☐	☐	☐	☐
Q21	Mental multiplication promotes number sense.	☐	☐	☐	☐	☐

Section 2

Write as many strategies as possible to solve each problem and code them.

No.	Question	Strategies	Coding (leave blank)
1	9 x 37		
2	23 x 25		
3	32 x 12		
4	14 x 17		
5	24 x 35		

Section 3

Q1. Did you attend the lecture? Yes ☐ No ☐

	Prompt	Of No Value	Of Little Value	Of Moderate Value	Of Considerable Value	Of Great Value
Q2.	If you attended, how would you rate the lecture?	☐	☐	☐	☐	☐

Q3. Did you study the lecture notes? Yes ☐ No ☐

	Prompt	Of No Value	Of Little Value	Of Moderate Value	Of Considerable Value	Of Great Value
Q4.	If you studied the lecture notes, how would you rate them??	☐	☐	☐	☐	☐

Question 5

Considering your coding sheet and given that Basic Number Facts, Derived facts and Counting Methods are encountered in earlier years, which of the following strategies would you teach to an ungraded Year 5? (tick if you would teach the strategy)

Code	Descriptor	Examples	Tick
BFS	Basic Fact Shortcuts – shortcuts for multipliers from 2 to 12	x5 = ½ x 10 x8 = double, double, double x9 = x10 – x1 x11 = x10 + x1 x12 = x10 + double x15 = x10 + x5 Trachtenberg, teen multipliers, squares ending in 5	☐
LRS	Left to right separated – using the Distributive Law	7x23: 7x20=140, 7x3=21, 140+21=161	☐
RLS	Right to left separated – visualising the formal algorithm	7x23: 7x3=21write 1 and carry 2, 7x2=14, 14+2=16. Answer=161	☐
FAC	Factors	6x32: 6=2x3, 32x3=96, 96x2=192	☐
PAR	Partitioning (Non-standard	8 x 21 = 8x10+8x11 = 80 + 88 = 168 19x34 = 20x34 – 1x34 = 680 – 34 = 646	☐
CMP	Compensation	17x25 = 17x100 ÷ 4 = 425 15x24 = 30x12 = 360	☐

Question 6

Considering your coding sheet, how would you rate each strategy?

Code	Strategy	Of No Value	Of Little Value	Of Moderate Value	Of Considerable Value	Of Great Value
BFS	Basic Fact Shortcuts – shortcuts for multipliers from 2 to 12	☐	☐	☐	☐	☐
LRS	Left to right separated – standard partitioning	☐	☐	☐	☐	☐
RLS	Right to left separated – visualising the formal algorithm	☐	☐	☐	☐	☐
FAC	Factors	☐	☐	☐	☐	☐
PAR	Partitioning (Non-standard)	☐	☐	☐	☐	☐
CMP	Compensation	☐	☐	☐	☐	☐

Question 7

Considering your coding sheet, what strategies, if any, would you **only** teach as extensions to a more able group of students? (tick as appropriate)

BFS ☐ LRS ☐ RLS ☐ FAC ☐ PAR ☐ CMP ☐

Question 8

How would you pace your instruction of mental computation strategies? (tick your preferred option)

As a block of 2-3 weeks ☐ Over 1 term ☐ Over 2 terms ☐ Over 3 or 4 terms

Question 9

Are there any general comments in relation to the research project that you would like to make?

____

Thank you for completing this questionnaire.

Peter Hall

Appendix 5 – Lecture Notes

Mental Multiplication by One- and Two-digit Numbers

Peter Hall

Basic Number Fact Shortcuts

Let's consider each multiplier from 2 to 12 separately as there are different shortcuts associated with each one.

Times 2: Doubling is, of course, the same as multiplying by two, or adding the same number. This may sound trivial but it can be achieved in two ways, which I'll call unders and overs.

Unders implies that the product doesn't need a carry to complete. For example, 14 x 2 is simply double 10 then double 4 to give 20 + 8 = 28. It's generally done in one go from left to right.

Other examples are

a) 13 x 2 b) 21 x 2 c) 43 x 2 d) 62 x 2 e) 74 x 2 f) 103 x 2 g) 121 x 2

Overs implies a carry of some sort, such as 27 x 2. Generally, I would double the tens digit first to give 40 and then 2 x 7 = 14 and add to get 54. This is the reverse of the formal way of doing the problem (right to left) but research suggests it works better when doing mental calculations.

Similarly with 68 x 2, I'd hold 60 x 2 = 120 in my head and double 8 (2x8=16) then add to get 136. When you get used to it, you simply double 60 and add 10 to get 130 because you know it is overs; then add 6 because you know double 8 ends in 6.

With something like 172 x 2: I'd do 2 x 170 = 200 + 140 = 340 and add 4 (2x2) to get 344. It pays to know the doubles up to 20 to cut down calculation time – if I know double 17 is 34 then double 170 is 340 and add double 2 to get 344.

Other examples are:

a) 18 x 2 b) 26 x 2 c) 48 x 2 d) 67 x 2 e) 79 x 2 f) 136 x 2 g) 161 x 2

Generally, when you double something in your head, you start with the highest place value and work back to digits. Students need to be proficient at doubling and halving as so many shortcuts rely primarily on those two skills.

Times 3: This can be seen as tripling if you are good at that or double + itself.

Try 32 x 3: double = 64 and add 32 =96
Or 3 x 30 = 90 3x2 = 6
90 + 6 = 96

57 x 3: I'd probably do 3 x 50 + 3 x 7 = 150 + 21 = 171

Again, I'd multiply the tens digit first, hold that in my head, then the units digit and add.

123 x 3 = 3 x 120 + 3 x 3 = 360 + 9 = 369 or
= 246 + 123 = 369 (using double + itself)
In my head I'll double 123 to get 246 then add 100 for 346, then 20 for 366, and finally 3 for 369.

47 x 3 = 3 x 40 + 3 x 7 =120 +21 = 141 or
= 94 + 47 = 134 + 7 = 141 (using double + itself)
You'll notice that I added the 40 to 94 using overs then finished by adding the 7, again using overs.

Like doubles, it pays to know the triples up to 20
Then 182 x 3 = 540 + 6 = 546

Now try

a) 13 x 3 b) 21 x 3 c) 43 x 3 d) 62 x 3 e) 74 x 3 f) 103 x 3 g) 121 x 3
h) 173 x 3

Times 4: Is of course x 2 x 2 or double double

13 x 4 =26 x 2 = 52

43 x 4 = 86 x 2= 160 + 12 = 172

Hold 160 in your head, add 10 for overs (170), then 2 for double 6 (172)

123 x 4 = 246 x 2 = 480 + 12 = 492

For a straight mental multiplication by 4:

13 x 4 = 40 + 12 = 52

43 x 4 = 160 + 12 = 172

123 x 4 = 400 + 80 + 12 = 492

Try these for yourself:

a) 18 x 4 b) 26 x 4 c) 43 x 4 d) 67 x 4 e) 72 x 4 f) 106 x 4 g) 161 x 4

Times 5: This is x 10 then halved or half and times by ten. Your choice depends on whether the multiplicand is odd or even.

For even numbers, half first then times by ten.

Examples:

16 x 5 = 8 x 10 = 80

24 x 5 = 12 x 10 = 120

38 x 5 = 19 x 10 = 190

114 x 5 = 57 x 10 = 570

Try these:

a) 18 x 5 b) 26 x 5 c) 48 x 5 d) 66 x 5 e) 72 x 5 f) 126 x 5 g) 164 x 5

For odd numbers do the reverse; times by ten and then half:

13 x 5 = 130 ÷ 2 = 65

17 x 5 = 170 ÷ 2 = 85

23 x 5 = 230 ÷ 2 = 115

45 x 5 = 450 ÷ 2 = 225

When you halve, think of the even ten multiple below, halve it and add 5.
So 130 ÷ 2 is 120 ÷ 2 = 60, add 5 = 165.

Try these for yourself
a) 15 x 5 b) 21 x 5 c) 43 x 5 d) 67 x 5 e) 71 x 5 f) 103 x 5 g) 121 x 5

For direct multiplication by 5, start with the highest digit (left to right).

16 x 5 = 50 + 30 = 80

24 x 5 = 100 + 20 = 120

45 x 5 = 200 + 25= 225

114 x 5 = 500 + 50 + 20 = 570

Try these for yourself
a) 19 x 5 b) 26 x 5 c) 42 x 5 d) 67 x 5 e) 74 x 5 f) 109 x 5 g) 122 x 5

Times 6: Either use its factors 2 and 3 or times 5 plus itself or do a straight multiplication by 6.

When using factors, the choice of doubling first or second is probably dependent on the question rather than a suggested order.

	Factors of 2 and 3	Times 5 + x1	x6
13 x 6	13 x 3 = 39	13 x 5 = 65	10 x 6 = 60

	39 x 2 = 78	65 + 13 = 78	3 x 6 = 18 60 + 18 = 78.
25 x 6	25 x 2 = 50 50 x 3 = 150	25 x 5 = 125 125 + 25 = 150	20 x 6 = 120 5 x 6 = 30 120 + 30 = 150.
83 x 6	83 x 3 = 249 249 x 2 =250 x 2 – 2 = 498	83 x 5 = 415 415 + 83 = 498	80 x 6 = 480 3 x 6 = 18 480 + 18 = 498.
132 x 6	132 x 3 = 396 396 x 2 = 400 x 2 - 8 = 792	132 x 5 = 660 660 + 132 = 792	100 x 6 = 600 30 x 6 = 180 2 x 6 = 12 total = 792

You could try:

a) 18 x 6 b) 26 x 6 c) 143 x 6 d) 670 x 6 e) 1234 x 6

Times 7: Done as x 5 + double or as straight x 7

	X 5 + double	Times 7
23 x 7	23 x 5 = 115 115 + 46 = 161	20 x 7 = 140 3 x 7 = 21 140 + 21 = 161
56 x 7	56 x 5 = 280 280 + 112 = 392	50 x 7 = 350 6 x 7 = 42 350 + 42 = 392.
124 x 7	124 x 5 = 620 620 + 248 = 868	100 x 7 = 700 20 x 7 = 140 4 x 7 = 28 total = 868

Here's some to try:

a) 22 x 7   b) 46 x 7   c) 76 x 7   d) 112 x 7   e) 134 x 7

Times 8: Double, double, double or x10 – double or straight x 8

	Double, double, double	Times 10 – double	x 8
13 x 8	13 x 2 = 26	13 x 10 = 130	10 x 8 = 80
	26 x 2 = 52	130 - 26 = 104	3 x 8 = 24
	52 x 2 = 104		80 + 24 = 104.
35 x 8	35 x 2 = 70	35 x 10 = 350	30 x 8 = 240
	70 x 2 = 140	350 - 70 = 280	5 x 8 = 40
	140 x 2 = 280		240 + 40 = 280.
73 x 8	73 x 2 = 146	73 x 10 = 730	70 x 8 = 560
	146 x 2 = 145 x 2 + 2	730 - 146 = 600 - 16	3 x 8 = 24
	292 x 2 = 300 x 2 - 16	= 584	560 + 24 = 584.
	= 584		
132 x 8	132 x 2 = 264	132 x 10 = 1320	100 x 8 = 800
	264 x 2 = 528	1320 - 264 = 1100 - 44	30 x 8 = 240
	528 x 2 = 1056	= 1056	2 x 8 = 16
			total = 1056

Try the following:

a) 23 x 8   b) 45 x 8   c) 67 x 8   d) 71 x 8   e) 120 x 8   f) 305 x 8

Times 9:   x10 – x1 or Hall's method or straight x9

Hall's method: add 1 to the tens digit and subtract this from the number. Now add a nine complement to the remaining digits.

For example: 36 x 9. Add one to the tens digit 3+1=4. Take this away from the number 36 - 4=32. Add a nine complement 3+2=5, 9-5=4. Hence add a 4 to get 324.

Another example: 87 x 9. Add one to the tens digit 8 + 1 = 9. Take this away from the number 87 – 9 = 78. Add a nine complement 7 + 8 = 15, 18 – 15 = 3. Hence 783

And another: 234 x 9. Add 1 to the tens digit 23 + 1 = 24. Take 24 from 234 = 210. Add a nine complement (6) to get 2106

	x10 – x1	Hall's method	x9
19 x 9	19 x 10 = 190 190 - 19 = 171	19 – (1+1) = 17 171	10 x 9 = 90 9 x 9 =81 90 + 81 = 171.
57 x 9	57 x 10 = 570 570 - 57 = 513	57 – (5+1) = 51 513	50 x 9 = 450 7 x 9 = 63 450 + 63 = 513.
112 x 9	112 x 10 = 1120 1120 -112 = 1008	112 – (11+1)= 100 1008	100 x 9 = 900 10 x 9 = 90 2 x 9 = 18 total = 1008

Note: Don't use Hall's method for multiples of 10 . Use either times tables or x10 – x1. For example 40 x 9 = 360 (4X9=36) or 40 x 10 – 40 = 400 – 40 = 360.

Try the following:
a) 23 x 9 b) 45 x 9 c) 63 x 9 d) 71 x 9 e) 120 x 9 f) 345 x 9

Times 10: Shift left and add a zero if whole numbers.

36 x 10 = 360 49 x 10 = 490 123 x 10 = 1230 102 x 10 = 1020

The concept of shift left and add zero for whole numbers is important. If you just say ‘add zero’, it causes problems later when students encounter decimal multiplication. For example, 3.4 x 10 can sometimes lead to incorrect answers of 3.40 or 30.4, if students have just been given the rule as ‘add zero’. Shift left tells them you are moving a place in the decimal system, which is an important concept for students to grasp. Always talk about 360 being ten times larger than 36 or 36 is tens times smaller than 360. Eventually this leads to saying that 36 is one tenth of 360.

For a better understanding of the research behind this, try Hurst, C., & Hurrell, D. (2016, December). Captain Zero... hero or villain?. In *2016 Mathematical Association of Victoria’s Annual Conference Proceedings*.

Times 11: x10 + x1 or Trachtenberg or x11

We’ll hear more of Trachtenberg later in the chapter, but for now his simple rule for multiplying by 11 is to add each number to its right hand neighbour.

For example: 342 x 11 = 03420 x 11 = (0+3)(3+4)(4+2)(2+0) = 3762
Note: Imagine a zero at the front and rear of the number. Trachtenberg calls this a nought sandwich. He defines a neighbour as the digit to the left.

For two digit numbers x 11, just add the two digits to get the middle digit. Leave the outside digits as they are except if a carry is required.

Examples:
24 x 11= 2(2+4)4 = 264
43 x 11 = 4(4+3)3 = 473
57 x 11 = 5(5+7)7 = 627 note the carry as 5 + 7 =12, hence carry 1.

Examples using all three methods:

	x10 + x1	Trachtenberg	x11
23 x 11	23 x 10 = 230	2(2+3)3=253	20 x 11 = 220

	230 + 23 = 253		3 x 11 = 33 220 + 33 = 253
56 x 11	56 x 10 = 560 560 + 56 = 616	5(5+6)6 = 616	50 x 11 = 550 6 x 11 = 66 550 + 66 = 616
73 x 11	73 x 10 = 730 730 + 73 = 803	7(7+3)3 = 803	70 x 11 = 770 3 x 11 = 33 770 + 33 = 803.
124 x 11	124 x 10 = 1240 1240 + 124 = 1364	1(1+2)(2+4)4 = 1364	120 x 11 = 1320 4 x 11 = 44 1320 + 44 = 1364

Try the following:

a) 23 x 11 b) 45 x 11 c) 67 x 11 d) 121 x 11 e) 320 x 11 f) 4135 x 11

Times 12: x10 + x2 or by factors of 3 and 4 or by x11 + x1 or Trachtenberg.

	Triple, double, double	x10 + x2	x11 + x1
13 x 12	13 x 3 = 39 39 x 2 = 78 78 x 2 = 156	13 x 10 = 130 130 + 26 = 156	13 x 11 = 143 143 + 13 = 156
35 x 12	35 x 2 = 70 70 x 3 = 210 210 x 2 = 420	35 x 10 = 350 350 + 70 = 420	35 x 11 = 385 385 + 35 = 420
57 x 12	57 x 2 =114 114 x 3 = 342 342 x 2 = 684	57 x 10 = 570 570 + 114 = 684	57 x 11 = 627 627 + 57 = 684

132 x 12	132 x 3 = 396	132 x 10 = 1320	132 x 11 = 1452
	396 x 2 = 800 - 8	1320 + 264 = 1584	1452 + 132
	792 x 2 = 1600 -16		= 1584
	= 1584		

Try the following:

a) 23 x 12 b) 45 x 12 c) 67 x 12 d) 121 x 12 e) 320 x 12 f) 135 x 12

Trachtenberg's Multiplication for 12

Rule: to multiply by 12:

Make a nought sandwich and starting from the right digit, double each digit and add to the neighbor

If the answer is greater than a single digit, simply carry over the extra digit (which will be a 1 or 2) to the next operation.

Examples:

316 x 12 = 03160 x 12

= (0x2+3=3) (3x2+1=7) (1x2+6+1=9) (6x2=12, write down 2, carry1)

= 3792

12345 x 12 = 0123450 x 12

= (0x2+1=1)(1x2+2=4)(2x2+3 +1=8) (3x2+4 +1=11, write 1, carry 1) (4x2+5+1=14, write 4, carry 1) (5x2=10, write down 0, carry1)

Answer = 148140

Higher Number Fact Shortcuts

Multiply by 15: x10 then add half again

For example:

26 x 15 = 260 + 130 = 390

46 x 15 = 460 + 230 = 690

87 x 15 = 870 + 435 = 1305

35 x 15 = 350 + 175 = 525

A few to try:

a) 18 x 15 b) 26 x 15 c) 43 x 15 d) 67 x 15 e) 124 x 15

Multiply by 20: double, shift left, add a place holder if needed.

For example:

34 x 20 = 68 x 10 = 680

52 x 20 = 104 x 10 = 1040

96 x 20 = 192 x 10 = 1920

A few to try:

a) 35 x 20 b) 29 x 20 c) 43 x 20 d) 67 x 20 e) 103 x 20 g) 121 x 20

Multiplying Numbers in the teens – from 10 to 19

Rule: Add the last digit of the multiplier to the first number. Multiply this by ten, then add the product of the two units digits.

Examples:

13 x 16

Add 6 to 13 (or 3 to 16) and x10 = (13 + 6) x 10 =190

Add 3 x 6 = 190 + 18 = 208

14 x 17

14 + 7 = 21 21 x 10 = 210, 4 x 7 = 28 210 + 28 = 238

16 x 19

16 + 9 = 25 25 x 10 = 250, 6 x 9 = 54 250 + 54 = 304

Try these for yourself:

a) 15 x 14 b) 12 x 17 c) 16 x 14 d) 15 x 18 e) 17 x 18 f) 18 x 19

As you get used to it, you start to skip steps and do more and more in your head. Eventually, the aim is to do it all mentally.

And in the twenties

Rule: Add the last digit of the multiplier to the first number. Multiply this by 20, then add the product of the two units digits.

Examples:

23 x 26

Add 6 to 23 (or 3 to 26) and x20 = (23 + 6) x 20 =29 x 20 = 580

Add 3 x 6 = 580 + 18 = 598

24 x 27

24 + 7 = 31 31 x 20 = 620, 4 x 7 = 28 620 + 28 = 648

26 x 29

26 + 9 = 35 35 x 20 = 700, 6 x 9 = 54 700 + 54 = 754

Try these for yourself:

a) 25 x 24 b) 22 x 27 c) 26 x 24 d) 25 x 28 e) 27 x 28

This method also works for any two numbers in the same decile. Use the same rule except multiply by the appropriate multiple of ten. In the thirties, multiply by 30; in the forties, multiply by 40; and so on.

Squaring Numbers Ending in 5: Multiply the ten multiples above and below the number and add 25.

Examples:

$25^2 = 20 \times 30 + 25 = 625$

$35^2 = 30 \times 40 + 25 = 1225$

$65^2 = 60 \times 70 + 25 = 4225$

$95^2 = 90 \times 100 + 25 = 9025$

Try a few:

a) 15^2 b) 45^2 c) 55^2 d) 75^2 e) 85^2 f) 105^2

Near Squares

Examples:

34 x 35 = 35^2 - 35 = 1225 – 35 = 1190

23 x 25 = 25^2 - 2 x 25 = 625 – 50 = 575

65 x 66 = 65^2 + 65 = 4225 + 65 = 4290

95 x 97 = 95^2 + 2 x 95 = 9025 + 200 -10 = 9215

Try a few:

a) 14 x 15 = 15^2 - 15 b) 43 x 45 = 45^2 - 90

c) 55 x 56 = 55^2 + 55 d) 77 x 75 = 75^2 + 150

There are many other shortcuts for squares and using squares, and different combinations of two-digit multipliers. You could try Handley, B. (2012). *Speed mathematics*. John Wiley & Sons for some more ideas.

Partitioning, Factor and Compensation Strategies and Combined Strategies

Standard Partitioning involves splitting one of the multipliers into its powers of ten and then multiplying left to right. This is also know as Left Right Separated (LRS)

For example:

12 x 135 = 10 x 135 + 2 x 135 or 100 x 12 + 30 x 12 + 5 x 12

= 1350 + 270 = 1200 + 360 + 60

= 1620 = 1620

Non-standard partitioning involves splitting one or both multipliers into parts and then multiplying each part and finally adding by the Distributive Law.

12 x 135 = 4 x 135 + 4 x 135 + 4 x 135		or	11 x 135 + 1 x 135
= 540 + 540 + 540			= 1485 +135
= 1620			= 1620
12 x 135 = 20 x 135 – 8 x 135		or	9 x 135 + 3 x 135
= 2700 – 1080			= 1215 + 405
= 1620			

The splits chosen reflect shortcuts mentioned in the last section.

Compensation covers a mix of strategies involving an adjustment to one multiplier and then compensation of some sort.

12 x 135 = 6 x 270
= 3 x 540
= 1500 + 120 = 1620

Now consider 23 x 21 and the range of strategies available to calculate the answer.

Standard Partitioning

23 x 21 = 20 x 23 + 1 x 23
As a mental calculation, you do 20 x 23 in your head (shift, double) and hold the answer of 460 there. Then add 1 x 23 to this and get 483.

Similarly we could partition 23:
21 x 23 = 21 x 20 + 21 x 3
As a mental exercise, do 21 x 20 = 420 (hold that in your head) and add 21 x 3 = 63 to get 483.

You could partition both numbers into tens and digits:
21 x 23 = (20 + 1) x (20 + 3)

This produces four partial products: 20 x 20 + 20 x 3 + 1 x 20 + 1 x 3
= 400 + 60 + 20 + 3 = 483

Non-Standard Partitioning

However, you could split 21 into 10 + 11.
Then do 23 x 10 = 230 (hold that in your head) and use the 11 trick (add to the neighbour) to get 23 x 11 = 253.
Finally, 253 + 230 = 483.

21 could also be split into 12 + 9 and use both the 12 trick (double + neighbour) and one of the many 9 tricks (x10 – x1).
This gives me 276 + 207 = 483.

What about splitting 21 into 15 + 6?
23 x 15 is x10 then half again = 230 + 115 = 345.
23 x 6 = 23 x 3 x 2 (using factors) = 69 x 2 = 138.
345 + 138 = 483.
Admittedly a little harder to hold the partial products in your head and do the addition.

One more? Try splitting 23 into 8 + 15 and multiply by 21.
21 x 15 = 210 + 105 = 315.
21 x 8 is double, double, double = 42, 84, 168.
168 + 315 = 483

So how many ways are there to multiply 21 x 23? Well, using partitioning, there are actually an **infinite** number of ways of doing it, most ridiculous, but all possible. Using whole numbers, there are 10 ways of partitioning 21 and 11 ways for 23. This implies that there are 10 + 11 = 21 ways of partitioning either number. But if you partition both, there are many more options.

This could be further extended if by considering a greater number of partitions.

For example:

Split 23 into 2 + 10 + 11.

21 x 23 = 21 x (2 + 10 + 11)

Expanding:

21 x 23 = 42 + 210 + 231 (using the 11 shortcut of add to neighbour)

= 483

All of the partial products can be done mentally through times tables or simple shortcuts.

Non-standard partitioning with negatives (Subtractive Distributive)

23 x (30 - 9) or 21 x (25 - 2) or 21 x (30 – 7)

So,

23 x 30 is triple, shift = 690.

9 x 23 = 23 – 3 + (9 complement) = 207

690 – 207 = 483

21 x 23 = 21 x 30 – 7 x 21

21 x 30 is triple, shift = 630.

7 x 21 = 147

630 – 130 - 17 = 500 – 17 = 483

Compensation

21 x 23 = 7 x 69 (one third of 21 x triple 23)

= 7 x 70 – 7 (subtractive distributive)

= 490 – 7 = 583

Can you think of other ways of distributing either 21 or 23 to get the same answer by a different route. Of course you can and that's what we want our students to be capable of. Not just a rote learned method but the ability to choose a pathway that suits their individual knowledge of multiplication and its short cuts.

Here's one using partitioning and factors

21 x (22 + 1) = 21 x 11 x 2 + 1 x 21

= 231 x 2 + 21 = 462 + 21 = 483

And another with partitioning and factors

split 21 into 22 – 1

23 x 22 = 23 x 11 x 2 = 253 x 2 = 506

506 – 23 = 483.

Another compensation strategy is used to multiply by 25:

Multiply by 25: As 25 is ¼ of 100, divide by 4 then x 100

16 x 25 = 4 x 100 = 400

28 x 25 = 7 x 100 = 700

36 x 25 = 9 x 100 = 900

52 x 25 = 13 x 100 = 1300

Harder examples:

19 x 25 = 19/4 x 100 = 4 ¾ x 100 = 475

19 divided by 4 goes four and remainder ¾ . That's 4 hundreds and 3 x 25 = 475

25 x 25 = 6 ¼ x 100 = 625

53 x 25 = 13 ¼ x 100 = 1325

85 x 25 = 21¼ x 100 = 2125

106 x 25 = 26 ½ x 100 = 2650

Some to try yourself:

a) 24 x 25 b) 48 x 25 c) 68 x 25 d) 72 x 25 e) 120 x 25 f) 304 x 25

And some harder ones:

a) 25 x 25 b) 43 x 25 c) 65 x 25 d) 71 x 25 e) 122 x 25 f) 306 x 25

This shortcut would then allow another alternate pathway for 21 x 23 mentioned above

21 x 23 = 21 x 25 – 2 x 21

21 x 25 = 5 ¼ x 100 = 525 and

2 x 21 = 42.

Therefore, 21 x 23 = 525 – 42 = 483

One last example using a variety of strategies

Try **55 x 90**

Let's start by partitioning 55 in different ways:

90 x 55 = 90 x 50 + 90 x 5 (Standard partitioning)
= 4500 + 450
= 4950

90 x 55 = 90 x 25 + 90 x 30 (Non-standard partitioning)
= 2250 + 2700
= 4950

90 x 55 = 90 x 10 + 90 x 45 (Non-standard partitioning and factors)
= 900 + 2 x 45 x 45
= 900 + 2 x 2025 (Note the trick for $45^2 = 40 \times 50 + 25 = 2025$)
= 900 + 4050
= 4950

90 x 55 = 90 x 100 – 90 x 45 (Non-standard partitioning and factors)
= 9000 – 2 x 45 x 45
= 9000 - 4050
= 4950

90 x 55 = 90 x 60 - 90 x 5 (Subtractive partitioning)
= 5400 – 450
= 4950

90 x 55 = 90 x 40 + 90 x 15 (Non-standard partitioning)

= 3600 + 900 + 450

= 4950

We could also partition 90 in a number of ways:

55 x 90 = 55 x 55 + 25 x 55 + 10 x 55

= 3025 + 1375 + 550 (Note the trick for $55^2 = 50 \times 60 + 25 = 3025$)

= 4400 + 550

= 4950

55 x 90 = 55 x 55 + 20 x 55 + 10 x 55 + 5 x 55

= 3025 + 1100 + 550 + 275

= 4125 + 825

= 4950

55 x 90 = 55 x 100 – 55 x 10

= 5500 – 550

= 4950

55 x 90 = 55 x 50 + 55 x 40

= 5500 ÷ 2 + 550 x 4

= 2750 + 2200

= 4950

55 x 90 = 55 x 10 + 55 x 80 (double, double, double, shift for 80)

= 550 + 440 x 10

= 550 + 4400

= 4950

55 x 90 = (55 x 10) x 9 (by the factors)

= 550 x 9

= 550 x 10 – 550 x 1

= 5500 – 550

= 4950

55 x 90 = 11 x 450 (÷5 and x5) (Compensation)
= 4950 (using the 11 shortcut)

Taxonomy of Strategies

The categorization to be used in this study has seven classifications of mental multiplication strategies listed with examples above. Added to this are the simpler strategies of Basic Derived Fact and Counting to form a taxonomy of nine strategies:

- Basic Fact (BDF) – a multiplication fact such as 5 x 8 or derived fact such 50 x 80.
- Counting (COU) – any counting strategy including repeated addition, skip counting, doubling and doubling plus or minus.
- Basic Fact Shortcuts (BFS) includes shortcuts for multipliers from 2 to 12 and include some of Trachtenberg's shortcuts, particularly x11 and x12, teen multiples and squares ending in 5.
- Right Left Separated (RLS) - use of the standard algorithm to multiply from right to left by place value. This is not mentioned above as it is regarded as a poor mental multiplication strategy suitable only for single-digit multipliers.
- Left Right Separated (LRS) use of the Distributive Law to multiply from the highest place value to the lowest. Also known as Standard Partitioning.
- Factors (FAC) – one or both numbers being split into factors to multiply successively.
- Partitioning (PAR) – non-standard additive and subtractive distribution
- Compensation (CMP) – includes Aliquot parts and Derived (see examples)
- Combined (CMB) – a combination of two or more of the above

Table A.3

Taxonomy of Strategies Used in this Study with Examples

Strategy	Coding	Examples
Basic or Derived Fact	BDF	5 x 8 = 40, 50 x 8 = 400, 50 x 80 = 4000 30 x 70 = 2100 40 x 600 = 24 000
Counting – skip counting, doubling +/-	COU	3 x 24 = 24+24+24 = 48+24 =72 3x24=2x24+24 = 48+24 = 72
Basic Fact Shortcuts	BFS	x5 = ½ x 10 x8 = double, double, double x9 = x10 – x1 x11 = x10 + x1 x12 = x 11 + x1 Trachtenberg shortcuts, numbers in the same decile, squares ending in 5, near squares.
Right Left Separated	RLS	7x23: 7x3=21write 1 and carry 2, 7x2=14, 14+2=16. Anwer=161 or 7x 3 + 7 x 20 = 21 + 140 = 161
Left Right Separated	LRS	7x23: 7x20=140, 7x3=21, 140+21=161 127 x 4 = 100 x 4 + 20 x 4 + 7 x 4 = 400 + 80 +28 = 508
Factors	FAC	6x32: 6=2x3, 32x3=96, 96x2=192 32 x 12 = 32 x 4 x 3 = 128 x 3 = 360 + 24 = 384
Partitioning (nonstandard)	PAR	8 x 21 = 8x10+8x11 = 80 + 88 = 168 21x23 = 11x23 + 10x23 = 253 +230 = 483 19x34 = 20x34 – 1x34 = 680 – 34 = 646
Compensation	CMP	17x25 = 17x100 ÷ 4 = 425 15x24 = 30x12 = 360
Combination	CMB	21 x 23 = 21 x 25 – 2 x 21 = 525 – 42 = 483 (CMP + PAR)

This completes the lecture notes although the PowerPoint presentation then went on to consider the 15 questions from the first test and relate them to the strategies discussed and the taxonomy.

Initial Test Questions and Suggested Strategies

This section was a feature of the tutorial that looked at the relative merits of different strategies in achieving a solution in the given time. This deals with the adaptivity of particular solutions or the appropriate choice of strategy. Alternatives were offered for each question and the participants were left to decide the best pathway for them.

40 x 90 = 4 x 9 x 10 x 10 = 36 x 100 (shift right two and add two place holders). This is a BDF strategy or FAC.

5 x 8000 = 5 x 8 x 1000 = 40 x 1000 (shift right three and add three placeholders), BDF or FAC.

18 x 6 = 18 x 3 x 2 = 54 x 2 = 108 (FAC)
= 9 x 12 = 108 (CMP)
= 6 x 10 + 6 x 8 = 60 + 48 = 108 (LRS)
= 18 x 5 + 18 = 90 + 18 = 108 (PAR)

5 x 19 = 190 ÷ 2 = 95 (BFS)
= 5 x 20 – 5 = 100 – 5 = 95 (PAR)
= 5 x 10 + 5 x 9 = 50 + 45 = 95 (LRS)

126 x 4= 100 x 4 + 20 x 4 + 6 x 4 = 400 + 80 + 24 = 504 (LRS)
= 125 x 4 + 4 = 500 (double double) + 4 = 504 (PAR)
= 252 x 2 = 504 (double double) (BFS)

9 x 45 = 45-5 + nine complement = 405 (BFS)
= 10 x 45 – 45 = 450 – 45 = 405 (PAR)
= 9 x 40 + 9x 5 = 360 + 45 = 405 (LRS)

3 x 195= 3 x 200 – 3 x 5 = 600 – 15 = 585 (PAR)

= 3 x 100 + 3 x 90 + 3 x 5 = 300 + 270 + 15 = 585 (LRS)

143 x 7= 100 x 7 + 40 x 7 + 3 x 7 = 700 + 280 + 21 = 1001 (LRS)

= 143 x 5 + 143 x 2 = 1430 ÷ 2 + 286 = 715 + 286 = 1001 (PAR + BFS)

= 150 x 7 – 7 x 7 = 1050 – 49 = 1001 (PAR + BFS)

19 x 25 = 20 x 25 – 25 = 400 – 25 = 375 (PAR + CMP)

= 4 ¾ x 100 = 475 (CMP)

= 10 x 25 + 9 x 25 = 250 + 225 = 475 (LRS + CMP)

23 x 12 = Trachtenberg = 276 (BFS)

= 23 x 11 + 23 = 253 (Trachtenberg) + 23 = 276 (PAR + BFS)

= 23 x 10 + 23 x 2 = 230 + 46 = 276 (LRS)

= 46 x 6 = 92 x 3 = 276 (CMP)

15 x 18 = 180 + 90 = 270 (BFS)

= 30 x 9 = 270 (CMP + BDF)

= 10 x 15 + 8 x 15 = 150 + 120 (double x3) = 270 (LRS + BFS)

21 x 23 = 10 x 23 + 11 x 23 = 230 + 253 = 283 (PAR + BFS)

= 20 x 23 + 1 x 23 = 460 + 23 = 483 (LRS)

123 x 11 = Trachtenberg = 1353 (BFS)

= 123 x 10 + 123 = 1230 + 123 = 1353 (LRS)

45 x 24 = 90 x 12 = 1080 (CMP)

= 45 x 20 + 45 x 4 = 900 + 180 = 1080 (LRS)

= 25 x 24 + 20 x 24 = 600 + 480 = 1080 (PAR + CMP)

34 x 35 = 39 x 30 + 4 x 5 = 40 x 30 – 30 + 20 = 1200 – 10 =1190 (BFS + PAR)

= 70 x 17 = 10 x 70 + 7 x 70 = 700 + 490 = 1190 (CMP + LRS)

= 35^2 - 35 = 1225 – 35 = 1190 (BFS + PAR)

Appendix 6 - Ethics Approval

Ethics Office
Research Development & Integrity
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HUMAN RESEARCH ETHICS COMMITTEE

MEMORANDUM TO: Dr Robert Whannell, A/Prof Penelope Serow & Mr Peter Hall

School of Education

This is to advise you that the Human Research Ethics Committee has approved the following:

PROJECT TITLE:	Investigating strategies for one- and two-digit mental multiplication used by pre-service teachers.
APPROVAL No.:	HE18-135
COMMENCEMENT DATE:	26 June, 2018
APPROVAL VALID TO:	26 June, 2019
COMMENTS:	Nil. Conditions met in full

The Human Research Ethics Committee may grant approval for up to a maximum of three years. For approval periods greater than 12 months, researchers are required to submit an application for renewal at each twelve-month period. All researchers are required to submit a Final Report at the completion of their project. The Progress/Final Report Form is available at the following web address: http://www.une.edu.au/research/research-services/rdi/ethics/hre/hrec-forms

The NHMRC National Statement on Ethical Conduct in Research Involving Humans requires that researchers must report immediately to the Human Research Ethics Committee anything that might affect ethical acceptance of the protocol. This includes adverse reactions of participants, proposed changes in the protocol, and any other unforeseen events that might affect the continued ethical acceptability of the project.

In issuing this approval number, it is required that all data and consent forms are stored in a secure location for a minimum period of five years. These documents may be required for compliance audit processes during that time. If the location at which data and documentation are retained is changed within that five year period, the Research Ethics Officer should be advised of the new location.

[redacted]

Jo-Ann Sozou
Secretary/Research Ethics Officer

11/05/2018 A18/52

Appendix 7 Excerpts From Relevant Curriculum Documents

The Australian Curriculum (ACARA, 2012)

The three-dimensional design of the Foundation – Year 10 Australian Curriculum recognises the importance of disciplinary knowledge, skills and understanding alongside general capabilities and cross-curriculum priorities.

Disciplinary knowledge, skills and understanding are described in the eight learning areas of the Australian Curriculum: English, Mathematics, Science, Health and Physical Education, Humanities and Social Sciences, The Arts, Technologies and Languages. The latter four learning areas have been written to include multiple subjects, reflecting custom and practice in the discipline. In each learning area or subject, content descriptions specify what young people will learn, and achievement standards describe the depth of understanding and the sophistication of knowledge and skill expected of students at the end of each year level or band of years.

General capabilities comprise one of the three dimensions of the Australian Curriculum. They encompass knowledge, skills, behaviours and dispositions that, together with curriculum content in each learning area and the cross-curriculum priorities, assist students to live and work successfully in the twenty-first century. They play a significant role in realising the goals set out in the Melbourne Declaration on Educational Goals for Young Australians (MCEETYA) 2008 that all young people in Australia should be supported to become successful learners, confident and creative individuals, and active and informed citizens.
The Australian Curriculum includes seven general capabilities. These are:

- Literacy
- Numeracy
- Information and communication technology capability
- Critical and creative thinking
- Personal and social capability
- Ethical understanding
- Intercultural understanding.

In the Australian Curriculum, general capabilities are addressed through the learning areas and are identified wherever they are developed or applied in content descriptions. They are also identified where they offer opportunities to add depth and richness to student learning in content elaborations.

The Australian Curriculum gives special attention to three cross-curriculum priorities:

- Aboriginal and Torres Strait Islander histories and cultures
- Asia and Australia's engagement with Asia
- Sustainability

The cross-curriculum priorities are embedded in the curriculum and will have a strong but varying presence depending on their relevance to each of the learning areas.

Of particular relevance to this study is the General Capability of Numeracy

Numeracy

In the Australian Curriculum, students become numerate as they develop the knowledge and skills to use mathematics confidently across other learning areas at school and in their lives more broadly. Numeracy encompasses the knowledge, skills, behaviours and dispositions that students need to use mathematics in a wide range of situations. It involves students recognising and understanding the role of mathematics in the world and having the dispositions and capacities to use mathematical knowledge and skills purposefully.

When teachers identify numeracy demands across the curriculum, students have opportunities to transfer their mathematical knowledge and skills to contexts outside the mathematics classroom. These opportunities help students recognise the interconnected nature of mathematical knowledge, other learning areas and the wider world, and encourage them to use their mathematical skills broadly.

▦ This icon shows where Numeracy has been identified in learning area content descriptions and elaborations.

Key ideas

The key ideas for Numeracy are organised into six interrelated elements in the learning continuum, as shown in Figure A.1

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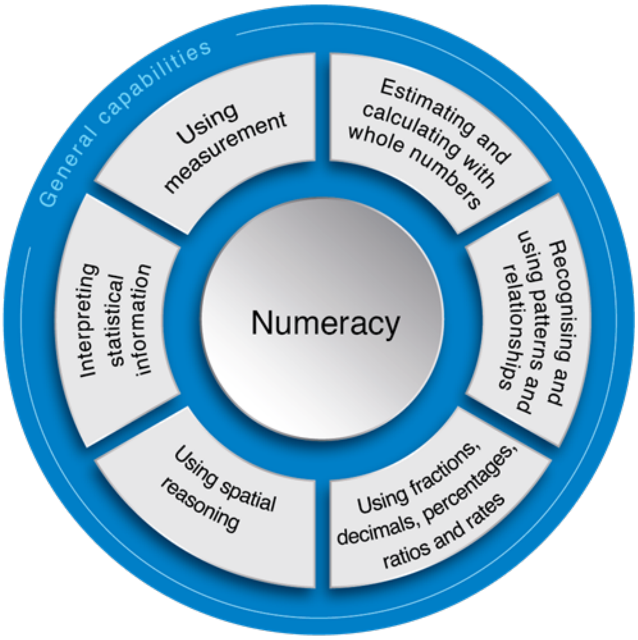


Figure A.1 Key Elements of Numeracy

At the Year 5 Level, the key element of estimating and calculating with whole numbers requires students to solve complex problems by estimating and calculating using efficient mental, written and digital strategies.

This is reflected in the Year 5 outcome ACMNA 100, which asks students to:

Solve problems involving multiplication of large numbers by one- or two-digit numbers using efficient mental, written strategies and appropriate digital technologies

Elaborations

- exploring techniques for multiplication such as the area model, the Italian lattice method or the partitioning of numbers
- applying the distributive law and using arrays to model multiplication and explain calculation strategies

The Australian Curriculum: Mathematics aims to be relevant and applicable to the 21st century. The inclusion of the proficiencies of understanding, fluency, problem-solving and reasoning in the curriculum is to ensure that student learning and student independence are at the centre of the curriculum. The curriculum focuses on developing increasingly sophisticated and refined mathematical understanding, fluency, reasoning, and problem-solving skills. These proficiencies enable students to respond to familiar and unfamiliar situations by employing mathematical strategies to make informed decisions and solve problems efficiently.

The proficiency strands describe the actions in which students can engage when learning and using the content of the Australian Curriculum: Mathematics.

Understanding

Students build a robust knowledge of adaptable and transferable mathematical concepts. They make connections between related concepts and progressively apply the familiar to develop new ideas. They develop an understanding of the relationship between the 'why' and the 'how' of mathematics. Students build understanding when they connect related ideas, when they represent concepts in different ways, when they identify commonalities and differences between aspects of content, when they describe their thinking mathematically and when they interpret mathematical information

Fluency

Students develop skills in choosing appropriate procedures; carrying out procedures flexibly, accurately, efficiently and appropriately; and recalling factual knowledge and concepts readily. Students are fluent when they calculate answers efficiently, when they recognise robust ways of answering questions, when they choose appropriate methods and approximations, when they recall definitions and regularly use facts, and when they can manipulate expressions and equations to find solutions.

Problem Solving

Students develop the ability to make choices, interpret, formulate, model and investigate problem situations, and communicate solutions effectively. Students formulate and solve problems when they use mathematics to represent unfamiliar or meaningful situations, when they design investigations and plan their approaches, when they apply their existing strategies to seek solutions, and when they verify that their answers are reasonable.

Reasoning

Students develop an increasingly sophisticated capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising. Students are reasoning mathematically when they explain their thinking, when they deduce and justify strategies used and conclusions reached, when they adapt the known to the unknown, when they transfer learning from one context to another, when they prove that something is true or false, and when they compare and contrast related ideas and explain their choices.

Fluency

In Years 3–6, students become fluent as they develop skills in choosing appropriate procedures; carrying out procedures flexibly and accurately; and recalling factual knowledge and concepts readily. Students are fluent when they calculate answers efficiently, when they recognise robust ways of answering questions, and when they recall definitions and regularly use facts.

The National Numeracy Learning Progression (ACARA, 2018)

Numeracy development influences student success in many areas of learning at school. The progression can be used to support students to successfully engage with the numeracy demands of the Foundation to Year 10 Australian Curriculum.

The National Numeracy Learning Progression outlines a sequence of observable indicators of increasingly sophisticated understanding of and skills in key numeracy concepts. By providing a comprehensive view of numeracy learning and how it develops over time, the progression gives teachers a conceptual tool that can assist them to develop targeted teaching and learning programs for students who are working above or below year-level expectations.

Of particular relevance to this study is the sub-element of multiplicative strategies which describes how a student becomes increasingly able to use multiplicative strategies in computation. The coordination of units multiplicatively involves using the values of one unit applied to each of the units of the other, the multiplier. This process of coordinating units is equally relevant to problems of division.

The indicators that are relevant to this study are contained in Mus6 and Mus7.

MuS6

Flexible strategies for multiplication

- draws on the structure of multiplication to use known multiples in calculating related multiples (uses multiples of 4 to calculate multiples of 8)
- uses known single-digit multiplication facts (7 boxes of 6 donuts is 42 donuts altogether because 7 x 6 = 42)
- applies known facts and strategies for multiplication to mentally calculate (3 sixes is 'double 6' plus 1 more row of 6, 5 x 19 is half of 10 x 19 or 5 x 19 is 5 x 20 take away 5)
- uses commutative properties of numbers (5 x 6 is the same as 6 x 5)

MuS7

Flexible number properties

- uses multiplication and division as inverse operations
- uses factors of a number to carry out multiplication and division (to multiply a number by 72, first multiply by 12 and then multiply the result by 6)
- uses knowledge of distributive property of multiplication over addition (7 x 83 equals 7 x 80 plus 7 x 3)
- uses decomposition into hundreds, tens and ones to calculate using partial products with numbers of any size (327 x 14 is equal to 4 x 327 plus 10 x 327)
- uses estimation and rounding to check the reasonableness of products and quotients

NSW Educational Standards Authority (NESA, 2015)

K to 10 Mathematics Syllabus
Course Description

Mathematics is used to identify, describe and apply patterns and relationships. It provides a precise means of communication and is a powerful tool for solving problems both within and beyond mathematics. Mathematical ideas are constantly developing, and mathematics is integral to scientific and technological advances in many fields of endeavour. Digital technologies provide access to new tools for continuing mathematical exploration and invention. In addition to its practical applications, the study of mathematics is a valuable pursuit in its own right, providing opportunities for originality, challenge and leisure.

Mathematics in Years 7–10 focuses on developing increasingly sophisticated and refined mathematical understanding, fluency, communication, logical reasoning, analytical thought and problem-solving skills. These capabilities enable students to respond to familiar and unfamiliar situations by employing strategies to make informed decisions and solve problems relevant to their further education and everyday lives.

Students develop understanding and fluency in mathematics through inquiry, exploring and connecting mathematical concepts, choosing and applying problem-solving skills and mathematical techniques, communication, and reasoning.

They study Number and Algebra, Measurement and Geometry, and Statistics and Probability. Within these strands they will cover a range of topic areas including: financial mathematics, algebraic techniques, equations, linear and non-linear relationships, surface area and volume, properties of geometrical figures, trigonometry, data collection and representation, data analysis, and probability.

Organisation of Content

For Kindergarten to Year 10, courses of study and educational programs are based on the outcomes of syllabuses. The content describes in more detail how the outcomes are to be interpreted and used, and the intended learning appropriate for the stage. In considering the intended learning, teachers will make decisions about the sequence, the emphasis to be given to particular areas of content, and any adjustments required based on the needs, interests and abilities of their students.

The knowledge, skills and understanding described in the outcomes and content provide a sound basis for students to successfully move to the next stage of learning.

Figure A.2 shows the scope of the strands and substrands, and illustrates the central role of Working Mathematically in Mathematics K–10 teaching and learning.

Figure A.2 *Strands and Sub-strands of the K-10 NESA Mathematics Syllabus*

The content presented in a stage represents the knowledge, skills and understanding that are to be acquired by a typical student by the end of that stage. It is acknowledged that students learn at different rates and in different ways, so that there will be students who have not achieved the outcomes for the stage(s) prior to that identified with their stage of schooling.

For example, some students will achieve Stage 3 outcomes during Year 5, while the majority will achieve them by the end of Year 6. Other students might not develop the same knowledge, skills and understanding until Year 7 or later.

The Mathematics K–10 Syllabus contains the syllabus content for Early Stage 1 to Stage 5. Within each stage, the syllabus is organised into the three content strands, Number and Algebra, Measurement and Geometry, and Statistics and Probability, with the components of Working Mathematically integrated into these strands. The syllabus is written with the flexibility to enable students to work at different stages in different strands. For example, students could be working on Stage 4 content in the Number and Algebra strand, while working on Stage 3 content in the Measurement and Geometry strand.

Outcomes, content, background information, and advice about language are organised into substrands within the three content strands. There are some substrands, mainly in Early Stage 1 to Stage 3, that contain the development of several concepts. To assist programming, the content in these substrands has been separated into two parts, '1' and '2', such as in 'Area 1' and 'Area 2'. The first part typically focuses on early concept development. Teachers and schools need to decide how to program the two parts of these substrands within a stage.

In Early Stage 1 to Stage 3, the language section of each substrand includes a word list. Words appearing for the first time in each substrand are listed in bold type. In Stage 4 and Stage 5, the background information includes the purpose/relevance of the substrands.

Course Performance Descriptors

Areas for Assessment

Knowledge, skills and understanding

Students:

Working Mathematically – develop understanding and fluency in mathematics through inquiry, exploring and connecting mathematical concepts, choosing and applying problem-solving skills and mathematical techniques, communication and reasoning

Number and Algebra – develop efficient strategies for numerical calculation, recognise patterns, describe relationships and apply algebraic techniques and generalisation

Measurement and Geometry – identify, visualise and quantify measures and the attributes of shapes and objects, and explore measurement concepts and geometric relationships, applying formulas, strategies and geometric reasoning in the solution of problems

Statistics and Probability – collect, represent, analyse, interpret and evaluate data, assign and use probabilities, and make sound judgements.

Strand overview: Number and Algebra

The knowledge, skills and understanding developed in the Number and Algebra strand are fundamental to the other strands of this syllabus and are developed across the stages from Early Stage 1 to Stage 5.3.

Numbers, in their various forms, are used to quantify and describe the world. From Early Stage 1 there is an emphasis on the development of number sense, and confidence and competence in using concrete materials and mental, written and calculator techniques for solving appropriate problems. Algorithms are introduced after students have gained a firm understanding of basic concepts, including place value, and have developed mental strategies for computing with two- and three-digit numbers. Approximation is important and the systematic use of estimation is to be encouraged at all times. Students should always check that their answers 'make sense' in the contexts of the problems that they are solving.

In the early stages, students explore number and pre-algebra concepts by pattern making, and by discussing, generalising and recording their observations. This demonstrates the importance of early number learning in the development of algebraic thinking and the algebra concepts
that follow.

The use of mental-computation strategies should be developed in all stages. Information and communication technology (ICT) can be used to investigate number patterns and relationships, and facilitate the solution of real problems involving measurements and quantities not easy to handle with mental or written techniques.

In Stage 2 to Stage 5, students apply their number skills to a variety of situations, including financial situations and practical problems, developing a range of life skills important for numeracy. Ratios and rates underpin proportional reasoning needed for problem solving and the development of concepts and skills in other aspects of mathematics, such as trigonometry, similarity and gradient.

Following the development of foundational number skills and pre-algebra concepts through patterning, a concrete approach to algebra is continued when students generalise their understanding of the number system to the algebraic symbol system. They develop an understanding of the notion of a variable, establish the equivalence of expressions, apply algebraic conventions, and graph relationships on the number plane.

Students recognise that graphing is a powerful tool that enables algebraic relationships to be visualised. The use of ICT for graphing provides an opportunity for students to compare and investigate these relationships dynamically. By the end of Stage 5.3, students have the opportunity to develop knowledge and understanding of the shapes of graphs of different relationships and the effects of performing transformations on these graphs.

Algebra has strong links with the other strands in the syllabus, particularly when situations are to be generalised symbolically.

Stage 3 Outcomes Related to This Study

A student:

MA3-1WM

describes and represents mathematical situations in a variety of ways using mathematical terminology and some conventions

MA3-2WM

selects and applies appropriate problem-solving strategies, including the use of digital technologies, in undertaking investigations

MA3-3WM

gives a valid reason for supporting one possible solution over another

MA3-6NA

selects and applies appropriate strategies for multiplication and division, and applies the order of operations to calculations involving more than one operation

MA3-6NA – Content

Students:

- Solve problems involving multiplication of large numbers by one- or two-digit numbers using efficient mental and written strategies and appropriate digital technologies(ACMNA100)

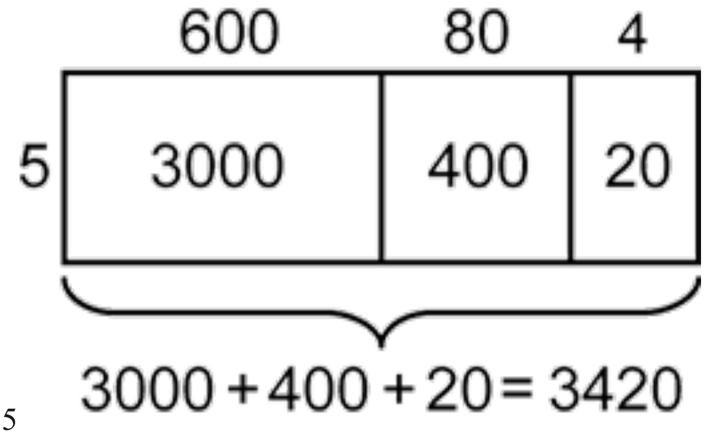
- use mental and written strategies to multiply three- and four-digit numbers by one-digit numbers, including:
- multiplying the thousands, then the hundreds, then the tens and then the ones, eg 673×4=(600×4)+(70×4)+(3×4)=2400+280+12=2692

- 673× =(600×4)+(70×4)+(3×4)

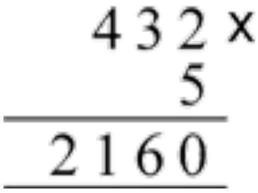
  4

  =2400+280+12

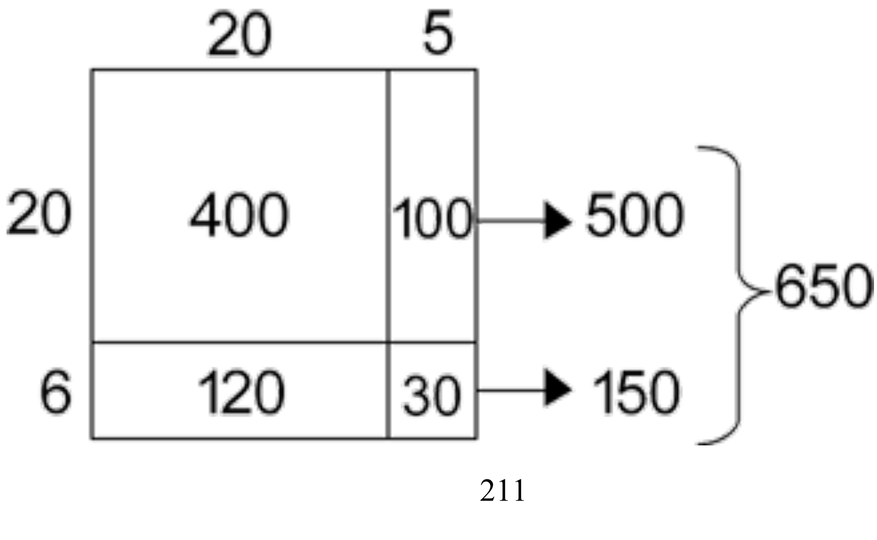
  =2692



- using an area model, eg 684 × 5



- using the formal algorithm, eg 432 × 5
- use mental and written strategies to multiply two- and three-digit numbers by two-digit numbers, including:
- using an area model for two-digit by two-digit multiplication, eg 25 × 26



- factorising the numbers, eg 12 × 25 = 3 × 4 × 25 = 3 × 100 = 300
- using the extended form (long multiplication) of the formal algorithm,

$$\begin{array}{r} 521\ \times \\ 22 \\ \hline 1042 \\ 10420 \\ \hline 11462 \\ \hline \end{array}$$

- use digital technologies to multiply numbers of up to four digits
- check answers to mental calculations using digital technologies (Problem Solving)
- apply appropriate mental and written strategies, and digital technologies, to solve multiplication word problems
- use the appropriate operation when solving problems in real-life situations (Problem Solving)
- use inverse operations to justify solutions (Problem Solving, Reasoning)
- record the strategy used to solve multiplication word problems
- use selected words to describe each step of the solution process (Communicating, Problem Solving)

Background Information

Students could extend their recall of number facts beyond the multiplication facts to 10 × 10 by memorising multiples of numbers such as 11, 12, 15, 20 and 25. They could also utilise mental strategies, eg 14 × 6 is 10 sixes plus 4 sixes'.

In Stage 3, mental strategies need to be continually reinforced.

Students may find recording (writing out) informal mental strategies to be more efficient than using formal written algorithms, particularly in the case of multiplication.

An inverse operation is an operation that reverses the effect of the original operation. Addition and subtraction are inverse operations; multiplication and division are inverse operations.

The area model for two-digit by two-digit multiplication in Stage 3 is a precursor to the use of the area model for the expansion of binomial products in Stage 5.

Language

Students should be able to communicate using the following language: multiply, multiplied by, product, multiplication, multiplication facts, **area**, **thousands**, **hundreds**, tens, ones, double, multiple, factor, divide, divided by, **quotient**, division, halve, remainder, **fraction**, **decimal**, equals, strategy, digit, **estimate**, **round to**.

In mathematics, 'quotient' refers to the result of dividing one number by another.

Teachers should model and use a variety of expressions for multiplication and division. They should draw students' attention to the fact that the words used for division may require the operation to be performed with the numbers in the reverse order to that in which they are stated in the question. For example, 'divide 6 by 2' and '6 divided by 2' require the operation to be performed with the numbers in the same order as they are presented in the question (ie $6 \div 2$). However, 'How many 2s in 6?' requires the operation to be performed with the numbers in the reverse order to that in which they are stated in the question (ie $6 \div 2$).

The terms 'ratio' and 'rate' are not introduced until Stage 4, but students need to be able to interpret problems involving simple rates as requiring multiplication or division.

Working Mathematically

Working Mathematically relates to the syllabus objective:

Students develop understanding and fluency in mathematics through inquiry, exploring and connecting mathematical concepts, choosing and applying problem-solving skills and mathematical techniques, communication and reasoning

As an essential part of the learning process, Working Mathematically provides students with the opportunity to engage in genuine mathematical activity and develop the skills to become flexible and creative users of mathematics.

In this syllabus, Working Mathematically encompasses five interrelated components:

1. **Communicating**

Students develop the ability to use a variety of representations, in written, oral or graphical form, to formulate and express mathematical ideas. They are communicating mathematically when they describe, represent and explain mathematical situations, concepts, methods and solutions to problems, using appropriate language, terminology, tables, diagrams, graphs, symbols, notation and conventions.

2. **Problem Solving**

Students develop the ability to make choices, interpret, formulate, model and investigate problem situations, and communicate solutions effectively. They formulate and solve problems when they use mathematics to represent unfamiliar or meaningful situations, design investigations and plan their approaches, apply strategies to seek solutions, and verify that their answers are reasonable.

3. **Reasoning**

Students develop an increasingly sophisticated capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising. They are reasoning mathematically when they explain their thinking, deduce and justify strategies used and conclusions reached, adapt the known to the unknown, transfer learning from one context to another, prove that something is true or false, and compare and contrast related ideas and explain their choices.

4. **Understanding**

Students build a strong foundation that enables them to adapt and transfer mathematical concepts. They make connections between related concepts and progressively apply the familiar to develop new ideas. Students develop an understanding of the relationship between the 'why' and the 'how' of mathematics.

They build understanding when they connect related ideas, represent concepts in different ways, identify commonalities and differences between aspects of content, describe their thinking mathematically, and interpret mathematical information.

5. **Fluency**

Students develop skills in choosing appropriate procedures, carrying out procedures flexibly, accurately, efficiently and appropriately, and recalling factual knowledge and concepts readily. They are fluent when they calculate answers efficiently, recognise robust ways of answering questions, choose appropriate methods and approximations, recall definitions and regularly use facts, and manipulate expressions and equations to find solutions.

The five components of Working Mathematically describe how content is explored or developed – that is, the thinking and doing of mathematics. They provide the language to build in the developmental aspects of the learning of mathematics. The components come into play when students are developing new skills and concepts, and also when they are applying their existing knowledge to solve routine and non-routine problems both within and beyond mathematics. At times the focus may be on a particular component of Working Mathematically or a group of the components, but often the components overlap. While not all of the Working Mathematically components apply to all content, they indicate the breadth of mathematical actions that teachers need to emphasise.

In addition to its explicit link to one syllabus objective, Working Mathematically has a separate set of outcomes for the components Communicating, Problem Solving and Reasoning. This approach has been adopted to ensure students' level of proficiency in relation to these components becomes increasingly sophisticated over the years of schooling.

Separate syllabus outcomes have not been developed for the Working Mathematically components Understanding and Fluency. These components are encompassed in the development of knowledge, skills and understanding across the range of syllabus strands, objectives and outcomes.

Teachers are able to extend students' level of proficiency in relation to the components of Working Mathematically by creating opportunities for their development through the learning experiences that they design.